Partial Types in O-Minimal Theories

Alex Johnson

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In this document we compute $f^n_T(\kappa, \lambda)$ for o-minimal T. In particular, we show that it is independent of n, and suspect this is the case in general but we were only able to show it simultaneous with computing f^n_T specifically in the case T is o-minimal. Of note, the arguments given also work when T is weakly o-minimal (definable sets are finite unions of convex sets, perhaps with endpoints only in some sufficiently saturated extension).

Definition. *For all $\lambda \geq \kappa \geq \aleph_0$ and $n \geq 1$, define*

$$f^n_T(\kappa, \lambda) = \sup\{|P| \mid P \textit{ is a family of pairwise incompatible } n\textit{-types } p, |p| \leq \kappa \\ \textit{taking parameters in some fixed } A, |A| \leq \lambda\}.$$

Definition. Given a linear order L, define a *partial left κ cut* as subset of L of size at most κ that is bounded above with no largest element. We define a *left κ cut* as a partial left κ cut that is downward closed. For all $\lambda \geq \kappa \geq \aleph_0$, define $\mathrm{ded}(\kappa, \lambda)$ as the supremum of the number of left κ cuts in a linear order of size λ.

An equivalent definition can be given in terms of branches through trees. Also "right κ cuts" are defined as one would expect. (In fact, we will use this below.) Cuts can clearly be coded by types, so we include them in the scope of the above definition.

Definition. *A theory T is o-minimal if and only if the language contains a binary relation $<$, T proves that $<$ is a dense linear order, and for every $\mathcal{M} \models T$ and $\varphi(x)$ perhaps with parameters in $\mathcal{M}$, $\varphi(\mathcal{M})$ is a finite union of points and intervals in $\mathcal{M}$ (with endpoints in $\mathcal{M}$) with respect to $<$.*

Theorem 1. *Suppose T is an o-minimal theory, $\lambda \geq \kappa \geq \aleph_0$, and $|T| \leq \lambda$. Then $f^n_T(\kappa, \lambda) = \mathrm{ded}(\kappa, \lambda)$, and in particular this function does not depend on n.*

Henceforth we assume T is a complete o-minimal theory witnessed by $<$. Whenever the cardinal parameter is considered for f^n_T, we also assume $|T| \leq \lambda$.

Definition. A 1-type p is *convex* if and only if $a < b < c$ and $a, c \models p$ implies $b \models p$.

Lemma 2. *Suppose P is a family of 1-types whose parameters are among A. Then there exists B, $|B| = |A| + |T|$, such that each $p \in P$ extends to a convex type q with parameters in B. Moreover, if p is nonprincipal, then q can be chosen to be equivalent to $q_L \cup q_R$, where q_L, q_R respectively define partial left and right $|p|$ cuts in B.*

Proof. Each $\varphi(x) \in \mathcal{L}(A)$ (parameters suppressed) is equivalent to a finite union of points an intervals. Let B be the set of all such endpoints for all such $\varphi(x)$ (this will automatically include A). We show that every $p \in P$ extends to a convex type with parameters in B.

Enumerate $p = \{\varphi_\alpha \mid \alpha < \kappa\}$. We define ψ_α by recursion such that $\{\psi_\beta \mid \beta < \alpha\} \cup \{\varphi_\beta \mid \alpha \leq \beta < \kappa\}$ is consistent, ψ_α defines a single point or interval with endpoints in B, and $\psi_\alpha \vdash \varphi_\alpha$. Once this is done, $q = p \cup \{\psi_\alpha \mid \alpha < \kappa\}$ will be the desired extension and is clearly convex.

Suppose this is done below α. We know φ_α defines a finite union of points and intervals with endpoints in B, one of which must be consistent with $\{\psi_\beta \mid \beta < \alpha\} \cup \{\varphi_\beta \mid \alpha < \beta < \kappa\}$. Define ψ_α any such consistent point or interval.

For the "moreover" part, note that p being nonprincipal means we can always take ψ_α as a nonempty open interval in the recursion above, and is therefore equivalent to either one or the conjunction of $b < x$, $x < c$ for some $b, c \in B$. The type q is clearly equivalent to the collection of all these, and can be partitioned into partial left and right $|p|$ cuts. (Note that even if one is not bounded in B, meaning the other is empty, it will be bounded in some larger linear order.) $\square$

Lemma 3. *For all $\lambda \geq \kappa \geq \aleph_0$, $f^1_T(\kappa, \lambda) = ded(\kappa, \lambda)$.*

Proof. Clearly $\mathrm{ded}(\kappa, \lambda) \leq f^1_T(\kappa, \lambda)$, since cuts can be coded by types. Take any family P of pairwise incompatible 1-types p taking parameters in some A such that $|p| \leq \kappa$ and $|A| \leq \lambda$. Applying Lemma 2 and discarding at most λ principal types, we may assume that each $p \in P$ is convex and decomposes into partial left and right κ cuts $p_L \cup p_R$. Let $\mathcal{M} \supseteq A$ be a model of size $|A|$ such that A is bounded from above and below, and let $\bar{p}_L, \bar{p}_R$ denote completions to left and right cuts in $\mathcal{M}$ (i.e. we close them downwards/upwards with respect to $<$).

The map $p \mapsto \langle \bar{p}_L, \bar{p}_R \rangle$ must be injective, since the $p \in P$ are pairwise incompatible, $\bar{p}_L \cup \bar{p}_R \equiv p_L \cup p_R \equiv p$, and two cuts are realized by the same elements if and only if they are equal. This shows $|P| \leq \mathrm{ded}(\kappa, \lambda) \cdot \mathrm{ded}(\kappa, \lambda) = \mathrm{ded}(\kappa, \lambda)$ as desired. (In fact, $p \mapsto \bar{p}_L$ is injective by itself, though mapping to a pair will be necessary for the proof of the main theorem.) $\square$

Proof of Theorem 1. The base case is given by Lemma 3. Assume the theorem holds at n, i.e. that $f^n_T(\kappa, \lambda) = \mathrm{ded}(\kappa, \lambda)$. It is clear that $\mathrm{ded}(\kappa, \lambda) = f^1_T(\kappa, \lambda) \leq f^{n+1}_T(\kappa, \lambda)$. Suppose for contradiction that $\mathrm{ded}(\kappa, \lambda) < f^{n+1}_T(\kappa, \lambda)$. Then there exists a family P of pairwise incompatible $n+1$-types p taking parameters in some A such that $|p| \leq \kappa$, $|A| \leq \lambda$, and $|P| > \mathrm{ded}(\kappa, \lambda)$.

We may extend each $p \in P$ to be closed under conjunctions and not alter their sizes or pairwise incompatibility. Label the free variables among the $p \in P$ by $x_1, \ldots, x_{n+1}$. For each $p \in P$, define $p^* = \{(\exists x_1, \ldots, x_n)\varphi \mid \varphi \in p\}$. These are consistent 1-types and the map $p \mapsto p^*$ is injective. Excluding at most λ many $p \in P$, we may assume none of them are principal.

The p^* may not be pairwise incompatible. However we can still map $p \mapsto p^* \mapsto \langle \bar{p}^*_L, \bar{p}^*_R \rangle$ as in the proof of Lemma 3. The range has size at most $\mathrm{ded}(\kappa, \lambda)$, so there must be $Q \subseteq P$, $|Q| > \mathrm{ded}(\kappa, \lambda)$, such that all $p \in Q$ map to the same pair. Shrinking P, we may assume $P = Q$.

In some sufficiently saturated extension, choose some c_{n+1} to fill the common cut defined by all p^*. Now, consider the family of n-types P' defined by $p' = \{\varphi(x_1, \ldots, x_n, c_{n+1}) \mid \varphi \in p\}$. Since each $p \in P$ is closed under conjunctions, these types are all consistent and $\langle c_1, \ldots, c_n \rangle \models p'$ if and only if $\langle c_1, \ldots, c_{n+1} \rangle \models p$. It follows that this family is pairwise incompatible, in bijective correspondence with P, and all types p' (each in bijective correspondence with p) have $|p'| \leq \kappa$ and take parameters in $A \cup \{c_{n+1}\}$ (a set of size at most λ). By the inductive hypothesis, $|P'| \leq \mathrm{ded}(\kappa, \lambda)$, but we just showed $|P'| = |P| > \mathrm{ded}(\kappa, \lambda)$, a contradiction. This completes the proof. $\square$