4

Logic, Lifting and Finality

The previous three chapters have introduced some basic elements of the theory of coalgebras, focusing on coalgebraic system descriptions, homomorphisms, behaviour, finality and bisimilarity. So far, only relatively simple coalgebras have been used, for inductively defined classes of polynomial functors, on the category **Sets** of sets and functions. This chapter will go beyond these polynomial functors and will consider other examples. But more important, it will follow a different, more systematic approach, relying not on the way functors are constructed but on the properties they satisfy – and work from there. Inevitably, this chapter will technically be more challenging, requiring more categorical maturity from the reader.

The chapter starts with a concrete description of two new functors, namely the multiset and distribution functors, written as $\mathcal{M}$ and $\mathcal{D}$, respectively. As we shall see, on the one hand, from an abstract point of view, they are much like powerset $\mathcal{P}$, but on the other hand they capture different kinds of computation: $\mathcal{D}$ is used for probabilistic computation and $\mathcal{M}$ for resource-sensitive computation.

Subsequently, Sections 4.2–4.5 will take a systematic look at relation lifting – used in the previous chapter to define bisimulation relations. Relation lifting will be described as a certain logical operation, which will be developed on the basis of a moderate amount of categorical logic, in terms of so-called factorisation systems. This will give rise to the notion of 'logical bisimulation' in Section 4.5. It is compared with several alternative formulations. For weak pullback-preserving functors on **Sets** these different formulations coincide. With this theory in place Section 4.6 concentrates on the existence of final coalgebras. Recall that earlier we skipped the proof of Theorem 2.3.9, claiming the existence of final coalgebras for finite Kripke polynomial functors. Here we present general existence results, for 'bounded' endofunctors on **Sets**. Finally, Section 4.7 contains another characterisation of simple polynomial functors in terms of size and preservation properties. It also contains a characterisation of more general 'analytical' functors, which includes for instance the multiset functor $\mathcal{M}$.

4.1 Multiset and Distribution Functors

A set is a collection of elements. Such an element, if it occurs in the set, occurs only once. This sounds completely trivial. But one can imagine situations in which multiple occurrences of the same element can be relevant. For instance, in a list it is quite natural that one and the same element occurs multiple times – but at different places in the list. A **multiset** – or what computer scientists usually call a **bag** – is a 'set' in which an element x may occur multiple times. One can write this for instance as $2x$, $10x$, $1x$, or even $0x$, where the n in nx describes that x occurs n times.

Thus, one can distinguish different operators for collecting elements according to whether the order of occurrence of elements matters or whether multiplicities of elements are relevant. The table in Figure 4.1 describes some of these collection types.

An important question is how to count occurrences of elements in a multiset. The obvious choice is to use natural numbers nx, like above. Instead of writing x, we like to write $|x\rangle$ in $n|x\rangle$. This 'ket' notation is meaningless syntactic sugar that is used to distinguis the element $x \in X$ from its occurrence in such counting expressions. It turns out to be convenient to allow also negative occurrences $-2|x\rangle$, describing for instance a 'lack' or 'deficit' of two elements x. But also one may want to allow $\frac{1}{2}|x\rangle$, so that the number in front of x may be interpreted as the probability of having x in a set. This is precisely what happens in the distribution functor $\mathcal{D}$; see below.

It thus makes sense to allow a quite general form of counting elements. We shall use an arbitrary commutative monoid $M = (M, +, 0)$ for this purpose and write multiple occurrences of an element x as $m|x\rangle$ for $m \in M$.

Operator	Order Relevant	Multiplicities Relevant
List $(-)^\star$	Yes	Yes
Powerset $\mathcal{P}$	No	No
Multiset $\mathcal{M}$	No	Yes
Distribution $\mathcal{D}$	No	Yes

Figure 4.1 Various collection types described as functors.

The expression $0|x\rangle$ is then used for non-occurrence of x. Later on, in Section 5.1 we shall see that it makes sense to assume more than an additive monoid structure on multiplicities, namely also multiplication (giving a semiring). But for now a monoid structure suffices.

The operation of forming (finite) multisets of a given set is functorial. The resulting functor will be called the multiset functor and written as $\mathcal{M}$. It is called the 'monoidal exponentiation functor' in [428] (see also [185, 179]).

Definition 4.1.1 For a commutative monoid $M = (M, +, 0)$ we define the **multiset functor** $\mathcal{M}_M\colon \mathbf{Sets} \to \mathbf{Sets}$ on a set X as

$$\mathcal{M}_M(X) = \{\varphi\colon X \to M \mid \mathrm{supp}(\varphi) \text{ is finite}\},$$

where $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ is called the support of φ.

On a function $f\colon X \to Y$ one has $\mathcal{M}_M(f)\colon \mathcal{M}_M(X) \to \mathcal{M}_M(Y)$ via

$$\mathcal{M}_M(f)(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x) = \sum\{\varphi(x) \mid x \in \mathrm{supp}(\varphi) \text{ with } f(x) = y\}.$$

This definition requires some explanation. A multiset $\varphi \in \mathcal{M}_M(X)$ is a function $\varphi\colon X \to M$ which is non-zero on only finitely many elements, say $\varphi(x_1) \neq 0, \ldots, \varphi(x_n) \neq 0$. The support of φ is then the subset $\mathrm{supp}(\varphi) = \{x_1, \ldots, x_n\} \subseteq X$ of those elements that occur in the multiset, with certain (non-zero) multiplicities. The multiplicity of $x_i \in \mathrm{supp}(\varphi)$ is $\varphi(x_i) \in M$. One conveniently writes such a multiset as formal sum $m_1|x_1\rangle + \cdots + m_n|x_n\rangle$, where $m_i = \varphi(x_i) \in M$ is the multiplicity of the element x_i. By convention, the $+$ in these formal sums is commutative and associative; $m|x\rangle + m'|x\rangle$ is the same as $(m + m')|x\rangle$, and $m|x\rangle + 0|y\rangle$ is $m|x\rangle$.

With this formal sum notation we can write the action of the functor $\mathcal{M}_M$ on functions as

$$\mathcal{M}_M(f)\Big(m_1|x_1\rangle + \cdots + m_n|x_n\rangle\Big) = m_1|f(x_1)\rangle + \cdots + m_n|f(x_n)\rangle.$$

This works by the above-mentioned conventions about formal sums. Preservation of composition holds simply by:

$$\begin{aligned} \mathcal{M}_M(g)(\mathcal{M}_M(f)(\textstyle\sum_i m_i|x_i\rangle)) &= \mathcal{M}_M(g)(\textstyle\sum_i m_i|f(x_i)\rangle) \\ &= \textstyle\sum_i m_i|g(f(x_i))\rangle \\ &= \mathcal{M}_M(g \circ f)(\textstyle\sum_i m_i|x_i\rangle). \end{aligned}$$

Thus, $\mathcal{M}_M(X)$ contains finite multisets, with elements from the set X and multiplicities in the monoid M. We restrict ourselves to *finite* multiset, to make sure that the sums $\sum$ in the definition of $\mathcal{M}_M(f)$ in Definition 4.1.1 exist.

Before describing examples, we mention some obvious properties.

Lemma 4.1.2 *The empty multiset and the join of multisets make the sets $\mathcal{M}_M(X)$ from Definition 4.1.1 commutative monoids. Moreover, the functions $\mathcal{M}_M(f)\colon \mathcal{M}_M(X) \to \mathcal{M}_M(Y)$ preserve this monoid structure.*

Further, for the initial (empty) set 0, for the final (singleton) set 1, and for a finite set V, there are isomorphisms:

$$\mathcal{M}_M(0) \cong 1 \qquad \mathcal{M}_M(1) \cong M \qquad \mathcal{M}_M(V) \cong M^V$$

(which are all isomorphisms of monoids).

Proof The monoid structure on $\mathcal{M}_M(X)$ can be described pointwise:

$$\varphi + \psi = \lambda x \in X.\, \varphi(x) + \psi(x) \quad \text{with zero} \quad \lambda x \in X.\, 0.$$

Alternatively, it may be described in terms of the formal sums: the operations represent the join of multisets and the empty multiset. The isomorphisms are obvious. ❑

Example 4.1.3 For some specific examples of monoids M we describe the multiset $\mathcal{M}_M(X)$ in some more detail. The characterisations we give as free structures can be checked per case but turn out to be instances of a more general result; see Example 5.4.3.2 later on.

1. We start with $M = \mathbb{N}$, the commutative monoid of natural numbers with addition $(0, +)$. The set $\mathcal{M}_{\mathbb{N}}(X)$ contains 'traditional' multisets (or bags), with natural numbers as multiplicities. This set $\mathcal{M}_{\mathbb{N}}(X)$ is the free commutative monoid on the set X.
2. For $M = \mathbb{Z}$ we can additionally have negative occurrences of elements in $\mathcal{M}_{\mathbb{Z}}(X)$. This $\mathcal{M}_{\mathbb{Z}}(X)$ is the free commutative (Abelian) group on X.
3. For the two-element monoid $M = 2 = \{0, 1\}$, with join $\vee$ (logical or) and unit 0 as monoid structure, there are no multiplicities except 0 and 1. Hence multisets over 2 are finite subsets: $\mathcal{M}_2(X) \cong \mathcal{P}_{\text{fin}}(X)$.
4. For $M = \mathbb{R}$ we get real numbers as multiplicities, and $\mathcal{M}_{\mathbb{R}}(X)$ is free vector space over $\mathbb{R}$ on X. Scalar multiplication is given by $r \bullet \varphi = \lambda x.\, r \cdot \varphi(x)$, where $\cdot$ is multiplication of real numbers.

The general description of multisets, over an arbitrary monoid, thus covers various mathematical structures. Next we illustrate how repeated multisets are of interest, namely in order to describe (multivariate) polynomials.

Example 4.1.4 Above we have used formal sums $m_1|x_1\rangle + \cdots + m_n|x_n\rangle$ as convenient notation for multisets. For monoids and groups one sometimes uses additive notation $(0, +)$ and sometimes multiplicative notation $(1, \cdot)$. Similarly, one may choose to use multiplicative notation for multisets, as in

$$x_1^{m_1} \cdots x_n^{m_n}.$$

Now let's consider the repeated multiset functor application

$$\mathcal{M}_M\Big(\mathcal{M}_{\mathbb{N}}(X)\Big), \tag{4.1}$$

where M is an arbitrary monoid. An element $\Phi \in \mathcal{M}_M(\mathcal{M}_{\mathbb{N}}(X))$ is described as a formal sum of multisets:

$$\Phi = \textstyle\sum_i m_i|\varphi_i\rangle \qquad \text{where} \qquad \varphi_i \in \mathcal{M}_{\mathbb{N}}(X).$$

Now it is convenient to use multiplicative notation for the inner multisets φ_i, so that we get:

$$\Phi = \textstyle\sum_i m_i|\, x_{i1}^{n_{i1}} \cdots x_{ik_i}^{n_{ik_i}}\,\rangle \qquad \text{for} \qquad k_i, n_{ij} \in \mathbb{N}.$$

Thus, elements of the double-multiset (4.1) are polynomials (as in algebra), with coefficients m_i from M. They are so-called multivariate polynomials, involving multiple variables x_{ij}, taken from a set of variables X. The univariate polynomials, with only one variable – say x – are obtained as special case of (4.1), namely by taking the singleton set $X = 1$. The element of $\mathcal{M}_M(\mathcal{M}_{\mathbb{N}}(1)) \cong \mathcal{M}_M(\mathbb{N})$ are formal sums $\sum_i m_i|n_i\rangle$, commonly written as $\sum_i m_i x^{n_i}$, where x is a chosen variable.

This concludes, for the time being, our investigation of multiset functors. In Section 5.1 we shall return to them and see that they carry a monad structure, provided the set of multiplicities is not only a monoid but also a semiring (with additionally multiplication, see Definition 5.1.4 and Lemma 5.1.5). In the remainder of this section we investigate the distribution functor $\mathcal{D}$, which involves probabilities as multiplicities.

Definition 4.1.5 The (discrete probability) **distribution functor** $\mathcal{D}\colon$ **Sets** $\to$ **Sets** is defined as

$$\mathcal{D}(X) = \{\varphi\colon X \to [0,1] \mid \operatorname{supp}(\varphi) \text{ is finite and } \textstyle\sum_x \varphi(x) = 1\},$$

where $[0,1] \subseteq \mathbb{R}$ is the unit interval of real numbers, and $\operatorname{supp}(\varphi) \subseteq X$ is the subset of $x \in X$ where $\varphi(x) \neq 0$, as in Definition 4.1.1. For a function $f\colon X \to Y$ the map $\mathcal{D}(f)\colon \mathcal{D}(X) \to \mathcal{D}(Y)$ is defined as for multisets, namely by

$$\mathcal{D}(f)(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x) = \sum\{\varphi(x) \mid x \in \operatorname{supp}(\varphi) \text{ with } f(x) = y\}.$$

A discrete probability distribution $\varphi \in \mathcal{D}(X)$ may be identified with a formal convex sum $r_1|x_1\rangle + \cdots + r_n|x_n\rangle$, where $\mathrm{supp}(\varphi) = \{x_1, \ldots, x_n\}$ and $r_i = \varphi(x_i) \in [0,1]$ is the probability associated with the element $x_i \in X$. These probabilities are required to add up to one: $\sum_i r_i = 1$. With this formal sum notation we can again describe the functor applied to a function f succinctly as $\mathcal{D}(f)(\sum_i r_i|x_i\rangle) = \sum_i r_i|f(x_i)\rangle$, as for multiset functors. This shows that the map $\mathcal{D}(f)$ is well defined, in the sense $\sum_y \mathcal{D}(f)(\varphi)(y) = 1$.

For distributions $\sum_i r_i|x_i\rangle \in \mathcal{D}(X)$ the probabilities $r_i \in [0,1]$ add up to 1. In some situations one wishes to be more flexible and allow $\sum_i r_i \leq 1$. Such 'sub' probability distributions give rise to a 'sub' probability functor $\mathcal{D}_{\leq 1}\colon \mathbf{Sets} \to \mathbf{Sets}$. These sets $\mathcal{D}_{\leq 1}(X)$ have more structure than sets $\mathcal{D}(X)$ of (proper) probabilities. For instance, they carry an action $[0,1] \times \mathcal{D}_{\leq 1}(X) \to \mathcal{D}_{\leq 1}(X)$. And if one allows non-finite support, as in Exercise 4.1.8, then subdistributions even form a dcpo (used in [206]).

Multiset functors $\mathcal{M}_M$ are parametrised by a monoid M. In contrast, distributions in $\mathcal{D}(X)$ take their values in the (fixed) set $[0,1]$ of probabilities. It is also possible to replace $[0,1]$ by a parameterised structure, namely a so-called effect monoid; see [250]. Such effect monoids are used in the (generalised) probability theory developed in the context of quantum mechanics (see e.g. [396, 95]). However, such generality is not needed here.

In analogy with Lemma 4.1.2 we have the following results for distribution functors.

Lemma 4.1.6 *For the distribution functor $\mathcal{D}$ there are the following isomorphisms*:

$$\mathcal{D}(0) \cong 0, \qquad \mathcal{D}(1) \cong 1, \qquad \mathcal{D}(2) \cong [0,1]. \qquad \square$$

We saw that sets $\mathcal{M}_M(X)$ are commutative monoids. Sets $\mathcal{D}(X)$ also carry algebraic structure in the form of convex sums; see [247]. Further investigation of the categorical structure of distribution functors will be postponed until Section 5.1 on monads. We conclude by briefly describing coalgebras of the functors $\mathcal{M}_M$ and $\mathcal{D}$ introduced in this section.

Coalgebras $c\colon X \to \mathcal{D}(X)$ of the distribution functor map a state x to a probability distribution $c(X) \in \mathcal{D}(X)$ over successor states. Such a distribution may be written as $c(x) = r_1|x_1\rangle + \cdots + r_n|x_n\rangle$, where $\sum_i r_i = 1$. A transition $x \to x_i$ takes place with probability $r_i \in [0,1]$. Sometimes this written as $x \xrightarrow{r_i} x_i$. Such a probabilistic transition system $X \to \mathcal{D}(X)$ is also known as Markov chain. This basic form may be extended in various ways; see Figure 4.2.

Here is a very simple example of a Markov chain. Assume an abstract political landscape where only lefties (L) and righties (R) are distinguished.

It appears that with each cycle (of one year, say) 80% of lefties remain lefties and 20% become righties. The righties are more stable: 90% of them remains loyal, and 10% become lefties. This may be written as a Markov chain, or probabilistic automaton, with two states L and R:

$$\begin{tikzcd} L \arrow[loop left, "0.8"] \arrow[r, bend left, "0.2"] & R \arrow[l, bend left, "0.1"] \arrow[loop right, "0.9"] \end{tikzcd} \tag{4.2}$$

This Markov chain can equivalently be described as a coalgebra $c\colon \{L, R\} \to \mathcal{D}(\{L, R\})$ of the distribution functor. The function c maps each state to a distribution, written as formal convex sum:

$$c(L) = 0.8|L\rangle + 0.2|R\rangle, \qquad c(R) = 0.1|L\rangle + 0.9|R\rangle. \tag{4.3}$$

Conventionally, such a system is described via a transition matrix

$$\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \tag{4.4}$$

and the analysis of the system proceeds via an analysis of the matrix (see [293, 227] for more information).

So far we have seen non-deterministic systems as coalgebras of the powerset functor $\mathcal{P}$ and probabilistic systems as coalgebras of the distribution functor $\mathcal{D}$. Many systems display both non-deterministic and probabilistic behaviour, in many combinations. Via a description in terms of coalgebras the differences can be seen clearly. Figure 4.2 gives an overview of the various systems that have been studied in the literature. The table is copied from [59, 434], to which we refer for more information. Here we focus on *discrete* probabilistic systems, in contrast to *continuous* ones, taking measurable spaces as state spaces. Such continuous systems can also be described as coalgebras, namely of the 'Giry' functor (or monad); see [157] and [369, 124, 252] for more information.

Coalgebras $c\colon X \to \mathcal{M}_M(X)$ of a multiset functor are known as multigraphs [115]. They can be understood more generally in terms of resources or costs $m \in M$ associated with a transition, as in $x \xrightarrow{m} x'$ when $c(x)(x') = m$. This is characteristic of a weighted automaton; see [421, 125, 90, 76].

4.1.1 Mappings between Collection Functors

In Figure 4.1 we have seen various functors, such as list $(-)^\star$, powerset $\mathcal{P}$, multiset $\mathcal{M}$ and distribution $\mathcal{D}$ that collect elements in a certain way.

Functor F	Name for $X \to F(X)$	Reference
$\mathcal{D}$	Markov chain	
$\mathcal{P}(A \times -) \cong \mathcal{P}^A$	Labelled transition system	
$(1 + \mathcal{D}(-))^A$	Reactive system	[328, 159]
$1 + \mathcal{D}(A \times -)$	Generative systems	[159]
$1 + (A \times -) + \mathcal{D}(-)$	Stratified system	[159]
$\mathcal{D}(-) + \mathcal{P}(A \times -)$	Alternating system	[197]
$\mathcal{D}(A \times -) + \mathcal{P}(A \times -)$	Vardi system	[456]
$\mathcal{P}(A \times \mathcal{D}(-))$	Simple Segala system	[426, 425]
$\mathcal{P}\mathcal{D}(A \times -)$	Segala system	[426, 425]
$\mathcal{D}\mathcal{P}(A \times -)$	Bundle system	[118]
$\mathcal{P}\mathcal{D}\mathcal{P}(A \times -)$	Pnueli–Zuck system	[391]
$\mathcal{P}\mathcal{D}\mathcal{P}(A \times (-) + (-))$	Most general systems	

Figure 4.2 An overview of (discrete) probabilistic system types. From [59, 434].

An obvious question that comes up is whether we can relate these functors via suitable mappings. This can be done via natural transformations. Recall from Definition 2.5.4 that they are mappings between functors which work in a uniform manner. Below it will be shown that naturality is quite a subtle matter in this setting.

Consider first the probability distribution functor $\mathcal{D}\colon \mathbf{Sets} \to \mathbf{Sets}$. Each distribution $\varphi \in \mathcal{D}(X)$ is a function $\varphi\colon X \to [0,1]$ with finite support $\mathrm{supp}(\varphi) \subseteq X$, given by $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ and probabilities adding up to 1: $\sum_x \varphi(x) = 1$. Thus, we have $\mathrm{supp}(\varphi) \in \mathcal{P}(X)$, or even $\mathrm{supp}(\varphi) \in \mathcal{P}_{\mathrm{fin}}(X)$. In order to emphasise that we take supports of distributions on X we label the support $\mathrm{supp}(\varphi)$ with this set X, as in $\mathrm{supp}_X(\varphi)$. Thus we have a collection of mappings:

$$\Big(\mathcal{D}(X) \xrightarrow{\;\mathrm{supp}_X\;} \mathcal{P}(X)\Big)_{X \in \mathbf{Sets}}.$$

This collection of functions is ‘natural in X’: for each function $f\colon X \to Y$ we have a commuting square:

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{D}(X) \arrow[r, "\mathrm{supp}_X"] \arrow[d, "\mathcal{D}(f)"'] & \mathcal{P}(X) \arrow[d, "\mathcal{P}(f)"] \\ \mathcal{D}(Y) \arrow[r, "\mathrm{supp}_Y"'] & \mathcal{P}(Y) \end{tikzcd}$$

We check in detail that this naturality diagram commutes. For a distribution $\varphi = \sum_{1\leq i\leq n} r_i|x_i\rangle$ in $\mathcal{D}(X)$, with non-zero probabilities $r_i \in (0,1]$, going east to south in the rectangle yields:

$$(\mathcal{P}(f) \circ \mathrm{supp}_X)(\varphi) = \mathcal{P}(f)(\{x_1,\ldots,x_n\}) = \{f(x_1),\ldots,f(x_n)\}.$$

Notice that this result is a set, in which elements $f(x_i) = f(x_j)$ for $i \neq j$ are not distinguished. Similarly, going south to east in the rectangle above gives

$$(\mathrm{supp}_Y \circ \mathcal{D}(f))(\varphi) = \mathrm{supp}_Y(\textstyle\sum_{1\leq i\leq n} r_i|f(x_i)\rangle) = \{f(x_1),\ldots,f(x_n)\}.$$

A subtle point is that when $f(x_i) = f(x_j) = y$, say, then this element occurs as $(r_i + r_j)y$ in the distribution $\mathcal{D}(f)(\varphi)$. This y appears in the resulting support $\mathrm{supp}_Y(\mathcal{D}(f)(\varphi))$ because $r_i+r_j \neq 0$. This is obvious for (non-zero) probabilities $r_i, r_j \in [0,1]$ but needs to be required explicitly for multiset functors.

Call a monoid M **zerosumfree** or **positive** if $x + y = 0$ in M implies $x = y = 0$. For a positive commutative monoid M, the multiset functor $\mathcal{M}_M$ comes with a natural transformation $\mathrm{supp}\colon \mathcal{M}_M \Rightarrow \mathcal{P}_{\mathrm{fin}}$.

If we take the non-positive monoid $M = \mathbb{Z}$ we can show that the support maps $\mathrm{supp}_X\colon \mathcal{M}_{\mathbb{Z}}(X) \to \mathcal{P}_{\mathrm{fin}}(X)$ are *not* natural. Take as example $A = \{a,b\}$ with multiset $\varphi = 1|a\rangle + -1|b\rangle \in \mathcal{M}_{\mathbb{Z}}(A)$. The unique function $!\colon A \to 1 = \{*\}$ then provides a counter example to naturality. On the one hand we have

$$(\mathcal{P}_{\mathrm{fin}}(!) \circ \mathrm{supp}_A)(\varphi) = \mathcal{P}_{\mathrm{fin}}(!)(\{a,b\}) = \{!(a),!(b)\} = \{*\}.$$

But by applying $\mathcal{M}_{\mathbb{Z}}$ first and then support we get a different result:

$$\begin{aligned}(\mathrm{supp}_1 \circ \mathcal{M}_{\mathbb{Z}}(!))(\varphi) &= \mathrm{supp}_1(1|!(a)\rangle + (-1)|!(b)\rangle) \\ &= \mathrm{supp}_1(1|*\rangle + (-1)|*\rangle) = \mathrm{sup}_1(0|*\rangle) = \emptyset.\end{aligned}$$

This phenomenon that certain occurrences cancel each other out is called interference.

In Section 2.5 we have already seen that turning a list into the set of its elements yields a natural transformation $(-)^\star \Rightarrow \mathcal{P}_{\mathrm{fin}}$ and that there are no 'natural' mappings in the other direction, turning finite sets into lists (by choosing some order). Here we show that the obvious mappings $\upsilon_X\colon \mathcal{P}_{\mathrm{fin}}(X) \to \mathcal{D}(X)$, choosing the uniform distribution on a finite subset, are also *not* natural. To be more specific, this mapping υ_X is given by:

$$\upsilon_X(\{x_1,\ldots,x_n\}) = \tfrac{1}{n}|x_1\rangle + \cdots + \tfrac{1}{n}|x_n\rangle.$$

Take for instance $A = \{a, b, c\}$ and $2 = \{\top, \bot\}$ with function $f : A \to 2$ given by $f(a) = f(b) = \top$ and $f(c) = \bot$. Then, for the subset $\{a, b, c\} \in \mathcal{P}_{\mathrm{fin}}(A)$ we have on the one hand:

$$\begin{aligned}(\mathcal{D}(f) \circ \upsilon_A)(\{a, b, c\}) &= \mathcal{D}(f)(\tfrac{1}{3}|a\rangle + \tfrac{1}{3}|b\rangle + \tfrac{1}{3}|c\rangle) \\ &= \tfrac{1}{3}|f(a)\rangle + \tfrac{1}{3}|f(b)\rangle + \tfrac{1}{3}|f(c)\rangle \\ &= \tfrac{1}{3}|\top\rangle + \tfrac{1}{3}|\top\rangle + \tfrac{1}{3}|\bot\rangle \\ &= \tfrac{2}{3}|\top\rangle + \tfrac{1}{3}|\bot\rangle.\end{aligned}$$

On the other hand:

$$\begin{aligned}(\upsilon_2 \circ \mathcal{P}_{\mathrm{fin}}(f))(\{a, b, c\}) &= \upsilon_2(\{f(a), f(b), f(c)\}) \\ &= \upsilon_2(\top, \bot\}) \\ &= \tfrac{1}{2}|\top\rangle + \tfrac{1}{2}|\bot\rangle.\end{aligned}$$

We conclude with a diagram describing some natural transformations between collection functors.

$$\begin{tikzcd} & (-)^{\star} \arrow[dr, bend left] & & \\ \mathcal{D} \arrow[r] & \mathcal{M}_{\mathbb{R}} \arrow[r] & \mathcal{P}_{\mathrm{fin}} \arrow[r] & \mathcal{P} \\ & \underset{(M \text{ positive})}{\mathcal{M}_M} \arrow[ur, bend right] & & \end{tikzcd} \tag{4.5}$$

The map $\mathcal{D} \to \mathcal{M}_{\mathbb{R}}$ has not been discussed explicitly but is the obvious inclusion, forgetting that probabilities add up to 1.

Exercises

4.1.1 Verify that the construction $\mathcal{M}_M(X)$ from Definition 4.1.1 is functorial not only in X, but also in M. Explicitly, for a homomorphism of monoids $g : M \to L$ there is a natural transformation $\mathcal{M}_M \Rightarrow \mathcal{M}_L$.

4.1.2 1. Prove that the multiset functor sends coproducts to products, in the sense that there is an isomorphism:

$$\mathcal{M}_M(X) \times \mathcal{M}_M(Y) \cong \mathcal{M}_M(X + Y).$$

2. Prove that this isomorphism is natural in X and Y.
3. Recall from Exercise 2.1.6 that the product $M \times L$ of (commutative) monoids M, L is at the same time a coproduct (a biproduct). Check that the isomorphism in point (1) is the cotuple

$$\mathcal{M}_M(X) \times \mathcal{M}_M(Y) \xrightarrow{[\mathcal{M}_M(\kappa_1), \mathcal{M}_M(\kappa_2)]} \mathcal{M}_M(X + Y)$$

in the category of commutative monoids.

4.1.3 Interpret, as in Example 4.1.4, what double-multisets in $\mathcal{M}_M(\mathcal{M}_{\mathbb{Z}}(X))$ are. They are usually called (multivariate) *Laurent* polynomials.

4.1.4 The Dirac distribution $\eta_X \colon X \to \mathcal{D}(X)$ is

$$\eta(x) = 1|x\rangle = \lambda y.\, \text{if } y = x \text{ then } 1 \text{ else } 0.$$

Prove that it forms a natural transformation $\eta\colon \mathrm{id} \Rightarrow \mathcal{D}$.

4.1.5 Prove that functions $X \to \mathcal{D}(Y)$ and $Y \to \mathcal{D}(Z)$ can be composed to a function $X \to \mathcal{D}(Z)$, in an associative manner, with the Diract distribution as neutral element. Conclude that Markov chains $X \to \mathcal{D}(X)$ on a set X form a monoid – in analogy with the situation for powersets in Exercise 1.1.2.

4.1.6 Consider the political Markov chain (4.2), as coalgebra (4.3).

1. Describe the coalgebra composed with itself, using composition from the previous exercise.
2. Check that this corresponds to matrix composition, for the corresponding transition matrix.

4.1.7 Calculate the eigenvector of the matrix (4.4) and show that it can be described as point $1 \to \mathcal{D}(\{L, R\})$, forming a fixed point of the coalgebra $c\colon \{L, R\} \to \mathcal{D}(\{L, R\})$ from (4.3), using composition from Exercise 4.1.5.

4.1.8 The distribution functor $\mathcal{D}$ as introduced in Definition 4.1.5 involves distributions $\varphi\colon X \to [0, 1]$ with *finite* support only. This finiteness restriction is not strictly needed. One can define

$$\mathcal{D}^\infty(X) = \{\varphi\colon X \to [0, 1] \mid \textstyle\sum_{x\in X} \varphi(x) \text{ exists and equals } 1\}.$$

1. Prove that in this case the support $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi \neq 0\}$ is necessarily a countable set.
2. Check that $\mathcal{D}^\infty$ is a functor $\mathbf{Sets} \to \mathbf{Sets}$.

(This generalisation involves the existence of sums over non-finite sets; they exist in the special case of the unit interval $[0, 1]$. This infinite generalisation is less natural for multiset functors $\mathcal{M}_M$.)

4.2 Weak Pullbacks

If one starts to develop the theory of coalgebras in full categorical generality, for endofunctors $F\colon \mathbb{C} \to \mathbb{C}$ on an arbitrary category $\mathbb{C}$ (and not just on **Sets**), so-called weak-pullback-preserving functors become relevant, for at least two reasons:

1. For these weak-pullback-preserving functors Aczel–Mendler bisimilarity coincides with equality on the final coalgebra – as in Theorem 3.3.2.
2. For such functors there is a well-behaved categorical version of relation lifting, giving us another way to define bisimulations, namely as coalgebras of these liftings.

The current section concentrates on this first point, and the subsequent Sections 4.3–4.5 on the second one. Hence we start by describing what (weak) pullbacks are.

Definition 4.2.1 Let $\mathbb{C}$ be an arbitrary category.

1. For two morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathbb{C}$ with a common codomain, a **pullback** is a commuting square as on the left:

$$\begin{tikzcd} & V \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} & V' \arrow[d, dashed, "\ell"] \arrow[ddl, bend right, "h'"'] \arrow[ddr, bend left, "k'"] & \\ & V \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

which is universal in the sense that for an arbitrary span $X \xleftarrow{h'} V' \xrightarrow{k'} Y$ with $f \circ h' = g \circ k'$ there is a unique 'mediating' map $\ell \colon V' \to V$ with $h \circ \ell = h'$ and $k \circ \ell = k'$, as on the right. In diagrams a pullback is often indicated via a small angle, as

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

A **weak pullback** is like a pullback except that only existence and not unique existence of the mediating map ℓ is required.

2. A functor $F \colon \mathbb{C} \to \mathbb{D}$ is called (**weak**) **pullback preserving** if it maps (weak) pullback squares to (weak) pullback squares; that is, if F applied a (weak) pullback square in the category $\mathbb{C}$ forms a (weak) pullback square in $\mathbb{D}$.

The pullback of two maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ in the category of sets can be described by the set

$$\mathrm{Eq}(f,g) = \{(x,y) \in X \times Y \mid f(x) = g(y)\},$$

which we used earlier in Lemma 3.2.5. It comes with obvious projections p_i to X and Y as in

$$\begin{tikzcd}
\{(x,y) \in X \times Y \mid f(x) = g(y)\} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Also, inverse images of predicates and relations can be described via pullbacks, as in

$$\begin{tikzcd}
h^{-1}(P) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, tail] \\
X \arrow[r, "h"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(h \times k)^{-1}(R) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail] \\
X \times U \arrow[r, "h \times k"'] & Y \times V
\end{tikzcd}$$

It is a general fact that a pullback of a mono yields a mono; see Exercise 4.2.2 below.

A pullback is a generalisation of a cartesian product of objects (as in Definition 2.1.1): in the presence of a final object $1 \in \mathbb{C}$, the product of two objects $X, Y \in \mathbb{C}$ is the same as the pullback:

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!"] \\
X \arrow[r, "!"'] & 1
\end{tikzcd} \tag{4.6}$$

Similarly, the following result exploits that being a monomorphism can be expressed via pullbacks.

Lemma 4.2.2 *An arbitrary map m in a category $\mathbb{C}$ is a monomorphism if and only if the square*

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, "m"'] & \bullet
\end{tikzcd}$$

is a (weak) pullback.

A (weak) pullback-preserving functor $F\colon \mathbb{C} \to \mathbb{D}$ thus preserves monomorphisms: if m is mono, then so is $F(m)$. ❑

Remark 4.2.3 In category theory it is quite common that structures, such as products and coproducts in Definitions 2.1.1 and 2.1.3, are described via the 'unique existence' of a certain map. The version with only existence, not necessarily unique existence, is typically called 'weak'. For instance, one can have a *weak final object* Z, such that for each object X there is a map $X \to Z$. Each non-empty set is a weak final object in the category **Sets**. There is a third notion in between, denoted by 'semi', involving 'natural existence'; see [208, 226]. For instance, an object Z is semi-final if for each object X there is a map $s_X\colon X \to Z$, and for each $f\colon X \to Y$ one has $s_Y \circ f = s_X$.

Even though 'weak' versions are not standard, weak pullbacks are quite natural in the theory of coalgebras. In particular, preservation of weak pullbacks is a common (and useful) property. It can often be characterised in different ways, as for instance in Proposition 4.2.9 below or in Theorem 4.4.6 later on. Also it plays a crucial role in the distinction between polynomial and analytical functors in Section 4.7.

As mentioned in the introduction to this section, the following result is one of the reasons why weak-pullback preservation is important. It generalises Theorem 3.3.2.

Theorem 4.2.4 *Assume a category $\mathbb{C}$ with pullbacks, and a weak-pullback-preserving functor $F\colon \mathbb{C} \to \mathbb{C}$ with a final coalgebra $Z \xrightarrow{\cong} F(Z)$. Let $c\colon X \to F(X)$ and $d\colon Y \to F(Y)$ be coalgebras. The pullback relation on X, Y in*

$$\begin{tikzcd}
\mathrm{Eq}(\mathrm{beh}_c, \mathrm{beh}_d) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "\langle \mathrm{id}, \mathrm{id}\rangle"] \\
X \times Y \arrow[r, "\mathrm{beh}_c \times \mathrm{beh}_d"'] & Z \times Z
\end{tikzcd} \tag{4.7}$$

is then the greatest Aczel–Mendler bisimulation on coalgebras c, d.

Proof First, if a relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$ is an Aczel–Mendler bisimulation, that is, if R carries a coalgebra $R \to F(R)$ in $\mathbb{C}$ making the legs r_i homomorphisms in

$$\begin{tikzcd}
F(X) & F(R) \arrow[l, "F(r_1)"'] \arrow[r, "F(r_2)"] & F(Y) \\
X \arrow[u, "c"] & R \arrow[l, "r_1"] \arrow[r, "r_2"'] \arrow[u] & Y \arrow[u, "d"']
\end{tikzcd}$$

then by finality $\mathrm{beh}_c \circ r_1 = \mathrm{beh}_d \circ r_2 \colon R \longrightarrow Z$. Hence there is a factorisation through the equality relation on Z, as in

$$\begin{tikzcd}
R \arrow[r, dashed] \arrow[d, tail, "\langle r_1, r_2\rangle"'] & Z \arrow[d, tail, "\langle \mathrm{id}, \mathrm{id}\rangle"] \\
X \times Y \arrow[r, "\mathrm{beh}_c \times \mathrm{beh}_d"'] & Z \times Z
\end{tikzcd}$$

This may be read as: pairs in R are equal when mapped to the final coalgebra. It means that R factors through the pullback $\mathrm{Eq}(\mathrm{beh}_c, \mathrm{beh}_d)$ from (4.7).

We are done if we show that the pullback relation $\mathrm{Eq}(\mathrm{beh}_c, \mathrm{beh}_d) \rightarrowtail X \times Y$ from (4.7) is an Aczel–Mendler bisimulation. Here we use that F preserves weak pullbacks (and Exercise 4.2.3) to obtain a coalgebra structure in

$$\begin{tikzcd}
\mathrm{Eq}(\mathrm{beh}_c, \mathrm{beh}_d) \arrow[rr] \arrow[d] \arrow[dr, dashed] & & Y \arrow[dr, bend left, "d"] & \\
X \arrow[dr, bend right, "c"'] & F(\mathrm{Eq}(\mathrm{beh}_c, \mathrm{beh}_d)) \arrow[rr] \arrow[d] & & F(Y) \arrow[d, "F(\mathrm{beh}_d)"] \\
& F(X) \arrow[rr, "F(\mathrm{beh}_c)"'] & & F(Z)
\end{tikzcd}$$

By construction, the two maps $X \leftarrow \mathrm{Eq}(\mathrm{beh}_c, \mathrm{beh}_d) \rightarrow Y$ are maps of coalgebras. ❑

We add another general result, this time involving preservation of *ordinary* (proper) pullbacks. Recall that a category of coalgebras inherits colimits (coproducts and coequalisers) from the underlying category (see Proposition 2.1.5 and Exercise 2.1.14). The situation for the dual notion of limit – including products, pullbacks and equalisers – is different. In general, they need not exist in categories of coalgebras, even if they exist in the underlying category – see also Section 6.3. The next result shows that pullbacks of coalgebras exist if the functor preserves ordinary pullbacks. As we shall see subsequently, in Proposition 4.2.6, this applies to simple polynomial functors.

Proposition 4.2.5 *Assume a functor $F \colon \mathbb{C} \to \mathbb{C}$ that preserves (ordinary) pullbacks. If the category $\mathbb{C}$ has pullbacks, then so has the category of coalgebras* $\mathbf{CoAlg}(F)$.

Proof Assume we have span of coalgebras as on the left below. Then we take the pullback in $\mathbb{C}$ of the underlying maps f, g, as on the right:

$$\begin{tikzcd} & \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix} \arrow[d, "g"] \\ \begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \arrow[r, "f"'] & \begin{pmatrix} F(W) \\ \uparrow e \\ W \end{pmatrix} \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & W \end{tikzcd}$$

We need a coalgebra structure on the object P. Here we use that F preserves pullbacks and obtain this structure in

$$\begin{tikzcd} P \arrow[dr, dashed, "b"] \arrow[drr, bend left, "d \circ p_2"] \arrow[ddr, bend right, "c \circ p_1"'] & & \\ & F(P) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"] \arrow[dr, phantom, "\lrcorner", very near start] & F(Y) \arrow[d, "F(g)"] \\ & F(X) \arrow[r, "F(f)"'] & F(W) \end{tikzcd}$$

The outer maps in this diagram commute:

$$\begin{aligned} F(f) \circ c \circ p_1 &= e \circ f \circ p_1 && \text{since } f \text{ is a map of coalgebras} \\ &= e \circ g \circ p_2 && \text{since the pullback square in } \mathbb{C} \text{ commutes} \\ &= F(g) \circ d \circ p_2 && \text{because } g \text{ is a map of coalgebras too.} \end{aligned}$$

Hence the dashed map (coalgebra) $b \colon P \to F(P)$ exists. Thus we have constructed a commuting square of coalgebra maps:

$$\begin{tikzcd} \begin{pmatrix} F(P) \\ \uparrow b \\ P \end{pmatrix} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix} \arrow[d, "g"] \\ \begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \arrow[r, "f"'] & \begin{pmatrix} F(W) \\ \uparrow e \\ W \end{pmatrix} \end{tikzcd}$$

In order to obtain this square, preservation of *weak* pullbacks would have been enough. But in order to show that it is a pullback in the category of coalgebras

CoAlg(F) we need that the functor F preserves proper pullbacks. Suppose we have two coalgebra homomorphisms

$$\begin{pmatrix} F(U) \\ \uparrow a \\ U \end{pmatrix} \xrightarrow{\;h\;} \begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} F(U) \\ \uparrow a \\ U \end{pmatrix} \xrightarrow{\;k\;} \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix}$$

with $f \circ h = g \circ k$. Then there is a unique mediating map $\ell \colon U \to P$ in $\mathbb{C}$ with $p_1 \circ \ell = h$ and $p_2 \circ \ell = k$. What remains is showing that ℓ is a homomorphism of coalgebras. Here we use that F preserves pullbacks. There are two parallel maps in

$$\begin{tikzcd}
U \arrow[dr, shift left] \arrow[dr, shift right] \arrow[drr, bend left, "F(k) \circ a"] \arrow[ddr, bend right, "F(h) \circ a"'] & & \\
& F(P) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"] \arrow[dr, phantom, "\lrcorner", very near start] & F(Y) \arrow[d, "F(g)"] \\
& F(X) \arrow[r, "F(f)"'] & F(W)
\end{tikzcd}$$

namely $b \circ \ell$ and $F(\ell) \circ a$. Thus, by uniqueness, $b \circ \ell = F(\ell) \circ a$, making ℓ a homomorphism of coalgebras. Uniqueness of ℓ in **CoAlg**(F) is left to the reader. ❑

The remainder of this section concentrates on preservation of (weak) pullbacks for specific functors. It is not hard to see that preservation of ordinary pullbacks implies preservation of weak pullbacks. This is left as exercise below. In Section 4.7 we shall see that preservation of (countably indexed) weak pullbacks and pullbacks is a key property of simple polynomial and analytical functors.

Proposition 4.2.6 1. *Every exponent polynomial functor* **Sets** $\to$ **Sets** *preserves pullbacks (and also weak ones).*

2. *The powerset functors $\mathcal{P}$ and $\mathcal{P}_{\mathit{fin}}$ preserve weak pullbacks.*

In particular, Kripke polynomial functors, finite or not, preserve weak pullbacks.

Proof 1. Trivially, identity and constant functors preserve pullbacks, so we look only at (set-indexed) coproducts and exponents. So assume for an index set I we have pullbacks in **Sets** as on the left below:

$$\begin{tikzcd} W_i \arrow[r, "k_i"] \arrow[d, "h_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_i \arrow[d, "g_i"] \\ X_i \arrow[r, "f_i"'] & Z_i \end{tikzcd} \qquad \begin{tikzcd} U \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & \coprod_{i\in I} W_i \arrow[r, "\coprod_{i\in I} k_i"] \arrow[d, "\coprod_{i\in I} h_i"] & \coprod_{i\in I} Y_i \arrow[d, "\coprod_{i\in I} g_i"] \\ & \coprod_{i\in I} X_i \arrow[r, "\coprod_{i\in I} f_i"'] & \coprod_{i\in I} Z_i \end{tikzcd}$$

We have to show that the square on the right is again a pullback. So assume we have a set U with maps $\alpha\colon U \to \coprod_{i\in I} X_i$ and $\beta\colon U \to \coprod_{i\in I} Y_i$, as indicated, satisfying $(\coprod_{i\in I} f_i) \circ \alpha = (\coprod_{i\in I} g_i) \circ \beta$. We can decompose each of the maps α, β into two parts, $\alpha = \langle \alpha_1, \alpha_2 \rangle$ and $\beta = \langle \beta_1, \beta_2 \rangle$, where

$$\alpha_1, \beta_1\colon U \longrightarrow I, \qquad \text{and for each } u \in U, \qquad \begin{cases} \alpha_2(u) \in X_{\alpha_1(u)} \\ \beta_2(u) \in Y_{\beta_1(u)}\,. \end{cases}$$

Because the diagram on the right commutes we have

$$\alpha_1 = \beta_1 \qquad \text{and for each } u \in U, \quad f_{\alpha_1(u)}(\alpha_2(u)) = g_{\alpha_1(u)}(\beta_2(u)).$$

Since the diagram on the left is a pullback there is for each $u \in U$ a unique element $\gamma(u) \in W_{\alpha_1(u)}$ with $h_{\alpha_1(u)}(\gamma(u)) = \alpha_2(u)$ and $k_{\alpha_1(u)}(\gamma(u)) = \beta_2(u)$. Thus we get a map $\langle \alpha_1, \gamma \rangle\colon U \to \coprod_{i\in I} W_i$ on the right. It is the unique mediating map.

For exponents assume again a pullback on the left in

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} U \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & W^A \arrow[r, "k^A"] \arrow[d, "h^A"] & Y^A \arrow[d, "g^A"] \\ & X^A \arrow[r, "f^A"'] & Z^A \end{tikzcd}$$

We need to find a mediating map on the right. This is straightforward by a pointwise construction: for each $u \in U$ and $a \in A$ we have

$$f(\alpha(u)(a)) = (f^A \circ \alpha)(u)(a) = (g^A \circ \beta)(u)(a) = g(\beta(u)(a)).$$

Hence there is a unique element $\gamma(u)(a) \in W$ with $h(\gamma(u)(a)) = \alpha(u)(a)$ and $k(\gamma(u)(a)) = \beta(u)(a)$. Thus we get a function $\gamma\colon U \to W^A$, as required.

2. We first consider the powerset functor, assuming a weak pullback as on the left below:

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} U \arrow[drr, bend left, "\beta"] \arrow[dr, bend right, "\alpha"'] & & \\ & \mathcal{P}(W) \arrow[r, "\mathcal{P}(k)"] \arrow[d, "\mathcal{P}(h)"] & \mathcal{P}(Y) \arrow[d, "\mathcal{P}(g)"] \\ & \mathcal{P}(X) \arrow[r, "\mathcal{P}(f)"'] & \mathcal{P}(Z) \end{tikzcd}$$

For the diagram on the right we assume $\mathcal{P}(f) \circ \alpha = \mathcal{P}(g) \circ \beta$. This means for each $u \in U$,

$$\{f(x) \mid x \in \alpha(u)\} = \{g(y) \mid y \in \beta(u)\}.$$

Thus, for each $x \in \alpha(u)$ there is an $y_x \in \beta(u)$ with $f(x) = g(y_x)$, and thus there is a $w_x \in W$ with $h(w_x) = x$ and $k(w_x) = y_x$. Similarly, for each $y \in \beta(u)$ there is an $x_y \in \alpha(u)$ with $f(x_y) = g(y)$ and so a $w_y \in W$ with $h(w_y) = x_y$ and $k(w_y) = y$. Now take $V(u) = \{w_x \mid x \in \alpha(u)\} \cup \{w_y \mid y \in \beta(u)\}$. Then

$$\begin{aligned} \mathcal{P}(h)(V(u)) &= \{h(w_x) \mid x \in \alpha(u)\} \cup \{h(w_y) \mid y \in \beta(u)\} \\ &= \{x \mid x \in \alpha(u)\} \cup \{x_y \mid y \in \beta(u)\} \\ &= \alpha(u), \end{aligned}$$

and similarly $\mathcal{P}(k)(V(u)) = \beta(u)$. ❑

The multiset functors $\mathcal{M}_M$ from Definition 4.1.1 do *not* preserve weak pullbacks in general. In [185] it is shown that this holds if and only if the commutative monoid M is both positive – so that $m + m' = 0$ implies $m = m' = 0$ – and a so-called refinement monoid (introduced in [123]).

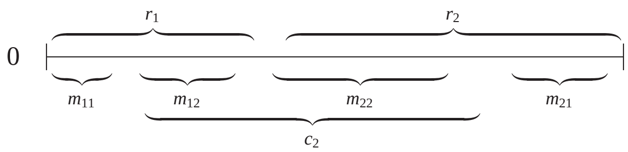
Definition 4.2.7 A commutative monoid $M = (M, 0, +)$ is called a **refinement monoid** if for each equation

$$r_1 + r_2 = c_1 + c_2$$

there is a 2×2 ‘matrix’ m_{ij} with

$$\begin{cases} r_i = m_{i1} + m_{i2} \\ c_j = m_{1j} + m_{2j} \end{cases} \quad \text{as depicted in} \quad \begin{array}{cc|c} m_{11} & m_{12} & r_1 \\ m_{21} & m_{22} & r_2 \\ \hline c_1 & c_2 & \end{array} .$$

The natural numbers $\mathbb{N}$ with addition are obviously a positive monoid. They are also a refinement monoid: assume $r_1 + r_2 = c_1 + c_2$ for $r_i, c_i \in \mathbb{N}$. The numbers m_{ij} that we need to find are written below the relevant segment of $\mathbb{N}$ in



$r_1 + r_2 =$
$c_1 + c_2.$

Clearly there are ways to shift this c_2 left or right, in such a way that the m_{ij} are fixed.

It is not hard to see that each abelian group is a refinement monoid: if $r_1 + r_2 = c_1 + c_2$, then we can take for example as matrix

$$\begin{array}{ccc} r_1 - c_1 & c_1 & r_1 \\ 2c_1 - r_1 & c_2 - c_1 & r_2 \\ \hline c_1 & c_2 & \end{array}$$

using (for the second row) that $(2c_1 - r_1) + (c_2 - c_1) = c_1 + c_2 - r_1 = r_1 + r_2 - r_1 = r_2$.

Lemma 4.2.8 (From [123]; also in [185]) *A commutative monoid is a positive refinement monoid if and only if the refinement property holds for each pair of numbers $(n, k) \in \mathbb{N}^2$. That is, if*

$$r_1 + \cdots + r_n = c_1 + \cdots + c_k,$$

then there is $n \times k$ matrix $(m_{ij})_{i \le n, j \le k}$ with

$$r_i = \textstyle\sum_j m_{ij} \qquad \textit{and} \qquad c_j = \textstyle\sum_i m_{ij}.$$

Proof For pairs $(0, k)$ and $(n, 0)$ positivity is used. For pairs of the form $(1, k)$ and $(n, 1)$ the result trivially holds. For the other cases it suffices to see how $(2, 3)$-refinement follows from $(2, 2)$-refinement. So assume we have $r_1 + r_2 = c_1 + c_2 + c_3$. We obtain the following successive refinements:

$$\begin{array}{ccc} m_{11} & m_{12} & r_1 \\ m_{21} & m_{22} & r_2 \\ \hline c_1 & c_2 + c_3 & \end{array} \quad \begin{array}{ccc} m'_{11} & m'_{12} & m_{12} \\ m'_{21} & m'_{22} & m_{22} \\ \hline c_2 & c_3 & \end{array} \quad \begin{array}{cccc} m_{11} & m'_{11} & m'_{12} & r_1 \\ m_{21} & m'_{21} & m'_{22} & r_2 \\ \hline c_1 & c_2 & c_3 & \end{array}.$$

$\square$

Now we can address preservation of weak pullbacks for the multiset functors $\mathcal{M}_M \colon \mathbf{Sets} \to \mathbf{Sets}$ from Definition 4.1.1.

Proposition 4.2.9 (From [185]) *A multiset functor $\mathcal{M}_M$ preserves weak pullbacks if and only if the commutative monoid M is both positive and a refinement monoid.*

Proof First, assume $\mathcal{M}_M$ preserves weak pullbacks. We show that M is a refinement monoid. So assume we have $r_1 + r_2 = c_1 + c_2$. For the set $2 = \{0, 1\}$ we consider the pullback in **Sets** on the left, and the resulting weak pullback on the right:

$$\begin{tikzcd} 2 \times 2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2 \arrow[d, "!"] \\ 2 \arrow[r, "!"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_M(2 \times 2) \arrow[r, "\mathcal{M}_M(\pi_2)"] \arrow[d, "\mathcal{M}_M(\pi_1)"'] & \mathcal{M}_M(2) \arrow[d, "\mathcal{M}_M(!)"] \\ \mathcal{M}_M(2) \arrow[r, "\mathcal{M}_M(!)"'] & \mathcal{M}_M(1) \cong M \end{tikzcd}$$

We use the numbers $r_i, c_i \in M$ in multisets $\varphi, \psi \in \mathcal{M}_M(2)$, namely via $\varphi(i) = r_i$ and $\psi(i) = c_i$, for $i \in 2 = \{0, 1\}$. The equation $r_1 + r_2 = c_1 + c_2$ yields $\mathcal{M}_M(!)(\varphi) = \mathcal{M}_M(!)(\psi)$. The weak pullback property gives a multiset $\chi \in \mathcal{M}_M(2\times 2)$ with $\mathcal{M}_M(\pi_1)(\chi) = \varphi$ and $\mathcal{M}_M(\pi_2)(\chi) = \psi$. Writing $\chi(i, j) = m_{ij}$ we get the required matrix, since, for instance

$$r_1 = \varphi(0) = \mathcal{M}_M(\pi_1)(\chi)(0) = \textstyle\sum_j \chi(0, j) = m_{00} + m_{01}.$$

Next we show that M is positive. If we have $m, m' \in M$ with $m + m' = 0$ we consider the pullback in **Sets** on the left, and the resulting weak pullback on the right:

$$\begin{tikzcd} 0 \arrow[r, "\mathrm{id}"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d, "!"] \\ 2 \arrow[r, "!"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_M(0) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathcal{M}_M(!)"'] & \mathcal{M}_M(0) \cong 1 \arrow[d, "\mathcal{M}_M(!) = 0"] \\ \mathcal{M}_M(2) \arrow[r, "\mathcal{M}_M(!)"'] & \mathcal{M}_M(1) \cong M \end{tikzcd}$$

The pair m, m' gives rise to the multiset $\varphi \in \mathcal{M}_M(2)$ given by $\varphi(0) = m$ and $\varphi(1) = m'$. It satisfies $\mathcal{M}_M(!)(\varphi) = m + m' = 0 = \mathcal{M}_M(!)(0)$, for the empty multiset $0 \in \mathcal{M}_M(0)$. The weak pullback property now yields an element in $\mathcal{M}_M(0)$, necessarily the empty multiset 0, with $\mathcal{M}_M(!)(0) = \varphi$. Thus $\varphi = 0$ and so $m = m' = 0$.

Conversely, assume M is a positive refinement monoid. If we have a weak pullback as on the left, we need to show that the square on the right is also a weak pullback:

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{M}_M(W) \arrow[r, "\mathcal{M}_M(k)"] \arrow[d, "\mathcal{M}_M(h)"'] & \mathcal{M}_M(Y) \arrow[d, "\mathcal{M}_M(g)"] \\ \mathcal{M}_M(X) \arrow[r, "\mathcal{M}_M(f)"'] & \mathcal{M}_M(Z) \end{tikzcd}$$

So suppose we have multisets $\varphi \in \mathcal{M}_M(X)$ and $\psi \in \mathcal{M}_M(Y)$ with $\mathcal{M}_M(f)(\varphi) = \mathcal{M}_M(g)(\psi)$. This means

$$\sum_{x\in f^{-1}(z)} \varphi(x) = \sum_{y\in g^{-1}(z)} \psi(y)$$

for each $z \in Z$. We can limit these sums to the finite subset $Z' \subseteq Z$, given by

$$Z' = \{f(x) \mid x \in \mathrm{supp}(\varphi)\} \cup \{g(y) \mid y \in \mathrm{supp}(\psi)\}.$$

For each $z \in Z'$ we now have $\sum_{x\in f^{-1}(z)} \varphi(x) = \sum_{y\in g^{-1}(z)} \psi(y)$. This has the shape of a general refinement problem, as in Lemma 4.2.8. Thus there is a matrix (m^z_{xy}), where $x \in f^{-1}(z) \cap \mathrm{supp}(\varphi)$ and $y \in g^{-1}(z) \cap \mathrm{supp}(\psi)$ with

$$\varphi(x) = \textstyle\sum_y m^z_{xy} \qquad \text{and} \qquad \psi(y) = \sum_x m^z_{xy}.$$

For each pair $x \in f^{-1}(z) \cap \mathrm{supp}(\varphi), y \in g^{-1}(z) \cap \mathrm{supp}(\psi)$ we use the weak pullback on the left and choose an element $w^z_{xy} \in W$ with $h(w^z_{xy}) = x$ and $k(w^z_{xy}) = y$. This yields a multiset $\chi_z = \sum_{x,y} m^z_{xy} w^z_{xy} \in \mathcal{M}_M(W)$.

By doing this for each $z \in Z'$ we can form $\chi = \sum_{z\in Z'} \chi_z \in \mathcal{M}_M(W)$. Notice that if $z_1 \neq z_2$, then $\mathrm{supp}(\chi_{z_1}) \cap \mathrm{supp}(\chi_{z_2}) = \emptyset$. Indeed, if $w \in \mathrm{supp}(\chi_{z_1}) \cup \mathrm{supp}(\chi_{z_2})$, then $z_1 = (f \circ h)(w) = z_2$, which is impossible. Finally,

$$\begin{aligned}
&\mathcal{M}_M(h)(\chi) \\
&= \textstyle\sum_{z\in Z'} \mathcal{M}_M(h)(\chi_z) \\
&= \textstyle\sum_{z\in Z'} \sum\{m^z_{xy}|h(w^z_{xy})\rangle \mid x \in f^{-1}(z) \cap \mathrm{supp}(\varphi), y \in g^{-1}(z) \cap \mathrm{supp}(\psi)\} \\
&= \textstyle\sum_{x\in\mathrm{supp}(\varphi)} \sum\{m^z_{xy}|h(w^z_{xy})\rangle \mid z = f(x), y \in g^{-1}(z) \cap \mathrm{supp}(\psi)\} \\
&= \textstyle\sum_{x\in\mathrm{supp}(\varphi)} \varphi(x)|x\rangle \\
&= \varphi.
\end{aligned}$$

Similarly one gets $\mathcal{M}_M(k)(\chi) = \psi$. ❑

For the distribution functor the situation is simpler.

Proposition 4.2.10 *The distribution functor* $\mathcal{D}$: **Sets** → **Sets** *from Definition 4.1.5 preserves weak pullbacks.*

Proof This result can be obtained in several ways. One can mimic the proof for the multiset functor $\mathcal{M}_M$, after observing that the unit interval $[0,1]$ of probabilities is a ‘partial commutative monoid’ that is positive and satisfies the refinement property (in a suitably generalised sense). This approach is followed (implicitly) in [359]. Alternatively, one can follow the appendix of [459] which contains a proof using the max-flow min-cut theorem from graph theory. ❑

The ‘continuous’ analogue of the (discrete) distribution functor $\mathcal{D}$ on **Sets** is the ‘Giry’ functor $\mathcal{G}$ on the category of measurable spaces; see e.g. [157, 117, 273, 369, 124, 252]. In [458] it is shown that this functor $\mathcal{G}$ does not preserve weak pullbacks. Another such example occurs in Exercise 4.2.11 below.

Exercises

4.2.1 Describe the graph Graph(f) of a function via a pullback in **Sets**.

4.2.2 Consider in an arbitrary category a pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Prove that if m is a mono, then so is m'.

4.2.3 Verify that taking a pullback on the left is the same as taking a pullback on the right:

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d, tail, "\langle p_1, p_2\rangle"'] & Z \arrow[d, tail, "\langle \mathrm{id}, \mathrm{id}\rangle"] \\ X \times Y \arrow[r, "f \times g"'] & Z \times Z \end{tikzcd}$$

4.2.4 Let $\mathbb{C}$ be a category with pullbacks. Prove that if a functor $F\colon \mathbb{C} \to \mathbb{C}$ preserves pullbacks, then it also preserves weak pullbacks.

4.2.5 Let F be a weak-pullback-preserving functor $\mathbb{C} \to \mathbb{C}$.

1. Prove that F preserves (ordinary) pullbacks of monos: if the diagram below on the left is a pullback, then so is the one on the right:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, tail, "m"'] & \bullet \arrow[d, tail, "n"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "F(f)"] \arrow[d, tail, "F(m)"'] & \bullet \arrow[d, tail, "F(n)"] \\ \bullet \arrow[r, "F(g)"'] & \bullet \end{tikzcd}$$

This result can be written as $F(g^{-1}(n)) \xrightarrow{\cong} F(g)^{-1}(F(n))$.

2. Conclude that F also preserves intersections $\wedge$ of monos, given as diagonal in a pullback square:

$$\begin{tikzcd} \bullet \arrow[r, tail] \arrow[d, tail] \arrow[dr, tail, "m \wedge n"] & \bullet \arrow[d, tail, "n"] \\ \bullet \arrow[r, tail, "m"'] & \bullet \end{tikzcd}$$

4.2.6 The following two results are known as the pullback lemmas. Prove them yourself.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(B)}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(A)}"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

1. If (A) and (B) are pullback squares, then the outer rectangle is also a pullback square.
2. If the outer rectangle and (A) are pullback squares, then (B) is a pullback square as well.

4.2.7 Let $F: \mathbb{C} \to \mathbb{C}$ be an endofunctor on a category $\mathbb{C}$ with pullbacks. Prove that the category **Alg**(F) of algebras also has pullbacks, constructed as in $\mathbb{C}$.

4.2.8 Assume a category $\mathbb{C}$ with pullbacks. Prove that each slice category $\mathbb{C}/I$ (see Exercise 1.4.3) then has

1. products (and trivially also a final object)
2. pullbacks.

4.2.9 Prove, using the axiom of choice, that in the category **Sets** a diagram

$$\begin{tikzcd} \bullet \arrow[r, "k"] \arrow[d, "h"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \quad \text{is a \emph{weak} pullback} \quad \text{iff} \quad \mathrm{Eq}(f, g) \subseteq \mathrm{Im}(\langle h, k \rangle).$$

4.2.10 Consider the following pullback in **Sets**:

$$\begin{tikzcd}
\{0,1,2,3,4,5\} = 6 \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{u,v,w\} \arrow[d, "g"] \\
\{a,b,c,d\} \arrow[r, "f"'] & 2 = \{0,1\}
\end{tikzcd}$$

where

$$\begin{cases} f(a) = f(b) = 0, f(c) = f(d) = 1 \\ g(u) = 0, g(v) = g(w) = 1 \\ h(0) = a, h(1) = b, h(2) = h(3) = c, h(4) = h(5) = d \\ k(0) = k(1) = u, k(2) = k(4) = v, k(3) = k(5) = w. \end{cases}$$

Let multisets $\varphi \in \mathcal{M}_{\mathbb{N}}(\{a,b,c,d\})$ and $\psi \in \mathcal{M}_{\mathbb{N}}(\{u,v,w\})$ be given by

$$\varphi = 2|a\rangle + 3|b\rangle + 4|c\rangle + 5|d\rangle, \qquad \psi = 5|u\rangle + 3|v\rangle + 6|w\rangle.$$

1. Check that $\mathcal{M}_{\mathbb{N}}(f)(\varphi) = \mathcal{M}_{\mathbb{N}}(g)(\psi)$.
2. Find a multiset $\chi \in \mathcal{M}_{\mathbb{N}}(6)$ with $\mathcal{M}_{\mathbb{N}}(h)(\chi) = \varphi$ and $\mathcal{M}_{\mathbb{N}}(k)(\chi) = \psi$.
3. There are four such χ; describe them all.

4.2.11 ([411]) Consider the neighbourhood functor $\mathcal{N}\colon \mathbf{Sets} \to \mathbf{Sets}$ from Exercise 2.2.7, given by the contravariant powerset functor composed with itself. That is, $\mathcal{N}(X) = \mathcal{P}(\mathcal{P}(X))$, and for $f\colon X \to Y$ we have $\mathcal{N}(f)\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(Y))$ by

$$\mathcal{N}(f)(A \subseteq \mathcal{P}(X)) = (f^{-1})^{-1}(A) = \{V \in \mathcal{P}(Y) \mid f^{-1}(V) \in A\}.$$

Later on, in Example 5.1.3.7, we shall recognise $\mathcal{N} = 2^{(2^{(-)})}$ as an example of a continuation monad.

Prove that $\mathcal{N}$ does *not* preserve weak pullbacks. *Hint*: Use the pullback of the following functions f, g (as in [411]):

$$\begin{cases} X = \{x_1, x_2, x_3\} \\ Z = \{z_1, z_2\} \end{cases} \qquad \begin{array}{rcl} X & \xrightarrow{f} & Z \\ x_1, x_2 & \longmapsto & y_1 \\ x_3 & \longmapsto & y_2 \end{array} \qquad \begin{array}{rcl} X & \xrightarrow{g} & Z \\ x_1 & \longmapsto & y_1 \\ x_2, x_3 & \longmapsto & y_2\,. \end{array}$$

4.3 Predicates and Relations

Recall from Section 3.1 that the notion of bisimulation (and hence of bisimilarity) is introduced via the lifting of a polynomial functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ to an endofunctor $\mathrm{Rel}(F)\colon \mathbf{Rel} \to \mathbf{Rel}$ on a category of relations. Bisimulations

R are then coalgebras $R \to \mathrm{Rel}(F)(R)$ of this functor, and congruences are $\mathrm{Rel}(F)$-algebras. In the next few sections we generalise this situation from **Sets** to more general categories $\mathbb{C}$. But first we need to better understand what predicates and relations are, in an arbitrary category $\mathbb{C}$. That is the topic of the present section. We shall describe these predicates and relations via 'logical' structure in $\mathbb{C}$, expressed in the form of a suitable factorisation system. As we shall see in Example 4.3.8 this factorisation structure corresponds to a predicate logic with finite conjunctions $\top, \wedge$, existential quantification $\exists$ and comprehension $\{x\colon \sigma \mid \varphi\}$.

First we need to establish some basic notions and terminology – some of which has already been used (implicitly). We start by ordering monomorphisms. Given two monos $m\colon U \rightarrowtail X$ and $n\colon V \rightarrowtail X$ one writes $m \leq n$ if there is a necessarily unique, dashed map φ in

$$\begin{tikzcd} U \arrow[rr, dashed, "\varphi"] \arrow[dr, tail, "m"'] & & V \arrow[dl, tail, "n"] \\ & X & \end{tikzcd} \qquad \text{with} \qquad n \circ \varphi = m.$$

This order $\leq$ is reflexive ($m \leq m$) and transitive ($m \leq n$ and $n \leq k$ implies $m \leq k$) and is thus a preorder. If we write $m \cong n$ for $m \leq n$ and $n \leq m$, then $\cong$ is an equivalence relation. An equivalence class $[m] = \{n \mid n \cong m\}$ of such monos is called a **subobject**. These subobjects are seen as predicates on the object X. They are partially ordered, via $\leq$ as described above.

In practice one often writes m for the corresponding equivalence class $[m]$. Thus, we often say things like: consider a subobject $m\colon U \rightarrowtail X$ with ...

In the category **Sets**, monos $U \rightarrowtail X$ are injections and subobjects $U \rightarrowtail X$ are subsets $U \subseteq X$. Thus, a relation $R \subseteq X \times Y$ is a subobject $R \rightarrowtail X \times Y$ in **Sets**. More generally, in a category $\mathbb{C}$, a relation is a subobject $R \rightarrowtail X \times Y$. Such relations carry a partial order $\leq$, as subobjects of $X \times Y$.

In practical situations it is sometimes more appropriate to consider certain subsets of monos as predicates. This will be illustrated in the case of directed complete partial orders (dcpos).

Example 4.3.1 Recall the category **Dcpo** of directed complete partial orders from Example 1.4.2.3d. These orders play an important role in the semantics of many programming constructs (see e.g. [188, 356, 34]). Here we take a closer look at predicates in **Dcpo**.

For a dcpo D a subset $U \subseteq D$ is called **admissible** if, with the order inherited from D, it is closed under directed joins – and thus a sub-dcpo. These admissible subsets look like subobjects (equivalence classes of monos) in **Dcpo**, but they are not. They correspond to a special class of monos, namely those monos that reflect the order. Thus, an admissible subset $U \subseteq D$ can be identified with an equivalence class of maps $m\colon E \to D$ in **Dcpo** that reflect the order: $m(x) \le m(x')$ in D iff $x \le x'$ in E. Such a map m is automatically injective: $m(x) = m(x')$ implies $m(x) \le m(x')$ and $m(x') \le m(x)$, and thus $x \le x'$ and $x' \le x$, so $x = x'$.

The appropriate notion to capture such situations where one wants to consider only particular subsets of monos is a factorisation system; see [56, 36, 146]. In general, such a factorisation system is given by two collections of morphisms, $\mathfrak{M}$ for 'abstract monos' and $\mathfrak{E}$ for 'abstract epis', satisfying certain properties. In the present 'logical' context we add three special requirements, in points (4)–(6) below, to the standard properties; so we will speak of a 'logical factorisation system'.

Definition 4.3.2 In an arbitrary category $\mathbb{C}$, a **logical factorisation system** is given by two collections of maps $\mathfrak{M}$ and $\mathfrak{E}$ satisfying the following properties:

1. Both $\mathfrak{M}$ and $\mathfrak{E}$ contain all isomorphisms from $\mathbb{C}$ and are closed under composition.
2. Each map $f\colon X \to Y$ in $\mathbb{C}$ can be factored as a map $\mathfrak{e}(f) \in \mathfrak{E}$ followed by a map $\mathfrak{m}(f) \in \mathfrak{M}$, as in

$$\begin{tikzcd} X \arrow[r, two heads, "\mathfrak{e}(f)"] \arrow[dr, "f"'] & \mathrm{Im}(f) \arrow[d, tail, "\mathfrak{m}(f)"] \\ & Y \end{tikzcd}$$

   In such diagrams we standardly write special arrows $\rightarrowtail$ for maps from $\mathfrak{M}$ and $\twoheadrightarrow$ for maps from $\mathfrak{E}$. Maps in $\mathfrak{M}$ and $\mathfrak{E}$ are sometimes called abstract monos and abstract epis, respectively.
3. The diagonal-fill-in property holds: in a commuting square as indicated below, there is a unique diagonal map as indicated, making the two triangles commute:

$$\begin{tikzcd} \bullet \arrow[r, two heads, "\text{in } \mathfrak{E}"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dashed] \\ \bullet \arrow[r, tail, "\text{in } \mathfrak{M}"'] & \bullet \end{tikzcd}$$

4. All maps in $\mathfrak{M}$ are monos.

5. The collection $\mathfrak{M}$ is closed under pullback: if $m \in \mathfrak{M}$, then the pullback $f^{-1}(m)$ along an arbitrary map f in $\mathbb{C}$ exists, and $f^{-1}(m) \in \mathfrak{M}$, as in

$$\begin{tikzcd} f^{-1}(V) \arrow[r] \arrow[d, tail, "f^{-1}(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, tail, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{4.8}$$

   Notice that we have overloaded the notation f^{-1} by writing both $f^{-1}(V)$ and $f^{-1}(n)$. This is often convenient.
6. For each $m \in \mathfrak{M}$ and $e \in \mathfrak{E}$ we have $m^{-1}(e) \in \mathfrak{E}$; this pullback exists by the previous point.

The standard example of a logical factorisation system is given by $\mathfrak{M}$ = (*injections*) and $\mathfrak{E}$ = (*surjections*) in **Sets**. But there are many other examples, for instance with the abstract monos $\mathfrak{M}$ given by admissible subsets on dcpos (Example 4.3.1), closed subsets of metric or topological spaces, or linearly closed subsets of vector spaces. The maps in $\mathfrak{E}$ are those whose 'obvious' factorisation has an isomorphism as monic part. Examples will be discussed explicitly in Example 4.3.7 below.

It can be shown that the collections $\mathfrak{M}$ and $\mathfrak{E}$ determine each other; see Exercise 4.3.5.

The above definition requires that the collection $\mathfrak{M}$ of abstract monos is closed under pullbacks. Sometimes it also happens that $\mathfrak{E}$ is closed under pullback, but such stability may fail (for a counter-example involving admissible subsets of dcpos, see [113, chapter 1, exercise 7]). It does hold in **Sets**, where surjections are indeed closed under pullback.

We now define categories of predicates and relations with respect to a logical factorisation system.

Definition 4.3.3 For a category $\mathbb{C}$ with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$ we define:

- the category $\mathrm{Pred}(\mathbb{C})$ of predicates in $\mathbb{C}$
- the category $\mathrm{Rel}(\mathbb{C})$ of relations in $\mathbb{C}$ – provided $\mathbb{C}$ has products $\times$.

Notice that in these notations $\mathrm{Pred}(\mathbb{C})$ and $\mathrm{Rel}(\mathbb{C})$ we leave $(\mathfrak{M}, \mathfrak{E})$ implicit. Usually it is clear for a given category what the relevant factorisation system is.

Objects of the category $\mathrm{Pred}(\mathbb{C})$ are subobjects/predicates $(m\colon U \rightarrowtail X)$ of maps $m \in \mathfrak{M}$. Morphisms from $(m\colon U \rightarrowtail X)$ to $(n\colon V \rightarrowtail Y)$ in $\mathrm{Pred}(\mathbb{C})$ are maps $f \colon X \to Y$ in $\mathbb{C}$ for which there is a necessarily unique dashed map as on the left below:

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d, tail, "m"'] & V \arrow[d, tail, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} R \arrow[r, dashed] \arrow[d, tail, "\langle r_1, r_2\rangle"'] & S \arrow[d, tail, "\langle s_1, s_2\rangle"] \\ X_1 \times X_2 \arrow[r, "f_1 \times f_2"] & Y_1 \times Y_2 \end{tikzcd}$$

Intuitively, this says $\forall x \in X.\, U(x) \Rightarrow V(f(x))$.

The category $\mathrm{Rel}(\mathbb{C})$ has relations/subobjects $\langle r_1, r_2\rangle \colon R \rightarrowtail X_1 \times X_2$ as objects, where the tuple $\langle r_1, r_2\rangle$ is in $\mathfrak{M}$. Morphisms in $\mathrm{Rel}(\mathbb{C})$ from the relation $\langle r_1, r_2\rangle \colon R \rightarrowtail X_1 \times X_2$ to $\langle s_1, s_2\rangle \colon S \rightarrowtail Y_1 \times Y_2$ are pairs of morphisms $f_1 \colon X_1 \to Y_1$, $f_2 \colon X_2 \to Y_2$ in $\mathbb{C}$ for which there is a necessarily unique morphism $R \to S$ making the diagram on the right commute. It says $R(x_1, x_2) \Rightarrow S(f_1(x_1), f_2(x_2))$.

For an object $X \in \mathbb{C}$ we sometimes write $\mathrm{Pred}(X)$ for the partial order of subobjects $U \rightarrowtail X$ of an object X, coming from $\mathfrak{M}$; it may be considered as the subcategory $\mathrm{Pred}(X) \hookrightarrow \mathrm{Pred}(\mathbb{C})$ with morphisms given by the identity map in $\mathbb{C}$. Similarly, we write $\mathrm{Rel}(X_1, X_2) \hookrightarrow \mathrm{Rel}(\mathbb{C})$ for the subcategory of relations $R \rightarrowtail X_1 \times X_2$ with pairs of identities as morphisms. Thus $\mathrm{Rel}(X_1, X_2) = \mathrm{Pred}(X_1 \times X_2)$.

Applying this construction for $\mathbb{C} = \mathbf{Sets}$ yields the category $\mathrm{Rel}(\mathbf{Sets})$, which is the category **Rel** as described earlier in Definition 3.2.3.

Lemma 4.3.4 *Let $\mathbb{C}$ be a category with a logical factorisation system* $(\mathfrak{M}, \mathfrak{E})$. *There are obvious forgetful functors that map predicates and relations to their carriers:*

$$\begin{array}{c} \mathrm{Pred}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{array} \qquad \begin{array}{c} U \rightarrowtail X \\ \mapsto \\ X \end{array} \qquad\qquad \begin{array}{c} \mathrm{Rel}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \times \mathbb{C} \end{array} \qquad \begin{array}{c} R \rightarrowtail X_1 \times X_2 \\ \mapsto \\ (X_1, X_2) \end{array}$$

1. *Each map $f \colon X \to Y$ in $\mathbb{C}$ gives rise to a pullback functor $f^{-1} \colon \mathrm{Pred}(Y) \to \mathrm{Pred}(X)$, using diagram (4.8). Similarly, each pair of maps $f_1 \colon X_1 \to Y_1$, $f_2 \colon X_2 \to Y_2$ gives rise to a pullback functor $(f_1 \times f_2)^{-1} \colon \mathrm{Rel}(Y_1, Y_2) \to \mathrm{Rel}(X_1, X_2)$. In this way we get two functors:*

$$\mathbb{C}^{\mathrm{op}} \xrightarrow{\mathrm{Pred}(-)} \mathbf{PoSets} \qquad \textit{and} \qquad (\mathbb{C} \times \mathbb{C})^{\mathrm{op}} \xrightarrow{\mathrm{Rel}(-)} \mathbf{PoSets}.$$

Such functors are also known as indexed categories; see e.g. *[239].*

2. *These posets* Pred(X) *and* Rel(X_1, X_2) *have finite meets* $\top, \wedge$, *given by the identity predicate and pullbacks of predicates*

$$\top_X = \left(\begin{array}{c} X \\ \downarrow \mathrm{id} \\ X \end{array}\right) \qquad \begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] \arrow[dr, tail, "m \wedge n"] & V \arrow[d, tail, "n"] \\ U \arrow[r, tail, "m"'] & X \end{tikzcd}$$

and similarly for relations. These meets are preserved by pullback functors f^{-1} *so the above indexed categories can be restricted to*

$$\mathbb{C}^{\mathrm{op}} \xrightarrow{\mathrm{Pred}(-)} \mathbf{MSL} \qquad \textit{and} \qquad (\mathbb{C} \times \mathbb{C})^{\mathrm{op}} \xrightarrow{\mathrm{Rel}(-)} \mathbf{MSL}$$

where **MSL** *is the category of meet semilattices (and monotone functions preserving* $\top, \wedge$ *as homomorphisms).*

3. *The mapping* $X \mapsto \top_X$ *yields a 'truth' functor* $\top : \mathbb{C} \to \mathrm{Pred}(\mathbb{C})$ *that is a right adjoint to the forgetful functor in*

$$\begin{tikzcd} \mathrm{Pred}(\mathbb{C}) \arrow[d, ""{name=A}] \\ \mathbb{C} \arrow[u, bend right=60, "\top"'{name=B}] \arrow[from=A, to=B, phantom, "\dashv"] \end{tikzcd}$$

4. *This truth functor* $\top : \mathbb{C} \to \mathrm{Pred}(\mathbb{C})$ *itself also has a right adjoint, mapping a predicate* $(U \rightarrowtail X)$ *to its domain* U. *This functor provides* ***comprehension*** *and will thus be written as* $\{-\}$ *in*

$$\begin{tikzcd} \mathrm{Pred}(\mathbb{C}) \arrow[d, ""{name=A}] \arrow[d, bend left=90, "\{-\}"{name=C}] \\ \mathbb{C} \arrow[u, bend right=60, "\top"'{name=B}] \arrow[from=A, to=B, phantom, "\dashv"] \arrow[from=B, to=C, phantom, "\dashv"] \end{tikzcd}$$

Since subobjects are equivalence classes, this domain functor $\{-\}$ *involves a choice of objects, from an isomorphism class.*

Proof 1. We first have to check that the pullback operations f^{-1} preserve the order between predicates. This is obvious. Further, we have $\mathrm{id}^{-1} = \mathrm{id}$ and also $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$; the latter holds by the pullback lemma; see Exercise 4.2.6. Notice that we get equalities (instead of isomorphisms) because predicates are subobjects and thus equivalence classes of maps.

2. The identity predicate $\top_X = (\mathrm{id} : X \rightarrowtail X)$ is obviously the top element in the poset Pred(X). Also, the above square defining $m \wedge n : P \rightarrowtail X$ satisfies $k \leq m$ and $k \leq n$ iff $k \leq m \wedge n$ in Pred(X). The top element (identity map) is clearly preserved under pullback. The same holds for the meet $m \wedge n$, but this requires some low-level reasoning with pullbacks (using Exercise

4.2.6 again). In a similar manner finite meets for relations exist; they are preserved by pullback.

3. We check the adjunction *forgetful* ⊣ *truth*. For a predicate $U \rightarrowtail X$ and an object $Y \in \mathbb{C}$ this involves a bijective correspondence:

$$\frac{(U \overset{m}{\rightarrowtail} X) \longrightarrow (Y \overset{\mathrm{id}}{\rightarrowtail} Y) \quad \text{in } \mathrm{Pred}(\mathbb{C})}{X \longrightarrow Y \quad \text{in } \mathbb{C}} \qquad \text{namely} \qquad \frac{\begin{tikzcd} U \arrow[r, dashed] \arrow[d, tail, "m"'] & Y \arrow[d, tail, "\mathrm{id}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}}{X \xrightarrow[f]{} Y}.$$

Clearly, the dashed map is uniquely determined as $f \circ m$.

4. We now prove the adjunction *truth* ⊣ *comprehension*. For an object $X \in \mathbb{C}$ and a predicate $n\colon V \rightarrowtail Y$ we have to prove bijective correspondences:

$$\frac{(X \overset{\mathrm{id}}{\rightarrowtail} X) \longrightarrow (V \overset{n}{\rightarrowtail} Y) \quad \text{in } \mathrm{Pred}(\mathbb{C})}{X \longrightarrow V \quad \text{in } \mathbb{C}} \qquad \text{namely} \qquad \frac{\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, tail, "\mathrm{id}"'] & V \arrow[d, tail, "n"] \\ X \arrow[r, "n \circ f"] & Y \end{tikzcd}}{X \xrightarrow[f]{} V}.$$

Clearly the maps above and under the double lines determine each other. ❑

We continue the investigation of the logical structure provided by factorisation systems. The next result shows that the existential quantifier ∃, written categorically as $\coprod$, exists for the predicates and relations defined in terms of a logical factorisation structure.

Proposition 4.3.5 *Let $\mathbb{C}$ be a category with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$.*

1. *Each map $f\colon X \to Y$ and each pair of maps $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$ in $\mathbb{C}$ give rise to functors (monotone functions) between posets:*

$$\mathrm{Pred}(X) \xrightarrow{\coprod_f} \mathrm{Pred}(Y) \qquad\qquad \mathrm{Rel}(X_1, X_2) \xrightarrow{\coprod_{f_1 \times f_2}} \mathrm{Rel}(Y_1, Y_2).$$

They are defined via factorisations: $\coprod_f(m) = \mathfrak{m}(f \circ m)$ and $\coprod_{f_1 \times f_2}(r) = \mathfrak{m}((f_1 \times f_2) \circ r)$ in

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "m"'] & \coprod_f(U) \arrow[d, tail, "\coprod_f(m)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} R \arrow[r] \arrow[d, tail, "r = \langle r_1, r_2 \rangle"'] & \coprod_{f_1 \times f_2}(R) \arrow[d, tail, "\coprod_{f_1 \times f_2}(r)"] \\ X_1 \times X_2 \arrow[r, "f_1 \times f_2"'] & Y_1 \times Y_2 \end{tikzcd}$$

Here we have used the same overloading for $\coprod_f$ *that we used for* f^{-1} *in (4.8), namely: we apply* $\coprod_f$ *both to maps and to their domain objects.*

2. *These functors* $\coprod$ *are left (Galois) adjoints to the pullback functors from Lemma 4.3.4:*

$$\begin{tikzcd} \mathrm{Pred}(X) \arrow[r, bend left, "\coprod_f"] \arrow[r, phantom, "\perp"] & \mathrm{Pred}(Y) \arrow[l, bend left, "f^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Rel}(X_1, X_2) \arrow[r, bend left, "\coprod_{f_1\times f_2}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(Y_1, Y_2). \arrow[l, bend left, "(f_1\times f_2)^{-1}"] \end{tikzcd}$$

They satisfy

$$\coprod\nolimits_{\mathrm{id}_X} = \mathrm{id}_{\mathrm{Pred}(X)} \quad \textit{and} \quad \coprod\nolimits_{g\circ f} = \coprod\nolimits_g \circ \coprod\nolimits_f\,. \tag{4.9}$$

3. *For each object* $X \in \mathbb{C}$ *the equality relation* $\mathrm{Eq}(X) \in \mathrm{Rel}(X, X)$ *is defined by factoring the diagonal* $\Delta = \langle \mathrm{id}, \mathrm{id}\rangle\colon X \rightarrowtail X \times X$, *in*

$$\begin{aligned} \mathrm{Eq}(X) &= \coprod\nolimits_{\langle \mathrm{id},\mathrm{id}\rangle}(\top) \\ &= \mathfrak{m}(\langle \mathrm{id}, \mathrm{id}\rangle) \end{aligned} \qquad \text{i.e.} \qquad \begin{tikzcd} X \arrow[r, two heads] \arrow[dr, "\langle \mathrm{id},\mathrm{id}\rangle"'] & \mathrm{Eq}(X) \arrow[d, tail, "\mathfrak{m}(\langle \mathrm{id},\mathrm{id}\rangle) = \mathrm{Eq}(X)"] \\ & X \times X \end{tikzcd}.$$

Here we deliberately overload the notation $\mathrm{Eq}(X)$. *This equality forms a functor in*

$$\begin{tikzcd} & \mathrm{Rel}(\mathbb{C}) \arrow[d] \\ \mathbb{C} \arrow[ur, "\mathrm{Eq}(-)"] \arrow[r, "\langle \mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{C}}\rangle"'] & \mathbb{C} \times \mathbb{C} \end{tikzcd}$$

If diagonals $\Delta = \langle \mathrm{id}, \mathrm{id}\rangle\colon X \rightarrowtail X \times X$ *are in the set* $\mathfrak{M}$ *of abstract monos, then* $\mathrm{Eq}(X)$ *is equal to this diagonal, and 'internal' equality in our predicate logic and 'external' equality coincide, in the sense that for two parallel maps* $f, g\colon Y \to X$ *one has*

$$\begin{aligned} f, g \textit{ are internally equal} &\overset{\mathit{def}}{\iff} \top \le \mathrm{Eq}(f, g) = \langle f, g\rangle^{-1}(\mathrm{Eq}(X)) \\ &\iff f = g \\ &\overset{\mathit{def}}{\iff} f, g \textit{ are externally equal.} \end{aligned}$$

(The direction $(\Longleftarrow)$ *of the equivalence in the middle always holds; only for* $(\Longrightarrow)$ *one needs that diagonals are in* $\mathfrak{M}$.*)*

4. *The inequalities* $\coprod_f(f^{-1}(n) \wedge m) \le n \wedge \coprod_f(m)$ *are isomorphisms, because the collection* $\mathfrak{E}$ *is closed under pullback along* $m \in \mathfrak{M}$ *– see Definition 4.3.2.6.*

Having an equality like in the last point is usually referred to as the ‘Frobenius’ condition. Logically, it corresponds to the equivalence of $\exists x.\,(\varphi \wedge \psi)$ and $\varphi \wedge (\exists x.\,\psi)$ if the variable x does not occur freely in the formula φ.

Proof We do the proofs for predicates since they subsume relations.

1. Assume $m \leq m'$ in $\mathrm{Pred}(X)$, say via a map φ, then we get $\coprod_f(m) \leq \coprod_f(m')$ in $\mathrm{Pred}(Y)$ via the diagonal-fill-in property:

$$\begin{tikzcd}
U \arrow[r, two heads] \arrow[d, "\varphi"'] & \coprod_f(U) \arrow[dd, tail, "\coprod_f(m)"] \arrow[ddl, dashed] \\
U' \arrow[d, two heads] & \\
\coprod_f(U') \arrow[r, tail, "\coprod_f(m')"'] & Y
\end{tikzcd}
\quad \text{i.e.} \quad
\begin{tikzcd}
& U' \arrow[rr, two heads] \arrow[ddl, tail, "m'"] & & \coprod_f(U') \arrow[ddl, tail, "\coprod_f(m')"] \\
U \arrow[ur, "\varphi"] \arrow[rr, two heads] \arrow[d, tail, "m"'] & & \coprod_f(U) \arrow[ur, dashed] \arrow[d, tail, "\coprod_f(m)"'] & \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

2. For predicates $m\colon U \rightarrowtail X$ and $n\colon V \rightarrowtail Y$ we have to produce a bijective correspondence:

$$\begin{array}{c} \coprod_f(m) \;\leq\; n \\ \hline\hline m \;\leq\; f^{-1}(n) \end{array}
\qquad \text{that is,} \qquad
\begin{array}{c}
\begin{tikzcd}
\coprod_f(U) \arrow[rr, dashed, "\varphi"] \arrow[dr, tail, "\coprod_f(m)"'] & & V \arrow[dl, tail, "n"] \\
& Y &
\end{tikzcd} \\ \hline\hline
\begin{tikzcd}
U \arrow[rr, dashed, "\psi"] \arrow[dr, tail, "m"'] & & f^{-1}(V) \arrow[dl, tail, "f^{-1}(n)"] \\
& Y &
\end{tikzcd}
\end{array}.$$

This works as follows. Given φ as indicated, we obtain ψ on the left below. The converse is sketched on the right:

$$\begin{tikzcd}
U \arrow[r, two heads] \arrow[dr, dashed, "\psi"] \arrow[ddr, tail, bend right, "m"'] & \coprod_f(U) \arrow[dr, bend left, "\varphi"] & \\
& f^{-1}(V) \arrow[r] \arrow[d, tail, "f^{-1}(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, tail, "n"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, two heads] \arrow[d, "\psi"'] & \coprod_f(U) \arrow[dd, tail, "\coprod_f(m)"] \arrow[ddl, dashed, "\varphi"'] \\
f^{-1}(V) \arrow[d] & \\
V \arrow[r, tail, "n"'] & Y
\end{tikzcd}$$

The functoriality equations (4.9) for $\coprod$ follow from the functoriality equations for pullback, via Galois connection (adjunction) $\coprod_f \dashv f^{-1}$.

3. For a map $f\colon X \to Y$ in $\mathbb{C}$ we have to show that the pair (f, f) is a morphism $\mathrm{Eq}(X) \to \mathrm{Eq}(Y)$ in $\mathrm{Rel}(\mathbb{C})$. This follows from diagonal-fill-in:

$$\begin{tikzcd}
X \arrow[rr, two heads] \arrow[d, "f"'] & & \mathrm{Eq}(X) \arrow[d, tail] \arrow[dll, dashed] \\
Y \arrow[d] & & X \times X \arrow[d, "f \times f"] \\
\mathrm{Eq}(Y) \arrow[rr, tail] & & Y \times Y
\end{tikzcd}$$

The outer rectangle commutes because $\langle \mathrm{id}, \mathrm{id}\rangle \circ f = (f \times f) \circ \langle \mathrm{id}, \mathrm{id}\rangle$.

Clearly, external equality $f = g$ implies internal equality, since the unit of the adjunction $\coprod_{\langle \mathrm{id},\mathrm{id}\rangle} \dashv \langle \mathrm{id}, \mathrm{id}\rangle^{-1}$ gives

$$\top \le \langle \mathrm{id}, \mathrm{id}\rangle^{-1} \coprod_{\langle \mathrm{id},\mathrm{id}\rangle}(\top) = \langle \mathrm{id}, \mathrm{id}\rangle^{-1}\mathrm{Eq}(X).$$

Hence by applying f^{-1} we get:

$$\begin{aligned}
\top = f^{-1}(\top) &\le f^{-1}\langle \mathrm{id}, \mathrm{id}\rangle^{-1}\mathrm{Eq}(X) \\
&= (\langle \mathrm{id}, \mathrm{id}\rangle \circ f)^{-1}\mathrm{Eq}(X) = \langle f, f\rangle^{-1}\mathrm{Eq}(X).
\end{aligned}$$

For the other direction, assume the diagonal $\Delta = \langle \mathrm{id}, \mathrm{id}\rangle$ is in $\mathfrak{M}$. Equality $\mathrm{Eq}(X)$ is then equal to this diagonal, and so $\langle f, g\rangle^{-1}(\mathrm{Eq}(X))$ is the pullback

$$\begin{tikzcd}
E \arrow[r, "p_2"] \arrow[d, tail, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "\langle \mathrm{id}, \mathrm{id}\rangle = \mathrm{Eq}(X)"] \\
Y \arrow[r, "\langle f, g\rangle"'] & X \times X
\end{tikzcd}$$

where the map p_1 is in fact the equaliser of f, g. Internal equality of f, g amounts to an inequality $\top \le \langle f, g\rangle^{-1}(\mathrm{Eq}(X)) = p_1$. It expresses that the map p_1 is an isomorphism, and so

$$\begin{aligned}
f = \pi_1 \circ \langle f, g\rangle &= \pi_1 \circ \langle \mathrm{id}, \mathrm{id}\rangle \circ p_2 \circ p_1^{-1} \\
&= \pi_2 \circ \langle \mathrm{id}, \mathrm{id}\rangle \circ p_2 \circ p_1^{-1} = \pi_2 \circ \langle f, g\rangle = g.
\end{aligned}$$

4. Assume a map $f\colon X \to Y$ with predicates $m\colon U \rightarrowtail X$ and $n\colon V \rightarrowtail Y$. The unit of the adjunction $\coprod_f \dashv f^{-1}$ gives an inequality $m \le f^{-1}(\coprod_f(m))$, from which we obtain

$$f^{-1}(n) \wedge m \le f^{-1}(n) \wedge f^{-1}(\textstyle\coprod_f(m)) = f^{-1}(n \wedge \coprod_f(m)).$$

The adjunction $\coprod_f \dashv f^{-1}$ now yields the required inequality $\coprod_f(f^{-1}(n) \wedge m) \le n \wedge \coprod_f(m)$. The second part of the statement requires more work and uses requirement (6) in Definition 4.3.2, which says that $\mathfrak{E}$ is closed under pullback along each $k \in \mathfrak{M}$. Consider the left side of the following diagram:

$$
\begin{tikzcd}
P \arrow[r, tail, "p_1"] \arrow[d, tail, "p_2"'] \arrow[dr, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & f^{-1}(V) \arrow[r] \arrow[d, tail, "f^{-1}(n)"] & V \arrow[d, tail, "n"] \\
U \arrow[r, tail, "m"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
U \arrow[r, "e"] \arrow[d, tail, "m"'] & \coprod_f(U) \arrow[d, tail, "\coprod_f(m)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

The diagonal is the conjunction $f^{-1}(n) \wedge m$. In order to get $\coprod_f(f^{-1}(n) \wedge m)$ we need to factorise the composite $f \circ (f^{-1}(n) \wedge m) = f \circ m \circ p_2$. This factorisation appears if we consider the other conjunct $n \wedge \coprod_f(m)$, arising as the diagonal on the right below:

$$
\begin{tikzcd}
P \arrow[r, "d"] \arrow[d, tail, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q \arrow[r, tail, "q_1"] \arrow[d, tail, "q_2"'] \arrow[dr, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, tail, "n"] \\
U \arrow[r, "e"'] \arrow[rr, bend right, "f \circ m"'] & \coprod_f(U) \arrow[r, tail, "\coprod_f(m)"'] & Y
\end{tikzcd}
$$

The outer rectangle is a pullback because $f \circ m = \coprod_f(m) \circ e$, as described above. Thus, by the pullback lemma (see Exercise 4.2.6), the rectangle on the left is a pullback. But then the map d arises as pullback of $e \in \mathfrak{E}$ along $q_2 \in \mathfrak{M}$. Hence, by assumption, $d \in \mathfrak{E}$. But this means that $\coprod_f(f^{-1}(n) \wedge m)$, which is by definition the $\mathfrak{M}$-part of $f \circ m \circ p_2$, is the diagonal map $n \wedge \coprod_f(m)$. Thus we are done. ❑

Remark 4.3.6 The third point of Proposition 4.3.5 deals with equality relations $\mathrm{Eq}(X)$, as $\mathfrak{M}$-part of the diagonal $\Delta = \langle \mathrm{id}, \mathrm{id} \rangle : X \rightarrowtail X \times X$. This diagonal Δ need not be in $\mathfrak{M}$. Its presence in $\mathfrak{M}$ is a non-trivial property, and therefore we have not made it part of the requirements for a 'logical factorisation system' in Definition 4.3.2. Recall, for instance, that one formulation of the Hausdorff property for topological spaces says: the diagonal relation Δ is closed. Closed subsets of topological spaces may indeed be used as abstract monos for a factorisation system. An alternative example will be elaborated in Example 4.3.7.1 below: in the category **SetsRel** of sets and relations diagonals Δ are not abstract monos.

The requirement that diagonals are in $\mathfrak{M}$ is equivalent to the requirement that all maps in $\mathfrak{E}$ are epis. This is left as an exercise below.

But if diagonals are in $\mathfrak{M}$ the resulting logic in the category satisfies the special property that internal and external equality coincide. This is for instance the case in every topos whose logic is described via *all* monos/subobjects (see e.g. [346]).

Example 4.3.7 1. The category **SetsRel** of sets and relations is defined in Example 1.4.2.4. Each relation $\langle r_1, r_2\rangle \colon R \hookrightarrow X \times Y$, seen as morphism $R \colon X \to Y$ in **SetsRel**, can be factored as

$$(X \xrightarrow{\;R\;} Y) = (X \xrightarrow{\;\mathfrak{e}(R)\;} Y' \xrightarrow{\;\mathfrak{m}(R)\;} Y),$$

where $Y' = \{y \mid \exists x.\, R(x,y)\}$ is the image of $r_2 \colon R \to Y$ and

$$\mathfrak{e}(R) = \left(\begin{tikzcd}[column sep=small] & R \arrow[dl, "r_1"'] \arrow[dr, two heads] & \\ X & & Y' \end{tikzcd}\right), \qquad \mathfrak{m}(R) = \left(\begin{tikzcd}[column sep=small] & Y' \arrow[dl, equal] \arrow[dr, tail] & \\ Y' & & Y \end{tikzcd}\right).$$

A factorisation system $(\mathfrak{M}, \mathfrak{E})$ exists on the category **SetsRel**: the collection $\mathfrak{M}$ consists of injections $Y' \rightarrowtail Y$, forming a relation as on the right above. And $\mathfrak{E}$ contains relations $\langle r_1, r_2\rangle \colon R \hookrightarrow X \times Y$ with the right leg r_2 forming a surjection. These maps in $\mathfrak{M}$ and $\mathfrak{E}$ are characterised in [220] as 'dagger kernels' and 'zero-epis', respectively.

The category **SetsRel** has products (actually biproducts) given by coproducts $+$ on sets. The diagonal (relation) $\Delta = \langle \mathrm{id}, \mathrm{id}\rangle \colon Y \rightarrowtail Y + Y$ in **SetsRel** is given by the subset:

$$\Delta \;=\; \{(y, \kappa_1 y) \mid y \in Y\} \cup \{(y, \kappa_2 y) \mid y \in Y\} \;\subseteq\; Y \times (Y + Y).$$

Thus, in span form it can be written as

$$\Delta = \left(\begin{tikzcd}[column sep=small] & Y+Y \arrow[dl, "{[\mathrm{id},\mathrm{id}]}"'] \arrow[dr, equal] & \\ Y & & Y+Y \end{tikzcd}\right)$$

with image given by the equality relation on $Y + Y$, obtained as

$$\mathrm{Eq}(Y) = \mathfrak{m}(\Delta) = \left(\begin{tikzcd}[column sep=small] & Y+Y \arrow[dl, equal] \arrow[dr, equal] & \\ Y+Y & & Y+Y \end{tikzcd}\right).$$

For parallel relations $R, S \colon X \to Y$ in **SetsRel** we then have

$$\langle R, S\rangle^{-1}(\mathrm{Eq}(Y)) \;=\; \{x \in X \mid \forall y, y'.\, R(x,y) \wedge S(x,y') \Rightarrow y = y'\}.$$

Hence internal equality $\top \leq \langle R, S\rangle^{-1}(\mathrm{Eq}(Y))$ is in **SetsRel** not the same as external equality.

2. Recall the category **Vect** of vector spaces (over the real numbers $\mathbb{R}$ or some other field). It carries a factorisation system $(\mathfrak{M}, \mathfrak{E})$ where

$$\mathfrak{M} \;=\; (\textit{injective linear maps}) \qquad \text{and} \qquad \mathfrak{E} \;=\; (\textit{surjective linear maps}).$$

The subobjects associated with $\mathfrak{M}$ are linearly closed subspaces. The product (actually biproduct) for two vector spaces is given by the product of the underlying sets, with coordinatewise structure. Hence the diagonal $\Delta = \langle \mathrm{id}, \mathrm{id} \rangle \colon V \rightarrowtail V \times V$ is the usual (set-theoretic) diagonal. Since these diagonals are in $\mathfrak{M}$, internal and external equality coincide in **Vect**.

3. We write **Hilb** for the category of Hilbert spaces (over the real or possibly also over the complex numbers). Morphisms are linear maps which are continuous (or equivalently bounded). The category **Hilb** also carries a factorisation system, where maps in $\mathfrak{M}$ correspond to subspaces which are both linearly and metrically closed. Also in this case diagonals are in $\mathfrak{M}$. More details can be found in [220].

In this book we use factorisation systems primarily for logical purposes, namely for predicate and relation lifting and for existential quantification. Factorisation systems are also used in coalgebra for minimal realisations, as in Exercise 2.5.14; see [77, 307] for more information.

We close this section with a special example of a category with a logical factorisation system. It is obtained from an ordinary logical calculus, by suitably organising the syntactic material into a category. It illustrates what kind of logical structure is relevant in this setting. Categories such as these are described as Lindenbaum–Tarski (or term model) constructions in [239]. These categories may also be described via an initiality property, but that goes beyond the current setting.

Example 4.3.8 We sketch a (multi-sorted, typed) logic, whose types, terms and formulas are given by the following BNF syntax:

$$\begin{array}{rrcl} \textbf{Types} & \sigma & := & B \mid 1 \mid \sigma \times \sigma \mid \{x \colon \sigma \mid \varphi\} \\ \textbf{Terms} & M & := & x \mid f(M, \ldots, M) \mid \pi_1 M \mid \pi_2 M \mid \langle M, M \rangle \\ \textbf{Formulas} & \varphi & := & P \mid \top \mid \varphi \wedge \varphi \mid M =_\sigma M \mid \exists x \colon \sigma.\, \varphi. \end{array}$$

A type is thus either a primitive type B, a singleton (or unit) type 1, a product type $\sigma \times \sigma'$ or a comprehension type $\{x \colon \sigma \mid \varphi\}$ for a formula φ. A term is either a variable x, a function application $f(M_1, \ldots, M_n)$ for a function symbol f with appropriate arity and type, a projection term $\pi_i M$ or a tuple $\langle M, M' \rangle$. Finally, a formula is one of the following: an atomic predicate P; a truth formula $\top$; a conjunction $\varphi \wedge \varphi'$; an equation $M =_\sigma M'$ for two terms M, M' both of type σ; or a (typed) existential formula $\exists x \colon \sigma.\, \varphi$. Here we assume that the function symbols f and atomic predicates P are somehow given, via an appropriate signature. Notice, by the way, that negation is not part of this logical language.

A proper account of our predicate logic now lists the typing rules for sequents of the form $x_1 \colon \sigma_1, \ldots, x_n \colon \sigma_n \vdash M \colon \tau$, expressing that term M, (possibly) containing typed variables $x_1, \ldots, x_n$, has type τ, and also the deduction rules for logical sequents written as $x_1 \colon \sigma_1, \ldots, x_n \colon \sigma_n \mid \varphi_1, \ldots, \varphi_n \vdash \psi$. Sometimes we write a term M as $M(x_1, \ldots, x_n)$ to make the variables occurring in M explicit (and similarly for formulas). We write $[M/x]$ for the (postfix) operation of substituting the term M for all (free) occurrences of the variable x (where M and x have the same type). We assume the reader is reasonably familiar with these rules; refer to [239] for further information. The exception we make is for the rules of the comprehension type (see below), because they are not so standard.

But first we show that we may restrict ourselves to terms and formulas containing at most one variable. Suppose a term $M(x, y)$ has two variables $x \colon \sigma, y \colon \tau$. Then we may replace x, y by a single variable $z \colon \sigma \times \tau$ in a product type. This z is placed in M via substitution: $M[\pi_1 z/x, \pi_2/y]$. This can of course be repeated for multiple variables. Similarly, we may replace entailments $\varphi_1, \cdots, \varphi_n \vdash \psi$ by an entailment $\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \psi$ between only two formulas (using $\top$ as antecedent if $n = 0$). If we keep track of the variable involved we write such sequents as $x \colon \sigma \mid \varphi \vdash \psi$.

Now we can be more explicit about comprehension types. If we have a formula $\varphi(x)$, involving a variable $x \colon \sigma$, we can form the type $\{x \colon \sigma \mid \varphi\}$. It comes with introduction and elimination rules, involving tags i for 'in' and o for 'out',

$$\frac{y \colon \tau \vdash M \colon \sigma \qquad y \colon \tau \mid \top \vdash \varphi[M/x]}{y \colon \tau \vdash \mathsf{i}_\varphi(M) \colon \{x \colon \sigma \mid \varphi\}}$$

$$\frac{y \colon \tau \vdash N \colon \{x \colon \sigma \mid \varphi\}}{y \colon \tau \vdash \mathsf{o}_\varphi(N) \colon \sigma} \qquad \frac{x \colon \sigma \mid \varphi, \psi \vdash \chi}{x' \colon \{x \colon \sigma \mid \varphi\} \mid \psi[\mathsf{o}_\varphi(x')/x] \vdash \chi[\mathsf{o}_\varphi(x')/x]}$$

with associated conversions $\mathsf{o}_\varphi(\mathsf{i}_\varphi(M)) = M$ and $\mathsf{i}_\varphi(\mathsf{o}_\varphi(N)) = N$. In the sequel we omit these subscripts φ for convenience.

We now form a category $\mathbb{L}$, for logic, with

objects types σ

morphisms $\sigma \to \tau$ which are equivalence classes $[M]$ of terms $x \colon \vdash M \colon \tau$. Two terms M, M' are equivalent when one can deduce $x \colon \sigma \mid \top \vdash M =_\sigma M'$. Via these equivalence classes we force 'internal' (provable) equality and 'external' equality (of maps) to coincide.

The identity map $\sigma \to \sigma$ is $[x_\sigma]$, where $x_\sigma \colon \sigma$ is a particular (chosen) variable of type σ. The composition of maps $[M(x)] \colon \sigma \to \tau$ and $[N(y)] \colon \tau \to \rho$ is given by the term $x \colon \sigma \vdash N[M(x)/y] \colon \rho$ obtained by substitution.

This category $\mathbb{L}$ has finite (categorical) products, via the type-theoretic products $1, \sigma \times \tau$.

The category $\mathbb{L}$ also carries a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$. The set $\mathfrak{M}$ contains the 'o-maps', given by a formula φ in

$$u\colon \{x\colon \sigma \,|\, \varphi\} \vdash \mathsf{o}(u)\colon \sigma.$$

(Actually, we take in $\mathfrak{M}$ all these o-maps composed with isomorphisms.)

These o-maps are clearly monic: if $\mathsf{o}(N) = \mathsf{o}(N')$, then $N = \mathsf{i}(\mathsf{o}(N)) = \mathsf{i}(\mathsf{o}(N')) = N'$. Also, they are closed under composition. This can be seen in the following diagram:

$$\begin{tikzcd}
\{y\colon \{x\colon \sigma \,|\, \varphi\} \,|\, \psi\} \arrow[r, tail, "{[\mathsf{o}]}"] \arrow[d, "{[P]}"', "\cong"] & \{x\colon \sigma \,|\, \varphi\} \arrow[d, tail, "{[\mathsf{o}]}"] \\
\{x\colon \sigma \,|\, \varphi \wedge \psi[\mathsf{i}(x)/y]\} \arrow[r, tail, "{[\mathsf{o}]}"'] & \sigma
\end{tikzcd}$$

The term $P(v)$ is $\mathsf{i}(\mathsf{o}(\mathsf{o}(v)))$ and is obtained in the following derivation:

$$\dfrac{\dfrac{\dfrac{x\colon \sigma \,|\, \varphi, \top \vdash \varphi}{y\colon \{x\colon \sigma \,|\, \varphi\} \,|\, \top \vdash \varphi[\mathsf{o}(y)/x]} \qquad \dfrac{y\colon \{x\colon \sigma \,|\, \varphi\} \,|\, \psi, \top \vdash \psi}{v\colon \{y\colon \{x\colon \sigma \,|\, \varphi\} \,|\, \psi\} \,|\, \top \vdash \psi[\mathsf{o}(v)/y]}}{\begin{array}{l} v\colon \{y\colon \{x\colon \sigma \,|\, \varphi\} \,|\, \psi\} \,|\, \top \vdash \varphi[\mathsf{o}(\mathsf{o}(v))/x] \wedge \psi[\mathsf{o}(v)/y] \\ \qquad\qquad\qquad\qquad\qquad = (\varphi \wedge \psi[\mathsf{i}(x)/y])[\mathsf{o}(\mathsf{o}(v))/x] \end{array}}}{v\colon \{y\colon \{x\colon \sigma \,|\, \varphi\} \,|\, \psi\} \vdash P(v) \stackrel{\text{def}}{=} \mathsf{i}(\mathsf{o}(\mathsf{o}(v)))\colon \{x\colon \sigma \,|\, \varphi \wedge \psi[\mathsf{i}(x)/y]\}} \; .$$

Notice that the i and os in $P(v) = \mathsf{i}(\mathsf{o}(\mathsf{o}(v)))$ are associated with different formulas. In a similar manner one obtains the inverse term P^{-1} of P as $P^{-1}(u) = \mathsf{i}(\mathsf{i}(\mathsf{o}(u)))$.

The set $\mathfrak{M}$ is also closed under pullback, via substitution in formulas:

$$\begin{tikzcd}
\{y\colon \tau \,|\, \varphi[M(y)/x]\} \arrow[r, "{[\mathsf{i}(M[\mathsf{o}(v)/y])]}"] \arrow[d, tail, "{[\mathsf{o}]}"'] & \{x\colon \sigma \,|\, \varphi\} \arrow[d, tail, "{[\mathsf{o}]}"] \\
\tau \arrow[r, "{[M(y)]}"'] & \sigma
\end{tikzcd}$$

The map on top is well typed since

$$\dfrac{\dfrac{y\colon \tau \,|\, \varphi[M(y)/x], \top \vdash \varphi[M(y)/x]}{v\colon \{y\colon \tau \,|\, \varphi[M(y)/x]\} \,|\, \top \vdash \varphi[M(y)/x][\mathsf{o}(v)/y] = \varphi[M[\mathsf{o}(v)/y]/x].}}{v\colon \{y\colon \tau \,|\, \varphi[M(y)/x]\} \vdash \mathsf{i}(M[\mathsf{o}(v)/y])\colon \{x\colon \sigma \,|\, \varphi\}}$$

We leave it to the reader to check that it forms a pullback in $\mathbb{L}$.

The set $\mathfrak{M}$ also contains the equality relations via the following isomorphism:

$$\begin{tikzcd} \sigma \arrow[rr, "{[\mathsf{i}(\langle x, x\rangle)]}", "\cong"'] \arrow[dr, "{\langle \mathrm{id}, \mathrm{id}\rangle = [\langle x, x\rangle]}"'] & & \{z\colon \sigma\times\sigma \mid \pi_1 z =_\sigma \pi_2 z\} \arrow[dl, "{[\mathsf{o}]}"] \\ & \sigma\times\sigma & \end{tikzcd}$$

The inverse in this diagram is given by $[\pi_1\mathsf{o}(v)] = [\pi_2\mathsf{o}(v)]$, where the variable v has type $\{z\colon \sigma\times\sigma \mid \pi_1 z =_\sigma \pi_2 z\}$.

We define $\mathfrak{E}$ so that it contains those maps of the form $[M]\colon \sigma \to \{y\colon \tau \mid \psi\}$ that satisfy $y\colon \tau \mid \psi \vdash \exists x\colon \sigma.\, \mathsf{o}(M(x)) = y$. We leave it to the reader to show that these maps are closed under composition and also closed under pullback along o-maps (from $\mathfrak{M}$).

We come to factorisation. For an arbitrary map $[M]\colon \sigma \to \tau$ in $\mathbb{L}$ we can consider the following predicate on the codomain type τ:

$$\mathrm{Im}([M]) \;=\; (y\colon \tau \vdash \exists x\colon \sigma.\, M(x) = y).$$

Thus we can factorise the map $[M]$ as

$$\begin{tikzcd} \sigma \arrow[r, "{[\mathsf{i}(M)]}"] \arrow[dr, "{[M]}"'] & \{\mathrm{Im}([M])\} = \{y\colon \tau \mid \exists x\colon \sigma.\, M(x) = y\} \arrow[d, tail, "{[\mathsf{o}]}"] \\ & \tau \end{tikzcd}$$

Finally, we check the diagonal-fill-in condition. Assume we have a commuting square:

$$\begin{tikzcd} \sigma \arrow[r, "{[M]}"] \arrow[d, "{[P]}"'] & \{y\colon \tau \mid \psi\} \arrow[d, "{[Q]}"] \\ \{z\colon \rho \mid \varphi\} \arrow[r, tail, "{[\mathsf{o}]}"'] & \rho \end{tikzcd} \tag{4.10}$$

Commutation says $\mathsf{o}(P) =_\rho Q[M/y']$, where $y'\colon \{y\colon \tau \mid \psi\}$. It may be clear that the only choice as diagonal $\{y\colon \tau \mid \psi\} \to \{z\colon \rho \mid \varphi\}$ is the term $\mathsf{i}(Q)$, since

$$\begin{array}{rcl} \mathsf{o}(\mathsf{i}(Q)) & = & Q \\ \mathsf{i}(Q)[M/y'] & = & \mathsf{i}(Q[M/y']) \\ & = & \mathsf{i}(\mathsf{o}(P)) \\ & = & P. \end{array}$$

The challenge is to show that the term $\mathsf{i}(Q)$ is appropriately typed. This is achieved via the derivation in Figure 4.3.

$$\boxed{\dfrac{\dfrac{y' : \{y\colon \tau \mid \psi\} \vdash \mathsf{o}(y') : \tau \qquad y\colon \tau \mid \psi, \top \vdash \exists x\colon \sigma.\, \mathsf{o}(M) = y}{y' : \{y\colon \tau \mid \psi\} \mid \top \vdash \exists x\colon \sigma.\, \mathsf{o}(M) = \mathsf{o}(y')}}{y' : \{y\colon \tau \mid \psi\} \mid \top \vdash \exists x\colon \sigma.\, M = y'}}$$

$$\boxed{\dfrac{x\colon \sigma \vdash P : \{z\colon \rho \mid \varphi\} \qquad \dfrac{z\colon \rho \mid \varphi, \top \mid \varphi}{z' : \{z\colon \rho \mid \varphi\} \mid \top \vdash \varphi[\mathsf{o}(z')/z]}}{x\colon \sigma \mid \top \vdash \varphi[\mathsf{o}(P)/z]}}$$

$$\dfrac{\qquad \dfrac{\dfrac{\qquad \dfrac{x\colon \sigma \mid \top \vdash \mathsf{o}(P) = Q[M/y'] \qquad \dfrac{x\colon \sigma \vdash M : \{y\colon \tau \mid \psi\} \qquad y', y'' : \{y\colon \tau \mid \psi\} \mid \varphi[Q[y''/y']/z], y'' = y' \vdash \varphi[Q/z]}{x\colon \sigma, y' : \{y\colon \tau \mid \psi\} \mid \varphi[Q[M/y']/z], M = y' \vdash \varphi[Q/z]}}{x\colon \sigma, y' : \{y\colon \tau \mid \psi\} \mid \varphi[\mathsf{o}(P)/z], M = y' \vdash \varphi[Q/z]}}{x\colon \sigma, y' : \{y\colon \tau \mid \psi\} \mid M = y' \vdash \varphi[Q/z]}}{y' : \{y\colon \tau \mid \psi\} \mid \exists x\colon \sigma.\, M = y' \vdash \varphi[Q/z]}}{\dfrac{y' : \{y\colon \tau \mid \psi\} \mid \top \vdash \varphi[Q/z]}{y' : \{y\colon \tau \mid \psi\} \vdash \mathsf{i}(Q) : \{z\colon \rho \mid \varphi\}}}$$

Figure 4.3 Well-typedness of the diagonal map for the rectangle (4.10).

Exercises

4.3.1 Let $(\mathfrak{M}, \mathfrak{E})$ be a logical factorisation system.

1. Show that a map $f \in \mathfrak{M} \cap \mathfrak{E}$ is an isomorphism.
2. Prove that if we can factor a map g both as $g = m \circ e$ and as $g = m' \circ e'$, where $m, m' \in \mathfrak{M}$ and $e, e' \in \mathfrak{E}$, then there is a unique isomorphism φ with $m' \circ \varphi = m$ and $\varphi \circ e = e'$.
3. Show for $m \in \mathfrak{M}$ and $e \in \mathfrak{E}$ that $\mathfrak{m}(m \circ f) = m \circ \mathfrak{m}(f)$ and $\mathfrak{e}(f \circ e) = \mathfrak{m}(f) \circ e$, where $\mathfrak{m}(-)$ and $\mathfrak{e}(-)$ take the $\mathfrak{M}$-part and $\mathfrak{E}$-part as in Definition 4.3.2.2.

4.3.2 Let $F: \mathbb{C} \to \mathbb{C}$ be an endofunctor on a category $\mathbb{C}$ with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$.

1. Assume that F preserves abstract epis, i.e. $e \in \mathfrak{E} \Rightarrow F(e) \in \mathfrak{E}$. Prove that the category $\mathbf{Alg}(F)$ of algebras also carries a logical factorisation system. Use that pullbacks in $\mathbf{Alg}(F)$ are constructed as in $\mathbb{C}$; see Exercise 4.2.7.
2. Check that every endofunctor $F: \mathbf{Sets} \to \mathbf{Sets}$ satisfies this assumption, i.e. preserves surjections – if the axiom of choice holds. *Hint*: Recall that the axiom of choice can be formulated as: each surjection has a section; see Section 2.1.

4.3.3 Define the category $\mathrm{EnRel}(\mathbb{C})$ of endorelations in a category $\mathbb{C}$ (with a logical factorisation system) via the following pullback of functors:

$$\begin{tikzcd}
\mathrm{EnRel}(\mathbb{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rel}(\mathbb{C}) \arrow[d] \\
\mathbb{C} \arrow[r, "\langle \mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{C}} \rangle"'] & \mathbb{C} \times \mathbb{C}
\end{tikzcd}$$

1. Describe this category $\mathrm{EnRel}(\mathbb{C})$ in detail.
2. Demonstrate that taking equality relations on an object forms a functor $\mathrm{Eq}(-): \mathbb{C} \to \mathrm{EnRel}(\mathbb{C})$.

4.3.4 Let $\mathbb{C}$ be a category with a logical factorisation system and finite coproducts $(0, +)$.

1. Show that the image of the unique map $!: 0 \to X$ is the least element $\perp$ in the poset $\mathrm{Pred}(X)$ of predicates on X.
2. Similarly, show that the join $m \vee n$ in $\mathrm{Pred}(X)$ of predicates $m: U \rightarrowtail X$ and $n: V \rightarrowtail Y$ is the image of the cotuple $[m, n]: U + V \to X$.

4.3.5 Two morphisms f, g in an arbitrary category $\mathbb{C}$ may be called orthogonal, written as $f \perp g$, if in each commuting square as below there is a unique diagonal making everything in sight commute:

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d] & \cdot \arrow[d] \arrow[dl, dashed] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

The diagonal-fill-in property for a factorisation system $(\mathfrak{M}, \mathfrak{E})$ in Definition 4.3.2 thus says that $e \perp m$ for each $m \in \mathfrak{M}$ and $e \in \mathfrak{E}$.

Now assume that a category $\mathbb{C}$ is equipped with a factorisation system $(\mathfrak{M}, \mathfrak{E})$, not necessarily 'logical'. This means that only properties (1)–(3) in Definition 4.3.2 hold.

1. Prove that $f \in \mathfrak{E}$ if and only if $f \perp m$ for all $m \in \mathfrak{M}$.
2. Similarly, prove that $g \in \mathfrak{M}$ if and only if $e \perp g$ for all $e \in \mathfrak{E}$.
3. Prove $e, d \circ e \in \mathfrak{E} \Rightarrow d \in \mathfrak{E}$.
4. Similarly (or dually), prove $m, m \circ n \in \mathfrak{M} \Rightarrow n \in \mathfrak{M}$.
5. Prove $m, n \in \mathfrak{M} \Rightarrow m \times n \in \mathfrak{M}$, assuming products exist in $\mathbb{C}$.
6. Show that diagonals $\Delta = \langle \mathrm{id}, \mathrm{id} \rangle$ are in $\mathfrak{M}$ if and only if all maps in $\mathfrak{E}$ are epis.

4.3.6 Prove the converse of Proposition 4.3.5.4: if $\coprod_f (f^{-1}(n) \wedge m) = n \wedge \coprod_f (m)$ holds for all appropriate f, m, n, then $\mathfrak{E}$ is closed under pullback along maps $m \in \mathfrak{M}$.

4.3.7 Let $(\mathfrak{M}, \mathfrak{E})$ be a factorisation system on a category $\mathbb{C}$ with finite products $1, \times$. Prove that the category of predicates $\mathrm{Pred}(\mathbb{C})$ also has finite products, via the following constructions:

1. The identity $(1 \rightarrowtail 1)$ on the final object $1 \in \mathbb{C}$ is final in $\mathrm{Pred}(\mathbb{C})$.
2. The product of predicates $(m \colon U \rightarrowtail X)$ and $(n \colon V \rightarrowtail Y)$ is the conjunction of the pullbacks $\pi_1^{-1}(m) \wedge \pi_2^{-1}(n)$, as predicate on $X \times Y$.

Show that also the category of relations $\mathrm{Rel}(\mathbb{C})$ has finite products.

4.3.8 Let $(\mathfrak{M}, \mathfrak{E})$ be a logical factorisation system on a category $\mathbb{C}$ with pullbacks. Prove that $\mathfrak{E}$ is closed under pullbacks along arbitrary maps if and only if the so-called Beck–Chevalley condition holds: for a pullback as on the left, the inequality on the right is an isomorphism:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "k"] \\ Z \arrow[r, "g"'] & W \end{tikzcd} \qquad \coprod_f h^{-1}(m) \;\leq\; g^{-1} \coprod_k(m)$$

4.4 Relation Lifting, Categorically

The previous section introduced predicates and relations in a category via a factorisation system, corresponding to conjunctions $\top, \wedge$, equality, existential quantification $\exists$ and comprehension $\{-\}$. In this section we use such factorisation systems to describe relation lifting for an arbitrary functor – not just for a polynomial one, as in Section 3.1. Predicate lifting with respect to such a factorisation system will be described later, in Section 6.1.2.

Definition 4.4.1 Let $\mathbb{C}$ be a category with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$ and an arbitrary endofunctor $F\colon \mathbb{C} \to \mathbb{C}$. Then we define a functor $\mathrm{Rel}(F)\colon \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbb{C})$ in the following way.

- Given a relation $r = \langle r_1, r_2\rangle\colon R \rightarrowtail X_1 \times X_2$ in $\mathrm{Rel}(\mathbb{C})$ we introduce a new relation $\mathrm{Rel}(F)(r)\colon \mathrm{Rel}(F)(R) \rightarrowtail F(X_1) \times F(X_2)$ via the following factorisation, describing the lifted relation as the right-hand-side leg of the triangle:

$$\begin{tikzcd} F(R) \arrow[rr, twoheadrightarrow] \arrow[dr, "{\langle F(r_1), F(r_2)\rangle}"'] & & \mathrm{Rel}(F)(R) \arrow[dl, rightarrowtail, "\mathrm{Rel}(F)(r)"] \\ & F(X_1) \times F(X_2) & \end{tikzcd}$$

- Assume a morphism $(f_1, f_2)\colon R \to S$ in $\mathrm{Rel}(\mathbb{C})$, as in

$$\begin{tikzcd} R \arrow[rr, dashed, "\varphi"] \arrow[d, rightarrowtail, "{r = \langle r_1, r_2\rangle}"'] & & S \arrow[d, rightarrowtail, "{s = \langle s_1, s_2\rangle}"] \\ X_1 \times X_2 \arrow[rr, "f_1 \times f_2"] & & Y_1 \times Y_2 \end{tikzcd}$$

The pair of maps $(F(f_1), F(f_2))$ forms a morphism $\mathrm{Rel}(F)(r) \to \mathrm{Rel}(F)(s)$ in $\mathrm{Rel}(\mathbb{C})$ by the diagonal-fill-in property from Definition 4.3.2.3:

$$\begin{tikzcd}
F(R) \arrow[r, two heads] \arrow[d, "F(\varphi)"'] \arrow[dr, bend left=40, "\langle F(r_1), F(r_2)\rangle"] & \mathrm{Rel}(F)(R) \arrow[d, tail] \arrow[dl, dashed, "\mathrm{Rel}(F)(f_1, f_2)"] \\
F(S) \arrow[d] \arrow[dr, bend right=40, "\langle F(s_1), F(s_2)\rangle"'] & F(X_1) \times F(X_2) \arrow[d, "F(f_1) \times F(f_2)"] \\
\mathrm{Rel}(F)(S) \arrow[r, tail] & F(Y_1) \times F(Y_2)
\end{tikzcd}$$

By uniqueness of such diagonals one verifies that $\mathrm{Rel}(F)(-)$ preserves identities and composition.

This definition of relation lifting generalises the situation in Lemma 3.3.1 for Kripke polynomial functors on **Sets**. We first establish some general properties of this relation lifting, much as in Section 3.2. But before we can do so we need to describe composition of relations.

For two relations $\langle r_1, r_2\rangle \colon R \rightarrowtail X \times Y$ and $\langle s_1, s_2\rangle \colon S \rightarrowtail Y \times Z$ we define their relational composition $S \circ R \rightarrowtail X \times Z$ via pullback and image: first form the object P by pullback in

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "s_2"] \arrow[d, "s_1"] & Z \\
R \arrow[r, "r_2"'] \arrow[d, "r_1"'] & Y \\
X
\end{tikzcd}
\quad \text{and take the image} \quad
\begin{tikzcd}
P \arrow[r, two heads, "e"] \arrow[dr, "\langle r_1 \circ p_1, s_2 \circ p_2\rangle"'] & (S \circ R) \arrow[d, tail] \\
& X \times Z
\end{tikzcd} \tag{4.11}$$

It is possible to turn objects with such relations between them into a category, say $\mathbb{C}$-**Rel** as in **SetsRel**, but this requires additional properties of logical factorisation systems (namely, diagonals are in $\mathfrak{M}$ and $\mathfrak{E}$ is closed under pullbacks, as in Exercise 4.3.8). See [295] for details.

Proposition 4.4.2 *Relation lifting as defined above forms a functor in a commuting diagram:*

$$\begin{tikzcd}
\mathrm{Rel}(\mathbb{C}) \arrow[r, "\mathrm{Rel}(F)"] \arrow[d] & \mathrm{Rel}(\mathbb{C}) \arrow[d] \\
\mathbb{C} \times \mathbb{C} \arrow[r, "F \times F"'] & \mathbb{C} \times \mathbb{C}
\end{tikzcd}$$

Then

1. *The functor* $\mathrm{Rel}(F)$ *preserves the order* $\leq$ *between relations (on the same objects):* $R \leq S \implies \mathrm{Rel}(F)(R) \leq \mathrm{Rel}(F)(()S)$.
2. *This* $\mathrm{Rel}(F)$ *also preserves reversal (also called daggers)* $(-)^{\dagger}$ *of relations, where*

$$(R \overset{\langle r_1, r_2\rangle}{\rightarrowtail} X \times Y)^{\dagger} = (R \overset{\langle r_2, r_1\rangle}{\rightarrowtail} Y \times X).$$

Moreover, there are inequalities:

3. $\mathrm{Eq}(F(X)) \leq \mathrm{Rel}(F)(\mathrm{Eq}(X))$ *and* $\mathrm{Eq}(F(X)) = \mathrm{Rel}(F)(\mathrm{Eq}(X))$ *in case either:*
   - *Diagonals are in* $\mathfrak{M}$*, or*
   - F *preserves abstract epis*, i.e. $e \in \mathfrak{E} \implies F(e) \in \mathfrak{E}$.
4. $\mathrm{Rel}(F)(S \circ R) \leq \mathrm{Rel}(F)(R) \circ \mathrm{Rel}(F)(S)$*, if* F *preserves abstract epis.*

Proof 1. Let $R \rightarrowtail X_1 \times X_2$ and $S \rightarrowtail X_1 \times X_2$ be two relations on the same objects. Then $R \leq S$ means that the pair of identities $(\mathrm{id}_{X_1}, \mathrm{id}_{X_2})$ is a morphism $R \to S$ in $\mathrm{Rel}(\mathbb{C})$. By applying $\mathrm{Rel}(F)$ we get a map of the form $(\mathrm{id}_{F(X_1)}, \mathrm{id}_{F(X_2)})\colon \mathrm{Rel}(F)(R) \to \mathrm{Rel}(F)(S)$ in $\mathrm{Rel}(\mathbb{C})$. This gives the required inequality $\mathrm{Rel}(F)(R) \leq \mathrm{Rel}(F)(S)$.

2. We write $\gamma = \langle \pi_2, \pi_1\rangle$ for the twist map. For a relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$, the reversed relation $R^{\dagger}$ is $\gamma \circ \langle r_1, r_2\rangle = \langle r_2, r_1\rangle\colon R \rightarrowtail Y \times X$. We write the image of $F(R)$ as $\langle s_1, s_2\rangle\colon \mathrm{Rel}(F)(R) \rightarrowtail F(X_1) \times F(X_2)$. The reversed lifted relation $\mathrm{Rel}(F)(R)^{\dagger}$ is $\langle s_2, s_1\rangle$. Thus we need to prove that the image of $\langle F(r_2), F(r_1)\rangle$ is $\langle s_2, s_1\rangle$. This is done via Exercise 4.3.1.3:

$$\begin{aligned}
\mathfrak{m}(\langle F(r_2), F(r_1)\rangle) &= \mathfrak{m}(\gamma \circ \langle F(r_1), F(r_2)\rangle) \\
&= \gamma \circ \mathfrak{m}(\langle F(r_1), F(r_2)\rangle) \quad \text{since } \gamma \text{ is an iso, thus in } \mathfrak{M} \\
&= \gamma \circ \langle s_1, s_2\rangle \\
&= \langle s_2, s_1\rangle.
\end{aligned}$$

3. The equality relation $\mathrm{Eq}(X)$ on an object X is given by the image of the diagonal, below on the left, giving rise to the relation lifting on the right:

$$\begin{tikzcd}
X \arrow[r, two heads, "e_X"] \arrow[dr, tail, "\Delta_X = \langle \mathrm{id}_X, \mathrm{id}_X\rangle"'] & \mathrm{Eq}(X) \arrow[d, tail, "\langle m_1, m_2\rangle"] \\
& X \times X
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mathrm{Eq}(X)) \arrow[rr, two heads, "d"] \arrow[dr, "\langle F(m_1), F(m_2)\rangle"'] & & \mathrm{Rel}(F)(\mathrm{Eq}(X)) \arrow[dl, tail, "\langle r_1, r_2\rangle"] \\
& F(X) \times F(X) &
\end{tikzcd}$$

The inequality $\mathrm{Eq}(F(X)) \leq \mathrm{Rel}(F)(\mathrm{Eq}(X))$ is obtained via diagonal-fill-in:

$$\begin{tikzcd}
F(X) \arrow[rr, tail, "e_{F(X)}"] \arrow[d, "F(e_X)"'] & & \mathrm{Eq}(F(X)) \arrow[dd, tail] \arrow[ddll, dashed] \\
F(\mathrm{Eq}(X)) \arrow[d, "d"'] & & \\
\mathrm{Rel}(F)(\mathrm{Eq}(X)) \arrow[rr, tail, "\langle r_1, r_2\rangle"'] & & F(X) \times F(X)
\end{tikzcd}$$

In case $F(e_X) \in \mathfrak{E}$ we have two factorisations of the diagonal $F(X) \rightarrowtail F(X) \times F(X)$, making the dashed map an isomorphism.

If diagonals $\Delta_X = \langle \mathrm{id}_X, \mathrm{id}_X\rangle \colon X \rightarrowtail X \times X$ are in $\mathfrak{M}$, then the equality relation $\mathrm{Eq}(X)$ on X is this diagonal Δ_X, and its lifting $\mathrm{Rel}(F)(\mathrm{Eq}(X))$ is the image of the tuple $\langle F(\mathrm{id}_X), F(\mathrm{id}_X)\rangle = \langle \mathrm{id}_{F(X)}, \mathrm{id}_{F(X)}\rangle \colon F(X) \rightarrowtail F(X) \times F(X)$. This image is the diagonal $\Delta_{F(X)}$ itself, which is the equality relation $\mathrm{Eq}(F(X))$ on $F(X)$.

4. Let $\langle r_1, r_2\rangle \colon R \rightarrowtail X \times Y$ and $\langle s_1, s_2\rangle \colon S \rightarrowtail Y \times Z$ be two relations with composition $S \circ R$ as described in (4.11). We write their liftings as two tuples $\langle r_1', r_2'\rangle \colon \mathrm{Rel}(F)(R) \rightarrowtail F(X) \times F(Y)$ and $\langle s_1', s_2'\rangle \colon \mathrm{Rel}(F)(S) \rightarrowtail F(Y) \times F(Z)$with composition:

$$\begin{tikzcd}
Q \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rel}(F)(S) \arrow[r, "s_2'"] \arrow[d, "s_1'"] & F(Z) \\
\mathrm{Rel}(F)(R) \arrow[r, "r_2'"'] \arrow[d, "r_1'"'] & F(Y) & \\
F(X) & &
\end{tikzcd}
\quad \text{and image} \quad
\begin{tikzcd}
Q \arrow[r, "d"] \arrow[dr, "\langle r_1' \circ q_1, s_2' \circ q_2\rangle"'] & \mathrm{Rel}(F)(S) \circ \mathrm{Rel}(F)(R) \arrow[d, tail] \\
& F(X) \times F(Z)
\end{tikzcd}$$

Consider the map φ obtained in

$$\begin{tikzcd}
F(P) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"'] \arrow[dr, dashed, "\varphi"] & F(S) \arrow[dr, bend left] & \\
F(R) \arrow[dr, bend right] & Q \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rel}(F)(S) \arrow[d, "s_1'"] \\
& \mathrm{Rel}(F)(R) \arrow[r, "r_2'"'] & F(Y)
\end{tikzcd}$$

Since $F(e) \in \mathfrak{E}$, by assumption, we obtain the required inequality $\leq$ as diagonal in

$$\begin{tikzcd}
F(P) \arrow[r, "F(e)"] \arrow[d, "\varphi"'] & F(S \circ R) \arrow[r, two heads] & \mathrm{Rel}(F)(S \circ R) \arrow[d, tail] \arrow[dll, dashed] \\
Q \arrow[d, "d"'] & & \\
\mathrm{Rel}(F)(S) \circ \mathrm{Rel}(F)(S) \arrow[rr, tail] & & F(X) \times F(Z)
\end{tikzcd}$$

$\square$

Stronger preservation results for lifted functors $\mathrm{Rel}(F)$ can be obtained if we assume that all abstract epis are split epis, i.e. that $\mathfrak{E} \subseteq \mathit{SplitEpis}$. This assumption in **Sets** is equivalent to the axiom of choice; see text before Lemma 2.1.7. Also in the category **Vect** of vector spaces surjective (linear) maps are split: if $f\colon V \to W$ in **Vect** is surjective, then there are isomorphisms $V \cong \ker(f) \oplus \mathrm{Im}(f)$ and $\mathrm{Im}(f) \cong W$. The resulting map

$$W \xrightarrow{\cong} \mathrm{Im}(f) \xrightarrow{\kappa_2} \ker(f) \oplus \mathrm{Im}(f) \xrightarrow{\cong} V$$

is a section of f.

Proposition 4.4.3 *Let* $(\mathfrak{M}, \mathfrak{E})$ *be a logical factorisation system on a category* $\mathbb{C}$ *where* $\mathfrak{E} \subseteq \mathit{SplitEpis}$. *In this case the lifting* $\mathrm{Rel}(F)\colon \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbb{C})$ *of a functor* $F\colon \mathbb{C} \to \mathbb{C}$ *preserves reversals* $(-)^{\dagger}$, *equality and coproducts* $\coprod$.

Moreover, if F preserves weak pullbacks, then $\mathrm{Rel}(F)$ *preserves composition of relations and inverse images:*

$$\begin{aligned}
\mathrm{Rel}(F)(S \circ R) &= \mathrm{Rel}(F)(S) \circ \mathrm{Rel}(F)(R) \\
\mathrm{Rel}(F)((f_1 \times f_2)^{-1}(S)) &= (f(f_1) \times F(f_2))^{-1}(\mathrm{Rel}(F)(S)).
\end{aligned}$$

In particular relation lifting preserves graph relations: $\mathrm{Rel}(F)(\mathrm{Graph}(f)) = \mathrm{Graph}(F(f))$, *since* $\mathrm{Graph}(f) = (f \times \mathrm{id})^{-1}(\mathrm{Eq}(Y)) \rightarrowtail X \times Y$ *for* $f\colon X \to Y$.

Proof Split epis are 'absolute': they are preserved by any functor F; see Lemma 2.1.7. As a result, equality and coproducts are preserved; see Proposition 4.4.2 and Exercise 4.4.3. Hence we concentrate on the second part of the proposition and assume that the functor F preserves weak pullbacks. We shall write sections t_R in a factorisation:

$$\begin{tikzcd}
F(R) \arrow[r, two heads, "e_R"'] \arrow[dr, "{\langle F(r_1), F(r_2)\rangle}"'] & \mathrm{Rel}(F)(R) \arrow[l, bend right, "t_R"'] \arrow[d, tail, "{\langle r_1', r_2'\rangle}"] \\
& F(X) \times F(Y)
\end{tikzcd}
\qquad \text{where} \qquad e_R \circ t_R = \mathrm{id}.$$

We then get $\langle F(r_1), F(r_2)\rangle \circ t_R = \langle r_1', r_2'\rangle$, and thus $F(r_i) \circ t_R = r_i'$, because e_R is a (split) epi:

$$\langle F(r_1), F(r_2)\rangle \circ t_R \circ e_R \;=\; \langle r_1', r_2'\rangle \circ e_R \circ t_R \circ e_R \;=\; \langle r_1', r_2'\rangle \circ e_R.$$

We first show that $\mathrm{Rel}(F)$ preserves composition of relations. We use pullbacks P and Q as in the proof of Proposition 4.4.2. Because F preserves weak pullbacks we obtain a map φ in

$$\begin{tikzcd}
Q \arrow[rr, "q_2"] \arrow[d, "q_1"'] \arrow[drr, dashed, "\varphi"] & & \mathrm{Rel}(F)(S) \arrow[dr, bend left, "t_S"] & \\
\mathrm{Rel}(F)(R) \arrow[drr, bend right, "t_R"'] & & F(P) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"'] & F(S) \arrow[d, "F(s_1)"] \\
& & F(R) \arrow[r, "F(r_2)"'] & F(Y)
\end{tikzcd}$$

Thus we obtain a diagonal:

$$\begin{tikzcd}
Q \arrow[rr, "d"] \arrow[d, "\varphi"'] & & \mathrm{Rel}(F)(S) \circ \mathrm{Rel}(F)(R) \arrow[dd, tail] \arrow[ddll, dashed] \\
F(P) \arrow[d, "F(e)"'] & & \\
F(S \circ R) \arrow[d] & & \\
\mathrm{Rel}(F)(S \circ R) \arrow[rr, tail] & & F(X) \times F(Z)
\end{tikzcd}$$

Preservation of inverse images is obtained as follows. Consider the pullback $(f(f_1) \times F(f_2))^{-1}(\mathrm{Rel}(F)(S))$ written simply as $\bullet$ in

$$\begin{tikzcd}
F((f_1 \times f_2)^{-1}(S)) \arrow[rr] \arrow[d, dashed, shift right, "g"'] \arrow[dd, bend right=60] & & F(S) \arrow[d] \\
\bullet \arrow[rr] \arrow[d, tail] \arrow[u, dashed, shift right, "h"'] \arrow[dr, phantom, "\lrcorner"] & & \mathrm{Rel}(F)(S) \arrow[d, tail] \\
F(X_1) \times F(X_2) \arrow[rr, "F(f_1) \times F(f_2)"'] & & F(Y_1) \times F(Y_2)
\end{tikzcd}$$

The map g is obtained because the lower square is a pullback. And h arises because F preserves weak pullbacks (and the map $F(S) \twoheadrightarrow \mathrm{Rel}(F)(S)$ is a split epi). Then $g \circ h = \mathrm{id}$, because the lower square is a (proper) pullback, so that

g is split epi. From Exercise 4.4.2 we can conclude that we have an appropriate factorisation giving $\bullet \cong \mathrm{Rel}(F)((f_1 \times f_2)^{-1}(S))$, as required. ❑

Corollary 4.4.4 *Let F be a weak-pullback-preserving endofunctor on a category $\mathbb{C}$ with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$ satisfying $\mathfrak{E} \subseteq$ SplitEpis. Then*

$$R \text{ is an equivalence relation} \implies \mathrm{Rel}(F)(R) \text{ is an equivalence relation.}$$

Writing $\mathrm{EqRel}(\mathbb{C})$ *for the category of equivalence relations* $R \rightarrowtail X \times X$ *in* $\mathbb{C}$*, we get an obvious restriction of the relation-lifting functor* $\mathrm{Rel}(F)$ *to* $\mathrm{EqRel}(F)$ *in*

$$\begin{tikzcd} \mathrm{EqRel}(\mathbb{C}) \arrow[r, "\mathrm{EqRel}(F)"] \arrow[d] & \mathrm{EqRel}(\mathbb{C}) \arrow[d] \\ \mathbb{C} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

Proof The fact that $R \rightarrowtail X \times X$ is an equivalence relation can be expressed via three inequalities $\Delta = \mathrm{Eq}(X) \leq R$, $R^\dagger \leq R$ and $R \circ R \leq R$. By the previous result relation lifting $\mathrm{Rel}(F)$ preserves equality, reversal and composition, making $\mathrm{Rel}(F)(R)$ an equivalence relation. ❑

The next definition captures some essential properties of the lifted functor $\mathrm{Rel}(F)$. Subsequently, a close connection with weak-pullback-preserving functors is established.

Definition 4.4.5 Let $\mathbb{C}$ be a category with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$. A **relator** for a functor $F\colon \mathbb{C} \to \mathbb{C}$, also known as an F-relator, is a functor $H\colon \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbb{C})$ that makes

$$\begin{tikzcd} \mathrm{Rel}(\mathbb{C}) \arrow[r, "H"] \arrow[d] & \mathrm{Rel}(\mathbb{C}) \arrow[d] \\ \mathbb{C} \times \mathbb{C} \arrow[r, "F \times F"'] & \mathbb{C} \times \mathbb{C} \end{tikzcd}$$

commute and preserves equality relations, relation reversal, relation composition and graph relations. (The latter means $H(\mathrm{Graph}(f)) = \mathrm{Graph}(F(f))$ and thus links H and F.)

There is some disagreement in the literature about the precise definition of an F-relator (see for instance [446, 409]), but the most reasonable requirements seem to be precisely those that yield the equivalence with weak-pullback-preservation by F in the next result. Often these relators are defined with respect to a category with relations as morphisms, such as the category **SetsRel** in Example 1.4.2.4. But the category Rel($\mathbb{C}$) with relations as objects (that we use) seems more natural in this context, for instance, because it contains bisimulations as coalgebras (see below). The situation is compared, in the set-theoretic case, more explicitly in Corollary 5.2.8 later on.

The essence of the following result comes from [450] and [94]. It applies in particular for $\mathbb{C} = \mathbf{Sets}$. A generalisation in enriched categories may be found in [70] (using preservation of exact squares instead of preservation of weak pullbacks).

Theorem 4.4.6 *Let $F\colon \mathbb{C} \to \mathbb{C}$ be a functor, where the category $\mathbb{C}$ carries a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$ with $\mathfrak{E} \subseteq$ SplitEpis. Then F has a relator if and only if F preserves weak pullbacks.*

Moreover, this relator, if it exists, is uniquely determined.

Proof Proposition 4.4.3 tells us that Rel(F) is a relator if F preserves weak pullbacks. Conversely, assume H is an F-relator. We use make extensive use of the equations in Exercise 4.4.4 for showing that F preserves weak pullbacks. So assume we have a weak pullback on the left below, and a pair of maps a, b making the outer diagram on the right commute:

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \qquad \begin{tikzcd} A \arrow[drrr, bend left, "b"] \arrow[dr, bend right, "a"'] & & & \\ & F(V) \arrow[rr, "F(k)"] \arrow[d, "F(h)"] & & F(Y) \arrow[d, "F(g)"] \\ & F(X) \arrow[rr, "F(f)"'] & & F(Z) \end{tikzcd}$$

Commutation of this diagram means

$$\begin{aligned} \top &\leq \langle F(f) \circ a, F(g) \circ b\rangle^{-1}(\mathrm{Eq}(F(Z))) \\ &= \langle a, b\rangle^{-1}(F(f) \times F(g))^{-1}(\mathrm{Eq}(F(Z))) \\ &= \langle a, b\rangle^{-1}(\mathrm{Eq}(F(f), F(g))) \\ &= \langle a, b\rangle^{-1}(H(\mathrm{Eq}(f, g))), \end{aligned} \qquad (*)$$

where the latter equation holds because in Exercise 4.4.4 equality is formulated in terms of graphs, composition and reversal, namely as $\mathrm{Eq}(f, g) = \mathrm{Graph}(g)^{\dagger} \circ \mathrm{Graph}(f)$. The two graph relations involved are obtained via pullbacks in

$$\begin{tikzcd} \mathrm{Graph}(f) \arrow[r, "m_2"] \arrow[d, tail, "\langle m_1, m_2\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "\Delta"] \\ X \times Z \arrow[r, "f \times \mathrm{id}"'] & Z \times Z \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Graph}(g) \arrow[r, "n_2"] \arrow[d, tail, "\langle n_1, n_2\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, tail, "\Delta"] \\ Y \times Z \arrow[r, "g \times \mathrm{id}"'] & Z \times Z \end{tikzcd}$$

And the relation composition $\mathrm{Graph}(g)^\dagger \circ \mathrm{Graph}(f) = \mathrm{Eq}(f,g)$ results from the following pullback and image:

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Graph}(g) \arrow[r, "n_1"] \arrow[d, "n_2"] & Y \\
\mathrm{Graph}(f) \arrow[r, "m_2"'] \arrow[d, "m_1"'] & Z & \\
X & &
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, two heads, "e"] \arrow[dr, "\langle m_1 \circ p_1, n_1 \circ p_2\rangle"'] & \mathrm{Eq}(f,g) \arrow[d, tail, "\langle r_1, r_2\rangle"] \\
& X \times Y
\end{tikzcd}$$

We then have

$$f \circ m_1 \circ p_1 \;=\; m_2 \circ p_1 \;=\; n_2 \circ p_2 \;=\; g \circ n_1 \circ p_2.$$

This line of reasoning started with a weak pullback. It yields a map $c\colon P \to V$, not necessarily unique, with $h \circ c = m_1 \circ p_1$ and $k \circ c = n_1 \circ p_2$. The image factorisation on the left below then gives a diagonal on the right:

$$\begin{tikzcd}
V \arrow[r, two heads, "d"] \arrow[d, tail, "\top = \mathrm{id}"'] & \coprod_{\langle h,k\rangle}(\top) \arrow[d, tail, "\ell"] \\
V \arrow[r, "\langle h, k\rangle"'] & X \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, two heads, "e"] \arrow[d, "c"'] & \mathrm{Eq}(f,g) \arrow[dd, tail, "\langle r_1, r_2\rangle"] \arrow[ddl, dashed] \\
V \arrow[d, two heads, "d"'] & \\
\coprod_{\langle h,k\rangle}(\top) \arrow[r, tail, "\ell"'] & X \times Y
\end{tikzcd}$$

Thus we have an inequality $\mathrm{Eq}(f,g) \leq \coprod_{\langle h,k\rangle}(\top)$. Via the equation $\coprod_{\langle h,k\rangle}(\top) = \mathrm{Graph}(k) \circ \mathrm{Graph}(h)^\dagger$ from Exercise 4.4.4 we can continue the reasoning from $(*)$ in

$$\begin{aligned}
\top \;&\leq\; \langle a,b\rangle^{-1}(H(\mathrm{Eq}(f,g))) \\
&\leq\; \langle a,b\rangle^{-1}(H(\textstyle\coprod_{\langle h,k\rangle}(\top))) \\
&=\; \langle a,b\rangle^{-1}(H(\mathrm{Graph}(k) \circ \mathrm{Graph}(h)^\dagger)) \\
&=\; \langle a,b\rangle^{-1}(\mathrm{Graph}(F(k)) \circ \mathrm{Graph}(F(h))^\dagger) \\
&=\; \langle a,b\rangle^{-1}(\textstyle\coprod_{\langle F(h),F(k)\rangle}(\top)).
\end{aligned}$$

This inequality can be described diagrammatically as

$$\begin{tikzcd}
F(V) \arrow[r, shift right, two heads, "d'"'] \arrow[d, tail, "\top = \mathrm{id}"'] & \coprod_{\langle F(h),F(k)\rangle}(\top) \arrow[l, shift right, "s"'] \arrow[d, tail, "\ell'"] & A \arrow[l, dashed, "j"'] \arrow[d, tail, "\mathrm{id} = \top"] \\
F(V) \arrow[r, "\langle F(h), F(k)\rangle"'] & F(X) \times F(Y) & A \arrow[l, "\langle a, b\rangle"]
\end{tikzcd}$$

where s satisfies $d' \circ s = \mathrm{id}$, using that d' is a split epi. The resulting map $s \circ j \colon A \to F(V)$ is the mediating map that proves that F preserves weak pullbacks:

$$\langle F(h), F(k)\rangle \circ s \circ j = \ell' \circ d' \circ s \circ j = \ell' \circ j = \langle a, b\rangle.$$

Finally, via the equations in Exercise 4.4.4 and the preservation properties of relators we can prove uniqueness: one gets $H = \mathrm{Rel}(F)$ from

$$\begin{aligned}
H(R) &= H(\mathrm{Graph}(r_2) \circ \mathrm{Graph}(r_1)^{\dagger}) \\
&= H(\mathrm{Graph}(r_2)) \circ H(\mathrm{Graph}(r_1)^{\dagger}) \\
&= H(\mathrm{Graph}(r_2)) \circ H(\mathrm{Graph}(r_1))^{\dagger} \\
&= \mathrm{Graph}(F(r_2)) \circ \mathrm{Graph}(F(r_1))^{\dagger} \\
&= \mathrm{Rel}(F)(\mathrm{Graph}(r_2)) \circ \mathrm{Rel}(F)(\mathrm{Graph}(r_1))^{\dagger} \\
&= \cdots = \mathrm{Rel}(F)(R).
\end{aligned}$$

❑

Exercises

4.4.1 Verify that composition of relations as defined categorically in (4.11) is in the set-theoretic case, with classes of maps $\mathfrak{M} = (\textit{injections})$ and $\mathfrak{E} = (\textit{surjections})$, the same as ordinary composition of relations.

4.4.2 Prove that split epis are orthogonal to all monos (where orthogonality is defined in Exercise 4.3.5).

Conclude that $\mathfrak{E} \subseteq \textit{SplitEpis}$, for a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$, implies $\mathfrak{E} = \textit{SplitEpis}$.

4.4.3 Use functoriality of relation lifting to obtain:

1. $\coprod_{F(f)\times F(g)}(\mathrm{Rel}(F)(R)) \leq \mathrm{Rel}(F)(\coprod_{f\times g}(R))$.
2. $\mathrm{Rel}(F)((f \times g)^{-1}(R)) \leq (F(f) \times F(g))^{-1}(\mathrm{Rel}(F)(R))$.

Prove that the inequality $\leq$ in (1) is an equality $=$ if the functor F preserves abstract epis.

4.4.4 Let $\mathbb{C}$ be a category with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$. Show that graph relations, composition and reversal are fundamental, in the sense that:

1. $\mathrm{Eq}(f, g) \stackrel{\text{def}}{=} \langle f, g\rangle^{-1}(\mathrm{Eq}(Z)) = \mathrm{Graph}(g)^{\dagger} \circ \mathrm{Graph}(f)$, for $f \colon X \to Z$ and $g \colon Y \to Z$.

And if $\mathfrak{E} \subseteq \textit{SplitEpis}$, show that

2. $R = \mathrm{Graph}(r_2) \circ \mathrm{Graph}(r_1)^{\dagger}$, for $\langle r_1, r_2\rangle \colon R \rightarrowtail X \times Y$.

3. $\mathrm{Im}(\langle h,k\rangle) \stackrel{\text{def}}{=} \coprod_{\langle h,k\rangle}(\top) = \mathrm{Graph}(k) \circ \mathrm{Graph}(h)^\dagger$, for $h\colon V \to X$ and $k\colon V \to Y$.

4.4.5 Let F be an endofunctor on a category $\mathbb{C}$. Use the pullback in Exercise 4.3.3 to define a lifting $\mathrm{EnRel}(F)\colon \mathrm{EnRel}(\mathbb{C}) \to \mathrm{EnRel}(\mathbb{C})$ of F to an endofunctor on endorelations in $\mathbb{C}$. Describe in detail how this functor works.

4.4.6 Let $\mathbb{C}$ be a category with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$.

1. Show that a natural transformation $\sigma\colon F \Rightarrow G$ gives rise to a 'lifted' natural transformation $\mathrm{Rel}(\sigma)\colon \mathrm{Rel}(F) \Rightarrow \mathrm{Rel}(G)$ in

$$\begin{tikzcd}
\mathrm{Rel}(\mathbb{C}) \arrow[rr, bend left, "\mathrm{Rel}(F)"] \arrow[rr, bend right, "\mathrm{Rel}(G)"'] \arrow[rr, phantom, "\Downarrow \mathrm{Rel}(\sigma)"] \arrow[d] & & \mathrm{Rel}(\mathbb{C}) \arrow[d] \\
\mathbb{C} \arrow[rr, bend left, "F\times F"] \arrow[rr, bend right, "G\times G"'] \arrow[rr, phantom, "\Downarrow \sigma\times\sigma"] & & \mathbb{C}
\end{tikzcd}$$

2. Prove that relation lifting is functorial, in the sense that it preserves identity natural transformations $\mathrm{Rel}(\mathrm{id}_F) = \mathrm{id}_{\mathrm{Rel}(F)}$ and composition of natural transformations: $\mathrm{Rel}(\tau \circ \sigma) = \mathrm{Rel}(\sigma) \circ \mathrm{Rel}(\tau)$.
3. Show that for two arbitrary endofunctors $F, G\colon \mathbb{C} \to \mathbb{C}$, there is a natural transformation $\mathrm{Rel}(FG) \Rightarrow \mathrm{Rel}(F)\mathrm{Rel}(G)$.

4.5 Logical Bisimulations

In the preceding sections we have first developed a general form of categorical logic using factorisation systems, and subsequently used this logic to introduce liftings $\mathrm{Rel}(F)$ of functors F to relations. This enables us to describe F-bisimulations as $\mathrm{Rel}(F)$-coalgebras, as in Lemma 3.2.4, but at a much more general level – not only for polynomial functors, not only on **Sets**.

Definition 4.5.1 Consider a functor $F\colon \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$ with a logical factorisation system, together with the resulting lifting $\mathrm{Rel}(F)\colon \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbb{C})$ as described in Definition 4.4.1.

In this setting a **logical F-bisimulation** is a $\mathrm{Rel}(F)$-coalgebra. It thus consists of a relation $R \rightarrowtail X \times Y$ with a pair of morphisms (coalgebras) $c\colon X \to F(X)$, $d\colon Y \to F(Y)$ in $\mathbb{C}$ forming a morphism in the category $\mathrm{Rel}(\mathbb{C})$:

$$\begin{tikzcd}
R \arrow[r, dashed] \arrow[d, tail] & \mathrm{Rel}(F)(R) \arrow[d, tail] \\
X \times Y \arrow[r, "c \times d"'] & F(X) \times F(Y)
\end{tikzcd}$$

For the record we add that a **logical F-congruence** is a $\mathrm{Rel}(F)$-algebra.

In particular, a coalgebra of the functor $\mathrm{EnRel}(F)\colon \mathrm{EnRel}(\mathbb{C}) \to \mathrm{EnRel}(\mathbb{C})$ from Exercise 4.4.5 is a bisimulation on a single coalgebra. It is an endorelation itself.

The notion of $\mathrm{Rel}(F)$-coalgebra thus already contains the two underlying coalgebras. It is more common that these two coalgebras c, d are already given and that a bisimulation is a relation R on their state spaces so that the pair (c, d) is a morphism $R \to \mathrm{Rel}(F)(R)$ in $\mathrm{Rel}(\mathbb{C})$. This is just a matter of presentation.

As is to be expected, equality relations are bisimulations. This can be formulated more abstractly as a lifting property, using the category $\mathrm{EnRel}(\mathbb{C})$ of endorelations from Exercise 4.3.3.

Lemma 4.5.2 *Let $F\colon \mathbb{C} \to \mathbb{C}$ be an endofunctor on a category $\mathbb{C}$ with a logical factorisation system. For each coalgebra $c\colon X \to F(X)$ the equality relation $\mathrm{Eq}(X) \rightarrowtail X \times X$ on its state is a bisimulation.*

As a result there is a lifting of the equality functor in

$$\begin{tikzcd}
\mathbf{CoAlg}(F) \arrow[r, "\mathrm{Eq}(-)"] \arrow[d] & \mathbf{CoAlg}(\mathrm{EnRel}(F)) \arrow[d] \\
\mathbb{C} \arrow[r, "\mathrm{Eq}(-)"'] \arrow[loop below, "F"] & \mathrm{EnRel}(\mathbb{C}) \arrow[loop below, "\mathrm{EnRel}(F)"]
\end{tikzcd}$$

This lifting sends

$$\left(X \xrightarrow{c} F(X) \right) \longmapsto \left(\begin{tikzcd}
\mathrm{Eq}(X) \arrow[r] \arrow[d, tail] & \mathrm{Eq}(F(X)) \le \mathrm{Rel}(F)(\mathrm{Eq}(X)) \arrow[d, tail] \\
X \times X \arrow[r, "c \times c"] & F(X) \times F(X)
\end{tikzcd} \right)$$

where we use that the functor $\mathrm{Rel}(F)$ and $\mathrm{EnRel}(F)$ coincide on endorelations. ❑

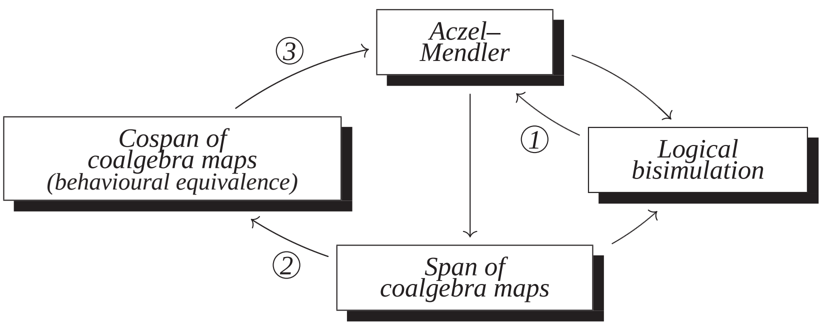
A similar lifting for algebras is more complicated, because in general, there is only an inequality $\mathrm{Eq}(F(X)) \le \mathrm{Rel}(F)(\mathrm{Eq}(X))$; see Proposition 4.4.2.3. But under additional assumptions guaranteeing $\mathrm{Eq}(F(X)) = \mathrm{Rel}(F)(\mathrm{Eq}(X))$ there is also a lifted functor $\mathrm{Eq}(-)\colon \mathbf{Alg}(F) \to \mathbf{Alg}(\mathrm{EnRel}(F))$. These additional assumptions are for instance that diagonals are abstract monos, or F preserves abstract epis.

In a next step we wish to compare notions of bisimulation (as in [436]):

- The above logical one involving a coalgebra $R \to \mathrm{Rel}(F)(R)$ in $\mathrm{Rel}(\mathbb{C})$;
- The span-based formulation – with the Aczel–Mendler notion involving a coalgebra $R \to F(R)$ in $\mathbb{C}$ as special case;
- The cospan-based formulation, also known as behavioural equivalence.

Earlier, in Theorems 3.3.2 and 3.3.3 it was shown that these notions coincide in the more restricted context of polynomial functors on **Sets**. In the present general setting they diverge – but they still coincide in **Sets**, for weak-pullback-preserving functors.

Theorem 4.5.3 *In the setting of Definition 4.5.1, with a logical factorisation system* $(\mathfrak{M}, \mathfrak{E})$ *on a category* $\mathbb{C}$*, there are the following implication arrows between notions of bisimulation:*



With additional side-conditions:

① *Abstract epis are split,* i.e. $\mathfrak{E} \subseteq$ *SplitEpis.*
② *The category* $\mathbb{C}$ *has pushouts.*
③ *The functor F preserves weak pullbacks and diagonals* $\Delta = \langle \mathrm{id}, \mathrm{id} \rangle$ *are in* $\mathfrak{M}$ *– or equivalently,* $\mathfrak{E} \subseteq$ *Epis; see Exercise 4.3.5.6.*

Proof Clearly, an Aczel–Mendler bisimulation $R \to F(R)$ on two coalgebras $c\colon X \to F(X)$ and $d\colon Y \to F(Y)$ forms a span of coalgebra maps $X \leftarrow R \to Y$. This implication is the (vertical) down arrow in the middle.

For the other unlabelled (unconditional) implication starting from an Aczel–Mendler bisimulation, write the relation as $\langle r_1, r_2 \rangle\colon R \rightarrowtail X \times Y$, carrying a coalgebra $R \to F(R)$ as on the left below. It gives rise to a $\mathrm{Rel}(F)$-coalgebra on the right via the factorisation that defines $\mathrm{Rel}(F)(R)$:

$$
\begin{tikzcd}
R \arrow[r] \arrow[d, tail, "\langle r_1, r_2 \rangle"'] & F(R) \arrow[d, "\langle F(r_1), F(r_2) \rangle"] \\
X \times Y \arrow[r, "c \times d"'] & F(X) \times F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r] \arrow[d, tail] \arrow[rr, dashed, bend left] & F(R) \arrow[r, two heads] \arrow[d, "\langle F(r_1), F(r_2) \rangle"] & \mathrm{Rel}(F)(R) \arrow[dl, bend left] \\
X \times Y \arrow[r, "c \times d"'] & F(X) \times F(Y) &
\end{tikzcd}
$$

(Here we assume that the relation R is already in $\mathfrak{M}$. If not, we have to consider it as a proper span and factorise it first; see below.)

In the other direction, for the implication with label/condition ①, assume a coalgebra $(c, d)\colon R \to \mathrm{Rel}(F)(R)$ in $\mathrm{Rel}(\mathbb{C})$ as in the outer diagram on the right above. If the abstract epi $F(R) \twoheadrightarrow \mathrm{Rel}(F)(R)$ is split, we obtain a map $R \to \mathrm{Rel}(F)(R) \to F(R)$ yielding a commuting diagram as on the left.

Next assume a general span of coalgebra maps:

$$
\begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \xleftarrow{\;f\;} \begin{pmatrix} F(W) \\ \uparrow b \\ W \end{pmatrix} \xrightarrow{\;g\;} \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix}.
$$

Clearly if the category $\mathbb{C}$ has pushouts, then so has the category $\mathbf{CoAlg}(F)$. The pushout of this diagram forms a cospan of coalgebras. This establishes the implication ②.

For the span above we consider the factorisations of $\langle f, g \rangle$ and $\langle F(r_1), F(r_2) \rangle$ in the diagram below. They make the image $\mathrm{Im}(\langle f, g \rangle)$ a logical bisimulation via diagonal-fill-in:

$$
\begin{tikzcd}
& F(W) \arrow[d, "F(e)"] \\
W \arrow[ur, "b"] \arrow[d, "e"'] \arrow[dd, bend right=60, "\langle f, g \rangle"'] & F(\mathrm{Im}(\langle f, g \rangle)) \arrow[d] \arrow[dd, bend left=60, "\langle F(r_1), F(r_2) \rangle"] \\
\mathrm{Im}(\langle f, g \rangle) \arrow[r, dashed] \arrow[d, tail, "\langle r_1, r_2 \rangle"'] & \mathrm{Rel}(F)(\mathrm{Im}(\langle f, g \rangle)) \arrow[d, tail] \\
X \times Y \arrow[r, "c \times d"'] & F(X) \times F(Y)
\end{tikzcd}
$$

Finally, for the implication with condition ③ assume we have a cospan of coalgebra maps:

$$
\begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \xrightarrow{\;f\;} \begin{pmatrix} F(W) \\ \uparrow b \\ W \end{pmatrix} \xleftarrow{\;g\;} \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix}.
$$

We claim that the pullback below on the left gives rise to a coalgebra map on the right:

$$\begin{tikzcd}
R \arrow[r] \arrow[d, tail, "{\langle p_1, p_2\rangle}"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, tail, "{\Delta = \langle \mathrm{id}, \mathrm{id}\rangle}"] \\
X \times Y \arrow[r, "f \times g"'] & W \times W
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "c"] \arrow[d, tail, "{\langle p_1, p_2\rangle}"'] & F(R) \arrow[d, "{\langle F(p_1), F(p_2)\rangle}"] \\
X \times Y \arrow[r, "c \times d"'] & F(X) \times F(Y)
\end{tikzcd}$$

By construction the pair $X \xleftarrow{p_1} R \xrightarrow{p_2} Y$ is the pullback of f, g. Hence because F preserves weak pullbacks there is a (not necessarily unique) map $c\colon R \to F(R)$ in

$$\begin{tikzcd}
R \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, dashed, "c"] & Y \arrow[dr, bend left, "d"] & \\
X \arrow[dr, bend right, "c"'] & F(R) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"] & F(Y) \arrow[d, "F(g)"] \\
& F(X) \arrow[r, "F(f)"'] & F(W)
\end{tikzcd}$$

It is precisely the map we seek. ❑

After this analysis of the abstract situation we become more concrete and characterise (logical) bisimulation for the two endofunctors from Section 4.1, namely multiset $\mathcal{M}_M$ (for a commutative monoid M) and distribution $\mathcal{D}$, both on **Sets**. We follow [459] where it was first shown that bisimulation equivalence for the distribution functor coincides with the (non-coalgebraic) formulation developed by Larsen and Skou [328].

For a relation $\langle r_1, r_2\rangle\colon R \hookrightarrow X \times Y$ the relation lifting $\mathrm{Rel}(\mathcal{M}_M)(R) \subseteq \mathcal{M}_M(X) \times \mathcal{M}_M(Y)$ is the image in **Sets** in the diagram

$$\begin{tikzcd}
\mathcal{M}_M(R) \arrow[r, two heads] \arrow[dr, "{\langle \mathcal{M}_M(r_1), \mathcal{M}_M(r_2)\rangle}"'] & \mathrm{Rel}(\mathcal{M}_M)(R) \arrow[d, tail] \\
& \mathcal{M}_M(X) \times \mathcal{M}_M(Y)
\end{tikzcd}$$

We can describe this image concretely as

$$\begin{aligned}
\mathrm{Rel}(\mathcal{M}_M)(R) &= \{(\varphi, \psi) \in \mathcal{M}_M(X) \times \mathcal{M}_M(Y) \mid \exists \chi \in \mathcal{M}_M(R). \\
&\qquad\qquad \mathcal{M}_M(r_1)(\chi) = \varphi \wedge \mathcal{M}_M(r_2)(\chi) = \psi\} \\
&= \{(\mathcal{M}_M(r_1)(\chi), \mathcal{M}_M(r_2)(\chi)) \mid \chi \in \mathcal{M}_M(R)\} \\
&= \{(\textstyle\sum_i m_i|x_i\rangle, \sum_i m_i|y_i\rangle) \mid \sum_i m_i|x_i, y_i\rangle \in \mathcal{M}_M(R)\}.
\end{aligned}$$

Thus, this relation R is a (logical) bisimulation for two $\mathcal{M}_M$-coalgebras $c: X \to \mathcal{M}_M(X)$ and $d: Y \to \mathcal{M}_M(Y)$ if for each pair $(x, y) \in R$ there is multiset $\sum_i m_i|\, x_i, y_i\,\rangle$ over R with $c(x) = \sum_i m_i|\, x_i\,\rangle$ and $d(x) = \sum_i m_i|\, y_i\,\rangle$.

It is not hard to see that bisimulations for the distribution functor $\mathcal{D}$ take precisely the same form, except that the multiplicities m_i must be in the unit interval $[0, 1]$ and add up to 1. For the distribution functor there is an alternative description of bisimulation equivalences (i.e. for relations that are at the same time bisimulations and equivalence relations).

Proposition 4.5.4 (From [459]) *Let $c, d: X \to \mathcal{D}(X)$ be two coalgebras of the distribution functor, with the same state space X. An equivalence relation $R \subseteq X \times X$ is then a logical bisimulation for $\mathcal{D}$-coalgebras c, d if and only if R is a 'probabilistic bisimulation' (as defined in [328]): for all $x, y \in X$,*

$$R(x, y) \implies c(x)[Q] = d(y)[Q], \qquad \textit{for each } R\textit{-equivalence class } Q \subseteq X$$

where for $\varphi \in \mathcal{D}(X)$ and $U \subseteq X$ we write $\varphi[U] = \sum_{x \in U} \varphi(x)$.

Proof First, assume R is a bisimulation equivalence with $R(x, y)$. As described above, there is then a formal distribution $\chi = \sum_i r_i|\, x_i, y_i\,\rangle \in \mathcal{D}(X \times X)$ with $R(x_i, y_i)$ for each i, and $c(x) = \sum_i r_i|\, x_i\,\rangle$ and $d(y) = \sum_i r_i|y_i\,\rangle$. Now let $Q \subseteq X$ be an R-equivalence class. Then $x_i \in Q$ iff $y_i \in Q$, since $R(x_i, y_i)$, and thus:

$$c(x)[Q] \;=\; \textstyle\sum_{x_i \in Q} r_i \;=\; \sum_{y_i \in Q} r_i \;=\; d(y)[Q].$$

Conversely, assume R is a probabilistic bisimulation with $R(x, y)$. We write $c(x) = \sum_i r_i|\, x_i\,\rangle$ and $d(y) = \sum_j s_j|y_j\,\rangle$. For each x_i and y_j in this sum, for which $R(x_i, y_j)$ holds, there is an equivalence class:

$$Q_{i,j} \stackrel{\text{def}}{=} [x_i]_R \;=\; [y_j]_R.$$

By assumption, $c(x)[Q_{i,j}] = d(y)[Q_{i,j}]$. These sums, say $t_{i,j} \in [0, 1]$, are non-zero because by definition $x_i \in \mathrm{supp}(c(x))$ and $y_j \in \mathrm{supp}(d(y))$. We now define $\chi \in \mathcal{D}(X \times X)$ by

$$\chi(u, v) \;=\; \begin{cases} \dfrac{c(x)(x_i) \cdot d(y)(y_j)}{t_{i,j}} & \text{if } (u, v) = (x_i, y_j) \text{ and } R(x_i, y_j) \\ 0 & \text{otherwise.} \end{cases}$$

We then have for $x_i \in \mathrm{supp}(c(x))$

$$\begin{aligned}
\mathcal{D}(\pi_1)(\chi)(x_i) &= \sum_{j, R(x_i, y_j)} \chi(x_i, y_j) \\
&= \sum_{j, R(x_i, y_j)} \frac{c(x)(x_i) \cdot d(y)(y_j)}{t_{i,j}} \\
&= c(x)(x_i) \cdot \frac{\sum_{j, R(x_i, y_j)} d(y)(y_j)}{t_{i,j}} \\
&= c(x)(x_i).
\end{aligned}$$

Similarly, $\mathcal{D}(\pi_2)(\chi) = d(y)$. Finally, the probabilities in χ add up to 1 since

$$\begin{aligned}
\sum_{i,j,R(x_i,y_j)} \chi(x_i, y_j) &= \sum_i \sum_{j, R(x_i, y_j)} \chi(x_i, y_j) \\
&= \sum_i c(x)(x_i) \qquad \text{as just shown} \\
&= 1.
\end{aligned}$$

❑

4.5.1 Logical Formulations of Induction and Coinduction

Earlier, in Theorem 3.1.4 we have stated that the familiar induction principle for initial algebras can be formulated in 'binary' form as: every congruence on the initial algebra is reflexive (i.e. contains the equality relation). In the present setting we can formulate this induction principle in far more general logical form, as preservation properties.

For the validity of these logical formulations we need the assumption that relation lifting preserves equality: $\mathrm{Eq}(F(X)) = \mathrm{Rel}(F)(\mathrm{Eq}(X))$. Recall from Proposition 4.4.2.3 that in general only the inequality $\leq$ holds. This property is used in the algebraic case to guarantee that equality lifts to a functor $\mathrm{Eq}(-)\colon \mathbf{Alg}(F) \to \mathbf{Alg}(\mathrm{EnRel}(F))$, analogously to Lemma 4.5.2, turning equality relations on carriers of algebras into logical congruences.

But first we have a quite general result.

Theorem 4.5.5 *Let $F\colon \mathbb{C} \to \mathbb{C}$ be an endofunctor, on a category $\mathbb{C}$ with a logical factorisation system, which has an initial algebra $\alpha\colon F(A) \xrightarrow{\cong} A$. Then each logical congruence is reflexive.*

More precisely, suppose we have two arbitrary algebras $a\colon F(X) \to X$ and $b\colon F(Y) \to Y$ and a relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$. Assume this R is a logical congruence, in the sense that the pair (a, b) forms an algebra $\mathrm{Rel}(F)(R) \to R$

in the category $\mathrm{Rel}(\mathbb{C})$ *of relations. Then there is a map* $\mathrm{Eq}(A) \to R$ *in* $\mathrm{Rel}(\mathbb{C})$, *namely:*

$$\begin{tikzcd}
\mathrm{Eq}(A) \arrow[r, dashed] \arrow[d, tail] & R \arrow[d, tail, "\langle r_1, r_2 \rangle"] \\
A \times A \arrow[r, "\mathrm{int}_a \times \mathrm{int}_b"'] & X \times Y
\end{tikzcd}$$

where $\mathrm{int}_a \colon A \to X$ *and* $\mathrm{int}_b \colon A \to Y$ *are the algebra homomorphisms obtained by initiality.*

Proof Exercise 4.5.1 says that the fact that R is a logical congruence may be described via a (necessarily unique) algebra structure $c \colon F(R) \to R$ with $\langle r_1, r_2 \rangle \circ c = (a \times b) \circ \langle F(r_1), F(r_2) \rangle$, as in the rectangle on the right, below. It yields an algebra map $\mathrm{int}_c \colon A \to R$ on the left, in

$$\begin{tikzcd}
F(A) \arrow[r, dashed, "F(\mathrm{int}_c)"] \arrow[d, "\cong"'] & F(R) \arrow[r, "\langle F(r_1), F(r_2) \rangle"] \arrow[d, "c"'] & F(X) \times F(Y) \arrow[d, "a \times b"] \\
A \arrow[r, dashed, "\mathrm{int}_c"'] & R \arrow[r, tail, "\langle r_1, r_2 \rangle"'] & X \times Y
\end{tikzcd}$$

By uniqueness we then get $r_1 \circ \mathrm{int}_c = \mathrm{int}_a$ and $r_2 \circ \mathrm{int}_c = \mathrm{int}_b$ in

$$\begin{tikzcd}
 & & X \\
A \arrow[urr, bend left, "\mathrm{int}_a"] \arrow[r, "\mathrm{int}_c"] \arrow[drr, bend right, "\mathrm{int}_b"'] & R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & \\
 & & Y
\end{tikzcd}$$

But then we obtain the map of relations $\mathrm{Eq}(A) \to R$ via diagonal-fill-in:

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "\mathrm{int}_c"'] & \mathrm{Eq}(A) \arrow[d, tail] \arrow[ddl, dashed] \\
 & A \times A \arrow[d, "\mathrm{int}_a \times \mathrm{int}_b"] \\
R \arrow[r, tail, "\langle r_1, r_2 \rangle"'] & X \times Y
\end{tikzcd}$$

❑

If we restrict ourselves to endorelations (on the same object), then we can formulate this binary induction principle more abstractly as a preservation property (as in [218]).

Corollary 4.5.6 *Assuming relation lifting preserves equality, the lifted equality functor* $\mathrm{Eq}(-) \colon \mathbf{Alg}(F) \to \mathbf{Alg}(\mathrm{EnRel}(F))$ *preserves initial objects.*

Thus: if $F(A) \xrightarrow{\cong} A$ *is initial in the category* $\mathbf{Alg}(F)$*, then the logical congruence* $\mathrm{Eq}(A) \rightarrowtail A \times A$ *is initial in* $\mathbf{Alg}(\mathrm{EnRel}(F))$, i.e. *is the initial logical congruence.*

Proof Let $b\colon \mathrm{EnRel}(F)(R) \to R$ be an $\mathrm{EnRel}(F)$–algebra, given by an F-algebra $b\colon F(X) \to X$ and a logical congruence relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times X$. Theorem 4.5.5 gives the unique map of relations $\mathrm{int}_b\colon \mathrm{Eq}(A) \to R$ in the category $\mathrm{EnRel}(\mathbb{C})$ of endorelations. This makes $\mathrm{Eq}(A)$ initial in $\mathbf{Alg}(\mathrm{EnRel}(F))$. Hence the functor $\mathrm{Eq}(-)\colon \mathbf{Alg}(F) \to \mathbf{Alg}(\mathrm{EnRel}(F))$ preserves initial objects. ❑

The formulation '$\mathrm{Eq}(-)\colon \mathbf{Alg}(F) \to \mathbf{Alg}(\mathrm{EnRel}(F))$ preserves initial objects' is used as a definition in [218]; it expresses that the logic involved satisfies the induction principle. The above result says that under mild assumptions (relation lifting preserves equality) the logic given by a logical factorisation system indeed satisfies the induction principle. In [218] logics are described more generally in terms of fibrations. Then it is shown that the crucial structure for this result is comprehension $\{-\}$. It is built into the kind of logics we consider here; see Section 4.3.

A bit more concretely, suppose we have a relation $R \subseteq A^\star \times A^\star$ on the initial algebra $A^\star$ of lists over a set A. Assume $R(\mathsf{nil}, \mathsf{nil})$ and $R(\sigma, \sigma') \Rightarrow R(\mathsf{cons}(a, \sigma), \mathsf{cons}(a, \sigma'))$ hold. These two assumptions express that R is a logical congruence, for the (initial) algebra $[\mathsf{nil}, \mathsf{cons}]\colon 1 + A \times A^\star \xrightarrow{\cong} A^\star$. The previous corollary then says that R must be reflexive, i.e. that $R(\sigma, \sigma)$ holds for all $\sigma \in A^\star$.

We turn to a similar logical formulation of coinduction. We can say, still following the approach of [218], that the coinduction principle is satisfied if the equality functor $\mathrm{Eq}(-)\colon \mathbf{CoAlg}(F) \to \mathbf{CoAlg}(\mathrm{EnRel}(F))$ from Lemma 4.5.2 preserves final objects. This is not automatically the case. The crucial structures we now need are quotients (instead of comprehension). We briefly explain how this works.

Definition 4.5.7 Let $\mathbb{C}$ be a category with a logical factorisation system. We say that it admits **quotients** if the equality functor $\mathrm{Eq}(-)\colon \mathbb{C} \to \mathrm{EnRel}(\mathbb{C})$ has a left adjoint, typically written as Q.

Intuitively, the above functor Q sends an endorelation $R \rightarrowtail X \times X$ to the quotient $X/\overline{R}$, where $\overline{R}$ is the least equivalence relation containing R. Exercise 4.5.5 will describe some conditions guaranteeing the existence of such quotients Q.

Theorem 4.5.8 *Assume a logical factorisation system with quotients on a category* $\mathbb{C}$*, and an endofunctor* $F\colon \mathbb{C} \to \mathbb{C}$ *whose relation lifting* $\mathrm{Rel}(F)$

preserves equality. Then the coinduction principle holds: the equality functor $\mathrm{Eq}(-)\colon \mathbf{CoAlg}(F) \to \mathbf{CoAlg}(\mathrm{EnRel}(F))$ *from Lemma 4.5.2 preserves final objects.*

Proof Let $\zeta\colon Z \xrightarrow{\cong} F(Z)$ be a final F-coalgebra. We have to prove that equality $\mathrm{Eq}(Z)$ on its carrier is the final logical bisimulation. So let $R \rightarrowtail X \times X$ be an arbitrary logical bisimulation on a coalgebra $c\colon X \to F(X)$. We have to produce a unique map of $\mathrm{Rel}(F)$-coalgebras:

$$\left(R \xrightarrow{c} \mathrm{Rel}(F)(R)\right) \longrightarrow \left(\mathrm{Eq}(Z) \xrightarrow{\zeta} \mathrm{Rel}(F)(\mathrm{Eq}(Z))\right).$$

Since such a map is by definition also a map $c \to \zeta$ in the category $\mathbf{CoAlg}(F)$ it can only be the unique map $\mathrm{beh}_c\colon X \to Z$ to the final coalgebra. Hence our task is reduced to showing that beh_c is a map of relations $R \to \mathrm{Eq}(Z)$. But since $\mathrm{Eq}(-)$ is right adjoint to quotients Q we need to find a map $Q(R) \to Z$. It arises by finality as soon as the object $Q(R)$ carries an F-coalgebra structure $Q(R) \to F(Q(R))$. Again we use the adjunction $Q \dashv \mathrm{Eq}(-)$ to obtain such a coalgebra map. It suffices to have a map of relations $R \to \mathrm{Eq}(F(Q(R)))$. The latter is obtained as from the unit $\eta\colon R \to \mathrm{Eq}(Q(R))$ of the adjunction, using that relation lifting preserves equality:

$$R \xrightarrow{\;c\;} \mathrm{Rel}(F)(R) \xrightarrow{\mathrm{Rel}(F)(\eta)} \mathrm{Rel}(F)(\mathrm{Eq}(Q(R))) = \mathrm{Eq}(F(Q(R))). \qquad \square$$

Exercises

4.5.1 Let F be an endofunctor on a category $\mathbb{C}$ with a logical factorisation system. Assume algebras $a\colon F(X) \to X$ and $b\colon F(Y) \to Y$ and a relation $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$. Prove that the pair (a, b) is a $\mathrm{Rel}(F)$-algebra $\mathrm{Rel}(F)(R) \to R$ in $\mathrm{Rel}(\mathbb{C})$ – making R a logical congruence – if and only if the object $R \in \mathbb{C}$ carries an F-algebra $c\colon F(R) \to R$ making the r_i algebra homomorphisms in

$$\begin{tikzcd}
F(X) \arrow[d, "a"'] & F(R) \arrow[l, "F(r_1)"'] \arrow[r, "F(r_2)"] \arrow[d, "c"'] & F(Y) \arrow[d, "b"] \\
X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & X
\end{tikzcd}$$

Check that this algebra c, if it exists, is unique.

4.5.2 Generalise Lemma 4.5.2 in the following manner. Assume an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$ with a logical factorisation system.

Consider two coalgebra homomorphisms f, g with the same domain. Prove that the image $\mathrm{Im}(\langle f, g\rangle) = \coprod_{\langle f,g\rangle}(\top)$ is a logical bisimulation.

4.5.3 Suppose we have two coalgebras $X \xrightarrow{c} F(X), Y \xrightarrow{d} F(Y)$ of an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$ with a logical factorisation system.

1. Prove that a relation $R \rightarrowtail X \times Y$ is a logical bisimulation for c, d if and only if $\coprod_{c\times d}(R) \leq \mathrm{Rel}(F)(R)$.
2. Assume that posets of relations have arbitrary joins $\bigvee$. Prove that logical bisimulations are closed under $\bigvee$, in the sense that if each R_i is a logical bisimulation, then so $\bigvee_i R_i$. *Hint*: Use that $\coprod_{c\times d}$, as left adjoint, preserves joins.

This shows that bisimilarity $\underline{\leftrightarrow}$, as join of all bisimulations, is a bisimulation itself.

4.5.4 Use Exercise 4.4.3.1 to prove that for coalgebra homomorphisms f, g one has: if R is a bisimulation, then so is $\coprod_{f\times g}(R)$.

4.5.5 Consider a category $\mathbb{C}$ with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$ with diagonals $\Delta = \langle \mathrm{id}, \mathrm{id}\rangle$ contained in $\mathfrak{M}$. Prove that if $\mathbb{C}$ has coequalisers, then its logic admits quotients – in the sense of Definition 4.5.7. *Hint*: Define the functor $Q\colon \mathrm{EnRel}(\mathbb{C}) \to \mathbb{C}$ via the coequaliser of the two legs of a relation.

4.5.6 Assume a logical factorisation system with quotients on a category $\mathbb{C}$, and an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$ whose relation lifting $\mathrm{Rel}(F)$ preserves equality. Prove that a bisimulation $R \rightarrowtail X \times X$ on a coalgebra $c\colon X \to F(X)$ yields a quotient coalgebra $c/R\colon Q(R) \to F(Q(R))$ and a map of coalgebras:

$$\begin{tikzcd}
F(X) \arrow[r, "F(q)"] & F(Q(R)) \\
X \arrow[u, "c"] \arrow[r, "q"'] & Q(R) \arrow[u, "c/R"']
\end{tikzcd}$$

This construction makes explicit what is used in the proof of Theorem 4.5.8; it generalises Theorem 3.3.4.1.

4.6 Existence of Final Coalgebras

Final coalgebras have already been used at various places in this text. They have been described explicitly for a number of special functors, such as for functors $(-)^A \times B$ for deterministic automata in Proposition 2.3.5. Often it is interesting to see what the elements of such final coalgebras are, but in actually using final coalgebras their universal property (i.e. finality) is most relevant; see for instance Section 3.5. Hence what is most important to know is whether or not a final coalgebra exists. Theorem 2.3.9 has mentioned, without proof, that a final coalgebra exists for each finite Kripke polynomial functor. It is the main aim in this section to prove a general result about the existence of final coalgebras in **Sets**, which implies the earlier mentioned Theorem 2.3.9. This general result says: bounded endofunctors on **Sets** have final coalgebras.

In this section we consider only final coalgebras for endofunctors on sets. There are more general results, applying to other categories. For instance, [469] shows that any accessible endofunctor on a locally presentable category admits a final coalgebra. Such results go beyond this introductory text. There is an extensive literature on final coalgebras [433, 12, 55, 291, 15, 18, 24, 469, 411, 176, 404, 170] that can be consulted for further information. The last two references [404, 170] describe a 'logical' construction of final coalgebras, via modal formulas (known as canonical models, in modal logic).

The section starts with a generalisation of the familiar construction of obtaining least fixed points of continuous endofunctions on directed complete posets (dcpo). It serves as a suitable introduction to the topic.

First we recall the basic fixed-point constructions for directed complete partial orders (dcpos). Assume D is a dcpo with a least element $\perp \in D$, and $f \colon D \to D$ is a continuous function – that is, an endomap in **Dcpo**, so f is not required to preserve $\perp$. The least fixed point $\mu f \in D$ can then be defined as a join of an ascending ω-chain of elements in D:

$$\perp \;\leq\; f(\perp) \;\leq\; f^2(\perp) \;\leq\; f^3(\perp) \;\leq\; \cdots \;\leq\; \mu f = \bigvee_{n\in\mathbb{N}} f^n(\perp),$$

where $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$. By continuity one obtains $f(\mu f) = \bigvee_n f(f^n(\perp)) = \bigvee_n f^{n+1}(\perp) = \mu f$. It is easy to see that μf is the least fixed point, or, better, the least *pre-fixed point*: if $f(x) \leq x$, then $\mu(f) \leq x$. By induction one obtains $f^n(\perp) \leq x$, and thus $\mu f = \bigvee_n f^n(\perp) \leq x$.

Aside: the Greek letter ω is often used in mathematical logic for the set $\mathbb{N}$ of natural numbers (considered as ordinal). It is standard in this context, in expressions such as ω-chain and ω-limit.

Since each poset is a category and a monotone function between posets is a functor, we can see $f \colon D \to D$ as a functor. The element μf is then the initial algebra. This construction can be generalised to categories (as in [433, lemma 2]), once the relevant notions have been suitably extended, both for algebras and for coalgebras.

Proposition 4.6.1 *Let $F : \mathbb{C} \to \mathbb{C}$ be an arbitrary endofunctor.*

1. *Suppose the following ω-chain starting from the initial object $0 \in \mathbb{C}$ has a colimit $A \in \mathbb{C}$:*

$$0 \xrightarrow{\;!\;} F(0) \xrightarrow{F(!)} F^2(0) \xrightarrow{F^2(!)} F^3(0) \longrightarrow \cdots \longrightarrow A. \tag{4.12}$$

   If the functor F is co-continuous, in the sense that it preserves colimits of ω-chains, then there is an initial algebra structure $F(A) \xrightarrow{\cong} A$.

2. *Dually, assume there is a limit $Z \in \mathbb{C}$ of the chain starting at the final object $1 \in \mathbb{C}$.*

$$1 \xleftarrow{\;!\;} F(1) \xleftarrow{F(!)} F^2(1) \xleftarrow{F^2(!)} F^3(1) \longleftarrow \cdots \longleftarrow Z. \tag{4.13}$$

   If F is continuous (preserves ω-limits), then we get a final coalgebra $Z \xrightarrow{\cong} F(Z)$ E.

Proof The two statements are each other's duals and we choose to prove only the second one; Exercise 4.6.1 elaborates on the first point. We shall make explicit what it means when an object X is a limit of a diagram:

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} \cdots \longleftarrow X\,.$$

It requires the presence of a 'universal cone': a collection of arrows $(X \xrightarrow{\zeta_n} X_n)_{n\in\mathbb{N}}$ satisfying $f_{n+1} \circ \zeta_{n+1} = \zeta_n$, with the following universal property. For each cone, given by an object $Y \in \mathbb{C}$ with arrows $g_n : Y \to X_n$ such that $f_{n+1} \circ g_{n+1} = g_n$, there is a unique map $h: Y \to Z$ with $\zeta_n \circ h = g_n$, for each $n \in \mathbb{N}$. In a diagram:

$$\begin{tikzcd}
X_0 & X_1 \arrow[l, "f_1"'] & X_2 \arrow[l, "f_2"'] & X_3 \arrow[l, "f_3"'] & \ldots \arrow[l, "f_4"'] & \cdots Z \arrow[lllll, bend right, "\zeta_0"'] \arrow[llll, bend right, "\zeta_1"'] \arrow[lll, bend right, "\zeta_2"'] \\
 & & & & \cdots & Y \arrow[ulllll, "g_0"] \arrow[ullll, "g_1"] \arrow[ulll, "g_2"] \arrow[ull, "g_3"] \arrow[u, dashed, "h"']
\end{tikzcd}$$

We now return to the situation in the proposition. Assume a limit (4.13) with maps $\zeta_n : Z \to F^n(1)$ satisfying $F^n(!) \circ \zeta_{n+1} = \zeta_n$. Applying F to the chain (4.13) and its limit Z yields another chain $(F(\zeta_n): F(Z) \to F^{n+1}(1))$ with limit $F(Z)$. Using the latter's universal property yields an isomorphism

$\zeta \colon Z \xrightarrow{\cong} F(Z)$ with $F(\zeta_n) \circ \zeta = \zeta_{n+1}$. It is a final coalgebra, since for an arbitrary coalgebra $c \colon Y \to F(Y)$ we can form a collection of maps $c_n \colon Y \longrightarrow F^n(1)$ via

$$\begin{aligned} c_0 &= (Y \xrightarrow{!} 1) \\ c_1 &= (Y \xrightarrow{c} F(Y) \xrightarrow{F(!)} F(1)) = F(c_0) \circ c \\ c_2 &= (Y \xrightarrow{c} F(Y) \xrightarrow{F(c)} F^2(1) \xrightarrow{F^2(!)} F^2(1)) = F(c_1) \circ c \\ &\vdots \\ c_{n+1} &= F(c_n) \circ c. \end{aligned}$$

The maps c_n commute with the arrows in the chain (4.13), as is easily seen by induction. This yields a unique map $h \colon Y \to Z$ with $\zeta_n \circ h = c_n$. It forms a homomorphism of coalgebras, i.e. satisfies $\zeta \circ h = F(h) \circ c$, by uniqueness of maps $Y \to F(Z)$ to the limit $F(Z)$:

$$\begin{aligned} F(\zeta_n) \circ \zeta \circ h &= \zeta_{n+1} \circ h \\ &= c_{n+1} \\ &= F(c_n) \circ c \\ &= F(\zeta_n) \circ F(h) \circ c. \end{aligned}$$ ❑

In order to make this ω-limit construction more concrete we consider some examples in **Sets**. The following result is then useful.

Lemma 4.6.2 *1. In **Sets** limits of ω-chains exist and are computed as follows. For a chain $(X_{n+1} \xrightarrow{f_n} X_n)_{n\in\mathbb{N}}$ the limit Z is a subset of the infinite product $\prod_{n\in\mathbb{N}} X_n$ given by*

$$Z = \{(x_0, x_1, x_2, \ldots) \mid \forall n \in \mathbb{N}.\, x_n \in X_n \wedge f_n(x_{n+1}) = x_n\}.$$

*2. Each exponent polynomial functor $F \colon$ **Sets** $\to$ **Sets** – without powerset – is continuous.*

Proof 1. The maps $\zeta_n \colon Z \to X_n$ are the nth projections. The universal property is easily established: given a set Y with functions $g_n \colon Y \to X_n$ satisfying $f_{n+1} \circ g_{n+1} = g_n$, the unique map $h \colon Y \to Z$ with $\zeta_n \circ h = g_n$ is given by the ω-tuple $h(y) = (g_0(y), g_1(y), g_2(y), \ldots)$.

2. By induction on the structure of F, using that products, coproducts and (constant) exponents preserve the relevant constructions. ❑

Corollary 4.6.3 *Each exponent polynomial functor $F \colon$ **Sets** $\to$ **Sets** has a final coalgebra, which can be computed as the limit of an ω-chain as in (4.13).* ❑

We show how to actually (re)calculate one such final coalgebra.

Example 4.6.4 In Corollary 2.3.6.2 we have seen that the final coalgebra for a (simple polynomial) functor $F(X) = X^A \times 2$ can be described as the set $2^{A^\star} = \mathcal{P}(A^\star) = \mathcal{L}(A)$ of languages with alphabet A. Here we shall reconstruct this coalgebra as the limit of an ω-chain.

Therefore we start by investigating what the chain (4.13) looks like for this functor F:

$$\begin{aligned}
F^0(1) &= 1 \cong \mathcal{P}(0) \\
F^1(1) &= 1^A \times 2 \cong 1 \times 2 \cong 2 \cong \mathcal{P}(1) \\
F^2(1) &\cong 2^A \times 2 \cong 2^{A+1} \cong \mathcal{P}(1 + A) \\
F^3(1) &\cong (2^{A+1})^A \times 2 \cong 2^{A\times(A+1)} \times 2 \cong 2^{A^2+A+1} \cong \mathcal{P}(1 + A + A^2) \quad \text{etc.}
\end{aligned}$$

One sees that

$$F^n(1) \cong \mathcal{P}(\textstyle\coprod_{i=0}^{n-1} A^i).$$

The maps $F^n(!)\colon F^{n+1}(1) \to F^n(1)$ in (4.13) are given by the inverse image κ_n^{-1} of the obvious coprojection function $\kappa_n\colon 1 + A + \cdots + A^{n-1} \longrightarrow 1 + A + \cdots + A^{n-1} + A^n$. An element $U \in Z$ of the limit Z as described in Lemma 4.6.2.1 consists of elements $U_n \subseteq 1{+}A{+}\cdots{+}A^{n-1} \longrightarrow 1{+}A{+}\cdots{+}A^{n-1}$, with the requirement that $\kappa_n^{-1}(U_{n+1}) = U_n$. The latter means that these $U_{n+1} \subseteq 1 + A + \cdots + A^{n-1} \longrightarrow 1 + A + \cdots + A^{n-1} + A^n$ can be identified with the set A^n of words of length n. Together they form a set of words, or language, $U \subseteq A^\star$, as in the original description in Corollary 2.3.6.2.

What we have done so far applies only to (co-)continuous endofunctors. But, for instance, the finite powerset is not continuous. Hence we shall need more powerful techniques to cover a larger class of functors. This is done via the notion of a *bounded* functor. It goes back to [291] and occurs regularly in the theory of coalgebras; see for instance [411, 187]. Here we shall use it for the description of final coalgebras. We first introduce the standard formulation, and then immediately introduce an equivalent alternative that is easier to work with in the current setting.

Definition 4.6.5 A functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ is called **bounded** if there is a set M such that for each coalgebra $X \to F(X)$ and state $x \in X$ there is a subcoalgebra on $S \hookrightarrow X$ with $x \in S$, where S is strictly smaller than M, i.e. $|S| < |M|$.

We use the notation $|X|$ for the cardinality of a set X. The notion of subcoalgebra $S \hookrightarrow X$ will be investigated more systematically in the next chapter in terms of invariants. Here the meaning should be obvious, namely a coalgebra $S \to F(S)$ making the inclusion $S \hookrightarrow X$ a homomorphism of coalgebras.

Example 4.6.6 Here is an example of a bounded functor that will play an important role. Deterministic automata are described as coalgebras of the functor $D(X) = X^A \times B$, for suitable sets A, B. This functor is bounded by the set $\mathcal{P}(A^\star)$. Indeed, given an arbitrary D-coalgebra $\langle \delta, \epsilon \rangle \colon X \to X^A \times B$ with a state $x \in X$ we can take as subset $S \hookrightarrow X$ the set of successor states of x, given by

$$S = \{\delta^*(x, \alpha) \mid \alpha \in A^\star\},$$

where the iterated transition function δ^* is introduced in (2.22). Clearly, $x \in S$ for $\alpha = \langle\rangle$. Also, S is closed under transitions and thus carries a subcoalgebra structure. Finally, $|S| \leq |A^\star| < |\mathcal{P}(A^\star|$.

The following alternative description of bounded functors is a combination of results from [449, 187, 28].

Proposition 4.6.7 *For a functor $F \colon$ **Sets** $\to$ **Sets** the following three statements are equivalent.*

1. *F is bounded.*
2. *F is **accessible**: there is a set M such that for each set X,*

$$F(X) = \bigcup\{F(U) \mid U \subseteq X \quad \text{and} \quad |U| < |M|\}.$$

3. *There are sets A, B with a natural transformation:*

$$(-)^A \times B \overset{\sigma}{\Longrightarrow} F$$

where for each set $X \neq \emptyset$ the component $\sigma_X \colon X^A \times B \to F(X)$ is surjective.

The equation in (2) is intuitively clear but a bit sloppy, since we have omitted inclusion functions $i \colon U \hookrightarrow X$, which turn elements $u \in F(U)$ into elements $F(i)(u) \in F(X)$.

A functor that is bounded by the set $\mathbb{N}$ of natural numbers is called **ω-accessible** or **finitary**. These ω-accessible/finitary functors are thus entirely determined by their behaviour on finite sets. They preserve ω-colimits (see Exercise 4.6.7), and thus have initial algebras. As we shall see, they also have final coalgebras.

Proof (1) $\Rightarrow$ (2) Assume that F is bounded, say via the set M (as in Definition 4.6.5). This same set can be used in the description of accessibility. The inclusion $(\supseteq)$ in (2) clearly holds, so we concentrate on $(\subseteq)$. If $X = \emptyset$ the result is obvious, so we may assume an element $x_0 \in X$. Let $w \in F(X)$; it yields a constant coalgebra $c = \lambda x \in X.\, w\colon X \to F(X)$. Since F is bounded by assumption, for the element $x_0 \in X$ there is a subcoalgebra $c_S : S \to F(S)$ of c, on a subset $i\colon S \hookrightarrow X$ with $|S| < |M|$, and an element $y_0 \in S$ with $i(y_0) = x_0$. We claim that $v = c_S(y_0) \in \bigcup\{F(U) \mid U \subseteq X \text{ and } |U| < |M|\}$ is the required element that is mapped to $w \in F(X)$:

$$\begin{aligned} F(i)(v) &= F(i)(c_S(y_0)) \\ &= c(i(y_0)) \qquad\qquad \text{since } c_S \text{ is a subcoalgebra} \\ &= w. \end{aligned}$$

(2) $\Rightarrow$ (3) Assume that F is accessible, say via the set M. We take $A = M$ and $B = F(M)$, and define $\sigma_X\colon X^M \times F(M) \to F(X)$ as $\sigma_X(f, b) = F(f)(b)$. It is easy to see that σ is natural, so we concentrate on showing that σ_X is surjective for $X \neq \emptyset$.

Let $w \in F(X)$. By accessibility of F there is a subset $i\colon U \hookrightarrow X$ with $|U| < |M|$ and an element $v \in F(U)$ with $w = F(i)(v)$. Since $|U| < |M|$ there is, by definition of the cardinal order, an injection $j\colon U \rightarrowtail M$. We distinguish two cases.

- $U = \emptyset$. In that case $i\colon U \rightarrowtail X$ is the unique map $!_X\colon \emptyset \to X$, since $\emptyset$ is initial in **Sets**. Thus $w = F(!_X)(v) \in F(X)$. We take $b = F(!_M)(v) \in F(M)$ and $f = \lambda m \in M.\, x_0\colon M \to X$, where $x_0 \in X$ is an arbitrary element. This pair $(f, b) \in X^M \times F(M)$ is mapped by σ_X to the element $w \in F(X)$ that we started from:

$$\begin{aligned} \sigma_X(f, b) = F(f)(b) = F(f)(F(!_M)(v)) &= F(f \circ {!_M})(v) \\ &= F(!_X)(v) = w. \end{aligned}$$

- $U \neq \emptyset$. Since $j\colon U \rightarrowtail M$ is injective there is a map $k\colon M \twoheadrightarrow S$ in the reverse direction with $k \circ j = \mathrm{id}$ – as noted at the end of Section 2.1. We now take $f = i \circ k\colon M \to X$ and $b = F(j)(v) \in F(M)$. Then

$$\begin{aligned} \sigma_X(f, b) = F(f)(b) = F(i \circ k)(F(j)(v)) &= F(i \circ k \circ j)(v) \\ &= F(i)(v) = w. \end{aligned}$$

(3) $\Rightarrow$ (1) This implication is easy using the previous example and Exercise 4.6.8. ❑

Lemma 4.6.8 *Each finite Kripke polynomial functor is bounded.*

Proof We shall use the third formulation from Proposition 4.6.7 and show by induction on the structure of a finite Kripke polynomial functor F that there are sets A, B with a suitable natural transformation $\sigma\colon (-)^A \times B \Rightarrow F$. We leave details to the interested reader.

- If F is the identity functor, we simply take $A = 1$ and $B = 0$.
- If F is the constant functor $X \mapsto C$, we take $A = 0$ and $B = C$.
- In case F is a product $F_1 \times F_2$ for which we have suitable natural transformations $\sigma_i \colon (-)^{A_i} \times B_i \Rightarrow F_i$, for $i = 1, 2$, we take $A = A_1 + A_2$ and $B = B_1 \times B_2$ and define $\sigma_X \colon X^A \times B \to F(X)$ by

$$\sigma_X(f, (b_1, b_2)) = \langle \sigma_1(\lambda a \in A_1.\, f(\kappa_1 a), b_1),\ \sigma_2(\lambda a \in A_2.\, f(\kappa_2 a), b_2)\rangle.$$

  It is clearly natural, and surjective for $X \neq \emptyset$.
- Similarly, if $F = \coprod_{i \in I} F_i$ with $\sigma_i \colon (-)^{A_i} \times B_i \Rightarrow F_i$, we take $A = \coprod_{i \in I} A_i$ and $B = \coprod_{i \in I} B_i$ and define $\sigma_X \colon X^A \times B \to F(X)$ by

$$\sigma_X(f, \kappa_j b) = \kappa_j \sigma_j(\lambda a \in A_j.\, f(\kappa_j a), b).$$

- Next consider $F = G^C$ and assume we have a suitable natural transformation $\tau \colon (-)^A \times B \Rightarrow G$. We then define $\sigma_X \colon X^{(C \times A)} \times B^C \to G(X)^C$ as

$$\sigma_X(f, g)(c) = \tau_X(\lambda a \in A.\, f(c, a), g(c)).$$

  Proving that σ_X is surjective, for $X \neq \emptyset$, involves the axiom of choice.
- Finally, assume $F = \mathcal{P}_{\text{fin}} G$, where G already comes with a natural transformation $\tau \colon (-)^A \times B \Rightarrow G$. Then we can define $\sigma_X \colon X^{(\mathbb{N} \times A)} \times B^\star \to \mathcal{P}_{\text{fin}}(GX)$ as the n-element set:

$$\begin{aligned} &\sigma_X(f, \langle b_1, \ldots, b_n\rangle) \\ &\quad = \{\tau_X(\lambda a \in A.\, f(1, a), b_1), \ldots, \tau_X(\lambda a \in A.\, f(n, a), b_n)\}. \end{aligned}$$ ❑

Theorem 4.6.9 *Each bounded functor* **Sets** $\to$ **Sets** *has a final coalgebra. In particular:*

- *Each finite Kripke polynomial functor, such as $\mathcal{P}_{\text{fin}}(A \times -)$, and*
- *Each finitary functor, such as multiset $\mathcal{M}_M$ or distribution $\mathcal{D}$,*

has a final coalgebra.

The existence of final coalgebras of finitary functors occurs already in [55].

Proof Let F be this bounded functor. The third, 'surjectivity' formulation of Proposition 4.6.7 yields a natural transformation $\sigma \colon (-)^A \times B \Rightarrow F$, for suitable sets A and B, with surjective components σ_X for $X \neq \emptyset$. Recall from Proposition 2.3.5 that the functor $(-)^A \times B$ has carrier $Z = B^{A^\star}$ of the final coalgebra $\zeta \colon Z \xrightarrow{\cong} Z^A \times B$. We define an F-coalgebra $\xi = \sigma_Z \circ \zeta \colon Z \to F(Z)$. We claim that it is 'weakly' final: for each F-coalgebra $c \colon X \to F(X)$ there is a (not necessarily unique) homomorphism of F-coalgebras $f \colon X \to Z$.

If X is the empty (initial) set $\emptyset$, there is obviously such a homomorphism $f\colon X \to Z$. Otherwise, we know that $\sigma_X\colon X^A \times B \twoheadrightarrow F(X)$ is surjective, and thus, using the axiom of choice, has a section $s\colon F(X) \to X^A \times B$ with $\sigma_X \circ s = \mathrm{id}_{F(X)}$. The coalgebra $s \circ c\colon X \to X^A \times B$ yields a homomorphism $f\colon X \to Z$ of $(-)^A \times B$-coalgebra by finality. It is then also a homomorphism of F-coalgebras in

$$\begin{tikzcd}
X^A \times B \arrow[r, "f^A \times \mathrm{id}_B"] \arrow[d, two heads, "\sigma_X"] & Z^A \times B \arrow[d, two heads, "\sigma_Z"'] \\
F(X) \arrow[r, "F(f)"] \arrow[u, bend left, "s"] & F(Z) \\
X \arrow[u, "c"] \arrow[r, "f"'] & Z \arrow[uu, bend right, "\zeta", "\cong"']
\end{tikzcd}$$

Explicitly:

$$\begin{aligned}
F(f) \circ c &= F(f) \circ \sigma_X \circ s \circ c && \text{because } \sigma_X \circ s = \mathrm{id} \\
&= \sigma_Z \circ (f^A \times \mathrm{id}_B) \circ s \circ c && \text{by naturality of } \sigma \\
&= \sigma_Z \circ \zeta \circ f && \text{since } f \text{ is a homomorphism} \\
&= \xi \circ f.
\end{aligned}$$

We now force the weakly final coalgebra $\xi\colon Z \to F(Z)$ to be truly final. The general theory of bisimulation from Section 4.4 will be used, for the standard logical factorisation system on **Sets**, with its quotients. Bisimilarity $\underline{\leftrightarrow}$ is a join of bisimulations, and thus a bisimulation itself by Exercise 4.5.3. Hence we can form a quotient coalgebra $\xi/\underline{\leftrightarrow}\colon W \to F(W)$ on $W = \mathcal{Q}(\underline{\leftrightarrow}) = Z/\underline{\leftrightarrow}$ by Exercise 4.5.6. This coalgebra $\xi/\underline{\leftrightarrow}$ is final: for each coalgebra $c\colon X \to F(X)$ there is a homomorphism $X \to W$, namely the composition of the map $X \to Z$ obtained by weak finality and the quotient map $q\colon Z \to W$. This is the only one, since if we have two such homomorphisms $f, g\colon X \to W$, then the image $\mathrm{Im}(\langle f, g\rangle) \rightarrowtail W \times W$ is a bisimulation by Exercise 4.5.2. Hence $\mathrm{Im}(\langle f, g\rangle) \leq \underline{\leftrightarrow}$, so that $f = g$. ❑

Weakly final (co)algebras, as used in this proof, may also be constructed in (second order) polymorphic type theory; see [191, 471]. Under suitable parametricity conditions, these constructions yield proper final coalgebras [199, 385, 49, 149].

In the end one may ask if there is a link between the final coalgebra constructions based on limits of chains and on boundedness. It is provided in [470], via transfinite induction, going beyond ω: for instance, for a finitary

(ω-accessible) functor F the final coalgebra can be reached in $\omega + \omega$ steps as the limit of the chain $F^{\omega+m}(1) = F^m(f^\omega(1))$, where $F^\omega(1)$ is the limit of the chain $F^n(1)$.

Exercises

4.6.1 1. Spell out the notions of colimit of ω-chain and of co-continuity.

2. Check that the colimit of an ω-chain $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \cdots$ in **Sets** can be described as the quotient of the disjoint union

$$\coprod_{n\in\mathbb{N}} X_n/\sim \;=\; \{(n,x) \mid n \in \mathbb{N} \wedge x \in X_n\}/\sim$$

where

$$(n,x) \sim (m,y) \iff \exists p \geq n, m.\ f_{np}(x) = f_{mp}(y),$$

with $f_{qp} = f_{p-1} \circ f_{p-2} \circ \cdots \circ f_q \colon X_q \to X_p$ for $q \leq p$.

3. Prove Proposition 4.6.1.1 in detail: the initial algebra of a co-continuous functor $F\colon \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$ with initial object $0 \in \mathbb{C}$ can be obtained as colimit A of the ω-chain

$$\begin{tikzcd}
0 \arrow[r, "!"] \arrow[drrr, "\alpha_0"'] & F(0) \arrow[r, "F(!)"] \arrow[drr, "\alpha_1"] & F^2(0) \arrow[r, "F^2(!)"] \arrow[dr, "\alpha_2"] & \cdots \\
& & & A
\end{tikzcd}$$

with the induced initial algebra $\alpha\colon F(A) \xrightarrow{\cong} A$ satisfying $\alpha \circ F(\alpha_n) = \alpha_{n+1}$.

4.6.2 Recall from Example 2.4.4 that the lift functor $\mathcal{L} = 1 + (-)$ on **Sets** has the natural numbers $\mathbb{N}$ as initial algebra. If we consider the functor $1 + (-)$ not on **Sets** but on **PoSets**, there are two obvious ways to add an element, namely at the bottom or at the top.

1. Check that $\mathbb{N}$ is the colimit of the chain (4.12) for the lift functor on **Sets**.
2. Write $\mathcal{L}_\perp$ for the functor which adds a bottom element $\perp$ to a poset X; prove that the natural numbers with the usual order $\leq$ form an initial algebra of the functor $\mathcal{L}_\perp\colon$ **PoSets** $\to$ **PoSets**, for instance via the chain construction (4.12).
3. Now write $\mathcal{L}_\top\colon$ **PoSets** $\to$ **PoSets** for the functor that adds an element $\top$ as top element. Check that the initial $\mathcal{L}_\top$-algebra is $(\mathbb{N}, \geq)$ – which has 0 as top element.

4.6.3 Prove that a left adjoint preserves colimits of ω-chains and, dually, that a right adjoint preserves limits of such chains.

4.6.4 Let $\alpha\colon F(A) \xrightarrow{\cong} A$ be an initial algebra constructed as colimit (4.12) for a functor F that preserves monomorphisms — like any weak-pullback-preserving, and hence any Kripke polynomial functor; see Lemma 4.2.2. Assume F also has a final coalgebra $\zeta\colon Z \xrightarrow{\cong} F(Z)$, and let $\iota\colon A \to Z$ be the unique (algebra and coalgebra) homomorphism with $\zeta \circ \iota = F(\iota) \circ \alpha^{-1}$. Prove that ι is injective. *Hint*: Define suitable $\zeta_n\colon Z \to F^n(1)$ and use that $F^n(!)\colon F^n(0) \to F^n(1)$ is mono.

4.6.5 Check that the list $(-)^\star$, multiset $\mathcal{M}_M$, distribution $\mathcal{D}$ and finite powerset $\mathcal{P}_{\text{fin}}$ functors are finitary (ω-accessible), but the ordinary powerset functor $\mathcal{P}$ is not.

4.6.6 Check that Lemma 4.6.8 specialises to: every simple polynomial functor is finitary. *Hint*: The easiest way is to use Proposition 2.2.3.

4.6.7 Prove that each finitary functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ preserves colimits of ω-chains (described explicitly in Exercise 4.6.1).

Conclude that such a functor has both an initial algebra, by Proposition 4.6.1, and a final coalgebra, by Theorem 4.6.9.

4.6.8 (See e.g. [187]) Let $F, G\colon \mathbf{Sets} \to \mathbf{Sets}$ be functors with a natural transformation $\sigma\colon G \Rightarrow F$ between them for which σ_X surjective for each $X \neq \emptyset$. Prove that F is bounded in case G is.

4.6.9 Show that if both $F, G\colon \mathbf{Sets} \to \mathbf{Sets}$ are bounded, then so is the composition GF.

4.7 Polynomial and Analytical Functors

This section will present another characterisation of simple polynomial endofunctors (on **Sets**) and also of the related class of analytical functors (introduced in [285]). The characterisation involves properties that we have seen earlier in this chapter – notably finitariness and (weak) pullbacks. Recall from Proposition 2.2.3 that simple polynomial functors are of the form $F(X) = \coprod_{i\in I} X^{\#i}$, where $\#\colon I \to \mathbb{N}$ is an 'arity'. Analytical functors are similar and have the form $F(X) = \coprod_{i\in I} X^{\#i}/G_i$ involving an additional quotient (see below for details). The main result of this section (Theorem 4.7.8) says that a functor is simple polynomial if and only if it is finitary and preserves (countable) pullbacks. Similarly, a functor is analytical if and only if it is finitary and preserves (countable) *weak* pullbacks.

These characterisation results will not be used elsewhere in this book but provide background theory about endofunctors. These results go back to [285] and have been digested and reformulated several times, as in [200] and [31], but see also [134]. The present section contains another minor re-digest, leaning heavily on [200]. Although the main result is relatively easy to formulate, its proof requires quite a bit of work.

First of all we need to generalise the notion of (weak) pullback, as introduced in Section 4.2. There, the *binary* (weak) pullback is defined for two maps $f_1 \colon X_1 \to Y$ and $f_2 \colon X_2 \to Y$ with common codomain. Here we generalise it to an arbitrary number of maps $f_i \colon X_i \to Y$, for indices i in an arbitrary index set I. In fact, we need only I to be countable, so we restrict ourselves to index set $I = \mathbb{N}$. The formulation for arbitrary sets I is then an obvious generalisation.

So assume we have a countable collection $(f_i \colon X_i \to Y)_{i \in \mathbb{N}}$ with common codomain (in an arbitrary category). The pullback of this collection is given by an object P together with maps $p_i \colon P \to X_i$ such that $f_i \circ p_i = f_j \circ p_j$, for all $i, j \in \mathbb{N}$, as in

$$\begin{tikzcd}
& & P \arrow[dll, "p_0"'] \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] \arrow[drr] & & \\
X_0 \arrow[drr, "f_0"'] & X_1 \arrow[dr, "f_1"] & X_2 \arrow[d, "f_2"] & X_3 \arrow[dl, "f_3"'] & \cdots \arrow[dll] \\
& & Y & &
\end{tikzcd}$$

This diagram is a (countable) pullback if it is universal in the obvious way: for each object Q with maps $g_i \colon Q \to X_i$ satisfying $f_i \circ g_i = f_j \circ g_j$, there is a unique map $g \colon Q \to P$ satisfying $p_i \circ g = g_i$. It is a *weak* pullback if such a g exists, without the uniqueness requirement. A functor F preserves such a (weak) pullback if the maps $F(p_1) \colon F(P) \to F(X_i)$ are a (weak) pullback of the collection $(f(f_i) \colon F(X_i) \to F(Y))_{i \in \mathbb{N}}$. These preservation properties play a crucial role in Theorem 4.7.8 below.

The present proof of this section's main result, Theorem 4.7.8, exploits the idea, as in [31], that each functor F can be written as coproduct $F = \coprod_i G_i$ of affine functors G_i (preserving the terminal object: $G_i(1) \cong 1$). This observation goes back to [449] and will be described first. We will use it only for the category **Sets**, but the construction involved can be performed more generally and so we describe it first in a separate lemma. This construction is important and may be understood as a form of reindexing/substitution – whence the $(-)^{-1}$ notation. This construction is used for instance in [276, 274] to define the notion of 'shapely' functor; see Proposition 4.7.9.

Lemma 4.7.1 *Let $\mathbb{C}$ be a category with finite limits (binary pullbacks and terminal object $1 \in \mathbb{C}$). Let $F \colon \mathbb{C} \to \mathbb{C}$ be an arbitrary endofunctor, with a map $u \colon A \to F(1)$ in $\mathbb{C}$. Form for each object $X \in \mathbb{C}$ the pullback*

$$\begin{tikzcd} u^{-1}(F)(X) \arrow[r, "\sigma_X"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, very near start, "\lrcorner"] & F(X) \arrow[d, "F(!_X)"] \\ A \arrow[r, "u"'] & F(1) \end{tikzcd} \tag{4.14}$$

1. *The mapping $X \mapsto u^{-1}(F)(X)$ extends to a functor $u^{-1}(F)\colon \mathbb{C} \to \mathbb{C}$, with a natural transformation $\sigma\colon u^{-1}(F) \Rightarrow F$.*
2. *For $X = 1$ the map $\pi_1\colon u^{-1}(F)(1) \to A$ is an isomorphism.*

Proof 1. For $f\colon X \to Y$ in $\mathbb{C}$ define $u^{-1}(F)(f)$ in

$$\begin{tikzcd} u^{-1}(F)(X) \arrow[rr, "\sigma_X"] \arrow[dr, dashed, "u^{-1}(F)(f)"] \arrow[ddr, bend right, "\pi_X"'] & & F(X) \arrow[dr, "F(f)"] \arrow[ddr, bend left, "F(!_X)"] & \\ & u^{-1}(F)(Y) \arrow[rr, "\sigma_Y"'] \arrow[d, "\pi_Y"] & & F(Y) \arrow[d, "F(!_Y)"'] \\ & A \arrow[rr, "u"'] & & F(1) \end{tikzcd}$$

The outer diagram commutes since $!_Y \circ f = \, !_X$. By construction this yields a natural transformation $\sigma\colon u^{-1}(F) \Rightarrow F$ that commutes with the πs.

2. For $X = 1$ we have $F(!_1) = \mathrm{id}\colon F(1) \to F(1)$ as vertical map on the right-hand side in (4.14). Hence its pullback $\pi_1\colon u^{-1}(F)(1) \to A$ is an isomorphism. ❑

Proposition 4.7.2 *Consider the previous lemma for $\mathbb{C} =$ **Sets** and $A = 1$.*

1. *For each functor $F\colon$ **Sets** $\to$ **Sets** one obtains a natural isomorphism*

$$\left(\coprod_{u \in F(1)} u^{-1}(F)\right) \stackrel{\cong}{\Longrightarrow} F$$

describing the functor F as coproduct of affine functors $u^{-1}(F)$ – where we use that there is an isomorphism $u^{-1}(F)(1) \cong 1$ by Lemma 4.7.1.2.

2. *If F preserves (weak) pullbacks, then so does each $u^{-1}(F)$.*
3. *If F is finitary, then so is $u^{-1}(F)$.*

Proof 1. In the previous lemma we found a natural transformation of the form $\sigma\colon u^{-1}(F) \Rightarrow F$. We left the dependence on the map u implicit. But if we make it explicit by writing σ^u instead of σ, then the above map

$\coprod_{u \in F(1)} u^{-1}(F) \Rightarrow F$ is the cotuple $[\sigma^u]_{u \in F(1)}$. It is an isomorphism since for each $w \in F(X)$ we get $u = F(!)(w) \in F(1)$ with $w' \in u^{-1}(F)(X)$ obtained via the following pullback:

$$\begin{tikzcd}
1 \arrow[dr, dashed, "w'"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, equal] & & \\
& u^{-1}(F)(X) \arrow[r, "\sigma_X"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(X) \arrow[d, "F(!)"] \\
& 1 \arrow[r, "u = F(!)(w)"'] & F(1)
\end{tikzcd}$$

This shows surjectivity. Injectivity is obvious by the uniqueness of maps into the pullback $u^{-1}(F)(X)$.

2. Fix $u \in F(1)$; for convenience we abbreviate $F' = u^{-1}(F)$. Assume a non-empty collection of maps $f_i : X_i \to Y$ in $\mathbb{C}$ with weak pullback $p_i : P \to X_i$. If we have map $g_i : Q \to F'(X_i)$ with $F'(f_i) \circ g_i = F'(f_j) \circ g_j$, then we get maps $\sigma_{X_i} \circ g_i : Q \to F(X_i)$ with

$$\begin{aligned} F(f_i) \circ \sigma_{X_i} \circ g_i &= \sigma_Y \circ F'(f_i) \circ g_i \\ &= \sigma_Y \circ F'(f_j) \circ g_j = F(f_j) \circ \sigma_{X_j} \circ g_j. \end{aligned}$$

Since F preserves weak pullbacks this yields a map $g : Q \to F(P)$ with $F(p_i) \circ g = \sigma_{X_i} \circ g_i$. Then we obtain a unique map $h : Q \to F'(P)$ in

$$\begin{tikzcd}
Q \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "!_Q"'] & & \\
& F'(P) \arrow[r, "\sigma_P"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(P) \arrow[d, "F(!_P)"] \\
& 1 \arrow[r, "u"'] & F(1)
\end{tikzcd}$$

In order to see that the outer diagram commutes, we pick an arbitrary index i_0; it exists because we assumed our collection of maps f_i is non-empty. Then

$$\begin{aligned} F(!_P) \circ g &= F(!_{X_{i_0}}) \circ F(p_{i_0}) \circ g \\ &= F(!_{X_{i_0}}) \circ \sigma_{X_{i_0}} \circ g_{i_0} \\ &= u \circ \; !_{F(X_{i_0})} \circ g_{i_0} \\ &= u \circ \; !_Q. \end{aligned}$$

The resulting map $h\colon Q \to F'(P)$ satisfies $F'(p_i) \circ h = g_i$ by uniqueness of mediating maps for the pullback defining $F'(X) = u^{-1}(F)$:

$$\begin{aligned} ! \circ F'(p_i) \circ h &= \,! \\ &= \,! \circ g_i \\ \sigma_{X_i} \circ F'(p_i) \circ h &= F(p_i) \circ \sigma_P \circ h \\ &= F(p_i) \circ g \\ &= \sigma_{X_i} \circ g_i. \end{aligned}$$

In case P is a proper (non-weak) pullback, uniqueness of h is obtained from uniqueness of $g = \sigma_P \circ h$.

3. Assume $F\colon \mathbf{Sets} \to \mathbf{Sets}$ is finitary. We need to show that $F' = u^{-1}(F)$ is finitary too. So assume an element $w \in F'(X)$. Then $\sigma_X(w) \in F(X)$. Hence there is a finite subset $\varphi\colon Y \hookrightarrow X$ with $v \in F(Y)$ such that $\sigma_X(w) = F(\varphi)(v)$. We obtain $v' \in F'(Y)$ via the pullback defining $F'(Y) = u^{-1}(f)(Y)$:

$$\begin{tikzcd} 1 \arrow[dr, dashed, "v'"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right] & & \\ & F'(Y) \arrow[r, "\sigma_Y"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(Y) \arrow[d, "F(!_Y)"] \\ & 1 \arrow[r, "u"'] & F(1) \end{tikzcd}$$

By uniqueness one obtains $F'(\varphi)(v') = w$, as required. ❑

If we wish to show that a functor is simple polynomial or analytical, we need to describe it as coproduct of elementary functors. The previous result is important for that goal. Another important ingredient will be described next.

Recall that a finitary functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ is determined by its behaviour on finite subsets. In fact, since each finite set Y is isomorphic to a natural number $Y \cong n$, we can say that such a finitary functor F is determined by the outcomes $F(n)$, for $n \in \mathbb{N}$ considered as n-element set. We shall make this a bit more precise via the operations

$$\begin{array}{rcl} F(n) \times X^n & \xrightarrow{\ \mathrm{ap}_n\ } & F(X) \\ (u, t) & \longmapsto & F(t)(u). \end{array} \tag{4.15}$$

The n-tuple $t \in X^n$ is identified with a function $t\colon n \to X$, to which the functor F is applied. The following result again relates a functor to a coproduct of simpler functors.

Lemma 4.7.3 *For an arbitrary functor* $F\colon \mathbf{Sets} \to \mathbf{Sets}$*, the cotuple of the* ap_n *maps in (4.15) yields a natural transformation:*

$$\Big(\coprod_{n\in\mathbb{N}} F(n) \times (-)^n\Big) \xRightarrow{\ \mathrm{ap} \,=\, [\mathrm{ap}_n]_{n\in\mathbb{N}}\ } F \tag{4.16}$$

all of whose components $\coprod_n F(n) \times X^n \Rightarrow F(X)$ *are surjective if and only if* F *is finitary.*

This functor $\coprod_{n\in\mathbb{N}} F(n) \times (-)^n$ *on the left-hand side is isomorphic to an arity functor* $F_{\#_F}$*, where the arity* $\#_F$ *associated with* F *is defined as* $\#_F = \pi_1\colon (\coprod_{n\in\mathbb{N}} F(n)) \to \mathbb{N}$.

Proof First, let F be finitary and u be an element of $F(X)$. Hence there is a finite subset $i\colon Y \hookrightarrow X$ and element $v \in F(Y)$ with $F(i)(v) = u$. Let $n = |Y|$ be the size of Y and choose an isomorphism $j\colon n \xrightarrow{\cong} Y$. Take $t = i \circ j\colon n \to X$ and $w = F(j^{-1})(v) \in F(n)$. Then

$$\mathrm{ap}_n(w, t) \;=\; F(t)(w) \;=\; (F(i \circ j) \circ F(j^{-1}))(v) \;=\; F(i)(v) \;=\; u.$$

Conversely, assume the map $\coprod_n F(n) \times X^n \Rightarrow F(X)$ is surjective for each set X, and let $u \in F(X)$. Then there is an $n \in \mathbb{N}$ and $(v, t) \in F(n) \times X^n$ with $\mathrm{ap}_n(v, t) = F(t)(v) = u$. Consider the image factorisation (in **Sets**)

$$\begin{tikzcd}
n \arrow[r, two heads, "e"] \arrow[dr, "t"'] & Y = \{t(i) \in X \mid i \in n\}. \arrow[d, hook, "m"] \\
& X
\end{tikzcd}$$

Now take $w = F(e)(v) \in F(Y)$. It satisfies $F(m)(w) = F(m \circ e)(v) = F(t)(v) = u$.

The arity functor $F_{\#_F}$ for the arity $\#_F$ is given by $F_\#(X) = \coprod_{n\in\mathbb{N}, u\in F(n)} X^n$. This is obviously isomorphic to $\coprod_{n\in\mathbb{N}} F(n) \times X^n$, as used above. ❑

A finitary functor F is thus a quotient, via the ap-map (4.16), of a simple polynomial functor $\coprod_n F(n) \times (-)^n$. We shall see that further (preservation) conditions on the functor F allow us to say more about this map ap. To this end the following category of elements is useful.

Definition 4.7.4 For a functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ let $\mathbf{Elt}(F)$ be the category of 'elements' of F, defined in the following manner. Objects of $\mathbf{Elt}(F)$ are pairs $X \in \mathbf{Sets}, u \in F(X)$. A morphism $(u \in F(X)) \to (v \in F(Y))$ is a map $f\colon X \to Y$ in **Sets** satisfying $F(f)(u) = v$. Composition and identities are inherited from **Sets**. There is thus an obvious forgetful functor $\mathbf{Elt}(F) \to \mathbf{Sets}$.

This category $\mathbf{Elt}(F)$ of elements is relevant in the current setting since an equation $\mathrm{ap}_n(u, t) = v \in F(X)$, for $u \in F(n)$ and $t \in X^n$, means that t is a morphism $t\colon (u \in F(n)) \to (v \in F(X))$ in $\mathbf{Elt}(F)$. We are interested in getting appropriately minimal versions of such maps.

Lemma 4.7.5 *Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ be a finitary functor.*

1. *For each $w \in F(X)$,*
   - *there is a map $(w' \in F(n)) \to (w \in F(X))$ in $\mathbf{Elt}(F)$, where $n \in \mathbb{N}$,*
   - *such that for each $f\colon (v \in F(Y)) \to (w' \in F(n))$ in $\mathbf{Elt}(F)$, the function $f\colon Y \to n$ is surjective.*
2. *If the functor F preserves countable weak pullbacks, then the previous point can be strengthened to: for each $w \in F(X)$,*
   - *there is a map $(w' \in F(n)) \to (w \in F(X))$ in $\mathbf{Elt}(F)$,*
   - *such that each $f\colon (v \in F(Y) \to (w' \in F(n))$ is a split epi in $\mathbf{Elt}(F)$.*

   If we write:

   $$F^{\circ}(n) = \{w \in F(n) \mid \textit{each } (v \in F(Y)) \to (w \in F(n)) \textit{ is a split epi}\},$$

   then this last result says: for each $w \in F(X)$ there is a map $(w' \in F^{\circ}(n)) \to (w \in F(X))$.

Proof 1. Let $w \in F(X)$. Because the component at X of the map ap in (4.16) is surjective, the set $\{(n, u, t) \mid n \in \mathbb{N}, u \in F(n), t \in X^n \text{ with } \mathrm{ap}_n(u, t) = w\}$ is non-empty. Among all these elements we take the one with the least number n and call it (n, w', t). Thus $\mathrm{ap}_n(w', t) = w$, so that $t\colon (w' \in F(n)) \to (w \in F(X))$ in $\mathbf{Elt}(F)$; additionally, for each $m \in \mathbb{N}$ with $\mathrm{ap}_m(u, s) = w$ we have $n \leq m$. Next, assume a map $f\colon (v \in F(Y) \to (w' \in F(n))$ in $\mathbf{Elt}(F)$. Factorise $f\colon Y \to n$ in $\mathbf{Sets}$ as $f = i \circ f'$ where $f'\colon Y \twoheadrightarrow m$ and $i\colon m \rightarrowtail n$ (so that $m \leq n$). We then have $v' = F(f')(v) \in F(m)$ and $t' = t \circ i \in X^m$ satisfying

$$\mathrm{ap}_m(v', t') = F(t')(v') = F(t \circ i \circ f')(v) = F(t \circ f)(v) = F(t)(w') = w.$$

But then $n \leq m$ and thus $m = n$. Hence $i\colon m \rightarrowtail n$ is an isomorphism, and $f = i \circ f'$ is surjective.

2. Assume towards a contradiction that for each $w \in F(X)$ and for all $(w' \in F(n)) \to (w \in F(X))$ there is a map $f\colon (v \in F(Y)) \to (w' \in F(n))$ that is not a split epi. We proceed in a number of steps.

   a. By (1) we do have a map $t_1\colon (w_1 \in F(n_1)) \to (w \in F(X))$ such that for each $f\colon (v \in F(Y)) \to (w_1 \in F(n_1))$ the function f is surjective. But, by assumption, there is a map $f_1 \colon (v_1 \in F(Y_1)) \to (w_1 \in F(n_1))$ which is not a split epi.

b. We now apply the assumption to $v_1 \in F(Y_1)$. It yields, as before, a map $t_2 \colon (w_2 \in F(n_2)) \to (v_1 \in F(Y_1))$ as in (1), together with $f_2 \colon (v_2 \in F(Y_2)) \to (w_2 \in F(n_2))$ that is not a split epi.

c. Continuing in this manner we obtain a chain of maps in **Elt**(F):

$$\begin{tikzcd}
(w \in F(X)) & (v_1 \in F(Y_1)) \arrow[dl, "f_1"'] & (v_2 \in F(Y_2)) \arrow[dl, "f_2"'] & \cdots \arrow[dl] \\
(w_1 \in F(n_1)) \arrow[u, "t_1"] & (w_2 \in F(n_2)) \arrow[u, "t_2"] & (w_3 \in F(n_3)) \arrow[u, "t_3"] &
\end{tikzcd} \tag{4.17}$$

d. By (1), for each of the resulting maps $f_i \circ t_{i+1} \colon (w_{i+1} \in F(n_{i+1})) \to (w_i \in F(n_i))$ in **Elt**(F) the underlying function $f_i \circ t_{i+1} \colon n_{i+1} \to n_i$ is surjective. Hence $n_{i+1} \geq n_i$. We also have $n_{i+1} \neq n_i$: if $n_{i+1} = n_i$, then the map $f_i \circ t_{i+1}$ is an isomorphism, say with inverse $s \colon n_i \to n_{i+1}$. As a result:

- s is a map $(w_i \in F(n_i)) \to (w_{i+1} \in F(n_{i+1}))$ in **Elt**(F) since $F(s)(w_i) = (F(s) \circ F(f_i \circ t_{i+1}))(w_{i+1}) = w_{i+1}$.
- $t_{i+1} \circ s \colon (w_i \in F(n_i)) \to (v_i \in F(Y_i))$ is a splitting for f_i, since $f_i \circ t_{i+1} \circ s = \mathrm{id}$.

Thus, an equality $n_{i+1} = n_i$ makes f_i a split epi in **Elt**(F) – which we know is not the case. Hence $n_{i+1} > n_i$.

e. Now write g_n for the resulting maps $g_i \colon (w_i \in F(n_i)) \to (w \in F(X))$, obtained as chain of ts and fs in (4.17). We take the (countable) pullback $P = \prod_X n_i$ of these $g_i \colon n_i \to X$ in **Sets**, with projections $p_i \colon P \to n_i$ satisfying $g_i \circ p_i = g_j \circ p_j$. Since F preserves weak pullbacks and $F(g_i)(w_i) = w$, there is an element $u \in F(P)$ with $F(p_i)(u) = w_i$. Hence $p_i \colon (u \in F(P)) \to (w_i \in F(n_i))$ in **Elt**(F).

f. Since F is finitary and $u \in F(P)$ we get a map $t \colon (u' \in F(k)) \to (u \in F(P))$, for some $k \in \mathbb{N}$. Recall that we have an ascending sequence $n_1 < n_2 < n_3 < \ldots$, so there is an index i with $k < n_i$. At the same time we have a map

$$(u' \in F(k)) \xrightarrow{\;t\;} (u \in F(P)) \xrightarrow{\;p_i\;} (w_i \in F(n_i))$$

whose underlying function $p_i \circ t \colon k \to n_i$ must be surjective – by construction of the objects $(w_i \in F(n_i))$, using (1). But surjectivity of this map implies $k \geq n_i$, which is impossible.

Hence our original assumption is wrong. ❑

If we apply these results to affine functors, things are beginning to fall into place.

Proposition 4.7.6 *Let* $F\colon \mathbf{Sets} \to \mathbf{Sets}$ *be a countable weak-pullback-preserving functor which is finitary and affine.*

1. *There is a unique* $n \in \mathbb{N}$ *for which* $\mathrm{ap}_n\colon F^\circ(n) \times (-)^n \Rightarrow F$ *is surjective.*
2. *This yields a (natural) isomorphism*

$$\left(\coprod_{u \in F^\circ(n)} X^n/\sim_u\right) \xrightarrow{\;\cong\;} F(X),$$

where $\sim_u$ *is the equivalence relation on* X^n *given by*

$$t \sim_u s \iff \exists \varphi\colon n \xrightarrow{\cong} n.\, t \circ \varphi = s \quad \textit{and} \quad F(\varphi)(u) = u.$$

3. *In case* F *preserves (proper) pullbacks, we get an isomorphism*

$$\left(\coprod_{u \in F^\circ(n)} X^n\right) \xrightarrow{\;\cong\;} F(X),$$

making F *a simple polynomial functor.*

Proof 1. Assume we have two elements $u \in F(X)$ and $v \in F(Y)$. We pick two maps $t\colon (u' \in F^\circ(n)) \to (u \in F(X))$ and $s\colon (v' \in F^\circ(m)) \to (v \in F(Y))$ with the properties of Lemma 4.7.5.2. The aim is to show $n = m$. The product $n \times m$ is a pullback over 1; applying F yields a weak pullback, as on the left below. Because F is affine we get $F(t)(u') = F(s)(v')$; hence there is an element $w \in F(n \times m)$ with $F(\pi_1)(w) = u'$ and $F(\pi_2)(w) = v'$. The maps π_i are then split epis, by Lemma 4.7.5.2, so we get a diagram in $\mathbf{Elt}(F)$ as on the right, with r_i splitting π_i:

$$\begin{tikzcd} F(n \times m) \arrow[r, "F(\pi_2)"] \arrow[d, "F(\pi_1)"'] & F(m) \arrow[d, "F(s)"] \\ F(n) \arrow[r, "F(t)"'] & F(1) \cong 1 \end{tikzcd} \qquad \begin{tikzcd} (w \in F(n \times m)) \arrow[r, two heads, "\pi_2"'] \arrow[d, two heads, "\pi_1"] & (v' \in F(m)) \arrow[l, bend right, "r_2"'] \arrow[d, "s"] \\ (u' \in F(n)) \arrow[u, bend left, "r_1"] \arrow[r, "t"'] & (* \in F(1)) \end{tikzcd}$$

Since $v' \in F^\circ(m)$ and $u' \in F^\circ(n)$ we obtain that the two resulting diagonal maps

$$(u' \in F(n)) \xrightarrow{r_1} (w \in F(n \times m)) \xrightarrow{\pi_2} (v' \in F(m))$$
$$(v' \in F(m)) \xrightarrow{r_2} (w \in F(n \times m)) \xrightarrow{\pi_1} (u' \in F(n))$$

are both split epis. Hence $n \leq m$ and $m \leq n$, and so $n = m$.

2. For $u \in F^\circ(n)$ we need to show

$$\mathrm{ap}_n(u, t) = \mathrm{ap}_n(u, s) \iff t \sim_u s.$$

The implication ($\Leftarrow$) is trivial: if $t \circ \varphi = s$ and $F(\varphi)(u) = u$, then

$$\begin{aligned}\mathrm{ap}_n(u, t) &= F(t)(u) = F(t)(F(\varphi)(u)) = F(t \circ \varphi)(u) \\ &= F(s)(u) = \mathrm{ap}_n(u, s).\end{aligned}$$

For the direction ($\Rightarrow$), assume $\mathrm{ap}_n(u, t) = \mathrm{ap}_n(u, s)$ in $F(X)$. We form the pullback

$$\begin{tikzcd} n \times_X n \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & n \arrow[d, "s"] \\ n \arrow[r, "t"'] & X \end{tikzcd}$$

Applying F yields a weak pullback, and thus an element $w \in F(n \times_X n)$ with maps $p_1, p_2 \colon (w \in F(n \times_X n)) \to (u \in F(n))$ in $\mathbf{Elt}(F)$. These maps p_i are both split epis, since $u \in F^\circ(n)$, say with splittings r_i. Then $\varphi = p_1 \circ r_2 \colon n \to n$ is a split epi, and thus an isomorphism. It satisfies

$$\begin{aligned} t \circ \varphi &= t \circ p_1 \circ r_2 = s \circ p_2 \circ r_2 = s \\ F(\varphi)(u) &= F(p_1 \circ r_2)(u) = F(p_1)(w) = u. \end{aligned}$$

3. Assume now that F preserves pullbacks. Since F is affine, i.e. $F(1) \cong 1$, F preserves all finite limits, and in particular products and equalisers. We first show that there is at most one map $(u \in F^\circ(n)) \to (v \in F(X))$. If we have two of them, say t, s, we can form the equaliser in $\mathbf{Sets}$:

$$\begin{tikzcd} E \arrow[r, tail, "e"] & n \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & X. \end{tikzcd}$$

This equaliser is preserved by F. Since $F(t)(u) = v = F(s)(u)$, there is a unique element $w \in F(E)$ with $F(e)(w) = u$. This map $e \colon (w \in F(E)) \to (u \in F(n))$ is then a split epi, and thus an isomorphism. Hence $t = s$.

With this observation we see that $\mathrm{ap}_n(u, t) = \mathrm{ap}_n(u, s)$ implies $s = t$. Thus the equivalence relation $\sim_u$ on X^n used in (2) is the equality relation. ❑

The equivalence relation $\sim_u$ used in point (2) involves a set of isomorphisms

$$G_u = \{\varphi \colon n \xrightarrow{\cong} n \mid F(\varphi)(u) = u\} \tag{4.18}$$

that is a subgroup of the symmetric group S_n of permutations on n. It induces an action:

$$G_u \times X^n \longrightarrow X^n \quad \text{by} \quad (\varphi, t) \longmapsto t \circ \varphi.$$

The set of equivalence classes $X^n/\sim_u$ can then also be described as the set of orbits

$$X^n/G_u = \{[t] \mid t \in X^n\} \qquad \text{with} \qquad [t] = \{t \circ \varphi \mid \varphi \in G_u\}.$$

This is used below.

Definition 4.7.7 ([285]) A functor $F \colon \mathbf{Sets} \to \mathbf{Sets}$ is called **analytical** if it can be written (via a natural isomorphism) as

$$F \cong \left(\coprod_{i \in I} X^{\#i}/G_i\right),$$

where $\# \colon I \to \mathbb{N}$ is an arity and $G_i \subseteq S_{\#i}$ is a subgroup of the symmetric group of permutations on $\#i \in \mathbb{N}$.

An example of an analytical functor is the multiset (or bag) functor $\mathcal{M}_{\mathbb{N}}$, since one can write

$$\mathcal{M}_{\mathbb{N}}(X) = 1 + X + X^2/S_2 + X^3/S_3 + \cdots$$

since each finite multiset, say with n elements in total, can be identified with an n-tuple in X^n up-to-the-order, that is, with an element of the quotient X^n/S_n that destroys the order. Intriguingly this formulation of $\mathcal{M}_{\mathbb{N}}(X)$ looks very similar to the Taylor expansion of the exponential function:

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots.$$

(Recall that the order of the symmetric group S_n is $n!$.) In [285] the concept of derivative of a functor (see also [4]) is defined, in such a way that $\mathcal{M}_{\mathbb{N}}$ is its own derivative; see Exercise 4.7.3 for a glimpse.

We finally come to the main characterisation result. The second point dealing with analytical functors comes from [285]. The first point is in a sense a restriction of the second and can be found in [200] (where simple polynomial functors are called normal, after [156]).

Theorem 4.7.8 *Let $F \colon$ **Sets** $\to$ **Sets** be an arbitrary functor.*

1. *F is simple polynomial if and only if it is finitary and preserves countable pullbacks.*
2. *F is analytical if and only if it is finitary and preserves countable weak pullbacks.*

Proof 1. We already know that a simple polynomial functor is finitary (Exercise 4.6.6) and preserves binary pullbacks (Proposition 4.2.6); it is not hard to see that the latter generalises to preservation of arbitrary pullbacks.

For the converse, assume F is finitary and preserves countable pullbacks. In order to see that F is simple polynomial we have to combine two results.

1. First, use Proposition 4.7.2 to write $F \cong \coprod_{u \in F(1)} u^{-1}(F)$ as coproduct of affine functors $u^{-1}(F)$, each of which is finitary and preserves countable pullbacks.
2. Next, use Proposition 4.7.6 to write these affine functors as $u^{-1}(F) \cong \coprod_{v \in I_u} X^{n_u}$, where $I_u = u^{-1}(F)^{\circ}(n_u)$ and $n_u \in \mathbb{N}$ is the unique number for which $I_u \times X^{n_u} \to u^{-1}(F)(X)$ is surjective.

Hence by taking $I = \coprod_{u \in F(1)} I_u$ and $\#(u, v) = n_u$ we obtain $F \cong \coprod_{i \in I} X^{\# i}$, making F a simple polynomial functor.

2. Following the same steps one shows that a finitary functor preserving countable *weak* pullbacks is of the form $\coprod_{i \in I} X^{\# i}/G_i$, where the subgroup G_i of the symmetric group on $\# i \in \mathbb{N}$ arises as in (4.18).

In the reverse direction, an analytical functor $F = \coprod_{i \in I} X^{\# i}/G_i$ is obviously finitary. It is not hard to see that it preserves countable weak pullbacks. This is left to the interested reader. ❑

We conclude this section with yet another characterisation of simple polynomial functors, namely as the 'shapely' functors from [276, 274] (see also [4, theorem 8.3]). For this result we need to know that a natural transformation is called **(weak) cartesian** if all its naturality squares are (weak) pullbacks. This terminology will also be used in the exercises below.

Proposition 4.7.9 *A functor $F \colon \mathbf{Sets} \to \mathbf{Sets}$ is simple polynomial if and only if it is* ***shapely****: it preserves binary pullbacks and comes with a cartesian natural transformation $F \Rightarrow (-)^{\star}$ to the list functor.*

Proof Assume F is polynomial: $F(X) = \coprod_{i \in I} X^{\# i}$ via an arity $\# \colon I \to \mathbb{N}$. We already know that it preserves pullbacks. Each tuple $t \in X^{\# i}$ can be seen as an $\# i$ element list $t = \langle t_0, t_1, \ldots, t_{\# i - 1}\rangle \in X^{\star}$. Thus we obtain a natural transformation $\sigma \colon F \Rightarrow (-)^{\star}$ as cotuple. We check that its naturality squares form pullbacks. So assume we have a function $f \colon X \to Y$ and a naturality square

$$\begin{tikzcd} F(X) \arrow[r, "\sigma_X"] \arrow[d, "F(f)"'] & X^{\star} \arrow[d, "f^{\star}"] \\ F(Y) \arrow[r, "\sigma_Y"'] & Y^{\star} \end{tikzcd}$$

If we have an element $v \in F(Y)$ and a list $\alpha = \langle x_1, \ldots, x_n \rangle \in X^\star$ with $\sigma_Y(v) = f^\star(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$, then $v \in F(Y) = \coprod_{i \in I} Y^{\# i}$ must be of the form $\langle i, \lambda k \leq n.\, f(x_k) \rangle$, where $\# i = n$. Hence there is precisely one mediating element $u \in F(X)$, namely $u = \langle i, \lambda k \leq n.\, x_k \rangle$.

Conversely, assume that F preserves pullbacks and that we have a cartesian natural transformation $\sigma \colon F \Rightarrow (-)^\star$. We take $I = F(1)$ with arity:

$$\# \stackrel{\text{def}}{=} \big(I = F(1) \xrightarrow{\sigma_1} 1^\star = \mathbb{N}\big).$$

Since σ is cartesian, the following naturality square is a pullback:

$$\begin{tikzcd}
F(X) \arrow[r, "\sigma_X"] \arrow[d, "F(!)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^\star \arrow[d, "!^\star = \mathsf{length}"] \\
I = F(1) \arrow[r, "\sigma_1 = \#"'] & 1^\star = \mathbb{N}
\end{tikzcd}$$

This means that $F(X)$ is the set of pairs $i \in I$ and $t \in X^\star$ with $\mathsf{length}(t) = \# i$. Hence such an element is a pair $i \in I, t \in X^{\# i}$. Thus $F(X) = \coprod_{i \in I} X^{\# i}$, making F simple polynomial. ❑

In the end we recall the description of a 'container' or 'dependent polynomial functor' from Exercise 2.2.6, as a functor of the form $F(X) = \coprod_{i \in I} X^{A_i}$, for an indexed collection $(A_i)_{i \in I}$ of not necessarily finite sets A_i. These containers are more general than simple polynomial functors. They capture the idea that many data types are given by a template that determines how data are stored. Their theory is developed, from a programming perspective, in [3, 2]. In particular, the idea of keeping track of a specific position within such datatypes (as one-hole contexts) can be formalised via derivatives of such functors; see [4]. Such a derivative of a functor goes back to [285]; it is described in Exercise 4.7.3 below, for simple polynomial functors.

Exercises

4.7.1 Assume two arities $\# \colon I \to \mathbb{N}$ and $\# \colon J \to \mathbb{N}$. Prove that there are bijective correspondences:

1.

$$\frac{\coprod_{i \in I} (-)^{\# i} \xRightarrow{\ \sigma\ } \coprod_{j \in J} (-)^{\# j}}{\overline{\quad I \xrightarrow[f]{} J \ \text{ with } \ \# f(i) \xrightarrow[\varphi_i]{} \# i \quad}} \,.$$

2. This correspondence restricts to

$$\frac{\text{(weak) cartesian } \coprod_{i\in I}(-)^{\#i} \overset{\sigma}{\Longrightarrow} \coprod_{j\in J}(-)^{\#j}}{I \xrightarrow[f]{} J \ \text{ with } \ \#f(i) \xrightarrow[\varphi_i]{\cong} \#i} .$$

4.7.2 For a functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ and a set X consider the coequaliser

$$\Big(\coprod_{n,m\in\mathbb{N}} F(m) \times n^m \times X^n\Big) \overset{d_1}{\underset{d_2}{\rightrightarrows}} \Big(\coprod_{n\in\mathbb{N}} F(n) \times X^n\Big) \xrightarrow{\ c\ } \widetilde{F}(X).$$

The two maps d_1, d_2 are given by $(u, t, s) \mapsto (F(t)(u), s)$ and $(u, t, s) \mapsto (u, s \circ t)$.

1. In this situation, describe a natural transformation $\widetilde{F} \Rightarrow F$, via the map ap: $\coprod_{n\in\mathbb{N}} F(n) \times X^n \to F(X)$ from (4.16).
2. Show that this $\widetilde{F} \Rightarrow F$ consists of monos if F preserves (binary) weak pullbacks.
3. Show also that it consists of epis if F is finitary.

4.7.3 (From [285]) For an arbitrary functor $F\colon \mathbb{C} \to \mathbb{C}$ define the **derivative**, if it exists, to the functor $F'\colon \mathbb{C} \to \mathbb{C}$ with a universal weak cartesian natural transformation $\rho\colon F' \times \mathrm{id}_{\mathbb{C}} \Rightarrow F$. Universality means that for an arbitrary functor G with a weak cartesian $\tau\colon G \times \mathrm{id}_{\mathbb{C}} \Rightarrow F$ there is a unique weak cartesian τ' making the following diagram commute:

$$\begin{tikzcd}
F' \times \mathrm{id}_{\mathbb{C}} \arrow[r, Rightarrow, "\rho"] & F \\
G \times \mathrm{id}_{\mathbb{C}} \arrow[u, dashed, "\tau' \times \mathrm{id}"] \arrow[ur, Rightarrow, "\tau"'] &
\end{tikzcd}$$

Prove that for a simple polynomial functor $F(X) = \coprod_{i\in I} X^{\#i}$ the derivative is

$$F'(X) = \coprod_{i\in I, \#i>0} \#i \times X^{\#i-1}.$$

Hint: Use Exercise 4.7.1.