This article covers the complete derivation of the liquid diffusivity equation. Special attention is placed on explaining how the derivation components are incorporated, along with the associated limitations of the described derivation. Intended for the young student or experienced engineer, the supplementary context and derivation walk-through is complete, descriptive, and most importantly, brief!

It seems trivial to say, but indeed it is necessary to recall: Oil and gas companies exist with the purpose of making money (read: profit) through the production and sale of hydrocarbon products in a responsible manner.

It is obvious, then, why petroleum engineers are overwhelmingly concerned with the projected volumes to be produced from a well. To determine reasonable volume projections, reservoir characterization along with reservoir-wellbore behaviors should be developed and studied.

In layman's terms, the path that a particle of fluid takes to travel from regions within the reservoir to the wellbore is of great value to engineers. The diffusivity equation describes this phenomenon in detail. Additionally, the reservoir-wellbore system can be further studied (and dare I say… "optimized") by analyzing the well deliverability effects as a result of differing drawdown strategies --- a true production engineer's dream. In summary, the diffusivity equation is a powerful tool for engineers, but it should be thoroughly understood, along with its assumptions and limitations.

Before delving into the complete diffusivity equation derivation, the general diffusivity equation for slightly compressible liquids is presented (**Eq. 1**). Following, a discussion of the components utilized in this derivation and their importance is developed.

$$\nabla^2 p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \quad (1)$$

To make sense of this, it is important that the reader recall the following three equations:

- Mass continuity equation
- Darcy's law for describing motion (originally for hydraulics)
- Fluid equation of state (assuming liquid fluid that is slightly compressible)

But how do these relate?

The continuity equation is a logical starting point for this derivation because it describes all mass transfer into and out of a closed volume (**Eq. 2**). The reader should note that this is precisely the Law of Conservation of Mass that is widely utilized in classical mechanics. For our purposes, the closed volume in consideration is the hydrocarbon-bearing reservoir and no assumptions will be made regarding the shape of the reservoir.

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial(\phi \rho)}{\partial t} \quad (2)$$

To describe the rate at which mass is transferred between the control volume and wellbore, an equation of motion is required. Darcy's law will be the equation utilized to describe the fluid velocity upon which mass passes through a control volume's area (**Eq. 3**). Those who are familiar with heat conduction or wave propagation will note that this is precisely the flux property.

$$\boldsymbol{v} = -\frac{k}{\mu}(\nabla p + \rho \boldsymbol{g}) \tag{3}$$

The assumptions that are inherent to Darcy's law still apply. Of primary concern are the following assumptions:

- Fluid is assumed to be Newtonian in behavior
- Gravity effects on the fluid will be neglected
- Fluid flow velocity will be dependent on pressure gradient
- Fluid viscosity is considered constant
- Permeability is considered constant

Finally, the fluid equation of state will allow for final reductions of the diffusivity equation. It is also this equation of state (EOS) that can impose great assumptions that may be limiting if the engineer is not careful in its application. Recall that for this derivation the liquid fluid case will be considered (**Eq. 4**). This case is most appropriate for a fluid that is slightly compressible (e.g. an oil reservoir that is being operated above the bubble point pressure). This point cannot be overstated, and the engineer should not overlook this consideration.

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \tag{4}$$

The reader will note that **Eq. 4** introduces the fluid compressibility definition as a function of both density and pressure. Furthermore, it is seen that the fluid density itself is a function of pressure. Taking it another step, notice that the fluid density is involved in both the mass continuity and fluid motion equation. Through sound calculus techniques, these fluid density terms will be accounted for with the defined EOS. This is yet one more proof of the importance of properly selecting fluid EOS once the engineer thoroughly understands the reservoir fluid.

With this brief orientation, we are now ready to apply fundamental calculus and algebra skills to formally derive the general diffusivity equation for the liquid case. Albeit one of the simplest derivations in petroleum engineering, this could be the most important one. This is the beauty of getting…

…Back to the Basics.

The derivation of the liquid diffusivity equation is based principally on the mass continuity equation. This is a powerful equation because, as we know, mass balance must always be honored.

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial(\phi\rho)}{\partial t}$$

To describe the fluid motion, Darcy's law (in gradient form) is substituted into the mass continuity equation

$$\nabla \cdot \left[\rho \frac{k}{\mu}(\nabla p + \rho \boldsymbol{g})\right] = \frac{\partial(\phi\rho)}{\partial t}$$

While in the reservoir, fluid may tend to flow horizontally to the wellbore. Therefore, it is reasonable to neglect the gravity term from Darcy's law

$$\nabla \cdot \left[\frac{\rho k}{\mu} \nabla p\right] = \frac{\partial(\phi\rho)}{\partial t}$$

Continuing, it is common to assume that the permeability and fluid viscosity are constant for liquid flow. Declaring this, the two parameters may be taken outside of the del operator as shown

$$\nabla \cdot [\rho \nabla p] = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Implementing the product rule, the left-hand side of the equation may be expanded

$$(\nabla \cdot \rho)\nabla p + \rho(\nabla \cdot \nabla p) = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Continuing focus on the left-hand side of the equation, the expression may be rewritten equivalently as

$$\left(\frac{\partial \rho}{\partial p}\nabla p\right)\nabla p + \rho \nabla^2 p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Further rewriting the left-hand side yields

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Shifting focus to the right-hand side of the equation now, the differential term may be expanded by way of the chain rule

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial p}\frac{\partial p}{\partial t}$$

Note that the use of the chain rule to expand the equation is very important. By doing so, we see that the porosity and fluid density are functions of pressure. Many studies have been focused around this area, so we should continue to work the right-hand side of the equation to get it to a form that we can identify. Thus, applying the product rule to the right-hand side yields

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\left[\phi \frac{\partial \rho}{\partial p} + \rho \frac{\partial \phi}{\partial p}\right]\frac{\partial p}{\partial t}$$

Maintaining focus on the right-hand side of the equation, the porosity and density terms may be factored out of the brackets in order to yield a familiar form that is recognizable

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\phi\rho\left[\frac{1}{\rho}\frac{\partial \rho}{\partial p} + \frac{1}{\phi}\frac{\partial \phi}{\partial p}\right]\frac{\partial p}{\partial t}$$

At this point, careful attention and inspection should be focused on the right-hand side of the equation. To this point, the mass continuity and fluid motion equations have been leveraged. However, the details surrounding the specific fluid and the reservoir conditions must be incorporated. For this liquid diffusivity derivation an equation of state for a slightly compressible liquid will be utilized.

Recall from hydrocarbon phase behavior and reservoir petrophysics, the equations for fluid compressibility and pore-volume compressibility, respectively,

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

$$c_f = \frac{1}{\phi}\frac{\partial \phi}{\partial p}$$

Substituting these two expressions into the working equation yields

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\phi\rho\left[c + c_f\right]\frac{\partial p}{\partial t}$$

Lumping the compressibility terms together, total compressibility (c_t) is now defined

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\phi\rho c_t\frac{\partial p}{\partial t}$$

Dividing through by the fluid density, ρ, yields

$$\frac{1}{\rho}\frac{\partial \rho}{\partial p}(\nabla p)^2 + \nabla^2 p = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

Notice that the first term of the left-hand side can be rewritten in terms of the fluid compressibility. This rewrite yields the final form of the diffusivity equation for slightly compressible liquids.

$$c(\nabla p)^2 + \nabla^2 p = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

Although this is the complete form, specific attention is directed to the left-hand side of the equation where the pressure gradient term is raised to the second power. Recalling properties of differential equations, it is noted that when a variable (i.e. pressure) is raised to a power other than 1, it is nonlinear. Further, being that the term in question is the gradient of pressure already shows that the expression is non-linear (gradient is not linear).

This key distinction is outlined because very few nonlinear systems can be solved directly through analytical methods. Thus, the following two assumptions are made:

- Liquid compressibility is small and constant (i.e. not a function of pressure)
- Small gradients

These two assumptions lead to the following rewrite of the diffusivity equation for liquids. Notice that now the nonlinearity is not present on the left-hand side. It is this form of the diffusivity equation (in its most general sense) that is utilized as a starting point in classic analytical solutions.

$$\nabla^2 p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t}$$

--- End of article

Appendix A. Mass Continuity Equation

Consider the following control volume:

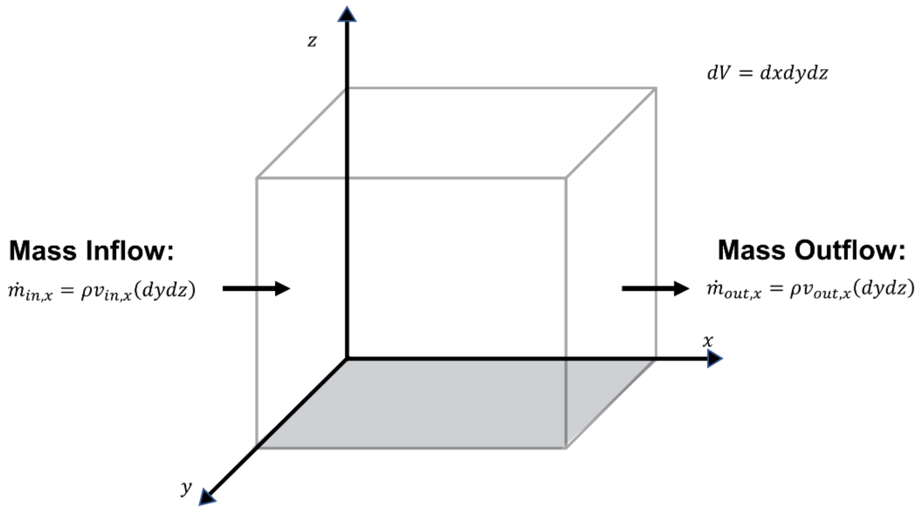

Fig. A1 – Control volume illustrating mass transport through 1-D system in the x-direction.

Defining the expression that governs the control volume magnitude

$$dV = dx * dy * dz$$

Recalling the form of density

$$\rho = \frac{mass}{volume} = \frac{m}{V}$$

Because fluid flow is dynamic, we must consider mass changes with respect to time. Recall that the product of velocity and area define volumetric flow rate as shown

$$\dot{V} = v * A$$

Being that mass flow rate is the primary target, relating density with volumetric flow rate yields the following useful expression

$$\dot{m} = \rho \dot{V}$$

Applying these concepts to the control volume depiction, consider first the mass flowing into the control volume from the left with respect to time

$$\dot{m}_{in,x} = \rho v_{in,x}(dydz)$$

Similarly, the mass flowing out of the control volume to the right

$$\dot{m}_{out,x} = \rho v_{out,x}(dydz)$$

Note that these two expressions describe flow in the x-direction. If the steady-state condition for the control volume is considered, the magnitude of mass transferred into the system would be equal to the mass transferred out of the system for each point in time. In form of an expression that is

$$\rho v_{in,x}(dydz) = \rho v_{out,x}(dydz)$$

Or similarly

$$\rho v_{in,x}(dydz) - \rho v_{out,x}(dydz) = 0$$

To reaffirm the concept of steady-state in this expression, notice that the left-hand side of the expression is set equal to zero. Physically, this means that there is zero accumulation present in the system. For completeness, this assumption will not be carried throughout this derivation.

Before continuing, the mass transfer equations for the y- and z-direction are presented as follows

$$\dot{m}_{in,y} = \rho v_{in,y}(dxdz)$$

$$\dot{m}_{out,y} = \rho v_{out,y}(dxdz)$$

$$\dot{m}_{in,z} = \rho v_{in,z}(dxdy)$$

$$\dot{m}_{out,z} = \rho v_{out,z}(dxdy)$$

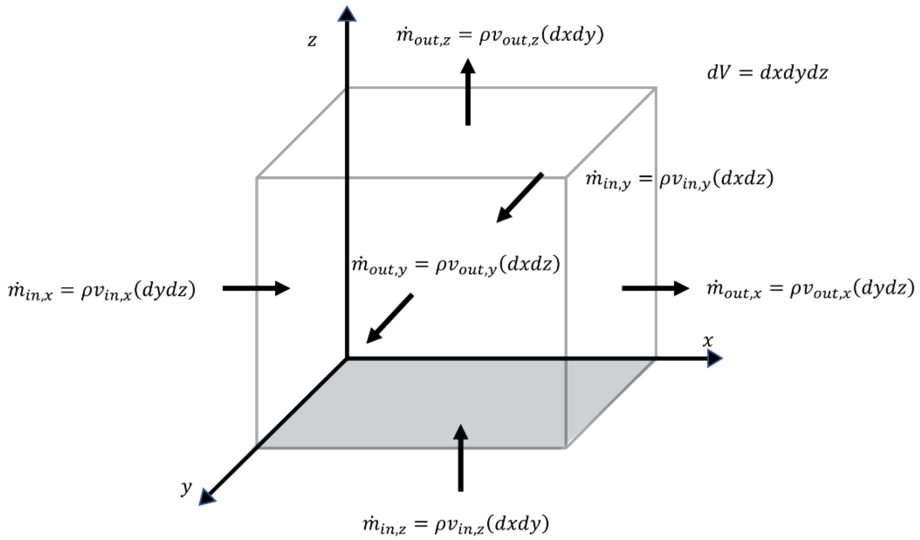

Fig. A2 – Control volume illustrating mass transport through 3-D system in all positive directions.

Accounting for mass transfer into and out of the system (+ in, - out) across all boundaries, the general form of the mass accumulation term with respect to time may be expressed as

$$\rho v_{in,x}(dydz) + \rho v_{in,y}(dxdz) + \rho v_{in,z}(dxdy) - \rho v_{out,x}(dydz) - \rho v_{out,y}(dxdz) - \rho v_{out,z}(dxdy) = \frac{\partial}{\partial t}(\rho dV)$$

Recall that $dV = dxdydz$. Dividing through by dV yields

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial \rho}{\partial t}$$

Note that this form now expresses fluid flow in terms of mass flux. Multiplying through by negative one yields an equivalent form that is familiar

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

Recall the gradient operator of the form

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Applying the gradient operator to the velocity vector defined by the fluid flow (divergence)

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Rewriting the mass continuity equation with the velocity divergence then becomes

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial \rho}{\partial t}$$

This is a simple way of expressing the mass continuity equation in its most general form. For petroleum engineers, the slight modification that is necessary relates to the mass accumulation term. Due to porous media being --- well --- porous, it is necessary to include the porosity term in the expression. Thus, the mass continuity equation utilized is

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial(\phi \rho)}{\partial t}$$

Appendix B. Equation of State (EOS) for Slightly Compressible Fluid

Beginning with the definition of fluid compressibility under isothermal conditions

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

For the slightly compressible liquid case, the fluid compressibility can be assumed constant. In other words, the fluid compressibility is not a function of pressure. Rewriting the fluid compressibility equation yields

$$c = \frac{1}{\rho}\frac{d\rho}{dp}$$

Note that the equation is now able to be expressed as a normal derivative with the prior assumptions implemented. Separating the differential terms yields

$$c\ dp = \frac{d\rho}{\rho}$$

Integration is then carried out between a reference pressure and a given pressure as shown below

$$\int_{p_0}^{p} c\ dp = \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$$

Integrating the expression yields the following expression. Note that for most purposes, the reference value of pressure is the bubble point pressure. Subsequently, the fluid density is that which is measured or computed at the bubble point pressure.

$$c(p - p_0) = \ln\left(\frac{\rho}{\rho_0}\right)$$

Rewriting the expression by implementing the inverse function of a natural log function

$$\exp[c(p - p_0)] = \exp\left[\ln\left(\frac{\rho}{\rho_0}\right)\right]$$

Reducing the expression and rearranging yields the final form of the density expression (EOS) for a slightly compressible liquid. Note that the expression is a weak, decaying exponential function with respect to pressure when above the bubble point (this is easily verifiable when plotting the function). For this reason, it is understandable why the compressibility term may be assumed constant in further derivations.

$$\rho = \rho_0 \exp[c(p - p_0)]$$

Appendix C. Fluid Geometry Descriptions for Diffusivity Equation

The reader is recommended to periodically review www.fa-jimenez.com as articles will be posted covering the vector calculus derivations associated with the following flow geometry descriptions.

Linear Flow:

Consider the gradient operator for cartesian coordinates

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Carrying out the dot product between the gradient operator and the gradient vector yields the Laplacian.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Note that the vector calculus carried out in this operation can be summarized as the divergence of the gradient function that is a vector. Rewriting the Laplacian for the pressure case yields the form necessary for the linear flow case diffusivity equation in cartesian coordinates:

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

As it relates to practicality and actual implementation, recall that a typical assumption of the diffusivity equation is horizontal flow. Additionally, there will be a coordinate axis that experiences the predominant flow of fluid (this is obviously dependent on the geometry of the well and the placement of the coordinate axes). Thus, when applied to the diffusivity equation, the geometry may be reduced and described as follows

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2}$$

Radial Flow:

First, it is noted that radial flow utilizes the cylindrical coordinate system to describe its geometry. That is,

- r = radial (Euclidean) distance extending outward from the z-axis
- θ = azimuth angle between a reference direction on a plane and a line's projection on the same plane
- z = height from the origin extending in the direction of the z-axis

Now, recall that the gradient operator for pressure in cylindrical coordinates is as shown

$$\nabla p = \frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{z}$$

Taking the gradient of the gradient operator yields the Laplacian operator as shown (i.e. divergence of gradient)

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}$$

For practicality and actual implementation, recall that a typical assumption of the diffusivity equation is horizontal flow. This will eliminate the need for the third term. Additionally, the tangential flow is normally not considered since it is assumed that the only heterogeneity that is experienced (if any) is based on depth into the reservoir from the wellbore. Thus, the working form of the geometry description for the radial flow case is as follows

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)$$

Relevant Readings

1. https://blasingame.engr.tamu.edu/
2. https://en.wikipedia.org/wiki/Cylindrical_coordinate_system
3. https://en.wikipedia.org/wiki/Del
4. https://www.projectrhea.org/rhea/index.php/Vector_Derivatives_Cylindrical_Coordinates
5. https://www.continuummechanics.org/continuityequation.html#:~:text=0%20are%20unknown.-,Conservation%20of%20Mass,change%20of%20mass%20within%20it.