**CS763, Assignment 1**

**Team Members**

* Ashray Malhotra (11D070002)
* Akshay Sarode (110260010)
* Rohan Prinja (110050011)

**Question 2**

Since OS, OQ and OR are given to be mutually perpendicular, we have OS · OQ = 0 and OS · OR = 0. Subtracting these two equations, we get OS · (OQ – OR) = 0, which means that OS is orthogonal to OR – OQ. This proves the first part of the question.

Next, we notice that OR – OQ is the same as RQ, which is a vector parallel to the image plane. We know that Oo is parallel to the z-axis and so it is orthogonal to the image plane. If a line is orthogonal to a plane it is perpendicular to all lines that lie in that plane. So Oo is orthogonal to QR i.e. Oo is orthogonal to OR – OQ. This proves the second part of the question.

Since OS and Oo are both orthogonal to OR – OQ, and further, OS is not parallel to Oo (since we can assume without loss of generality that S is on the image plane at a different point than o), therefore OS and Oo span a plane. Any line in this plane will be a linear combination of OS and Oo. So an arbitrary line in this plane is of the form aOS + bOo where a and b are some real numbers. This line must be orthogonal to OR – OQ since

(aOS + bOo) · (OR – OQ)  
 = a(OS · (OR – OQ)) + b(Oo · (OR – OQ))  
 = a(0) + b(0)  
 = 0

This means that every line in the plane of triangle OSo is orthogonal to OR – OQ. This in turn means that the plane itself is orthogonal to OR – OQ. As a side effect, this also proves that oS is orthogonal to OR – OQ, since oS is a linear combination of OS and Oo (with a = 1 and b = –1). This proves the third part of the question.

Now we modify the above proof to account for the case in which the three perpendicular lines meet at a point other than O. Suppose the lines meet at a point T. The first part of the proof now becomes TS · (OR – OQ) = 0. The second part does not change as it has nothing to do with the meeting point of the three perpendicular lines. So Oo · (OR – OQ) = 0 still holds.

The third part of the proof is the same as before, except that now we are taking a linear combination of TS and Oo instead of OS and Oo. Here, we have to be careful that TS is not parallel to Oo. If it is, then these two vectors span a line instead of a plane, and oS is definitely not on that line, since oS is in the image plane which is perpendicular to Oo (and thus, perpendicular to the line spanned by TS and Oo).

So we need an **additional condition** that **TS is not parallel to Oo**. Only then can we get line oS as a linear combination of TS and Oo. Similarly, **TR and TQ should also not be parallel to Oo**. The rest of the proof follows exactly as before. Before, there was no need to explicitly enforce this condition since the only that OS could be parallel to Oo was if S coincided with O, which we assumed was not the case. Similarly for OR and OQ.

**Question 3**

The vanishing point of a line depends only on its direction vector. So we will ignore the precise location of the three coplanar lines and instead only look at their direction vectors. Let the direction vectors be v1 = (x1, y1, z1), v2 = (x2, y2, z2) and v3 = (x3, y3, z3) respectively. Then, since the three lines are coplanar,

v1 · (v2 × v3) = 0  
⇔ (x1, y1, z1) · (y2z3 – y3z2, x3z2 – x2z3, x2y3 – x3y2) = 0  
⇔ x1(y2z3 – y3z2) + y1(x3z2 – x2z3) + z1(x2y3 – x3y2) = 0 [1]

The vanishing point of a line with direction vector (x, y, z) is (fx/z, fy/z, f). So, the vanishing points are (fx1/z1, fy1/z1, f), (fx2/z2, fy2/z2, f) and (fx3/z3, fy3/z3, f) respectively. To prove that these points are collinear is equivalent to proving that the points wi = (xi/zi, yi/zi, 1) are collinear for i = 1, 2, 3 (we are simply scaling each point by 1/f, so the collinearity is not affected). Now, w1, w2 and w3 are collinear iff

(w3 – w2) × (w2 – w1) = 0.  
 ⇔ (x3z2 – x2z3, y3z2 – y2z3, 0)/(z2z3) × (x2z1 – x1z2, y2z1 – y1z2, 0)/(z1z2) = 0

Expanding the above cross product and simplifying, we get the following equation,

⇔ z2(x1y3 – x3y1) + z3(x2y1 – x1y2) + z1(x3y2 – x2y3) = 0 [2]

It can be seen that [2] is just a rearranged version of [1]. Thus, the coplanarity of the three lines in the real world implies [2] which is equivalent to [1], which is equivalent to the collinearity of the vanishing points of the three lines. Hence proved.