Ultraproducts and $\mathcal{U}$

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Goals of the Talk

Are you...		A logician?	
		Yes	No
Familiar with ultraproducts?	Yes	See new applications	See logician's POV
	No	Add to toolbox	Wild ride!

Preview

What is an Ultraproduct?

An ultraproduct is a quotient of a direct product that:

1. Preserves the "average" structure of the factors
2. Fills "gaps" not realized by the factors
3. Is concrete enough to study specific examples

Products

Definition

Given an index set I and sets $\mathcal{M}_i$, $i \in I$, we define the *direct product* as

$$\prod_{i \in I} \mathcal{M}_i = \{f \mid f \text{ is a function with domain } I \text{ and } (\forall i \in I) f(i) \in \mathcal{M}_i\}$$

To denote specific elements we will write $f = \langle a_i \rangle_i$ where $a_i = f(i)$.

Ultrafilters

Definition

A *filter* on a set I is a collection $F \subseteq \mathcal{P}(I)$ such that:

1. $I \in F$
2. If $A \in F$ and $A \subseteq B$, then $B \in F$
3. If $A, B \in F$, then $A \cap B \in F$

If in addition F satisfies

4. For all $A \subseteq I$, exactly one of $A \in F$ or $(I - A) \in F$ holds

then we call F an *ultrafilter*.

Examples

- Fix $a_0 \in I$. Then $F = \{A \subseteq I \mid a_0 \in A\}$ is an ultrafilter.
- For any infinite I, $F = \{A \subseteq I \mid (I - A) \text{ is finite}\}$ is the *cofinite filter*.

Ultrafilters Continued

Facts about Ultrafilters

- If $F = \{A \subseteq I \mid a_0 \in A\}$ for some fixed $a_0 \in I$, we say F is *principal*.
- Principal ultrafilters yield trivial ultraproducts (isomorphic to one of their factors) and so they are usually not considered.
- Zorn's Lemma implies that any filter can be extended to an ultrafilter.
- The existence of nonprincipal ultrafilters is not provable from ZF.
- Any nonprincipal ultrafilter will contain all cofinite sets.

Ultraproducts I

Definition

Fix an index set I and an ultrafilter $\mathcal{U}$ on I. Given $f, g \in \prod_I \mathcal{M}_i$, define $f \sim_{\mathcal{U}} g$ if and only if $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$. We write $f_{\mathcal{U}}$ for the equivalence class of f, and define the *ultraproduct* $\prod_{\mathcal{U}} \mathcal{M}_i = \prod_I \mathcal{M}_i / \sim_U$.

Example

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $I = \mathbb{N}$ and set $\mathcal{M}_i = \mathbb{N}$ for all $i \in I$. We can form the ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} \mathbb{N}$. (We call $\mathcal{M}$ an ultra*power*.)

Cardinality of Ultraproducts

Proposition

If $\mathcal{U}$ is a nonprincipal ultrafilter on $I = \mathbb{N}$, then $\prod_{\mathcal{U}} \mathbb{N}$ is uncountable.

Languages and Models

Definitions

A *language* $\mathcal{L}$ consists of constant symbols, function symbols, and relation symbols. A *model* $\mathcal{M}$ for a language $\mathcal{L}$ is a set together with interpretations for the symbols in $\mathcal{L}$ (i.e. actual constants, functions, and relations in the context of $\mathcal{M}$).

Examples

- $\mathcal{L} = \{0, 1, +, \cdot\}$ The Language of Rings
    - $\langle \mathbb{C}, 0, 1, +, \cdot \rangle$
    - $\langle M_n(\mathbb{R}), 0_n, I_n, +, \circ \rangle$
- $\mathcal{L} = \{0, S, +, \cdot\}$ The Language of Peano Arithmetic
    - $\langle \mathbb{N}, 0, S, +, \cdot \rangle$

Ultraproducts II

Definition

Given $\mathcal{L}$-models $\mathcal{M}_i$, $i \in I$, and an ultrafilter $\mathcal{U}$ on I, we give the set $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ the structure of an $\mathcal{L}$-model as follows:

- For constants C, $C^{\mathcal{M}} = \langle C^{\mathcal{M}_i} \rangle_{\mathcal{U}}$.
- For functions F, $F^{\mathcal{M}}(f^1_{\mathcal{U}} \ldots f^n_{\mathcal{U}}) = \langle F^{\mathcal{M}_i}(f^1(i) \ldots f^n(i)) \rangle_{\mathcal{U}}$.
- For relations R, $R^{\mathcal{M}}(f^1_{\mathcal{U}} \ldots f^n_{\mathcal{U}}) \Leftrightarrow \{ i \mid R^{\mathcal{M}_i}(f^1(i) \ldots f^n(i)) \} \in \mathcal{U}$.

Example: Nonstandard Naturals

Example

Let $\mathcal{L} = \{0, +, <\}$ and $\mathcal{M} = \prod_{\mathcal{U}} \mathbb{N}$, $\mathcal{U}$ a nonprincipal ultrafilter on $I = \mathbb{N}$.

- $0^{\mathcal{M}} = \langle 0, 0, 0, \ldots \rangle_{\mathcal{U}}$
- $\langle 1, 2, 3, \ldots \rangle_{\mathcal{U}} +^{\mathcal{M}} \langle 1, 1, 1, \ldots \rangle_{\mathcal{U}} = \langle 2, 3, 4, \ldots \rangle_{\mathcal{U}}$
- The naturals embed by $n \mapsto \langle n, n, n, \ldots \rangle_{\mathcal{U}}$.
- For all $n \in \mathbb{N}$, $\langle n, n, n, \ldots \rangle_{\mathcal{U}} <^{\mathcal{M}} \langle 1, 2, 3, \ldots \rangle_{\mathcal{U}}$

Example: Hyperreals

Example

Let $\mathcal{L} = \{0, 1, +, \cdot, <\}$, let $\mathcal{U}$ be a nonprincipal ultrafilter on $I = \mathbb{N}$, and define $\mathbb{R}^* = \prod_{\mathcal{U}} \mathbb{R}$. We call $\mathbb{R}^*$ the *hyperreals*.

- The reals embed into the hyperreals by $r \mapsto \langle r, r, r, \ldots \rangle_{\mathcal{U}}$.
- $\omega = \langle 1, 2, 3, 4, \ldots \rangle_{\mathcal{U}}$ is greater than all $n = \langle n, n, n, \ldots \rangle_{\mathcal{U}}$
- $\varepsilon = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle_{\mathcal{U}}$ is positive but less than every $\frac{1}{n} = \langle \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots \rangle_{\mathcal{U}}$

The Fundamental Theorem of Ultraproducts

Łos's Theorem

Suppose $\mathcal{U}$ is an ultrafilter on I and $\mathcal{M}_i$, $i \in I$, are $\mathcal{L}$-models, and take the ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$. For any first-order sentence φ,

$$\mathcal{M} \models \varphi \Leftrightarrow \{i \mid \mathcal{M}_i \models \varphi\} \in \mathcal{U}.$$

Examples of First-Order Properties

Examples

1. An ultraproduct of domains/fields is a domain/field.
2. An ultraproduct of algebraically closed fields (ACF) is an ACF.
3. An ultraproduct of models of Peano Arithmetic (PA) is a model of PA.

Regarding Other Properties

Discussion

- $\mathbb{R}$ has no infinitesimals, but $\mathbb{R}^* = \prod_{\mathcal{U}} \mathbb{R}$ does.
    - The Archimedean Property is not expressible as a first-order sentence.
    - Being Dedekind/Cauchy complete quantifies over sets of reals.
- $\prod_{\mathcal{U}} \mathbb{N}$ is an uncountable model of number theory
    - No sentence, or set of sentences, can say "my models are countable".
    - No first-order formula $\varphi(x)$ of PA can say "$x = S^n 0$ for some n".

Application: Ax-Grothendieck Theorem

Theorem

Every injective polynomial map $P : \mathbb{C}^n \to \mathbb{C}^n$ is surjective.

Application: Ax-Grothendieck Theorem Continued

Claim 1

Every injective polynomial map $P : (\mathbb{F}_p^{alg})^n \to (\mathbb{F}_p^{alg})^n$ is surjective.

Application: Ax-Grothendieck Theorem Continued

Claim 2

Let I be the set of primes and let $\mathcal{U}$ be a nonprincipal ultrafilter on I. Then $\prod_{\mathcal{U}} \mathbb{F}_p^{alg}$ is an algebraically closed field of characteristic zero.

Application: Ax-Grothendieck Theorem Continued

Claim 3

$\mathbb{C} \cong \prod_{\mathcal{U}} \mathbb{F}_p^{alg}$ and the theorem holds for $\mathbb{C}$.

Types

Definition

A *type* is a collection of formulas (with a common free variable) such that every finite subcollection is satisfiable (with respect to some model).

Examples

- Dedekind cuts: $\{\text{“}x > q\text{”} \mid q \in \mathbb{Q}, q < \pi\} \cup \{\text{“}x < r\text{”} \mid r \in \mathbb{Q}, r > \pi\}$
- Type of an infinitesimal: $\{\text{“}x > 0\text{”}\} \cup \{\text{“}x < \frac{1}{n}\text{”} \mid n \in \mathbb{N}^+\}$
- Type of a nonstandard natural: $\{x \neq 0, x \neq S0, x \neq SS0, \ldots\}$
- Type of a transcendental number: $\{\text{“}p(x) \neq 0\text{”} \mid p \in \mathbb{Q}[t]\}$

ω_1-Saturation

Theorem

Suppose $\mathcal{M}_i$, $i \in I$, are $\mathcal{L}$-models, $\mathcal{U}$ is a nonprincipal ultrafilter on I, and there exist $A_n \in \mathcal{U}$, $n \in \mathbb{N}$, such that $\bigcap A_n = \emptyset$. Then the ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ satisfies every countable type. (We say $\mathcal{M}$ is ω_1-*saturated*.)

ω_1-Saturation Continued

Countably Complete Ultrafilters?

Warning!

The existence of nonprincipal ultrafilters closed under countable intersections is equivalent to a large cardinal hypothesis!

Ultralimits

Definition

Suppose $\mathcal{U}$ is an ultrafilter on I and (X, d) is a metric space. Define the *ultralimit* of a sequence $\langle x_i \rangle_i \subseteq X$ to be $\lim_{\mathcal{U}} x_i = x$ if and only if $\{i \mid d(x_i, x) < \varepsilon\} \in \mathcal{U}$ for all $\varepsilon > 0$.

An Analyst's Ultraproduct

Definition

Let E_i, $i \in I$, be a family of Banach spaces. Define

$$\ell_\infty(I, E_i) = \left\{ \langle x_i \rangle \in \prod_I E_i \,\middle|\, \sup_I ||x_i|| < \infty \right\}$$

and

$$N_\mathcal{U} = \left\{ \langle x_i \rangle \in \ell_\infty(I, E_i) \,\middle|\, \lim_\mathcal{U} ||x_i|| = 0 \right\}.$$

It turns out $N_\mathcal{U}$ is a closed subspace of the Banach space $\ell_\infty(I, E_i)$, so we define the *Banach space ultraproduct* $\prod_\mathcal{U} E_i = \ell_\infty(I, E_i)/N_\mathcal{U}$.

Facts about Banach Space Ultraproducts

Facts

1. The quotient norm turns out to be equivalent to $||\langle x_i \rangle_{\mathcal{U}}|| = \lim_{\mathcal{U}} ||x_i||$
2. An ultrapower of Banach spaces with respect to a countably incomplete ultrafilter is either finite-dimensional or non-separable.
3. Suppose Y is a separable Banach space, finitely representable in X. Suppose $\mathcal{U}$ is a countably incomplete ultrafilter. Then Y embeds isometrically into the ultrapower $\prod_{\mathcal{U}} X$.

Banach Spaces as Models

"Definition"

If you ask a model theorist what a Banach space is, they will answer:

$$\langle B, \mathbb{R}, 0_B, +_B, 0_{\mathbb{R}}, 1_{\mathbb{R}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, <_{\mathbb{R}}, c_r, ||\cdot||, \cdot_{\text{scalar}} \rangle_{r \in \mathbb{R}}$$

a structure in sorts $B, \mathbb{R}$ satisfying various axioms.

The Γ-Ultraproduct

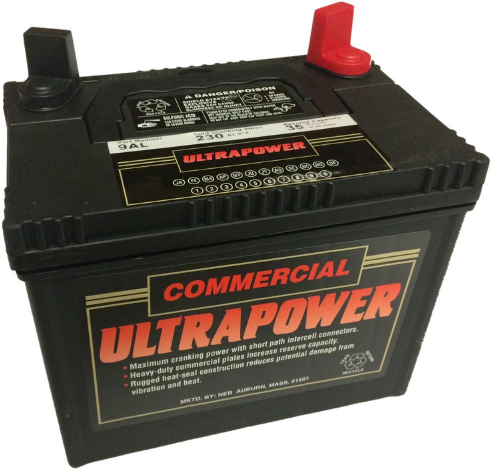
Definition

Given an ultrafilter $\mathcal{U}$ on I, $\mathcal{L}$-models $\mathcal{M}_i$, $i \in I$, and a set Γ of types that are each omitted on all $\mathcal{M}_i$, define

$$\prod_I^{\Gamma} \mathcal{M}_i = \left\{ f \in \prod_I \mathcal{M}_i \,\middle|\, (\forall p \in \Gamma)(\exists \varphi(x) \in p)\{i \mid \mathcal{M}_i \models \neg\varphi(f(i))\} \in \mathcal{U} \right\}$$

Now quotient by $\mathcal{U}$ as usual to define the Γ-*ultraproduct* $\prod_{\mathcal{U}}^{\Gamma} \mathcal{M}_i$.

Thanks!


DANGER/POISON
Part Number
9AL
Cold Cranking Amps
230
Reserve Capacity
35
ULTRAPOWER
COMMERCIAL
ULTRAPOWER
• Maximum cranking power with short path intercell connectors.
• Heavy-duty commercial plates increase reserve capacity.
• Rugged heat-seal construction reduces potential damage from vibration and heat.
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