PHYSICAL INTERPRETATIONS OF CURL AND DIVERGENCE

1. Physical Interpretation of the Curl

Let $\mathbf{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field. We can think of $\mathbf{F}$ as representing the velocity field of some fluid in space. We want to give a physical meaning to the curl $\nabla \times \mathbf{F}$ of this vector field at a point. Choose coordinates so that our point of interest is the origin. Consider a small circle γ of radius h centred at the origin, in the x-y plane, oriented counterclockwise. We can parametrize this curve as:

$$\gamma : \mathbf{r}(t) = (h\cos(t), h\sin(t), 0) \qquad 0 \le t \le 2\pi$$

The line integral of the vector field $\mathbf{F}$ over this path γ measures the *circulation* of the vector field along this path, or the tendency of the field to follow the path. A positive circulation means that as we traverse the path γ, we tend to move in the direction of the vector field $\mathbf{F}$, while a negative circulation means we tend to move in the opposite direction of $\mathbf{F}$. Since we assume the circle to be very small, and $\mathbf{F}$ is assumed to be differentiable, we will take the first order (linear) approximation of the vector field near the origin $\mathbf{0} = (0,0,0)$:

$$\begin{aligned}
P(x,y,z) &= P(\mathbf{0}) + P_x(\mathbf{0})x + P_y(\mathbf{0})y + P_z(\mathbf{0})z \\
Q(x,y,z) &= Q(\mathbf{0}) + Q_x(\mathbf{0})x + Q_y(\mathbf{0})y + Q_z(\mathbf{0})z \\
R(x,y,z) &= R(\mathbf{0}) + R_x(\mathbf{0})x + R_y(\mathbf{0})y + R_z(\mathbf{0})z
\end{aligned}$$

Since we are computing a line integral of $\mathbf{F}$ over the curve γ, we substitute $x = h\cos(t)$, $y = h\sin(t)$, and $z = 0$ in the above to obtain:

$$\begin{aligned}
P(\mathbf{r}(t)) &= P(\mathbf{0}) + P_x(\mathbf{0})h\cos(t) + P_y(\mathbf{0})h\sin(t) \\
Q(\mathbf{r}(t)) &= Q(\mathbf{0}) + Q_x(\mathbf{0})h\cos(t) + Q_y(\mathbf{0})h\sin(t) \\
R(\mathbf{r}(t)) &= R(\mathbf{0}) + R_x(\mathbf{0})h\cos(t) + R_y(\mathbf{0})h\sin(t)
\end{aligned}$$

Note that if we were taking quadratic or higher order approximations the extra terms would all have at least an h^2 factor in them. The velocity vector field of the curve γ is

$$\mathbf{r}'(t) = (-h\sin(t), h\cos(t), 0) \tag{1.1}$$

So the integrand in the line integral becomes (after some rearranging):

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = {} & h\left(Q(\mathbf{0})\cos(t) - P(\mathbf{0})\sin(t)\right) \\
& + h^2\left(Q_x(\mathbf{0})\cos^2(t) - P_y(\mathbf{0})\sin^2(t) + (Q_y(\mathbf{0}) - P_x(\mathbf{0}))\sin(t)\cos(t)\right) \\
& + h^3(\cdots)
\end{aligned}$$

The term $(\cdots)$ represents all the leftover parts from a higher order approximation. Now we integrate this from 0 to 2π in t, and use the fact that:

$$\int_0^{2\pi} \sin(t)dt = \int_0^{2\pi} \cos(t)dt = \int_0^{2\pi} \sin(t)\cos(t)dt = 0$$
$$\int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} \cos^2(t)dt = \pi$$

Hence we obtain the circulation of $\mathbf{F}$ around γ:

$$\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)dt = \pi h^2 \left(Q_x(\mathbf{0}) - P_y(\mathbf{0})\right) + h^3 \left(\cdots\right)$$

If we divide the circulation around this path by the area πh^2 of the circle, and take the limit as $h \to 0$, we obtain:

$$\lim_{h \to 0} \frac{1}{\pi h^2} \oint_\gamma \mathbf{F} \cdot d\mathbf{r} = Q_x(\mathbf{0}) - P_y(\mathbf{0}) \tag{1.2}$$

which is the z-component of the curl $\nabla \times \mathbf{F}$ of $\mathbf{F}$. If we had performed the same calculation using a small circular path in the y-z or z-x planes, then we would have obtained the x- and y- components of the curl, respectively. Thus we see that the component of the curl of a vector field at a point, in a given direction, is equal to the *infinitesmal circulation of the field per unit area* around a circular path centred at that point, in a plane whose normal vector points in the given direction, with orientation given by the right hand rule. This is why we say the curl measures the tendency of the vector field to 'curl' around a given point in a direction given by the right hand rule.

2. Physical Interpretation of the Divergence

Let $\mathbf{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field. Again $\mathbf{F}$ should be thought of as the velocity field of some fluid in space. We want to give a physical meaning to the divergence $\nabla \cdot \mathbf{F}$ of this vector field at a point. Choose coordinates so that our point of interest is the origin.

Consider a small sphere M of radius h centred at the origin, with the outward pointing unit normal vector field $\mathbf{n}$. We can parametrize this sphere using spherical coordinates:

$$M : \mathbf{X}(u,v) = (h\sin(u)\cos(v), h\sin(u)\sin(v), h\cos(u)) \qquad 0 \le u \le \pi, 0 \le v \le 2\pi$$

The flux integral of the vector field $\mathbf{F}$ through this surface S measures the net amount of fluid moving out of the sphere per unit time. Since we assume the sphere to be very small, and $\mathbf{F}$ is assumed to be differentiable, we again take the first order (linear) approximation of the vector field near the origin $\mathbf{0} = (0,0,0)$:

$$P(x,y,z) = P(\mathbf{0}) + P_x(\mathbf{0})x + P_y(\mathbf{0})y + P_z(\mathbf{0})z$$
$$Q(x,y,z) = Q(\mathbf{0}) + Q_x(\mathbf{0})x + Q_y(\mathbf{0})y + Q_z(\mathbf{0})z$$
$$R(x,y,z) = R(\mathbf{0}) + R_x(\mathbf{0})x + R_y(\mathbf{0})y + R_z(\mathbf{0})z$$

Since we are computing a flux integral of $\mathbf{F}$ through the surface S, we substitute $x = h\sin(u)\cos(v)$, $y = h\sin(u)\sin(v)$, and $z = h\cos(u)$ in the above to obtain:

$$
\begin{aligned}
P(\mathbf{X}(u,v)) &= P(\mathbf{0}) + P_x(\mathbf{0})h\sin(u)\cos(v) + P_y(\mathbf{0})h\sin(u)\sin(v) + P_z(\mathbf{0})h\cos(u)\\
Q(\mathbf{X}(u,v)) &= Q(\mathbf{0}) + Q_x(\mathbf{0})h\sin(u)\cos(v) + Q_y(\mathbf{0})h\sin(u)\sin(v) + Q_z(\mathbf{0})h\cos(u)\\
R(\mathbf{X}(u,v)) &= R(\mathbf{0}) + R_x(\mathbf{0})h\sin(u)\cos(v) + R_y(\mathbf{0})h\sin(u)\sin(v) + R_z(\mathbf{0})h\cos(u)
\end{aligned}
$$

Note that if we were taking quadratic or higher order approximations the extra terms would all have at least an h^2 factor in them.

The normal vector field $\mathbf{X}_u \times \mathbf{X}_v$ of the surface M can be computed to be:

$$
\mathbf{X}_u \times \mathbf{X}_v = \big(h^2\sin^2(u)\cos(v), h^2\sin^2(u)\sin(v), h^2\sin(u)\cos(u)\big) \tag{2.1}
$$

So the integrand in the line integral becomes (after some rearranging):

$$
\begin{aligned}
&\mathbf{F}\left(\mathbf{X}(u,v)\right)\cdot\left(\mathbf{X}_u \times \mathbf{X}_v\right)\\
&= h^2\left(P(\mathbf{0})\sin^2(u)\cos(v) + Q(\mathbf{0})\sin^2(u)\sin(v) + R(\mathbf{0})\sin(u)\cos(u)\right)\\
&+ h^3\big(P_x(\mathbf{0})\sin^3(u)\cos^2(v) + P_y(\mathbf{0})\sin^3(u)\sin(v)\cos(v) + P_z(\mathbf{0})\sin^2(u)\cos(u)\cos(v)\\
&+ Q_x(\mathbf{0})\sin^3(u)\sin(v)\cos(v) + Q_y(\mathbf{0})\sin^3(u)\sin^2(v) + Q_z(\mathbf{0})\sin^2(u)\cos(u)\sin(v)\\
&+ R_x(\mathbf{0})\sin^2(u)\cos(u)\cos(v) + R_y(\mathbf{0})\sin^2(u)\cos(u)\sin(v) + R_z(\mathbf{0})\sin(u)\cos^2(u)\big)\\
&+ h^4\left(\cdots\right)
\end{aligned}
$$

The term $(\cdots)$ represents all the leftover parts from a higher order approximation. Now we integrate this from 0 to π in u and 0 to 2π in v. Most of the integrals turn out to be zero, and all that remains is:

$$
\begin{aligned}
\oiint_M \mathbf{F}\cdot\mathbf{n}dS &= \int_0^{\pi}\int_0^{2\pi}\mathbf{F}\left(\mathbf{X}(u,v)\right)\cdot\left(\mathbf{X}_u\times\mathbf{X}_v\right)dudv\\
&= \frac{4}{3}\pi h^3\left(P_x(\mathbf{0}) + Q_y(\mathbf{0}) + R_z(\mathbf{0})\right) + h^4\left(\cdots\right)
\end{aligned}
$$

If we divide the flux through this surface by the volume $\frac{4}{3}\pi h^3$ of the sphere, and take the limit as $h \to 0$, we obtain:

$$
\lim_{h\to 0}\frac{1}{\frac{4}{3}\pi h^3}\oiint_M \mathbf{F}\cdot\mathbf{n}dS = P_x(\mathbf{0}) + Q_y(\mathbf{0}) + R_z(\mathbf{0}) \tag{2.2}
$$

which is the divergence $\nabla\cdot\mathbf{F}$ of $\mathbf{F}$. Thus we see that the divergence of a vector field at a point is equal to the *infinitesimal flux of the field per unit volume* through a sphere centred at that point. This is why we say the divergence measures the tendency of the vector field to ‘diverge’ from a point.