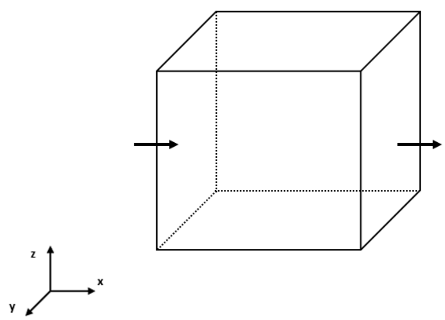
Mass Continuity Equation (Reviewed):

Consider the following control volume



Defining the expression that governs the control volume magnitude

$dV = dx * dy * dz$

Recalling the form of density

$\rho = \frac{mass}{volume} = \frac{m}{V}$

Because fluid flow is dynamic, we must consider mass changes with respect to time. Recall that the product of velocity and area define volumetric flow rate as shown

$\dot{V} = v * A$

Being that mass flow rate is the primary target, relating density with volumetric flow rate yields the following useful expression

$\dot{m} = \rho\dot{V}$

Applying these concepts to the control volume depiction, consider first the mass flowing into the control volume from the left with respect to time

$\dot{m}_{in,x} = \rho v_{in,x}(dydz)$

Similarly, the mass flowing out of the control volume to the right

$\dot{m}_{out,x} = \rho v_{out,x}(dydz)$

Note that these two expressions describe flow in the x-direction. If the steady-state condition for the control volume is considered, the magnitude of mass transferred into the system would be equal to the mass transferred out of the system for each point in time. In form of an expression that is

$\rho v_{in,x}(dydz) = \rho v_{out,x}(dydz)$

Or similarly

$\rho v_{in,x}(dydz) - \rho v_{out,x}(dydz) = 0$

To reaffirm the concept of steady-state in this expression, notice that the left-hand side of the expression is set equal to zero. Physically, this means that there is zero accumulation present in the system. For completeness, this assumption will not be carried throughout this derivation. Before continuing, the mass transfer equations for the y- and z-direction are presented as follows

$$\dot{m}_{in,y} = \rho v_{in,y}(dxdz)$$

$$\dot{m}_{out,y} = \rho v_{out,y}(dxdz)$$

$$\dot{m}_{in,z} = \rho v_{in,z}(dxdy)$$

$$\dot{m}_{out,z} = \rho v_{out,z}(dxdy)$$

[INSERT] drawing here to show arrows of mass transfer for full 3-D

Accounting for mass transfer into and out of the system (+ in, - out) across all boundaries, the general form of the mass accumulation term with respect to time may be expressed as

$$\rho v_{in,x}(dydz) + \rho v_{in,y}(dxdz) + \rho v_{in,z}(dxdy) - \rho v_{out,x}(dydz) - \rho v_{out,y}(dxdz) - \rho v_{out,z}(dxdy) = \frac{\partial}{\partial t}(\rho dV)$$

Recall that $dV = dxdydz$. Dividing through by dV yields

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial \rho}{\partial t}$$

Note that this form now expresses fluid flow in terms of mass flux. Multiplying through by negative one yields an equivalent form that is familiar

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

Recall the gradient operator of the form

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Applying the gradient operator to the velocity vector defined by the fluid flow (divergence)

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Rewriting the mass continuity equation with the velocity divergence then becomes

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial \rho}{\partial t}$$

This is a simple way of expressing the mass continuity equation in its most general form. For petroleum engineers, the slight modification that is necessary relates to the mass accumulation term. Due to porous media being --- well --- porous, it is necessary to include the porosity term in the expression. Thus, the mass continuity equation utilized is

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial(\phi \rho)}{\partial t}$$

Equation of State (EOS) Derivation Review:

Beginning with the definition of fluid compressibility under isothermal conditions

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

For the slightly compressible liquid case, the fluid compressibility can be assumed constant. In other words, the fluid compressibility is not a function of pressure. Rewriting the fluid compressibility equation yields

$$c = \frac{1}{\rho}\frac{d\rho}{dp}$$

Note that the equation is now able to be expressed as a normal derivative with the prior assumptions implemented. Separating the differential terms yields

$$c\ dp = \frac{d\rho}{\rho}$$

Integration is then carried out between a reference pressure and a given pressure as shown below

$$\int_{p_0}^{p} c\ dp = \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$$

Integrating the expression yields the following expression. Note that for most purposes, the reference value of pressure is the bubble point pressure. Subsequently, the fluid density is that which is measured or computed at the bubble point pressure.

$$c(p - p_0) = \ln\left(\frac{\rho}{\rho_0}\right)$$

Rewriting the expression by implementing the inverse function of a natural log function

$$\exp[c(p - p_0)] = \exp\left[\ln\left(\frac{\rho}{\rho_0}\right)\right]$$

Reducing the expression and rearranging yields the final form of the density expression (EOS) for a slightly compressible liquid. Note that the expression is a weak, decaying exponential function with respect to pressure when above the bubble point (this is easily verifiable when plotting the function). For this reason, it is understandable why the compressibility term may be assumed constant in further derivations.

$$\rho = \rho_0 \exp[c(p - p_0)]$$

Solving the Laplacian for varying geometries:

Linear Flow:

Consider the gradient operator for cartesian coordinates

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Applying the gradient operator to itself is next to be carried out. However, note that this operation has a formal name – the Laplacian operator – and is outlined as follows

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\hat{\imath}\right]\hat{\imath} + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\hat{\jmath}\right]\hat{\jmath} + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\hat{k}\right]\hat{k}$$

Alternatively, the reduced form of the Laplacian operator is written as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Noting that the diffusivity equation calls for the Laplacian of pressure, the final form of the expression for linear flow is written as

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

As it relates to practicality and actual implementation, recall that a typical assumption of the diffusivity equation is horizontal flow. Additionally, there will be a coordinate axis that experiences the predominant flow of fluid (this is obviously dependent on the geometry of the well and the placement of the coordinate axes). Thus, when applied to the diffusivity equation, the geometry may be sufficiently described as follows

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2}$$

Radial Flow:

First, it is noted that radial flow utilizes the cylindrical coordinate system to describe its geometry. That is,

- r = radial (Euclidean) distance extending outward from the z-axis
- θ = azimuth angle between a reference direction on a plane and a line's projection on the same plane
- z = height from the origin extending in the direction of the z-axis

Now, recall that the gradient operator for pressure in cylindrical coordinates is as shown (refer to **Back to the Basics: Vector Calculus Derivations** for complete formulation)

$$\nabla p = \frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{z}$$

Taking the gradient of the gradient operator yields the Laplacian operator as shown (i.e. divergence of gradient)

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}$$

For practicality and actual implementation, recall that a typical assumption of the diffusivity equation is horizontal flow. This will eliminate the need for the third term. Additionally, the tangential velocity is normally not considered since it is assumed that the only heterogeneity that is experienced (if any) is based on depth into the reservoir from the wellbore. Thus, the working form of the geometry description for the radial flow case is as follows

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)$$

Liquid Diffusivity Equation:

[INSERT] Introduction paragraphs explaining and outlining the 3 equations we base ourselves from

The derivation of the liquid diffusivity equation is based principally on the mass continuity equation. This is a powerful equation because, as we know, mass balance must always be honored.

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial(\phi\rho)}{\partial t}$$

To describe the fluid motion, Darcy's law (in gradient form) is substituted into the mass continuity equation

$$\nabla \cdot \left[\rho \frac{k}{\mu}(\nabla p + \rho \boldsymbol{g})\right] = \frac{\partial(\phi\rho)}{\partial t}$$

While in the reservoir, fluid may tend to flow horizontally to the wellbore. Therefore, it is reasonable to neglect the gravity term from Darcy's law

$$\nabla \cdot \left[\frac{\rho k}{\mu} \nabla p\right] = \frac{\partial(\phi\rho)}{\partial t}$$

[INSERT] Short discussion about the LHS being the positional part of the equation

Continuing, it is common to assume that the permeability and fluid viscosity are constant for liquid flow. Declaring this, the two parameters may be taken outside of the del operator as shown

$$\nabla \cdot [\rho \nabla p] = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Implementing the product rule, the left-hand side of the equation may be expanded

$$(\nabla \cdot \rho)\nabla p + \rho(\nabla \cdot \nabla p) = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Continuing focus on the left-hand side of the equation, the expression may be rewritten equivalently as

$$\left(\frac{\partial \rho}{\partial p}\nabla p\right)\nabla p + \rho \nabla^2 p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Further rewriting the left-hand side yields

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial t}$$

Shifting focus to the right-hand side of the equation now, the differential term may be expanded by way of the chain rule

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\frac{\partial(\phi\rho)}{\partial p}\frac{\partial p}{\partial t}$$

Note that the use of the chain rule to expand the equation is very important. By doing so, we see that the porosity and fluid density are functions of pressure. Many studies have been focused around this area, so we should continue to work the right-hand side of the equation to get it to a form that we can identify. Thus, applying the product rule to the right-hand side yields

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\left[\phi\frac{\partial \rho}{\partial p} + \rho\frac{\partial \phi}{\partial p}\right]\frac{\partial p}{\partial t}$$

Maintaining focus on the right-hand side of the equation, the porosity and density terms may be factored out of the brackets in order to yield a familiar form that is recognizable

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\phi\rho\left[\frac{1}{\rho}\frac{\partial \rho}{\partial p} + \frac{1}{\phi}\frac{\partial \phi}{\partial p}\right]\frac{\partial p}{\partial t}$$

At this point, careful attention and inspection should be focused on the right-hand side of the equation. To this point, the mass continuity and fluid motion equations have been leveraged. However, the details surrounding the specific fluid and the reservoir conditions must be incorporated. For this liquid diffusivity derivation an equation of state for a slightly compressible liquid will be utilized.

Recall from hydrocarbon phase behavior and reservoir petrophysics, the equations for fluid compressibility and pore-volume compressibility, respectively,

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

$$c_f = \frac{1}{\phi}\frac{\partial \phi}{\partial p}$$

Substituting these two expressions into the working equation yields

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\phi\rho\left[c + c_f\right]\frac{\partial p}{\partial t}$$

Lumping the compressibility terms together, total compressibility (c_t) is now defined

$$\frac{\partial \rho}{\partial p}(\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k}\phi\rho c_t\frac{\partial p}{\partial t}$$

Dividing through by the fluid density, ρ, yields

$$\frac{1}{\rho}\frac{\partial \rho}{\partial p}(\nabla p)^2 + \nabla^2 p = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

Notice that the first term of the left-hand side can be rewritten in terms of the fluid compressibility. This rewrite yields the final form of the diffusivity equation for slightly compressible liquids.

$$c(\nabla p)^2 + \nabla^2 p = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

Although this is the complete form, specific attention is directed to the left-hand side of the equation where the pressure gradient term is raised to the second power. Recalling properties of differential equations, it is noted that when a variable (i.e. pressure) is raised to a power other than 1, it is nonlinear. Further, being that the term in question is the gradient of pressure already shows that the expression is non-linear (gradient is not linear).

This key distinction is outlined because very few nonlinear systems can be solved directly through analytical methods. Thus, the following two assumptions are made:

- Liquid compressibility is small and constant (i.e. not a function of pressure)
- Small gradients

These two assumptions lead to the following rewrite of the diffusivity equation for liquids. Notice that now the nonlinearity is not present on the left-hand side. It is this form of the diffusivity equation (in its most general sense) that is utilized as a starting point in classic analytical solutions.

$$\nabla^2 p = \frac{\phi\mu c_t}{k}\frac{\partial p}{\partial t}$$

Read more information:

https://en.wikipedia.org/wiki/Cylindrical_coordinate_system

https://en.wikipedia.org/wiki/Del

https://blasingame.engr.tamu.edu/

https://www.projectrhea.org/rhea/index.php/Vector_Derivatives_Cylindrical_Coordinates