Green's Function

① $y'' + 2y' + y = 2x \quad t \geq 0,\ y(0) = y'(0) = 0$

$\lambda^2 + 2\lambda + 1 = 0 \longrightarrow \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm \sqrt{0}}{2}$

$\therefore \lambda_1 = -1,\ \lambda_2 = -1$

Case 3: $y_h = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$

$\boxed{y_h = c_1 e^{-x} + c_2 x e^{-x}}$ $\qquad Ly = 0;\ a \leq x \leq b$

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

$y_1(x) = e^{-x}$

$y_2(x) = x e^{-x}$

$G(t,s) = \begin{cases} 0; & t < 0 \\ c_1 e^{-t} + c_2 t e^{-t}; & t \geq s \end{cases}$

$g(s,s) = 0;\ \frac{d}{dt} g(t,s) \Big|_{t=s} = 1$

$g(s,s) = C_1 e^{-s} + C_2 s e^{-s} = 0$

$= C_1 e^{-s} = -C_2 s e^{-s};\quad C_1 = -C_2 S$

$\frac{d}{dt} g(t,s) \Big|_{t=s} = 1 \rightarrow \frac{d}{dt}(c_1 e^{-t} + c_2 t e^{-t}) = -c_1 e^{-t} - c_2 t e^{-t} + c_2 e^{-t}$

$= -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$= -c_1 e^{-s} + c_2 e^{-s} - c_2 s e^{-s} = 1$

$= C_2 s e^{-s} + C_2 e^{-s} - C_2 s e^{-s} = 1$

$= C_1 e^{-s} = 1 \therefore C_1 = e^{s}$

$G(t,s) = \begin{cases} 0, & t < s \\ c_1 e^{-t} + c_2 t e^{-t}, & t \geq s \end{cases}$

$$= -se^{s-t} + te^{s-t}$$

$$y = \int_0^t g(t,s)\, r(s)\, ds = \int_0^t (-se^{s-t} + te^{s-t})\, r(s)\, ds$$

$$r(s) = 2x$$

$$y = \int_0^t (-se^{s-t} + te^{s-t})\, 2t\, ds$$

$$y = 2\int_0^t tse^{s-t}\, ds + 2\int_0^t t^2 e^{s-t}\, ds$$

$$= -2[(s-1)e^{s-t}]_0^t + 2t[te^{s-t}]_0^t$$

$$= -2[(t-1)e^{t-t} - (0-1)e^{-t}] + 2[te^{0} - te^{-t}]$$

$$= -2[(t-1) + e^{-t}] + 2t(1-e^{-t})$$

$$= -2[e^{-t} + t - 1] + 2t(1-e^{-t})$$

$$= -2e^{-t} - 2t + 2t + 2t - te^{-t}$$

$$= 2 - 2e^{-t} - 2te^{-t}$$

$$= 2;\ \boxed{y(x) = 2x - 2e^{-t} - 2te^{-t}}$$

Undetermined Coefficients

$$y'' + 2y' + y = 2x \longrightarrow \boxed{y_h = C_1 e^{-x} + C_2 x e^{-x}}$$

$$y_p = A_2x^2 + A_1 x + A_0$$

$$y_p' = 2A_2 x + A_1 \ ; \quad y_p'' = 2A_2 \ ; \quad \theta(x) = 2x$$

$$2A_2 + 2x(2A_2 + A_1) + A_2x^2 + A_1 x + A_0 = 0x^2 + 0x + 2x$$

$$A_2 = 0 \ ; \ 4A_2 + A_1 = 0 \ ; \quad A_1 = 0$$

$$A_0 + 2A_1 + 2A_2 = 2x \ ; \quad A_0 = 2x$$

$$y_p = A_2x^2 + A_1x + A_0 = 2x \ \therefore \ y_p = 2x$$

$$y(x) = C_1e^{-x} + C_2xe^{-x} + 2x$$

② $y'' + y = x^2; \quad t \geq 0; \quad y(0) = y'(0) = 0$

$\lambda^2 + \lambda = x^2 \rightarrow \frac{-1 \pm \sqrt{1-(4\cdot 1\cdot 0)}}{2} < \begin{matrix} \frac{-1-1}{2} = -1 = \lambda_1 \\ \frac{-1+1}{2} = 0 = \lambda_2 \end{matrix}$

Case 1: $y_h = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$\boxed{y_h = C_1 e^{-x} + C_2 e^{0}}$ $\qquad Ly = 0; \ a < x < b$

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

$y_1(x) = e^{-x} \qquad y_2(x) = 1$

$$G(t,s) = \begin{cases} 0; & t < s \\ \underbrace{\boxed{C_1} e^{-x} + \boxed{C_2}}_{g(t,s)}; & t \geq s \end{cases}$$

$g(s,s) = 0; \ \frac{d}{dt} g(t,s)\Big|_{t=s} = 1$

$g(s,s) = C_1 e^{-s} + C_2 = 0$

$C_2 = -C_1 e^{-s}$

$\frac{d}{dt} = (C_1 e^{-t} + C_2) = -C_1 e^{-t} + C_2$

$= -C_1 e^{-s} + C_1 e^{-s}$

$\therefore C_1 = 0$

$$G(t,s) = \begin{cases} 0; & t < s \\ C_1 e^{-t} + C_2; & t \geq s \end{cases}$$

$0 e^{-t} - C_1 e^{-s} = -C_1 e^{-s}$

$\boxed{g(t,s) = -C_1 e^{-s}}$

$y(s) = \int_0^t g(t,s) r(s)\, ds \qquad r(s) = x^2$

$= \int_0^t (-s e^{-s-t}) x^2$

$= x^2 (-s e^{-s-t})\, ds$

$$= x^2 \left[(-s-1)e^{s-t}\right]_t^0$$

$$= x^2\left[(-t-1)e^{t-t} - (0-1)e^{-t}\right]$$

$$= x^2\left[(-t-1) + e^{-t}\right]$$

$$= x^2e^{-t} - x^2t + 1 +$$

$$\boxed{y = 1 + x^2e^{-t} - x^2t}$$

Checking with an undetermined coefficient

Case I: $y_p = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0$

$\theta(x) = \int n(x)$

$y'' + y = x^2 \longrightarrow \theta(x) = x^2$

$y_p = A_2x^2 + A_1x + A_0$

$y'_p = 2A_2x + A_1$

$y''_p = 2A_2$

$\theta(x) = x^2$

$2A_2 + 2(2A_2x + A_1) + A_2x^2 + A_1x + A_0 = x^2 + 0x$

$2A_2 + 4A_2x + 2A_1 + A_2x^2 + A_1x + A_0 = x^2 + 0x$

$A_2x^2 + x(4A_2 + A_1) + A_0 + 2A_1 + 2A_2 = x^2 + 0x$

$A_2 = \cancel{0}$; $A_2 + A_1 = 0$ $\quad A_0 + 2A_1 + 2A_2 = x^2$

$A_1 = 0 \quad A_2 = 0$

$y_p = A_2x^2 + A_1x + A_0 = 0x^2 + 0x + x^2$

$y_p = x^2$

general solution: $y(x) = C_1e^{-x} + C_2e^0 + x^2$

$y(0) = C_1e^0 + C_2 + x^2$

$= C_1 + C_2 + x^2 = 0$

$C_1 + 0 + x^2 = 0 \quad \therefore C_1 = -x^2$

$y'(x) = -C_1 e^{-x} + 2x$

$y'(0) = -C_1 e^{0} + 1$

$\quad = -C_1, \quad \therefore C_1 = -1$

$y(x) = 1 + x^2 e^{-t} - x^2 t$

$\boxed{y(t) = 1 + t^2 e^{-t} - t^2 t}$