# An Upper Bound for the Partial Type Counting Function in Simple Theores

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In this document we prove an upper bound for the partial type counting function for simple theories. This is done because the proof given in Casanovas's paper (Theorem 2.8 (1) $\Rightarrow$ (2)) uses the claim $f^n_T(\kappa, \lambda) \leq \prod_\varphi f^n_\varphi(\kappa, \lambda)$ which is not obvious, and perhaps false in general (though true a fortiori in the simple case). The given argument generalizes the crucial Lemma 2.4 present there to get the desired result directly.

**Definition.** *For all $\lambda \geq \kappa \geq \aleph_0$ and $n \geq 1$, define*

$$f^n_T(\kappa, \lambda) = \sup\{|P| \mid P \textit{ is a family of pairwise incompatible } n\textit{-types } p, |p| \leq \kappa \\ \textit{taking parameters in some fixed } A, |A| \leq \lambda\}.$$

**Theorem 1.** *Suppose $T$ is simple and $|T| \leq \kappa \leq \lambda$. Then for all $n$, $f^n_T(\kappa, \lambda) \leq \lambda^{|T|} + 2^\kappa$.*

*Proof.* Suppose for contradiction that $P$ is a family of pairwise incompatible types $p$, $|p| \leq \kappa$, all with parameters in some fixed $A$, $|A| \leq \lambda$, and $|P| > \lambda^{|T|} + 2^\kappa$. Shrinking $P$, we may assume $|P| = \mu$ is regular. We also assume that each $p \in P$ is closed under conjunctions. Enumerate $P = \{p_\alpha \mid \alpha < \mu\}$ and each $p_\alpha = \{\varphi^\alpha_i(\bar{x}, \bar{a}^\alpha_i) \mid i < \kappa\}$. By local character of simplicity, each $p_\alpha$ does not fork over some $A_\alpha \subseteq A$, $|A_\alpha| \leq |T|$. Since $\mu > \lambda^{|T|}$ and is regular, we may shrink $P$ (still size $\mu$) and get that all $A_\alpha = A_0$.

Now each $p_\alpha$ induces a type $q_\alpha = \mathrm{tp}(\bar{a}^\alpha / A_0)$ where $\bar{a}^\alpha$ is the sequence formed by concatenating all $\bar{a}^\alpha_i$, $i < \kappa$. So $q_\alpha$ is a type in $\kappa$ many variables over $|A_0| \leq |T|$. There are at most $2^{|T|}$ $n$-types over $A_0$ and $\kappa \geq |T|$ so there are at most $(2^{|T|})^\kappa = 2^\kappa$ types $q_\alpha$, and $\mu > 2^\kappa$ is regular so again we may assume that all $q_\alpha = q_0$. In other words, for all $\alpha, \beta < \mu$, $\bar{a}^\alpha \equiv_{A_0} \bar{a}^\beta$.

For $\alpha < \beta < \mu$, let $h(\alpha, \beta) = (i, j)$ such that $\{\varphi^\alpha_i(\bar{x}, \bar{a}^\alpha_i), \varphi^\beta_j(\bar{x}, \bar{a}^\beta_j)\}$ is inconsistent (this using the fact that each type is closed under conjunctions). This gives $\kappa$ many colors on $\mu > 2^\kappa$ many nodes, so by Erdös-Rado there is some $I \subseteq \mu$, $|I| \geq \kappa^+$, such that $h(\alpha, \beta) = (i_0, j_0)$ for all $\alpha < \beta$ both in $I$. Since $|I| > |T|$ and can be assumed regular, we can shrink $I$ (maintaining $|I| > |T|$ but losing $|I| > \kappa$) and assume that the maps $\alpha \mapsto \varphi^\alpha_{i_0}(\bar{x}, \bar{y}^\alpha_{i_0})$ and $\alpha \mapsto \varphi^\alpha_{j_0}(\bar{x}, \bar{y}^\alpha_{j_0})$ are constant $= \varphi(\bar{x}, \bar{y})$. In particular, for all $\alpha \in I$ we get $\varphi(\bar{x}, \bar{a}^\alpha_{i_0}) \in p_\alpha$ and also $\{\varphi(\bar{x}, \bar{a}^\alpha_{i_0}) \mid \alpha \in I\}$ is infinite and 2-inconsistent. We also have that $\bar{a}^\alpha_{i_0} \equiv_{A_0} \bar{a}^\beta_{i_0}$ for all $\alpha, \beta$, which means that in fact every $p_\alpha$, $\alpha \in I$ divides over $A_0$. But this contradicts the choice of $A_0$. We conclude that no such family $P$ can exist, and the upper bound on the partial type counting function holds for simple theories. $\square$