THE STABILITY OF THEORIES FROM CATEGORICITY TO THEIR SPECTRUM

by

Alexander Johnson

A Thesis

Submitted to the Division of Natural Sciences New College of Florida in partial fulfillment of the requirements for the degree Bachelor of Arts in Mathematics under the co-sponsorship of Professor Patrick McDonald and Professor Karsten Henckell

Sarasota, Florida May 2018

Acknowledgements

It is my pleasure to thank my sponsors Professor McDonald and Professor Henckell for their continual support throughout my thesis and undergraduate education. In particular, Professor Henckell taught me set theory my first semester and has sponsored tutorials in logic ever since - thank you for introducing me to the subject I love! Professor McDonald has given excellent feedback and advice in applying to graduate school - thank you for being my advisor these last three years!

I am also very grateful for my correspondence with Professor Malliaris at the University of Chicago, who has graciously extended her mentorship past last summer's REU. Her explanations and examples have been invaluable in understanding the material. Thank you for your encouragement and guidance!

Finally, I am incredibly lucky to have such a wonderful family rooting for me in Illinois, as well as many friends, here and elsewhere, mathematically inclined or not, who have shared in my journey as I have shared in theirs.

Contents

Acknowledgements . . . ii

Abstract . . . vi

1 Introduction **1**

2 Preliminaries **4**

2.1 Notation . . . 4

2.2 First Order Logic . . . 5

2.3 Set Theory . . . 7

2.4 Model Theory . . . 16

2.5 The Monster Model . . . 23

3 Uncountable Categoricity **28**

3.1 Background on Morley's Theorem . . . 28

3.2 Proof Idea . . . 30

3.3 Stability and Indiscernible Sequences . . . 31

3.4 Transcendence Rank . . . 42

3.5 Proving Morley's Theorem . . . 49

4 Order and Instability **58**

4.1 The Order Property . . . 58

4.2 More on Transcendence Rank . . . 63

4.3 The Unstable Formula Theorem . . . 66

4.4 Unstable Theories . . . 69

4.5 More Results . . . 72

5 Forking **74**

5.1 Motivation . . . 74

5.2 Strong Splitting, Dividing, and Forking . . . 74

5.3 Results on Forking . . . 78

5.4 The Average Type . . . 83

6 Finite Equivalence Relations **91**

6.1 Utility for the Stability Spectrum . . . 91

6.2 Definitions . . . 91

6.3 Basic Properties of Finite Equivalence . . . 92

6.4 The Finite Equivalence Relation Theorem . . . 99

6.5 Useful Consequences . . . 101

7 The Stability Spectrum **103**

7.1 Statement and Proof Idea . . . 103

7.2 $\kappa(T)$ . . . 104

7.3 Computing the Stability Spectrum . . . 107

8 Conclusion **111**

Bibliography . . . 112

List of Figures

3.1 An indiscernible set in infinitely many infinite equivalence classes . . 33

3.2 A type p splitting over X 36

3.3 The type $v = v$ has rank at least 2. 44

4.1 Order in the theory of random graphs 59

5.1 A type p splitting strongly over X 75

5.2 The formula $w^1 < v < w^2$ divides over X. 77

5.3 A type p forking over X 77

5.4 Sets I_1 and I_2 equivalent via J whose union is not indiscernible . . . 85

The Stability of Theories from Categoricity to their Spectrum

by

Alexander Johnson

Submitted to the Division of Natural Sciences on May 14th, 2018 in partial fulfillment of the requirements for the degree Bachelor of Arts in Mathematics

Abstract

This thesis is an expository work on major results in stability theory, a subfield of mathematical logic. We begin by proving Morley's Categoricity Theorem, which states that a complete countable theory has one model in some uncountable power if and only if it has one model in every uncountable power. A necessary condition for this uncountable categoricity to hold is stability in some infinite power λ, which states that for every set of size less than λ there are fewer than λ types thereover. We then develop a rich toolbox including transcendence rank, indiscernible sequences, and forking with the goal of proving the Stability Spectrum Theorem, which states that for every complete theory T, $|T|$ arbitrary, there exist cardinals $\kappa(T)$ and μ_0 such that T is stable in μ if and only if $\mu = \mu_0 + \mu^{<\kappa(T)}$. All these results are taken from Morley's original paper [8], and Shelah's *Classification Theory* [9]. A secondary goal of the thesis is to provide examples, exposition, and discussion absent in their parent sources, and to translate Morley's "topological" language into Shelah's.

Professor Patrick McDonald: ________________________________

Professor Karsten Henckell: ________________________________

Chapter 1

Introduction

The aim of this thesis is to exposit on the Stability Spectrum Theorem, a result in mathematical logic, and acquire the necessary tools to do so. Before introducing the topic, let us say a few words about the field in which it lies. Logic is an umbrella term which encompasses set theory, model theory, recursion theory, category theory, type theory, and other areas. Many of these regard the foundations of mathematics, metamathematics, and a high degree of abstraction. One might be drawn to study logic for this reason; arguably metamathematics is highly relevant and applicable, since it tells us what mathematicians can and can not do. Another virtue of logic is the insight that high levels of abstraction can reveal. Just as abstract algebra gives insight over number theory by generalizing $\mathbb{Z}$ and $\mathbb{Z}_n$ as groups, so does model theory give insight over abstract algebra by generalizing groups, rings, and fields as models.

A formalized, *realized* notion of infinity lies at the center of many fields of logic.[1] A triumph of set theory, transfinite objects and methods are what drew the author to this field, and for many are a rich source of beauty and mystery. The structure and immensity of the infinite manifest in a variety of ways: large cardinals from inaccessibility upwards test the strength of our axiom systems, and can gauge the relative

[1]Though some areas of logic, such as constructive mathematics or intuitionistic logic, emphasize finite mathematics.

consistency of adding new statements. Independence proofs spring up naturally and quickly[2] when one first begins their study of the infinite, and one turns to model theory to show such statements are independent by showing that there are many possible universes of set theory. This thesis will make reference to inaccessible cardinals and the Continuum Hypothesis, but otherwise will only use the transfinite machinery of set theory to pursue problems in model theory.

The centerpiece of this thesis, the Stability Spectrum Theorem, is best motivated through its history. We begin with Morley's Categoricity Theorem, proven in Chapter 3, which states that for a complete countable theory T, T has one model (up to isomorphism) in some uncountable power (i.e. size) if and only if T has one model in every uncountable power. In [8], a necessary condition for this to occur is introduced called *stability*, which roughly states that a theory can define few subsets of its models over a certain number of parameters. This means the theory lacks a certain kind of descriptive ability that would have otherwise generated multiple models in uncountable powers.

Stability is measured with respect to a cardinal λ. The Stability Spectrum Theorem characterizes the cardinals in which a theory can be stable. We give the proof in [9] up to a computationally weaker form which nonetheless implies the usual statement. To situate this theorem in the realm of mathematics, see the following subset relation:

$$\begin{aligned}\text{Stability Spectrum Theorem} &\in \text{Stability Theory} \cap \text{Classification Theory}\\ &\subset \text{Model Theory} \subset \text{Mathematical Logic}\end{aligned}$$

Prerequisite to all of these is a solid understanding of set theory and logic. Lest we obliterate our reader base, Chapter 2 provides an introduction to what will be

[2]For example, the Continuum Hypothesis spawns many independent statements regarding cardinal arithmetic.

required in the sequel. By no means will we develop the theory from the ground up, nor do we provide proofs, but our hope is to convey intuition for the transfinite techniques used. See [6] for a concise reference, or [4] for a self-study book. For model theory, see [3].

Chapter 3 covers Morley's Theorem. Chapter 4 covers the Unstable Formula Theorem and its corollaries, which give many equivalent conditions to instability regarding a single pesky formula with much descriptive power. A critical result—the fact that indiscernible sequences and sets align precisely in stable theories—will be used without reference throughout.

Chapter 5 develops forking, a core tool used in proving the Stability Spectrum Theorem. Chapter 6 introduces finite equivalence classes, which are used to derive a few useful corollaries at the end of the chapter. Finally, in Chapter 7 we prove the Stability Spectrum Theorem, and compare it to a computationally stronger version given in [9].

Since we follow [8] and [9] so closely, throughout the work we will retrieve the theorem number as they appear in their parent sources using footnotes in our text. This furthers the secondary goal of this thesis: to produce a document that can allow one to better understand the theory developed in these sources. Specifically, we "de-topologize" [8], i.e. we present arguments having paraphrased the topological language dominant in Morley's original paper, and with regard to [9] we add details, examples, and exposition.

Chapter 2

Preliminaries

2.1 Notation

In this section we introduce notation regarding familiar mathematical objects. More specialized notation (i.e. to logic) will be introduced later in the chapter with the concepts they denote. Overall, our notation will closely follow [9], but with some departures. In particular, we use the Gothic lettering for models found in [3], (see model theory section).

The letters k, l, m, and n will be used for natural numbers. A finite tuple of variables, constants, or anything else will be denoted by a bar, $\bar{v}$. The length of a tuple or sequence f, finite or infinite, will be denoted $\ell(f)$, (see Section 2.3 for a discussion on ordinals, which formalizes this notion for various sizes, or rather lengths, of infinity). Concatenation of sequences will be denoted $f^\frown g$, and angle brackets are used to denote sequences or tuples with specific elements. Given sets A and B, we write $[A]^{<\omega}$ to denote the set of all finite subsets of A, ${}^{<\omega}A$ to denote the set of finite tuples, and ${}^{B}A$ to denote the set of all functions $f : B \to A$.

Despite this, we will almost always write $\bar{a} \in A$ to mean $\bar{a} \in {}^{<\omega}A$. We also will associate $\bar{a}$ with $\{a_1 \dots a_n\}$ as a set when we wish to speak about the coordinates of

$\bar{a}$. For example, we may write $\bar{a} \cup \bar{b}$ to refer to the set $\{a_1 \dots a_n, b_1 \dots b_m\}$, or perhaps write things like $\bigcup_n \bar{a}_n$. [1]

2.2 First Order Logic

In logic[2], formulas and proofs are well-defined (meta)mathematical objects. We will not belabor the details behind these constructions, but rest assured that they follow precisely those thought to be valid; for example, the formal definition of a proof is tedious, but unsurprising. What we will note in this section is notation.

Notation 2.2.1. The symbols $\neg$, $\wedge$, and $\vee$ denote negation, conjunction ('and'), and disjunction ('or') respectively. We may use large versions of the latter two, $\bigwedge$ and $\bigvee$, to denote a finite conjunction or disjunction in the obvious way. We will also make use of implications $\rightarrow, \leftarrow, \leftrightarrow$ and quantifiers $\exists, \forall$ in our formulas. Instead of writing $\exists v_1 \exists v_2 \dots \exists v_n$, we may write $\exists \bar{v}$, where $\bar{v} = v_1^\frown \cdots ^\frown v_n$, and similarly for $\forall$. We use the letters v and w to denote free variables in formulas, which we sometimes omit. A formula φ with free variables among $\bar{v}$ will be written $\varphi(\bar{v})$. If in addition there are relevant parameters $\bar{a}$, we will write $\varphi(\bar{v}; \bar{a})$. If we write $\varphi(\bar{v}; \bar{w})$, we imply that parameters will typically replace variables $\bar{w}$. On that matter, we disregard the formalism regarding parameters as constant symbols in an expanded language, and will not refer to the expansion for this purpose, (see Section 2.5).

Ellipses are an informal notion (e.g. the formula $\varphi_1 \wedge \cdots \wedge \varphi_n$), but using induction or some other formal metamathematical scheme, we can construct formulas with the desired property. In other words, formulas do not actually contain ellipses, but we can produce formulas that behave the same way; to save time, we omit the details, and assume that any desire for additional rigor will correlate with the ability to supply

[1] The utility of these abuses of notation will become clear as finite tuples become ubiquitous.

[2] See the first few sections of [3] for a concise introduction to logic as needed for model theory.

it. We assume throughout that all formulas are well-formed, (i.e. are constructed via valid recursive steps, or in looser terms, are meaningful and make sense).

Our formulas may include constants (e.g. 0) relation symbols (e.g $<$) and function symbols (e.g. $+$) in addition to the logical symbols outlined above. A *language* $\mathcal{L}$ is a set of constants, relation symbols, and function symbols, together with countably many variables $\{v_n \mid n < \omega\}$. (As such, a language will always be infinite.) We will typically give these variables other names (i.e. we use metavariables to denote formal variables). It is not expected that the average reader worries about such things, we just point them out to pay tribute to the disregarded rigor. We associate a language with the set of all formulas expressible therein, (which will have the same cardinality—see Section 2.3). In an abuse of notation, we will use the symbol $\mathcal{L}$ to denote both. And indeed, we allow our languages to have infinitely many non-logical symbols, even uncountably many.

Notation 2.2.2. If φ is a formula, $\varphi^0 = \varphi$ whereas $\varphi^1 = \neg\varphi$. On the other hand, if σ is a short, easily verifiable statement (e.g. $\sigma =$ "if $k \leq n$" where k and n are indices somewhere), then $\varphi^\sigma = \varphi$ if σ is true, and $\varphi^\sigma = \neg\varphi$ if σ is false.[3]

Definition 2.2.3. A *sentence* is a formula, all of whose free variables are bound under a quantifier.

Definition 2.2.4. If Λ is a set of sentences and φ another sentence, we write $\Lambda \vdash \varphi$ to mean that there is a proof of φ using only sentences in Λ as axioms, (together with "logical axioms", e.g. modus ponens). We call φ a consequence of Λ. We say Λ is consistent if and only if there does not exist φ such that $\Lambda \vdash \varphi \wedge \neg\varphi$.

The following lemma we will use a few times.

Lemma 2.2.5. *Suppose Λ is consistent but $\Lambda \cup \{\varphi\}$ is inconsistent. Then $\Lambda \vdash \neg\varphi$.*

[3]This is the notation used in [9]. It is a bit confusing, because in Boolean logic 1 is truth and 0 falsehood, but we decided switching the notation would be even more confusing for those trying to cross reference.

We will refer to a consistent set of sentences as a *theory* T. If $T = \{\varphi\}$ we may omit braces and refer to the theory as φ. In the model theory section, we will define the notion of a consistent set of formulas (called a *type*). When we say a theory has a certain cardinality (for example, "a countable theory T") we also imply its language has this cardinality. [4]

Definition 2.2.6. A set Λ of sentences is *maximally consistent* if and only if Λ is consistent and has no proper consistent extension. Λ is *complete* if and only if its set of consequences is maximally consistent.

We will use the following definition mainly when talking about metamathematical concerns, with $\Lambda = \text{ZFC}$.

Definition 2.2.7. If Λ is a set of sentences and φ another sentence for which $\Lambda \nvdash \varphi$ and $\Lambda \nvdash \neg\varphi$, we say φ is *independent* of Λ.

One can show that if $\Lambda \nvdash \varphi$, then $\Lambda \cup \neg\varphi$ is consistent. As such, complete theories have no independent statements (in their language). Zorn's Lemma can be used to show existence of complete extensions to any theory, and studying this complete extension is often sufficient to learn about the base theory. All theories will be assumed complete unless otherwise stated.[5]

2.3 Set Theory

We take set theory as our foundation. As such, everything is a set—even numbers and formulas—which may seem philosophically repugnant to some, but offers great

[4] As one might notice, already we're using set theory in the logic section! Indeed, the "correct" way to learn this material is as follows: start with a small amount of logic, then a small amount of set theory, then learn more serious logic, and then more serious set theory. Dependencies are interwoven, and this close to the ocean floor of mathematics things become murky if one moves too quickly.

[5] Of course this does not apply to metamathematical discussions about the theory ZFC, which is incomplete (assuming it is consistent, which of course we all do).

utility. In this section we go over the main ideas and results of set theory needed for this thesis, though to save time we omit most of the proofs.[6]

We assume the axioms of ZFC (Zermelo-Fraenkel with Choice) set theory, which is to say we assume the typical axioms for doing mathematics. Included is the Axiom of Choice (C), which is equivalent to Zorn's Lemma. We will not make remark of its usage—indeed, we will use it quite frequently. It is of great interest to the author which results hold without the assumption of the Axiom of Choice, but such a task was too great for this thesis. (Such questions are often quite difficult, rich with independence results.) The other axioms (Z) are typical (e.g. existence of powersets and unions), though the Axiom Schema of Replacement gets its own letter (F).[7]

Set theory tackles the infinite with the following definition.

Definition 2.3.1. Suppose there exists a bijection $f : X \to Y$. Then we say X and Y have the same cardinality, and write $|X| = |Y|$. If f is merely an injection, we write $|X| \leq |Y|$. If $|X| \leq |Y|$ but it is not the case that $|X| = |Y|$ we write $|X| < |Y|$.

We will shortly discuss what $|X|$ is on its own. Before then, we have the following theorem.

Theorem 2.3.2 (Schröder-Bernstein Theorem)**.** *Suppose* $|X| \leq |Y|$ *and* $|Y| \leq |X|$. *Then* $|X| = |Y|$.

Theorem 2.3.3 (Cardinal Comparability)**.** *Given any two sets* X *and* Y *either* $|X| \leq |Y|$ *or* $|Y| \leq |X|$.

Now we turn to ordinal numbers, the backbone of the set-theoretic universe, and

[6] See [4] for a highly accessible development of set theory from the beginning. A close reading of the entire book will prepare the reader well to study model theory, or a more advanced set theory book such as [6].

[7] Both set theory texts appearing in the bibliography contain a formal list of the ZFC axioms.

the base our transfinite arguments.[8] They begin

$$0, 1, 2, \ldots, \omega, \omega+1, \ldots, \omega \cdot 2, \ldots, \omega \cdot \omega = \omega^2, \ldots, \omega^\omega, \ldots, \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} = \varepsilon_0, \ldots$$

where the ellipses make further and further jumps. They never stop; every time a limiting process presents itself, a new ordinal sits atop, and then you add one. The totality of the ordinal line is too big to be a set; it is called a *proper class.*

We denote ordinals by lowercase Greek letters, (though we sometimes use η or ν to denote sequences; context will make the usage clear). Sometimes we use i and j for ordinals as well. Every ordinal is a set of previous ordinals. For example, $0 = \emptyset$, $2 = \{0, 1\}$, $\omega = \mathbb{N}$, and so on. This lets us give ordinals the following ordering.

Definition 2.3.4. Given ordinals α and β, we say $\alpha < \beta$ if and only if $\alpha \in \beta$.

An equivalent condition is $\alpha \subset \beta$, (proper). This also justifies the notation ${}^{\alpha}A$ to denote the set of all functions $f : \alpha \to A$. We interpret ${}^{<\alpha}A$ and ${}^{\leq\alpha}A$ in the obvious way.

Theorem 2.3.5. *The ordering $<$ on any ordinal $\alpha = \{\beta \mid \beta < \alpha\}$ and also the totality of all ordinals is a well-ordering. That is to say, given any nonempty class of ordinals, there exists a least element.*

Theorem 2.3.6. *Given any set of ordinals X, there exists an ordinal $\alpha = \sup(X)$. (Here* sup *is the usual supremum as defined for ordered sets.)*

We will make use of these two theorems often and without reference. But note that we must have X a set! The converse also holds; indeed, a class of ordinals is a set if and only if it is bounded in the ordinal line. We will also use this next theorem frequently without reference.

[8] See [4] or [6] for a definition and construction of the ordinal numbers. From these we can prove their salient properties, though we do no need that level of machinery in this thesis.

Theorem 2.3.7. *If X is well-ordered set, then there exists a unique ordinal α and a unique order isomorphism $f : X \to \alpha$. We say α is the order type of X.*

This next theorem requires the Axiom of Choice, and is quite controversial among intuitionists, (take $X = \mathbb{R}$).

Theorem 2.3.8 (Well-Ordering Theorem)**.** *Given any set X, there exists a well-ordering of X, and hence, an ordinal α such that $|X| = |\alpha|$.*

We will use this theorem when assigning cardinalities to sets. This is also what we refer to as an "enumeration" of a set. If we say an "enumeration perhaps with repeats" we mean merely a surjection of an ordinal onto a set.

Theorem 2.3.9 (Hartog's Theorem)**.** *Given any set X, there exists an ordinal α such that $|X| \ngeq |\alpha|$. With Cardinal Comparability, $|X| < |\alpha|$.*[9]

Hartog's Theorem is the first concrete result regarding the vastness of the ordinal line. Even so, all ordinals come in one of three flavors, (the first being zero).

Definition 2.3.10. Suppose $\alpha > 0$ is an ordinal. If there exists β such that $\beta + 1 = \alpha$, then we say α is a *successor ordinal.* Otherwise, we say α is a *limit ordinal.* If $\alpha \geq \omega$, we say α is a infinite ordinal. Otherwise, α is finite. (This agrees with α being finite or infinite as a set.) [10]

For example, ω and $\omega \cdot 2$ are limit ordinals. All limit ordinals are infinite. In general, we always use δ to denote limit ordinals.

Definition 2.3.11. We say a sequence of sets $\{A_\alpha \mid \alpha < \gamma\}$ is *increasing* if and only if $\alpha < \beta < \gamma$ implies $A_\alpha \subseteq A_\beta$, and *continuous* if and only if $\bigcup_{\alpha<\delta} A_\alpha = A_\delta$ for every limit ordinal $\delta < \gamma$.

[9] Although Cardinal Comparability is equivalent to the Axiom of Choice, Hartog's Theorem can be proven from ZF alone!

[10] Some authors list 0 as a limit ordinal. We do not follow this practice. We also remark that both limit ordinals and successor ordinals are unbounded in the ordinal line.

Now let's talk about some basic ordinal arithmetic. The definition will be imprecise.

Definition 2.3.12. If α and β are ordinal numbers, then $\alpha + \beta$ is the order type corresponding to "α, then β", and $\alpha \cdot \beta$ is the order type corresponding to "α, β-times".[11]

We now come to transfinite induction and recursion.

Theorem 2.3.13 (Transfinite Induction)**.** *Suppose $P(\alpha)$ is a statement about ordinals. If $P(0)$, $P(\beta) \Rightarrow P(\beta+1)$, and if $(\forall \alpha < \delta\, P(\alpha)) \Rightarrow P(\delta)$ for limit ordinals δ, then $P(\alpha)$ holds for all ordinals α.*

Proof. No least counterexample by well-orderedness. □

In practice, the zero step and limit step will usually be the easiest, leaving the successor step. For example, one might define a set X_0, X_{n+1} given X_n, and then take $X_\omega = \bigcup_{n<\omega} X_n$. In fact, we can compress the base case, successor step, and limit step into a single line!

$$(\forall \beta < \alpha\, P(\beta)) \Rightarrow P(\alpha)$$

We also have transfinite recursion.[12]

Theorem 2.3.14 (Transfinite Recursion)**.** *Suppose we have a scheme to define A_0, $A_{\alpha+1}$ given A_α, and A_δ given A_α for all $\alpha < \delta$ a limit ordinal. Then there exists sets A_α for all ordinals α satisfying our scheme.*

Comparatively speaking, the proof is nontrivial! We make this point to show that, despite frequent interchange of the words "induction" and "recursion", they are in

[11] Ordinal exponentiation is more complex, and does not have a nice intuitive definition like above. Thankfully, we do not need it in our thesis. Note however that ordinal arithmetic is non-commutative; $\omega + 1 \neq 1 + \omega = \omega$.

[12] Here we state Transfinite Recursion (as well as Transfinite Induction) in very informal terms. These have much more precise—and general—statements, see [6].

fact two separate results. To summarize, induction is a proof technique about objects that have already been defined, whereas recursion is a construction of new objects to satisfy some scheme.

We can use transfinite induction/recursion on the entirety of the ordinal line, or stop at any ordinal along the way. Since we could have stopped at ω, this makes ordinary induction and recursion a special case. On that note, we will henceforth drop the word "transfinite"; context, in particular the observance of a limit step, will indicate if we actually go past ω.

Now let's talk about cardinal numbers.

Definition 2.3.15. A cardinal number κ is an ordinal number such that no $\alpha < \kappa$ has $|\alpha| = |\kappa|$. From the Well-Ordering Theorem, every set X has a least ordinal κ such that $|X| = |\kappa|$, so we define $|X| = \kappa$. We call such κ the *cardinality* or *size* of X. We may also say that X has *power* κ.

The letters κ, λ, and μ will be used for cardinals, and are assumed infinite unless otherwise stated. Hartog's Theorem actually gives us the least cardinal number κ such that $|X| < \kappa$. This we call $|X|^+$, the cardinal successor, not to be confused with ordinal successor![13] A hierarchy of cardinal numbers lies embedded in the ordinals. Transfinite recursion gives us the following definition that enumerates the cardinals.

Definition 2.3.16. $\aleph_0 = \omega$

$\aleph_{\alpha+1} = \aleph_\alpha^+$

$\aleph_\delta = \sup_{\alpha<\delta} \aleph_\alpha$ for limit ordinals δ

For example, $|\mathbb{N}| = |\mathbb{Q}| = \aleph_0$. We say such sets are *countable* if they are finite or have size $\aleph_0$, and are *uncountable* otherwise.[14]

[13]Unless they are finite, which is the only case where they agree. If κ is an infinite cardinal, then κ is necessarily a limit ordinal.

[14]To avoid ambiguity, some authors say a set is *countably infinite* to mean it is countable and infinite. We may sometimes use this practice, though context and the assumptions regarding the size of theories and models summarized at the end of this chapter will make the distinciton clear.

Notation 2.3.17. We define $\omega_\alpha = \aleph_\alpha$. The ω_α numbers we think of as ordinals, and the $\aleph_\alpha$ numbers we think of as cardinals, (though of course the second is a special case of the first). We may sometimes use ω_α instead of $\aleph_\alpha$ if, for example, we want to index another $\aleph$ number or as a base for transfinite induction or recursion. Or, we may use $\aleph_\alpha$ instead of ω_α to emphasize cardinal arithmetic (to be defined shortly) instead of ordinal arithmetic.

One can show that $\aleph_\delta$ will always be a cardinal for limit ordinals δ. Cardinal numbers inherit their (well!) order from the ordinals. With our new notation, the cardinal numbers begin:[15]

$$0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots, \aleph_\omega, \aleph_{\omega+1} \ldots, \aleph_{\omega_1}, \ldots, \aleph_{\aleph_{\aleph_{\ldots}}}, \ldots$$

The cardinals are also unbounded in the ordinal line. If α is a successor ordinal, we say that $\aleph_\alpha$ is a *successor cardinal*, (equivalently, $\kappa = \lambda^+$ for some λ). Otherwise we say $\aleph_\alpha$ is a *limit cardinal*.

We now come to cardinal arithmetic.[16]

Definition 2.3.18. Given cardinal numbers κ and λ, let X and Y be disjoint sets of cardinalities κ and λ respectively. Then:

$\kappa + \lambda = |X \cup Y|$ (this is where disjointness is necessary)

$\kappa \cdot \lambda = |X \times Y|$ (Cartesian product)

$\kappa^\lambda = |\{\text{all functions } f : Y \to X\}| = |{}^Y X|$

$\kappa^{<\lambda} = \sup_{\mu<\lambda} \kappa^\mu$

Note that this definition agrees with usual arithmetic for finite cardinals. However, addition and multiplication turn out to be somewhat trivial.

Theorem 2.3.19 (Fundamental Theorem of Cardinal Arithmetic)**.** *Given an infinite cardinal κ and a nonzero, potentially finite cardinal λ, $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

[15] The last entry might seem a bit silly. Indeed, this is a cardinal κ such that $\aleph_\kappa = \kappa$.

[16] In particular, $0^0 = 1$, $0 \cdot \lambda = 0$, and $1^\lambda = 1$, for all λ, finite or infinite.

On the other hand, cardinal exponentiation is highly nontrivial. First, a famous result of Cantor.

Theorem 2.3.20. *For all* λ, $2^\lambda > \lambda$.

In particular, $|\mathbb{R}| = 2^{\aleph_0} > \aleph_0$. Note that the above theorem only implies $2^\lambda \geq \lambda^+$, *not* that they are equal. The *Continuum Hypothesis* (CH) is the statement $2^{\aleph_0} = \aleph_1$, while the *Generalized Continuum Hypothesis* (GCH) is the statement $2^\lambda = \lambda^+$ for all infinite λ. Both CH and GCH are independent of ZFC, (and GCH is independent of ZFC+CH). It is useful to keep this in mind when considering cardinal arithmetic.

We also have theorems involving inequalities.

Theorem 2.3.21. *Suppose* λ, κ, *and* μ *are cardinals and* $\lambda \leq \kappa$. *Then:*

$\lambda + \mu \leq \kappa + \mu$

$\lambda \cdot \mu \leq \kappa \cdot \mu$

$\lambda^\mu \leq \kappa^\mu$

$\mu^\lambda \leq \mu^\kappa$

These inequalities cannot be made strict in general; counterexamples are easy to construct.

Theorem 2.3.22. *Suppose* λ *is an infinite cardinal and* $2 \leq \mu \leq \lambda$. *Then* $2^\lambda = \mu^\lambda$.

The proof is so much fun that we give it.

Proof. $2^\lambda \leq \mu^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$, whence equality holds throughout. □

As one might anticipate, we also have infinitary sums and products.

Definition 2.3.23. Suppose κ_i are cardinal numbers for $i \in I$ some index set, and suppose $|X_i| = \kappa_i$ are disjoint. Then:

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} \kappa_i \right|$$

$$\prod_{i \in I} \kappa_i = \Big| \prod_{i \in I} X_i \Big|,$$

the second product being the usual (arbitrary) Cartesian product.

The following result is used frequently at limit steps in constructions.

Theorem 2.3.24. *Given sets* $\{A_\alpha \mid \alpha < \lambda\}$, *we have*

$$\Big| \bigcup_{\alpha < \lambda} A_\alpha \Big| \leq (\sup_{\alpha < \lambda} |A_\alpha|) \cdot \lambda.$$

We use this inequality in the proof of Theorem 2.5.3 to bound the cardinality of unions at limit ordinals. If $|A_\alpha|, \lambda \leq \aleph_0$, then the right hand side is a cardinal $\leq \aleph_0$ and the theorem can be summarized as "a countable union of countable sets is countable".

We now turn to cofinality, an important concept.

Definition 2.3.25. Given a limit ordinal δ, we define $\mathrm{cf}(\delta)$ (the *cofinality* of δ) to be the smallest cardinal λ such that δ is the supremum of λ smaller ordinals.

Usually, but not always, δ is a cardinal. Intuitively, the cofinality of a set is how many stepping stones you need to get to the top. For example, $\mathrm{cf}(\aleph_\omega) = \mathrm{cf}(\aleph_0) = \omega$. As a generalization, $\mathrm{cf}(\aleph_{\alpha+\omega}) = \omega$ for any α. Clearly $\mathrm{cf}(\delta) \leq |\delta|$. The following theorem gives an equivalent definition of cofinality for cardinals.

Theorem 2.3.26. *Given a cardinal* κ, *cf*(κ) *is the smallest cardinal* λ *such that* κ *is a union of* λ *sets of size* $< \kappa$.

Definition 2.3.27. A cardinal λ is said to be *regular* if and only if $\mathrm{cf}(\lambda) = \lambda$. Otherwise if $\mathrm{cf}(\lambda) < \lambda$, λ is said to be *singular*.

Theorem 2.3.28. *For every infinite cardinal* λ, *both cf*(λ) *and* λ^+ *are regular cardinals.*

This shows $\mathrm{cf}(\aleph_1) = \omega_1$, the first uncountable ordinal. For an example of a singular cardinal with uncountable cofinality, we have $\mathrm{cf}(\aleph_{\omega_1}) = \omega_1$.

Theorem 2.3.29 (König's Theorem)**.** *For every infinite cardinal κ, $\kappa^{cf(\kappa)} > \kappa$.*

As a corollary, $\mathrm{cf}(2^\kappa) > \kappa$. This strengthens the result $2^\kappa > \kappa$.

Definition 2.3.30. An uncountable cardinal $\bar{\kappa}$ is said to be *strongly inaccessible* if and only if $\bar{\kappa}$ is regular, and for all $\lambda < \bar{\kappa}$, $2^\lambda < \bar{\kappa}$.

One can verify that $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$ for strongly inaccessible $\bar{\kappa}$. ZFC cannot show that strong inaccessibles exist, and it cannot even show that their existence is relatively consistent.[17] Nevertheless, we will make careful use of inaccessible cardinals in the monster model section.

One final result we will need is Ramsey's Theorem.[18]

Theorem 2.3.31 (Ramsey's Theorem, Infinite Version)**.** *Suppose X is an infinite set, and $A_1 \cup A_2 = [X]^2$ is a disjoint union of all pairs of elements from X. Then there exists an infinite $Y \subseteq X$ such that for some $i = 1, 2$, $[Y]^2 \subseteq A_i$*

Graph theory represents this theorem well: Every two-colored infinite complete graph has an infinite complete monochromatic subgraph. Indeed, we will use the word "coloring" to refer to the sets A_1, A_2 above.

2.4 Model Theory

Definition 2.4.1. A *model* $\mathfrak{A}$ for a language $\mathcal{L}$ is a set A together with an interpretation function mapping constant symbols to elements in A, and relation/function symbols to relations/functions on A. If necessary, we may say that $\mathfrak{A}$ is an $\mathcal{L}$-model.

[17] "Cannot show" meaning "cannot show unless ZFC is inconsistent".

[18] The proof of this theorem (which is classical) is not included in [4], but instead in [6].

Examples are omnipresent: anything from algebra, linear orders, $\mathbb{N}$ for number theory, etc. We use $\mathfrak{A}$ and $\mathfrak{B}$ to denote models, with universes A and B, respectively. The monster model (see that section) will be denoted $\mathfrak{M}$ with universe M. All models in this thesis are assumed to be infinite.

The following (rough) definition gives a semantic definition for truth.

Definition 2.4.2. We write $\mathfrak{A} \models \varphi$ ("$\mathfrak{A}$ models φ"), φ a sentence, to mean that φ is true when its symbols are interpreted by the model $\mathfrak{A}$. We write $\mathfrak{A} \models T$ if and only if $\mathfrak{A} \models \varphi$ for all $\varphi \in T$. If $\varphi(\bar{v})$ is a formula and $\bar{a} \in A$, we can write $\mathfrak{A} \models \varphi[\bar{a}]$ with the obvious meaning.[19]

Theorem 2.4.3 (Completeness Theorem)**.** *A set of sentences T is consistent if and only if T has a model.*

Theorem 2.4.4 (Compactness Theorem)**.** *If every finite subset of T has a model, then T has a model.*

Definition 2.4.5. If $\mathfrak{A}$ and $\mathfrak{B}$ are models of the same language with $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$ for every sentence φ, we write $\mathfrak{A} \equiv \mathfrak{B}$ and say $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent*.

Definition 2.4.6. Given models $\mathfrak{A}$, $\mathfrak{B}$ of a common language, a function f is an $(\mathfrak{A}, \mathfrak{B})$-*partial elementary mapping* if and only if $\mathrm{dom}(f) \subseteq A$, $\mathrm{ran}(f) \subseteq B$ and for every $a_0, \ldots, a_n \in \mathrm{dom}(f)$ and formula $\varphi(v_0 \ldots v_n)$, we have

$$\mathfrak{A} \models \varphi[a_0 \ldots a_n] \text{ iff } \mathfrak{B} \models \varphi[f(a_0) \ldots f(a_n)].$$

In particular, even if $f = \emptyset$, we have $\mathfrak{A} \equiv \mathfrak{B}$. If $\mathrm{dom}(f) = A$, we say f is an *elementary embedding*. If in addition $\mathrm{ran}(f) = B$, then we say f is an isomorphism $\mathfrak{A} \cong \mathfrak{B}$.

[19] Indeed, $\models$ is a metamathematical notion. Truth is determined by the metatheory ZFC in this case. Confusion may arise when the metatheory *is* the target theory, but this will never come up in this thesis.

When context is clear we omit $(\mathfrak{A}, \mathfrak{B})$.

Definition 2.4.7. Given two models $\mathfrak{A}$ and $\mathfrak{B}$ of the same language, we say $\mathfrak{A}$ is a *submodel* of $\mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{B}$, if and only if $A \subseteq B$ and inclusion is a homomorphism, (preserves constants, relations, and functions, but no more). We say $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$, $\mathfrak{A} \preccurlyeq \mathfrak{B}$, if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and inclusion is an elementary mapping.

Theorem 2.4.8 (Tarski-Vaught Criterion)**.** *$\mathfrak{A} \preccurlyeq \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and for all formulas $\exists v \varphi(v; \bar{w})$ and $\bar{a} \in A$,*

$$\textit{If } \mathfrak{B} \models \exists v \varphi[v; \bar{a}], \textit{ then there is } a \in A \textit{ such that } \mathfrak{B} \models \varphi[a; \bar{a}].$$

Theorem 2.4.9. *Suppose $\mathfrak{A}$ is a model for the language $\mathcal{L}$, and $\{c_a \mid a \in A\}$ are new constant symbols not in $\mathcal{L}$. Then $\mathfrak{B} \succcurlyeq \mathfrak{A}$ if and only if $\mathfrak{B} \supseteq \mathfrak{A}$ and the expansion of $\mathfrak{B}$ interpreting c_a as a models*

$$\Gamma_A = \{\varphi(c_{a_1} \dots c_{a_n}) \mid \mathfrak{A} \models \varphi[a_1 \dots a_n]\}.$$

Theorem 2.4.10 (Downward Löwenheim-Skolem-Tarski Theorem)**.** *Suppose $\mathfrak{A}$ is an infinite model for the language $\mathcal{L}$ and $X \subseteq A$. Then for every λ such that*

$$|X| + |\mathcal{L}| \leq \lambda \leq |A|,$$

there exists an elementary submodel $\mathfrak{B} \preccurlyeq \mathfrak{A}$ of cardinality λ with $B \supseteq X$.

Theorem 2.4.11 (Upward Löwenheim-Skolem-Tarski Theorem)**.** *Suppose T is a theory with infinite models. Then T has a model in all $\lambda \geq |T|$.*

We will refer to these last two results as DLST and ULST respectively. Together these theorems imply that if a theory T has infinite models, then T has infinite models in all $\lambda \geq |T|$.[20] Since having only finitely many elements is expressible in first order logic, a complete theory either has only finite models or only infinite models. Since we desire to study the infinite, all theories in this thesis are assumed to have infinite models.

Theorem 2.4.12. *Suppose* $\{\mathfrak{A}_\alpha \mid \alpha < \lambda\}$ *are models for a common language such that* $\beta < \alpha$ *implies* $\mathfrak{A}_\beta \preccurlyeq \mathfrak{A}_\alpha$. *Define their union* $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ *as the model with universe* $\bigcup_{\alpha<\lambda} A_\alpha$, *relations (and functions)* $\mathcal{I}(R) = \bigcup_{\alpha<\lambda} \mathcal{I}_\alpha(R)$ *where* $\mathcal{I}_\alpha$ *is the interpretation function for* $\mathfrak{A}_\alpha$, *and the same constants. Then* $\mathfrak{A}_\lambda$ *is a well-defined model and is an elementary extension of each* $\mathfrak{A}_\alpha$.

We use this theorem frequently when recursively constructing elementary chains of models to ensure that their unions (at limit ordinals) are well defined elementary extensions.

Definitions 2.4.13. Let $\mathfrak{A}$ be a model for the language $\mathcal{L}$ and $X \subseteq A$.

1. A *type* or *m-type* over X in $\mathfrak{A}$ is a consistent (i.e. finitely satisfiable in $\mathfrak{A}$) set of formulas with at most m free variables. A *complete type* is a type that is maximal consistent.

2. $\mathrm{tp}_\Delta(\bar{b}, X, \mathfrak{A}) = \{\varphi(\bar{v}; \bar{a}) \mid \bar{a} \in X,\ \varphi \in \Delta, \text{ and } \mathfrak{A} \models \varphi[\bar{b}; \bar{a}]\}$ denotes the complete type realized by $\bar{b}$ in $\mathfrak{A}$ with parameters from X and formulas from Δ.

3. $\mathbf{S}^m_\Delta(X, \mathfrak{A}) = \{tp_\Delta(\bar{b}, X, \mathfrak{A}) \mid \bar{b} \in {}^mA\}$ denotes the set of all complete types in at most m free variables with parameters in X and formulas from Δ realized in $\mathfrak{A}$.

Types will be denoted by p or q, and will not be assumed complete. As with theories, if $p = \{\varphi\}$, we may omit braces and refer to the type as φ. If p is a type and X a parameter set, $p \restriction X$ denotes the subtype consisting of all formulas containing

[20] In particular, if ZFC is consistent, then it has countable models! This situation is called Skolem's Paradox, since ZFC asserts the existence of uncountable objects. (Of course, there is no actual contradiction; the meaning of "countable" changes from model to model.)

only parameters from X. As one might guess, $\mathrm{param}(p)$ denotes the set of parameters in p.

We omit mention of $\mathfrak{A}$ in (2) or (3) to refer to consistent but not necessarily realized types. (Equivalently, once we have defined the monster model, we omit $\mathfrak{A}$ when $\mathfrak{A} = \mathfrak{M}$—see monster model section.) In (3), we omit m when $m = 1$. If $\Delta = \{\varphi\}$ a singleton, we may just write φ. We omit $\Delta = \mathcal{L}$.

Example 2.4.14. Consider the language $\mathcal{L} = \{+, \cdot, 0, 1\}$ of fields. A formula in this language might express being a square root of two, which we would write $v \cdot v = 1+1$. This information is expressible with a single formula, but types are needed to express more complicated information. For example, the type of a transcendental number will contain infinitely many formulas, each saying that v is not the root of a certain rational polynomial.

Example 2.4.15. Let us consider another example to show that types have a level of resolution greater than single formulas. Using the ULST, we can show that there exist uncountable models of number theory elementarily equivalent to $\mathbb{N}$. As such, a single formula (preceded by $\exists v$ to make it a sentence) cannot distinguish between standard and nonstandard models[21], but the infinite set of formulas

$$\{v \neq 0, v \neq 0^+, v \neq 0^{++}, \dots\},$$

will be realized precisely in nonstandard models. Every finite subset is realizable in $\mathbb{N}$, so this set is a type.

Notations 2.4.16. Given types p and q, we write $p \vdash q$ to mean that every tuple realizing p also realizes q.[22] If $p \vdash q$ and $q \vdash p$, we write $p \equiv q$.

[21] The phrase "nonstandard model" is loosely defined as any model not isomorphic to the "typical" model for a given theory. For example, uncountable models of number theory or non-Archimedean ordered fields are often said to be nonstandard.

[22] Indeed, this is a semantic notion, but by completeness it doesn't matter which turnstile we use.

Definition 2.4.17. A model $\mathfrak{A}$ is *λ-saturated* if and only for if for all $m < \omega$ and every parameter set $X \subseteq A$, $|X| < \lambda$, we have that $\mathfrak{A}$ realizes every m-type over X. We say $\mathfrak{A}$ is *saturated* if and only if it is $|A|$-saturated.

Proposition 2.4.18. *In the above definition it suffices to consider $m = 1$ when λ is infinite.*

Example 2.4.19. As a dense linear order, $\langle \mathbb{Q}, < \rangle$ is saturated. However $\langle \mathbb{R}, < \rangle$ is not saturated, since one can write down the type of an infinitesimal using $\aleph_0$ parameters as follows.

$$\{v > 0, v < \frac{1}{2}, v < \frac{1}{3}, v < \frac{1}{4}, \dots\}$$

Any dense linear order $\mathfrak{A}$ of cardinality $\aleph_1$ will fail to be saturated if the Continuum Hypothesis is false, since if we take $\mathbb{Q} \subset A$ as parameters we get $2^{\aleph_0}$ Dedekind cuts.

Example 2.4.20. As an algebraically closed field of characteristic zero, the (complex) algebraic numbers $\langle \mathbb{A}, 0, 1, +, \cdot \rangle$ are not saturated, since using only symbols in the language we can write down the type of an element solving no rational polynomial. However, any countable extension field of transcendence degree ω is saturated.

Theorem 2.4.21. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent λ-saturated models of cardinality λ. Then $\mathfrak{A} \cong \mathfrak{B}$.*

A direct proof can be found in [3]. We will instead derive the result from a lemma involving partial elementary mappings. This approach will turn out to be more economical, as we will use the required lemma again later on.

Lemma 2.4.22. *Suppose $D \subseteq C \subseteq A$, $|D| < \lambda$, $|C| \leq \lambda$, and f is an $(\mathfrak{A}, \mathfrak{B})$-partial elementary mapping with domain D. If $\mathfrak{B}$ is λ-saturated, then we can extend f to a partial elementary mapping g with domain C.*

Proof. Enumerate $C = \{c_\alpha \mid \alpha < |C|\}$, and let $C_\alpha = \{c_\beta \mid \beta < \alpha\} \cup D$ for each $\alpha < |C|$. We define an increasing sequence $\{f_\alpha \mid \alpha \leq |C|\}$ of partial elementary mappings such that each $\mathrm{dom}(f_\alpha) = C_\alpha$. Once this is done, we let $g = f_{|C|}$ and the proof is complete.

Let $f_0 = f$, and $f_\delta = \bigcup_{\alpha<\delta} f_\alpha$ at limit ordinals. For the successor step, suppose f_α is defined. Consider the type $p(v) = tp(c_\alpha, C_\alpha, \mathfrak{A})$. Even if $f = \emptyset$, we still have $\mathfrak{A} \equiv \mathfrak{B}$ from the assumption that f is a partial elementary mapping, and so $p(v)$ is consistent with $\mathfrak{B}$. Since $|C_\alpha| < |C| \leq \lambda$, we have by λ-saturation that $\mathfrak{B}$ realizes $p(v)$, say at b_α. Now define $f_{\alpha+1} \supset f_\alpha$ by mapping $f_{\alpha+1}(c_\alpha) = b_\alpha$. It is clear that $f_{\alpha+1}$ is a partial elementary mapping with domain $C_{\alpha+1}$. □

Proof of Theorem 2.4.21. We first remark that every ordinal α is uniquely expressible in the form $\alpha = \delta + n$, where δ is a limit ordinal and $n < \omega$.

Enumerate $A = \{a_\alpha \mid \alpha < \lambda\}$ and $B = \{b_\alpha \mid \alpha < \lambda\}$. Let $A_\alpha = \{a_\beta \mid \beta < \alpha\}$ and $B_\alpha = \{b_\beta \mid \beta < \alpha\}$. We define a (non-strictly) increasing sequence $\{f_\alpha \mid \alpha \leq \lambda\}$ of $(\mathfrak{A}, \mathfrak{B})$-partial elementary mappings such that

1. If $\alpha = \delta + 2n$, then $\mathrm{dom}(f_\alpha) = A_\alpha$,

2. If $\alpha = \delta + 2n + 1$ or $\alpha = \delta$, then $\mathrm{ran}(f_\alpha) = B_\alpha$, and

3. If $\alpha < \lambda$, then $|f_\alpha| < \lambda$.

From the remark above, α is uniquely expressible in one of these two forms, so (1) and (2) induce a dichotomy on ordinals.

We begin by letting $f_0 = \emptyset$, a partial elementary mapping by the assumption that $\mathfrak{A} \equiv \mathfrak{B}$. Take unions at limit ordinals as typical; note that $\mathrm{dom}(f_\delta) = A_\delta$ and also $\mathrm{ran}(f_\delta) = B_\delta$ at this stage. Now suppose we have defined f_α for some $\alpha = \delta + 2n + 1$. To define $f_{\alpha+1}$, we apply Lemma 2.4.22. If instead $\alpha = \delta + 2n$, apply the lemma to f_α^{-1} (a $(\mathfrak{B}, \mathfrak{A})$-partial elementary mapping) and then take the inverse of the result to get $f_{\alpha+1}$. This is well-defined, since any partial elementary mapping must be injective by considering the formula $v \neq w$.

Finally let $g = f_\lambda$, a partial elementary mapping with domain A and range B. Then g is an isomorphism $\mathfrak{A} \cong \mathfrak{B}$. □

2.5 The Monster Model

We now prove an existence theorem of sorts for saturated models. This is quite a delicate issue, but once the details have been worked out it will be of paramount importance in all that follows. This is also where we begin to cross-reference results from [9].

Lemma 2.5.1. *Suppose $\mathfrak{A}$ is an $\mathcal{L}$-model and p a type in $\mathfrak{A}$, possibly with parameters. Then there exists an elementary extension $\mathfrak{B} \succcurlyeq \mathfrak{A}$ of cardinality no larger than $|A| + |\mathcal{L}|$ that realizes p. $\mathfrak{B}$ can be taken to be a proper extension.*[23]

Proof. Let $\{c_a \mid a \in A\}$ and $\{c_\varphi \mid \varphi \in p\}$ be two sets of new constant symbols not in $\mathcal{L}$. Consider the theory

$$T = \{\psi(c_{a_1} \dots c_{a_n}) \mid \mathfrak{A} \models \psi[a_1 \dots a_n]\} \cup \{\varphi(c_\varphi) \mid \varphi \in p\}.$$

To ensure a proper extension, add to T the axioms

$$\{c_a \neq c \mid a \in A\},$$

where c is another new constant symbol. Every finite subset of T has some expansion of $\mathfrak{A}$ as a model, and so by compactness, T has a model $\mathfrak{B}'$. By the Downward Löwenheim-Skolem-Tarski Theorem, one can find $\mathfrak{B}'' \equiv \mathfrak{B}'$ of size no larger than $|A| + |\mathcal{L}|$ with universe at least (the interpretations of)

$$\{c_a \mid a \in A\} \cup \{c_\varphi \mid \varphi \in p\} \cup \{c\}.$$

[23] Cf. Lemma I.1.6 in [9].

Then $\mathfrak{B} = \mathfrak{B}'' \upharpoonright \mathcal{L}$ (the *reduct* of $\mathfrak{B}''$ to the language $\mathcal{L}$) is an elementary extension of $\mathfrak{A}$ of the desired cardinality that realizes p. □

The next lemma will allow us to bound the number of types with parameters, given a bound on the number of types without parameters.

Lemma 2.5.2. *Suppose $\mathfrak{A}$ is an infinite $\mathcal{L}$-model and $|\mathfrak{A}| + |\mathcal{L}| \leq \lambda = \lambda^{<\kappa}$. If there are no more than λ complete m-types without parameters in $\mathfrak{A}$, then there are no more than λ complete m-types with fewer than κ parameters in $\mathfrak{A}$.*

Proof. Our hypothesis states that $|\mathbf{S}^m(\emptyset)| \leq \lambda$ for all $m < \omega$. In fact, we know more, namely that $|\mathbf{S}^m(Y)| \leq \lambda$ for each finite $Y \subseteq A$, since if $|Y| = k$, then $|\mathbf{S}^m(Y)| \leq |\mathbf{S}^{m+k}(\emptyset)| \leq \lambda$ by substituting in free variables v_i for each $y_i \in Y$. Now, we consider $|\mathbf{S}^m(X)|$ for some infinite, but fixed $X \subseteq A$, $|X| = \mu < \kappa$. (This is sufficient, since there are only $|[A]^{<\kappa}| \leq \lambda^{<\kappa} = \lambda$ such parameter sets.)

We now make our critical observation: each type p is determined uniquely by its subtypes $p \upharpoonright Y$ for each finite $Y \subseteq X$. To see this, note that each formula φ has only finitely many parameters, so there is some Y such that $\varphi \in p \upharpoonright Y$, and thus the union of all such $p \upharpoonright Y$ recovers p. Together with the observation that $|[X]^{<\omega}| = |X|$, this gives us

$$|\mathbf{S}^m(X)| \leq \prod_{\substack{Y \subseteq X \\ Y \text{finite}}} |\mathbf{S}^m(Y)| \leq \lambda^\mu = \lambda,$$

as desired.

□

Theorem 2.5.3. *Suppose that the hypotheses of Lemma 2.5.2 hold, including the bound on types. Then there exists an elementary extension $\mathfrak{B} \succcurlyeq \mathfrak{A}$ of cardinality λ that is κ-saturated.*[24]

Proof. We define an elementary chain $\{\mathfrak{A}_\alpha \mid \alpha \leq \lambda\}$ with $\mathfrak{A}_0 = \mathfrak{A}$ such that for all α:

[24]Cf. Theorem I.1.7 in [9].

(i) $\mathfrak{A}_\alpha \subsetneq \mathfrak{A}_{\alpha+1}$

(ii) $|\mathfrak{A}_\alpha| \leq \lambda$

(iii) $\mathfrak{A}_{\alpha+1}$ realizes all types over fewer than κ parameters in $\mathfrak{A}_\alpha$.

If this can be done, we claim that $\mathfrak{B} = \mathfrak{A}_\lambda$ is our desired elementary extension. Conditions (i) and (ii) will ensure that $\mathfrak{A}_\lambda$ has cardinality precisely λ. Now suppose p is a type over $\mathfrak{A}_\lambda$ with fewer than κ parameters. Since $\lambda^{<\kappa} = \lambda$, we must have $\kappa \leq \mathrm{cf}(\lambda)$, and so the parameter set will not be cofinal in the elementary chain, say contained in A_α for some $\alpha < \lambda$. By (iii) our type will be realized in $\mathfrak{A}_{\alpha+1}$, and hence also in $\mathfrak{A}_\lambda$.

We now construct the chain. Set $\mathfrak{A}_0 = \mathfrak{A}$, and let $\mathfrak{A}_\delta = \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$ at limit ordinals δ. For the succesor step, suppose $\mathfrak{A}_\alpha$ has been defined. The fact that $\mathfrak{A}_\alpha \equiv \mathfrak{A}$ implies that there are no more than λ types without parameters in $\mathfrak{A}_\alpha$. Now we use Lemma 2.5.2 to conclude that there are no more that λ types over fewer than κ parameters in $\mathfrak{A}_\alpha$. So we can enumerate them (perhaps with repeats) as $\{p_\beta \mid \beta < \lambda\}$. We then define *another* elementary chain $\{\mathfrak{A}_\alpha^\beta \mid \beta \leq \lambda\}$, where each $\mathfrak{A}_\alpha^{\beta+1}$ realizes type p_β. Now set $\mathfrak{A}_\alpha^0 = \mathfrak{A}_\alpha$, take unions at limit ordinals, and apply Lemma 2.5.1 at successor ordinals. To satisfy condition (1), let $\mathfrak{A}_\alpha^1$ be a proper extension. Once finished let $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha^\lambda$. This concludes the proof. $\square$

We want to use the above theorem to produce *saturated* models, which requires $\kappa = \kappa^{<\kappa}$. This equation holds at $\kappa = \aleph_0$, but then we could only use it on countable models. Instead of considering less effective solutions, we introduce the following.

Axiom 2.5.4. For every cardinal κ, there exists a strongly inaccessible $\bar{\kappa} \geq \kappa$.

Cardinal arithmetic then shows that $\bar{\kappa} = \bar{\kappa}^{<\bar{\kappa}}$, so we can use the above theorem on any model. In the context of a given model or theory, we call this the monster model $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$ with universe M. To make this well-defined, we (implicitly) take the smallest $\bar{\kappa}$ needed for a given argument to work.[25]

Notations 2.5.5. Whenever we consider a set X without reference to a specific model, we mean $X \subseteq M$ for the monster model $\mathfrak{M}$. We always assume $|X| < \bar{\kappa}$, so actually $X \subseteq A \subseteq M$ for some $\mathfrak{A}$. We write $\models \varphi$ in place of $\mathfrak{M} \models \varphi$.

In practice, the existence of $\mathfrak{M}$ will only simplify our arguments and notation. For example, our rule to omit $\mathfrak{A}$ from Definitions 2.4.13(2) and (3) when we want to consider a type consistent but not necessarily realized can now be restated as "we omit $\mathfrak{A}$ when $\mathfrak{A} = \mathfrak{M}$."

Conversely, given a model $\mathfrak{A}$ of size $< \bar{\kappa}$, we have $\mathfrak{A} \preccurlyeq \mathfrak{M}$ (this by saturation), so we can speak about $\mathfrak{M}$ instead of quantifying over all models of a bounded size. In other words, our theorems that use $\mathfrak{M}$ can be restated and proven from ZFC alone.

Since this is so important, we will look at an example. Consider Lemma 3.3.16, (one can either proceed to this point then turn back to this discussion, or preview the theorem briefly and then read this paragraph). We prove the Lemma for $\mathfrak{M}$, and claim this suffices to prove the lemma for all models, (of cardinality $< \bar{\kappa}$, which we could have chosen large enough for a given model from the start). Let $\mathfrak{A}$ be an arbitrary model. We never reference $|M|$ in any theorem, but we do reference its saturation, so choose $\bar{\kappa} > |A|$ large enough such that $\mathfrak{M}$ has enough saturation for the argument at hand. Now the proof goes through, and we conclude $\mathfrak{M}$ satisfies the condition (†). But in fact, inspecting this condition will show that any failure in $\mathfrak{A}$ would, by elementary embedding, translate into a failure in $\mathfrak{M}$. So we conclude that $\mathfrak{A}$ satisfies (†) as well, and $\mathfrak{A}$ was arbitrary. (Of course, this is unique to this argument—other arguments need their own justifications.)

[25] In [9], Shelah denotes the monster model by $\mathfrak{C}$ (an allusion to $\mathbb{C}$), and only assumes the existence of a strongly inaccessible cardinal "larger than all the cardinalities [his book] will deal with." We make a stronger assumption here so that it can be stated more formally and therefore be immune to apparent circularity. See the section on set theory for the definition of strongly inaccessible cardinals and a brief discussion about their relationship to ZFC.

Let's look at one more example, this time the usage of the notation $\models \varphi[\bar{a}]$, (we omit the $\mathfrak{M}$). What does this mean? For this to make any sense at all, we must be taking $\bar{a} \in A$ for some model $\mathfrak{A}$, which embeds in $\mathfrak{M}$. But then $\mathfrak{M} \models \varphi[\bar{a}]$ if and only if $\mathfrak{A} \models \varphi[\bar{a}]$, and so there is no problem evaluating this. Looking forward, our definition of an indiscernible sequence I over $\mathfrak{M}$ will require evaluating $\models \varphi[\bar{a}]$ for various $\bar{a} \in I$. We can find a model $\mathfrak{A} \preccurlyeq \mathfrak{M}$ containing I, so in this model (or any other model embedded in $\mathfrak{M}$ containing I) it suffices to evaluate $\mathfrak{A} \models \varphi[\bar{a}]$. Lemma 4.5.2 is another good example of this; it is used in Definition 5.4.1 wherein $\bar{c}, \bar{a}$ quantify over sets, which embed in $\mathfrak{M}$. On that matter, when we speak of a "set", we really speak of a subset of a model; the elements have no structure on their own.

We will not belabor the point further, but we hope to have made it clear that any reasonable usage of the monster model can be formalized in ZFC, and that we have good reason to do so in the first place.

Finally, we summarize the assumptions stated in this chapter to be used throughout the rest of the work:

- The axioms of ZFC set theory,
- All formulas are well-formed,
- All theories are consistent and complete,
- All theories have (only) infinite models, and finally
- All models are infinite.

Chapter 3

Uncountable Categoricity

3.1 Background on Morley's Theorem

One question model theorists seek to answer is the number of models up to isomorphism that a theory admits. If a countable theory has infinite models, then it has models of all infinite cardinalities by the Upward and Downward Löwenheim-Skolem-Tarski Theorems. Within a given cardinality, however, it is possible that there is only one model of that size (up to isomorphism).

Definition 3.1.1. Suppose that a theory T has only one model up to isomorphism of size κ. Then we say T is *κ-categorical*; sometimes we say *categorical in κ*.

More generally, we define the *spectrum function* $I(T, \kappa)$ as the number of models of T of size κ up to isomorphism. Consider the following examples of various spectra for complete theories.

Example 3.1.2. Pure identity theory (all tautologies using only logical symbols) is categorical in every power, since with no structure to preserve, any bijection *is* an isomorphism.

Example 3.1.3. The theory of dense linear orders (DLO) can be shown to be $\aleph_0$-categorical by a back-and-forth argument. However, this theory is not categorical in any uncountable power, since one can always alter a given uncountable DLO to get a new one. For example, one could append either $\mathbb{Q}$ or a DLO of size $\aleph_1$ to the end of an existing DLO to get a distinct DLO of the same size.

Example 3.1.4. The theory of algebraically closed fields of characteristic zero (ACF_0) is categorical in every uncountable power. In fact, any uncountable ACF_0 is determined by its transcendence degree, (which is determined by its cardinality). Given two such fields, map the subfield of (complex) algebraic elements identically from one to the other, and then recur on the independent transcendental elements to define an isomorphism. However, this theory isn't $\aleph_0$-categorical; the complex algebraics form one ACF_0, and the algebraic closure of, say, $\mathbb{A}[\pi]$ will be countable and nonisomorphic.

Example 3.1.5. The theory of real closed fields $T = \mathrm{Th}(\mathbb{R})$ (RCF) is not categorical in any power. Using a standard compactness argument with DLST, one can obtain Archimedean and non-Archimedean ordered fields elementarily equivalent to $\mathbb{R}$ in every infinite power.

It was conjectured (by Łos) that all complete countable theories fall into one of four categories: categoricity in all infinite powers, categoricity in $\aleph_0$ only, categoricity in uncountable powers only, and categoricity in no power. This conjecture was proved by Michael Morley in his 1965 dissertation:

Theorem 3.1.6 (Morley's Categoricity Theorem)**.** *Suppose T is a complete countable theory that is categorical in some uncountable power. Then T is categorical in every uncountable power.*

We will present essentially his original argument, though with different notation and methods prominent in Shelah's work on the stability spectrum. In doing so, we will better motivate the results to follow.

3.2 Proof Idea

The idea behind the proof of Morley's Theorem is to show that every uncountable model is saturated, and then use Theorem 2.4.21 to get categoricity in every uncountable power. The proof is by contradiction; we assume that in some uncountable power there is a model that is not saturated, and produce two different models of cardinality $\kappa > \aleph_0$, (where we assumed categoricity held).

We first need a notion that allows us to go from a local property (categoricity at κ) to a more global property. This notion is *stability*, which roughly says that a theory admits the minimum number of types over parameter sets of a certain size. We show that if a countable theory is categorical in one uncountable power, then it is stable in all powers. Doing so requires the development of *transcendence rank*, which computes how many times we can use types to partition realization sets for other types. This tool we use throughout the thesis.

Our first model (of power κ) will be $\aleph_1$-saturated (Theorem 3.5.2). This will be relatively easy to construct by a straightforward recursion using stability to ensure the model stays at size κ.

Our second model will fail $\aleph_1$-saturation, and is much more difficult to construct (Theorem 3.5.3). We take our uncountable model that is not saturated, and shrink to a countable model that misses a single type, (a straightforward application of the DLST). Then, we grow this model to power κ in such a way that this type is not realized. This will involve large subsets of the model being *indiscernible*, meaning one cannot distinguish two n-tuples by formula. The abundance of indiscernible sequences in models of stable theories is established in Theorem 3.3.11 and is a major result with widespread application to later topics.

3.3 Stability and Indiscernible Sequences

In proving Morley's Theorem, we will need the following result.

Theorem 3.3.1. *Given any complete countable theory T and cardinal κ, there exists a model $\mathfrak{A}$ of power κ for which every countable parameter set $X \subseteq A$ admits at most countably many realized types thereover.*[1]

We remark that any parameter set X begets at least $|X|$ realized types, but in this case, this is also the maximum number. In general, we will later consider finitely satisfiable but perhaps not realized types, and we will call models that similarly beget a minimal number *stable*. The condition above is a bit weaker, however, since we are only concerned with those types that are realized.

The idea behind proving this theorem involves writing down a model whose elements can be expressed by functions[2] drawing from some fixed input set. The elements in this set will all look alike, which will limit the number of distinct elements definable thereover. Once we make precise what "look alike" means, we will call these elements *indiscernible*. A detailed proof of this theorem is readily available, and so we will not reproduce it here.

Definition 3.3.2. Given a model $\mathfrak{A}$, a set of formulas closed under permutations of variables Δ, and a parameter set $X \subseteq A$, a sequence $\langle \bar{a}_\xi \mid \xi < \alpha \rangle$ of (finite tuples of) elements in A is an (n, Δ)-*indiscernible sequence over X in $\mathfrak{A}$* if and only if for each

$$\xi_0 < \cdots < \xi_{n-1} < \alpha, \quad \eta_0 < \cdots < \eta_{n-1} < \alpha$$

we have

$$\mathrm{tp}_\Delta(\langle \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_{n-1}} \rangle, X, \mathfrak{A}) = \mathrm{tp}_\Delta(\langle \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_{n-1}} \rangle, X, \mathfrak{A}).$$

[1] Cf. Theorem 3.7 in [8], or Corollary 3.3.14 in [3].

[2] These are called *Skolem functions.* However interesting, they will not make a second appearance in the thesis, and so we will not spend time on them.

We say that the sequence is Δ-*indiscernible* (over X in $\mathfrak{A}$) if and only if it is (n, Δ)-indiscernible for all n. We omit $\mathfrak{A}$ when $\mathfrak{A} = \mathfrak{M}$, and Δ when $\Delta = \mathcal{L}$. Omitting the requirement that the ξ_k and η_k are ordered yields the definition for an *indiscernible set*.

Example 3.3.3. Consider any ordered sequence of elements in $\mathbb{Q}$. Without parameters, there is no way to distinguish any two n-tuples of these elements, so this sequence is indiscernible. However, the presence of any parameter equal to one of the elements of this sequence, or lying between two elements of this sequence, would allow us to discern between them. Of course, these do not form an indiscernible *set*, since the rationals can see their order.

Example 3.3.4. Consider now a collection of pairs of rational numbers, $\bar{a}_\xi = (a_\xi, b_\xi)$. Depending on how these pairs are arranged, they may or may not be indiscernible. They are indiscernible if they are nested or pairwise disjoint, (interpreted as intervals). A mixture of these two, however, will not yield indiscernibility.

Ramsey's Theorem can be used to produce models containing indiscernible sequences of any size and in fact of any order type, (see Theorem 3.3.17 below). In fact, this is also part of the proof of Theorem 3.3.1. A related result is proven below using the same technique, which we will use later on.

Theorem 3.3.5. *Suppose I is an infinite set of finite tuples of the same size, Δ, X finite, and $n < \omega$. Then there exists an infinite subsequence $J \subseteq I$ which forms an (n, Δ)-indiscernible sequence over X.*

Proof. Since Δ and X are finite, there are only finitely many formulas $\psi(\bar{v}; \bar{b})$ for $\bar{b} \in X$. Consider one of these formulas. Let I_1 be the set of all n-tuples from I that satisfy $\psi(\bar{v}; \bar{b})$, and I_2 be the set of all n-tuples that do not. By Ramsey's theorem, there exists an infinite $J \subseteq I$ such that either $J^n \subseteq I_1$ or $J^n \subseteq I_2$. Now induction will complete the proof. $\square$

It is natural to ask how the above theorem generalizes; indeed, its conditions are quite strict! In certain models, indiscernible sequences are hard to come by. As we saw above, any dense linear order can easily distinguish between its elements with little information. In contrast, consider the following example, (which we will reference many times!)

Example 3.3.6. Consider the language $\mathcal{L} = \{R\}$, R a binary relation, and T the theory whose axioms specify that R is an equivalence relation with infinitely many infinite equivalence classes. Suppose X is some parameter set. Indiscernible sequences, say of single elements, must lie either all in the same equivalence class (and be unequal to any $a \in X$), or each be in their own equivalence class without any parameters, (see Figure 3.1 for the second case). As a preview of Theorem 3.3.11, if I is any set with $|X| < |I|$, we will be able to find $J \subseteq I$, $|X| < |J| = |I|$ meeting one of these conditions.

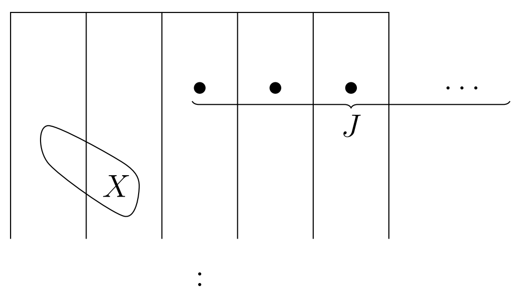

Figure 3.1: An indiscernible set in infinitely many infinite equivalence classes

What makes this example different from dense linear orders? The idea is that parameters carry much less weight here. In dense linear orders, countably many parameters can be used to define uncountably many Dedekind cuts, which we can represent as types. We will devote great study to the case where this doesn't happen—that is, when a model admits a minimum number of types for a given parameter set.

Definition 3.3.7. A model $\mathfrak{A}$ is λ-*stable* if and only if for every parameter set $X \subseteq A$ of size $|X| \leq \lambda$, we have that for all $m < \omega$, $|\mathbf{S}^m(X, \mathfrak{A})| \leq \lambda$. A theory T is λ-*stable* if and only if each of its models is. We say T is *stable* when T is λ-stable for some λ, T is *unstable* when it is not stable, and T is *superstable* when there exists κ such that T is λ-stable for every $\lambda \geq \kappa$.

Example 3.3.8. Consider again the theory in Example 3.3.6. Suppose that we are given countably many parameters $\{a_n \mid n \in \omega\}$ scattered throughout, together with our equivalence relation R. What types are expressible in this language? A moment's thought will confirm that there are only three forms the (complete) type can take:

- $p(v) \supset \{v = a_k\}$
- $p(v) \supset \{vRa_k\}$
- $p(v) \supset \{\neg vRa_0, \neg vRa_2, \neg vRa_2, \dots\}$

Respectively, these express equality to a particular parameter, relation to a particular parameter, or relation to none of the parameters. By our stipulation that there are infinitely many infinite equivalence classes, all three will define a consistent type. In the first two we have a choice of k, yielding a total of $\aleph_0 + \aleph_0 + 1 = \aleph_0$ types.

Although we have defined stability in terms of realized m-types, it suffices to consider the number of consistent 1-types.

Theorem 3.3.9. *A theory T is μ-stable if and only if for all X, $|X| \leq \mu$, we have $|\mathbf{S}(X)| \leq \mu$.*[3]

Lemma 3.3.10. *Suppose λ is a regular cardinal, $\mathfrak{A}$ a model, $X \subseteq A$ a parameter set, and $|\mathbf{S}^m(X, \mathfrak{A})| \geq \lambda$. Then there exists a finite $B \subseteq A$ such that $|\mathbf{S}(X \cup B)| \geq \lambda$.*[4]

Proof. We induct on m. The case $m = 1$ is trivial, so suppose the result holds for $m = k$. We suppose $|\mathbf{S}^{k+1}(X, \mathfrak{A})| \geq \lambda$; if it happens that also $|\mathbf{S}^k(X, \mathfrak{A})| \geq \lambda$, then

[3] Cf. Corollary I.2.2 in [9].
[4] Cf. Lemma I.2.1 in [9].

the result holds by the inductive hypothesis. Otherwise $|\mathbf{S}^k(X, \mathfrak{A})| < \lambda$. For each $q \in \mathbf{S}^{k+1}(X, \mathfrak{A})$, define $q^* = \{\exists v_k \varphi(v_0 \dots v_k) \mid \varphi(v_0 \dots v_k) \in q\}$, and then let q^+ be the unique extension of q^* to a complete type in $\mathbf{S}^k(X, \mathfrak{A})$.

This gives us a map $\mathbf{S}^{k+1}(X, \mathfrak{A}) \to \mathbf{S}^k(X, \mathfrak{A})$ from a set of size $\geq \lambda$ to a set of size $< \lambda$. Since λ is regular, there must exist some $p \in \mathbf{S}^k(X, \mathfrak{A})$ in the range of the mapping whose pullback has size $\geq \lambda$. Otherwise λ could be written as a union of fewer than λ sets of size less than λ, (an equivalent condition for λ being singular, see Theorem 2.3.26). Symbolically, $|\{q \in \mathbf{S}^{k+1}(X, \mathfrak{A}) \mid q^+ = p\}| \geq \lambda$.

Now let $b_0, \dots, b_n$ satisfy p in $\mathfrak{A}$. This also satisfies each q^*, but we are not guaranteed satisfaction of q in $\mathfrak{A}$. No matter, we only need to show that there are at least λ types in a single variable consistent with $\mathfrak{A}$. Then let $B = \{b_0 \dots b_{k-1}\}$, and then substituting the parameter b_i for v_i in the $\geq \lambda$ many q such that $q^+ = p$ will yield $|\mathbf{S}(X \cup B)| \geq \lambda$, as desired. $\square$

Proof of Theorem 3.3.9. The forward direction is obvious. For the reverse direction, we use the previous lemma at $\lambda = \mu^+$, a regular cardinal. If for contradiction $|\mathbf{S}^m(X, \mathfrak{A})| \geq \mu^+$ for some m, then we would have $|\mathbf{S}(X \cup B)| \geq \mu^+ > \mu$ for some finite B, in which case $|X \cup B| = |X| \leq \mu$ and we contradict the hypothesis. $\square$

This brings us to the main theorem of this section. We follow closely the development in the first chapter of [9].

Theorem 3.3.11. *Suppose T is a λ-stable theory. If a parameter set X and an arbitrary set I satisfy $|I| > \lambda \geq |X|$, then there exists $J \subseteq I$ an indiscernible sequence over X of size $|J| > \lambda$.*[5]

To prove this theorem, we introduce a new notion called splitting. We then prove a sufficient condition for a sequence being indiscernible, show how to find sequences

[5] Cf. Theorem I.2.8 in [9].

satisfying the condition given a hypothesis on a model, and then show that every λ-stable model satisfies that hypothesis.

Definition 3.3.12. A type p (Δ_1, Δ_2)-*splits* over a parameter set X if and only if there exist tuples $\bar{b}$, $\bar{c}$ such that $\mathrm{tp}_{\Delta_1}(\bar{b}, X) = \mathrm{tp}_{\Delta_1}(\bar{c}, X)$, but for some $\varphi \in \Delta_2$ we have both $\varphi(\bar{v}; \bar{b}), \neg\varphi(\bar{v}; \bar{c}) \in p$. We omit $\Delta_i = \mathcal{L}$, $i = 1, 2$.

See Figure 3.2 below for a visual representation of splitting.

Example 3.3.13. To gain some intuition for splitting, suppose G is a graph and b, c are vertices. Suppose that $X \subset G$ has the property that b and c have the same edge relations to all $a \in X$. But suppose some $d \in G - X$ has an edge to b and not to c. Then any type containing the formulas $\mathrm{Edge}(v, b)$, $\neg\mathrm{Edge}(v, c)$ would split over X.[6]

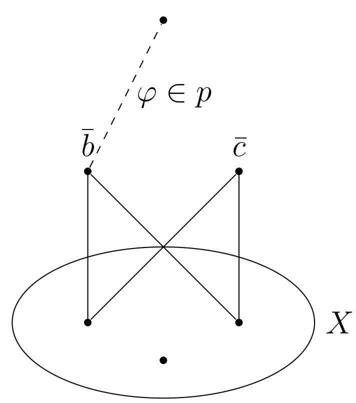

Figure 3.2: A type p splitting over X

In practice, we will typically consider the negation of splitting. If p does *not* split over X, then for every $\bar{b}$ and $\bar{c}$ we have either $\mathrm{tp}(\bar{b}, X) \neq \mathrm{tp}(\bar{c}, X)$ or for every formula φ,

$$\varphi(\bar{v}; \bar{b}) \in p \text{ iff } \varphi(\bar{v}; \bar{c}) \in p.$$

In proofs we will choose $\bar{b}$ and $\bar{c}$ to satisfy $\mathrm{tp}(\bar{b}, X) = \mathrm{tp}(\bar{c}, X)$, which then forces the second condition above to hold.

[6] This is an example of splitting in a particular language, but is also an abstraction of the general case where an "edge" represents the truth of any formula with parameters in X.

Lemma 3.3.14. *Suppose $I = \langle \bar{a}_\xi \mid \xi < \alpha \rangle$ and $X_\xi = \bigcup \{\bar{a}_\eta \mid \eta < \xi\} \cup X$. Suppose also that for all $\eta < \xi < \alpha$, $p_\xi = tp(\bar{a}_\xi, X_\xi)$ does not split over X and $p_\eta \subseteq p_\xi$. Then I is an indiscernible sequence over X.*[7]

Proof. We show that I is an n-indiscernible sequence over X for all n by induction. Let

$$\xi_0 < \cdots < \xi_{n-1} < \alpha, \quad \eta_0 < \cdots < \eta_{n-1} < \alpha$$

be ordinals. Our task is to show that

$$\mathrm{tp}(\langle \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_{n-1}} \rangle, X) = \mathrm{tp}(\langle \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_{n-1}} \rangle, X).$$

Suppose $n = 1$, and $\varphi(\bar{v}) \in \mathrm{tp}(\bar{a}_{\xi_0}, X) \subseteq p_{\xi_0}$. Then $\neg\varphi(\bar{v}) \notin p_{\xi_0} \supseteq p_0$, so $\neg\varphi(\bar{v}) \notin p_0$ and it follows that $\varphi(\bar{v}) \in p_0 \subseteq p_{\eta_0}$. The parameters still range over X so $\varphi(\bar{v}) \in \mathrm{tp}(\bar{a}_{\eta_0}, X)$ as desired.

Now assume the result for $n = k$, and consider $n = k + 1$. Let $\beta = \max(\xi_k, \eta_k)$. Since the tuples $\langle \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_{k-1}} \rangle$ and $\langle \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_{k-1}} \rangle$ realize the same type but p_β does not split over X, it must be the case that for any formula $\varphi(\bar{v}_0 \ldots \bar{v}_k; \bar{c})$

$$\varphi(\bar{v}_k; \bar{c}, \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_{k-1}}) \in p_\beta \text{ iff } \varphi(\bar{v}_k; \bar{c}, \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_{k-1}}) \in p_\beta.$$

Now since $p_{\xi_k}, p_{\eta_k} \subseteq p_\beta$, we get

$$\begin{aligned}
\varphi(\bar{v}_0 \ldots \bar{v}_k; \bar{c}) \in \mathrm{tp}(\langle \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_k} \rangle, X) &\text{ iff } \varphi(\bar{v}_k; \bar{c}, \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_{k-1}}) \in p_{\xi_k} \\
&\text{ iff } \varphi(\bar{v}_k; \bar{c}, \bar{a}_{\xi_0} \ldots \bar{a}_{\xi_{k-1}}) \in p_\beta \\
&\text{ iff } \varphi(\bar{v}_k; \bar{c}, \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_{k-1}}) \in p_\beta \\
&\text{ iff } \varphi(\bar{v}_k; \bar{c}, \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_{k-1}}) \in p_{\eta_k} \\
&\text{ iff } \varphi(\bar{v}_0 \ldots \bar{v}_k; \bar{c}) \in \mathrm{tp}(\langle \bar{a}_{\eta_0} \ldots \bar{a}_{\eta_k} \rangle, X)
\end{aligned}$$

[7]Cf. Lemma I.2.5 in [9].

as desired. $\square$

Lemma 3.3.15. *Let $\mathfrak{A}$ be a λ-stable model in which*

$$\begin{array}{l}\text{There is no increasing } \{X_\alpha \mid \alpha \leq \lambda\} \text{ and } p \in S(X_\lambda, \mathfrak{A}) \text{ such that} \\ p \restriction X_{\alpha+1} \text{ splits over } X_\alpha \text{ for all } \alpha < \lambda.\end{array} \tag{$\dagger$}$$

If $X \subseteq A$, $I \subseteq A$, and $|I| > \lambda \geq |X|$, then there exists $J \subseteq I$, $|J| > \lambda$ such that J is an indiscernible sequence over X.[8]

Proof. We first prove the following

$$\begin{array}{l}\text{There exist } B, C \text{ such that } X \subseteq B \subseteq C \subseteq A, |C| \leq \lambda \text{ and } p \in S(C, \mathfrak{A}) \\ \text{such that both} \\ \text{(1) For all } C', C \subseteq C' \subseteq A, |C'| \leq \lambda, p \text{ has an extension } p' \in S(C') \\ \text{realized in } I - C' \text{ that does not split over } B. \text{ (In particular neither} \\ \text{does } p.) \\ \text{(2) For all } \bar{c} \in {}^{<\omega}A, \text{ there exists } \bar{c}\,' \in {}^{<\omega}C \text{ such that } \mathrm{tp}(\bar{c}, B) = \mathrm{tp}(\bar{c}\,'B).\end{array} \tag{$\dagger\dagger$}$$

Suppose for the sake of contradiction $\neg(\dagger\dagger)$. We will define an increasing sequence $\{B_\alpha \mid \alpha \leq \lambda\}$ with the aim of contradicting ($\dagger$) such that each $B_\alpha \subseteq A$ and $|B_\alpha| \leq \lambda$. Let $B_0 = X$, and $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$ at limit ordinals.

Now for the successor step, suppose B_α is defined and $|B_\alpha| \leq \lambda$. Since $\mathfrak{A}$ is λ-stable, there are at most λ realized types over parameters B_α. So we can define $C_\alpha \subseteq A$ as consisting of (the elements of) representative tuples that realize each type. Then $|C_\alpha| \leq \lambda$, and for all $\bar{c} \in A$ there exists $\bar{c}\,' \in C$ such that $\mathrm{tp}(\bar{c}, B_\alpha) = \mathrm{tp}(\bar{c}\,', B_\alpha)$. This satisfies (2) of ($\dagger\dagger$), so (1) must fail.

[8] Cf. Lemma I.2.6 in [9].

For every $p \in \mathbf{S}(C_\alpha, \mathfrak{A})$ that does not split over B_α, there is a C'_p such that $C_\alpha \subseteq C'_p \subseteq A$, $|C'_p| \leq \lambda$, and every extension of p in $\mathbf{S}(C'_p)$ realized in $I - C'_p$ splits over B_α. Now let

$$B_{\alpha+1} = \bigcup \{C'_p \mid p \in \mathbf{S}(C_\alpha, \mathfrak{A}) \text{ does not split over } B_\alpha\} \cup C_\alpha.$$

By λ-stability, $|B_{\alpha+1}| \leq \lambda$.

Now let $\bar{c} \in I - B_\lambda$ and $p \in \mathrm{tp}(\bar{c}, B_\lambda)$. We show $p \restriction B_{\alpha+1}$ splits over B_α for all $\alpha < \lambda$, which contradicts (†). If this is not the case, then neither does $q = p \restriction C_\alpha$ since restricting parameters makes splitting harder. Then $q \in \mathbf{S}(C_\alpha, \mathfrak{A})$, so $C'_q \subseteq B_{\alpha+1}$. But $p \restriction C'_q$ (an extension of q) is realized by $\bar{c} \in I - C'_q \supseteq I - B_\lambda$, and does not split over B_α, contradicting the construction of B_α. This gives us our contradiction of (†), and so we conclude that (††) must hold.

Now we construct our indiscernible sequence from I. We define tuples $\bar{c}_\alpha \in I$ by recursion on $\alpha < \lambda^+$. Suppose $\bar{c}_\beta$ has been defined for all $\beta < \alpha$. Define $C_\alpha = \{\bar{c}_\beta \mid \beta < \alpha\} \cup C$ (note $|C_\alpha| \leq \lambda$), and using (††)(1) let $p_\alpha \in \mathbf{S}(C_\alpha)$ be an extension of p realized by $\bar{c}_\alpha \in I - C_\alpha$ that does not split over B.

We wish to apply Lemma 3.3.14 to show that $I' = \{\bar{c}_\alpha \mid \alpha < \lambda^+\}$ is an indiscernible sequence over X. The only hypothesis in question is whether $p_\beta \subseteq p_\alpha$ for $\beta < \alpha$. So, suppose $\varphi(\bar{v}; \bar{b}) \in p_\beta$. Choose $\bar{b}' \in C$ such that $\mathrm{tp}(\bar{b}, B) = \mathrm{tp}(\bar{b}', B)$. We showed that p_β does not split over B, and so $\varphi(\bar{v}; \bar{b}) \in p_\beta$ implies $\varphi(\bar{v}; \bar{b}') \in p_\beta$. Now p_β extends to p, and so $\varphi(\bar{v}; \bar{b}') \in p$. We cannot have $\neg\varphi(\bar{v}; \bar{b}') \in p_\alpha$ since it also extends to p, so we must have $\varphi(\bar{v}; \bar{b}') \in p_\alpha$. By non-splitting, we get $\varphi(\bar{v}; \bar{b}) \in p_\alpha$, as desired. □

In proving the following lemma, we will assume ¬(†), which gives us parameter sets A_α, $\alpha < \lambda$ such that $p \restriction A_\alpha$ splits over A_α. Our goal is to use this to define at least 2^λ types, which contradicts stability. As an analogy, we are taking a sheet of paper, and folding it λ times. Now, when we cut our splitting type into the page and unfold, we get a snowflake with 2^λ types carved into it.

Lemma 3.3.16. *Suppose T is λ-stable. Then every model $\mathfrak{A}$ of T satisfies* $(\dagger)$ *from Lemma 3.3.15.*[9]

Proof. We prove the above for the monster model $\mathfrak{M}$, which will imply the result for all models of cardinality $< \bar{\kappa}$. Suppose $\neg(\dagger)$. Let $\{A_\alpha \mid \alpha \leq \lambda\}$ be an increasing sequence and $p \in S(A_\lambda)$. Choose $\bar{b}_\alpha, \bar{c}_\alpha \in A_{\alpha+1}$ such that $\mathrm{tp}(\bar{b}_\alpha, A_\alpha) = \mathrm{tp}(\bar{c}_\alpha, A_\alpha)$ but both $\varphi_\alpha(\bar{v}; \bar{b}_\alpha), \neg\varphi_\alpha(\bar{v}; \bar{c}_\alpha) \in p \restriction A_{\alpha+1} \subseteq p$ for some φ_α. Let $\bar{a}$ realize p, and $\mu = \min\{\mu \mid 2^\mu > \lambda\}$.

We will now define for each $g \in {}^{\leq\mu}2$ an elementary mapping F_g with domain $A_{\ell(g)}$ by recursion on $\ell(g)$. For $\ell(g) = 0$, let $F_0 = \mathrm{id}_{A_0}$. If $\ell(g) = \delta$ for δ a limit ordinal, let $F_g = \bigcup_{\alpha<\delta} F_{g\restriction\alpha}$.

Suppose now that F_g is defined for all g of length α. For any such g, we have two more definitions to make: $F_{g^\frown\langle 0\rangle}$ and $F_{g^\frown\langle 1\rangle}$. The former will just be an arbitrary extension of F_g to $A_{\alpha+1}$, which is possible by saturation of $\mathfrak{M}$ and Lemma 2.4.22. For the other mapping, set $F_{g^\frown\langle 1\rangle}(b^k_\alpha) = F_{g^\frown\langle 0\rangle}(c^k_\alpha)$ for each b^k_α, c^k_α in the tuples $\bar{b}_\alpha$,$\bar{c}_\alpha$ respectively. We then extend to an elementary mapping on all of $A_{\alpha+1}$, using the fact that $\mathrm{tp}(\bar{b}_\alpha, A_\alpha) = \mathrm{tp}(\bar{c}_\alpha, A_\alpha)$.

We now define

$$B = \{F_g(b^k_\alpha), F_g(c^k_\alpha) \mid g \in {}^\alpha 2, 0 < \alpha < \mu, b^k_\alpha \in \bar{b}_\alpha, c^k_\alpha \in \bar{c}_\alpha\}.$$

Note that $|B| \leq \sum_{\alpha<\mu} 2^\alpha \leq \lambda$. For each $g \in {}^\mu 2$, extend F_g to an elementary mapping F'_g with domain $A_\mu \cup \{\bar{a}\}$, and let $p_g = \mathrm{tp}(F'_g(\bar{a}), B)$. We claim that this defines $2^\mu > \lambda$ complete types p_g over $\leq \lambda$ parameters, contradicting λ-stability.

Suppose $g \neq h \in {}^\mu 2$. Let $\alpha = \min\{\beta \mid g(\beta) \neq h(\beta)\}$; without loss of generality, assume $g(\alpha) = 0$. By definition of $\bar{b}_\alpha$, we have $\varphi_\alpha(\bar{v}; \bar{b}_\alpha) \in p$ and hence $\models \varphi_\alpha[\bar{a}; \bar{b}_\alpha]$.

[9] Cf. Lemma I.2.7 in [9].

Thus $\models \varphi_\alpha[F'_h(\bar{a}); F'_h(b^0_\alpha) \dots F'_h(b^k_\alpha)]$, so $\varphi_\alpha(\bar{v}; F'_h(b^0_\alpha) \dots F'_h(b^k_\alpha)) \in p_h$. The same argument shows $\neg\varphi_\alpha(\bar{v}; F'_g(c^0_\alpha) \dots F'_g(c^k_\alpha)) \in p_g$ and since $g(\alpha) = 0$ and g and h agree below α we have $F'_h(b^k_\alpha) = F'_g(c^k_\alpha)$. Thus $\neg\varphi_\alpha(\bar{v}; F'_h(b^0_\alpha) \dots F'_h(b^k_\alpha)) \in p_g$. We conclude that $p_g \neq p_h$, which finishes the proof. $\square$

The preceding two lemmas clearly imply Theorem 3.3.11. Without much more effort, we can improve this result.

Theorem 3.3.17. *Suppose T is $\aleph_0$-stable. Then any infinite indiscernible sequence I (over X) is an indiscernible set (over X).*

Proof. Suppose that for some formula $\varphi(v_1 \dots v_n; \bar{c})$, there was $a_1 < \dots < a_n$ in I and permutations σ, τ of $\{1 \dots n\}$ such that

$$\models \varphi(a_{\tau(1)} \dots a_{\tau(n)}; \bar{c}) \wedge \neg\varphi(a_{\sigma(1)} \dots a_{\sigma(n)}; \bar{c}).$$

We may assume that $\sigma = (k, k+1)\tau$ for some k. Let $\psi(v_1 \dots v_n; \bar{c}) = \varphi(v_{\tau(1)} \dots v_{\tau(n)})$. Then we have

$$\models \psi(a_1 \dots a_n; \bar{c}) \wedge \neg\psi(a_1 \dots a_{k-1}, a_{k+1}, a_k, a_{k+2} \dots a_n; \bar{c}). \tag{$\star$}$$

By order indiscernibility this holds for any ordered n-tuple in I. By a standard application of completeness and compactness, there exists a set Y order-isomorphic to $\mathbb{R}$ such that any ordered n-tuple from Y satisfies ($\star$), (embedded in a model of T). Let $Z \subseteq Y$ be a countable dense subset. We claim that $|\mathbf{S}(Z \cup \bar{c})| \geq 2^{\aleph_0}$, contradicting $\aleph_0$-stability.

Suppose $y < y'$ in Y. Choose $n-1$ elements in Z such that

$$z_1 < \dots < z_{k-2} < y < z_{k-1} < y' < z_{k+1} < \dots < z_n$$

for k above. Then y has

$$\models \psi(z_1 \dots z_{k-2}, y, z_{k-1} \dots z_n; \bar{c}),$$

while y' has

$$\models \neg\psi(z_1 \dots z_{k-2}, y', z_{k-1} \dots z_n; \bar{c}),$$

since the order of y' and z_{k-1} in indices k and $k+1$ are flipped. This means that y and y' satisfy distinct types. Thus $|\mathbf{S}(Z \cup \bar{c})| \geq 2^{\aleph_0}$. □

3.4 Transcendence Rank

Before we relate stability to uncountable categoricity, we need a better handle on how types behave with respect to their theories. Specifically, this tool will measure the degree to which we can increase the resolution of a type by adjoining new types, (see Example 3.4.5). This will lead us to global properties of a theory, which will be a key ingredient in the proof of both Morley's Theorem and the Stability Spectrum Theorem.

Notation 3.4.1. We use the symbol ∞ to denote the totality of the ordinal line, or more philosophically, the "absolute infinite". Ranging over all $\alpha < \infty$ is the same as ranging over all α.

Definition 3.4.2 (Rank of a Type)**.** Given a type p, $m < \omega$, Δ a set of m-formulas, and $2 \leq \lambda \leq \infty$, we define $R^m(p, \Delta, \lambda)$ as follows.

(i) If p is consistent, then $R^m(p, \Delta, \lambda) \geq 0$.

(ii) If $R^m(p, \Delta, \lambda) \geq \alpha$ for all $\alpha < \delta$ a limit ordinal, then $R^m(p, \Delta, \lambda) \geq \delta$.

(iii) We say that $R^m(p, \Delta, \lambda) \geq \alpha + 1$ provided the following holds:

For each finite $p' \subseteq p$ and all $\mu < \lambda$, there exists Δ-types q_β, $\beta \leq \mu$ (whose formulas are of the form $\varphi(\bar{v};\bar{a})$ or $\neg\varphi(\bar{v};\bar{a})$ for $\varphi(\bar{v};\bar{w}) \in \Delta$) such that if $\beta_1 \neq \beta_2$ then q_{β_1} and q_{β_2} are explicitly contradictory (one contains a formula negated in the other) and each $R^m(p' \cup q_\beta, \Delta, \lambda) \geq \alpha$.

If $R^m(p, \Delta, \lambda) \geq \alpha$ but $R^m(p, \Delta, \lambda) \not\geq \alpha + 1$, we say $R^m(p, \Delta, \lambda) = \alpha$. Otherwise, if $R^m(p, \Delta, \lambda) \geq \alpha$ for all α, we write $R^m(p, \Delta, \lambda) = \infty$. We call this the *transcendence rank* of a type p and may omit Δ and λ if they are clear from context.

Remark. *The definition of $R^m(p, \Delta, \lambda)$ is made for a particular theory T. We will often omit reference to T, though if we do we may refer to it as the* ambient theory.

To get some intuition, we first preview some results to come. We will show that condition (iii) above can be reduced to considering q_{β_1} and q_{β_2} only be contradictory (not necessarily explicitly so) and the quantification over finite p' can be eliminated altogether and replaced by p itself. So, condition (iii) is just measuring the ability for the type to be "broken up" into λ pieces. The rank of a type is the number of times you can do the breaking. The most interesting cases we will consider are $\lambda = 2, \aleph_0$, and ∞.

Once we show a few basic properties, we will compute the ranks of a few types from a few different theories.

Lemma 3.4.3. *Suppose $p_1 \vdash p_2$, (for instance, when $p_2 \subseteq p_1$). Then $R^m(p_1, \Delta, \lambda) \leq R^m(p_2, \Delta, \lambda)$.*[10]

Proof. We show by recursion that if $R^m(p_1, \Delta, \lambda) \geq \alpha$ then $R^m(p_2, \Delta, \lambda) \geq \alpha$. If $R^m(p_1, \Delta, \lambda) \geq 0$, then p_1 is consistent, so p_2 is also. The limit case is trivial, so for the successor case we assume the implication holds at α, and consider $R^m(p_1, \Delta, \lambda) \geq \alpha + 1$. Let $p_2' \subseteq p_2$ be finite and $\mu < \lambda$. Then $p_1 \vdash p_2'$. By an application of compactness, let $p_1' \subseteq p_1$ be finite such that $p_1' \vdash p_2'$. Since $R^m(p_1, \Delta, \lambda) \geq \alpha + 1$, we

[10] Cf. Theorem II.1.1 in [9].

have explicitly contradictory types q_β, $\beta < \mu$ such that each $R^m(p_1' \cup q_\beta, \Delta, \lambda) \geq \alpha$. Certainly we have $p_1' \cup q_\beta \vdash p_2' \cup q_\beta$, so by inductive assumption, $R^m(p_2' \cup q_\beta, \Delta, \lambda) \geq \alpha$. Thus the same q_β show that $R^m(p_2, \Delta, \lambda) \geq \alpha + 1$ as desired. □

Additionally, it is evident that the function R^m is monotonic with Δ, and anti-monotonic with λ. Since we have $p \vdash \bar{v} = \bar{v}$ for any type p, $R^m(\bar{v} = \bar{v}, \mathcal{L}, 2)$ is the maximum transcendence rank of any type for a given theory.

Definition 3.4.4. If $R^m(\bar{v} = \bar{v}, \mathcal{L}, 2) < \infty$, we say that the ambient theory T is *totally transcendental*.

This means that there is an upper bound on how far you can break up any type, depending only on the theory and not a particular model! We can think of totally transcendental theories as having poor "resolution"; they have trouble scrutinizing elements in their models.

Example 3.4.5. Consider $R^m(\bar{v} = \bar{v}, \mathcal{L}, 2)$ for the theory of dense linear orders. We claim $R^m(\bar{v} = \bar{v}, \mathcal{L}, 2) \geq \omega$. To see this, note that one can show $\geq n$ for all $n < \omega$ by taking $2^n + 1$ parameters, and defining a binary tree of types that progressively narrow down membership to a certain (multi) interval. Every step up the tree decreases the rank by 1, any leaf is consistent, hence the root $\bar{v} = \bar{v}$ has rank $\geq n$. (See Figure 3.3 for the case $n = 2$.) Note that in fact we could have taken the single formula $\bar{w}_1 < \bar{v} < \bar{w}_2$ instead of $\mathcal{L}$.

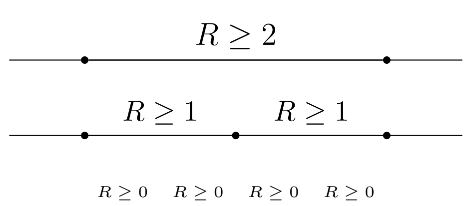

Figure 3.3: The type $v = v$ has rank at least 2.

Example 3.4.6. Now consider the theory of infinitely many infinite equivalence classes. We show $R^m(\bar{v} = \bar{v}, \varphi, 2) = 1$ where $\varphi(\bar{v}, \bar{w})$ is the formula $\bar{v}E\bar{w}$. Certainly the rank is at least 1, since the type $\{\bar{v} = \bar{v}\}$ is consistent, and then one can branch into types containing $\bar{v}E\bar{a}$ and $\neg\bar{v}E\bar{a}$. But the type containing $\bar{v}E\bar{a}$ can do no further branching; restrained to a single equivalence class, it is not possible to have both $\bar{v}E\bar{b}$ and $\neg\bar{v}'E\bar{b}$ while still having $\bar{v}E\bar{a}$, $\bar{v}'E\bar{a}$. Using elimination of quantifiers, one can see how this generalizes to showing finite rank for any single formula φ, in contrast to the previous paragraph where we had no problem getting infinite rank. This phenomenon will be explored further in Chapter 4.

It is well established that an infinite collection of formulas can capture more information than a finite collection. But it turns out that transcendence rank is not enhanced by infinitary data of this kind. (The following lemma also lets us simplify condition (iii) of our definition.)

Lemma 3.4.7. *For any type p, there exists a finite $q \subseteq p$ such that $R^m(q, \Delta, \lambda) = R^m(p, \Delta, \lambda)$.*[11]

Proof. For any $q \subseteq p$, we know that $R^m(q, \Delta, \lambda) \geq R^m(p, \Delta, \lambda)$. If $R^m(p, \Delta, \lambda) = \infty$, then equality holds and we can take $q = \emptyset$. Otherwise, $R^m(p, \Delta, \lambda) = \alpha < \infty$. So there exists a finite $q \subseteq p$ and $\mu < \lambda$ such that no explicitly contradictory Δ-types q_β, $\beta < \mu$ has $R^m(q \cup q_\beta, \Delta, \lambda) \geq \alpha$. This means $R^m(q, \Delta, \lambda) \not\geq \alpha + 1$, and hence $R^m(q, \Delta, \lambda) = \alpha = R^m(p, \Delta, \lambda)$. $\square$

We also have the following uniqueness result, (provided existence).[12]

Lemma 3.4.8. *For any type q whose parameters lie in X, if there exists $p \in \mathbf{S}^m_\Delta(X)$, $q \subseteq p$, such that $R^m(p, \Delta, 2) = R^m(q, \Delta, 2)$, then p is unique.*[13]

[11] Cf. Theorem II.1.2 in [9].

[12] For an existence result that sadly cannot be used in conjunction with this lemma, see Theorem 4.2.3.

[13] Cf. Corollary II.1.5 in [9].

Proof. We show that for any $\varphi(\bar{v}; \bar{a})$, whether or not $\varphi(\bar{v}; \bar{a}) \in p$ is controlled by q. Indeed, given $\varphi(\bar{v}; \bar{w}) \in \Delta$ and $\bar{a} \in X$, we know that $\varphi^t(\bar{v}; \bar{a}) \in p$ for one of $t = 0, 1$. Since $q \subseteq q \cup \{\varphi^t(\bar{v}; \bar{a})\} \subseteq p$ we have that $R^m(q \cup \{\varphi^t(\bar{v}; \bar{a})\}, \Delta, 2) = \alpha$. But if this held for both $t = 0, 1$, we would get $R^m(q, \Delta, 2) \geq \alpha + 1$. Thus, only one of these can hold; whichever one it is decides which formula p contains. $\square$

We now relate stability to total transcendence.

Theorem 3.4.9. *The following are equivalent for complete countable theories T:*

(i) T is $\aleph_0$-stable.

(ii) T is κ-stable for all κ.

(iii) T is totally transcendental.[14]

Proof. The direction (ii)⇒(i) is trivial. For (iii)⇒(ii), suppose T is totally transcendental and let X be a set of size κ. Every $p \in \mathbf{S}(X)$ begets a finite $q_p \subseteq p$ such that $R^m(q_p, \mathcal{L}, 2) = R^m(p, \mathcal{L}, 2)$. By Lemma 3.4.8, the map $p \mapsto q_p$ is injective. But the number of finite types over X is equal to $|X| = \aleph_0$, and hence $|\mathbf{S}(X)| \leq \aleph_0$, (of course equality holds).

Finally, we show (i)⇒(iii). We assume that T is not totally transcendental, and show that T is not $\aleph_0$-stable. It is worth thinking about how one might attempt to prove this before reading argument to follow. For example, one might want to use the fact that $R^m(\bar{v} = \bar{v}, \mathcal{L}, 2) > \omega$ and define a binary tree of types, at each node being offered φ or $\neg\varphi$ as an addition. However, recursion directly from the rank of $\bar{v} = \bar{v}$ would require taking ω steps *backwards* through the ordinal line. One might want to use compactness as a quick fix, but since $\mathcal{L}$ has more than a single formula, the set on which we use compactness will not be able to represent all types at the same time.[15]

[14] This result is a special case of a combination of results from [9]; cf. Theorems II.3.1, II.3.2, and Conclusion II.3.3. Alternatively, cf. Theorems 2.7 and 2.8 in [8].

[15] It is worth noting, however, that this works in the case $\Delta = \{\psi\}$ a singleton. We will use this technique later on.

The idea is to show the consistency of the sets

$$q_g = \{\varphi_{g\restriction n}(\bar{v};\bar{w}_{g\restriction n})^{g(n)} \mid n \in \omega\}$$

for appropriately chosen $\varphi_{g\restriction n}$ for all $g \in {}^{\omega}2$. Once countably many parameters $\bar{a}_{g\restriction n}$ have been assigned for the variables $\bar{w}_{g\restriction n}$, we will have uncountably many types using only countably many parameters. In order to show consistency, we use compactness and consider finite approximations of the q_g, constructed by recursion. In order to carry the recursion to successive steps, we will need more than consistency; we need positive rank to split the type in the binary tree.

Hence, we recur on $n < \omega$ and show that there exists $u_n \subseteq \omega_1$, $|u_n| = \aleph_1$, $\varphi_g \in \mathcal{L}$, and $\bar{a}_g^{i,n}$ for all $i \in u_n$ and $g \in {}^{<n}2$ such that $R^m(p_g^i, \mathcal{L}, 2) \geq i$ for all $i \in u_n$ and $g \in {}^{n}2$, where

$$p_g^i = \{\varphi_{g\restriction k}(\bar{v};\bar{a}_{g\restriction k}^{i,n})^{g(k)} \mid k < \ell(g)\}.$$

Once this recursion is complete, we can show the consistency of any finite subset of q_g by choosing n large enough, and i arbitrary. (We remark that ω_1 is chosen to be larger than $|\mathcal{L}| \leq \aleph_0$; this is where the proof can be generalized. Note also that $\infty > \omega_1$.)

For the base case $n = 0$, let $u_0 = \omega_1$; there is nothing more to verify or define. Now, suppose the recursion has been completed through stage n, and consider $n+1$. For each $i \in \omega_1$, choose $j = j(i)$ such that $i < j \in u_n$, (possible since $|u_n| = \aleph_1$ so is cofinal in ω_1). This yields $R^m(p_g^j, \mathcal{L}, 2) \geq j > i$ for all $g \in {}^{n}2$. So by definition, some φ_g^i and $\bar{a}_g^i$ have $R^m(p_g^j \cup \{\varphi_g^i(\bar{v};\bar{a}_g^i)^t\}, \mathcal{L}, 2) \geq i$, $t = 0, 1$. Note that we are mapping ω_1 into a countable set of formulas φ_g^i, so by regularity some uncountable set $u_{n+1} \subseteq \omega_1$ maps all i to the same formula $\varphi_g^i = \varphi_g$. To complete the recursion, we just need to define $\bar{a}_g^{i,n+1}$ for all $i \in u_{n+1}$ and $g \in {}^{\leq n}2$. Let $\bar{a}_g^{i,n+1} = \bar{a}_g^{j(i),n}$ if $g \in {}^{<n}2$, and $\bar{a}_g^i$ otherwise. The recursion is complete, and with it, the proof. $\square$

In fact, we can generalize some parts of the above theorem to work for uncountable languages. We do not need this for proving Morley's Theorem (which concerns only countable $|T|$), but we will need it later on. Since the proof is so similar, we sketch the argument here.

Theorem 3.4.10. *Suppose T is stable in some λ, $|\mathcal{L}|$ arbitrary. Then T is totally transcendental.*[16]

Proof. Suppose T is not totally transcendental. We show that T is not stable in $\aleph_0$, (hence, T is not stable in any infinite power). The proof is the same as the direction (i)⇒(iii) above with a few minor modifications. Instead of $\aleph_1$ (and also ω_1), use $|\mathcal{L}|^+ + \aleph_0$, a regular cardinal. The rest of the proof is the same. □

We now relate the results of this section back to uncountable categoricity.

Theorem 3.4.11. *Suppose a countable theory T is categorical in some uncountable power κ. Then T is $\aleph_0$-stable, (and hence also totally transcendental, and κ-stable for all κ).*[17]

Proof. Suppose T is not $\aleph_0$-stable. Then there exists a countable set X such that $\mathbf{S}(X)$ is uncountable. By embedding in the monster model and then using the DLST, X can be taken as a subset of a model $\mathfrak{A}$ of cardinality κ where uncountably many types in $\mathbf{S}(X)$ are realized. But by Theorem 3.3.1, there exists a model $\mathfrak{B}$ of power κ such that any countable subset admits only countably many realized types. Certainly $\mathfrak{A} \ncong \mathfrak{B}$, and so we contradict categoricity at power κ. □

Example 3.4.12. The converse of the above theorem is false. Consider the theory of infinitely many infinite equivalence classes, (the same theory in Example 3.3.8). As we remarked earlier, this theory is $\aleph_0$-stable. However, categoricity fails in all uncountable powers; uncountably many countable classes, or countably many uncountable classes give two distinct models for any uncountable power.

[16]Cf. Theorems II.3.1, II.3.2, and Conclusion II.3.3 in [9].
[17]Cf. Theorem 3.8 in [8].

3.5 Proving Morley's Theorem

We are still a few lemmas away from Morley's Theorem. The new ideas involved (isolated types, models prime over sets) will not appear later in the thesis, so we collect them here. The results of this section correspond closely to those in Morley's original paper, [8]. These will culminate in the following.

Theorem 3.5.1. *Suppose T is a complete countable theory, categorical in some uncountable power κ. Then every uncountable model of T is saturated.*[18]

Together with the uniqueness theorem for saturated models, this implies Morley's Theorem. We prove this theorem by contradiction, using a method that takes in an uncountable model that is not saturated, and produces models of any uncountable size that fail $\aleph_1$-saturation. This will contradict categoricity at power κ by the following result.[19]

Theorem 3.5.2. *Suppose T is κ-stable. Then for every uncountable κ, T has a model of power κ that is $\aleph_1$-saturated.*[20]

Proof. Unfortunately, we cannot apply Theorem 2.5.3 because the cardinal equation in its hypothesis may not hold. No matter, the direct proof is similar and not difficult with stability. We start with a model $\mathfrak{A}$ of T of power κ, and produce chain of extensions ω_1 long that realize all types over countably many parameters appearing prior. That is, we construct by recursion on $\alpha \leq \omega_1$ models $\mathfrak{A}_\alpha$ of size κ such that

(i) $\mathfrak{A}_0 = \mathfrak{A}$

(ii) $\mathfrak{A}_{\alpha+1} \succcurlyeq \mathfrak{A}_\alpha$ and realizes all types over any number of parameters from $\mathfrak{A}_\alpha$

(iii) $\mathfrak{A}_\delta = \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$ for limit ordinals δ

[18] Cf. Theorem 5.5 in [8].
[19] A more general statement holds, which we will return to later.
[20] Cf. Theorem 5.2 in [8].

To justify the successor step, we note that $|\mathfrak{A}_\alpha| = \kappa$ and so by κ-stability we get $|\mathbf{S}(A_\alpha)| = \kappa$. Thus, we apply Lemma 2.5.1 κ times yielding a model $\mathfrak{A}_{\alpha+1}$ of cardinality κ that realizes every type over $\mathfrak{A}_\alpha$. (This may seem like overkill, but considering only countable parameter sets is problematic, as there may be more than κ such sets. This is where the cardinal equation from Theorem 2.5.3 could fail!)

The model $\mathfrak{A}_{\omega_1}$ will be $\aleph_1$-saturated, since any countable parameter set will not be cofinal in the chain, say lying in $\mathfrak{A}_\alpha$, and hence any type thereover will be realized in $\mathfrak{A}_{\alpha+1}$. □

That's one model. To contradict categoricity at power κ, we need one more.

Theorem 3.5.3. *Suppose T is totally transcendental and has an uncountable model that is not saturated. Then for every uncountable κ, T has a model of power κ that is not $\aleph_1$-saturated.*[21]

We prove this theorem from a series of lemmas, and a few definitions.

Notation 3.5.4. Given a formula φ, perhaps with parameters in $\mathbf{S}(X)$, we define $U_\varphi = \{p \in \mathbf{S}(X) \mid \varphi \in p\}$.[22]

Definition 3.5.5. A type $p \in \mathbf{S}(X)$ is *isolated* if and only if there exists a formula $\varphi(\bar{v}; \bar{a}) \in p$ such that no other type in $\mathbf{S}(X)$ contains φ. In other words, U_φ is a singleton containing p.

Lemma 3.5.6. *Suppose T is totally transcendental. Then for any formula ψ perhaps with parameters from X, there exists an isolated type $q \in \mathbf{S}(X)$ containing ψ. In fact, any type of least rank in U_ψ will be isolated.*[23]

[21] Cf. Theorem 5.4 in [8].

[22] In [8], a topology is defined on $\mathbf{S}(X)$ with these sets as a basis. We are not using topological language in this thesis, and so we give an equivalent definition for isolation. Morley also defined transcendence rank topologically. As such, although we follow Morley's paper closely here, the arguments have been modified to match our definitions.

[23] Cf. Theorem 4.2 in [8].

Proof. Consider the set U_ψ defined above. By total transcendence, all these types have ordinal rank, so let $q \in U_\psi$ have least rank α. By Lemma 3.4.7, there exists a finite $q' \subseteq q$ with rank α. Let $\varphi = \bigwedge q' \wedge \psi$; we claim that q is isolated with respect to this formula, (note by completeness $\varphi \in q$, since it is certainly consistent with q). Suppose $\varphi \in p$, another type in $\mathbf{S}(X)$. Then by completeness again, $\psi \in p \in U_\psi$, (and also $q' \subseteq p$). This gives $R^m(p, \mathcal{L}, 2) \leq R^m(q', \mathcal{L}, 2) = \alpha$, so by leastness in U_ψ, p has rank α. By Lemma 3.4.8, $p = q$. This shows that q is isolated. □

Definition 3.5.7. Suppose $\mathfrak{B}$ is a model of T, and $A \subseteq B$. We say $\mathfrak{B}$ is *prime over* A if and only if for every model $\mathfrak{B}'$ of T and monomorphism $f : A \to \mathfrak{B}'$, f extends to $g : \mathfrak{B} \to \mathfrak{B}'$.

Remark. *In the above definition, A inherits the interpretation function from $\mathfrak{B}$ insofar as it is defined. In the theorem below, we start with an arbitrary set A. Although Morley did not take this approach in his paper, we continue to think of A as a subset of the monster model to formalize adjoining elements.*

Lemma 3.5.8. *Suppose T is complete. Then a set A is a model of T if and only if every nonempty $U_\varphi \subseteq \mathbf{S}(A)$ contains a type realized by A.*[24]

Proof. For the forward direction, let $\varphi(v; \bar{c})$ be a consistent formula, perhaps with parameters from A. By completeness, the sentence $\exists v \varphi(v; \bar{c})$ holds in the model A, so any $a \in A$ such that $A \models \varphi(a; \bar{c})$ will define a type containing φ.

For the converse, take A as a subset of a model $\mathfrak{B}$ of T. We claim that in fact $A \preccurlyeq \mathfrak{B}$, of course implying that $A \equiv \mathfrak{B} \models T$. To do so, we induct on formulas. The steps regarding atomic formulas and logical connectives are straightforward since $A \subseteq \mathfrak{B}$. For the existential quantifier, we verify the Tarski-Vaught criterion; suppose $\mathfrak{B} \models \exists v \varphi(v; \bar{c})$, where $\bar{c}$ is a tuple from A. By assumption, A will realize a type containing $\varphi(v; \bar{c})$, say at a, whence $A \models \varphi(a; \bar{c})$. In particular, this step verifies closure under function and constant symbols. □

[24] Cf. Theorem 4.1 in [8].

So, if T is complete and totally transcendental, the above two lemmas imply that $A \models T$ if and only if A realizes every isolated type in $\mathbf{S}(A)$.

Theorem 3.5.9. *Suppose T is totally transcendental. Then any set A has a model $\mathfrak{B}$, $A \subseteq B$, such that $\mathfrak{B}$ is prime over A.*[25]

Proof. Let $\kappa = |A| + \aleph_0$. By counting formulas, we see that $\mathbf{S}(A)$ has at most κ isolated types. Let $\{p_\alpha \mid \alpha < \kappa\}$ list these (perhaps with repetitions) where p_α is isolated with respect to the formula φ_α. Now, we define an increasing chain A_α by recursion on $\alpha \leq \kappa$ such that:

(i) $A_0 = A$

(ii) $A_{\alpha+1} = A_\alpha$ if A_α realizes p_α

(iii) $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$ if A_α does not realize p_α, and a_α is a point realizing some isolated $q \in \mathbf{S}(A_\alpha)$, $q \supseteq p_\alpha$

(iv) $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ for limit ordinals δ

To justify the successor step, we note that by Lemma 3.5.6, we can always find an isolated type $q \in \mathbf{S}(A_\alpha)$ containing the formula for which p_α is isolated. Then $q \restriction A$ still contains this formula, so by isolation, this is just p_α and hence $p_\alpha \subseteq q$.

Now, given a model $\mathfrak{B}'$ of T and monomorphism $f_0 : A \to \mathfrak{B}'$, we will construct an increasing sequence of monomorphisms $f_\alpha : A_\alpha \to \mathfrak{B}'$ by recursion on $\alpha \leq \kappa$. The only ambiguous step of the construction is case (iii) above. For this, let consider the point a_α realizing the isolated type $q \in \mathbf{S}(A_\alpha)$, and let $U = \{p \in \mathbf{S}(B') \mid f_\alpha(q) \subseteq p\}$. Let $\psi_\alpha = f_\alpha(\varphi_\alpha)$; we claim $U_{\psi_\alpha} \subseteq U$. To see this, note that $f_\alpha(q)$ is isolated with respect to ψ_α, and so if $\psi_\alpha \in p \in \mathbf{S}(B')$, then $f_\alpha(q) \subseteq p$. Now by Lemma 3.5.6, there exists a type $p \in U$ isolated with respect to ψ_α. Since $\mathfrak{B}'$ is a model, this type

[25] Cf. Theorem 4.3 in [8].

is realized by Lemma 3.5.8; map $f_{\alpha+1}(a_\alpha)$ to any such realization point. Since we started with $q = \mathrm{tp}(a_\alpha, A_\alpha)$, it is clear that $f_{\alpha+1}$ is a monomorphism.

We are almost done, but it might be the case that A_κ is not a model of T. So we repeat this construction another ω times, forming $A_{\kappa\cdot n}$ for all $n \leq \omega$, such that $A_{\kappa\cdot(n+1)}$ realizes every isolated type over $A_{\kappa\cdot n}$. Now we claim $A_{\kappa\cdot\omega}$ realizes every isolated type in $\mathbf{S}(A_{\kappa\cdot\omega})$, which will imply it is a model of T by Lemma 3.5.8. Indeed, if p is isolated with respect to the formula φ, then φ is a formula perhaps with parameters in $A_{\kappa\cdot n}$ for some $n < \omega$. Let $q \in \mathbf{S}(A_{\kappa\cdot n})$ be isolated containing φ, realized in $A_{\kappa\cdot(n+1)}$, say at a. Now the type of a in $\mathbf{S}(A_{\kappa\cdot\omega})$ will contain φ, and hence be equal to p by isolation. In other words, $A_{\kappa\cdot\omega}$ realizes p at a, and we are finished. □

We will shortly prove an extension of this theorem regarding a sequence of sets, but first we need a lemma that will tell us how to handle sequences of isolated types.

Lemma 3.5.10. *Suppose T is totally transcendental. Then the following holds:*[26]

(a) Suppose $\{A_\alpha \mid \alpha < \gamma\}$ is an increasing sequence of sets, $A = \bigcup_{\alpha<\gamma} A_\alpha$, and $\{p_\alpha \mid \alpha < \gamma\}$ is an increasing sequence of types $p_\alpha \in \mathbf{S}(A_\alpha)$ such that for all $\alpha \leq \beta < \gamma$, $p_\beta \restriction A_\alpha = p_\alpha$. Then:

(i) There exists $\alpha_0 < \gamma$ and ρ such that for all $\alpha_0 \leq \alpha < \gamma$, $R^m(p_\alpha, \mathcal{L}, 2) = \rho$.

(ii) There exists a unique type $p \in S(A)$ of rank ρ such that for all $\alpha < \gamma$, $p \restriction A_\alpha = p_\alpha$.

(b) If $\{A_\alpha \mid \alpha < \gamma\}$ is any increasing sequence of sets with p isolated in $\mathbf{S}(A_0)$, then there exists an increasing sequence of types $\{p_\alpha \mid \alpha < \gamma\}$, $p_0 = p$, where p_α is isolated in $\mathbf{S}(A_\alpha)$ and for all $\alpha \leq \beta < \gamma$, $p_\beta \restriction A_\alpha = p_\alpha$.

Proof. We will prove part (a), and then use it to prove part (b). If $\gamma = \alpha + 1$ a successor ordinal then part (i) is trivial, and for part (ii) just take $p = p_\alpha$ (uniqueness

[26] Cf. Lemma 4.4 in [8].

in part (ii) from Lemma 3.4.8). Otherwise, for (i), we note that by Lemma 3.4.3, the ranks of the p_α will be monotone decreasing. So, it must be constant past some α_0, say at rank η. For (ii), we have no other choice but $p = \bigcup_{\alpha<\gamma} p_\alpha$. To see p has rank η, note that there exists a finite subtype $p' \subseteq p$ of the same rank, thus there exists $\beta \geq \alpha_0$ such that $p' \subseteq p_\beta \subseteq p$, whence all three must have rank η.

Now we show part (b), constructing the sequence of types p_α by recursion on α. (We assume $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ at limit ordinals δ. If not, interpolate these sets into the given sequence.) Set $p_0 = p$. Having defined up to p_α, we take $p_{\alpha+1}$ as the type of least transcendence rank in $\{q \in S(A_{\alpha+1}) \mid p_\alpha \subseteq q\}$, shown to be isolated by a similar argument as in the proof of Lemma 3.5.6.

For the limit step, we use (a)(ii) to get p_δ. It remains to show that p_δ is isolated. As before, it suffices to show it has least rank in $\{q \in \mathbf{S}(A_\delta) \mid p_{\alpha_0} \subseteq p\}$. Suppose some q in this set had rank $< \eta = R^m(p_\delta, \mathcal{L}, 2)$. Choose $\beta \geq \alpha_0$ such that $q \restriction A_\beta \neq p_\delta \restriction A_\beta (= p_\beta)$, with α_0 from (a)(i) where the rank reaches its minimum at η. Then since $p_{\alpha_0} \subseteq q \restriction A_\beta$, we have $R^m(q \restriction A_\beta, \mathcal{L}, 2) \leq R^m(p_{\alpha_0}, \mathcal{L}, 2) = \eta$. If equality held, then p_{α_0} could be extended either to $q \restriction A_\beta$ or p_β, two distinct types of rank η, and then $R^m(p_{\alpha_0}, \mathcal{L}, 2) \geq \eta+1$. To avoid this contradiction, we must have $R^m(q \restriction A_\beta, \mathcal{L}, 2) < \eta$, but this contradicts leastness of rank when we chose p_β. We conclude that p_δ has least rank extending p_{α_0}, and hence it is isolated. □

Theorem 3.5.11. *Suppose T is totally transcendental, and $\{A_\alpha \mid \alpha < \gamma\}$ is an increasing continuous sequence of sets. Then there exists an increasing continuous sequence of models*[27] *$\{\mathfrak{B}_\alpha \mid \alpha < \gamma\}$ of T with $\mathfrak{B}_\alpha$ prime over A_α for all $\alpha < \gamma$.*[28]

Proof. We define a continuous sequence of models $\{\mathfrak{B}_\alpha \mid \alpha < \gamma\}$ such that each $\mathfrak{B}_\alpha \supseteq A_\alpha$ and if $\alpha = \beta+1$, $\alpha' \geq \alpha$, and $\mathfrak{B}' \models T$, then any monomorphism $f : A_{\alpha'} \cup \mathfrak{B}_\beta \to \mathfrak{B}'$ extends to a monomorphism $g : A_{\alpha'} \cup \mathfrak{B}_\alpha \to \mathfrak{B}'$. (Here we allow $\beta = -1$ and stipulate

[27]By this, we mean that the universes of the models form an increasing continuous sequence as in Definition 2.3.11.

[28]Cf. Theorem 4.5 in [8].

$\mathfrak{B}_{-1} = \emptyset$.) Once completed, induction will let us conclude that $\mathfrak{B}_\alpha$ is prime over A_α for all α. To help us with the proof, we let $A = \bigcup_{\alpha<\gamma} A_\alpha$ and as they are defined, $C_\alpha = A \cup \mathfrak{B}_\alpha$.

For the base case, use Theorem 3.5.9 to get $\mathfrak{B}_0$ prime over A_0. At limit ordinals δ, let $\mathfrak{B}_\delta = \bigcup_{\alpha<\delta} \mathfrak{B}_\alpha$. For the successor step $\alpha = \beta + 1$, we first let $\{p_\xi \mid \xi \leq \kappa\}$ list the isolated types of $\mathbf{S}(A_\alpha \cup \mathfrak{B}_\beta)$. By Lemma 3.5.10, there exists a sequence of types $\{p_{0,\eta} \mid \alpha \leq \eta < \gamma\}$ such that $p_{0,\alpha} = p_0$, and each $p_{0,\eta}$ is isolated in $\mathbf{S}(A_\eta \cup \mathfrak{B}_\beta)$. Let $q_0 = \bigcup_{\alpha\leq\eta<\gamma} p_{0,\eta}$. If C_β has an element realizing q_0, denote it a_0. Otherwise, adjoin an element a_0, (for the construction of $\mathfrak{B}_\alpha$). We now iterate this process $\kappa \cdot \omega$ times, defining a_ξ, $\xi < \kappa \cdot \omega$, such that $A_\alpha \cup \mathfrak{B}_\beta \cup \{a_\xi \mid \xi \leq \kappa \cdot \omega\}$ is a model of T (by the argument at the end of Lemma 3.5.9) and for each $\alpha \leq \eta < \gamma$, a_ξ realizes an isolated point of $\mathbf{S}(A_\eta \cup \mathfrak{B}_\beta \cup \{a_{\xi'} \mid \xi' < \xi\})$. We define $\mathfrak{B}_\alpha = \mathfrak{B}_\beta \cup \{a_\xi \mid \xi < \kappa \cdot \omega\}$. Again by the proof of Lemma 3.5.9, this shows that the monomorphism extension condition holds for α. □

Now we have the machinery necessary to slowly grow a model and not realize new types. First, we show how to go from an arbitrary uncountable model that is not saturated, and produce a countable model missing a single type.

Lemma 3.5.12. *Suppose T is totally transcendental and $\mathfrak{B}$ is an uncountable model of T that is not saturated. Then there exists a countable model $\mathfrak{A}$ of T, $\mathfrak{A} \subseteq \mathfrak{B}$, with a subset $A' \subseteq \mathfrak{A}$ such that:*

(i) There exists an infinite set Y indiscernible over A' in $\mathfrak{A}$, and

(ii) there exists a type $q \in \mathbf{S}(A')$ not realized in $\mathfrak{A}$.[29]

Proof. Let $C \subseteq B$, $|C| < |B|$, be a parameter set such that some $p \in \mathbf{S}(C)$ is not realized in $\mathfrak{B}$. By Theorem 3.3.11 and Theorem 3.3.17, there exists a countably infinite *set* Y indiscernible over C in $\mathfrak{B}$. Now use the DLST to find a countable

[29] Cf. Lemma 5.3 in [8].

submodel $\mathfrak{A}_0 \subseteq \mathfrak{B}$ containing Y. We note that for every $a \in A_0$, $\mathrm{tp}(a, C) \neq p$, lest p be realized in $\mathfrak{B}$. Thus for each such a, there is a formula $\psi_a \in \mathrm{tp}(a, C)$, $\neg\psi_a \in p$.

Let $A'_1 = \bigcup_{a \in A_0} \mathrm{param}(\psi_a)$, a countable subset of C. Then no $a \in A_0$ realizes $p \restriction A'_1$ either. Use the DLST again to obtain a countable submodel $\mathfrak{A}_1 \subseteq \mathfrak{B}$ containing $A_0 \cup A'_1$. Iterate this process ω times, yielding a countable sequence of models $\mathfrak{A}_n$ containing sets A'_n such that $A'_n \subseteq A_n \cap C$ and no $a \in A_n$ realizes $p \restriction A'_{n+1}$. We let $A = \bigcup_{n<\omega} A_n$ and $A' = \bigcup_{n<\omega} A'_n$. Then Y is an indiscernible set over A' in $\mathfrak{A}$ (since $A' \subseteq C$), and the type $q = p \restriction A'$ is unrealized in $\mathfrak{A}$. □

We can finally produce our second model of power κ.

Proof of Theorem 3.5.3. Use Lemma 3.5.12 to obtain a countable model $\mathfrak{A}$, a set A' and indiscernible set Y over A' in $\mathfrak{A}$, and $q \in \mathbf{S}(A')$ not realized in $\mathfrak{A}$. We first claim that there exists a set $Y_\kappa \supseteq A' \cup Y$, $|Y_\kappa| = \kappa$, such that $Y_\kappa - A'$ is indiscernible over A'. To see this, note that the existence of such a set is equivalent to the consistency of a particular infinite set of sentences expressing indiscernibility of any two ordered n-tuples of κ constant symbols. The set Y shows this theory to be consistent by compactness. Let $\{y_\alpha \mid \alpha < \kappa\}$ well-order $Y_\kappa - A'$. Let $A_\alpha = A' \cup \{y_\beta \mid \beta < \alpha\}$ and apply Theorem 3.5.11 to get an increasing continuous sequence $\{\mathfrak{B}_\alpha \mid \alpha \leq \kappa\}$ of models of T, $\mathfrak{B}_\alpha$ prime over A_α for $\alpha < \kappa$. By construction, $|\mathfrak{B}_\kappa| = \kappa$, (or else we could just use the DLST).

We claim that q is unrealized in $\mathfrak{B}_\kappa$. It suffices to show by induction on α that q is unrealized in all $\mathfrak{B}_\alpha$. For $n < \omega$, we can map A_n by monomorphism into $\mathfrak{A}$ (supposing that the first ω elements of Y_κ lie in Y) and then extend to a monomorphism of $\mathfrak{B}_n$ into $\mathfrak{A}$. These are models, so this is an elementary mapping which shows $\mathfrak{B}_n$ does not realize q.

The limit step is trivial. Our last case is $\alpha = \beta + 1 > \omega$, where we suppose $\mathfrak{B}_\beta$ does not realize q. Since $Y_\kappa - A'$ is an indiscernible set over A', we can map A_α by isomorphism onto A_β, leaving A' fixed. This yields a monomorphism $\mathfrak{B}_\alpha$ into $\mathfrak{B}_\beta$

that fixes A'. We assumed that $\mathfrak{B}_\beta$ does not realize q, and so neither can $\mathfrak{B}_\alpha$. The induction is complete. □

Proof of Theorem 3.5.1. If there exists an uncountable model that was not saturated, we would have by Theorem 3.5.3 a model of power κ which is not $\aleph_1$-saturated. We know by Theorem 3.5.2 that there exists a model of power κ which *is* $\aleph_1$-saturated, hence these cannot be isomorphic. This contradicts categoricity at power κ, so we conclude that every uncountable model is saturated. □

Proof of Morley's Theorem 3.1.6. By Theorem 2.4.21, any two saturated models of the same power are isomorphic. Since every uncountable model is saturated, we conclude that there is only one model up to isomorphism in any given uncountable power. □

Going further, Shelah extended Morley's theorem to theories of arbitrary size.

Theorem 3.5.13. *Suppose T is a complete theory, $|T|$ arbitrary. Then T is categorical in some $\lambda > |T|$ if and only if T is categorical in every $\lambda > |T|$.*

We will not pursue the proof in this thesis. It can be found in [9].

Chapter 4

Order and Instability

4.1 The Order Property

The primary result of this section is the following.

Theorem 4.1.1. *Suppose $\varphi(\bar{v}, \bar{w})$ is a formula, $\ell(\bar{v}) = m$, $\ell(\bar{w}) = n$, $\mathfrak{A}$ a model and $X \subseteq A$ such that $|\mathbf{S}^m_\varphi(X, \mathfrak{A})| > |X| + \aleph_0$. Then:*

(i) Letting $\theta(\bar{v}_1, \bar{v}_2, \bar{v}_3; \bar{w}_1, \bar{w}_2, \bar{w}_3) = [\varphi(\bar{v}_1; \bar{w}_2) \leftrightarrow \varphi(\bar{v}_1; \bar{w}_3)], \ell(v_i) = \ell(w_i)$ for $i = 1, 2, 3$, there exist in A sequences $\bar{a}_i$, $i < \omega$ such that $\mathfrak{A} \models \theta[\bar{a}_i; \bar{a}_j]$ if and only if $i < j$.

(ii) There exist in A sequences $\bar{a}_n, \bar{b}_n$, $n < \omega$, and $t \in \{0, 1\}$ such that $\mathfrak{A} \models \varphi[\bar{b}_n; \bar{a}_k]^t$ if and only if $n < k$.[1]

Before jumping into the proof, let's give a few examples regarding the punchline (ii), (we use (i) to prove it).

Example 4.1.2. Consider $\mathbb{R}$ as a dense linear order, and let $\varphi(v, w)$ be the formula $v < w$. We know this formula to be unstable in $\mathbb{R}$ as it defines uncountably many

[1] Cf. Theorem I.2.10 in [9], where it is stated in greater generality. We instead prove a simpler version that will suffice for our purposes, which corresponds to $\lambda = \chi = \aleph_0$, λ and χ being cardinal parameters in the more general version.

Dedekind cuts over $\mathbb{Q}$. The obvious choices for the a_i and b_i are $a_i = b_i = i$, but in fact there are many such choices. In fact, we can choose a_i arbitrary such that $i < j$ implies $a_i < a_j$, and then let b_i realize the appropriate cuts. This example may seem trivial, but it reveals the purest form of the above theorem.

Example 4.1.3. For a more complicated example, consider the edge relation $E(v, w)$ in the theory of random graphs.[2] If G is such a graph, say saturated over a countable subset H, the edge relation we know to be unstable in G; every collection of edge relations and non-edge relations to a finite subset of H will be realized by definition, so all types of the form $\{E(v, h_n)^{\eta(n)} \mid n < \omega\}$ for some $\eta \in {}^{\omega}2$ will be consistent. To satisfy condition (ii) of the above theorem, we let a_i enumerate H, and then choose the b_i to realize the appropriate types. Figure 4.1 below illustrates this.

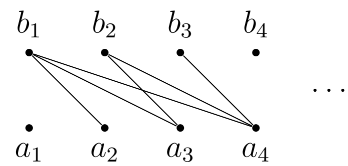

Figure 4.1: Order in the theory of random graphs

Note that we have more power in G than we do in $\mathbb{R}$; indeed, our types do not need to respect the order on H (there is none) and so we can have, for example, an edge to every h_{2k} and no h_{2k+1}. This is clearly impossible in $\mathbb{R}$ if we take $h_i = i$ and the formula $v < w$.

Proof of Theorem 4.1.1. We may assume that $|X| \geq \aleph_0$; if not, just add elements, and the instability condition will remain. We start by proving (i), and use it to prove (ii). Let $\kappa = |X|$. For each $i < \kappa^+$, choose $\bar{c}_i \in A$ that realize distinct types $\mathrm{tp}_\varphi(\bar{c}_i, X, \mathfrak{A}) \in \mathbf{S}_\varphi(X, \mathfrak{A})$. Let $\psi(\bar{v}; \bar{w}) = \varphi(\bar{w}; \bar{v})$.

[2]The theory of random graphs states that given any two finite subsets of vertices, there exists a vertex with edges to every vertex in the first set and to none in the second set. The theory is named such because if $\mathbb{N}$ is your vertex set and an edge between each pair of vertices $n < m$ is decided by a coin flip, the result will be a random graph with probability 1.

Now we define an increasing sequence of sets $X_n \subseteq A$ by recursion on $n < \omega$ such that for every $p \in \mathbf{S}^m_\varphi(X_n, \mathfrak{A}) \cup \mathbf{S}^n_\psi(X_n, \mathfrak{A})$ and finite $Y \subseteq X_n$, $p \restriction Y$ is realized in X_{n+1}. Also we require that $|X_n| = |X|$. To verify this is possible, we just need to count the number of finite m-types over X_n. Indeed, there are only $|X_n| = |X|$ finite subsets of X_n, and only finitely many types definable thereover with only the formulas ϕ or ψ. This culminates in $|X|$ many finite types $p \restriction Y$ to satisfy, so we can define X_{n+1} such that $|X_{n+1}| = |X|$. Let $X_\omega = \bigcup_{n<\omega} X_n$; then $|X_\omega| = \kappa$.

Now we prove

$$\begin{array}{l}\text{There exists } i < \kappa^+ \text{ such that for every } n < \omega \text{ and finite } Y \subseteq X_n, \\ \mathrm{tp}_\varphi(\bar{c}_i, X_{n+1}, \mathfrak{A}) \ (\psi, \phi)\text{-splits over } Y.\end{array} \tag{$\star$}$$

Suppose not. Then for every $i < \kappa^+$, there exists an n and finite $Y \subseteq X_n$ in violation of ($\star$). Since κ^+ is regular, there exists X_n that is chosen κ^+ times. We restrict consideration to these κ^+ indices. Moreover, since there are only $|X| < \kappa^+$ subsets of $|X_n|$, some finite $Y \subseteq X_n$ is chosen κ^+ times, and we restrict consideration again. Let $Z \subseteq X_{n+1}$ realize every $p \in \mathbf{S}^n_\psi(Y, \mathfrak{A})$, ($Y$ is already finite). Since $|\mathbf{S}_\varphi(Z, \mathfrak{A})| \leq |X| < \kappa^+$, there are κ^+ indices such that $\mathrm{tp}_\varphi(\bar{c}_i, Z, \mathfrak{A}) = p$ where p some fixed type. To recap, there are κ^+ indices i for which for which $\mathrm{tp}_\varphi(\bar{c}_i, X_n, \mathfrak{A})$ (ψ, φ)-splits over Y, and these types are all the same restricted to Z. In particular, this is the case for two indices, without loss of generality we suppose they are 0 and 1.

The $\bar{c}_i$ realize distinct types, so some $\bar{a} \in X$ gives $\mathfrak{A} \models (\varphi[\bar{c}_0; \bar{a}] \leftrightarrow \neg\varphi[\bar{c}_1; \bar{a}])$. Choose $\bar{a}' \in Z$ such that $\mathrm{tp}_\psi(\bar{a}, Y, \mathfrak{A}) = \mathrm{tp}_\psi(\bar{a}', Y, \mathfrak{A})$, (possible by definition). Now, we have assumed that $\mathrm{tp}_\varphi(\bar{c}_l, X_{n+1}, \mathfrak{A})$ does not (ψ, φ)-split over Y for $l = 0, 1$, so it must be the case that $\mathfrak{A} \models (\varphi[\bar{c}_l; \bar{a}] \leftrightarrow \varphi[\bar{c}_l; \bar{a}'])$. Then we have $\mathfrak{A} \models (\varphi[\bar{c}_0; \bar{a}'] \leftrightarrow \neg\varphi[\bar{c}_1; \bar{a}'])$, but this contradicts $\mathrm{tp}_\varphi(\bar{c}_l, Z, \mathfrak{A}) = p$, since this would imply that the same type contains both formulas. So ($\star$) must hold.

Now we define $\bar{d}_n, \bar{e}_n, \bar{f}_n \in X_{2n+2}$ by recursion on $n < \omega$. Suppose this is done for all $k < n$, (this also covers the base case). Let $Y_n = \bigcup_{k<n}(\bar{d}_k \cup \bar{e}_k \cup \bar{f}_k)$, so by $(\star)$ $\mathrm{tp}_\varphi(\bar{c}_i, X_{2n+1}, \mathfrak{A})$ (ψ, φ)-splits over Y_n, (since $k < n$ implies $2k + 2 \leq 2n$, so $Y_n \subseteq X_{2n}$). Choose $\bar{d}_n, \bar{e}_n \in X_{2k+1}$ such that $\mathrm{tp}_\psi(\bar{d}_n, Y_n, \mathfrak{A}) = \mathrm{tp}_\psi(\bar{e}_n, Y_n, \mathfrak{A})$ but $\mathfrak{A} \models \varphi[\bar{c}_i; \bar{d}_n] \wedge \neg\varphi[\bar{c}_i; \bar{e}_n]$. Choose $\bar{f}_n \in X_{2k+2}$ that realizes $\mathrm{tp}_\varphi(\bar{c}_i, Y_n \cup \bar{d}_k \cup \bar{e}_n, \mathfrak{A})$.

Let $\bar{a}_n = \bar{f}_n{}^\frown\bar{e}_n{}^\frown\bar{d}_n$. We claim that this satisfies (i). Indeed, if $k \leq n$, then $\mathfrak{A} \models \varphi[\bar{c}_i; \bar{d}_k] \wedge \neg\varphi[\bar{c}_i; \bar{e}_k]$, $\mathfrak{A} \models \varphi[\bar{c}_i; \bar{d}_k] \leftrightarrow \varphi[\bar{f}_n; \bar{d}_k]$, and $\mathfrak{A} \models \varphi[\bar{c}_i; \bar{e}_k] \leftrightarrow \varphi[\bar{f}_n; \bar{e}_k]$, hence we have $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{d}_k] \not\leftrightarrow \varphi[\bar{f}_n; \bar{e}_k]$, (in fact, $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{d}_k] \wedge \neg\varphi[\bar{f}_n; \bar{e}_k]$).

Now suppose $n < k$. We have $\mathrm{tp}_\psi(\bar{d}_k, Y_k, \mathfrak{A}) = \mathrm{tp}_\psi(\bar{e}_k, Y_k, \mathfrak{A})$, but also $\bar{f}_n \in Y_k$, so $\mathfrak{A} \models \psi[\bar{d}_k; \bar{f}_n] \leftrightarrow \psi[\bar{e}_k; \bar{f}_n]$ i.e. $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{d}_k] \leftrightarrow \varphi[\bar{f}_n; \bar{e}_k]$. This proves (i).

For (ii) we apply Ramsey's Theorem. We color pairs n, m of natural numbers as follows: $\mathrm{Col}(n, m) = 0$ if $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{d}_k]$, $n < k$, and $\mathrm{Col}(n, m) = 1$ otherwise. Then there exists an infinite monochromatic subset of ω, which by renaming, we assume is all of ω. If the color is 1, then we have $\mathfrak{A} \models \neg\varphi[\bar{f}_n; \bar{d}_k]$ for $n < k$, and if $k \leq n$ we still have $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{d}_k]$, so we let $\bar{a}_k = \bar{d}_k$ and $\bar{b}_k = \bar{f}_k$, $t = 1$. Otherwise the color is 0, so for $n < l$ we have $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{d}_k]$, which is equivalent to $\mathfrak{A} \models \varphi[\bar{f}_n; \bar{e}_k]$ by (i). If $k \leq n$ then $\mathfrak{A} \models \neg\varphi[\bar{f}_n; \bar{e}_k]$, so we let $\bar{a}_k = \bar{e}_k$ and $\bar{b}_k = \bar{f}_k$, $t = 0$. $\square$

From this, we derive the following corollary.

Corollary 4.1.4. *Suppose φ (i.e. the theory $\{\varphi\}$) is unstable in some infinite power λ. Then there exists $\bar{a}_n$, $n < \omega$, such that for every $k < \omega$, the type*

$$p_k = \{\varphi(\bar{v}; \bar{a}_n)^{\mathit{if}\ k \leq n} \mid n < \omega\}$$

is consistent.[3] *In this case, we say φ satisfies the order property.*[4]

[3] Cf. Notation 2.2.2.
[4] Cf. Lemma II.2.3 in [9].

Proof. We actually show consistency of the single type

$$p = \{\varphi(\bar{v}_k; \bar{w}_n)^{\text{if } k \leq n} \mid n, k < \omega\},$$

where the $\bar{v}_k$ and $\bar{w}_n$ are all distinct variables. (Really we should be working with $p \cup T$, but we omit the ambient theory to avoid clutter.) Then we let $\bar{a}_n$ realize $\bar{w}_n$ and split the type into $\aleph_0$ subtypes for each variable $\bar{v}_k$. This way, there is no ambiguity that the same $\bar{a}_n$ work for all k.[5]

By λ-instability, we have $|\mathbf{S}^m_\varphi(X)| > \lambda \geq |X| \geq \aleph_0$ for some X, and so we apply Theorem 4.1.1(ii). If $t = 0$, the theorem applies directly, with $\bar{b}_k$ realizing $\bar{v}_k$ and $\bar{a}_{n+1}$ realizing $\bar{w}_n$, (the shift needed because we want $\varphi(\bar{v}_k, \bar{w}_k)$ to be true).

If $t = 1$, then we have to use compactness. Consider some finite subset of p, which we can expand to be of the form

$$\left\{\begin{array}{l}
\neg\varphi(\bar{v}_0; \bar{w}_0), \varphi(\bar{v}_0; \bar{w}_1), \ldots, \varphi(\bar{v}_0; \bar{w}_m) \\
\neg\varphi(\bar{v}_1; \bar{w}_0), \neg\varphi(\bar{v}_1; \bar{w}_1), \varphi(\bar{v}_1; \bar{w}_2), \ldots, \varphi(\bar{v}_1; \bar{w}_m) \\
\vdots \\
\neg\varphi(\bar{v}_k; \bar{w}_0), \ldots, \neg\varphi(\bar{v}_k; \bar{w}_k), \varphi(\bar{v}_k; \bar{w}_{k+1}), \ldots, \varphi(\bar{v}_k; \bar{w}_m)
\end{array}\right\}.$$

Now we just flip the picture[6], and let $\bar{a}_{m-n}$ realize $\bar{w}_n$, and $\bar{b}_{m-k}$ realize $\bar{v}_k$. Then $\mathfrak{A} \models \varphi(\bar{b}_{m-k}; \bar{a}_{m-n})$ if and only if $m - n \leq m - k$ if and only if $k \leq n$ as desired. By compactness, the type p is consistent. □

We will later prove the converse, and in fact, equivalence of many conditions to φ being unstable.

[5] Ordinarily we formalize parameters as constant symbols with a canonical interpretation. Here, it is paramount that these parameters are all interpreted in the same way in the same model, (a submodel of the monster model). We bring this up because the proof given in [9] seems to omit this subtlety.

[6] A finite initial segment of Figure 4.1 is one representation.

4.2 More on Transcendence Rank

This section includes more results regarding the rank function R^m. Perhaps not of independent interest, we will use these results later on. The reader may skip this section and refer back to it as needed.

Lemma 4.2.1. *Suppose f is an automorphism of $\mathfrak{M}$. Then for every p, Δ, and λ we have $R^m(p, \Delta, \lambda) = R^m(f(p), \Delta, \lambda)$, where $f(p) = \{\varphi(\bar{v}; f(\bar{a})) \mid \varphi(\bar{v}; \bar{a}) \in p\}$.*[7]

Proof. We show that $R^m(p, \Delta, \lambda) \geq \alpha$ if and only if $R^m(f(p), \Delta, \lambda) \geq \alpha$ by induction on α. It suffices to prove just the forward direction, since f^{-1} is also an automorphism.

The zero case is easy, and the limit case is trivial. For the successor step, unpack the definition; nothing clever is required. $\square$

Lemma 4.2.2. *There is λ_T depending only on T such that for every type p and formula set Δ, if $\lambda \geq \lambda_T$ then $R^m(p, \Delta, \lambda) = R^m(p, \Delta, \infty)$.*[8]

Proof. The core of this proof is a simple counting argument. For a single formula $\theta(\bar{v}; \bar{a})$ and fixed Δ, $R^m(\theta, \Delta, \lambda)$ is antimonotonic with λ. Thus, there is some $\lambda = \lambda(\theta, \bar{a}, \Delta)$ such that for all $\mu \geq \lambda$, $R^m(\theta(\bar{v}; \bar{a}), \Delta, \mu) = R^m(\theta(\bar{v}; \bar{a}), \Delta, \lambda)$. By Lemma 4.2.1, this rank is independent of $\mathrm{tp}(\bar{a}, \emptyset)$. So λ actually depends on the triple $(\theta, \mathrm{tp}(\bar{a}, \emptyset), \Delta)$, of which there are at most $2^{|T|}$ (a *set* of choices). Let $\lambda_T = \sup\{\lambda(\theta, \bar{a}, \Delta) \mid \theta, \bar{a}, \Delta\} < \infty$.

We verify that this λ_T works. Let p be any type, and $\lambda \geq \lambda_T$. We first show that $R^m(p, \Delta, \lambda) = R^m(p, \Delta, \lambda_T)$. Let $p' \subseteq p$ be a finite subtype such that $R^m(p', \Delta, \lambda) = R^m(p, \Delta, \lambda)$. Since $\bigwedge p' \equiv p'$, we can replace p' with a single formula $\theta = \bigwedge p'$. Now we see that

$$R^m(p, \Delta, \lambda) = R^m(\theta, \Delta, \lambda) = R^m(\theta, \Delta, \lambda_T) \geq R^m(p, \Delta, \lambda_T) \geq R^m(p, \Delta, \lambda),$$

[7]Cf. Exercise II.1.1 in [9].
[8]Cf. Lemma II.1.11 in [9].

whence equality holds throughout.

Finally, we show by induction on α that if $R^m(p, \Delta, \lambda) \geq \alpha$ then $R^m(p, \Delta, \infty) \geq \alpha$ for any $\lambda \geq \lambda_T$. It follows that $R^m(p, \Delta, \lambda) \leq R^m(p, \Delta, \infty)$, but by antimonotonicity, equality holds. The zero and limit cases are trivial. Suppose $R^m(p, \Delta, \lambda) \geq \alpha + 1$, and let μ be a cardinal. In particular, $R^m(p, \Delta, \infty) \geq \alpha$ from the inductive assumption. If $\mu \leq \lambda$, then $R^m(p, \Delta, \lambda) \leq R^m(p, \Delta, \mu)$. Otherwise, if $\mu \geq \lambda (\geq \lambda_T)$, then $R^m(p, \Delta, \lambda) = R^m(p, \Delta, \mu)$. So either way, $R^m(p, \Delta, \mu) \geq \alpha + 1$. Since this works for all μ, we conclude (from the definition) that $R^m(p, \Delta, \infty) \geq \alpha + 1$ as desired. $\square$

Now we prove an existence result for complete types of the same rank, (compare Lemma 3.4.8). Note that these two results do not give existence and uniqueness simultaneously, since one uses $\lambda = 2$, and the other uses $\lambda \geq \aleph_0$.

Theorem 4.2.3. *Suppose p is an m-type perhaps with parameters from X and $\lambda \geq \aleph_0$. Then there exists $q \in \mathbf{S}^m(X)$, $p \subseteq q$, such that $R^m(q, \Delta, \lambda) = R^m(p, \Delta, \lambda)$.*[9]

Remark. *For $\lambda \geq \aleph_0$, this lets us replace "explicitly contradictory" with merely "contradictory" in the definition of transcendence rank.*

For the proof, we need the following lemma.

Lemma 4.2.4. *For any type r, formulas $\psi_i = \psi_i(\bar{v}; \bar{a}_i)$, $i < n$, Δ, and $\lambda \geq \aleph_0$, we have*

$$R^m(\{\bigvee_{i<n} \psi_i\} \cup r, \Delta, \lambda) = \max_{i<n} R^m(\{\psi_i\} \cup r, \Delta, \lambda).$$[10]

Proof. By Lemma 3.4.3, we have $R^m(\{\psi_i\} \cup r, \Delta, \lambda) \leq R^m(\{\bigvee_{i<n} \psi_i\} \cup r, \Delta, \lambda)$, hence $\geq$ holds in the statement we wish to show. So it suffices to show, by induction on β, that if $R^m(\{\bigvee_{i<n} \psi_i\} \cup r, \Delta, \lambda) \geq \beta$, then $\max_{i<n} R^m(\{\psi_i\} \cup r, \Delta, \lambda) \geq \beta$.

The case $\beta = 0$ is easy, and as usual the limit case is trivial. For the successor case, suppose $R^m(\{\bigvee_{i<n} \psi_i\} \cup r, \Delta, \lambda) \geq \beta + 1$, but $\max_{i<n} R^m(\{\psi_i\} \cup r, \Delta, \lambda) = \beta$.

[9] Cf. Theorem II.1.6 in [9].
[10] Cf. Claim II.1.7 in [9].

Then for each $i < n$, there are $\mu_i < \lambda$ and finite $q_i \subseteq \{\psi_i\} \cup r$ such that no pairwise explicitly contradictory $\{r_{j,i} \mid j \leq \mu_i\}$ have $R^m(q_i \cup r_{j,i}, \Delta, \lambda) \geq \beta$ for $j \leq \mu_i$, $i < n$. Let $q = \bigcup_{i<n} q_i$ and $\mu = n \cdot (1 + \max_i \mu_i)$.[11] Since $\lambda \geq \aleph_0$ we have $\mu < \lambda$. Since $\{\psi_i\} \cup q \cup r_{j,i} \supseteq q_i \cup r_{j,i}$, we also have that no pairwise explicitly contradictory $\{r_{j,i} \mid j \leq \mu_i\}$ can have $R^m(\{\psi_i\} \cup q \cup r_{j,i}, \Delta, \lambda) \geq \beta$.

However, since $R^m(\{\bigvee_{i<n} \psi_i\} \cup r, \Delta, \lambda) \geq \beta + 1$, for the $\mu < \lambda$ and finite q above there exists pairwise explicitly contradictory $\{r_j \mid j \leq \mu\}$, such that $R^m(\{\bigvee_{i<n} \psi_i\} \cup q \cup r_j, \Delta, \lambda) \geq \beta$, $j \leq \mu$. By the induction hypothesis with $q \cup r_j$ instead of r, we have $\max_{i<n} R^m(\{\psi_i\} \cup q \cup r_j, \Delta, \lambda) \geq \beta$, $j \leq \mu$. Thus for each such j, there exists $i(j)$ such that $R^m(\{\psi_{i(j)}\} \cup q \cup r_j, \Delta, \lambda) \geq \beta$. Simply, this is a map $i : \mu \to n$. By choice of μ, there will be some index i_0 chosen for at least $\max_{i<n} \mu_i + 1$ different j. By reindexing, we may assume $i_0 = i(j)$ for all $j \leq \mu_{i_0}$, (indeed, only the number of such $r_{i,j}$ is important). But then we have $R^m(\{\psi_{i_0}\} \cup q \cup r_{j,i_0}, \Delta, \lambda) \geq \beta$ for $j \leq \mu_{i_0}$, contradicting choice of μ_{i_0} above. This finishes the successor step, and with it, the proof. $\square$

Proof of Theorem 4.2.3. Let $\Gamma = \{\neg\psi(\bar{v}; \bar{a}) \mid R^m(\psi(\bar{v}; \bar{a}), \Delta, \lambda) < \alpha, \bar{a} \in X\}$. First we claim that it is sufficient to show $p \cup \Gamma$ consistent. For if this is done, let $q \in \mathbf{S}^m(X)$ extend $p \cup \Gamma$ to a complete type. Then $R^m(q, \Delta, \lambda) \leq R^m(p, \Delta, \lambda) = \alpha$. There is no finite $q' \subseteq q$ of rank $< \alpha$, as then $R^m(\bigwedge q', \Delta, \lambda) < \alpha$ so $\neg \bigwedge q' \in \Gamma \subseteq q$, making q inconsistent. Lest we contradict Lemma 3.4.7, we conclude that q has rank α.

So we verify that $p \cup \Gamma$ is consistent. Otherwise, there is a finite subset that is inconsistent. We may assume this consists of a finite $r \subseteq p$ of the same rank α, and $\psi_i = \psi_i(\bar{v}; \bar{a}_i)$, $i < n$, of rank $< \alpha$, (where $r \cup \{\neg\psi_i \mid i < n\}$ is inconsistent). By Lemma 2.2.5, we have $r \vdash \bigvee_{i<n} \psi_i$, so by the above lemma $\alpha \leq R^m(\{\bigvee_{i<n} \psi_i\} \cup r, \Delta, \lambda) = \max_{i<n} R^m(\{\psi_i\} \cup r, \Delta, \lambda) < \alpha$, a contradiction. Thus $p \cup \Gamma$ is consistent. $\square$

[11] So μ is just $\max_i \mu_i$ if any are infinite. Otherwise, we include the n and $1+$ for the case that all are finite.

4.3 The Unstable Formula Theorem

Before coming to the main theorem of this section, we need one more definition.

Notation 4.3.1. Given a formula $\varphi(\bar{v}, \bar{w})$, $\ell(\bar{v}) = m$, and an ordinal α, we define

$$\Gamma(\varphi, m, \alpha) = \{\varphi(\bar{v}_\eta; \bar{w}_{\eta\restriction\beta})^{\eta(\beta)} \mid \eta \in {}^{\alpha}2, \beta < \alpha\}.$$

We frequently omit m.

This is a clear generalization of the tree arguments we've seen earlier. When restricted to a single variable $\bar{v}_\eta$, we will get a unique type for each η.

Theorem 4.3.2 (The Unstable Formula Theorem)**.** *Suppose $\varphi(\bar{v}; \bar{w})$ is a formula, $\ell(\bar{v}) = m$. Then the following are equivalent, (relative to a given theory T).*[12]

(i) φ *is unstable in all infinite powers* λ.

(ii) φ *is unstable in some infinite power* λ.

(iii) φ *has the order property, (see Corollary 4.1.4).*

(iv) For every $n < \omega$, *the set* $\Gamma(\varphi, n)$ *is consistent.*

(v) $\Gamma(\varphi, \alpha)$ *is consistent for every ordinal* α.

(vi) $R^m(\bar{v} = \bar{v}, \varphi, \infty) = \infty$

(vii) $R^m(\bar{v} = \bar{v}, \varphi, 2) \geq \omega$

Following [9], we prove the Unstable Formula Theorem in a series of lemmas.

Lemma 4.3.3. *(i)⇒(ii)⇒(iii)⇒(iv)⇒(v)*[13]

[12]Cf. Theorem II.2.2 [9], wherein there are two more conditions regarding "definability" of a type. We will not use them in this thesis, so we omit them. On that note, the proof given in [9] is split into multiple lemmas, similar to what we do here.

[13]Cf. Lemmas II.2.3 through II.2.5 in [9].

Proof. The first implication is trivial, and the second is Corollary 4.1.4.

For the implication (iii)⇒(iv), we will use the compactness theorem. First we define a linear order on ${}^{\omega}2$. If $\eta, \nu \in {}^{<\omega}2$ and $\eta \restriction k = \nu \restriction k$, $\eta(k) = 0$, $\nu(k) = 1$, then $\eta < \nu$. If $\eta \subset \nu$ and $\nu(\ell(\eta)) = 1$, then $\eta < \nu$. Otherwise if $\nu(\ell(\eta)) = 0$, $\nu < \eta$. This is a linear order, as a minute or two of concentration will verify. Now, we consider a finite subset of $\Gamma(\varphi, n)$, expanded to be of the form

$$\{\varphi(\bar{v}_\eta; \bar{w}_\nu)^{\text{if } \eta<\nu} \mid \ell(\eta) = m, \ell(\nu) < m\}.$$

This will include all formulas of the form $\varphi(\bar{v}_\eta; \bar{w}_{\eta\restriction k})$ for which $\eta(k) = 0$, and also formulas $\neg\varphi(\bar{v}_\eta; \bar{w}_{\eta\restriction k})$ for which $\eta(k) = 1$. It also has a bunch of other stuff, but it doesn't matter. The order property tells us this set is consistent, so by compactness, so is $\Gamma(\varphi, n)$, (for all $n < \omega$).

Finally, the implication (iv)⇒(v) is just an application of compactness. $\square$

Lemma 4.3.4. *(v)⇒(i), (completing the equivalence proof of the first five properties of the Unstable Formula Theorem).*[14]

Proof. Let λ be an infinite cardinal, and let μ be least such that $2^\mu > \lambda$, (in particular, $\mu \leq \lambda$). Let $\mathfrak{A} \models \Gamma(\varphi, \mu)$ with $\bar{a}_\nu$ realizing $\bar{w}_\nu$ and $\bar{c}_\eta$ realizing $\bar{v}_\eta$. We take as our parameter set $X = \bigcup\{\bar{a}_\nu \mid \ell(\nu) < \mu\}$. We have $|X| \leq |{}^{<\mu}2| \cdot \ell(\bar{w}) \leq \mu \cdot \lambda = \lambda$, (since each $|{}^{\alpha}2| \leq \lambda$ for $\alpha < \mu$, and $\ell(\bar{w})$ is finite). For each $\eta \in {}^{\mu}2$, let $p_\eta = \mathrm{tp}_\varphi(\bar{c}_\eta, X)$. We claim that if $\eta \neq \nu$, then $p_\eta \neq p_\nu$, yielding at least $2^\mu > \lambda$ types over $\leq \lambda$ parameters X. Indeed, if $\eta(\alpha) = 0$ and $\nu(\alpha) = 1$ is the first point of disagreement, then p_η will contain the formula $\varphi(\bar{v}; \bar{a}_{\eta\restriction\alpha})$ whereas p_ν will contain the formula $\neg\varphi(\bar{v}; \bar{a}_{\eta\restriction\alpha})$. $\square$

Now we turn to rank considerations.

Lemma 4.3.5. *(v)⇒(vi)⇒(vii)*[15]

[14] Cf. Lemma II.2.6 in [9].
[15] Cf. Lemmas II.2.7 and II.2.8 in [9].

Proof. The implication (vi)⇒(vii) is trivial; indeed, R^m is antimonotonic with λ, and $\infty > \omega$.

Now we do (v)⇒(vi). We show for arbitrary λ that $R^m(\bar{v} = \bar{v}, \varphi, \lambda) = \infty$. Then by Lemma 4.2.2, we will conclude that $R^m(\bar{v} = \bar{v}, \varphi, \infty) = \infty$. To verify this, it suffices to show $R^m(p, \varphi, \lambda) = \infty$ for some type p, by Lemma 3.4.3.

We are assuming that $\Gamma(\varphi, \alpha)$ is consistent for all α. So, for each α, let $\bar{a}^\alpha_\rho$ realize $\bar{w}_\rho$ and $\bar{c}^\alpha_\eta$ realize $\bar{v}_\eta$ where $\rho \in {}^{<\alpha}2$ and $\eta \in {}^{\alpha}2$. Then for each $\eta \in {}^{<\lambda}2$, $p_\eta = \{\varphi(\bar{v}; \bar{a}^\lambda_{\eta\restriction\beta})^{\eta(\beta)} \mid \beta < \ell(\eta)\}$ is a consistent φ-m-type (as a subtype of $\Gamma(\varphi, \lambda)$). Let p_ν have minimal rank among the p_η, say with $R^m(p_\nu, \varphi, \lambda) \geq \alpha$.

We show that $R^m(p_\nu, \varphi, \lambda) \geq \alpha + 1$, so since α was arbitrary, we conclude it has rank ∞. For each $\gamma < \lambda$, let $\bar{0}_\gamma$ be a sequence of zeroes of length γ. Consider the types $p^\gamma = p_{\nu^\frown \bar{0}_\gamma^\frown 1}$, (note that $\ell(\nu) + \gamma + 1 < \lambda$ as ordinals). These types are explicitly contradictory (if $\gamma < \gamma'$, then p^γ will contain a negated formula not negated in $p^{\gamma'}$) and extend p_ν. Also, by minimality of the rank of p_ν, they must have rank $\geq \alpha$. So by definition, $R^m(p_\nu, \varphi, \lambda) \geq \alpha + 1$. This works at all α, so we conclude that $R^m(p_\nu, \varphi, \lambda) = \infty$ as desired. □

We now generalize our notion of $\Gamma(\varphi, \alpha)$ slightly. (This is not needed to complete the proof of the Unstable Formula Theorem, but we will use it later on.)

Notation 4.3.6. For $n < \omega$, type p , and formula φ we define

$$\Gamma_p(\varphi, n) = \{\psi(\bar{v}_\eta; \bar{a}) \mid \psi(\bar{v}; \bar{a}) \in p, \eta \in {}^n2\} \cup \{\varphi(\bar{v}_\eta; \bar{w}_{\eta\restriction k})^{\eta(k)} \mid \eta \in {}^n2, k < n\}$$

Lemma 4.3.7. *Given an m-type p, $R^m(p, \varphi, 2) \geq n$ if and only if $\Gamma_p(\varphi, n)$ is consistent.*[16]

Proof. First we show (⇒). By compactness, it suffices to prove $\Gamma_p(\varphi, n)$ consistent for all finite p. We define $\bar{a}_\rho$ for each $\rho \in {}^{<k}2$ by recursion on $k \leq n$ such that for

[16]Cf. Lemma II.2.9 in [9].

all $\eta \in {}^k2$, $R^m(p_\eta, \varphi, 2) \geq n - k$ where $p_\eta = p \cup \{\varphi(\bar{v}; \bar{a}_{\eta\restriction j})^{\eta(j)} \mid j < k\}$. Then for any $\eta \in {}^n2$, p_η will be consistent (rank ≥ 0), and if $\bar{c}_\eta$ realizes p_η, then $\bar{c}_\eta$ and $\bar{a}_\rho$ will realize $\Gamma_p(\varphi, n)$.

At $k = 0$, we set $p_\emptyset = p$, so by assumption $R^m(p, \varphi, 2) \geq n$, and there are no other parameters to define. Now suppose the recursion is completed up through $k < n$, and consider $k+1$. For any $\eta \in {}^k2$, we have that $R^m(p_\eta, \varphi, 2) \geq n - k > 0$, so there must be $\bar{a}_\eta$ such that $p_\eta \cup \{\varphi(\bar{v}; \bar{a}_\eta\}$ and $p_\eta \cup \{\neg\varphi(\bar{v}; \bar{a}_\eta)\}$ both have rank $\geq n-k-1 = n-(k+1)$. So with this $\bar{a}_\eta$, these types form $p_{\eta^\frown 0}$ and $p_{\eta^\frown 1}$ respectively of rank $\geq n - (k+1)$.

Now we show ($\Leftarrow$). Let $\bar{a}_\rho$ realize $\bar{w}_\rho$ in $\Gamma_p(\varphi, n)$. We will show by induction on $k \leq n$ that $R^m(p_\eta, \varphi, 2) \geq k$ for all $\eta \in {}^{n-k}2$. At $k = 0$ we have simply that p_η is consistent, which follows since p_η is embedded in $\Gamma_p(\varphi, n)$. Suppose this holds at some $k < n$, and consider $k+1$. If $\eta \in {}^{n-(k+1)}2$, then $p_{\eta^\frown 0}$ and $p_{\eta^\frown 1}$ are mutually contradictory types which we assume have rank $\geq k$. Since these types extend p_η, we have that p_η is of rank $\geq k+1$ as desired. At $k = n$, we obtain $R^m(p, \varphi, 2) \geq n$. □

Corollary 4.3.8. *(vii)⇒(iv). This completes the proof of the Unstable Formula Theorem.*[17]

Proof. Since $\Gamma_{\bar{v}=\bar{v}}(\varphi, n) \equiv \Gamma(\varphi, n)$ and $R^m(p, \varphi, 2) \geq \omega > n$, we use the above lemma for all $n < \omega$. □

4.4 Unstable Theories

The Unstable Formula Theorem can be extended to theories as follows. This result is not called "The Unstable Theory Theorem", since we still have much more classification to do.

[17] Cf. Lemma II.2.10 in [9].

Theorem 4.4.1. *Given a theory T (and, for (iv), given m) the following are equivalent:*

(i) T is unstable (in every infinite power).

(ii) T is unstable in some infinite power $\lambda = \lambda^{|T|}$.

(iii) Some formula $\varphi(v; \bar{w})$ is unstable, (with respect to T).

(iv) Some formula $\varphi(\bar{v}; \bar{w})$ is unstable, $\ell(\bar{v}) = m$.

(v) There exists a formula $\varphi(\bar{v}; \bar{w})$, $\ell(\bar{v}) = \ell(\bar{w})$, and a sequence $\bar{a}_n$, $n < \omega$, such that $\models \varphi[\bar{a}_n; \bar{a}_k]$ if and only if $n < k$.

(vi) There exists an infinite indiscernible sequence that is not an indiscernible set.[18]

As before, we prove this theorem in a series of lemmas. But first, note the following.

Remark. *The direction (i)⇒(vi), or rather its contrapositive, provides a partial converse to Theorem 3.3.17, (not necessarily stable at $\aleph_0$, but somewhere).*

Lemma 4.4.2. *Suppose $|\mathbf{S}^m_\Delta(X)| \geq \lambda$ and λ has the property that if $\lambda_i < \lambda$ are chosen for $i < |\Delta|$, then $\prod_{i<|\Delta|} \lambda_i < \lambda$. Then there exists $\varphi \in \Delta$ such that $\mathbf{S}^m_\varphi(X) \geq \lambda$.*[19]

Proof. Enumerate $\Delta = \{\varphi(\bar{v}; \bar{w}_i) \mid i < |\Delta|\}$, and consider the mapping $g : \mathbf{S}^m_\Delta(X) \to \prod_{i<|\Delta|} \mathbf{S}^m_{\varphi_i}(X)$ defined by

$$g(p) = \langle \ldots, p \restriction \varphi_i, \ldots \rangle_{i<|\Delta|}.$$

[18] Cf. Theorem II.2.13 in [9], wherein a few more equivalent properties are listed. Again, the ones covered here are sufficient for our purposes. Unlike the Unstable Formula Theorem, however, Shelah proves a few lemmas first and then gives the proof all at once.

[19] Cf. Lemma II.2.15 in [9].

This mapping is injective, since if $p \neq q$ are complete Δ-types, then they disagree at some formula $\varphi_i \in \Delta$. This gives

$$\lambda \leq |\mathbf{S}^m_\Delta(X)| \leq \Big|\prod_{i<|\Delta|} \mathbf{S}^m_{\varphi_i}(X)\Big| = \prod_{i<|\Delta|} |\mathbf{S}^m_{\varphi_i}(X)|.$$

So, letting $\lambda_i = |\mathbf{S}^m_\varphi(X)|$, we have shown $\lambda \leq \prod_{i<|\Delta|} \lambda_i$. Lest we contradict our assumption, there must be some $\lambda_i \geq \lambda$ which finishes the proof. $\square$

Lemma 4.4.3. *(i)⇒(ii)⇒(iii)⇒(iv)⇒(i), so the first four properties are equivalent.*

Proof. The implication (i)⇒(ii) is trivial (take $\lambda = 2^{|T|}$ for example). For (ii)⇒(iii), we use Lemma 3.3.9 and then the previous lemma at $(\lambda^{|T|})^+$. This shows $\varphi(v;\bar{w})$ unstable in λ, but then the Unstable Formula Theorem gives instability in all infinite powers. Showing (iii)⇒(iv) just involves adding dummy variables to $\varphi(v;\bar{w})$, and (iv)⇒(i) follows from noting that $|\mathbf{S}^m(X)| \geq |\mathbf{S}^m_\varphi(X)|$. $\square$

Lemma 4.4.4. *(iii)⇒(v)⇒(vi)⇒(iii), which adds properties (v) and (vi) to the equivalence and completes the proof of Theorem 4.4.1.*

Proof. First we show (iii)⇒(v). By the Unstable Formula Theorem, $\varphi(v;\bar{w})$ has the order property, so there exist $\bar{a}_n$, $n < \omega$, such that for each $k < \omega$ the type $p_k = \{\varphi(v;\bar{a}_n)^{\text{if } k\leq n} \mid n < \omega\}$ is consistent. Let b_k realize p_k, and let $\bar{c}_n = b_n{}^\frown\bar{a}_n$. Now define $\psi(v_1,\bar{w}_1;v_2,\bar{w}_2) = [\varphi(v_1;\bar{w}_2) \wedge v_1 \neq v_2]$. Then $\models \psi[\bar{c}_k;\bar{c}_n]$ if and only if $k < n$ as desired.

Now we do (v)⇒(vi). Let $\varphi(\bar{v};\bar{w})$ and $\{\bar{a}_n \mid n < \omega\}$ be as in (v). For each finite Δ and n, Theorem 3.3.5 implies the existence of an infinite subset of $\{\bar{a}_n \mid n < \omega\}$ that forms a Δ-n-indiscernible sequence, (which is not an indiscernible set if $\varphi \in \Delta$). Since this works for all finite Δ and n, the compactness theorem implies the existence of an infinite indiscernible sequence I for which some $\bar{a},\bar{b} \in I$ have $\models \varphi[\bar{a};\bar{b}] \leftrightarrow \neg\varphi[\bar{b};\bar{a}]$. That is to say, an infinite indiscernible sequence that is not an indiscernible set.

Finally, see that (vi)⇒(iii). In fact, the proof of Theorem 3.3.17 showed that if there exists an indiscernible sequence that is not an indiscernible set, then some formula is unstable in $\aleph_0$. Then the Unstable Formula Theorem gives instability in all infinite powers. □

4.5 More Results

As a consequence of the Unstable Formula Theorem (in particular the equivalence (i)⇔(iii)) we get the following corollary.

Corollary 4.5.1. *Let $\varphi(\bar{v};\bar{w})$ be a formula, and $\psi(\bar{v};\bar{w}) = \varphi(\bar{w};\bar{v})$. Then φ is unstable if and only if ψ is unstable.*[20]

Proof. Suppose φ is unstable. Instability is equivalent to the order property, so suppose $\bar{b}_k$, $k < \omega$, realize the types

$$p_k = \{\varphi(\bar{v};\bar{a}_n)^{\text{if } k \leq n} \mid n < \omega\}.$$

So we have $\models \varphi[\bar{b}_k;\bar{a}_n]$ if and only if $k \leq n$. By a compactness argument similar to the proof of Corollary 4.1.4, we verify consistency of the types

$$q_n = \{\psi(\bar{v};\bar{b}_k)^{\text{if } n \leq k} \mid k < \omega\},$$

and hence ψ is unstable.

□

We will use the following result later on.

Lemma 4.5.2. *Suppose $\varphi(\bar{v};\bar{w})$ is stable. Then there exist finite $\Delta = \Delta(\varphi)$ and $n = n(\varphi) < \omega$ such that if I is a Δ-n-indiscernible set (of sequences of length $\ell(w)$)*

[20] Cf. Exercise II.2.8 in [9].

and $\bar{a}$ has $\ell(\bar{a}) = \ell(\bar{v})$, then either

$$|\{\bar{c} \in I \mid \models \varphi[\bar{a}; \bar{c}]\}| < n$$

or

$$|\{\bar{c} \in I \mid \models \neg\varphi[\bar{a}; \bar{c}]\}| < n.^{21}$$

Proof. Suppose not, and that for every finite Δ and n both of the above sets have size $\geq n$ for some I and $\bar{a}$. By compactness (ranging also over finite Δ), there would exist an infinite indiscernible set I and $\bar{a}$ such that both of these sets are in fact infinite.

Let $\{\bar{c}_n \mid n < \omega\}$ enumerate a countable subset of I. By compactness and indiscernibility, we have consistency of the type

$$p_s = \{\varphi(\bar{v}; \bar{c}_n)^{\text{if } n \in s} \mid n < \omega\}$$

for every $s \subseteq \omega$. This gives $2^{\aleph_0}$ φ-types over only $|\bigcup_{n<\omega} \bar{c}_n| = \aleph_0$ parameters, contradicting stability of φ.

We conclude that there must exist such $\Delta = \Delta(\varphi)$ and $n = n(\varphi)$. $\square$

[21] Cf. Lemma II.2.20 in [9].

Chapter 5

Forking

5.1 Motivation

In proving the Stability Spectrum Theorem, we seek to generalize Lemma 3.3.16, wherein we are given an increasing sequence X_α, $\alpha \leq \lambda$, and $p \in \mathbf{S}^m(X_\lambda)$ such that $p \restriction X_{\alpha+1}$ splits over X_α, and deduce instability at λ. We wish to derive instability at μ whenever $\mu < \mu^{<\lambda}$, but the given approach does not generalize when $\mu \geq 2^\lambda$, since it only produces 2^λ types over $\leq \lambda$ parameters. Using a new notion called *strong splitting* we will obtain μ^λ types over $\leq \mu$ parameters, giving instability at μ.

Strong splitting fails to have the right extension properties, so we introduce *dividing* and *forking* (the former, a notion regarding formulas, used to define the latter, a notion regarding types) to get results such as Theorem 5.3.4. We then prove equivalences such as Theorem 7.2.1 to link forking and strong splitting for the existence of certain increasing sequences used to produce μ^λ types and get instability results.

5.2 Strong Splitting, Dividing, and Forking

Here we introduce the major definitions of this section, and give a few examples. So far we have been considering only m-types, i.e. types with $m < \aleph_0$ free variables.

However, now we will want to consider types that may have infinitely many free variables, and will assume this unless otherwise stated. We use the notation $\bar{v}_U$ to mean that $\bar{v}$ is a tuple, perhaps infinite, indexed by $u \in U$. For example, $\bar{v} = \bar{v}_{\bar{b}}$ means $\ell(\bar{v}) = \ell(\bar{b})$.

Note also that we will frequently assume T to be stable, (in some power). This is for our work on the stability spectrum later on.

Notations 5.2.1. Given an m-type p and set of sequences I, each $\bar{a} \in I$ having $\ell(\bar{a}) = m$, we define

$$p(I) = \{\bar{a} \in I \mid \bar{a} \text{ realizes } p\}.$$

We define $\varphi(I; \bar{c})$ for formulas similarly. We also define the following:

$$\mathrm{tp}_*(B, X) = \{\varphi(\bar{v}_{\bar{b}}; \bar{a}) \mid \bar{b} \in B, \bar{a} \in X, \models \varphi[\bar{b}; \bar{a}]\}$$

$$\mathrm{tp}_*(\bar{b}, X) = \{\varphi(\bar{v}_{\bar{c}}; \bar{a}) \mid \bar{c} \subseteq \bar{b}, \bar{a} \in X, \models \varphi[\bar{b}; \bar{a}]\}$$

Definition 5.2.2. A type p *splits strongly* over X if and only if there exists a sequence $I = \{\bar{a}_n \mid n < \omega\}$ indiscernible over X and formula φ with $\varphi(\bar{v}; \bar{a}_0), \neg\varphi(\bar{v}; \bar{a}_1) \in p$.

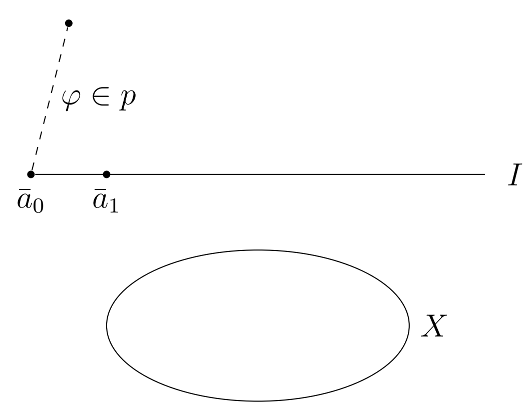

Figure 5.1: A type p splitting strongly over X

So clearly strong splitting is a stronger condition than splitting. The following examples show that it is strictly stronger.

Example 5.2.3. Consider the theory of three infinite equivalence classes C_i, $i = 1, 2, 3$, and the type of any element $a \in C_1$, $p = \mathrm{tp}(a, X)$, with $X \subseteq C_3$. Then taking $b \in C_1$ and $c \in C_2$, and the formula vEw gives splitting over X. But clearly strong splitting is impossible, since the only infinite indiscernible sequences lie entirely in one C_i; all three choices of i would fail in some way.

Definition 5.2.4. The formula $\varphi(\bar{v}; \bar{a})$ *divides* over X if and only if there exist $\bar{a}_l$ for $l < \omega$ such that

(i) $\mathrm{tp}(\bar{a}, X) = \mathrm{tp}(\bar{a}_l, X)$

(ii) $\{\varphi(\bar{v}; \bar{a}_l) \mid l < \omega\}$ is n-inconsistent for some $n < \omega$, (i.e. any n formulas from this set are inconsistent together).

Dividing can be thought of as "weak strong splitting", (once we show that we can equivalently take $\bar{a}_l$ as elements of an indiscernible sequence instead of condition (i), see Lemma 5.3.1(iii)). Then, instead of the formula appearing in a type, dividing is a property of the formula itself, where only finitely many can hold at once. Figure 5.2 illustrates an example of this.

Example 5.2.5. Consider the theory of infinitely many infinite equivalence classes, and an indiscernible sequence I with one element in a separate class. Then the formula vEw divides over $\emptyset$ (or over any set X lying outside of I's reach) via 2-inconsistency in condition (ii) of the definition.

Example 5.2.6. Consider now tuples $\bar{a}_n = (a_n^1, a_n^2)$ in a dense linear order, and the formula $\varphi(v; \bar{w}) \equiv w^1 < v < w^2$. Depending on how $I = \{\bar{a}_n \mid n < \omega\}$ are arranged, this formula might or might not divide, (over $\emptyset$, or any set over which I is indiscernible). If they are nested, $a_n^1 < a_m^2$ for all n, m, then we do not get division. If they are disjoint (as intervals), then we do get division, again by 2-inconsistency (this is the case illustrated in the figure below).

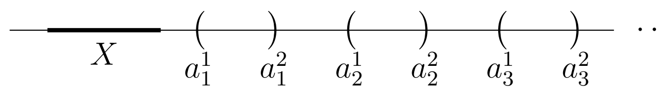

Figure 5.2: The formula $w^1 < v < w^2$ divides over X.

Example 5.2.7. Suppose $\varphi(v, \bar{w})$ asserts that v is the zero of a polynomial of degree k with coefficients in $\bar{w}$. Then any set $I = \{\bar{a}_n \mid n < \omega\}$ indiscernible over X will yield $\{\varphi(v, \bar{a}_n) \mid n < \omega\}$ as k-inconsistent, so $\varphi(v, \bar{a}_0)$ divides over X.

We go from single formulas to types with the following definition.[1]

Definition 5.2.8. A type p *forks* over X if and only if there exist formulas $\varphi_k(\bar{v}_k; \bar{a}_k)$, $k < n$, where the $\bar{v}_k$ appear free in p such that:

(i) $p \vdash \bigvee_{k<n} \varphi_k(\bar{v}_k; \bar{a}_k)$

(ii) $\varphi_k(\bar{v}_k; \bar{a}_k)$ divides over X for all $k < n$

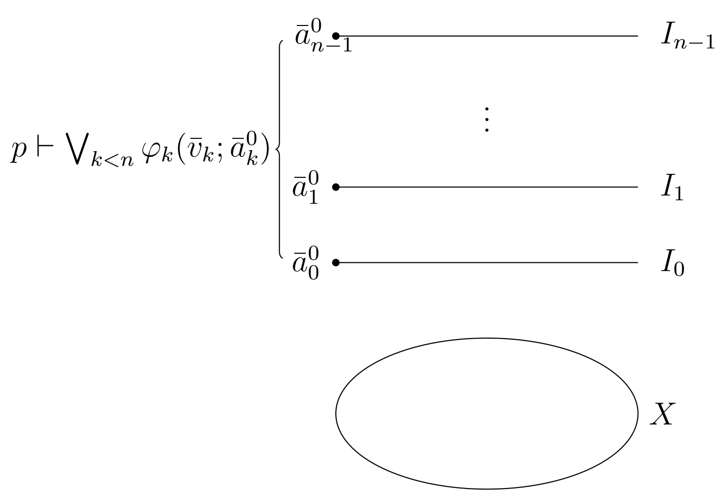

Figure 5.3: A type p forking over X

As in splitting, we will often consider "nonforking" as a desirable property. In this case, it means a certain kind of generality of the extension: given $p \in \mathbf{S}^m(X)$

[1] In [1], an alternative definition is given. Baldwin proves the equivalence for stable theories in section V.III of his book, calling Shelah's notion of forking (the notion presented in this thesis) "implicit division".

and $Y \subseteq X$, if p does not fork over Y then $p \restriction Y$ and p have the same rank, ($R^m(p, \Delta, \lambda) = R^m(p \restriction Y, \Delta, \lambda)$ for finite Δ and $2 \leq \lambda \leq \aleph_0$).[2]

Example 5.2.9. In general forking does not imply dividing, even for single formulas. Consider the theory of the unit circle, with a ternary relation $R(v_1, v_2, v_3)$ expressing that $v_1 < v_2 < v_3$ ordered clockwise. For any $\bar{a} = \langle a_1, a_3 \rangle$, the formula $\varphi(v_2; \bar{a}) = R(a_1, v_2, a_3)$ divides over $\emptyset$, since one can choose indiscernible tuples $\bar{a}^n$ whose arcs $\{c \mid R(a_1^n, c, a_3^n)\}$ are disjoint on the unit circle. Now choose finitely many tuples $\bar{a}_i$, $i < k$, such that $v_2 = v_2 \vdash \bigvee_{i<k} \varphi(v_2; \bar{a}_i)$. This shows that the formula $v_2 = v_2$ forks over $\emptyset$. But clearly no type can divide over its own parameter set, (here $\emptyset$).

We will later show that no type can fork over its own parameter set in stable theories, (Corollary 5.3.3). So the above example shows also that we cannot generalize this result to unstable theories.

5.3 Results on Forking

Lemma 5.3.1. *(i) Suppose $\bar{v} \subseteq \bar{w}$ and $\psi(\bar{w}; \bar{c}) \vdash \varphi(\bar{v}; \bar{c})$. Then if $\varphi(\bar{v}; \bar{c})$ divides over X, then so does $\psi(\bar{w}; \bar{c})$.*

(ii) Suppose p is a type in finitely many variables $\bar{v}$. Then p forks over X if and only if there exist formulas $\varphi_k(\bar{v}; \bar{a}_k)$, $k < n$, such that $p \vdash \bigvee_{k<n} \varphi_k(\bar{v}; \bar{a}_k)$ and $\varphi_k(\bar{v}; \bar{a}_k)$ divides over X for all $k < n$. (In other words, we can assume the dividing formulas φ_k have the same free variables as p.)

(iii) The formula $\varphi(\bar{v}; \bar{a})$ divides over X if and only if there exists $I = \{\bar{a}_l \mid l < \omega\}$ an indiscernible sequence over X with $\bar{a}_0 = \bar{a}$ such that $\{\varphi(\bar{v}; \bar{a}_l) \mid l < \omega\}$ is n-inconsistent for some $n < \omega$.

[2]We do not give the proof here. The proof is in [9] section III.4, but there are copious details left to the reader. The author of this thesis was unwilling to work them all out.

(iv) If $\varphi(\bar{v};\bar{a})$ divides over X, then $\varphi(\bar{v};\bar{a})$ forks over X.

(v) A type p forks over X if and only of some finite $q \subseteq p$ forks over X. (In particular, q is an m-type for some finite m.)

(vi) If $p \cup \{\psi_l(\bar{v};\bar{a}_l)\}$ forks over X for $l < n$, then $p \cup \{\bigvee_{l<n} \psi_l(\bar{v};\bar{a}_l)\}$ forks over X.

(vii) Suppose $X \subseteq Y$, $q \vdash p$ are types of the same variables, and q does not fork over X. Then p does not fork over Y. If $p \subseteq q$, then the same holds with "fork" being replaced by "split strongly".[3]

Proof. Part (i) is immediate from the definition, and (ii) follows from (i) by the addition of "dummy variables".

The condition in (iii) is seemingly stronger than dividing, so one implication is easy. Suppose that $\varphi(\bar{v};\bar{a})$ divides over X as defined. By an application of Ramsey's theorem and compactness we can find $\{\bar{b}_l \mid l < \omega\}$ indiscernible over X such that $\mathrm{tp}(\bar{b}_l, X) = \mathrm{tp}(\bar{a}, X)$ and $\{\varphi(\bar{v};\bar{b}_l) \mid l < \omega\}$ is n-inconsistent. The only setback is that we probably don't have $\bar{a} = \bar{b}_0$. No matter, since these elements have the same type over X, we can find an X-automorphism F of $\mathfrak{M}$ sending $F(\bar{b}_0) = \bar{a}$. Then the sequence $\{F(\bar{b}_l) \mid l < \omega\}$ will have the desired properties.

The claims (iv) and (v) are immediate, (the latter from syntactic compactness).

For (vi), take all dividing φ for each $p \cup \{\psi_l\}$.

Finally, (vii) follows from the definitions. $\square$

Lemma 5.3.2. *Suppose T is stable, p, q are m-types and p perhaps has parameters in X.*

(i) If $\varphi(\bar{v};\bar{w}) \in \Delta$, $\ell(\bar{v}) = m$, and $\varphi(\bar{v};\bar{a})$ divides over X, then

$$R^m(p, \Delta, \aleph_0) > R^m(p \cup \{\varphi(\bar{v};\bar{a})\}, \Delta, \aleph_0).$$

[3] Cf. Lemma III.1.1 in [9].

(ii) *Suppose $q \supseteq p$ and q forks over X. Then there exists a finite Δ_0 such that every finite $\Delta \supseteq \Delta_0$ has $R^m(q, \Delta, \aleph_0) < R^m(p, \Delta, \aleph_0)$.*[4]

Proof. First we show (i). Let $I = \{\bar{a}_l \mid l < \omega\}$ be an indiscernible sequence over X, $\bar{a} = \bar{a}_0$, such that $\{\varphi(\bar{v}; \bar{a}_l) \mid l < \omega\}$ is n-inconsistent (by Lemma 5.3.1(iii)). We take $p_l = p \cup \{\varphi(\bar{v}; \bar{a}_l)\}$. Since T is stable, T is totally transcendental by Theorem 3.4.10, so $\alpha = R^m(p, \Delta, \aleph_0) < \infty$. Then also $\alpha \geq R^m(p_l, \Delta, \aleph_0) = R^m(p_0, \Delta, \aleph_0)$, (the last equality from Lemma 4.2.1; we state it to show that they all have the same rank).

Suppose for contradiction that $\alpha = R^m(p_l, \Delta, \aleph_0)$ for each $l < \omega$. We would like to use the p_l to show that p has been extended $\aleph_0$ ways and conclude its rank must be greater than α, but the p_l are only n-inconsistent, (perhaps for $n > 2$, otherwise we'd be done). Let $Y = X \cup \bigcup I$. By Theorem 4.2.3, there exist $q_l \in \mathbf{S}^m(Y)$ extending p_l with the same rank α. Lest q_k be contradictory, we have $|\{l < \omega \mid \varphi(\bar{v}; \bar{a}_l) \in q_k\}| < n$ for each $k < \omega$. Therefore, since each $\varphi(\bar{v}; \bar{a}_l) \in q_l$, we also have $|\{q_l \mid q_l = q_k\}| < n$ for each $k < \omega$. These types are complete, so some infinite collection must be pairwise inconsistent. Since $p \subseteq q_l$, this shows that $R^m(p, \Delta, \aleph_0) \geq \alpha + 1$, a contradiction.

Now we prove (ii). Let $q \vdash \bigvee_{k<n} \varphi_k(\bar{v}; \bar{a}_k)$ where each $\varphi_k(\bar{v}; \bar{a}_k)$ divides over X, (using Lemma 5.3.1(ii) to make the variables match up). We claim $\Delta_0 = \{\varphi_k(\bar{v}; \bar{w}_k) \mid k < n\}$ works. Take $\Delta \supseteq \Delta_0$ finite. We have $q \vdash p \cup \{\bigvee_{k<n} \varphi_k(\bar{v}; \bar{a}_k)\} \vdash p$, so

$$R^m(q, \Delta, \aleph_0) \leq R^m(p \cup \{\bigvee_{k<n} \varphi_k(\bar{v}; \bar{a}_k), \Delta, \aleph_0)\} = \max_{k<n} R^m(p \cup \{\varphi_k(\bar{v}\bar{a}_k)\}, \Delta, \aleph_0),$$

(the last equality from Lemma 4.2.4). By part (i), $R^m(p \cup \{\varphi_k(\bar{v}; \bar{a}_k)\}, \Delta, \aleph_0) < R^m(p, \Delta, \aleph_0)$, so we get $R^m(q, \Delta, \aleph_0) < R^m(p, \Delta, \aleph_0)$ as desired. □

From this we derive the following corollary. Curiously, this took a lot more work than, say, showing types do not split over their own parameter sets!

[4]Cf. Lemma III.1.2 in [9].

Corollary 5.3.3. *In a stable theory, no type forks over its own parameter set.*[5]

Proof. Suppose some p forks over $X = \mathrm{param}(p)$. We may assume p is an m-type by Lemma 5.3.1(v). Then by Lemma 5.3.2(ii), we would have $R^m(p, \Delta, \aleph_0) < R^m(p, \Delta, \aleph_0)$ for some appropriately chosen Δ, which is absurd. $\square$

Theorem 5.3.4. *Suppose p is a type, perhaps in infinitely many free variables $\bar{v}_U$, over parameters B nonforking over X. Then there exists a complete type $q \supseteq p$ in the same variables over B nonforking over X.*[6]

Proof. Let $\Gamma = \{\psi(\bar{v}_{\bar{u}}; \bar{a}) \mid \bar{u} \in U \text{ (finite)}, \bar{a} \in B, \psi(\bar{v}_{\bar{u}}; a) \text{ forks over } X\}$ and define $q' = p \cup \{\neg\psi \mid \psi \in \Gamma\}$. We first show that q' is consistent. If not, let $p' \cup \{\neg\psi_k(\bar{v}_k; \bar{a}_k) \mid k < n\}$ be a finite inconsistent subset, ($p' \subseteq p$). This means $p \vdash \bigvee_{k<n} \psi_k$. Since each ψ_k forks over X, we have $\bigvee_{k<n} \psi_k$ also forks over X by Lemma 5.3.1(vi). But then p forks over X by Lemma 5.3.1(vii), contradicting the hypothesis. So q' is consistent.

Let $q \supseteq q'$ be any complete extension in $\bar{v}_U$ over parameters B. We claim that q does not fork over X. Suppose it did, where $q \vdash \bigvee_{k<n} \varphi_k(\bar{v}_k; \bar{a}_k)$, each $\varphi_k(\bar{v}_k; \bar{a}_k)$ a dividing formula over X. Let $q^* \subseteq q$ be a finite forking subtype, and let $\theta(\bar{v}; \bar{b}) = \bigwedge q^*$. By completeness, $\theta \in q$. But θ forks over X, so $\theta \in \Gamma$, hence $\neg\theta \in q' \subseteq q$, an impossibility. We conclude that q does not fork over X. $\square$

Corollary 5.3.5. *Suppose $X_1 \subseteq Y_1$, $X_2 \subseteq Y_2$, Z_1, and Z_2 are sets, and F a surjective elementary mapping $F : Y_1 \twoheadrightarrow Y_2$ such that $F[X_1] = X_2$ and $\mathrm{tp}_*(Z_1, Y_1)$ does not fork over X_1. Then we can extend F to an elementary mapping F' with domain $Y_1 \cup Z_1$ such that $\mathrm{tp}_*(F'[Z_1], Y_2 \cup Z_2)$ does not fork over X_2.*[7]

Proof. Let $p = \mathrm{tp}_*(Z_1, Y_1)$. Certainly we have $F(p)$ does not fork over $F(X_1) = X_2$. By the previous theorem, extend $F(p)$ to a complete type q (with the same variables) in parameters $Y_2 \cup Z_2$ nonforking over X_2. Then, for each $\bar{c} \in Z_1$ let $F'(\bar{c})$ realize the

[5] Cf. Corollary III.1.3 in [9].
[6] Cf. Theorem III.1.4 in [9].
[7] Cf. Corollary III.1.5 in [9].

variable $\bar{v}_{\bar{c}}$ in q. Then $q = \mathrm{tp}_*(F'(Z_1), Y_2 \cup Z_2)$ does not fork over X_2, and F' will be an elementary mapping by the equation $F(\mathrm{tp}_*(Z_1, Y_1)) = \mathrm{tp}_*(F'(Z_1), Y_2)$. $\square$

Theorem 5.3.6. *Suppose T is stable.*

(i) If p strongly splits over X, then p forks over X.

(ii) p forks over X if and only if there exists a set $B \supseteq$ param(p) such that for every $q \supseteq p$ (in the same variables) complete over parameters B, q strongly splits over X.[8]

Proof. We start by proving (i). Let $I = \{\bar{a}_n \mid n < \omega\}$ be an indiscernible sequence over X (also an indiscernible set by stability) and $\varphi(\bar{v}; \bar{w})$ a formula with $\varphi(\bar{v}; \bar{a}_0), \neg\varphi(\bar{v}; \bar{a}_1) \in p$. By Lemma 4.5.2, there exists $n = n(\varphi) < \omega$ such that for all $\bar{c}$, either $|\{k \mid \models \varphi[\bar{c}; \bar{a}_k]\}| < n$ or $|\{k \mid \models \neg\varphi[\bar{c}; \bar{a}_k]\}| < n$. Let $\bar{b}_k = \bar{a}_{2k}\frown\bar{a}_{2k+1}$, and $\psi(\bar{v}; \bar{b}_k) = \varphi(\bar{v}; \bar{a}_{2k}) \wedge \neg\varphi(\bar{v}; \bar{a}_{2k+1})$. Note that $\{\bar{b}_n \mid n < \omega\}$ is also indiscernible over X. Moreover, $\{\psi(\bar{v}; \bar{b}_k) \mid k < \omega\}$ is n-inconsistent, since otherwise we would contradict the choice of $n(\varphi)$ above. So by Lemma 5.3.1(iii), $\psi(\bar{v}; \bar{b}_0)$ divides over X. Since $\varphi(\bar{v}; \bar{a}_0), \neg\varphi(\bar{v}; \bar{a}_1) \in p$, we have $p \vdash \psi(\bar{v}; \bar{b}_0)$, so p forks over X.

We now prove the forward direction of (ii). Suppose $p \vdash \bigvee_{k<n} \varphi_k(\bar{v}; \bar{a}_k)$ where each $\varphi_k(\bar{v}; \bar{a}_k)$ divides over X. By Lemma 5.3.1(iii) there exist sequences (by stability, sets) $I_k = \{\bar{a}_{i,k} \mid i < \omega\}$ indiscernible over X, $\bar{a}_{0,k} = \bar{a}_k$, and $n(k)$ such that $\{\varphi_k(\bar{v}; \bar{a}_{i,k}) \mid i < \omega\}$ is $n(k)$-inconsistent. Let $B = \mathrm{param}(p) \cup \{\bar{a}_{i,k} \mid i < n(k), k < n\}$, and let $q \in \mathbf{S}^m(B)$ extend p. Then since $p \vdash \bigvee_{k<n} \varphi_k(\bar{v}; \bar{a}_k)$, we have for some $k < n$ $\varphi_k(\bar{v}; \bar{a}_k) \in q$. Now by $n(k)$-inconsistency, there must be some i such that $\neg\varphi_k(\bar{v}; \bar{a}_{i,k}) \in q$. This shows q splits strongly over X.

We finish (ii) by showing the reverse direction. Suppose p is a type over B such that for every $q \in \mathbf{S}^m(B)$ extending p, q strongly splits over X. By Theorem 5.3.4, if

[8] Cf. Theorem III.1.6 in [9].

p does not fork over X then there would exists a complete $q \supseteq p$ with parameters in B nonforking over X. But by part (i), strong splitting implies forking. $\square$

5.4 The Average Type

Now we define a notion inspired by Lemma 4.5.2.

Definition 5.4.1. Given an infinite indiscernible set I and any other set X, we define $\mathrm{Av}_\Delta(I, X) = \{\varphi(\bar{v}; \bar{a})^t \mid \varphi \in \Delta, t = 0, 1, \bar{a} \in X,$ and for all but finitely many $\bar{c} \in I$ we have $\models \varphi[\bar{c}; \bar{a}]^t\}$. This we call the *average type*, since it represents the typical element of I with respect to parameters X. As usual we omit $\Delta = \mathcal{L}$.

We think of the average type as describing the most typical element in an indiscernible sequence I. Consider the following example.

Example 5.4.2. Suppose we are given an indiscernible set I within one of the equivalence classes, say C, in a model of the theory of infinitely many infinite equivalence classes. Let X be a set, say somewhere in C. Then the average type will be satisfied by all of $C - X$, since all but finitely many $a \in I$ relate (but are unequal to) X. Now if X is disjoint from C, then all $a \in I$ do not relate, so the average type will be satisfied by all elements in any class disjoint from X.

Lemma 5.4.3. *Suppose T is stable.*

1. *$Av(I, X)$ is consistent and complete.*
2. *If I is an infinite indiscernible set over X then $\bar{c}$ realizes $Av(I, X \cup \bigcup I)$ if and only if $I \cup \{\bar{c}\}$ is indiscernible over X.*[9]

Proof. We first prove (i). For consistency, we use compactness; note that any finite subset is realized by all but finitely many members of I. Now we show completeness.

[9] Cf. Lemma III.1.7 in [9].

If $\bar{a} \in X$ and $\varphi(\bar{v}; \bar{w})$ is a formula, then by Lemma 4.5.2 either $\varphi(I; \bar{a})$ or $\neg\varphi(I; \bar{a})$ is finite (see Notations 5.2.1). Respectively, these give either $\neg\varphi(\bar{v}; \bar{a}) \in \mathrm{Av}(I, X)$ or $\varphi(\bar{v}; \bar{a}) \in \mathrm{Av}(I, X)$, showing completeness.

Now we prove (ii), starting with the forward direction. It suffices to show that for any $\bar{a} \in X$, formula $\varphi(\bar{v}_1 \dots \bar{v}_{n+1}; \bar{w})$, $\bar{b}_1 \dots \bar{b}_n \in I$, and $\bar{d}_1 \dots \bar{d}_{n+1} \in I \cup \{\bar{c}\}$, we have

$$\models \varphi[\bar{c}, \bar{b}_1 \dots \bar{b}_n; \bar{a}] \leftrightarrow \varphi[\bar{d}_1 \dots \bar{d}_{n+1}; \bar{a}].$$

Replace $\bar{c}$ (including possible occurrence among the $\bar{d}_k$) with a free variable $\bar{v}$, and call this formula $\psi(\bar{v}; \bar{e})$. Then we get a formula which, by indiscernibility of I over X, is realized everywhere in I. By definition, we have $\psi(\bar{v}; \bar{e}) \in \mathrm{Av}(I, X \cup \bigcup I)$, and so by assumption, $\bar{c}$ realizes $\psi(\bar{v}; \bar{e})$.

We finish (ii) by showing the reverse direction. Suppose $\varphi(\bar{v}; \bar{a})$ is a formula realized at all but finitely many points in I, $\bar{a} \in X \cup \bigcup I$. If it were the case that $\models \neg\varphi[\bar{c}; \bar{a}]$, then by indiscernibility we would also have $\models \neg\varphi[\bar{d}; \bar{a}]$ for all $\bar{d} \in I$, a contradiction. We conclude $\models \varphi[\bar{c}; \bar{a}]$, and since this works for all such formulas φ, we have that $\bar{c}$ realizes $\mathrm{Av}(I, X \cup \bigcup I)$. $\square$

However, it need not be the case that $\mathrm{Av}(I, X)$ is realized anywhere in I. Indeed, if $X = \bigcup I$, then the formulas $\bar{v} \neq \bar{a}$ for each $\bar{a} \in \bigcup I$ will ensure that no $\bar{c} \in I$ satisfies the average type.

Definition 5.4.4. Two indiscernible sets I_1 and I_2 are said to be *equivalent* if and only if there exists an infinite set J such that $I_1 \cup J$ and $I_2 \cup J$ are both still indiscernible. (Here indiscernibility is taken over $\emptyset$.)

Note that although this implies $\mathrm{tp}(\bar{a}_1, \emptyset) = \mathrm{tp}(\bar{a}_2, \emptyset)$ for any $\bar{a}_1 \in I_1$ and $\bar{a}_2 \in I_2$, we are *not* saying that $I_1 \cup I_2$ is indiscernible, as we would have to consider the case $\bar{a} \in I_1 \cap I_2$. Figure 5.4 gives an example of equivalent indiscernible sets whose union is not indiscernible in the theory of infinitely many infinite equivalence classes.[10]

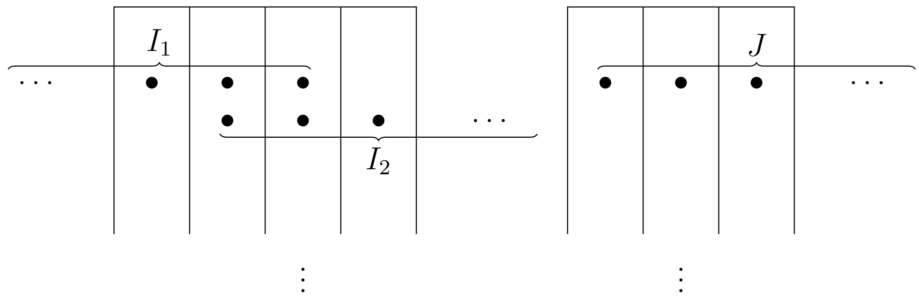

Figure 5.4: Sets I_1 and I_2 equivalent via J whose union is not indiscernible

In this example, note that the monster model is always large enough to find room for J in new equivalence classes.

Lemma 5.4.5. *Suppose T is stable, and I_1, I_2 are indiscernible sets. Then the following are equivalent:*

(i) *I_1 and I_2 are equivalent.*

(ii) *For any set X, $\mathrm{Av}(I_1, X) = \mathrm{Av}(I_2, X)$.*

(iii) *There is a model $\mathfrak{A}$, $A \supseteq I_1 \cup I_2$ such that $\mathrm{Av}(I_1, A) = \mathrm{Av}(I_2, A)$.*[11]

Proof. First we do (i)⇒(ii). If J establishes the equivalency of I_1 and I_2, we have $\mathrm{Av}(I_i, X) \supseteq \mathrm{Av}(I_i \cup J, X)$ for $i = 1, 2$. But by completeness, equality holds. Similarly, $\mathrm{Av}(J, X) = \mathrm{Av}(I_i \cup J, X)$ for $i = 1, 2$. We arrive at $\mathrm{Av}(I_1, X) = \mathrm{Av}(I_2, X)$.

The direction (ii)⇒(iii) is trivial by embedding in the monster model. For the reverse, suppose (iii) holds but (ii) does not, so some set X and formula $\varphi(\bar{v}; \bar{a})$, $\bar{a} \in X$, have $\varphi(\bar{v}; \bar{a}) \in \mathrm{Av}(I_1, X)$, $\neg\varphi(\bar{v}; \bar{a}) \in \mathrm{Av}(I_2, X)$. By Lemma 4.5.2, there exists $n(\varphi)$ such that for any indiscernible set I and $\bar{c}$, we have $|\varphi(I; \bar{c})| < n(\varphi)$ or $|\neg\varphi(I; \bar{c})| < n(\varphi)$. (So the second holds for $I_1, \bar{a}$ and the first holds for $I_2, \bar{a}$.) In light

[10]Cf. Example 3.3.6.
[11]Cf. Lemma III.1.8 in [9].

of this, choose $\bar{b}_0 \ldots \bar{b}_{n(\varphi)-1} \in I_1$ and $\bar{d}_0 \ldots \bar{d}_{n(\varphi)-1} \in I_2$ such that

$$\models \bigwedge_{k<n(\varphi)} \varphi[\bar{b}_k, \bar{a}] \wedge \neg\varphi[\bar{d}_k; \bar{a}].$$

We can replace $\bar{a}$ with an existentially quantified $\bar{v}$, and then our model $\mathfrak{A}$ containing I_1, I_2 will model this sentence. We can then instantiate $\bar{v}$ at $\bar{a}^* \in A$. So without loss of generality, we can assume $\bar{a} \in A$. But then $\varphi(\bar{v}; \bar{a}) \in \mathrm{Av}(I_1, A)$ and $\neg\varphi(\bar{v}; \bar{a}) \in \mathrm{Av}(I_2, A)$, and these types are equal. This contradiction lets us conclude that (ii)$\Leftrightarrow$(iii).

Finally, we show (ii)$\Rightarrow$(i). We use Lemma 5.4.3 (ii) and define $\bar{c}_n$ by recursion on $n < \omega$ realizing $\mathrm{Av}(I_1, C_n) = \mathrm{Av}(I_2, C_n)$ where $C_n = \bigcup I_1 \cup \bigcup I_2 \cup \bigcup_{i<n} \bar{c}_i$. By the referenced lemma, we will have indiscernibiltiy of $I_i \cup \{\bar{c}_n \mid n < \omega\}$ for $i = 1, 2$, and hence $J = \{\bar{c}_n \mid n < \omega\}$ shows the equivalence of I_1 and I_2. $\square$

Definition 5.4.6. A type p is *stationary* over X provided p does not fork over X, and also p has no two contradictory extensions that do not fork over X.

Remark. *By Theorem 5.3.4, an equivalent definition for p being stationary over X is that p has a unique extension to a complete nonforking type in B for each $B \supseteq$ param(p).*

Lemma 5.4.7. *Suppose $X \subseteq Y$, and $p \restriction Y$ is stationary over X. Then p does not split over Y.*[12]

Proof. Suppose p splits over Y; let $\bar{b}, \bar{c}$ have $\mathrm{tp}(\bar{b}, Y) = \mathrm{tp}(\bar{c}, Y)$ but $\varphi(\bar{v}; \bar{b}), \neg\varphi(\bar{v}; \bar{c}) \in p$. Our goal is to define two contradictory nonforking extensions; note that (by Lemma 5.3.1(vii)) neither $p \restriction Y \cup \{\varphi(\bar{v}; \bar{b})\}$ nor $p \restriction Y \cup \{\neg\varphi(\bar{v}; \bar{c})\}$ fork over X. So, let F be an automorphism of $\mathfrak{M}$ identity on Y sending $\bar{c}$ to $\bar{b}$. Then $F(p \restriction Y \cup \{\neg\varphi(\bar{v}; \bar{c})\}) = p \restriction Y \cup \{\neg\varphi(\bar{v}; \bar{b})\}$ does not fork over X either. This gives two distinct nonforking extensions, contradicting $p \restriction Y$ being stationary over X. $\square$

[12] Cf. Lemma III.1.9 in [9].

Definition 5.4.8. An infinite indiscernible set I is *based on* X if and only if for every set Y, $\mathrm{Av}(I, Y)$ does not fork over X.

Example 5.4.9. I is based on $\bigcup I$, see [1] for a proof.

Lemma 5.4.10. *Suppose T is stable. Then:*

(i) Suppose p is an m-type over $Y \supseteq X$ stationary over X, $Y_j = Y \cup \bigcup_{i<j} \bar{a}_i$, $p \subseteq p_j = \mathrm{tp}(\bar{a}_j, Y_j)$, and p_j does not fork over X. Then $I = \{\bar{a}_i \mid i < i_0\}$ (for any $i_0 \geq \omega$) is an indiscernible set over Y based on X.

(ii) With the notation from (i), for all $Z \supseteq Y$, $\mathrm{Av}(I, Z)$ is the unique extension of p in $\mathbf{S}^m(Z)$ nonforking over X.[13]

Proof. In proving (i), we first show I is an indiscernible set over Y. We go all the way back to Lemma 3.3.14 to verify this, (along with stability to conclude it is indiscernible as a set and not just a sequence). First, we see that p_j does not split over Y by Lemma 5.4.7. Since p is stationary and the p_j are complete, we must have $p_i \subseteq p_j$ for $i \leq j < i_0$. So Lemma 3.3.14 applies, yielding indiscernibiltiy of I over Y.

Now we show I is based on X. Let Y' be any set; we show $\mathrm{Av}(I, Y')$ does not fork over X. We have p_j defined for $j < i_0$. We define more $\bar{a}_j$ for $i_0 \leq j < i_0 + \omega$ (ordinal addition). Let $Z = Y_{i_0} \cup Y'$, and define by induction new $\bar{a}_j$ such that $p_j = \mathrm{tp}(\bar{a}_j, Z \cup \bigcup_{i<j} \bar{a}_j)$ extends p and does not fork over X. This recursion is done via Theorem 5.3.6, (both parts and also the strong splitting clause in Lemma 5.3.1(vii)).

Once this is done, note that $p_j \upharpoonright B_j$ (for all $j < i_0 + \omega$) satisfies the conditions described in (i), and so $I' = \{\bar{a}_j \mid j < i_0 + \omega\}$ is also an indiscernible set over X. So this indiscernible sequence is equivalent to I, which is also equivalent to $I' - I$. From

[13] Cf. Lemma III.1.10 in [9].

Lemma 5.4.5 we have

$$\mathrm{Av}(I, Y') \subseteq \mathrm{Av}(I, Z) = \mathrm{Av}(I', Z) = \mathrm{Av}(I' - I, Z) = p_{i_0},$$

where the last equality follows since $p_{i_0} = \mathrm{tp}(\bar{a}_j, Z)$ for all $i_0 \leq j < i_0 + \omega$ (this because we still have $p_i \subseteq p_j$ for $i \leq j$, so restricted to the same parameters they must be equal). Now, p_{i_0} does not fork over X, so neither can $\mathrm{Av}(I, Y')$. This is what we wished to show, so the proof of (i) is complete.

We claim (ii) follows from (i). Indeed, if $Z \supseteq Y$ is arbitrary, let $Y' = Z$ above. We showed $\mathrm{Av}(I, Y')$ was a nonforking extension of p, so by stationarity, this is the unique nonforking extension in parameters $Z = Y'$. $\square$

Lemma 5.4.11. *Suppose T is stable and $X \subseteq Y$. Let $\mu = (|X| + 2)^{|T|}$. Then:*

(i) If $p \in \mathbf{S}^m(A)$ does not fork over $X \subseteq A$ and $\mathfrak{A}$ is μ^+-saturated, then p is stationary over X.

(ii) If $\Gamma_1 = \{p \in \mathbf{S}^m(Y) \mid p$ does not fork over $X\}$, then $|\Gamma_1| \leq \mu$.

(iii) If $\Gamma_2 = \{p \in \mathbf{S}^m(Y) \mid p$ does not strongly split $X\}$, then $|\Gamma_2| \leq \mu$.

(iv) There are at most μ non-equivalent infinite indiscernible sets based on X.[14]

Proof. First we prove (i). Suppose p is not stationary over X. Since it does not fork, it has two distinct complete nonforking (over X) extensions $q_1 \neq q_2$ in $\mathbf{S}^m(Y)$ for some $Y \supseteq A$. Say $\varphi(\bar{v}; \bar{b}) \in q_2$ while $\neg\varphi(\bar{v}; \bar{b}) \in q_2$. We define an increasing sequence $X_i \subseteq A$, $|X_i| \leq \mu$, elements $\bar{b}_i \in A$, and types $p_i \in \mathbf{S}^m(A)$ by recursion on $i < \mu^+$. We let $X_0 = X$ and take unions and limit ordinals. Suppose X_i is defined. We define $\bar{b}_i$, p_i, and also X_{i+1}. By saturation, we take $\bar{b}_i \in A$ such that $\mathrm{tp}(\bar{b}_i, X_i) = \mathrm{tp}(\bar{b}, X_i)$. As subtypes of $\bar{q}_i$, we have that $p \restriction X_i \cup \{\varphi(\bar{v}; \bar{b}_i)\}$ and $p \restriction X_i \cup \{\neg\varphi(\bar{v}; \bar{b}_i)\}$ do not

[14]Cf. Lemma III.1.11 in [9].

fork over X. Suppose (by completeness) that $\varphi(\bar{v};\bar{b}_i) \in p$. (If the negated formula is in p, the argument is similar.) Choose p_i to be a complete nonforking extension of $p \upharpoonright X_i \cup \{\neg\varphi(\bar{v};\bar{b}_i)\}$ in $\mathbf{S}^m(A)$, (Theorem 5.3.4). Finally, take $X_{i+1} = X_i \cup \bar{b}_i$.

By Theorem 4.4.1, we may assume T is stable in all $\lambda = \lambda^{|T|}$. In particular, $\lambda = \mu$. By Theorem 3.3.11 we can find an infinite indiscernible sequence (set) $J \subseteq \{\bar{b}_i \mid i < \mu^+\}$. Now take any $i_0 < i_1$ such that $\bar{b}_{i_0}, \bar{b}_{i_1} \in J$. Assume that $\varphi(\bar{v};\bar{b}_{i_0}) \in p$, (if the negated formula is in p, the argument is similar). Then $\varphi(\bar{v};\bar{b}_{i_0}) \in p_{i_1}$. But lest p split strongly over X, we must have $\varphi(\bar{v};\bar{b}_{i_1}) \in p$ as well, yielding $\neg\varphi(\bar{v};\bar{b}_{i_1}) \in p_{i_1}$. This means p_{i_1} splits strongly over X, a contradiction by Theorem 5.3.6(i).

Since strong splitting implies forking (by Theorem 5.3.6(i)), we see that $\Gamma_1 \subseteq \Gamma_2$. Thus the bound in (ii) will follow from the bound in (iii). To prove (iii), we define an equivalence relation $\sim$ on finite tuples from Y as follows. We say that $\bar{b} \sim \bar{c}$ provided there exists $\bar{b} = \bar{a}_0, \ldots, \bar{a}_n = \bar{c}$ and infinite indiscernible sets $I_0 \ldots I_{n-1}$ (over X) such that $\bar{a}_i, \bar{a}_{i+1} \in I_i$, $i < n$. Reflexivity and transitivity are easy to see, and symmetry just involves flipping indices. Choose $Y' \subseteq Y$ such that each tuple has exactly one representative. Then see that the map $p \mapsto p \upharpoonright Y'$ is an injection from Γ_2 into $\mathbf{S}^m(Y')$. Indeed, if $p_1 \neq p_2$ did not strongly split over X while $p_1 \upharpoonright Y' = p_2 \upharpoonright Y'$, then suppose $\varphi(\bar{v};\bar{b}) \in p_1$, $\neg\varphi(\bar{v};\bar{b}) \in p_2$. Suppose $\bar{b} \sim \bar{c} \in Y'$. Then if $\varphi(\bar{v};\bar{c}) \in p_1$, then also $\varphi(\bar{v};\bar{c}) \in p_2$ so p_2 would split strongly over X. Otherwise, if $\neg\varphi(\bar{v};\bar{c}) \in p_1$, we would have p_1 splitting strongly over X. This yields a contradiction either way, so the map must be injective, showing $|\Gamma_2| \leq |\mathbf{S}^m(Y')|$.

By the same reasoning as in the proof of (i), we have that T is stable in μ. So, it suffices to show that $|Y'| \leq \mu$, i.e., that there are at most that many equivalence classes, since then we'd have $|\mathbf{S}^m(Y')| \leq |Y'|$. Suppose not; let $I = \{\bar{b}_i \mid i < \mu^+\}$ be a set of tuples (of the same length) such that $\bar{b}_i \not\sim \bar{b}_j$ for $i \neq j$. Again by Theorem 3.3.11 and stability, there exists $J \subseteq I$, $|J| = |I|$, indiscernible over X. Of course, this would imply all of J is equivalent, a contradiction. This shows (iii), and as remarked above, (ii).

Finally, we show (iv). Suppose we have $\{I_i \mid i < \mu^+\}$ infinite pairwise nonequivalent indiscernible sets based on X. Choose a model $\mathfrak{A}$, $A \supseteq X \cup \bigcup_{i<\mu} I_i$. Let $p_i = \mathrm{Av}(I_i, X)$. Then Lemma 5.4.5 implies $p_i \neq p_j$ for $i \neq j$. But since I_i is based on X, this implies these (complete in A) types do not fork over X. This contradicts (ii). $\square$

Theorem 5.4.12. *Suppose T is stable. For every tuple $\bar{b}$ and set X, there exists an infinite indiscernible set I over X based on X with $\bar{b} \in I$. If $\mathfrak{A}$ is a $(|X| + \aleph_0)^+$-saturated model with $X \cup \bar{b} \subseteq A$, we can take $I \subseteq A$. If $\mathrm{tp}(\bar{b}, X)$ does not fork over $Y \subseteq X$, then we can choose I based on Y.*[15]

Proof. Let $\mathfrak{A}'$ be a $((|X| + 2)^{|T|})^+$-saturated model, $A' \supseteq X \cup \bar{b}$. We know $\mathrm{tp}(\bar{b}, X)$ does not fork over X, (by Corollary 5.3.3). So by Theorem 5.3.4 choose $r \supseteq \mathrm{tp}(\bar{b}, X)$, $r \in \mathbf{S}^m(A')$, nonforking over X. Then by Lemma 5.4.11(i) r is stationary over X. Now we define $\bar{c}_i \in A'$ and r_i by recursion on $i < \omega$. Let $\bar{c}_0 \in A'$ realize r, (so in addition $\mathrm{tp}(\bar{c}, X) = \mathrm{tp}(\bar{b}, X)$). Suppose we have defined up to $\bar{c}_i$. Let r_i extend r to a complete nonforking type in $A' \cup \bigcup_{j<i} \bar{c}_j$. Choose $\bar{c}_{i+1} \in A'$ realizing r_i.

By Lemma 5.4.10, $J = \{\bar{c}_i \mid i < \omega\}$ is an indiscernible set over $A' \supseteq X$ based on X, and also each $\bar{c}_i$ realizes r. Let F be an automorphism of $\mathfrak{M}$ fixing X satisfying $F(\bar{c}_0) = \bar{b}$. Then $I = F[J]$ satisfies the first part of the theorem. If we are given $\mathfrak{A}$ from the start, we may choose F such that $F(\bar{c}_i) \in A$. Finally, if $\mathrm{tp}(\bar{b}, X)$ does not fork over $Y \subseteq X$, we can choose r also nonforking over Y, which will then imply I is based on Y, (by Lemma 5.4.10). $\square$

[15]Cf. Conclusion III.1.12 in [9].

Chapter 6

Finite Equivalence Relations

6.1 Utility for the Stability Spectrum

Finite Equivalence Relations are necessary to prove the results in the fifth section, titled "Useful Consequences". These are needed for proving the Stability Spectrum Theorem; otherwise we will not reference the concepts defined in this chapter. As such, if one does not wish to be encumbered by even more definitions, they can read the results at the end and move on.

6.2 Definitions

Notation 6.2.1. We let $\mathrm{FE}^m(X)$ denote the set of formulas $\varphi(\bar{v}, \bar{w}; \bar{a})$ such that $\ell(\bar{v}) = \ell(\bar{w}) = m$, $\bar{a} \in X$, and φ is an equivalence relation with a finite number of equivalence classes. (This provable from the ambient theory T.) Define $\mathrm{FE}(X) = \bigcup_{m<\omega} \mathrm{FE}^m(X)$, and denote elements therein as $E(\bar{v}; \bar{w})$. We denote the number of equivalence classes for a given equivalence relation E as $n(E)$.

Definition 6.2.2. We say that a formula $\varphi(\bar{v}; \bar{b})$ is *almost over X* if and only if there exists $E(\bar{v};\bar{w}) \in \mathrm{FE}(X)$ such that

$$\forall \bar{v}, \bar{w} : (E(\bar{v};\bar{w}) \to (\varphi(\bar{v};\bar{b}) \leftrightarrow \varphi(\bar{w};\bar{b})).$$

In this case we say φ *depends* on E. We say that a type p is almost over X provided every $\varphi \in p$ is.

Definition 6.2.3. We define the *strong type* of $\bar{a}$ over X as

$$\mathrm{stp}(\bar{a}, X) = \{E(\bar{v};\bar{a}) \mid E(\bar{v};\bar{w}) \in \mathrm{FE}^m(X)\},$$

where $m = \ell(\bar{a})$. We also define

$$\mathrm{stp}_*(B, X) = \{E(\bar{v};\bar{a}) \mid \bar{a} \in B, E(\bar{v};\bar{w}) \in \mathrm{FE}(X)\},$$

(compare Notations 5.2.1).

6.3 Basic Properties of Finite Equivalence

Lemma 6.3.1. *(i) If $\bar{b} \in X$, then $\varphi(\bar{v};\bar{b})$ is almost over X.*

(ii) $\bar{a}$ realizes $\mathrm{stp}(\bar{a}, X)$

(iii) $\mathrm{stp}(\bar{a}, X) \vdash \mathrm{tp}(\bar{a}, X)$

(iv) $\mathrm{stp}(\bar{a}, X)$ *is almost over X*

(v) If $\mathrm{stp}(\bar{a}, X) \equiv \mathrm{stp}(\bar{b}, X)$ *and $\varphi(\bar{v};\bar{c})$ is almost over X, then* $\models \varphi[\bar{a};\bar{c}] \leftrightarrow \varphi[\bar{b};\bar{c}]$.

(vi) If $\bar{a}$ realizes p and p is almost over X, then $\mathrm{stp}(\bar{a}, X) \vdash p$.[1]

Proof. For (i), use the formula $\varphi(\bar{v};\bar{b}) \leftrightarrow \varphi(\bar{w};\bar{b})$, (two equivalence classes).

[1] Cf. Lemma III.2.1, and the remark after Definition III.2.1 in [9].

Claim (ii) is trivial. For (iii), suppose $\bar{c}$ realizes $\operatorname{stp}(\bar{a}, X)$ and consider $\varphi(\bar{v}; \bar{b}) \in \operatorname{tp}(\bar{a}, X)$. Then, as in (i), we have $\varphi(\bar{v}; \bar{b}) \leftrightarrow \varphi(\bar{w}; \bar{b}) \in \mathrm{FE}^m(X)$, and since $\models \varphi[\bar{a}; \bar{b}]$, we conclude $\models \varphi[\bar{c}; \bar{b}]$ as desired.

Claims (iv), (v), and (vi) are straightforward. $\square$

Lemma 6.3.2. *(i) The set $\{\varphi(\bar{v}; \bar{b}) \mid \varphi(\bar{v}; \bar{b})$ is almost over $X\}$ is closed under all connectives and quantifiers, addition of dummy variables, and permutation of variables.*

(ii) The number of formulas almost over X up to logical equivalence is at most $|X| + |T|$. As a consequence, the number of non-equivalent types almost over X is at most $2^{|X|+|T|}$.

(iii) Suppose $\operatorname{stp}(\bar{a}, X) \equiv \operatorname{stp}(\bar{b}, X)$. Then there is an automorphism F of $\mathfrak{M}$ such that $F(\bar{a}) = \bar{b}$, F identity on X, and F preserves formulas (up to logical equivalence) almost over X.[2]

Proof. In proving (i), the connective $\neg$ is trivial. For $\wedge$ and the operations on variables, note that $\mathrm{FE}(X)$ is closed under these as well. Now we show closure under the existential quantifier. Suppose $\varphi(v_0, \bar{v}; \bar{b})$ is almost over X depending on $E(v_0, \bar{v}; w_0, \bar{w})$. We define the equivalence relation

$$E'(\bar{v}; \bar{w}) = (\forall v_0 \exists w_0 E(v_0, \bar{v}; w_0, \bar{w})) \wedge (\forall w_0 \exists v_0 E(v_0, \bar{v}; w_0, \bar{w})).$$

Then $E' \in \mathrm{FE}(X)$ and $\exists v_0 \varphi(v_0, \bar{v}; \bar{b})$ is almost over X dependent on E'.

Now we show (ii). First we note that if φ and ψ are almost over X dependent on E and F respectively, then $\neg\varphi$ depends on E, and $\varphi \wedge \psi$ depends on $E \wedge F$, (this buried in details omitted in the proof of (i)). If one writes it all out, we can see that,

[2]Cf. Lemma III.2.2 in [9]. An analogous result holds for stp_* with a set instead of tuples.

if φ and ψ both depend on E, then so does $\varphi \leftrightarrow \psi$. This means

$$E(\bar{v}, \bar{w}) \to ((\varphi(\bar{v}) \leftrightarrow \psi(\bar{v})) \leftrightarrow (\varphi(\bar{w}) \leftrightarrow \psi(\bar{w})),$$

where we have reduced clutter by omitting parameters. At $\bar{v} = \bar{w}$, we have $\varphi \leftrightarrow \psi$. In other words, if two formulas depend on the same equivalence relation, they are equivalent. Thus it suffices to count $|\mathrm{FE}(X)| \leq |X| + |T|$.

Finally, we prove (iii). In particular, $\mathrm{tp}(\bar{a}, X) = \mathrm{tp}(\bar{b}, X)$ by Lemma 6.3.1(ii),(iii). So we automorph $\mathfrak{M}$ as usual, sending $F(\bar{a}) = \bar{b}$ and fixing X. If $\varphi(\bar{v}; \bar{c})$ is almost over X depending on $E(\bar{v}; \bar{d})$, $\bar{d} \in X$, then F will not change E, and $F(\varphi) = \varphi(\bar{v}; F(\bar{c}))$ will be almost over X depending on $F(E) = E$. As remarked in the proof if (ii), we will then have $F(\varphi) \leftrightarrow \varphi$ as desired. $\square$

Lemma 6.3.3. *Let* $\Gamma = \{\varphi(\bar{v}; F(\bar{a})) \mid F$ *is an automorphism of* $\mathfrak{M}$ *identity on* $X\}$. *Then the following are equivalent:*

(i) $\varphi(\bar{v}; \bar{a})$ *is almost over* X

(ii) Γ *has finitely many formulas up to logical equivalence*

(iii) Γ *has size* $< |\mathfrak{M}|$ *(up to logical equivalence)*

Moreover, $\varphi(\bar{v}; \bar{a})$ *is equivalent to a formula with parameters* X *if and only if* Γ *has only one formula up to* $\leftrightarrow$.[3]

Remark. *Lest we actually evoke* $\bar{\kappa}$, *condition (iii) above says* Γ *stops growing eventually as we take larger and larger approximations to the monster model, (actually it is finite, by the equivalence). The negation says* Γ *grows arbitrarily with the size of* $\mathfrak{M}$. *As typical, we work as if we have* $\bar{\kappa}$ *in the proof. With regards to sloppiness, we note that the proof of the "moreover" part includes some serious slippage between target theory and metatheory reasoning. This is to avoid clutter with the symbol* $\models$.

[3] Cf. Lemma III.2.3 in [9].

Proof. The direction (ii)⇒(iii) is trivial. For the reverse direction, we use compactness, showing consistency of the theory

$$T \cup \{\varphi(\bar{v};\bar{c}_i) \not\leftrightarrow \varphi(\bar{v};\bar{c}_j) \mid i \neq j < |\mathfrak{M}|\} \cup \Sigma,$$

where Σ expresses indiscernibility of $\bar{a}$ and $\bar{c}_i$ over X for all i. This allows us to map $F(\bar{a}) = \bar{c}_i$ fixing X, achieving $|\mathfrak{M}|$ nonequivalent models.

Since each $E \in \mathrm{FE}(X)$ has only finitely many equivalence classes, we have (i)⇒(ii). As our last direction, we show the reverse. Assume Γ is finite. Then there exists a (least) n such that the set

$$\Gamma_1 = \{\theta(\bar{w}_i;\bar{c}) \mid \theta(\bar{w}_i;\bar{c}) \in \mathrm{tp}(\bar{a},X), i < n\} \cup \{\neg\forall\bar{v}(\varphi(\bar{v};\bar{w}_i) \leftrightarrow \varphi(\bar{v};\bar{w}_j)) \mid i < j < n\}$$

is inconsistent, (the first half asserts that $\bar{w}_i$ is the image $F(\bar{a})$ for some automorphism, and the second half asserts nonequivalence).

We may assume the contradiction is in Γ_2, which is the same as Γ_1 except remove the condition $\theta(\bar{w}_i,\bar{c}) \in \mathrm{tp}(\bar{a},X)$ and use $\theta(\bar{w},\bar{c})$ such that $\models \theta[\bar{a};\bar{c}]$, (still with different variables $\bar{w}_i$). This is accomplished by taking the conjunction of the finitely many θ present, and $\bar{c} \in X$ will account for all parameters.

By leastness of n, we can find $n-1$ elements to satisfy Γ_2, meaning they all satisfy $\theta(\bar{w}_i;\bar{c})$, they are all φ-inequivalent, and any nth element found satisfying $\theta(\bar{w};\bar{c})$ must be φ-equivalent to one of these. Symbolically (and indeed a priori),

$$\models \exists\bar{w}_0\ldots\bar{w}_{n-2}\Big[\bigwedge_{i<n-1}\theta(\bar{w}_i;\bar{c}) \wedge \bigwedge_{i\neq j}\neg\forall\bar{v}(\varphi(\bar{v};\bar{w}_i) \leftrightarrow \varphi(\bar{v};\bar{w}_j))$$
$$\wedge\, \forall\bar{w}\big[\theta(\bar{w};\bar{c}) \rightarrow \bigvee_{i<n-1}\forall v(\varphi(\bar{v};\bar{w}_i) \leftrightarrow \varphi(\bar{v};\bar{w}))\big]\Big].$$

Let $E(\bar{v}_1; \bar{v}_2) = \forall \bar{w}[\theta(\bar{w}; \bar{c}) \to (\varphi(\bar{v}_1, \bar{w}) \leftrightarrow \varphi(\bar{v}_2, \bar{w}))]$. One can see that E is an equivalence relation, and if $E \in \mathrm{FE}(X)$, then $\varphi(\bar{v}; \bar{a})$ depends on E. The whole point is to show that E has finitely many equivalence classes, which is established in the above block of logic.

Finally, we prove the "moreover". The forward direction is obvious. For the reverse, we note that for some $\theta(\bar{w}; \bar{c}) \in \mathrm{tp}(\bar{a}, X)$ we have

$$\models \exists \bar{w}_0 \Big[\theta(\bar{w}_0; \bar{c}) \wedge \forall w \big[\theta(\bar{w}; \bar{c}) \to \forall v(\varphi(\bar{v}; \bar{w}_0) \leftrightarrow \varphi(\bar{v}; \bar{w}))\big]\Big].$$

Let $\psi(\bar{v}; \bar{c}) = \forall w[\theta(\bar{w}; \bar{c}) \to \varphi(\bar{v}; \bar{w})]$. We claim that $\varphi(\bar{v}; \bar{a}) \leftrightarrow \psi(\bar{v}; \bar{c})$. Indeed, since $\models \theta[\bar{a}; \bar{c}]$, we instantiate $\bar{w}$ at $\bar{a}$ and conclude $\psi(\bar{v}; \bar{c}) \to \varphi(\bar{v}; \bar{a})$. Conversely, if $\varphi(\bar{v}; \bar{a})$ and consider $\bar{w}$ such that $\theta(\bar{w}; \bar{c})$. Then we have $\varphi(\bar{v}; \bar{a}) \leftrightarrow \varphi(\bar{v}; \bar{w}_0) \leftrightarrow \varphi(\bar{v}; \bar{w})$. This shows $\varphi(\bar{v}; \bar{a}) \to \psi(\bar{v}; \bar{c})$ as desired. □

We derive two corollaries of this lemma, (and other results). The first (or at least, its proof) has the same flavor as Galois theory.

Corollary 6.3.4. *Suppose $E(\bar{v}; \bar{w})$ is a finite equivalence relation with arbitrary parameters almost over X. Then there exists $E' \in \mathrm{FE}(X)$ refining E, i.e.*

$$\forall \bar{v}, \bar{w}[E'(\bar{v}; \bar{w}) \to E(\bar{v}; \bar{w})].$$ [4]

Proof. We consider $E(\bar{v}; \bar{w})$ as a formula and use the above lemma. Since Γ is finite, there exist finite equivalence relations $E_0 \dots E_{n-1}$ over some parameters, such that any automorphism F of $\mathfrak{M}$ fixing X (in particular, identity) will take E to one of the E_i. Then $E'(\bar{v}; \bar{w}) = \bigwedge_{i<n} E_i(\bar{v}; \bar{w})$ is a finite equivalence relation invariant under any X-automorphism of $\mathfrak{M}$. By the "moreover" part of the previous lemma, we conclude E' in fact has parameters in X and clearly E' refines E. □

[4] Cf. Lemma III.2.4 in [9].

Corollary 6.3.5. *Suppose T stable, and $I = \{\bar{a}_i \mid i < \omega\}$ an infinite indiscernible set based on X. Let $\varphi(\bar{v}; \bar{w})$ be a formula, and for $n < \omega$ let*

$$\varphi_n(\bar{v}; \bar{a}^n) = \bigvee_{\substack{s \subseteq 2n \\ |s| = n}} \bigwedge_{i \in s} \varphi(\bar{v}; \bar{a}_i),$$

($\bar{a}^n$ includes all $\bar{a}_0 \ldots \bar{a}_{2n-1}$). Then for all large enough n (size depending only on φ), φ_n is almost over X.[5]

Proof. By Lemma 4.5.2, there exists n such that for any indiscernible set I and all $\bar{a}$, $\ell(\bar{a}) = \ell(\bar{v})$, we have either $|\varphi(\bar{a}; I)| < n$ or $|\neg\varphi(\bar{a}; I)| < n$. (We flip the variables around in our notation, so really we use the lemma at $\psi(\bar{v}; \bar{w}) = \varphi(\bar{w}; \bar{v})$.) In particular, this is true of all $n \geq n(\varphi)$. So if $\bar{b}$ has $\models \varphi_n[\bar{b}; \bar{a}^n]$, then there exists $s \subseteq 2n$, $|s| = n$, with $\models \varphi[\bar{b}; \bar{a}_i]$ for all $i \in s$, meaning $|\varphi(\bar{b}; I)| \geq n$, so $|\neg\varphi(\bar{b}; I)| < n$, hence $\varphi(\bar{b}; \bar{w}) \in \mathrm{Av}(I, \bar{b})$. The converse also holds; membership to the average type implies $|\varphi(\bar{b}; I)| = \aleph_0$. We cannot immediately conclude that there exists $s \subseteq 2n$, $|s| = n$, with $\models \varphi[\bar{b}; \bar{a}_i]$ for all $i \in s$, since although infinitely elements of I work, we might not be able to find n in the first $2n$. However, we can say $|\neg\varphi(\bar{a}; I)| < n$ for some n, and then we can conclude $\models \varphi_n[\bar{b}; \bar{a}^n]$. We have shown that

$$\models \varphi_n[\bar{b}; \bar{a}^n] \text{ if and only if } \varphi(\bar{b}; \bar{w}) \in \mathrm{Av}(I, \bar{b}).$$

So, by Lemma 6.3.3, if $\varphi_n(\bar{v}; \bar{a}_n)$ is not almost over X, $|\Gamma|$ can be as large as we want, say of size $\mu = ((|X|+2)^{|T|})^+$ via X-automorphisms F_i, $i < \mu$, sending $F_i(\bar{a}^n) = \bar{a}_i^n$, $\varphi_n(\bar{v}; \bar{a}_i^n)$ pairwise inequivalent. Let $\bar{b}_{i,j}$, $i < j < \mu$, establish $\models \varphi_n[\bar{b}_{i,j}; \bar{a}_i^n] \not\leftrightarrow \varphi_n[\bar{b}_{i,j}; \bar{a}_j^n]$. Therefore, $\varphi(\bar{b}_{i,j}; \bar{w}) \in \mathrm{Av}(F_i(I), \bar{b})$ if and only if $\varphi(\bar{b}_{i,j}; \bar{w}) \notin \mathrm{Av}(F_j(I), \bar{b})$. In other words, $F_i(I)$ and $F_j(I)$ are inequivalent, (by Lemma 5.4.5). Also, since I is based on X and F_i is an X-automorphism, we have $F_i(I)$ is also based on X. But this

[5] Cf. Lemma III.2.5 in [9].

gives μ inequivalent indiscernible sets based on X in contradiction to Lemma 5.4.11. We conclude $\varphi_n(\bar{v};\bar{a}^n)$ is almost over X for all $n \geq n(\varphi)$. □

Lemma 6.3.6. *Suppose $p \in \mathbf{S}^m(Y)$ does not fork over $X \subseteq Y$ and either:*

(i) $q \supseteq p$ is a type in the same variables almost over Y, or

(ii) $E(\bar{v};\bar{w}) \in \mathrm{FE}^m(Y)$ and $q = p \cup \{E(\bar{v};\bar{a})\}$ is consistent (for some $\bar{a}$).

Then in either case, q does not fork over p, and $R^m(q,\Delta,\aleph_0) = R^m(p,\Delta,\aleph_0)$ for all finite Δ.[6]

Proof. We first show that (i) will follow from (ii), (i.e. it suffices to prove the lemma assuming (ii) and then the lemma will hold also assuming (i)). Suppose $q \supseteq p$ is almost over Y and forks over X. Let $q' \supseteq q$ be a finite forking subtype (see Lemma 5.3.1(v)) which is also almost over Y. Then by Lemma 6.3.2(i), $\varphi(\bar{v};\bar{b}) = \bigwedge q'$ is almost over Y, say depending on $E(\bar{v};\bar{w}) \in \mathrm{FE}^m(Y)$. (Also, since $\varphi \equiv q'$, φ forks over X.) Let $\bar{a}$ realize q. Then since $\models \varphi[\bar{a};\bar{b}]$, we have $\models \forall v(E(\bar{v};\bar{a}) \to \varphi(\bar{v};\bar{b}))$. Thus $p \cup \{E(\bar{v};\bar{a})\} \vdash \varphi$, so $p \cup \{E(\bar{v};\bar{a})\}$ forks over X. Similarly, if we let q' be a finite subtype of the same rank as q assumed to have $R^m(q,\Delta,\aleph_0) < R^m(p,\Delta,\aleph_0)$, then we'd get $R^m(p \cup \{E(\bar{v};\bar{a})\},\Delta,\aleph_0) \leq R^m(\varphi,\Delta,\aleph_0) = R^m(q,\Delta,\aleph_0) < R^m(p,\Delta,\aleph_0)$. So it suffices to prove the lemma assuming (ii), and then this argument will end in contradiction, proving the lemma assuming (i).

Assume (ii). Choose $\bar{a}_0 \ldots \bar{a}_{n-1}$ such that $\mathrm{tp}(\bar{a}_k,Y) = \mathrm{tp}(\bar{a},Y)$ but $\models \neg E(\bar{a}_l,\bar{a}_k)$ for all $l \neq k < n = n(E)$, (hence this set is maximal). This means that the set $\mathrm{tp}(\bar{a},Y) \cup \{\neg E(\bar{v};\bar{a}_k) \mid k < n\}$ is inconsistent. Since tp is closed under finite conjunctions we can choose $\varphi(\bar{v};\bar{c}) \in \mathrm{tp}(\bar{a},Y)$ such that $\varphi(\bar{v};\bar{c}) \vdash \bigvee_{k<n} E(\bar{v};\bar{a}_k)$.

Assume for contradiction that $p \cup \{E(\bar{v};\bar{a})\}$ forks over X. We can choose Y-automorphisms that send $\bar{a}$ to $\bar{a}_k$, and so $p \cup \{E(\bar{v};\bar{a}_k)\}$ also forks over X (note $X \subseteq Y$) for each $k < n$. By Lemma 5.3.1(vi), $p \cup \{\bigvee_{k<n} E(\bar{v};\bar{a}_k)\}$ forks over X. Since $\bar{a}$ does not necessarily realize p but it does realize $\varphi(\bar{v};\bar{c})$, we have $\exists \bar{w}(\varphi(\bar{w};\bar{c}) \wedge$

[6] Cf. Lemma III.2.6 in [9].

$E(\bar{v};\bar{w})) \in p$ by completeness (and a change of variable in φ), lest $p \cup \{E(\bar{v};\bar{a})\}$ be inconsistent. Thus, if $\bar{b}$ realizes p, there exists $\bar{a}^*$ realizing $\varphi(\bar{w};\bar{c}) \wedge E(\bar{b};\bar{w})$, hence also $\bigvee_{k<n} E(\bar{a}^*;\bar{a}_k)$ and $\bigvee_{k<n} E(\bar{b};\bar{a}_k)$ by transitivity of E. Since the $\bar{b}$ realizing p was arbitrary, we get $p \vdash p \cup \{\bigvee_{k<n} E(\bar{v};\bar{a}_k)\}$. This means p forks over X in contradiction to the hypothesis.

Now we assume for contradiction that $R^m(q,\Delta,\aleph_0) = \alpha < R^m(p,\Delta,\aleph_0)$. The Y-automorphisms sending $\bar{a}$ to $\bar{a}_k$ preserve rank by Lemma 4.2.1, so each $p \cup \{E(\bar{v};\bar{a}_k)\}$ has rank α. By Lemma 4.2.4, the rank of $p \cup \{\bigvee_{k<n} E(\bar{v};\bar{a}_k)\}$ is also α. But from the previous paragraph, we have $R^m(p,\Delta,\aleph_0) \leq R^m(p \cup \{\bigvee_{k<n} E(\bar{v};\bar{a}_k)\},\Delta,\aleph_0) = \alpha$, a contradiction. □

Corollary 6.3.7. *Suppose p is almost over X. Then p does not fork over X.*[7]

Proof. Let $q \supseteq p$ be a complete m-type over $X \cup \mathrm{param}(p)$. We know $q \upharpoonright X$ does not fork over X by Corollary 5.3.3. Then $q \upharpoonright X \cup p \supseteq p$ is almost over X, so by the previous lemma, it does not fork over X. Hence neither does p. □

6.4 The Finite Equivalence Relation Theorem

Theorem 6.4.1. *Suppose T is stable, $p, q \in \mathbf{S}^m(A)$, $p \neq q$ and neither forks over $X \subseteq A$, and $\mathfrak{A}$ is $((|X|+2)^{|T|})^+$-saturated. Then there exists $E(\bar{v};\bar{w}) \in \mathrm{FE}^m(X)$ such that*

$$p(\bar{v}) \cup q(\bar{w}) \vdash \neg E(\bar{v};\bar{w}).^{8}$$

Proof. Let $\varphi(\bar{v};\bar{b}) \in p$, $\neg\varphi(\bar{v};\bar{b}) \in q$, $\bar{b} \in A$. By Theorem 5.4.12, there exists a set $I = \{\bar{b}_i \mid i < \omega\}$ indiscernible over X based on X with $\bar{b}_0 = \bar{b}$. Since not forking implies not strongly splitting, it must be the case that $\varphi(\bar{v};\bar{b}_i) \in p$ and $\neg\varphi(\bar{v};\bar{b}_i) \in q$ for all i.

[7] Cf. Corollary III.2.7 in [9].
[8] Cf. Theorem III.2.8 in [9].

By Lemma 6.3.5, there exists $n < \omega$ such that $\varphi_n(\bar{v}; \bar{b}^n)$ is almost over X, say dependant on $E(\bar{v}; \bar{w}) \in FE(X)$.[9] Since $\varphi_n(\bar{v}; \bar{b}_n) \in p$ and $\neg\varphi_n(\bar{v}; \bar{b}_n) \in q$, we have

$$p(\bar{v}) \cup q(\bar{w}) \vdash (\varphi_n(\bar{v}; \bar{b}^n) \wedge \neg\varphi_n(\bar{w}; \bar{b}^n)) \vdash \neg E(\bar{v}; \bar{w}).$$

$\square$

We can eliminate the saturation condition as follows.

Corollary 6.4.2. *Suppose T is stable. Then:*

(i) For all $\bar{a}$ and X, $\operatorname{stp}(\bar{a}, X)$ is stationary over X.

(ii) If $p, q \in \mathbf{S}(Y)$, $p \neq q$ and neither forks over $X \subseteq Y$, then there exists $E(\bar{v}; \bar{w}) \in \mathrm{FE}^m(X)$ such that

$$p(\bar{v}) \cup q(\bar{w}) \vdash \neg E(\bar{v}; \bar{w}).\text{[10]}$$

Proof. To prove (i), let $p = \operatorname{stp}(\bar{a}, X)$. We know p is almost over X, so by Corollary 6.3.7 p does not fork over X. Now suppose p_1, p_2 are two contradictory extensions nonforking over X. We will derive a contradiction, from which we conclude p is stationary over X. Let $\mathfrak{A}$ be a $((|X| + 2)^{|T|})^+$-saturated model, $A \supseteq \operatorname{param}(p_1) \cup \operatorname{param}(p_2)$. We may assume, by taking nonforking extensions, that in fact $p_1, p_2 \in \mathbf{S}^m(A)$. Let $E(\bar{v}; \bar{w}) \in \mathrm{FE}^m(X)$ have $p_1(\bar{v}) \cup p_2(\bar{w}) \vdash \neg E(\bar{v}; \bar{w})$. But since $E(\bar{v}; \bar{a}) \in p \subseteq p_1, p_2$, we get

$$p_1(\bar{v}) \cup p_2(\bar{w}) \vdash \neg E(\bar{v}; \bar{w}) \wedge E(\bar{v}; \bar{a}) \wedge E(\bar{w}; \bar{a}),$$

an impossibility since E must be transitive.

We prove (ii) by contradiction. If there is no such E, then the set

$$\Gamma = p(\bar{v}) \cup q(\bar{w}) \cup \{E(\bar{v}; \bar{w}) \mid E \in \mathrm{FE}^m(X)\}$$

[9] We use the notation of Lemma 6.3.5 for the definition of $\varphi_n(\bar{v}; \bar{b}^n)$.

[10] Cf. Corollary III.2.9 in [9]. This is the result we will reference later on.

would be consistent, (finitely consistent since FE^m is closed under finite conjunction). Let $\bar{a}$ realize $\bar{v}$ and $\bar{b}$ realize $\bar{w}$ in Γ. This gives $\mathrm{stp}(\bar{a}, X) \equiv \mathrm{stp}(\bar{b}, X)$, $\bar{a}$ realizes p and $\bar{b}$ realizes q. But this means $\mathrm{stp}(\bar{a}, X)$ is stationary over X, contradicting (i). $\square$

6.5 Useful Consequences

Lemma 6.5.1. *Suppose I is an infinite indiscernible set over X and $\varphi(\bar{v}_0 \ldots \bar{v}_{n-1}; \bar{c})$ is almost over X. Then for some $t = 0, 1$ we have that for all distinct $\bar{a}_0 \ldots \bar{a}_{n-1} \in I$, $\models \varphi[\bar{a}_0 \ldots \bar{a}_{n-1}; \bar{c}]^t$.*[11]

Proof. Suppose for contradiction that for $t = 0, 1$ there was $\bar{a}^t_0 \ldots \bar{a}^t_{n-1} \in I$ with $\models \varphi[\bar{a}^t_0 \ldots \bar{a}^t_{n-1}\bar{c}]^t$. By indiscernibility, we may assume that $\bar{a}^0_i \neq \bar{a}^1_i$ for all $i < n$. Since φ is almost over X, say dependant on $E(\bar{v}; \bar{w})$, we have $\models \neg E(\bar{a}^0_0 \ldots a^0_{n-1}; \bar{a}^1_1 \ldots a^1_{n-1})$. But by indiscernibility (note E perhaps takes parameters in X) and disjointness of $\bar{a}^t_i$ for $i < n$, $t = 0, 1$, this implies that there exists infinitely many equivalence classes, contradicting the definition of FE. $\square$

Corollary 6.5.2. *Let I an infinite indiscernible set over $X \subseteq Y$. Then:*

(i) Suppose T is stable and for all $\bar{c} \in \bigcup I$ (i.e. of all finite lengths), $\mathrm{tp}(\bar{c}, Y)$ *does not fork over X. Then I is indiscernible over Y.*

(ii) Suppose that for all $\bar{b} \in Y$, $\mathrm{tp}(\bar{b}, X \cup \bigcup I)$ *does not strongly split over X, then I is indiscernible over Y.*[12]

Proof. (i) Let $\bar{a}_0, \ldots \bar{a}_{n-1}, \bar{b}_0 \ldots \bar{b}_{n-1} \in I$. It suffices to show $\mathrm{tp}(\bar{a}_0 \ldots \bar{a}_{n-1}, Y) = \mathrm{tp}(\bar{b}_0 \ldots \bar{b}_{n-1}, Y)$. Indeed, by the proof of the above lemma their strong types are equivalent (lest there exist $E \in \mathrm{FE}^m(Y)$ which we can show has infinitely many equivalence classes). By assumption, neither type forks, so they must be the same type lest the strong type not be stationary, which would contradict Corollary 6.4.2.

[11] Cf. Lemma III.2.10 in [9]. Also recall Notation 2.2.2 for the exponent t.
[12] Cf. Corollary III.2.11 in [9].

(ii) Let $\bar{a}_0 \dots \bar{a}_n, \bar{c}_0 \dots \bar{c}_n \in I$, $n \geq 0$. If we let I^n be the set of all n-tuples from I, then I^n is also indiscernible over X. Since $\mathrm{tp}(\bar{b}, X \cup \bigcup I)$ does not strongly split over X, we must have for every formula $\varphi(\bar{v}; \bar{b})$, $\ell(\bar{v}) = (n+1) \cdot \ell(\bar{a}_0)$,

$$\models \varphi[\langle \bar{a}_0 \dots \bar{a}_n \rangle; \bar{b}] \leftrightarrow \varphi[\langle \bar{c}_0 \dots \bar{c}_n \rangle; \bar{b}]$$

as desired.[13] □

Lemma 6.5.3. *Let A be the universe of a model $\mathfrak{A}$.*

(i) Suppose $\varphi(\bar{v}; \bar{b})$ is almost over A. Then for some $\bar{a} \in A$ and formula $\psi(\bar{v}; \bar{w})$, $\vdash \varphi(\bar{v}; \bar{b}) \leftrightarrow \psi(\bar{v}; \bar{a})$.

(ii) Suppose $p \in \mathbf{S}^m(A)$ does not fork over $X \subseteq A$ and let $\bar{a}$ realize p. Then p is stationary over X and $p \equiv \mathrm{stp}(\bar{a}; A)$.[14]

Proof. (i) Let $\varphi(\bar{v}; \bar{b})$ depend on $E \in \mathrm{FE}(A)$. Let $\bar{a}_0 \dots \bar{a}_{n-1} \in A$ be representatives (in the *model* $\mathfrak{A}$) for the $n = n(E)$ equivalence classes of E. Depending on whether or not $\models \varphi[\bar{a}_i, \bar{b}]$, we have $\models \forall \bar{v}(E(\bar{v}; \bar{a}_i) \rightarrow \varphi(\bar{v}; \bar{b})^t)$ for some $t = 0, 1$. By reordering, we suppose $t = 0$ for $i < m \leq n$ and $t = 1$ for $m \leq i < n$. Since $\models \forall \bar{v} \bigvee_{i<n} E(\bar{v}; \bar{a}_i)$, we will have $\vdash \varphi[\bar{v}; \bar{b}] \leftrightarrow \bigvee_{i<m} E(\bar{v}; \bar{a}_i)$ if $m > 0$, and $\vdash \varphi[\bar{v}; \bar{b}] \leftrightarrow (\bar{v} \neq \bar{v})$ if $m = 0$.

(ii) By Corollary 6.4.2(i), it suffices to prove $p \vdash \mathrm{stp}(\bar{a}, X)$. This will prove p stationary over X, and then since $\mathrm{stp}(\bar{a}, A) \vdash \mathrm{tp}(\bar{a}, A) = p$ we get equivalence. Take $E \in \mathrm{FE}^m(A)$. As in the proof of (i), take $\bar{a}_0 \dots \bar{a}_{n-1} \in A$ such that $\models \forall \bar{v} \bigvee_{i<n} E(\bar{v}; \bar{a}_i)$. For some j we have $\models E(\bar{a}; \bar{a}_j)$, hence $p \vdash E(\bar{v}; \bar{a}_j)$ (since p is complete over A and we'd otherwise have $\neg E(\bar{v}; \bar{a}_j) \in p$ hence $\models \neg E(\bar{a}; \bar{a}_j)$) and so $p \vdash E(\bar{v}; \bar{a})$. □

[13] Here $\langle \bar{a}_0 \dots \bar{a}_n \rangle$ and $\langle \bar{c}_0 \dots \bar{c}_n \rangle$ represent singular members of I^{n+1} to match the definition of strong splitting.

[14] Cf. Lemma III.2.15 in [9].

Chapter 7

The Stability Spectrum

7.1 Statement and Proof Idea

Theorem 7.1.1 (The Stability Spectrum Theorem)**.** *Suppose T is stable. Then:*

(i) If μ_0 is the first cardinal $\geq |T|$ such that T is stable, then $\mu_0 \leq 2^{|T|}$. T is stable in $\mu \geq |T|$ if and only if $\mu = \mu_0 + \mu^{<\kappa(T)}$. In fact, $\mu_0 = \sup_{|X| \leq |T|} |\mathbf{S}(X)|$.

(ii) T is superstable (i.e. λ-stable for all large enough λ) if and only if $\kappa(T) = \aleph_0$.[1]

We sketch the proof, which will be given in full by the end of this chapter. As we will note later, this differs slightly from Shelah's statement in [9], (specifically, we stop at a statement whose computation is more complex, and do not develop the machinery needed to simplify it).

The first order of business will be defining the cardinal $\kappa(T)$, which is roughly a measure of how much strong splitting can occur in models of T. In order to bound $\kappa(T)$ we will need to establish a few equivalent definitions. We also show how to find a small[2] subset of the parameters of a type p over which p does not fork.

Then we will have two main lemmas: one involving stability, and one involving in-

[1] Cf. Corollary III.3.8 in [9]. In fact, the requirement that $\mu \geq |T|$ is omitted there, so a fortiori T is never stable below $|T|$. We were unable to justify this step without additional machinery found in [1]; Cf. Theorem 4.34 and its preceding lemmas.

[2] Small relative to $\kappa(T)$

stability. For the first lemma, we will have a few cardinal equations that are sufficient for stability. We assume that T is unstable, and reach a contradiction by counting nonforking types. The second lemma proceeds similarly to 3.3.16, constructing a snowflake mapping and then unfolding to get instability, (see the preceding paragraph to that lemma).

7.2 $\kappa(T)$

We begin with the following.

Theorem 7.2.1. *Suppose T is stable. Then the following are equivalent, (for any T, κ, m, and λ):*

(i) There is an increasing sequence X_i, $i \leq \kappa$, and $p \in \mathbf{S}^m(X_\kappa)$ such that $p \restriction X_{i+1}$ forks over X_i for all $i < \kappa$.

(ii) As in (i), but with "forks" replaced by "splits strongly".

(iii)$_\lambda$ There exists an increasing sequence X_i, $i \leq \kappa$, and $q \in \mathbf{S}^m(X_\kappa)$, sets $I_i = \{\bar{a}^i_j \mid j < \lambda\} \subseteq X_{i+1}$ indiscernible over X_i and $\varphi_i(\bar{v};\bar{a}^i_0) \in q$ such that $\neg\varphi_i(\bar{v};\bar{a}^i_j) \in q$ for $j > 0$ and there exists $m_i < \omega$ such that $\{\varphi_i(\bar{v};\bar{a}^i_j) \mid j < \lambda\}$ is m_i-inconsistent.

(iv)$_\lambda$ There exists an increasing sequence of $|A_i|^+$-saturated models $\mathfrak{A}_i$, $i \leq \kappa$, and $q \in \mathbf{S}^m(A_\kappa)$, sets $I_i = \{\bar{a}^i_j \mid j < \lambda\} \subseteq A_{i+1}$ indiscernible over A_i and $\varphi_i(\bar{v};\bar{a}^i_0) \in q$ such that $\neg\varphi_i(\bar{v};\bar{a}^i_j) \in q$ for $j > 0$ and there exists $m_i < \omega$ such that $\{\varphi_i(\bar{v};\bar{a}^i_j) \mid j < \lambda\}$ is m_i-inconsistent.[3]

Using Theorem 7.2.1 we define $\kappa(T)$.

Definition 7.2.2. Let $\kappa^m(T)$ be the first infinite cardinal for which the above fails, and $\kappa(T) = \sup_{m<\omega} \kappa^m(T)$. (We will show later that $\kappa(T) < \infty$ for stable T.) In the case T is unstable, we stipulate $\kappa(T) = \infty$.

[3]Cf. Theorem III.3.1 in [9].

Proof of Theorem 7.2.1. The direction $(\mathrm{iv})_\lambda \Rightarrow(\mathrm{iii})_\lambda$ is trivial, and $(\mathrm{iii})_\lambda \Rightarrow$(ii) (for any infinite λ) is easy to see. Theorem 5.3.6(i) shows (ii)$\Rightarrow$(i). We complete the proof by showing (i)$\Rightarrow(\mathrm{iv})_\lambda$.

By the definition of forking, we have $p \restriction X_{i+1} \vdash \bigvee_{k<n_i} \varphi^i_k(\bar{v};\bar{a}^i_k)$ with each $\varphi^i_k(\bar{v};\bar{a}^i_k)$, $k < n_i$, dividing over X_i. So (using Lemma 5.3.1(iii)) there exists a set $\{\bar{a}^i_{k,l} \mid l<\omega\}$ indiscernible over X_i, $\bar{a}^i_{k,0} = \bar{a}^i_k$ and $\{\varphi^i_k(\bar{v};\bar{a}^i_{k,l}) \mid l<\omega\}$ m_i-inconsistent for some $m_i<\omega$. We may assume that the sequence X_i is continuous at limit ordinals, (otherwise shrink the limit sets X_δ to make it continuous, and (i) will hold with this new sequence).

Let $Y_i = \bigcup_{k,l} \bar{a}^i_{k,l}$. We will define elementary mappings F_i and models $\mathfrak{A}_i$ by recursion on $i \leq \kappa$, such that $\mathrm{Dom}(F_\delta) = X_\delta$, $\mathfrak{A}_\delta = \bigcup_{i<\delta} \mathfrak{A}_i$, $\mathrm{Dom}(F_{i+1}) = X_{i+1} \cup Y_i$, $\mathfrak{A}_0$ is $(\lambda + |X_\kappa|)^+$-saturated, $\mathfrak{A}_{i+1}$ is $|A_i|^+$-saturated, $F_j \restriction X_i = F_i \restriction X_i$ and $\mathfrak{A}_i \subseteq \mathfrak{A}_j$ for $i<j$, $\mathrm{Ran}(F_i) \subseteq A_i$, and for all $\bar{a} \in X_{i+1} \cup Y_i$, $\mathrm{tp}(F_{i+1}(\bar{a}), A_i)$ does not fork over $F_i(X_i)$.

At $i=0$, take F_0 as the identity on X_0 and $\mathfrak{A}_0$ to contain X_0 with enough saturation. At limit steps $i=\delta$, let $F_\delta = \bigcup_{i<\delta} F_i \restriction X_i$ and $\mathfrak{A}_\delta = \bigcup_{i<\delta} \mathfrak{A}_i$. For the successor step, suppose F_i, $\mathfrak{A}_i$ have been defined. We use Corollary 5.3.5 to define an elementary mapping $F_{i+1} : X_{i+1} \cup Y_i \to A_i \cup F_{i+1}[X_{i+1} \cup Y_i]$ with the nonforking property above, and then extend $A_i \cup F_{i+1}[X_{i+1} \cup Y_i]$ to an $|A_i|^+$-saturated model $\mathfrak{A}_{i+1}$.

Choose $q \in \mathbf{S}^m(A_\kappa)$ such that $F_\kappa(p) \subseteq q$. We have $p \restriction X_{i+1} \vdash \bigvee_{k<n_i} \varphi^i_k(\bar{v};\bar{a}^i_k)$, and so $F_{i+1}(p \restriction X_{i+1}) \vdash \bigvee_{k<n_i} \varphi^i_k(\bar{v};F_{i+1}(\bar{a}^i_k))$. Since $F_\kappa \restriction X_{i+1} = F_{i+1} \restriction X_{i+1}$, we get $q \vdash F_\kappa \restriction X_{i+1} \vdash \bigvee_{k<n_i} \varphi^i_k(\bar{v};\bar{a}^i_k)$. By completeness, we have for each $i<\kappa$ some $k(i)<n_i$ such that $\varphi^i_{k(i)}(\bar{v};F(\bar{a}^i_{k(i)})) \in q$. Furthermore, $\{\bar{a}^i_{k(i),l} \mid l<\omega\}$ is indiscernible over X_i, so $\{F_{i+1}(\bar{a}^i_{k(i),l}) \mid l<\omega\}$ is indiscernible over $F_{i+1}[X_i]$. By Corollary 6.5.2(i) and the definition of $\mathfrak{A}_i$, we have that $\{F_{i+1}(\bar{a}^i_{k(i),l}) \mid l<\omega\}$ is indiscernible over A_i.

Let $\bar{b}^i_j = F_{i+1}(\bar{a}^i_{k(i),j})$. Since $\mathfrak{A}_{i+1}$ is $|A_i|^+$-saturated and $\lambda < |A_i|^+$, we can define additional $\bar{b}^i_j \in A_{i+1}$ for $\omega \leq j < \lambda$ such that $I^i = \{\bar{b}^i_j \mid j < \lambda\}$ is indiscernible over A_i. (Just write down the type satisfied by an additional indiscernible element.) We have that $\{\varphi^i_{k(i)}(\bar{v}; \bar{b}^i_j) \mid j < \omega\}$ is $m^i_{k(i)}$-inconsistent, so only finitely many can appear in q. By omitting if necessary, we may assume that only $\varphi^i_{k(i)}(\bar{v}; \bar{b}^i_0) \in q$, hence $j > 0$ implies $\neg\varphi^i_{k(i)}(\bar{v}; \bar{b}^i_j) \in q$. We have shown all the necessary components of (iv)$_\lambda$, so our work is done. □

Corollary 7.2.3. *Suppose T is stable. For all $p \in \mathbf{S}^m(Y)$, there exists $X \subseteq Y$, $|X| < \kappa^m(T)$ such that p does not fork over X.*[4]

Remark. *Of course p does not fork over Y, but this shows we can find a smaller nonforking set, size determined by the theory at hand. As a second, unrelated remark, note that we need a stricter cardinal bound during the recursion below (instead of just $< \kappa^m(T)$) to ensure the size stays small at limit steps, (for instance, in the event this cardinal is not regular).*

Proof of Corollary 7.2.3. Suppose there was no such X. We will define $X_i \subseteq Y$ by recursion on $i < \kappa^m(T)$ such that $|X_i| < |i|^+ + \aleph_0$ and $p \restriction X_{i+1}$ forks over X_i. Then we will set $X_{\kappa^m(T)} = Y$, and contradict (i) in the theorem above, (since then $\kappa^m(T) < \kappa^m(T)$). Let $X_0 = \emptyset$ and take unions at limit ordinals. Having defined up to X_i, let p' be a finite forking (over X_i) subtype of p, (which forks over X_i since $|X_i| < \kappa^m(T)$). Then let $X_{i+1} = X_i \cup \mathrm{param}(p')$. Since p' forks over X_i, $p \restriction X_{i+1} \supseteq p'$ does as well. □

Corollary 7.2.4. *For stable T, $\kappa(T) \leq |T|^+$.*[5]

Proof. Suppose not. By Theorem 7.2.1(i), there exists an increasing sequence X_i, $i \leq |T|^+$, $m < \omega$, and $p \in \mathbf{S}^m(X_{|T|^+})$ such that for all $i < |T|^+$, $p \restriction X_{i+1}$ forks over X_i. As a result of Lemma 3.4.7, for every Δ there exists $\alpha(\Delta) < |T|^+$ such that

[4] Cf. Theorem III.3.2 in [9].

[5] Cf. Theorem III.3.3 in [9].

$R^m(p, \Delta, \aleph_0) = R^m(p \restriction X_{\alpha(\Delta)}, \Delta, \aleph_0)$ (chosen such that $p' \subseteq p \restriction X_{\alpha(\Delta)}$ where p' is a finite subtype of the same rank). There are $|T|$ such finite Δ, so $\alpha = \sup_\Delta \alpha(\Delta) < |T|^+$. But $p \subseteq p \restriction X_\alpha$ forks over X_α, and so by Lemma 5.3.2(ii) there exists Δ_0 such that $R^m(p \restriction X_\alpha, \Delta_0, \aleph_0) > R^m(p, \Delta_0, \aleph_0) (= R^m(p \restriction X_{\alpha(\Delta_0)}, \Delta_0, \aleph_0))$, a contradiction since $p \restriction X_\alpha \supseteq p \restriction X_{\alpha(\Delta_0)}$. □

7.3 Computing the Stability Spectrum

In this section we prove lemmas providing conditions for stability and instability, the latter of which generalizes Lemma 3.3.16 as promised at the beginning of Chapter 5. We then knit them together and prove the Stability Spectrum Theorem.

Lemma 7.3.1. *Suppose T is stable, and $\mu = \mu^{<\kappa^1(T)}$. Then T is μ-stable if any of the following hold:*

(i) $\mu \geq 2^{|T|}$,

(ii) $\mu \geq \sup_{|X| \leq |T|} |\mathbf{S}(X)|$, *or*

(iii) there exists μ_0 such that $|T| \leq \mu_0 \leq \mu$ and T is μ_0-stable.[6]

Proof. Suppose T is not μ-stable. Then there exists X and m such that $|X| \leq \mu < \mathbf{S}^m(X)$. By Lemma 3.3.9, we may assume $m = 1$. By Corollary 7.2.3, there is for every $p \in \mathbf{S}(X)$ a set $Y_p \subseteq X$, $|Y_p| < \kappa^1(T)$, such that p does not fork over Y_p. There are at most $\mu^{<\kappa^1(T)}$ such Y_p, so for some Y we have $|\{p \in \mathbf{S}(X) \mid Y_p = Y\}| > \mu$, (we are mapping a set of size $\geq \mu^+$, a regular cardinal, into a set of size $\leq \mu$).

Corollary 7.2.4 says $\kappa(T) \leq |T|^+$ and so $|Y| \leq |T|$. Take $Y \subseteq A$ for some model

[6] Cf. Lemma III.3.6 in [9].

$\mathfrak{A}$, $|A| = |T|$, (an application of embedding in $\mathfrak{M}$ and then the DLST). Then

$$\begin{aligned}\mu < |\{p \in \mathbf{S}(X) \mid Y_p = Y\}| &\leq |\{p \in \mathbf{S}(X) \mid p \text{ does not fork over } Y\}| \\ &\leq |\{p \in \mathbf{S}(A \cup X) \mid p \text{ does not fork over } Y\}| \\ &= |\{p \in \mathbf{S}(A) \mid p \text{ does not fork over } Y\}| \leq |\mathbf{S}(A)|,\end{aligned}$$

where the second to last equality (specifically $\geq$) follows from Lemma 6.5.3. Now if $\mu \geq 2^{|T|}$, we get a contradiction since easily $\mathbf{S}(A) \leq 2^{|T|}$. If $\mu \geq \sup_{|X| \leq |T|} |\mathbf{S}(X)|$, then in particular $\mu \geq \mathbf{S}(A)$, a contradiction. If T is stable in μ_0, $|A| = |T| \leq \mu_0$, then $|\mathbf{S}(A)| \leq \mu_0$ which contradicts $\mu_0 \leq \mu$. So any of the three assumptions will yield T being μ-stable. $\square$

Lemma 7.3.2. *Suppose T is stable, but $\mu < \mu^{<\kappa(T)}$. Then T is not stable in μ.*[7]

Proof. Let $\kappa < \kappa(T)$ be least such that $\mu^\kappa > \mu$. Then for some m, $\kappa < \kappa^m(T)$, so by Theorem 7.2.1(iii)$_\mu$ there exists an increasing sequence X_i, $i \leq \kappa$, $p \in \mathbf{S}^m(X_\kappa)$, $I_i = \{\bar{a}^i_j \mid j < \mu\} \subseteq X_{i+1}$ indiscernible over X_i, and $\varphi_i(\bar{v}; \bar{a}^i_0) \in p$ such that $j > 0$ implies $\neg\varphi_i(\bar{v}; \bar{a}^i_j) \in p$.

We will define elementary mappings F_η for all $\eta \in {}^{\leq\kappa}\mu$ by recursion on $\ell(\eta)$ such that:

(i) $\operatorname{dom}(F_\eta) = \bigcup\{\bar{a}^i_j \mid j < \mu, i < \ell(\eta)\}$

(ii) Whenever $\eta = \rho \restriction i$, F_ρ extends F_η

(iii) If $\ell(\eta) = i$, then for every j, $F_{\eta\frown j}(\bar{a}^i_0) = F_{\eta\frown 0}(\bar{a}^i_j)$, $F_{\eta\frown j}(\bar{a}^i_j) = F_{\eta\frown 0}(\bar{a}^i_0)$, and otherwise if $\alpha \neq 0, j$, then $F_{\eta\frown j}(\bar{a}^i_\alpha) = F_{\eta\frown 0}(\bar{a}^i_\alpha)$

Let $F_\emptyset = \emptyset$, and if $\ell(\eta)$ is a limit ordinal, let $F_\eta = \bigcup_{\gamma < \ell(\eta)} F_{\eta\restriction\gamma}$. For the successor step $\ell(\eta) = \alpha + 1$, we take $F_{\eta\restriction\alpha\frown 0}$ as an arbitrary extension of $F_{\eta\restriction\alpha}$ with the right domain, and $F_{\eta\restriction\alpha\frown\eta(\alpha)}$ is defined to satisfy (iii) above.

[7]Cf. Lemma III.3.7 in [9].

Once this is done, take $Y = \bigcup\{F_\eta(\bar{a}^i_j) \mid \eta \in {}^{<\kappa}\mu, i < \ell(\eta), j < \mu\}$. Note that $|Y| \leq \mu \cdot \kappa \cdot \sum_{\gamma<\kappa} \mu^\gamma \leq \mu$. For each $\eta \in {}^\kappa\mu$, let $q_\eta \in \mathbf{S}^m(Y)$ be a complete type extending $F_\eta(p \restriction \mathrm{dom}(F_\eta))$. We claim that we have defined $\mu^\kappa > \mu$ types, thereby showing instability at μ.

We show that $\eta \neq \rho$ implies $p_\eta \neq p_\rho$. Indeed, let α be least such that $\eta(\alpha) \neq \rho(\alpha)$, where $\eta(\alpha) = j_1 \neq \rho(\alpha) = j_2$. Without loss of generality, take $j_2 \neq 0$. If $j_1 \neq 0$ as well, then $F_\rho(\bar{a}^\alpha_{j_1}) = F_{\rho\restriction\alpha+1}(\bar{a}^\alpha_{j_1}) = F_{\rho\restriction\alpha\frown 0}(\bar{a}^\alpha_{j_1}) = F_{\rho\restriction\alpha\frown j_1}(\bar{a}^\alpha_0) = F_\eta(\bar{a}^\alpha_0)$. Since $\varphi_\alpha(\bar{v}; \bar{a}^\alpha_0) \in p$ $\varphi_\alpha(\bar{v}; F_\eta(\bar{a}^\alpha_0)) \in q_\eta$, and since $\neg\varphi_\alpha(\bar{v}; \bar{a}^\alpha_{j_1}) \in p$, $\neg\varphi(\bar{v}; \bar{F}_\rho(\bar{a}^\alpha_{j_1})) \in q_\rho$. So these types are distinct.

Now suppose $j_1 = 0$. Then we can check similarly that $F_\rho(\bar{a}^\alpha_0) = F_\eta(\bar{a}^\alpha_{j_2})$, and conclude $\varphi_\alpha(\bar{v}; F(\bar{a}^\alpha_0)) \in q_\rho$ while $\neg\varphi_\alpha(\bar{v}; F(\bar{a}^\alpha_{j_2})) \in q_\eta$. Again these types are distinct; this finishes the proof. $\square$

We finally prove the main result of the thesis.

Proof of the Stability Spectrum Theorem 7.1.1. First we prove (i). Let μ_0 be the first cardinal $\geq |T|$ in which T is stable. We have $\kappa^1(T) \leq \kappa(T) \leq |T|^+$, yielding $2^{|T|} = (2^{|T|})^{<\kappa^1(T)} \leq 2^{|T|\cdot|T|} = 2^{|T|}$ (split into cases depending on $\kappa^1(T) = |T|^+$ or $\kappa^1(T) \leq |T|$), so by Lemma 7.3.1(i) at $\mu = 2^{|T|}$, T is stable in $2^{|T|}$, hence $\mu_0 \leq 2^{|T|}$. If T is stable in $\mu \geq |T|$, then $\mu = \mu^{<\kappa(T)} = \mu^{<\kappa(T)} + \mu_0$ (since $\mu_0 \leq \mu$) by Lemma 7.3.2. Conversely, if $\mu = \mu^{<\kappa(T)} + \mu_0$, then clearly $\mu \geq \mu_0$. If equality holds, T is stable in μ. If not, then it must be the case that $\mu = \mu^{<\kappa(T)}$, hence by Lemma 7.3.2 again, T is stable in μ.

To get the explicit computation of μ_0, note that if $\mu_0 < \sup_{|X|\leq|T|} |\mathbf{S}(X)|$, then for some $|X| \leq |T| \leq \mu_0$, $\mu_0 < |\mathbf{S}(X)|$, contradicting stability. So it suffices to show that T is stable in $\mu = \sup_{|X|\leq|T|} |\mathbf{S}(X)| (\geq |T|)$ and then this will be least. Indeed, take $|X| \leq \mu$. If $|\mathbf{S}(X)| > \mu$, then $|\mathbf{S}(X)| > |\mathbf{S}(X)|$, a contradiction. So $|\mathbf{S}(X)| \leq \mu$, so T is stable in μ.

The reverse direction of (ii) is easy, since $\mu = \mu^{<\aleph_0}$ for all infinite μ, (this also implies T is countable by the remark above). So T is stable in all $\mu \geq 2^{|T|}$ by Lemma 7.3.1(i), hence T is superstable. For the forward direction, suppose $\kappa(T) \geq \aleph_0$. From cardinal arithmetic, we have that $\aleph_{\alpha+\omega}^{<\kappa(T)} \geq \aleph_{\alpha+\omega}^{\aleph_0} > \aleph_{\alpha+\omega}$, and so stability fails in arbitrarily large cardinals (by Lemma 7.3.2) so T is not superstable. □

In reference [9], Shelah calls another result the Stability Spectrum Theorem, which is stated a bit differently. Shelah also spends much more time proving it, (reaching the above theorem as a mere corollary on the way). Here is Shelah's version of the Stability Spectrum Theorem.

Theorem 7.3.3 (Shelah's Stability Spectrum Theorem)**.** *Given any stable theory T, there exists cardinals $\kappa(T) \leq |T|^+$ and $D(T)$ such that either T is stable in λ if and only if $\lambda = \lambda^{<\kappa(T)} + |D(T)|$ or T is stable in λ if and only if $\lambda = \lambda^{<\kappa(T)} + 2^{\aleph_0}$.*

We could just let $D(T) = \mu_0$, the first cardinal in which T is stable (and then ignore the second clause), but Shelah's version has the advantage that the computation $D(T)$ is simpler (it is defined as $\sum_{m<\omega} |\mathbf{S}^m(\emptyset)|$). This is desirable because given only that T is stable in *some* power, there may be no easy way to find the *least* such power μ_0.

Chapter 8

Conclusion

The Stability Spectrum Theorem is but a small part of classification theory. There exist other cardinal properties of theories and models, such as λ-homogeneity,[1] that have their own spectrum theorems. The techniques developed here are used throughout this program.

Since this thesis devoted an entire chapter to Morley's Theorem, it is appropriate to say a bit more about how this problem in particular generalizes. One can ask: If uncountable categoricity fails, how badly does it fail? What can be said about the spectrum of a theory $I(T, \kappa)$? In fact this is the main result in [9], called the Main Gap Theorem, which roughly says that there are either few models in each uncountable power, or the maximum number.[2]

It is of considerable regret that ultraproducts do not appear in this thesis; they are highly relevant (though elaboration would have taken us too far off course). Briefly, one can linearly order all complete theories *across any countable language* with respect to the ease with which ultrapowers of their models are saturated. This is called Keisler's Order, and is used in part to detect classification properties for theories. Stable theories constitute the first two classes. For a brief introduction, see [7].

[1] A model $\mathfrak{A}$ is *λ-homogeneous* if and only if every partial elementary mapping $f : X \to \mathfrak{A}$ from $X \subseteq A$, $|X| < \lambda$, extends to a partial elementary mapping $f : X \cup \{a\} \to \mathfrak{A}$ for every $a \in A$.

[2] $I(T, \kappa) \leq 2^\kappa$ assuming $|T| \leq \kappa$.

Bibliography

[1] J. Baldwin. *Fundamentals of Stability Theory*. Springer-Verlag, Berlin, 1988.

[2] J. Baldwin. "The Monster Model". 2010. `http://homepages.math.uic.edu/~jbaldwin/pub/monster.pdf`

[3] C. C. Chang and H. J. Keisler. *Model Theory*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[4] H. Enderton. *Elements of Set Theory*. Academic Press, San Diego, 1977.

[5] V. Harnik. "On the Existence of Saturated Models of Stable Theories." Proceedings of the American Mathematical Society 52, no. 1 (1975): 361-67.

[6] T. Jech. *Set Theory*. Springer-Verlag, Berlin, third edition, 2002.

[7] A. Johnson. "Counting and Realizing Types: A Survey of Stability and Saturation." 2017. `http://math.uchicago.edu/~may/REU2017/REUPapers/Johnson.pdf`

[8] M. Morley. "Categoricity in Power." Transactions of the American Mathematical Society 114, no. 2 (1965): 514-538.

[9] S. Shelah. *Classification Theory*. North-Holland Publishing Co., Amsterdam, second edition, 1990.