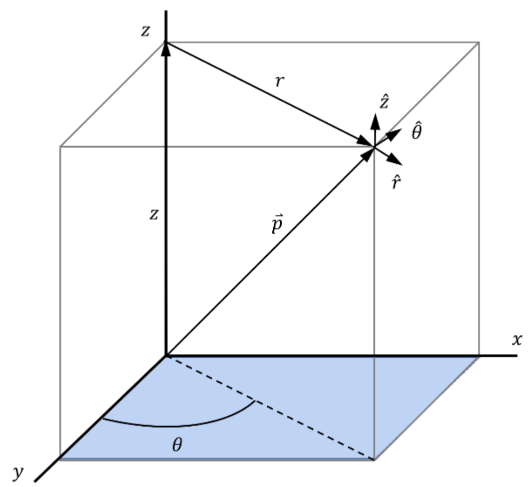
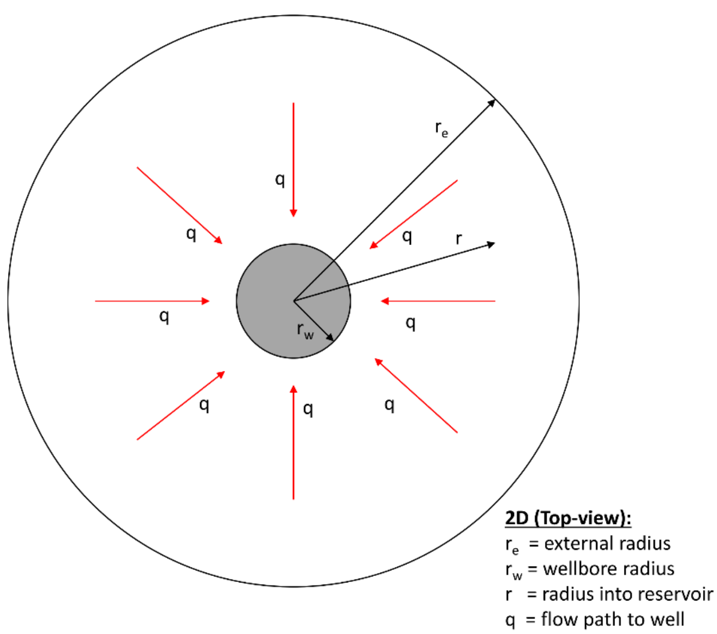
Vector Calculus for Cylindrical (Radial) Coordinate System

Properly describing the flow geometry within a reservoir is essential for petroleum engineers. This article thoroughly explains the calculus derivation for flow geometry that is incorporated in many solutions to the diffusivity equation.

Most people who have conducted work in 3-D space are likely familiar with the cartesian coordinate system. Here, the axes are most commonly denoted as: x, y, z. However, this is not the only 3-D coordinate system that is widely used. Cylindrical and spherical are two other very common coordinate systems. Each system has strengths and weaknesses depending on the use case.



For petroleum engineers, the cylindrical coordinate system is particularly useful when describing fluid flow geometry through a reservoir. In fact, many well testing solutions were originally derived with the cylindrical system imposed.



The exact reasons for this coordinate system's implementation are outside of the scope of this article but suffice it to say that the way fluid flows from distances in the reservoir to the well (under certain conditions) can be modelled as a converging radial flow. The expression that will be derived is the Laplacian in cylindrical coordinates as outlined below, but the gradient operator will also be derived along the way.

$$\nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}$$

Without delay, let's begin the derivation for radial geometry.

Cylindrical Coordinate Derivation:

Having experience with the cartesian coordinate system, a reasonable starting point is to transform the cartesian axes to the cylindrical system. The following parametric equations describe this transformation and outline the three parameters necessary to describe a point in space (r, θ, z)

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

Although the following equations will not be required explicitly in this derivation, the properties that they describe can be useful in other works

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Recall that the position vector in the cartesian coordinate system is defined as

$$\vec{s} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

Modifying the position vector through substitution of cylindrical parametric equation definitions yields (note: the unit vectors present are still those from the cartesian system. The position vector in cylindrical system will be shown later)

$$\vec{s} = r\cos(\theta)\ \hat{\imath} + r\sin(\theta)\ \hat{\jmath} + z\,\hat{k}$$

It is now useful if unit vectors can be generated for each dimension parameter in the coordinate system. Recalling the definition of a unit vector

$$\hat{r} = \frac{\partial\vec{s}/\partial r}{\|\partial\vec{s}/\partial r\|}$$

$$\hat{\theta} = \frac{\partial\vec{s}/\partial \theta}{\|\partial\vec{s}/\partial \theta\|}$$

$$\hat{z} = \frac{\partial\vec{s}/\partial z}{\|\partial\vec{s}/\partial z\|}$$

Where the denominator is the magnitude of the partial derivative taken with respect to each vector dimension. Note that the form of the magnitude function is as follows for the generalized 3-D vector case $\vec{v} = \langle a, b,\ c\rangle$

$$\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$$

For brevity, only the first unit vector is shown in detail as

$$\hat{r} = \frac{\partial \vec{s}/\partial r}{\|\partial \vec{s}/\partial r\|} = \frac{\cos(\theta)\,\hat{\imath} + \sin(\theta)\,\hat{\jmath}}{\left\|\sqrt{(\cos(\theta)\,\hat{\imath})^2 + (\sin(\theta)\,\hat{\jmath})^2}\right\|}$$

Evaluating the partial derivative in the denominator yields (note: the magnitude of a vector results in a scalar)

$$\hat{r} = \frac{\cos(\theta)\,\hat{\imath} + \sin(\theta)\,\hat{\jmath}}{\sqrt{\cos^2(\theta) + \sin^2(\theta)}}$$

Recall that the denominator is familiar in form to the following Pythagorean identity

$$1 = \cos^2(\theta) + \sin^2(\theta)$$

Substituting the identity into the expression of the unit vector, and simplifying, results in

$$\hat{r} = \cos(\theta)\ \hat{\imath} + \sin(\theta)\ \hat{\jmath}$$

Extending the unit vector procedure to the other dimension parameters yields

$$\hat{\theta} = \frac{\partial \vec{s}/\partial \theta}{\|\partial \vec{s}/\partial \theta\|} = \frac{-r\sin(\theta)\,\hat{\imath} + r\cos(\theta)\,\hat{\jmath}}{\sqrt{r^2[\sin^2(\theta) + \cos^2(\theta)]}} = \frac{1}{r}[-r\sin(\theta)\,\hat{\imath} + r\cos(\theta)\,\hat{\jmath}] = -\sin(\theta)\ \hat{\imath} + \cos(\theta)\ \hat{\jmath}$$

$$\hat{z} = \frac{\hat{k}}{\|\hat{k}\|} = \hat{k}$$

At this point, it is important that the reader realizes that the magnitude of the unit vector $\hat{r}$ and $\hat{z}$ are equal to one. In other words, the partial derivative of the position vector is already normalized to unit length. In contrast, the magnitude of the $\hat{\theta}$ unit vector is equal to the scalar r. This stands to show that the partial derivative taken with respect to the θ-dimension required normalization.

With the unit vectors now defined, attention is focused on the development of the gradient operator in cylindrical coordinates. To begin, the first-order Taylor series (linear approximation) is recalled in cylindrical coordinates

$$f(r + \Delta r, \theta + \Delta\theta, z + \Delta z) = f(r, \theta, z) + \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial z}\Delta z$$

Rearrangement of the Taylor series expression allows for the function difference based on the displacement of the dimension parameters

$$\Delta f = f(r + \Delta r, \theta + \Delta\theta, z + \Delta z) - f(r, \theta, z)$$

$$\Delta f = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial z}\Delta z$$

Taking a moment to step back, a displacement vector will be developed and later incorporated with the Taylor series approximation. Beginning with the definition of the position vector in cylindrical coordinates (note: the θ-dimension is implicitly incorporated in the unit vector, $\hat{r}$)

$$\vec{p} = r\,\hat{r} + z\,\hat{z}$$

The incremental position vector can then be denoted as

$$d\vec{p} = d(r\,\hat{r} + z\,\hat{z})$$

Leveraging the product rule for derivatives allows for the following rewrite

$$d\vec{p} = \hat{r}dr + r\,d\hat{r} + \hat{z}dz + z\,d\hat{z}$$

Expanding the derivative terms corresponding to the unit vectors

$$d\vec{p} = \hat{r}dr + r\left[\frac{\partial\hat{r}}{\partial r}dr + \frac{\partial\hat{r}}{\partial\theta}d\theta + \frac{\partial\hat{r}}{\partial z}dz\right] + \hat{z}dz + z\left[\frac{\partial\hat{z}}{\partial r}dr + \frac{\partial\hat{z}}{\partial\theta}d\theta + \frac{\partial\hat{z}}{\partial z}dz\right]$$

Substituting the known partial derivatives of the unit vectors yields

$$d\vec{p} = \hat{r}dr + r[(0)dr + (-\hat{\imath}\sin(\theta) + \hat{\jmath}\cos(\theta))d\theta + (0)dz] + \hat{z}dz + z[(0)dr + (0)d\theta + (0)dz]$$

Noting that the second term within the first brackets is the definition of a unit vector, the incremental position vector is reduced to the following expression

$$d\vec{p} = \hat{r}dr + \hat{\theta}rd\theta + \hat{z}dz$$

At this point, all necessary expressions have been defined. What is left to do is incorporate the incremental position vector with the Taylor series linear approximation. Recalling the property of limits from calculus, the Taylor series can be expressed as follows for an infinitesimal displacement

$$df = \lim_{\Delta\to 0}\Delta f = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial z}dz$$

With the knowledge of the gradient operator being the vector derivative of a scalar field, the following expression is applicable

$$df = \vec{\nabla}f \cdot d\vec{p}$$

Rewriting this expression with the previously developed terms yields

$$\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial z}dz = \left[\hat{r}\left(\vec{\nabla}f\right)_r + \hat{\theta}\left(\vec{\nabla}f\right)_\theta + \hat{z}\left(\vec{\nabla}f\right)_z\right]\cdot[\hat{r}dr + \hat{\theta}rd\theta + \hat{z}dz]$$

Evaluating the dot product on the right-hand side

$$\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial z}dz = \left(\vec{\nabla}f\right)_r dr + \left(\vec{\nabla}f\right)_\theta rd\theta + \left(\vec{\nabla}f\right)_z dz$$

Upon inspection, the following gradient components can be defined

$$\left(\vec{\nabla}f\right)_r = \frac{\partial f}{\partial r}$$

$$\left(\vec{\nabla}f\right)_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta}$$

$$\left(\vec{\nabla}f\right)_z = \frac{\partial f}{\partial z}$$

It is at this point that the gradient operator can now be expressed for the general case

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{z}\frac{\partial}{\partial z}$$

This is an important point in the derivation of the flow geometry in cylindrical coordinates. Before we extend this derivation to the Laplacian, the reader should note the following:

- The θ-dimension is an angle and not a length. As this angle increases, its direction follows a circle. Therefore, the arc length formula is imposed to reflect it as a length ($Arc\ Length = r\theta$).
- The gradient operator requires that the second term include a normalization factor of r to take care of the arc length point discussed above.
- Another method to understand the necessary normalization factor of r is to take the determinant of the Jacobian, where it is seen that the result is r. This is a formal manner in determining normalization factors.
- The cylindrical coordinate system is unique in that its unit vectors are not constant whereas in the cartesian system they are.

To begin the derivation of the Laplacian, it must be understood that it is nothing more than the dot product of the gradient operator and vector gradient. For the purposes of this derivation, the following vector gradient is presented

$$\vec{\nabla} a = \hat{r}\frac{\partial a}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial a}{\partial \theta} + \hat{z}\frac{\partial a}{\partial z}$$

The dot product to be evaluated is as follows

$$\vec{\nabla}\cdot\vec{\nabla} a = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{r}\frac{\partial a}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial a}{\partial \theta} + \hat{z}\frac{\partial a}{\partial z}\right)$$

Expanding the right-hand side results in the following expression

$$\vec{\nabla}\cdot\vec{\nabla} a = \hat{r}\frac{\partial}{\partial r}\cdot\left[\hat{r}\frac{\partial a}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial a}{\partial \theta} + \hat{z}\frac{\partial a}{\partial z}\right] + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta}\cdot\left[\hat{r}\frac{\partial a}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial a}{\partial \theta} + \hat{z}\frac{\partial a}{\partial z}\right] + \hat{z}\frac{\partial}{\partial z}\cdot\left[\hat{r}\frac{\partial a}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial a}{\partial \theta} + \hat{z}\frac{\partial a}{\partial z}\right]$$

The strategy for evaluating this expression will be to take all the partial derivatives first and then reduce. Continuing in this manner, and applying the product rule, the expression may be written as

$$\begin{aligned}\vec{\nabla}\cdot\vec{\nabla} a = \hat{r}\cdot&\left[\left(\frac{\partial}{\partial r}\left(\frac{\partial a}{\partial r}\right)\hat{r} + \frac{\partial a}{\partial r}\frac{\partial \hat{r}}{\partial r}\right) + \left(\frac{\partial}{\partial r}\left(\frac{\partial a}{\partial \theta}\right)\frac{\hat{\theta}}{r} + \frac{\partial a}{\partial \theta}\frac{\partial}{\partial r}\left(\frac{\hat{\theta}}{r}\right)\right) + \left(\frac{\partial}{\partial r}\left(\frac{\partial a}{\partial z}\right)\hat{z} + \frac{\partial a}{\partial z}\frac{\partial \hat{z}}{\partial r}\right)\right]\\
+\frac{\hat{\theta}}{r}\cdot&\left[\left(\frac{\partial}{\partial \theta}\left(\frac{\partial a}{\partial r}\right)\hat{r} + \frac{\partial a}{\partial r}\frac{\partial \hat{r}}{\partial \theta}\right) + \left(\frac{\partial}{\partial \theta}\left(\frac{\partial a}{\partial \theta}\right)\frac{\hat{\theta}}{r} + \frac{\partial a}{\partial \theta}\frac{\partial}{\partial \theta}\left(\frac{\hat{\theta}}{r}\right)\right) + \left(\frac{\partial}{\partial \theta}\left(\frac{\partial a}{\partial z}\right)\hat{z} + \frac{\partial a}{\partial z}\frac{\partial \hat{z}}{\partial \theta}\right)\right]\\
+\hat{z}\cdot&\left[\left(\frac{\partial}{\partial z}\left(\frac{\partial a}{\partial r}\right)\hat{r} + \frac{\partial a}{\partial r}\frac{\partial \hat{r}}{\partial z}\right) + \left(\frac{\partial}{\partial z}\left(\frac{\partial a}{\partial \theta}\right)\frac{\hat{\theta}}{r} + \frac{\partial a}{\partial \theta}\frac{\partial}{\partial z}\left(\frac{\hat{\theta}}{r}\right)\right) + \left(\frac{\partial}{\partial z}\left(\frac{\partial a}{\partial z}\right)\hat{z} + \frac{\partial a}{\partial z}\frac{\partial \hat{z}}{\partial z}\right)\right]\end{aligned}$$

Rewriting the expression to substitute unit vector partial derivatives and reduce where possible yields

$$\begin{aligned}\vec{\nabla}\cdot\vec{\nabla} a = \hat{r}\cdot&\left[\left(\hat{r}\frac{\partial^2 a}{\partial r^2} + \frac{\partial a}{\partial r}(0)\right) + \left(\frac{\hat{\theta}}{r}\frac{\partial^2 a}{\partial r\partial\theta} - \frac{\hat{\theta}}{r^2}\frac{\partial a}{\partial \theta}\right) + \left(\hat{z}\frac{\partial^2 a}{\partial r\partial z} + \frac{\partial a}{\partial z}(0)\right)\right]\\
+\frac{\hat{\theta}}{r}\cdot&\left[\left(\hat{r}\frac{\partial^2 a}{\partial\theta\partial r} + \frac{\partial a}{\partial r}(-\hat{\imath}\sin(\theta) + \hat{\jmath}\cos(\theta))\right) + \left(\frac{\hat{\theta}}{r}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial a}{\partial \theta}\frac{1}{r}(-\hat{\imath}\cos(\theta) - \hat{\jmath}\sin(\theta))\right) + \left(\hat{z}\frac{\partial^2 a}{\partial\theta\partial z} + \frac{\partial a}{\partial z}(0)\right)\right]\\
+\hat{z}\cdot&\left[\left(\hat{r}\frac{\partial^2 a}{\partial z\partial r} + \frac{\partial a}{\partial r}(0)\right) + \left(\frac{\hat{\theta}}{r}\frac{\partial^2 a}{\partial z\partial\theta} + \frac{\partial a}{\partial \theta}(0)\right) + \left(\hat{z}\frac{\partial^2 a}{\partial z^2} + \frac{\partial a}{\partial z}(0)\right)\right]\end{aligned}$$

Similar to the approach taken with the gradient derivation, the unit vector definitions can be substituted to further reduce the expression. Following this suit, further rewrite and reduction yields

$$\vec{\nabla}\cdot\vec{\nabla}a = \hat{r}\cdot\left[\hat{r}\frac{\partial^2 a}{\partial r^2}+\frac{\hat{\theta}}{r}\frac{\partial^2 a}{\partial r\partial\theta}-\frac{\hat{\theta}}{r^2}\frac{\partial a}{\partial\theta}+\hat{z}\frac{\partial^2 a}{\partial r\partial z}\right]+\frac{\hat{\theta}}{r}\cdot\left[\hat{r}\frac{\partial^2 a}{\partial\theta\partial r}+\hat{\theta}\frac{\partial a}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial^2 a}{\partial\theta^2}-\frac{\hat{r}}{r}\frac{\partial a}{\partial\theta}+\hat{z}\frac{\partial^2 a}{\partial\theta\partial z}\right]$$

$$+\hat{z}\cdot\left[\hat{r}\frac{\partial^2 a}{\partial z\partial r}+\frac{\hat{\theta}}{r}\frac{\partial^2 a}{\partial z\partial\theta}+\hat{z}\frac{\partial^2 a}{\partial z^2}\right]$$

With this refined form, the dot product is evaluated and the Laplacian is yielded

$$\vec{\nabla}\cdot\vec{\nabla}a = \frac{\partial^2 a}{\partial r^2}+\frac{1}{r}\frac{\partial a}{\partial r}+\frac{1}{r^2}\frac{\partial^2 a}{\partial\theta^2}+\frac{\partial^2 a}{\partial z^2}$$

Noticing that there are two differential terms that are taken with respect to r, one last rewrite is possible, and this is the form that is most commonly used in vector calculus and diffusivity equations.

$$\nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 a}{\partial\theta^2}+\frac{\partial^2 a}{\partial z^2}$$