The Stability of Theories from Categoricity to their Spectrum

Alexander Johnson

April 24, 2018

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- First Order Logic
- Set Theory
- Model Theory

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A tuple $\mathfrak{A} = (A, \mathcal{I})$ *models* φ (written $\mathfrak{A} \models \varphi$) if and only if φ holds true when relativized to A via $\mathcal{I}$. We write $\mathfrak{A} \models T$ if and only if $\mathfrak{A} \models \varphi$ for every $\varphi \in T$ and say $\mathfrak{A}$ is a model of T.

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Examples

$(\mathbb{Q}, <) \models \mathrm{DLO} \quad (\mathbb{C}, 0, 1, +, *) \models \mathrm{ACF}_0 \quad (\mathbb{R}, 0, 1, +, *, <) \models \mathrm{RCF}$

Löwenheim-Skolem-Tarski Theorem

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If T has exactly one model up to isomorphism of size κ, we say T is *categorical* in power κ.

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4. The theory RCF $(= Th(\mathbb{R}))$ fails categoricity in all infinite powers, (one can form Archimedean and non-Archimedean RCFs at each power by a compactness argument.)

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Question

Must every complete theory in a countable language fall into one of these four categories?

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We will sketch a proof.

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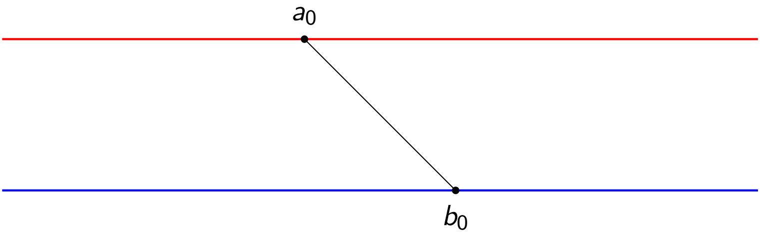

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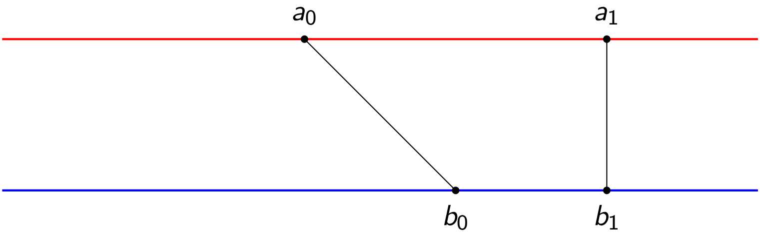

Now map b_1 to any point in A respecting order, (WLOG a_1.)

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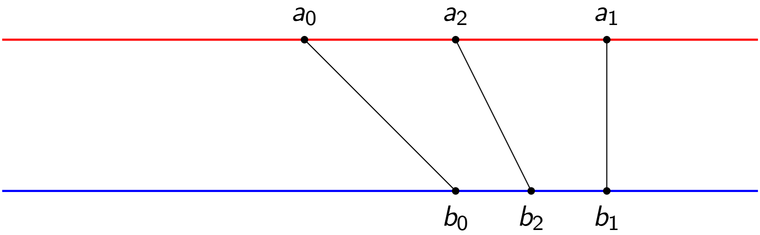

Next take the first point in A not yet mapped to, and map it into B respecting order.

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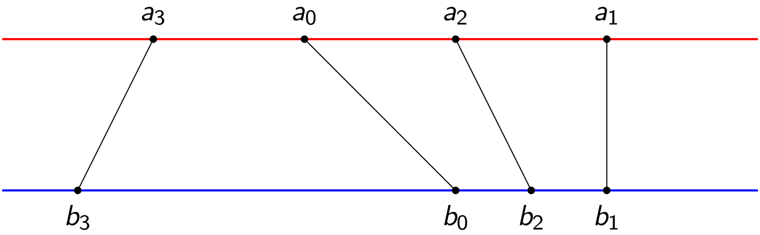

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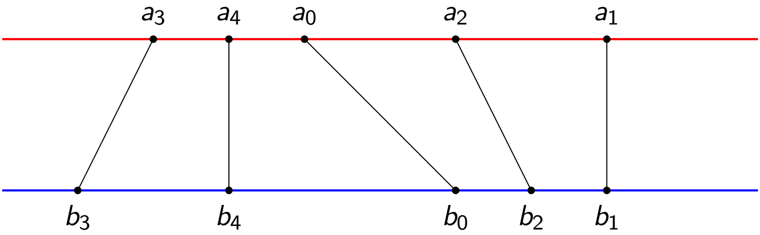

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A model $\mathfrak{A}$ is *κ-saturated* if and only if every type p in $< \kappa$ parameters is realized in $\mathfrak{A}$.

Examples

$(\mathbb{Q}, <)$ is $\aleph_0$-saturated. $(\mathbb{C}, 0, 1, +, *)$ is $2^{\aleph_0}$-saturated, $(|\mathbb{C}| = 2^{\aleph_0}.)$

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The proof uses transfinite recursion for the uncountable case, but otherwise is exactly the same as for countable DLOs!

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This contradiction finishes our sketch of Morley's Theorem.

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Examples

- ACF_0 is $\aleph_0$-stable (types express roots or non-roots)
- DLO is not stable in any power. A countable dense set can define uncountably many Dedekind cuts.

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Even when T is countable, the answer is complex.

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For example, $\lambda = 2^{|T|}$ satisfies this equation when $|T|$ is infinite, $(2^{|T|})^{|T|} = 2^{|T|\cdot|T|} = 2^{|T|}$.

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Suppose T is stable in some least power μ_0. Then there exists a cardinal $\kappa(T)$ for which T is stable in μ if and only if $\mu = \mu_0 + \mu^{<\kappa(T)}$.

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(In general we define $\mu^{<\kappa} = \sup_{\lambda<\kappa} \mu^{\lambda}$.)

The proof takes quite a lot of machinery. A very course sketch is that we define a tree of types with a certain "forking" property allowing us to contradict stability if the tree grows to a certain size.

I thank my sponsors

Professor McDonald and Professor Henckell

and the rest of my committee

Professor Poimenidou and Professor Kottke

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