COUNTING AND REALIZING TYPES: A SURVEY OF STABILITY AND SATURATION

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Abstract. We explore the rich interplay between the number of types in a model and the propensity for the model to admit extensions that realize them. We begin with the Keisler-Shelah theorem (proven here assuming the Generalized Continuum Hypothesis) which gives a semantic characterization of elementary equivalence via ultrapowers. This motivates Keisler's Order, a tool that allows one to compare the complexity of complete countable theories. We then construct the monster model using strongly inaccessible cardinals, which can be justified in ZFC to serve as a useful proof technique. Finally we define stability and indiscernible sequences, and prove a result that relates the two.

Contents

1. Introduction

There are many structural properties of models that cannot be distinguished by checking satisfaction with respect to sets of sentences. Whether or not a model realizes a set of *formulas* (i.e. it realizes each formula at the same spot) reveals a much richer structure.

A set of formulas for which it is possible to find a model satisfying all finite subsets is called a *type*. We say a type is *realized* in a model if the model satisfies every formula at once. If we wish to emphasize some finite bound m on the number of free variables, we call it an m-type. For example, in the language of Peano Arithmetic the 1-type

$$\{v \neq 0, v \neq S0, v \neq SS0, \dots\}$$

is realized precisely by nonstandard models, i.e. models not isomorphic to $\mathbb{N}$. Although types with infinitely many free variables are of great importance, (e.g. one can express the failure of being well-ordered using a type in $\aleph_0$ many free variables,) we will not discuss them in this paper.

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We allow our m-types to have parameters from the universe of some model. Formally, we add constant symbols to our language for each parameter, and then interpret them back to the elements they signified in the base model. We call a model λ-*saturated* if and only if it realizes all types with fewer than λ parameters, where λ is some infinite cardinal.

An early question one might ask is how to construct λ-saturated models. In particular, given a model $\mathfrak{A}$, can we find $\mathfrak{B} \succcurlyeq \mathfrak{A}$ such that $\mathfrak{B}$ is λ-saturated? Do we have any say about its cardinality? A general answer to these two questions is given in Section 4 when we construct the monster model $\mathfrak{M}$, a $\bar{\kappa}$-saturated model of cardinality $\bar{\kappa}$ for some strongly inaccessible cardinal $\bar{\kappa}$.[1]

First, however, we present a construction using ultraproducts. Most of our work will be needed to construct a peculiar kind of ultrafilter called *good*, necessary to conclude that the ultraproduct is λ^+-saturated. The applications of good ultrafilters and saturated ultraproducts are numerous. Combined with uniqueness and an instantiation of the Generalized Continuum Hypothesis (GCH) we obtain the Keisler-Shelah Theorem, which states that two models are elementarily equivalent if and only if they have isomorphic ultrapowers.

The tools developed thus far also suggest an ordering on models based on the ease with which an ultrafilter can saturate them. Given two models in countable languages, we define $\mathfrak{A} \trianglelefteq_\lambda \mathfrak{B}$ if and only if every regular ultrafilter D on λ has the following property.

$$\text{If } \mathfrak{B}^\lambda/D \text{ is } \lambda^+\text{-saturated, then } \mathfrak{A}^\lambda/D \text{ is } \lambda^+\text{-saturated.}$$

This property is invariant under elementary equivalence, which induces the Keisler Order on complete countable theories instead of particular models. What is exciting about this notion is that it allows one to compare the complexity of theories across different languages; this suggests the order's interest for classification theory.

In the final section we make use of the monster model in introducing the notions of stability and indiscernible sequences. Roughly, stable theories permit the minimum number of types in their models, and indiscernible sequences are those for which any two n-tuples therein satisfy the same formulas. Our main result of this section shows that in any sufficiently large subset of a model for any stable theory, we can find large indiscernible sequences. This formalizes and proves our suspicion that stable theories lack a certain kind of descriptive power.

The following notation is derived primarily from [6], with a few simplifications derived from [1]. This will come at the cost of some generality, but our theorems proven from it will not change much.

Notations 1.1. We use $\mathcal{L}$ to denote a first-order language and make no assumptions on its cardinality. In an abuse of notation we will associate the language $\mathcal{L}$ with the set of all formulas of $\mathcal{L}$ containing any number of free variables. When we say a theory T is countable, we mean that both T and its language are countable. We use $\mathfrak{A}$ and $\mathfrak{B}$ to denote models, with universes A and B, respectively. The monster model (defined in Section 5) will be denoted $\mathfrak{M}$ with universe M. If the Gothic letter for a model is not currently in use, the Latin letter may be used for another purpose.

[1]We use $\bar{\kappa}$ to denote a strongly inaccessible cardinal throughout the paper, and will not use a bar for any other purpose. For sequences, we will use a vector arrow - see Notations 1.1.

We use the letters v and w to denote free variables in formulas, which we sometimes omit. A finite tuple of variables, constants, or anything else will be denoted by a vector, $\vec{v}$. A formula φ with free variables among $\vec{v}$ will be written $\varphi(\vec{v})$. If in addition there are relevant parameters $\vec{a}$, we will write $\varphi(\vec{v};\vec{a})$. On that matter, we disregard the formalism regarding parameters as constant symbols in an expanded language, and will not refer to the expansion for this purpose.

Types will be denoted by p or q, and will not be assumed complete. If p is a type and X a parameter set, $p \upharpoonright X$ denotes the subtype consisting of all formulas containing only parameters from X.

The ordinal length of a tuple or sequence f, finite or infinite, will be denoted $\ell(f)$. Concatenation of sequences will be denoted $f^\frown g$, and angle brackets are used to denote sequences or tuples with specific elements. Given sets A and B, we write $[A]^{<\omega}$ to denote the set of all finite subsets of A, and ${}^{B}A$ to denote the set of all functions $f : B \to A$. Given an ordinal α, we interpret ${}^{<\alpha}A$ and ${}^{\leq\alpha}A$ in the obvious way.

The ultraproduct of the models $\{\mathfrak{A}_\beta \mid \beta \in I\}$ modulo an ultrafilter D on I will be denoted $\prod_D \mathfrak{A}_\beta$. If all $\mathfrak{A}_\beta = \mathfrak{A}$ (as in the case of an ultra*power*,) we write $\mathfrak{A}^\lambda/D$. We write f_D to refer to an element in an ultraproduct's universe, and sometimes omit the D. We write f_β to refer to the projection onto the β^{th} factor model once a representative f has been chosen for f_D.

The letters κ, λ, and μ will be used to for cardinals, infinite unless otherwise stated. The letters $\alpha, \beta, \eta, \nu, \xi$, and ζ will be used for ordinals, and δ specifically for limit ordinals. We use the $\aleph$ operation in the usual way to denote specific cardinals. The letters k, n, and m will be used for natural numbers. Context will make these designations clear.

Definition 1.2. Given models $\mathfrak{A}$, $\mathfrak{B}$ of a common language, a function f is an $(\mathfrak{A},\mathfrak{B})$-*partial elementary mapping* if and only if $\mathrm{dom}(f) \subseteq A$, $\mathrm{ran}(f) \subseteq B$ and for every $a_0, \ldots, a_n \in \mathrm{dom}(f)$ and formula $\varphi(v_0 \ldots v_n)$, we have

$$\mathfrak{A} \models \varphi[a_0 \ldots a_n] \text{ iff } \mathfrak{B} \models \varphi[f(a_0) \ldots f(a_n)].$$

In particular, even if $f = \emptyset$, we have $\mathfrak{A} \equiv \mathfrak{B}$. Note that if $\mathrm{dom}(f) = A$, then f is an elementary embedding $\mathfrak{A} \preccurlyeq \mathfrak{B}$ in the usual sense. If in addition $\mathrm{ran}(f) = B$, then f is an isomorphism $\mathfrak{A} \cong \mathfrak{B}$.

When context is clear we omit $(\mathfrak{A},\mathfrak{B})$.

Definitions 1.3. Let $\mathfrak{A}$ be a model for the language $\mathcal{L}$ and $X \subseteq A$.

(1) A *type* or m-*type* over X in $\mathfrak{A}$ is a consistent (i.e. finitely satisfiable in $\mathfrak{A}$) set of formulas with at most m free variables. A *complete type* is a type that is maximal consistent.
(2) $tp(\vec{b}, X, \mathfrak{A}) = \{\varphi(\vec{v};\vec{a}) \mid \vec{a} \in {}^{<\omega}X,\ \varphi \in \mathcal{L}, \text{ and } \mathfrak{A} \models \varphi[\vec{b};\vec{a}]\}$ denotes the complete type realized by $\vec{b}$ in $\mathfrak{A}$ with parameters from X.
(3) $\mathbf{S}^m(X,\mathfrak{A}) = \{tp(\vec{b}, X, \mathfrak{A}) \mid \vec{b} \in {}^{m}A\}$ denotes the set of all complete types in at most m free variables with parameters in X realized in $\mathfrak{A}$.

We omit mention of $\mathfrak{A}$ in (2) or (3) to refer to consistent but not necessarily realized types. (Equivalently, once we have defined the monster model, we omit $\mathfrak{A}$ when $\mathfrak{A} = \mathfrak{M}$.) In (3), we omit m when $m = 1$.

Having established notation, we state some standard definitions and results in model theory that are used throughout the paper. It is assumed that the reader is comfortable with these, and as such, we will seldom stop to mention their usage. Proofs (as well as a development of model theory from the ground up) can be found in [1].

Theorem 1.4 (Compactness Theorem). *If every finite subset of T has a model, then T has a model.*

Theorem 1.5. *Suppose $\mathfrak{A}$ is a model for the language $\mathcal{L}$, and $\{c_a \mid a \in A\}$ are new constant symbols not in $\mathcal{L}$. Then $\mathfrak{B} \succcurlyeq \mathfrak{A}$ if and only if $\mathfrak{B} \supseteq \mathfrak{A}$ and the expansion of $\mathfrak{B}$ interpreting c_a as a models*

$$\Gamma_A = \{\varphi(c_{a_1} \dots c_{a_n}) \mid \mathfrak{A} \models \varphi[a_1 \dots a_n]\}.$$

Theorem 1.6 (Downward Löwenheim-Skolem-Tarski Theorem). *Suppose $\mathfrak{A}$ is an infinite model for the language $\mathcal{L}$ and $X \subseteq A$. Then for every λ such that*

$$|X| + |\mathcal{L}| \leq \lambda \leq |A|,$$

there exists an elementary submodel $\mathfrak{B} \preccurlyeq \mathfrak{A}$ of cardinality λ with $B \supseteq X$.

Theorem 1.7. *Suppose $\{\mathfrak{A}_\alpha \mid \alpha \leq \lambda\}$ are models for a common language such that $\beta < \alpha$ implies $\mathfrak{A}_\beta \preccurlyeq \mathfrak{A}_\alpha$. Define their union $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$ as the model with universe $\bigcup_{\alpha<\lambda} A_\alpha$, relations (and functions) $\mathcal{I}(R) = \bigcup_{\alpha<\lambda} \mathcal{I}_\alpha(R)$ where $\mathcal{I}_\alpha$ is the interpretation function for $\mathfrak{A}_\alpha$, and the same constants. Then $\mathfrak{A}_\lambda$ is a well-defined model and is an elementary extension of each $\mathfrak{A}_\alpha$.*

We use this theorem frequently when recursively constructing elementary chains of models to ensure that their unions (at limit ordinals) are well defined elementary extensions.

Theorem 1.8 (Fundamental Theorem of Ultraproducts). *Suppose $\{\mathfrak{A}_\beta \mid \beta \in I\}$ are models for a common language and D is an ultrafilter on I. Then given any formula $\varphi(v_1 \dots v_n)$ and $a_1, \dots, a_n \in \prod_D A$ we have*

$$\prod_D \mathfrak{A}_\beta \models \varphi[a_1 \dots a_n] \text{ iff } \{\beta \in I \mid \mathfrak{A}_\beta \models \varphi[a_{1_\beta} \dots a_{n_\beta}]\} \in D.$$

Definition 1.9. An ultrafilter D on I is λ-*regular* if and only if there exists $E \subseteq D$ such that $|E| = \lambda$ and for all $i \in I$ there are at most finitely many $e \in E$ such that $i \in e$. Such an E is called a *regularizing family*. If $\lambda = \aleph_0$, we also say D is *countably incomplete*. We say that D is *regular* to mean that D is λ-regular on an index set of size λ.

Theorem 1.10. *If D is a countably incomplete ultrafilter, then there exists a regularizing family in the form of a descending chain.*

Theorem 1.11. *Given any set I of cardinality λ, there exists a λ-regular ultrafilter D on I.*

Theorem 1.12. *Suppose D is a regular ultrafilter on λ, and $\mathfrak{A}$ is an infinite model. Then $|\mathfrak{A}^\lambda/D| = |A|^\lambda$.*

Definition 1.13. An ultrafilter D on I is *uniform* if and only if every $e \in D$ has $|e| = |I|$.

On the subject of prerequisites, we assume no more set theory than is typically required for model theory, (i.e. transfinite induction and recursion, König's Theorem on cofinalities, and cardinal arithmetic.) See the appendix for a complete list of nontrivial results from set theory used in this paper.

Finally, we mention that although there are no original results in this paper, many of the proofs we present contain details omitted from their sources. The results in Section 2 were derived from [1], in Section 3 from [4], and in Sections 4 and 5 from [6].

2. Saturation by Ultrapowers

Definition 2.1. A model $\mathfrak{A}$ is λ-*saturated* if and only for if for all $m < \omega$ and every parameter set $X \subseteq A$, $|X| < \lambda$, we have that $\mathfrak{A}$ realizes every m-type over X. We say $\mathfrak{A}$ is *saturated* if and only if it is $|A|$-saturated.

Proposition 2.2. *In the above definition it suffices to consider $m = 1$ when λ is infinite.*

Proof. See [1], Proposition 2.3.6 for the (not so special) case $\lambda = \aleph_0$. □

Examples 2.3. As a dense linear order, $\langle \mathbb{Q}, < \rangle$ is saturated. However, a dense linear order $\mathfrak{A}$ of cardinality $\aleph_1$ will fail to be saturated if the Continuum Hypothesis (CH) is false, since if we take $\mathbb{Q} \subset A$ as parameters we get $2^{\aleph_0}$ Dedekind cuts.

As an algebraically closed field of characteristic zero, the (complex) algebraic numbers $\langle \mathbb{A}, 0, 1, +, \cdot \rangle$ are not saturated, since using only symbols in the language we can write down the type of an element solving no rational polynomial. However, any countable extension field of transcendence degree ω is saturated.

Theorem 2.4. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent λ-saturated models of cardinality λ. Then $\mathfrak{A} \cong \mathfrak{B}$.*

A direct proof can be found in [1]. We will instead derive the result from a lemma involving partial elementary mappings. This approach will turn out to be more economical, as we will use Lemma 2.5 again later on.

Lemma 2.5. *Suppose $D \subseteq C \subseteq A$, $|D| < \lambda$, $|C| \leq \lambda$, and f is an $(\mathfrak{A}, \mathfrak{B})$-partial elementary mapping with domain D. If $\mathfrak{B}$ is λ-saturated, then we can extend f to a partial elementary mapping g with domain C.*

Proof. Enumerate $C = \{c_\alpha \mid \alpha < |C|\}$, and let $C_\alpha = \{c_\beta \mid \beta < \alpha\} \cup D$ for each $\alpha < |C|$. We define an increasing sequence $\{f_\alpha \mid \alpha \leq |C|\}$ of partial elementary mappings such that each $\mathrm{dom}(f_\alpha) = C_\alpha$. Once this is done, we let $g = f_{|C|}$ and the proof is complete.

Let $f_0 = f$, and$f_\delta = \bigcup_{\alpha<\delta} f_\alpha$ at limit ordinals. For the successor step, suppose f_α is defined. Consider the type $p(v) = tp(c_\alpha, C_\alpha, \mathfrak{A})$. Even if $f = \emptyset$, we still have $\mathfrak{A} \equiv \mathfrak{B}$ from the assumption that f is a partial elementary mapping, and so $p(v)$ is consistent with $\mathfrak{B}$. Since $|C_\alpha| < |C| \leq \lambda$, we have by λ-saturation that $\mathfrak{B}$ realizes $p(v)$, say at b_α. Now define $f_{\alpha+1} \supset f_\alpha$ by mapping $f_{\alpha+1}(c_\alpha) = b_\alpha$. It is clear that $f_{\alpha+1}$ is a partial elementary mapping with domain $C_{\alpha+1}$. □

Proof of Theorem 2.4. We first remark that every ordinal α is uniquely expressible in the form $\alpha = \delta + n$, where δ is a limit ordinal and $n < \omega$.

Enumerate $A = \{a_\alpha \mid \alpha < \lambda\}$ and $B = \{b_\alpha \mid \alpha < \lambda\}$. Let $A_\alpha = \{a_\beta \mid \beta < \alpha\}$ and $B_\alpha = \{b_\beta \mid \beta < \alpha\}$. We define a (non-strictly) increasing sequence $\{f_\alpha \mid \alpha \leq \lambda\}$ of $(\mathfrak{A}, \mathfrak{B})$-partial elementary mappings such that

(1) If $\alpha = \delta + 2n$, then $\mathrm{dom}(f_\alpha) = A_\alpha$,
(2) If $\alpha = \delta + 2n + 1$ or $\alpha = \delta$, then $\mathrm{ran}(f_\alpha) = B_\alpha$, and
(3) If $\alpha < \lambda$, then $|f_\alpha| < \lambda$.

From the remark above, α is uniquely expressible in one of these two forms, so (1) and (2) induce a dichotomy on ordinals.

We begin by letting $f_0 = \emptyset$, a partial elementary mapping by the assumption that $\mathfrak{A} \equiv \mathfrak{B}$. Take unions at limit ordinals as typical; note that $\mathrm{dom}(f_\delta) = A_\delta$ and also $\mathrm{ran}(f_\delta) = B_\delta$ at this stage. Now suppose we have defined f_α for some $\alpha = \delta + 2n + 1$. To define $f_{\alpha+1}$, we apply Lemma 2.5. If instead $\alpha = \delta + 2n$, apply the lemma to f_α^{-1} (a $(\mathfrak{B}, \mathfrak{A})$-partial elementary mapping) and then take the inverse of the result to get $f_{\alpha+1}$. This is well-defined, since any partial elementary mapping must be injective by considering the formula $v \neq w$.

Finally let $g = f_\lambda$, a partial elementary mapping with domain A and range B. Then g is an isomorphism $\mathfrak{A} \cong \mathfrak{B}$. □

Now we turn to the question of existence. In particular, we want to take the ultrapower of a model and have it turn out to be λ^+-saturated. For motivation, let's jump into trying to prove this, and see what we need to assume along the way.

Conjecture 2.6 (Potentially False). *Let $\mathfrak{A}$ be a model for a language $\mathcal{L}$, $|\mathcal{L}| \leq \lambda$, and let D be an ultrafilter on λ. Then the ultrapower $\mathfrak{A}^\lambda/D$ is λ^+-saturated.*

Proof Attempt. Suppose $p(v)$ a type consistent with $\mathfrak{A}^\lambda/D$, perhaps with parameters $X \subseteq A^\lambda/D$, $|X| \leq \lambda$. We want to conclude that the ultrapower realizes the type. One guess is to use finite satisfaction and the Fundamental Theorem of Ultraproducts to choose elements $h_\beta \in \mathfrak{A}$ that satisfy a set of formulas, and then hope that $h = \langle h_\beta \mid \beta \in \lambda \rangle_D$ satisfies the type in the ultrapower.

Indeed, fix a representative $a \in {}^\lambda A$ for each element $a_D \in X$. Enumerate (perhaps with repeats) $p = \{\varphi_\alpha \mid \alpha < \lambda\}$. Now, define $f : [\lambda]^{<\omega} \to D$ by

$$f(\sigma) = \{\beta \in \lambda \mid \mathfrak{A} \models \exists v \bigwedge_{\alpha \in \sigma} \varphi_\alpha(v; a_{1_\beta} \ldots a_{k_\beta})\},$$

where $\sigma \in [\lambda]^{<\omega}$ and $a_1 \ldots a_k$ are the parameters (if any) of φ_α.

Naïvely, we may want to ensure that at index β, $h(\beta)$ satisfies as many formulas as possible. But if we take every σ for which $\beta \in f(\sigma)$, we have no guarantee that $\bigcap_\sigma f(\sigma)$ is in D. We correct this by assuming that D is a regular ultrafilter on λ, with $\{E_\alpha \mid \alpha < \lambda\}$ a regularizing family. (See appendix for the definition of a regular ultrafilter.) Now consider

$$f(\sigma) = \{\beta \in \lambda \mid \mathfrak{A} \models \exists v \bigwedge_{\alpha \in \sigma} \varphi_\alpha(v; a_{1_\beta} \ldots a_{k_\beta})\} \cap \bigcap_{\alpha \in \sigma} E_\alpha.$$

We still have $f(\sigma) \in D$, but now there are only finitely many σ such that $\beta \in f(\sigma)$, the intersection of which is also in D. But there is still a problem: just because $\beta \in f(\sigma_1) \cap \cdots \cap f(\sigma_n)$, it might not be the case that $\beta \in f(\sigma_1 \cup \cdots \cup \sigma_n)$. One can construct specific ultrafilters and models for which the analogous problem is incorrigible. As stated, the conjecture is false. ⊠

To solve this problem, we examine properties of functions $f : [\lambda]^{<\omega} \to D$. Consider the following definition.

Definitions 2.7. Suppose A and I are nonempty sets. Given two functions
$$f, g : [A]^{<\omega} \to \mathcal{P}(I),$$
we write $g \leq f$ ("g *refines* f") if and only if for all $u \in [A]^{<\omega}$, $g(u) \subseteq f(u)$.

A function $f : [A]^{<\omega} \to \mathcal{P}(I)$ is *antimonotonic* if and only if for all $u, s \in [A]^{<\omega}$,
$$u \subseteq s \to f(u) \supseteq f(s).$$

A function $f : [A]^{<\omega} \to \mathcal{P}(I)$ is *multiplicative* if and only if for all $u, s \in [A]^{<\omega}$,
$$f(u \cup s) = f(u) \cap f(s).$$

(In practice, A will frequently be an infinite cardinal.)[2]

It is not hard to show that the function f defined above is antimonotonic, and multiplicitivity was the missing property. So, one might hope that any antimonotonic function can be refined to a multiplicitive function. This leads us naturally to the following.

Definition 2.8. An ultrafilter D is called λ-*good* if and only if for all cardinals $\kappa < \lambda$ and for all antimonotonic $f : [\kappa]^{<\omega} \to D$, there exists an multiplicative $g : [\kappa]^{<\omega} \to D$ refining f.

Lemma 2.9. *An ultrafilter D is λ^+-good if and only if for every antimonotonic $f : [\lambda]^{<\omega} \to D$, there exists a multiplicative $g : [\lambda]^{<\omega} \to D$ refining f.*

Proof. The forward direction is clear. For the reverse, let $f : [\kappa]^{<\omega} \to D$ be an antimonotonic function for some $\kappa < \lambda$. Define $f' : [\lambda]^{<\omega} \to D$ by
$$f'(s) = f(s \cap \kappa).$$
One can show that f' is also antimonotonic, and so it admits a multiplicative refinement $g' : [\lambda]^{<\omega} \to D$. Now if we define $g : [\kappa]^{<\omega} \to D$ by $g = g' \upharpoonright [\kappa]^{<\omega}$, we get a multiplicative refinement of f. $\square$

The existence of λ^+-good regular ultrafilters on λ is sufficient for proving our main theorem. In fact, we can do with slightly less.

Lemma 2.10. *Every λ^+-good countably incomplete (i.e. $\aleph_0$-regular) ultrafilter on an index set I, $|I| \leq \lambda$ is regular.*

Proof. Let $\{Y_n \mid n \in \omega\}$ be a descending chain of sets $Y_n \in D$ with empty intersection, which we are given by countable incompleteness. Let $f : [I]^{<\omega} \to D$ map $f(u) = Y_{|u|}$. It is easy to see that f is antimonotonic. Therefore it admits a multiplicative refinement g. Letting $X_i = g(\{i\})$ gives an $|I|$-regularizing family, since if $\beta \in X_{i_1} \cap \cdots \cap X_{i_n}$, then $\beta \in g(\{i_1 \ldots i_n\}) \subseteq Y_n$, and hence infinite intersections of X_i must be empty. This also shows that there can only be finitely many j such that $X_j = X_i$ for some fixed i. The union of $< |I|$ finite sets has size $< |I|$, and so there must be $|I|$ many distinct X_i, as desired. $\square$

To prove the existence of λ^+-good regular ultrafilters on λ, we will need several set theoretic lemmas, and the following definition.

[2] The canonical terminology is "monotonic" in place of "antimonotonic." In [1], the term "additive" is used in place of "multiplicative."

Definition 2.11. Suppose Π is a nonempty collection of partitions of λ, where each partition has λ equivalence classes. Let F be a nontrivial filter over λ. Then we say $\langle \Pi, F\rangle$ is *consistent* if and only if for all $X \in F$ and $X_i \in P_i \in \Pi$, where $1 \leq i \leq n$ and the P_i's are distinct, we have $X \cap \bigcap_{i=1}^n X_i \neq \emptyset$.[3]

One might arrive at this definition by thinking about how to construct an ultrafilter from a filter. If one considers finitely many sets $\{X_i \mid 1 \leq i \leq n\}$ such that either X_i or X_i^c can be added to the filter, one might hope that a choice for $i < j$ would not force a decision for j. A necessary condition to prevent this is that the intersection of X_i or X_i^c with either X_j or X_j^c (and also with any element of the filter already chosen) must be nonempty. The pair $\{X_i, X_i^c\}$ forms a partition of the index set, and so the above definition can be thought of as a generalization of this idea to any set of partitions whose elements are candidates for inclusion to the filter.

Lemma 2.12. *Suppose $\{X_\alpha \mid \alpha < \lambda\}$ is a collection of sets such that each $|X_\alpha| = \lambda$. Then there exists a pairwise disjoint collection $\{Y_\alpha \mid \alpha < \lambda\}$ of sets $Y_\alpha \subseteq X_\alpha$ such that each $|Y_\alpha| = \lambda$ as well.*

Proof. For each $\alpha \leq \lambda$, let $T_\alpha = \{\langle \xi, \eta\rangle \mid \xi \leq \eta < \alpha\}$. Note that each $|T_\alpha| < \lambda$ for $\alpha < \lambda$. Now we define an injection with domain T_λ such that

$$\xi \leq \eta < \lambda \to f(\xi, \eta) \in X_\xi.$$

The sets $Y_\xi = \{f^{-1}(\xi, \eta) \mid \xi \leq \eta < \lambda\}$ then satisfy the lemma.

We define f by transfinite recursion, defining f_α on T_α for all $\alpha < \lambda$. The union will then be the desired function. Suppose f_ξ has been defined on T_ξ for all $\xi < \beta$, forming a chain of injective functions. We only have to assign new values on the horizontal strip $\beta \times \{\beta\}$. Since each $|T_\xi| < \lambda$ while each $|Y_\xi| = \lambda$, one can always choose $f_\beta(\xi, \beta) \in X_\xi$ for $\xi < \beta$ unequal to previously chosen values, (which themselves define the rest of f_β.) This defines f_β on T_β and completes the induction.

□

Notation 2.13. If F is a filter and $F \cup E$ has the finite intersection property, then $\langle F, E\rangle$ will denote the filter generated by $F \cup E$.

Lemma 2.14. *Let λ be an infinite cardinal. Then*

- *i. Given a uniform filter F over λ generated by $E \subseteq F$ such that $|E| \leq \lambda$, there exists a collection Π of 2^λ partitions of λ such that $\langle \Pi, F\rangle$ is consistent.*
- *ii. Suppose $\langle \Pi, F\rangle$ is consistent and $J \subseteq \lambda$. Then either $\langle \Pi, \langle F, \{J\}\rangle\rangle$ is consistent, or $\langle \Pi', \langle F, \{\lambda \setminus J\}\rangle\rangle$ is consistent for some cofinite $\Pi' \subseteq \Pi$.*
- *iii. Suppose $\langle \Pi, F\rangle$ is consistent, $P \in \Pi$, and $p : [\lambda]^{<\omega} \to F$ is antimonotonic. Then there exists a filter $F' \supset F$ and multiplicative $q : [\lambda]^{<\omega} \to F'$ such that $q \leq p$ and $\langle \Pi \setminus \{P\}, F'\rangle$ is consistent.*

Proof. (i) Let $\{K_\alpha \mid \alpha < \lambda\}$ enumerate (perhaps with repeats) all finite intersections of elements of E. Note that each $|K_\alpha| = \lambda$. By Lemma 2.12, there exist pairwise disjoint $I_\alpha \subseteq K_\alpha$ for $\alpha < \lambda$ with each $|I_\alpha| = \lambda$. Consider now the set

$$B = \{\langle s, r\rangle \mid s \in [\lambda]^{<\omega} \text{ and } r : \mathcal{P}(s) \to \lambda\}.$$

[3] In the literature the notion of an independent family of sets is used in proofs in a similar way.

This set has size λ, and so each I_α can enumerate it such that

$$B = \{\langle s_\xi, r_\xi\rangle \mid \xi \in I_\alpha\}$$

for each $\alpha < \lambda$.

For each $J \subseteq \lambda$, define $f_J : \lambda \to \lambda$ by

$$f_J(\xi) = \begin{cases} r_\xi(s_\xi \cap J) & \text{if } \xi \in \bigcup_{\alpha<\lambda} I_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Note that this is well-defined by disjointness of the sets I_α. Now we define our set of partitions as

$$\Pi = \{\{f_J^{-1}(\eta) \mid \eta < \lambda\} \mid J \subseteq \lambda\}.$$

Our first claim is that $|\Pi| = 2^\lambda$, which we will prove by showing that $J_1 \neq J_2$ beget different partitions. Without loss of generality, let $x \in J_1 \setminus J_2$. Consider $s = \{x\}$ and $r = \{\emptyset \mapsto 1, \{x\} \mapsto 0\}$. At some ζ we have $\langle s, r\rangle = \langle s_\zeta, r_\zeta\rangle$, whence $f_{J_1}(\zeta) = 0$ and $f_{J_2}(\zeta) = 1$. Now repeat the same argument for $s = \emptyset$ and $r = \{\emptyset \mapsto 0\}$. Some ν has $\langle s, r\rangle = \langle s_\nu, r_\nu\rangle$, whence $f_{J_1}(\nu) = f_{J_2}(\nu) = 0$. This shows that $\zeta \neq \nu$, (though we also could have just specified that ζ and ν were chosen from different I_α,) and so the sets $f_{J_1}^{-1}(\{0\})$ and $f_{J_2}^{-1}(\{0\})$ intersect but are unequal. As such, they cannot belong to the same partition. Note that this also shows $\operatorname{ran}(f_J) = \lambda$ for each $J \subseteq \lambda$, since r could have mapped to any $\eta \in \lambda$, which implies that each partition contains λ nonempty equivalence classes.

Now we show that $\langle \Pi, F\rangle$ is consistent. Let $X \in F$ and $X_i \in \{f_{J_i}^{-1}(\eta) \mid \eta < \lambda\}$ for distinct $J_1, \dots, J_n \subseteq \lambda$. Choose $\eta_i \in X_i$ for all $1 \leq i \leq n$. Find s such that $s \cap J_i \neq s \cap J_j$ for $1 \leq i < j \leq n$, which then lets us find $r : \mathcal{P}(s) \to \lambda$ such that $r(s \cap J_i) = \eta_i$. Now $X \in F$ means there exists some α such that $I_\alpha \subseteq X$. Find $\xi \in I_\alpha$ such that $\langle s, r\rangle = \langle s_\xi, r_\xi\rangle$. Then $f_{J_i}(\xi) = \eta_i$, which implies $\xi \in X \cap \bigcap_{i=1}^n X_i \neq \emptyset$, as desired.

(ii) Suppose that $\langle \Pi, F\rangle$ is consistent, but that $\langle \Pi, \langle F, \{J\}\rangle\rangle$ is not. Then for some $\tilde{X} \in \langle F, \{J\}\rangle$ and $X_i \in P_i \in \Pi$ for $P_1, \dots, P_n$ distinct partitions have

$$\tilde{X} \cap \bigcap_{i=1}^n X_i = \emptyset.$$

Since $\tilde{X} \supseteq X \cap J$ for some $X \in F$, this implies that

$$X \cap J \cap \bigcap_{i=1}^n X_i = \emptyset.$$

We claim that $\langle \Pi \setminus \{P_1 \dots P_n\}, \langle F, \{\lambda \setminus J\}\rangle\rangle$ is consistent. Suppose $\tilde{Y} \in \langle F, \{\lambda \setminus J\}\rangle$ and $Y_j \in Q_j \in \Pi$ for $Q_1, \dots, Q_m$ distinct partitions unequal to any $P_1, \dots, P_n$. Then $\tilde{Y} \supseteq Y \cap (\lambda \setminus J)$ for some $Y \in F$. Our assumption that $\langle \Pi, F\rangle$ is consistent gives

$$(X \cap Y) \cap \bigcap_{j=1}^m Y_j \cap \bigcap_{i=1}^n X_i \neq \emptyset.$$

Any element of the above set is necessarily not in J, and so it is an element of $\lambda \setminus J$. This gives

$$\tilde{Y} \cap \bigcap_{j=1}^{m} Y_j \supseteq (\lambda \setminus J) \cap Y \cap \bigcap_{j=1}^{m} Y_j \neq \emptyset$$

as desired.

(iii) Suppose $\langle \Pi, F \rangle$ is consistent, $P \in \Pi$, and $p : [\lambda]^{<\omega} \to F$ is antimonotonic. Let $\{X_\xi \mid \xi < \lambda\}$ enumerate P and $\{t_\xi \mid \xi < \lambda\}$ enumerate $[\lambda]^{<\omega}$. For each $\xi < \lambda$, define $q_\xi : [\lambda]^{<\omega} \to \mathcal{P}(\lambda)$ by

$$q_\xi(s) = \begin{cases} p(t_\xi) \cap X_\xi & \text{if } s \subseteq t_\xi \\ \emptyset & \text{if } s \not\subseteq t_\xi \end{cases}.$$

Then $q_\xi(s) \subseteq p(t_\xi) \subseteq p(s)$ by antimonotonicity of p in the first case, and the same containment holds trivially in the second. Since $\langle \Pi, F \rangle$ is consistent, $q_\xi(s) \neq \emptyset$ whenever $s \subseteq t_\xi$. Note also that $s_1 \cup s_2 \subseteq t_\xi$ if and only if both $s_1 \subseteq t_\xi$ and $s_2 \subseteq t_\xi$, and so $q_\xi(s_1 \cup s_2) = q_\xi(s_1) \cap q_\xi(s_2)$. Now define $q(s) = \bigcup_{\xi<\lambda} q_\xi(s)$. Clearly $q \leq p$; we must show q is multiplicative. Given $s, u \in [\lambda]^{<\lambda}$, we have

$$\bigcup_{\xi<\lambda} q_\xi(s \cup u) = \bigcup_{\xi<\lambda} q_\xi(s) \cap q_\xi(u) \subseteq \bigcup_{\xi<\lambda} q_\xi(s) \cap \bigcup_{\xi<\lambda} q_\xi(u).$$

We must show that $(\supseteq)$ holds as well. Note however that if $x \in q_{\xi_1}(s) \cap q_{\xi_2}(u)$, then $x \in X_{\xi_1} \cap X_{\xi_2}$, whence $\xi_1 = \xi_2$ and $x \in q_{\xi_1}(s) \cap q_{\xi_1}(u)$ on the left hand side as well. This establishes $q(s \cup u) = q(s) \cap q(u)$.

One may want to verify that $F \cup \operatorname{ran}(q)$ has the finite intersection property before proceeding, but if that failed the generated filter would contain $\emptyset$ and we would not be able to make the following argument. So let $F' = \langle F, \operatorname{ran}(q) \rangle$, and we show that $\langle \Pi, F' \rangle$ is consistent. Let $\tilde{X} \in F'$ and $X_i \in P_i \in \Pi$ for distinct partitions $P_1, \ldots, P_n$ unequal to P. For some $X \in F$ and $s_1, \ldots, s_m \in [\lambda]^{<\omega}$ we have

$$\tilde{X} \supseteq X \cap q(s_1) \cap \cdots \cap q(s_m) = X \cap q(s_1 \cup \cdots \cup s_m).$$

We then find ξ such that $t_\xi = s_1 \cup \cdots \cup s_m$ and show

$$X \cap q(t_\xi) \cap \bigcap_{i=1}^{n} X_i \neq \emptyset.$$

Note that $q(t_\xi) \supseteq q_\xi(t_\xi) \cap X_\xi$. Now from the assumption that $\langle \Pi, F \rangle$ is consistent we have

$$X \cap p(t_\xi) \cap X_\xi \cap \bigcap_{i=1}^{n} X_i \neq \emptyset$$

which gives us the desired result. □

Theorem 2.15. *There exists a countably incomplete λ^+-good ultrafilter D on I, provided $|I| = \lambda$.*

Proof. Take $I = \lambda$, and for each $n \in \omega$ let

$$I_n = \{\delta + k \mid \delta + k < \lambda, \text{ where } \delta \text{ is zero or a limit ordinal and } k \geq n\}.$$

Then $I = I_0 \supseteq I_1 \supseteq \cdots$ is a descending chain of subsets of λ of size λ with empty intersection. Let F_0 be the uniform filter generated by $\{I_n \mid n \in \omega\}$. By Lemma 2.14 (i), there exists a set Π_0 of 2^λ partitions of λ such that $\langle \Pi_0, F_0 \rangle$ is consistent.

Now we define by transfinite recursion filters F_ξ and sets of partitions Π_ξ for $\xi < 2^\lambda$ such that:

(1) $|\Pi_\xi| = 2^\lambda$
(2) $|\Pi_\xi \setminus \Pi_{\xi+1}| < \omega$
(3) Each $\langle \Pi_\xi, F_\xi \rangle$ is consistent
(4) If $\eta \le \xi < 2^\lambda$, then $\Pi_\eta \supseteq \Pi_\xi$ and $F_\eta \subseteq F_\xi$.

Let $\{J_\xi \mid \xi < 2^\lambda\}$ enumerate $\mathcal{P}(\lambda)$ and let $\{p_\xi \mid \xi < 2^\lambda\}$ enumerate all antimonotonic functions $p : [\lambda]^{<\omega} \to \mathcal{P}(\lambda)$. We use the previously mentioned fact that every ordinal can be written uniquely in the form $\delta + k$ where δ is zero or a limit, and $k \in \omega$. Our recursion will alternate between using Lemma 2.14 (ii) to ensure the eventual filter is an ultrafilter, and (iii) to ensure all antimonotonic functions admit multiplicative extensions.

Suppose we have defined Π_η, F_η for all $\eta < \xi < 2^\lambda$. At limit ordinals δ, let $\Pi_\delta = \bigcap_{\eta<\delta} \Pi_\eta$ and $F_\delta = \bigcup_{\eta<\delta} F_\eta$; the four properties above will still be satisfied, with (2) implying (1). If ξ is of the form $\delta + 2n + 1$, let J be the first element of the enumeration of $\mathcal{P}(\lambda)$ not in $F_{\xi-1}$. Using part (ii) of the lemma, there exist Π_ξ, F_ξ satisfying (1)-(4) with F_ξ equal to either $\langle F_{\xi-1}, \{J\}\rangle$ or $\langle F_{\xi-1}, \{\lambda \setminus J\}\rangle$. Otherwise if ξ is of the form $\delta + 2n + 2$, let p be the first antimonotonic function of the enumeration with codomain $F_{\xi-1}$ not yet dealt with. Using part (iii) of the lemma, there exist Π_ξ, F_ξ satisfying (1)-(4) and also multiplicative $q \le p$ with codomain F_ξ.

Now, let $F = \bigcup_{\xi<2^\lambda} F_\xi$. This is an ultrafilter from the odd steps of the recursion, and is countably incomplete since it still contains the sets I_n. Now suppose $p : [\lambda]^{<\omega} \to F$ is antimonotonic. Since $|[\lambda]^{<\omega}| = \lambda$ and by König's Theorem $\mathrm{cf}(2^\lambda) > \lambda$, the codomain of p lies in some F_ξ for $\xi < 2^\lambda$. At each even step we dealt with a new antimonotonic function, so we can define a sequence of those functions that we made sure admit multiplicative refinements. This creates an initial segment of length 2^λ in the enumeration of all such antimonotonic functions, which is not possible unless our sequence contains them all. And so p will eventually get refined to a multiplicative q with codomain F. This shows that F is an λ^+-good ultrafilter, as desired. □

Now that we have good ultrafilters, we can restate Conjecture 2.6 as a theorem and give a formal proof. In fact, we can consider arbitrary ultraproducts and not just ultrapowers.

Theorem 2.16 (Good Ultrafilters Saturate Ultraproducts). *Suppose $\{\mathfrak{A}_\beta \mid \beta \in \lambda\}$ are models for $\mathcal{L}$ where $|\mathcal{L}| \le \lambda$. Then for any λ^+-good regular ultrafiter D on λ, the ultraproduct $\prod_D \mathfrak{A}_i$ is λ^+-saturated.*

Proof. Let $p(v)$ be a type over $\le \lambda$ parameters consistent with $\prod_D \mathfrak{A}_\beta$. We show it is satisfied there too. As in the proof (attempt) of Conjecture 2.6, we have $|p| \le \lambda$, and so we can enumerate (perhaps with repeats) $p = \{\varphi_\alpha \mid \alpha < \lambda\}$. In fact, this was the only point where we referenced parameters, namely their number to bound $|p|$, and so we omit mention of them henceforth.[4] Let $\{E_\alpha \mid \alpha < \lambda\}$ be a regularizing

[4]Had we included them, we would have had to choose representatives $a_\beta \in A_\beta$ for each parameter a_D in the ultraproduct, as we did in the attempted proof of Conjecture 2.6. The rest of the proof would proceed the same way.

family for D, and define $f : [\lambda]^{<\omega} \to D$ by

$$f(\sigma) = \{\beta \in \lambda \mid \mathfrak{A}_\beta \models \exists v \bigwedge_{\alpha\in\sigma} \varphi_\alpha(v)\} \cap \bigcap_{\alpha\in\sigma} E_\alpha.$$

It is not hard to see that f is antimonotonic, and by goodness of D it admits a multiplicative refinement g.

For each β, let $\sigma(\beta) = \sigma_1 \cup \cdots \cup \sigma_n$ be the union of the finitely many σ such that $\beta \in g(\sigma)$. Then $\beta \in g(\sigma(\beta))$ by multiplicitivity, so we can choose $h_\beta \in A_\beta$ such that $\mathfrak{A}_\beta \models \bigwedge_{\alpha\in\sigma(\beta)} \varphi_\alpha[h_\beta]$. We claim that $h_D = \langle h_\beta \mid \beta \in \lambda\rangle_D$ realizes the type in the ultraproduct. So, let $\varphi(v) \in p(v)$. If $\beta \in g(\{\varphi\})$, then $\varphi(v) \in \sigma(\beta)$, and so $\mathfrak{A}_\beta \models \varphi[h_\beta]$ on a large set. Therefore $\prod_D \mathfrak{A}_\beta \models \varphi[h_D]$. This completes the proof. □

In fact, we can prove a partial converse of the above theorem relating to multiplicative refinements.

Theorem 2.17. *Suppose $\{\mathfrak{A}_\beta \mid \beta \in \lambda\}$ are models for $\mathcal{L}$ and D is a regular ultrafilter on λ with regularizing family $\{X_\alpha \mid \alpha < \lambda\}$. Then $\prod_D \mathfrak{A}_\beta$ realizes the type $p(v) = \{\varphi_\alpha(v) \mid \alpha < \lambda\}$ (perhaps with parameters) if and only if the function $f : [\lambda]^{<\omega} \to D$ defined by*

$$f(\sigma) = \{\beta \in \lambda \mid \mathfrak{A}_\beta \models \exists v \bigwedge_{\alpha\in\sigma} \varphi_\alpha(v)\} \cap \bigcap_{\alpha\in\sigma} X_\alpha$$

admits a multiplicative refinement.

Proof. The backwards direction was where we used goodness in the proof of the previous theorem. For the forwards direction, let h_D realize the type. Then, let

$$g(\sigma) = \{\beta \in \lambda \mid \mathfrak{A}_\beta \models \bigwedge_{\alpha\in\sigma} \varphi_\alpha[h_\beta]\} \cap \bigcap_{\alpha\in\sigma} X_\alpha.$$

One can easily verify that $g \leq f$ is a multiplicative refinement. □

To conclude this section, we prove the Keisler-Shelah Theorem using the GCH. This surprising theorem says that elementary equivalence can be given an entirely semantic characterization using ultrapowers. It should be noted that the same theorem can be proven from ZFC alone with a bit more work.[5]

Theorem 2.18 (Keisler-Shelah)**.** *Suppose λ is an infinite cardinal and $\mathfrak{A}$, $\mathfrak{B}$ are infinite models of the language $\mathcal{L}$ such that $|A|, |B|, |\mathcal{L}| \leq \lambda$. Suppose also that $2^\lambda = \lambda^+$. Then for any λ^+-good regular ultrafilter D on λ, (a non-vacuous condition,) the following are equivalent:*

(1) $\mathfrak{A} \equiv \mathfrak{B}$
(2) $\mathfrak{A}^\lambda/D \cong \mathfrak{B}^\lambda/D$

Proof. We assume the GCH. For the nontrivial direction, let D be an λ^+-good regular ultrafilter on λ. Then the ultrapowers $\mathfrak{A}^\lambda/D$ and $\mathfrak{B}^\lambda/D$ are elementarily equivalent by the assumption that $\mathfrak{A} \equiv \mathfrak{B}$, and both λ^+-saturated by goodness. Because of this, we have the lower bounds $\lambda^+ \leq |\mathfrak{A}^\lambda/D|$ and $\lambda^+ \leq |\mathfrak{B}^\lambda/D|$. We can also bound these above by $|\mathfrak{A}^\lambda/D| \leq \lambda^\lambda$ and $|\mathfrak{B}^\lambda/D| \leq \lambda^\lambda$. Finally, the fact that $\lambda^\lambda = 2^\lambda$ and our assumption that $2^\lambda = \lambda^+$ shows that both ultrapowers have cardinality λ^+, so by uniqueness, they are isomorphic. □

[5] This was proven by Shelah after Keisler's original proof in ZFC+GCH.

3. Keisler's Order

In the previous section we went to great lengths to construct good ultrafilters. This was to guarantee saturation in the ultraproduct, but perhaps not all models are so difficult to saturate. Consider the complete theory of algebraically closed fields of characteristic zero, denoted ACF_0.

Theorem 3.1. *ACF_0 is categorical in every uncountable power, i.e. any two models of the same uncountable cardinality are isomorphic.*

Proof using facts from algebra. Any two algebraically closed fields of characteristic zero of the same transcendence degree are isomorphic, and the transcendence degree is the same as the cardinality when they are uncountable. □

Theorem 3.2. *If $\mathfrak{A}$ is an ACF_0 and D a regular ultrafilter on λ, then the ultrapower $\mathfrak{A}^\lambda/D$ is λ^+-saturated.*

Proof. By regularity, $|\mathfrak{A}^\lambda/D| = 2^\lambda$. An ultrapower $\mathfrak{A}^\lambda/D'$ modulo a λ-good regular ultrafilter D' is λ^+-saturated and also has cardinality 2^λ. So by categoricity, $\mathfrak{A}^\lambda/D \cong \mathfrak{A}^\lambda/D'$ and both are λ^+-saturated. □

So in some cases, goodness is not necessary at all![6] We would like to make explicit "how necessary" goodness is. Specifically, we want a definition that tells us when one model is easier to saturate than another.

Definition 3.3. Given a cardinal λ and two models $\mathfrak{A}$ and $\mathfrak{B}$ in perhaps distinct languages of maximum size λ, we write $\mathfrak{A} \trianglelefteq_\lambda \mathfrak{B}$ if and only if for every regular ultrafilter D on λ, we have

$$\text{If } \mathfrak{B}^\lambda/D \text{ is } \lambda^+\text{-saturated, then } \mathfrak{A}^\lambda/D \text{ is } \lambda^+\text{-saturated.}$$

We write $\mathfrak{A} \trianglelefteq \mathfrak{B}$ if and only if $\mathfrak{A} \trianglelefteq_\lambda \mathfrak{B}$ for all λ, (thereby necessitating a countable language.)

This definition is already quite useful, since it allows us to compare models of any language. The following theorem improves this dramatically, and will allow us to compare entire theories.

Theorem 3.4. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent models for $\mathcal{L}$, $|\mathcal{L}| \leq \lambda$, and that D is a regular ultrafilter on λ. Then*

$$\mathfrak{A}^\lambda/D \textit{ is } \lambda^+\textit{-saturated if and only if } \mathfrak{B}^\lambda/D \textit{ is } \lambda^+\textit{-saturated.}$$

Proof. Suppose $\mathfrak{A}^\lambda/D$ is λ^+-saturated. Let $X \subseteq B^\lambda/D$, $|X| \leq \lambda$ be a parameter set, and $p(v)$ a type over X in $\mathfrak{B}^\lambda/D$. Fix representatives $b \in {}^\lambda B$ for each element $b_D \in X$. We henceforth drop the subscript D on parameters for readability.

Let $E = \{E_\alpha \mid \alpha < \lambda\}$ be a regularizing family for D. Note that by our bounds on $|X|$ and $|\mathcal{L}|$ we have $|p| \leq \lambda$, so there exists an injection $H : p \hookrightarrow E$. For each $\beta \in \lambda$, let $\varphi(\beta) = \{\varphi \in p \mid \beta \in H(\varphi)\}$, a finite set. There are only finitely many parameters therein, so we write $\varphi(\beta) = \varphi(\beta)(b_1 \ldots b_n)$, omitting mention of the possible free variable v.

[6]Even the result above could have been proven without goodness using elimination of quantifiers. Ultrapowers in the theory ACF_0 are easy to saturate.

Now we look at the parts of $\varphi(\beta)$ that $\mathfrak{B}$ does and doesn't satisfy at β. Given a specific parameter b_i from the ultrapower, let b_{i_β} denote the image in the β^{th} factor model. Define

$$\tilde{\varphi}_0(\beta) = \{\exists v \bigwedge \theta \mid \theta \subseteq \varphi(\beta) \text{ and } \mathfrak{B} \models \exists v \bigwedge \theta[b_{1_\beta} \dots b_{n_\beta}]\},$$

$$\tilde{\varphi}_1(\beta) = \{\neg\exists v \bigwedge \theta \mid \theta \subseteq \varphi(\beta) \text{ and } \mathfrak{B} \models \neg\exists v \bigwedge \theta[b_{1_\beta} \dots b_{n_\beta}]\},$$

and $\tilde{\varphi}(\beta) = \tilde{\varphi}_0(\beta) \cup \tilde{\varphi}_1(\beta)$. Substituting in variables $\vec{w}$ for parameters $\vec{b}$, we have

$$\mathfrak{A} \equiv \mathfrak{B} \models \exists \vec{w} \bigwedge \tilde{\varphi}(\beta)(\vec{w}),$$

since $\tilde{\varphi}(\beta)$ is finite. (Note that $\exists v$ is *not* on the outside of this expression.)

For each parameter b_β appearing in $\tilde{\varphi}(\beta)$, there exists some $f_\beta(b_\beta) \in A$ such that $\mathfrak{A} \models \bigwedge \tilde{\varphi}(\beta)[f_\beta(b_{1_\beta}) \dots f_\beta(b_{n_\beta})]$. For other b_β, $f_\beta(b_\beta)$ can be defined arbitrarily. Here β refers to the index model and is arbitrary but fixed, whereas b_β is an arbitrary parameter in $\tilde{\varphi}(\beta)$, itself given as the β^{th} coordinate of $b \in X$, so from here on we write $f_\beta(b)$. Note that for any $\theta \subseteq \varphi(\beta)$, we have both

$$\mathfrak{B} \models \exists v \bigwedge \theta[b_{1_\beta} \dots b_{n_\beta}] \to \mathfrak{A} \models \exists v \bigwedge \theta[f_\beta(b_{1_\beta}) \dots f_\beta(b_{n_\beta})]$$

and

$$\mathfrak{B} \models \neg\exists v \bigwedge \theta[b_{1_\beta} \dots b_{n_\beta}] \to \mathfrak{A} \models \neg\exists v \bigwedge \theta[f_\beta(b_{1_\beta}) \dots f_\beta(b_{n_\beta})],$$

whence the reverse implication also holds in both expressions.

If we let

$$f(b) = \langle f_\beta(b) \mid \beta \in \lambda \rangle_D$$

for each $b \in X$, we get a map $f : X \to A^\lambda/D$. Consider the type

$$q(v) = \{\varphi(v; f(b_1) \dots f(b_n)) \mid \varphi(v; b_1 \dots b_n) \in p(v)\}$$

over parameters $Y = f[X]$. We claim that this type is finitely satisfiable in $\mathfrak{A}^\lambda/D$. Indeed, let $\theta \subset q(v)$ be finite. Whenever $\theta \subseteq \varphi(\beta)$, we have

$$\mathfrak{B} \models \exists v \bigwedge \theta[b_{1_\beta} \dots b_{n_\beta}] \leftrightarrow \mathfrak{A} \models \exists v \bigwedge \theta[f_\beta(b_1) \dots f_\beta(b_n)].$$

By finite satisfiability in $\mathfrak{B}^\lambda/D$, the left hand side holds on a large set, and the indices β for which $\theta \subseteq \varphi(\beta)$ are precisely $\bigcap H[\theta]$, another large set. Thus the right hand side holds on a large set, so $\mathfrak{A}^\lambda/D \models \exists v \bigwedge \theta[f_\beta(b_1) \dots f_\beta(b_n)]$ as desired.

By λ^+-saturation and the fact that $|Y| \leq |X| \leq \lambda$, $q(v)$ is realized in $\mathfrak{A}^\lambda/D$, say at a_D. For each $\beta \in \lambda$, let

$$\psi(\beta) = \{\psi \in \varphi(\beta) \mid \mathfrak{A} \models \psi[a_\beta, f_\beta(b_1) \dots f_\beta(b_n)]\}.$$

Note the evaluation of the the (previously omitted) free variable v at a_β. Now we have

$$\mathfrak{A} \models \exists v \bigwedge \psi(\beta)[f_\beta(b_1) \dots f_\beta(b_n)]$$

and since $\psi(\beta) \subseteq \varphi(\beta)$,

$$\mathfrak{B} \models \exists v \bigwedge \psi(\beta)[b_{1_\beta} \dots b_{n_\beta}].$$

Choose $c_\beta \in B$ such that

$$\mathfrak{B} \models \bigwedge \psi(\beta)[c_\beta, b_{1_\beta} \dots b_{n_\beta}].$$

We claim that $c = \langle c_\beta \mid \beta < \lambda\rangle_D$ satisfies $p(v)$ in $\mathfrak{B}^\lambda/D$. To show this, take $\psi \in p(v)$. If

$$\beta \in H(\psi) \cap \{\beta \in \lambda \mid \mathfrak{A} \models \psi[a_\beta, f_\beta(b_1)\dots f_\beta(b_n)]\},$$

then $\psi \in \psi(\beta)$, and so our choice of c_β gives

$$\mathfrak{B} \models \psi[c_\beta, b_{1_\beta} \dots b_{n_\beta}]$$

as desired. This holds on a large set, and so the ultrapower $\mathfrak{B}^\lambda/D$ realizes the type and our work is done. □

At last we can make the following definition, which serves as a powerful tool to classify theories by their complexity.

Definition 3.5 (Keisler's Order)**.** Suppose T_1 and T_2 are two complete countable theories. Then we say $T_1 \trianglelefteq T_2$ if and only if some (equivalently every) $\mathfrak{A}_1 \models T_1$ and $\mathfrak{A}_2 \models T_2$ have $\mathfrak{A}_1 \trianglelefteq \mathfrak{A}_2$.

Example 3.6. ACF_0 is minimum in Keisler's Order by Theorem 3.2.

Some known structural properties of Keisler's Order can be found in [5]. There are many other ramifications of Keisler's Order to topics in this paper and classification theory as a whole, but we will not explore them further here.

4. The Monster Model

We have already seen how to use ultrapowers to get λ^+-saturated models provided $|\mathcal{L}| \leq \lambda$. Here we present a different approach which allows us to better tailor their size. The following lemma will be needed for this construction.

Lemma 4.1. *Suppose $\mathfrak{A}$ is an $\mathcal{L}$-model and p a type in $\mathfrak{A}$, possibly with parameters. Then there exists an elementary extension $\mathfrak{B} \succcurlyeq \mathfrak{A}$ of cardinality no larger than $|A| + |\mathcal{L}|$ that realizes p. $\mathfrak{B}$ can be taken to be a proper extension.*

Proof. Let $\{c_a \mid a \in A\}$ and $\{c_\varphi \mid \varphi \in p\}$ be two sets of new constant symbols not in $\mathcal{L}$. Consider the theory

$$T = \{\psi(c_{a_1} \dots c_{a_n}) \mid \mathfrak{A} \models \psi[a_1 \dots a_n]\} \cup \{\varphi(c_\varphi) \mid \varphi \in p\}.$$

To ensure a proper extension, add to T the axioms

$$\{c_a \neq c \mid a \in A\},$$

where c is another new constant symbol. Every finite subset of T has some expansion of $\mathfrak{A}$ as a model, and so by compactness, T has a model $\mathfrak{B}'$. By the Downward Löwenheim-Skolem-Tarski Theorem, one can find $\mathfrak{B}'' \equiv \mathfrak{B}'$ of size no larger than $|A| + |\mathcal{L}|$ with universe at least (the interpretations of)

$$\{c_a \mid a \in A\} \cup \{c_\varphi \mid \varphi \in p\} \cup \{c\}.$$

The reduct $\mathfrak{B} = \mathfrak{B}'' \restriction \mathcal{L}$ is then an elementary extension of $\mathfrak{A}$ of the desired cardinality that realizes p. □

The next lemma will allow us to bound the number of types with parameters, given a bound on the number of types without parameters.

Lemma 4.2. *Suppose $\mathfrak{A}$ is an infinite $\mathcal{L}$-model and $|\mathfrak{A}| + |\mathcal{L}| \leq \lambda = \lambda^{<\kappa}$. If there are no more than λ complete m-types without parameters in $\mathfrak{A}$, then there are no more than λ complete m-types with fewer than κ parameters in $\mathfrak{A}$.*

Proof. Our hypothesis states that $|\mathbf{S}^m(\emptyset)| \leq \lambda$ for all $m < \omega$. In fact, we know more, namely that $|\mathbf{S}^m(Y)| \leq \lambda$ for each finite $Y \subseteq A$, since if $|Y| = k$, then $|\mathbf{S}^m(Y)| \leq |S^{m+k}(\emptyset)| \leq \lambda$ by substituting in free variables v_i for each $y_i \in Y$. Now, we consider $|\mathbf{S}^m(X)|$ for some infinite, but fixed $X \subseteq A$, $|X| = \mu < \kappa$. (This is sufficient, since there are only $|[A]^{<\kappa}| \leq \lambda^{<\kappa} = \lambda$ such parameter sets.)

We now make our critical observation: each type p is determined uniquely by its subtypes $p \restriction Y$ for each finite $Y \subseteq X$. To see this, note that each formula φ has only finitely many parameters, so there is some Y such that $\varphi \in p \restriction Y$, and thus the union of all such $p \restriction Y$ recovers p. Together with the observation that $|[X]^{<\omega}| = |X|$, this gives us

$$|S^m(X)| \leq \prod_{\substack{Y \subseteq X \\ Y \text{finite}}} |S^m(Y)| \leq \lambda^\mu = \lambda,$$

as desired.

□

Theorem 4.3. *Suppose that the hypotheses of Lemma 4.2 hold, including the bound on types. Then there exists an elementary extension $\mathfrak{B} \succcurlyeq \mathfrak{A}$ of cardinality λ that is κ-saturated.*

Proof. We define an elementary chain $\{\mathfrak{A}_\alpha \mid \alpha \leq \lambda\}$ with $\mathfrak{A}_0 = \mathfrak{A}$ such that for all α:

(1) $\mathfrak{A}_\alpha \subsetneq \mathfrak{A}_{\alpha+1}$
(2) $|\mathfrak{A}_\alpha| \leq \lambda$
(3) $\mathfrak{A}_{\alpha+1}$ realizes all types over fewer than κ parameters in $\mathfrak{A}_\alpha$.

If this can be done, we claim that $\mathfrak{B} = \mathfrak{A}_\lambda$ is our desired elementary extension. Conditions (1) and (2) will ensure that $\mathfrak{A}_\lambda$ has cardinality precisely λ. Now suppose p is a type over $\mathfrak{A}_\lambda$ with fewer than κ parameters. Since $\lambda^{<\kappa} = \lambda$, we must have $\kappa \leq \mathrm{cf}(\lambda)$, and so the parameter set will not be cofinal in the elementary chain, say contained in A_α for some $\alpha < \lambda$. By (3) our type will be realized in $\mathfrak{A}_{\alpha+1}$, and hence also in $\mathfrak{A}_\lambda$.

We now construct the chain. Set $\mathfrak{A}_0 = \mathfrak{A}$, and let $\mathfrak{A}_\delta = \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$ at limit ordinals δ. For the succesor step, suppose $\mathfrak{A}_\alpha$ has been defined. The fact that $\mathfrak{A}_\alpha \equiv \mathfrak{A}$ implies that there are no more than λ types without parameters in $\mathfrak{A}_\alpha$. Now we use Lemma 4.2 to conclude that there are no more that λ types over fewer than κ parameters in $\mathfrak{A}_\alpha$. So we can enumerate them (perhaps with repeats) as $\{p_\beta \mid \beta < \lambda\}$. We then define *another* elementary chain $\{\mathfrak{A}_\alpha^\beta \mid \beta \leq \lambda\}$, where each $\mathfrak{A}_\alpha^{\beta+1}$ realizes type p_β. Now set $\mathfrak{A}_\alpha^0 = \mathfrak{A}_\alpha$, take unions at limit ordinals, and apply Lemma 4.1 at successor ordinals. To satisfy condition (1), let $\mathfrak{A}_\alpha^1$ be a proper extension. Once finished let $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha^\lambda$. This concludes the proof. □

We want to use the above theorem to produce *saturated* models, which requires $\kappa = \kappa^{<\kappa}$. This equation holds at $\kappa = \aleph_0$, but then we could only use it on countable models. Instead of considering less effective solutions, we introduce the following.

Axiom 4.4. For every cardinal κ, there exists a strongly inaccessible $\bar{\kappa} \geq \kappa$.

Cardinal arithmetic then shows that $\bar{\kappa} = \bar{\kappa}^{<\bar{\kappa}}$, so we can use the above theorem on any model. In the context of a given model or theory, we call this the monster model $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$ with universe M. To make this well-defined, we (implicitly) take the smallest $\bar{\kappa}$ needed for a given argument to work.[7]

Notations 4.5. Whenever we consider a set X without reference to a specific model, we mean $X \subseteq M$ for the monster model $\mathfrak{M}$. Unless otherwise specified, assume $|X| < \bar{\kappa}$. We write $\models \varphi$ in place of $\mathfrak{M} \models \varphi$.

In practice, the existence of $\mathfrak{M}$ will only simply our arguments and notation. For example, our rule to omit $\mathfrak{A}$ from Definitions 1.3(2) and (3) when we want to consider a type consistent but not necessarily realized can now be restated as "we omit $\mathfrak{A}$ when $\mathfrak{A} = \mathfrak{M}$."

Conversely, given a model $\mathfrak{A}$ of size $< \bar{\kappa}$, we have $\mathfrak{A} \preccurlyeq \mathfrak{M}$, so we can speak about $\mathfrak{M}$ instead of quantifying over all models of a bounded size. In other words, our theorems that use $\mathfrak{M}$ can be carefully restated and then proven from ZFC alone.

5. Stability and Indiscernibles

The main results here follow Chapter I, Section 2 of Shelah's *Classification Theory*, [6]. We will state the theorem number from [6] in parentheses next to the numbers here for those results.

So far we have been pursuing models that realize types over a bounded number of parameters. In this section we consider the number of types realized. If $|X| = \lambda$, then there will be at least λ 1-types over X, namely those containing $v \neq a$ for each $a \in X$. (As before, we assume λ is an infinite cardinal.) This motivates the following.

Definition 5.1. A model $\mathfrak{A}$ is λ-*stable* if and only if for every parameter set $X \subseteq A$ of size $|X| \leq \lambda$, we have that for all $m < \omega$, $|\mathbf{S}^m(X, \mathfrak{A})| \leq \lambda$. A theory T is λ-*stable* if and only if each of its models is. We say T is *stable* when T is λ-stable for some λ.

Although we have defined stability in terms of realized m-types, it suffices to consider the number of consistent 1-types.

Theorem 5.2 (Shelah I.2.2). *A theory T is μ-stable if and only if for all X, $|X| \leq \mu$, we have $|\mathbf{S}(X)| \leq \mu$.*

Lemma 5.3 (Shelah I.2.1). *Suppose λ is a regular cardinal, $\mathfrak{A}$ a model, $X \subseteq A$ a parameter set, and $|\mathbf{S}^m(X, \mathfrak{A})| \geq \lambda$. Then there exists a finite $B \subseteq A$ such that $|\mathbf{S}(X \cup B)| \geq \lambda$.*

Proof. We induct on m. The case $m = 1$ is trivial, so suppose the result holds for $m = k$. We suppose $|\mathbf{S}^{k+1}(X, \mathfrak{A})| \geq \lambda$; if it happens that also $|\mathbf{S}^k(X, \mathfrak{A})| \geq \lambda$, then the result holds by the inductive hypothesis. Otherwise $|\mathbf{S}^k(X, \mathfrak{A})| < \lambda$. For each $q \in \mathbf{S}^{k+1}(X, \mathfrak{A})$, define $q^* = \{\exists v_k \varphi(v_0 \ldots v_k) \mid \varphi(v_0 \ldots v_k) \in q\}$, and then let q^+ be the unique extension of q^* to a complete type in $\mathbf{S}^k(X, \mathfrak{A})$.

This gives us a map $\mathbf{S}^{k+1}(X, \mathfrak{A}) \to \mathbf{S}^k(X, \mathfrak{A})$ from a set of size $\geq \lambda$ to a set of size $< \lambda$. Since λ is regular, there must exist some $p \in \mathbf{S}^k(X, \mathfrak{A})$ in the range

[7] In [6], Shelah denotes the monster model by $\mathfrak{C}$, and only assumes the existence of a strongly inaccessible cardinal "larger than all the cardinalities [his book] will deal with." We make a stronger assumption here so that it can be stated more formally and therefore be immune to apparent circularity. See the appendix for the definition of strongly inaccessible cardinals and a brief discussion about their relationship to ZFC.

of the mapping whose pullback has size $\geq \lambda$. Otherwise λ could be written as a union of fewer than λ sets of size less than λ, (an equivalent condition for λ being singular–see appendix.) Symbolically, $|\{q \in \mathbf{S}^{k+1}(X, \mathfrak{A}) \mid q^+ = p\}| \geq \lambda$.

Now let $b_0, \ldots, b_n$ satisfy p in $\mathfrak{A}$. This also satisfies each q^*, but we are not guaranteed satisfaction of q in $\mathfrak{A}$. No matter, we only need to show that there are at least λ types in a single variable consistent with $\mathfrak{A}$. Then let $B = \{b_0 \ldots b_{k-1}\}$, and then substituting the parameter b_i for v_i in the $\geq \lambda$ many q such that $q^+ = p$ will yield $|\mathbf{S}(X \cup B)| \geq \lambda$, as desired. □

Proof of Theorem 5.2. The forward direction is obvious. For the reverse direction, we use the previous lemma at $\lambda = \mu^+$, a regular cardinal. If for contradiction $|\mathbf{S}^m(X, \mathfrak{A})| \geq \mu^+$ for some m, then we would have $|\mathbf{S}(X \cup B)| \geq \mu^+ > \mu$ for some finite B, in which case $|X \cup B| = |X| \leq \mu$ and we contradict the hypothesis. □

Examples 5.4. $\langle \mathbb{R}, < \rangle$ is not $\aleph_0$-stable, since it only takes countably many parameters $\mathbb{Q}$ to define uncountably many Dedekind cuts.

For a positive example, consider the theory over countably many unary relations $\{U_n \mid n \in \omega\}$ with axioms stating that each U_n is infinite, and if $n \neq m$ then U_n and U_m are disjoint. To see that this theory is $\aleph_0$-stable, consider the types definable over countably many parameters in any model. The type can either mandate equality to one of the parameters, membership to one of their equivalence classes, or membership to none of their equivalence classes. These define only $\aleph_0$ many types.

The following two theorems, found in [1], give many more positive examples.

Theorem 5.5. *If a countable theory T is categorical in some uncountable power κ, then T is $\aleph_0$-stable.*

In particular, ACF_0 is $\aleph_0$-stable. Note that the converse is not true, since the theory of countably many disjoint infinite sets given above is not categorical in any power.

Theorem 5.6. *If a countable theory T is $\aleph_0$-stable, then T is λ-stable for all infinite λ.*

The following result connects stable theories to saturated models. The result is proved in [1] for regular λ, and in [2] for singular λ.

Theorem 5.7. *Suppose T is an $\aleph_0$-stable theory in a countable language with infinite models. Then for every uncountable λ and $\kappa \geq \lambda$, T has a λ-saturated model of cardinality κ.*

One interpretation is that λ-stable theories realize few types because they have trouble making distinctions among elements in their models. The following definition makes this precise.

Definition 5.8. Given a model $\mathfrak{A}$ and parameter set $X \subseteq A$, sequence $\langle a_\xi \mid \xi < \alpha \rangle$ of elements in A is an *n-indiscernible sequence over X in $\mathfrak{A}$* if and only if for each

$$\xi_0 < \cdots < \xi_{n-1} < \alpha, \quad \eta_0 < \cdots < \eta_{n-1} < \alpha$$

we have

$$tp(\langle a_{\xi_0} \ldots a_{\xi_{n-1}} \rangle, X, \mathfrak{A}) = tp(\langle a_{\eta_0} \ldots a_{\eta_{n-1}} \rangle, X, \mathfrak{A}).$$

We say that the sequence is *indiscernible* (over X in $\mathfrak{A}$) if and only if it is n-indiscernible for all n. We omit $\mathfrak{A}$ when $\mathfrak{A} = \mathfrak{M}$.[8]

This brings us to the main theorem of this section.

Theorem 5.9 (Shelah I.2.8)**.** *Suppose T is a λ-stable theory. If a parameter set X and an arbitrary set I satisfy $|I| > \lambda \geq |X|$, then there exists $J \subseteq I$ an indiscernible sequence over X of size $|J| > \lambda$.*

To prove this theorem, we introduce a new notion called splitting. We then prove a sufficient condition for a sequence being indiscernible, show how to find sequences satisfying the condition given a hypothesis on a model, and then show that every λ-stable model satisfies that hypothesis.

Definition 5.10. A type p *splits* over a parameter set X if and only if there exist tuples $\vec{b}$, $\vec{c}$ such that $\mathrm{tp}(\vec{b}, X) = \mathrm{tp}(\vec{c}, X)$, but for some φ we have both $\varphi(\vec{v}; \vec{b}), \neg\varphi(\vec{v}; \vec{c}) \in p$.

To gain some intuition for splitting, suppose G is a graph and b, c are vertices. Suppose that $X \subset G$ has the property that b and c have the same edge relations to all $a \in X$. But suppose some $d \in G \setminus X$ has an edge to b and not to c. Then any type containing the formulas $\mathrm{Edge}(v, b)$, $\neg\mathrm{Edge}(v, c)$ would split over X.

In practice, we will typically consider the negation of splitting. If p does *not* split over X, then for every $\vec{b}$ and $\vec{c}$ we have either $\mathrm{tp}(\vec{b}, X) \neq \mathrm{tp}(\vec{c}, X)$ or for every formula φ,

$$\varphi(\vec{v}; \vec{b}) \in p \text{ iff } \varphi(\vec{v}; \vec{c}) \in p.$$

In proofs we will choose $\vec{b}$ and $\vec{c}$ to satisfy $\mathrm{tp}(\vec{b}, X) = \mathrm{tp}(\vec{c}, X)$, which then forces the second condition above to hold.

Lemma 5.11 (Shelah I.2.5)**.** *Suppose $I = \langle a_\xi \mid \xi < \alpha \rangle$ and $X_\xi = \bigcup\{a_\eta \mid \eta < \xi\} \cup X$. Suppose also that for all $\eta < \xi < \alpha$, $p_\xi = tp(a_\xi, X_\xi)$ does not split over X and $p_\eta \subseteq p_\xi$. Then I is an indiscernible sequence over X.*

Proof. We show that I is an n-indiscernible sequence over X for all n by induction. Let

$$\xi_0 < \cdots < \xi_{n-1} < \alpha, \quad \eta_0 < \cdots < \eta_{n-1} < \alpha$$

be ordinals. Our task is to show that

$$\mathrm{tp}(\langle a_{\xi_0} \ldots a_{\xi_{n-1}} \rangle, X) = \mathrm{tp}(\langle a_{\eta_0} \ldots a_{\eta_{n-1}} \rangle, X).$$

Suppose $n = 1$, and $\varphi(v) \in \mathrm{tp}(a_{\xi_0}, X) \subseteq p_{\xi_0}$. Then $\neg\varphi(v) \notin p_{\xi_0} \supseteq p_0$, so $\neg\varphi(v) \notin p_0$ and it follows that $\varphi(v) \in p_0 \subseteq p_{\eta_0}$. The parameters still range over X so $\varphi(v) \in \mathrm{tp}(a_{\eta_0}, X)$ as desired.

Now assume the result for $n = k$, and consider $n = k + 1$. Let $\beta = \max(\xi_k, \eta_k)$. Since the tuples $\langle a_{\xi_0} \ldots a_{\xi_{k-1}} \rangle$ and $\langle a_{\eta_0} \ldots a_{\eta_{k-1}} \rangle$ realize the same type but p_β does not split over X, it must be the case that for any formula $\varphi(v_0 \ldots v_k; \vec{c})$

$$\varphi(v_k; \vec{c}, a_{\xi_0} \ldots a_{\xi_{k-1}}) \in p_\beta \text{ iff } \varphi(v_k; \vec{c}, a_{\eta_0} \ldots a_{\eta_{k-1}}) \in p_\beta.$$

[8]This definition is the special case of a more general definition given in [6], where for instance each a_ξ is allowed to be a finite tuple of some fixed length. Our main theorem and its proof remain valid with this more general definition, but we consider this special case because the notation is less cumbersome. It should be noted that further work in classification theory makes full use of the more general definition.

Now since $p_{\xi_k}, p_{\eta_k} \subseteq p_\beta$, we get

$$\begin{aligned}
\varphi(v_0 \dots v_k; \vec{c}) \in \operatorname{tp}(\langle a_{\xi_0} \dots a_{\xi_k} \rangle, X) &\text{ iff } \varphi(v_k; \vec{c}, a_{\xi_0} \dots a_{\xi_{k-1}}) \in p_{\xi_k} \\
&\text{ iff } \varphi(v_k; \vec{c}, a_{\xi_0} \dots a_{\xi_{k-1}}) \in p_\beta \\
&\text{ iff } \varphi(v_k; \vec{c}, a_{\eta_0} \dots a_{\eta_{k-1}}) \in p_\beta \\
&\text{ iff } \varphi(v_k; \vec{c}, a_{\eta_0} \dots a_{\eta_{k-1}}) \in p_{\eta_k} \\
&\text{ iff } \varphi(v_0 \dots v_k; \vec{c}) \in \operatorname{tp}(\langle a_{\eta_0} \dots a_{\eta_k} \rangle, X)
\end{aligned}$$

as desired. □

Lemma 5.12 (Shelah I.2.6). *Let $\mathfrak{A}$ be a λ-stable model in which*

(†) *There is no increasing $\{X_\alpha \mid \alpha \leq \lambda\}$ and $p \in S(X_\lambda, \mathfrak{A})$ such that $p \restriction X_{\alpha+1}$ splits over X_α for all $\alpha < \lambda$.*

If $X \subseteq A$, $I \subseteq A$, and $|I| > \lambda \geq |X|$, then there exists $J \subseteq I$, $|J| > \lambda$ such that J is an indiscernible sequence over X.

Proof. We first prove the following

(††) There exist B, C such that $X \subseteq B \subseteq C \subseteq A, |C| \leq \lambda$ and $p \in S(C, \mathfrak{A})$ such that both
(1) For all $C', C \subseteq C' \subseteq A, |C'| \leq \lambda, p$ has an extension $p' \in S(C')$ realized in $I \setminus C'$ that does not split over B. (In particular neither does p.)
(2) For all $\vec{c} \in {}^{<\omega}A$, there exists $\vec{c}\,' \in {}^{<\omega}C$ such that $\operatorname{tp}(\vec{c}, B) = \operatorname{tp}(\vec{c}\,' B)$.

Suppose for the sake of contradiction $\neg$(††). We will define an increasing sequence $\{B_\alpha \mid \alpha \leq \lambda\}$ with the aim of contradicting (†) such that each $B_\alpha \subseteq A$ and $|B_\alpha| \leq \lambda$. Let $B_0 = X$, and $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$ at limit ordinals.

Now for the successor step, suppose B_α is defined and $|B_\alpha| \leq \lambda$. Since $\mathfrak{A}$ is λ-stable, there are at most λ realized types over parameters B_α. So we can define $C_\alpha \subseteq A$ as consisting of (the elements of) representative tuples that realize each type. Then $|C_\alpha| \leq \lambda$, and for all $\vec{c} \in {}^{<\omega}A$ there exists $\vec{c}\,' \in {}^{<\omega}C$ such that $\operatorname{tp}(\vec{c}, B_\alpha) = \operatorname{tp}(\vec{c}\,', B_\alpha)$. This satisfies (2) of (††), so (1) must fail.

For every $p \in S(C_\alpha, \mathfrak{A})$ that does not split over B_α, there is a C'_p such that $C_\alpha \subseteq C'_p \subseteq A$, $|C'_p| \leq \lambda$, and every extension of p in $\mathbf{S}(C'_p)$ realized in $I \setminus C'_p$ splits over B_α. Now let

$$B_{\alpha+1} = \bigcup \{C'_p \mid p \in S(C_\alpha, \mathfrak{A}) \text{ does not split over } B_\alpha\} \cup C_\alpha.$$

Note that we use λ-stability again to ensure $|B_{\alpha+1}| \leq \lambda$.

Now let $c \in I \setminus B_\lambda$ and $p \in \operatorname{tp}(c, B_\lambda)$. We show $p \restriction B_{\alpha+1}$ splits over B_α for all $\alpha < \lambda$, which contradicts (†). If this is not the case, then neither does $q = p \restriction C_\alpha$ since restricting parameters makes splitting harder. Then $q \in S(C_\alpha, \mathfrak{A})$, so $C'_q \subseteq B_{\alpha+1}$. But $p \restriction C'_q$ (an extension of q) is realized by $c \in I \setminus C'_q \supseteq I \setminus B_\lambda$, and does not split over B_α, contradicting the construction of B_α. This gives us our contradiction of (†), and so we conclude that (††) must hold.

Now we construct our indiscernible sequence from I. We define elements $c_\alpha \in I$ by recursion on $\alpha < \lambda^+$. Suppose c_β has been defined for all $\beta < \alpha$. Define

$C_\alpha = \{c_\beta \mid \beta < \alpha\} \cup C$ (note $|C_\alpha| \leq \lambda$), and using (††)(1) let $p_\alpha \in S(C_\alpha)$ be an extension of p realized by $c_\alpha \in I \setminus C_\alpha$ that does not split over B.

We wish to apply Lemma 5.11 to show that $I' = \{c_\alpha \mid \alpha < \lambda^+\}$ is an indiscernible sequence over X. The only hypothesis in question is whether $p_\beta \subseteq p_\alpha$ for $\beta < \alpha$. So, suppose $\varphi(v; \vec{b}) \in p_\beta$. Choose $\vec{b}\,' \in {}^{<\omega}C$ such that $\mathrm{tp}(\vec{b}, B) = \mathrm{tp}(\vec{b}\,', B)$. We showed that p_β does not split over B, and so $\varphi(v; \vec{b}) \in p_\beta$ implies $\varphi(v; \vec{b}\,') \in p_\beta$. Now p_β extends to p, and so $\varphi(v; \vec{b}\,') \in p$. We cannot have $\neg\varphi(v; \vec{b}\,') \in p_\alpha$ since it also extends to p, so we must have $\varphi(v; \vec{b}\,') \in p_\alpha$. By non-splitting, we get $\varphi(v; \vec{b}) \in p_\alpha$, as desired. □

Lemma 5.13 (Shelah I.2.7)**.** *Suppose T is λ-stable. Then every model $\mathfrak{A}$ of T satisfies* (†) *from Lemma 5.12.*

Proof. We prove the above for the monster model $\mathfrak{M}$, which will imply the result for all models of cardinality $< \bar{\kappa}$. Suppose $\neg$(†). Let $\{A_\alpha \mid \alpha \leq \lambda\}$ be an increasing sequence and $p \in S(A_\lambda)$. Choose $\vec{b}_\alpha, \vec{c}_\alpha \in {}^{<\omega}A_{\alpha+1}$ such that $\mathrm{tp}(\vec{b}_\alpha, A_\alpha) = \mathrm{tp}(\vec{c}_\alpha, A_\alpha)$ but both $\varphi_\alpha(v; \vec{b}_\alpha), \neg\varphi_\alpha(v; \vec{c}_\alpha) \in p \restriction A_{\alpha+1} \subseteq p$ for some φ_α. Let a realize p, and $\mu = \min\{\mu \mid 2^\mu > \lambda\}$.

We will now define for each $g \in {}^{\leq\mu}2$ an elementary mapping F_g with domain $A_{\ell(g)}$ by recursion on $\ell(g)$. For $\ell(g) = 0$, let $F_0 = \mathrm{id}_{A_0}$. If $\ell(g) = \delta$ for δ a limit ordinal, let $F_g = \bigcup_{\alpha<\delta} F_{g\restriction\alpha}$.

Suppose now that F_g is defined for all g of length α. For any such g, we have two more definitions to make: $F_{g^\frown\langle 0\rangle}$ and $F_{g^\frown\langle 1\rangle}$. The former will just be an arbitrary extension of F_g to $A_{\alpha+1}$, which is possible by saturation of $\mathfrak{M}$ and Lemma 2.5. For the other mapping, set $F_{g^\frown\langle 1\rangle}(b^k_\alpha) = F_{g^\frown\langle 0\rangle}(c^k_\alpha)$ for each b^k_α, c^k_α in the tuples $\vec{b}_\alpha$,$\vec{c}_\alpha$ respectively. We then extend to an elementary mapping on all of $A_{\alpha+1}$, using the fact that $\mathrm{tp}(\vec{b}_\alpha, A_\alpha) = \mathrm{tp}(\vec{c}_\alpha, A_\alpha)$.

We now define

$$B = \{F_g(b^k_\alpha), F_g(c^k_\alpha) \mid g \in {}^\alpha 2, 0 < \alpha < \mu, b^k_\alpha \in \vec{b}_\alpha, c^k_\alpha \in \vec{c}_\alpha\}.$$

Note that $|B| \leq \sum_{\alpha<\mu} 2^\alpha \leq \lambda$. For each $g \in {}^\mu 2$, extend F_g to an elementary mapping F'_g with domain $A_\mu \cup \{a\}$, and let $p_g = \mathrm{tp}(F'_g(a), B)$. We claim that this defines $2^\mu > \lambda$ complete types p_g over $\leq \lambda$ parameters, contradicting λ-stability.

Suppose $g \neq h \in {}^\mu 2$. Let $\alpha = \min\{\beta \mid g(\beta) \neq h(\beta)\}$; without loss of generality, assume $g(\alpha) = 0$. By definition of $\vec{b}_\alpha$, we have $\varphi_\alpha(v; \vec{b}_\alpha) \in p$ and hence $\models \varphi_\alpha[a; \vec{b}_\alpha]$. Thus $\models \varphi_\alpha[F'_h(a); F'_h(b^0_\alpha) \dots F'_h(b^k_\alpha)]$, so $\varphi_\alpha(v; F'_h(b^0_\alpha) \dots F'_h(b^k_\alpha)) \in p_h$. The same argument shows $\neg\varphi_\alpha(v; F'_g(c^0_\alpha) \dots F'_g(c^k_\alpha)) \in p_g$ and since $g(\alpha) = 0$ and g and h agree below α we have $F'_h(b^k_\alpha) = F'_g(c^k_\alpha)$. Thus $\neg\varphi_\alpha(v; F'_h(b^0_\alpha) \dots F'_h(b^k_\alpha)) \in p_g$. We conclude that $p_g \neq p_h$, which finishes the proof. □

The preceding two lemmas clearly imply Theorem 5.9.

6. Conclusion

Let us reflect on the work we have done. We have covered two main methods for constructing saturated extensions: ultrapowers modulo good ultrafilters, and elementary chains. The first of these methods gave us the Keisler-Shelah Theorem and forged another link between semantics and syntax. The second led us quite naturally to questions in cardinal arithmetic which, once resolved, gave us the monster model.

We then considered how many types a model realizes, and defined a theory to be stable when that number was minimal for each of its models. By the contrapositives to Theorems 5.5 and 5.6, unstable theories are not categorical in any uncountable power. However, Keisler's Order sheds light on this, and is used to find new properties to classify theories.

There is much exploration to be done regarding Keisler's Order and the consequences of saturation and stability, both by the student and the researcher. We hope that this paper provides the groundwork for future study in these directions.

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7. Appendix

Here we list some definitions and results from set theory that we use frequently in our arguments above. On the topic of set theory, we also freely use the Axiom of Choice and its implications throughout the paper. See [3] for proofs.

Theorem 7.1 (Fundamental Theorem of Cardinal Arithmetic)**.** *Given an infinite cardinal κ and a nonzero, potentially finite cardinal λ, $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

Theorem 7.2. *Suppose λ is an infinite cardinal. Then $2^\lambda = \lambda^\lambda$.*

Definition 7.3. Given an ordinal α, we define $\mathrm{cf}(\alpha)$, (the *cofinality* of α,) to be the smallest cardinal λ such that α is the supremum of λ smaller ordinals.

Clearly $\mathrm{cf}(\alpha) \leq |\alpha|$. The following theorem gives an equivalent definition of cofinality for cardinals.

Theorem 7.4. *Given a cardinal κ, $cf(\kappa)$ is the smallest cardinal λ such that κ is a union of λ sets of size $< \kappa$.*

Definition 7.5. A cardinal λ is said to be *regular* if and only if $\mathrm{cf}(\lambda) = \lambda$. Otherwise if $\mathrm{cf}(\lambda) < \lambda$, λ is said to be *singular*.

Theorem 7.6. *For every infinite cardinal λ, both $cf(\lambda)$ and λ^+ are regular cardinals.*

Theorem 7.7 (König's Theorem)**.** *For every infinite cardinal κ, $\kappa^{cf(\kappa)} > \kappa$.*

As a corollary, $\mathrm{cf}(2^\kappa) > \kappa$. We use this fact in the proof of Theorem 2.15.

Theorem 7.8. *Given sets $\{A_\alpha \mid \alpha < \lambda\}$, we have*

$$|\bigcup_{\alpha<\lambda} A_\alpha| \leq (\sup_{\alpha<\lambda} |A_\alpha|) \cdot \lambda.$$

We use this inequality in the proof of Theorem 4.3 to bound the cardinality of unions at limit ordinals.

Definition 7.9. Given cardinals κ and λ, we define

$$\kappa^{<\lambda} = \sup_{\mu<\lambda} \kappa^\mu.$$

Cardinal exponentiation is quite nontrivial in general, rife with independence results. The only point where we engaged with it, however, was when we constructed the monster model via strongly inaccessible cardinals.

Definition 7.10. An uncountable cardinal $\bar{\kappa}$ is said to be *strongly inaccessible* if and only if $\bar{\kappa}$ is regular, and for all $\lambda < \bar{\kappa}$, $2^\lambda < \bar{\kappa}$.

Now we can verify that $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$ for strongly inaccessible $\bar{\kappa}$. ZFC cannot show that strong inaccessibles exist, and it cannot even show that their existence is relatively consistent. Nevertheless, we assert Axiom 4.4 without hesitation. The skeptic is directed to the discussion thereafter.

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