Three + One Classes of Keisler's Order

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This document summarizes the picture of Keisler's Order for countable theories as it was known in the 20th Century. Specifically, we show that there are at least three classes of Keisler's Order, with a fourth being consistent (following from Martin's Axiom). Of course it was later shown that Keisler's Order has size continuum and is nonlinear between the third and final class in ZFC. We will reiterate this point several times throughout so that readers do not get confused thinking certain problems are still open, or that certain statements are independent.

The first two classes consist of the stable theories where the minimal class has NFCP, and the next class has FCP. Two more (unstable) classes we identify will be the third and the last classes of the order. The last class contains all SOP theories but is not classified by this property. We show these classes are distinct assuming Martin's Axiom.

This document will focus on proofs that involve compactness and incompactness of various ultraproducts, and how this yields results for Keisler's Order. We will blackbox any results from stability theory as well as all ultrafilter constructions. (Rest assured I have worked through these results separately, with the exception of the ultrafilter construction requiring Martin's Axiom.) Everything here is from Shelah's *Classification Theory*.

1 Preliminaries

Definition. A formula $\varphi(\bar{x};\bar{y})$ has the *finite cover property* (FCP) if and only if for all $n < \omega$ there exists an inconsistent set of φ-formulas (i.e. φ and its negation, with parameters) such that every subset of size n is consistent. A theory has the FCP if and only if some formula $\varphi(x;\bar{y})$ does.

The above definition requires the formula witnessing the FCP to have a single variable x, but by the FCP theorem below this is requirement can be dropped.

Lemma 1.1. *Any unstable theory has the FCP.*

We will only list below the statements appearing in the FCP theorem that we need or that give insight to its application.

Theorem 1.2 (The FCP Theorem)**.** *Suppose T is stable. Then the following are equivalent.*

(i) T has the FCP.

(ii) There exists a formula $\varphi(\bar{x};\bar{y})$ with the FCP.

(iii) There exists a formula $\varphi(x, y; \bar{z})$ such that every $\bar{z}$ defines an equivalence relation on x, y, and for all n there exists $\bar{c}_n$ such that $\varphi(x, y; \bar{c}_n)$ has $\geq n$ but finitely many equivalence classes.

As with typical model theory convention, NFCP means "not the finite cover property".

Lemma 1.3. *Suppose T has NFCP and Δ is finite. Then there exists $m = m(\Delta)$ such that whenever I is a Δ-n-indiscernible set of size at least n, I extends to arbitrarily large Δ-n-indiscernible sets.*

The following theorem shows that saturation is equivalent to the existence of sufficiently long indiscernible sequences. We'll need some definitions and lemmas first.

Definition. Two indiscernible sets I_1 and I_2 are *equivalent* if and only if there exists an indiscernible set J such that both $I_1 \cup J$ and $I_2 \cup J$ are indiscernible.

In particular, I_1 and I_2 are equivalent if $I_1 \cap I_2$ is infinite.

Definition. Define

$$\dim(I, \mathcal{M}) = \min\{|J| : J \subseteq \mathcal{M} \text{ is a maximal indiscernible set equivalent to } I\}.$$

Of course, equivalent indiscernible sets have the same dimension.

Definition. Define $\kappa^m(T)$ to be the first cardinal κ for which there is no increasing sequence A_i, $i \leq \kappa$, and $p \in S^m(A_\kappa)$ such that for all $i < \kappa$, $p \upharpoonright A_{i+1}$ forks over A_i. We stipulate $\kappa^m(T) = \infty$ if no such cardinal exists. Define $\kappa(T) = \sup_{m<\omega} \kappa^m(T)$.

Given that we are blackboxing the proofs of the following lemmas, we don't actually need to know what $\kappa(T)$ is, except that it is an infinite cardinal.

Lemma 1.4. *Suppose T is stable. Then $\kappa(T) \leq |T|^+$.*

Lemma 1.5. *Suppose T is stable. If J is a maximal indiscernible set in $\mathcal{M}$, then $|J| + \kappa(T) = \dim(J, \mathcal{M}) + \kappa(T)$.*

Lemma 1.6. *Suppose T is stable, λ is infinite, and $\mathcal{M}$ is $(\aleph_1 + \kappa(T))$-saturated. Then $\mathcal{M}$ is λ-saturated if and only if every infinite indiscernible $I \subseteq \mathcal{M}$ has $\dim(I, \mathcal{M}) \geq \lambda$.*

The following corollary is what we will use for Keisler's Order in the case T is stable and countable.

Corollary 1.7. *Suppose T is countable and stable, $\mathcal{N}$ is $\aleph_1$-saturated, and λ is uncountable. If every countable indiscernible set in $\mathcal{N}$ can be extended to an indiscernible set of size λ in $\mathcal{N}$, then $\mathcal{N}$ is λ-saturated.*

Proof. Lemma 1.4 gives $\kappa(T) \leq \aleph_1$, so the hypotheses of Lemma 1.6 are satisfied. So we must show that every infinite indiscernible $I \subseteq \mathcal{N}$ has $\dim(I, \mathcal{N}) \geq \lambda$. No countable indiscernible sequence is maximal by hypothesis, so $\dim(I, \mathcal{N}) \geq \aleph_1 \geq \kappa(T)$. Passing to a countable subset and then extending to a maximal indiscernible set gives $J \subseteq \mathcal{N}$ equivalent to I, $|J| \geq \lambda$ by hypothesis. By Lemma 1.5 we have $\lambda \leq |J| + \kappa(T) = \dim(J, \mathcal{N}) + \kappa(T) = \dim(I, \mathcal{N}) + \kappa(T) = \dim(I, \mathcal{N})$ as desired. $\square$

2 Regular and Good Ultrafilters

In this section we gather results regarding good and regular ultrafilters as they pertain to Keisler's Order. First, some cardinality results. Of note, the first two can be found in Chang and Keisler's book and aren't that hard to prove. The third one requires a bit more fiddling and was left as an exercise in Shelah's book with a generous hint.

Lemma 2.1. *If D is a regular ultrafilter on I, then $\lambda^I/D = \lambda^{|I|}$ for all infinite λ.*

Lemma 2.2. *If D is an $\aleph_1$-incomplete ultrafilter on I and $\lambda = \kappa^I/D$ for some infinite κ, then $\lambda^{\aleph_0} = \lambda$.*

Lemma 2.3. *If D is an $\aleph_1$-incomplete ultrafilter on I and $\lambda = \prod_i n_i/D \geq \aleph_0$, then $\lambda^{\aleph_0} = \lambda$.*

And finally, some preliminary results on compactness of ultrapowers.

Lemma 2.4. *Any ultrapower modulo an $\aleph_1$-incomplete ultrafilter is $\aleph_1$-compact.*

Definition. *An ultrafilter D is called λ-good if and only if for every $\alpha < \lambda$ and $f : [\alpha]^{<\omega} \to D$ such that $u \subseteq w \Rightarrow f(w) \subseteq f(u)$, there exists a refinement g such that $g(u) \subseteq f(u)$ and $g(u \cup w) = g(u) \cap g(w)$. An ultrafilter D on I is good if and only if it is $|I|^+$-good.*

Lemma 2.5. *For all λ, there exists a good $\aleph_1$-incomplete ultrafilter on λ.*

It should be noted that a good $\aleph_1$-incomplete ultrafilter is regular, but for whatever reason the above result is usually stated without this strengthening. The proof of this fact is sometimes embedded in whatever argument otherwise needed full regularity of the ultrafilter. In particular, the following lemma will be used when D is a regular ultrafilter on an uncountable cardinal, which implies the given hypothesis. We remark that this was pieced together from several lemmas in Shelah, and the resulting statement can be split into several slightly stronger lemmas.

Lemma 2.6. *Suppose D is an $\aleph_1$-incomplete ultrafilter on I and λ is an uncountable cardinal. Then the following are equivalent:*

(i) D is λ-good.

(ii) Every ultraproduct modulo D is λ-compact.

(iii) $\mathcal{M}^I/D$ is λ-compact for any dense linear order $\mathcal{M}$.

Proof. (i)$\Rightarrow$(ii) is a standard fact about good ultrafilters and is already written up in several places and (ii)$\Rightarrow$(iii) is trivial. To establish the equivalence we will show (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i).

First we show (ii)$\Rightarrow$(i). Fix $\mu < \lambda$ and let $\mathcal{M} = ([\mu]^{<\omega}, \subseteq)$ (finite subsets ordered by inclusion). Let $\mathcal{M}'$ be a λ-compact elementary extension of $\mathcal{M}$. Note that being empty is definable in this structure, as are finite intersections and unions. Let $f : [\mu]^{<\omega} \to D$ be monotonic; we seek a multiplicative refinement. We claim that for each $t \in f(\emptyset)$ and $\alpha < \mu$, there exists $u_\alpha[t] \in \mathcal{M}'$ such that for all $w \in [\mu]^{<\omega}$ we have

$$\mathcal{M}' \models \bigcap_{\alpha \in w} u_\alpha[t] \neq \emptyset \Leftrightarrow t \in f(w).$$

We can do this for fixed t and all $w \in [A]^{<\omega}$ when $A \subseteq \mu$ is finite by recursion on $|A|$ as follows. If $t \notin f(\{\alpha\})$ set $u_\alpha[t] = \emptyset$, so it suffices to consider the case $t \in \bigcap_{\alpha\in A} f(\alpha)$. To help us with the recursion we will show that the claim holds when choosing $u_\alpha[t]$ to be infinite sets with infinite complement, such that any nonempty intersection of the $u_\alpha[t]$ and their complements $u_\alpha[t]^c$ is infinite. This will give us enough wiggle room to add another set in the inductive step. (Then, since there are only a finite number of required intersections, the sets can be cut down to finite sizes.) When $|A| = \emptyset$ there is nothing to do. When $A = \{\alpha\}$ just make $u_\alpha[t]$ to be infinite with infinite complement. If we have $u_\alpha[t]$ for all $\alpha \in A$ and want to define one more $u_\beta[t]$, simply choose an infinite set (with infinite complement) from every intersection of the $u_\alpha[t]$ and $u_\alpha[t]^c$ that $u_\beta[t]$ must intersect. The fact that f is monotonic will ensure this is possible. Finally, for all of μ we use λ-compactness of $\mathcal{M}'$.

Moving forward, set $u_\alpha = \langle u_\alpha[t]\rangle_D$ and $p(x) = \{x \subseteq u_\alpha \mid \alpha < \mu\} \cup \{x \neq \emptyset\}$. Any finite subtype mentioning $\alpha \in w$ for some finite w is realized in $\mathcal{M}'$ on the big set $f(w)$. So the type is consistent and realized in $\mathcal{M}'^I/D$, say by c. Now $g(w) = \{t \in I \mid \emptyset \neq c[t] \subseteq \bigcap_{\alpha\in w} u_\alpha[t]\}$ will be a multiplicative refinement of f.

The last part of this theorem is (iii)$\Rightarrow$(ii). The proof is four and a half pages in Shelah. I don't really feel like writing it all up right now, so I think I'll skip it. Sorry... $\square$

The following two lemmas follow from a more general result in Shelah. I'm going to blackbox all the ultrafilter constructions.

Lemma 2.7. *There exists a regular ultrafilter D on $2^{\aleph_0}$ and $n_i < \omega$ such that $\prod_i n_i/D = 2^{\aleph_0}$.*

Lemma 2.8. *There exists a regular ultrafilter D on $2^{\aleph_0}$ and $n_i < \omega$ such that $\prod_i n_i/D = 2^{2^{\aleph_0}}$ is the least infinite $\prod_i m_i/D$, and also $lcf(\omega, D) = 2^{\aleph_0}$.*

Finally, the following result will be used to establish the consistency of a fourth class. (Again, the fact that there are at least four classes, and in fact uncountably many, is provable in ZFC. But this is what was used and known in the 20th Century.)

Lemma 2.9. *Assume Martin's Axiom and $\aleph_1 < 2^{\aleph_0}$. Then there exists a regular ultrafilter on $\lambda = \aleph_1$ that is not good such that every model $\mathcal{M}$ of the theory of random graphs has that $\mathcal{M}^\lambda/D$ is $\aleph_2$-saturated.*

3 Incompactness Results

Here we collect results about when various ultrapowers are not compact.

Lemma 3.1. *Suppose $\mathcal{M}$ models an unstable theory T, D is an ultrafilter over I, and m_i, $i \in I$ are finite such that $\aleph_0 \leq \prod_D m_i < 2^\lambda$. Then $\mathcal{N} = \mathcal{M}^I/D$ is not λ^+-compact.*

Proof. We know that either T has the strict order property or T has the independence property. We will split the proof into two cases based on these.

Case 1: T has the strict order property. Since $\mathcal{M}$ is a model, there exists $\bar{a}_i^n \in \mathcal{M}$, $0 \leq i < n < \omega$, and strict partial order $\varphi(\bar{x}; \bar{y})$ such that for all $0 \leq i, j < n$, $i < j$ if and only if $\models \varphi[\bar{a}_i^n; \bar{a}_j^n]$. Let $P_i = \{\bar{a}_j^{m_i} \mid j < m_i\}$ and P be the predicate such that $(\mathcal{N}, P) = \prod_D(\mathcal{M}, P_i)$. By hypothesis,

$\aleph_0 \leq |P| < 2^\lambda$. Each P_i, together with φ, defines a discrete linear order in $(\mathcal{M}, P_i)$ of size m_i, thus P, φ defines an infinite discrete linear order in $(\mathcal{N}, P)$. Moreover, each $(\mathcal{M}, P_i)$ believes that for every $\bar{x}$, if there exists $\bar{y}, \bar{y}' \in P_i$ such that $\varphi(\bar{x};\bar{y})$ and also all $\bar{y}'' \in P_i, \varphi(\bar{y}'';\bar{y}')$ have $\neg\varphi(\bar{x};\bar{y}'')$, then there exists $\bar{y} \in P_i$ such that for all $\bar{z} \in P_i$, $\varphi(\bar{x};\bar{z}) \equiv \varphi(\bar{y};\bar{z})$. So the same holds true of (N, P). In other words, we can take strict infimums (and dually, strict infimums) in P.

In this paragraph we replace $\varphi(\bar{x};\bar{y})$ with a (partial) order relation $<$, and also drop the tuple bars. In (N, P), we can show the following property: If $a, b \in P$ and there are infinitely many $c \in P$ such that $a < c < b$, then there exists $b' < a'$ such that there are infinitely many c, d such that $a < c < b' < a' < d < b$. This follows from successively taking strict supremums and infimums ω many times to obtain a sequence in P of order type $\omega + \omega^*$. Then, one takes one more infimum to plug the hole, which must have an immediate successor. These points will be b' and a'.

Suppose for contradiction that $\mathcal{N}$ is λ^+-compact. Define $\bar{a}_\eta, \bar{b}_\eta \in P$, $\eta \in 2^{\leq\lambda}$ by recursion on $\ell(\eta)$ such that

(i) For all η, $\mathcal{N} \models \varphi[\bar{a}_\eta, \bar{b}_\eta]$.

(ii) For all η and $n < \omega$, $(\mathcal{N}, P) \models (\exists \bar{x}_0, \dots, \bar{x}_n)\bar{x}_0 = \bar{a}_\eta \wedge \bar{x}_n = \bar{b}_\eta \wedge \bigwedge_{i<n} \varphi[\bar{x}_i; \bar{x}_{i+1}] \wedge \bigwedge_{i\leq n} P(\bar{x}_i)$.

(iii) For all $\eta \subsetneq \nu$, $\mathcal{N} \models \varphi[\bar{a}_\eta; \bar{a}_\nu] \wedge \varphi[\bar{b}_\nu; \bar{b}_\eta] \wedge \varphi[\bar{b}_{\eta^\frown 0}; \bar{a}_{\eta^\frown 1}]$.

Let $\bar{a}_\emptyset$ be the first element of P, and let $\bar{b}_\emptyset$ be the last. We use the claimed property above (using and maintaining (ii)) to handle both the limit and successor steps. At the end of the day, we have chosen at most $|P| < 2^\lambda$ many points $\bar{a}_\eta$ (and also $\bar{b}_\eta$), but they are distinct so there are also at least 2^λ many, a contradiction.

Case 2: T has the independence property. Suppose this is witnessed by $\varphi(x;\bar{y})$ and $\bar{a}^k_l, \in \mathcal{M}$ has that for all $w \subseteq k < \omega$ there exists $b^k_w \in \mathcal{M}$ such that

$$\mathcal{M} \models \bigwedge_{l<k} \varphi[b^k_w; \bar{a}^k_l]^{l\in w}.$$

Let μ be the least infinite $\prod_i n_i/D$, $n_i < \omega$. Note that $\mu < 2^\lambda$ by hypothesis. Define $P_i = \{\bar{a}^{k_i}_l \mid l < k_i\}$ and $Q_i = \{b^{k_i}_w \mid w \subseteq k_i\}$, where $k_i < \omega$ is to be defined. So $|P_i| \leq |Q_i| = 2^{k_i}$, and if we add these as predicates we get $(\mathcal{N}, P, Q) = \prod_i (\mathcal{M}, P_i, Q_i)/D$ with $|P| \leq |Q| = \prod_i |Q_i|/D$. Now, if we set $k_i = \lfloor \log_2(n_i) \rfloor$, then for all $l < \omega$ we have

$$\{i \mid \ |P_i| \geq l\} \supseteq \{i \mid k_i \geq l\} \supseteq \{i \mid n_i \geq 2^l\} \in D$$

because $\prod_i n_i/D$ is infinite. So $|P|$ is infinite, and thus $|P| \geq \mu$. But also $2^{k_i} \leq n_i$ so also $|Q| \leq \mu$ and thus $|P| = |Q| = \mu$.

Let $\lambda_1 = \min\{\lambda, \mu\}$. Let $I \subseteq P$ have $|I| = \lambda_1$. Note that for every $J \subseteq I$, $p_J = \{\varphi(x;\bar{a})^{\bar{a}\in J} \mid \bar{a} \in I\}$ is consistent by compactness and choice of the P_i. If it is realized in $\mathcal{N}$, then it is realized by some element of Q, so there are at most $|Q| = \mu$ types realized in $\mathcal{N}$. There are 2^{λ_1} types p_J. Obviously $2^\mu > \mu$, but also $2^\lambda > \mu$ by hypothesis. So $2^{\lambda_1} > \mu$, and there must be a type p_J omitted in $\mathcal{N}$. Since $|I| \leq \lambda$, we conclude that $\mathcal{N}$ is not λ^+ compact. □

Lemma 3.2. *Suppose T is stable and has the FCP, $\mathcal{M} \models T$, and D is an ultrafilter on I. If there exists $m_i < \omega$ such that $\aleph_0 \leq \mu = \prod_i m_i/D$ then $\mathcal{M}^I/D$ is not μ^+ compact.*

Proof. We use the characterization of FCP that there is $\varphi(x, y; \bar{z})$ such that every $\bar{c} \in \mathcal{M}$ defines an equivalence relation $\varphi(x, y; \bar{c})$ and for all $n < \omega$ there exists $\bar{c}_n$ defining at least n but finitely many equivalence classes. Let $N(\bar{c}, \mathcal{M})$ be the number of equivalence classes defined by $\bar{c}$, so $N(\bar{c}, \mathcal{M}^I/D) = \prod_i N(\bar{c}[i], \mathcal{M})/D$. If we choose $\bar{c} \in \mathcal{M}^I/D$ such that $\aleph_0 \leq N(\bar{c}, \mathcal{M}^I/D) := \lambda \leq \prod_i m_i/D := \mu$, then $\mathcal{M}^I/D$ will not be λ^+-compact (and hence not μ^+-compact) since one could write down a type of an element in no equivalence class.

For each i, we will set $\bar{c}[i] = \bar{c}_{n(i)}$ where $n(i) = \max\{l \mid N(\bar{c}_l, \mathcal{M}) \leq m_i\}$ if such l exists, and otherwise $n(i) = 1$. Note that for each $k < \omega$ there is a big set of i for which $m_i \geq k$, in particular when $k = N(\bar{c}_l, \mathcal{M})$, so for each $l < \omega$ $n(i)$ will almost always be defined in the first case where $l \leq N(\bar{c}[i], \mathcal{M}) \leq m_i$. Thus $\aleph_0 \leq \prod_i N(\bar{c}[i], \mathcal{M})/D \leq \prod_i m_i/D$ as desired. $\square$

Lemma 3.3. *Suppose T is unstable, $\mathcal{M} \models T$, and D is an $\aleph_1$-incomplete ultrafilter on I. Then $\mathcal{M}^I/D$ is not κ^+-compact where $\kappa = lcf(\omega, D)$.*

Proof. Suppose $\varphi(x; \bar{y})$ and $\bar{a}^n_l \in \mathcal{M}$ witness the order property, i.e. $\{\varphi(x; \bar{a}^n_k)^{m<k} \mid k < n\}$ is consistent for all $m < n < \omega$. Choose any distinct $\bar{b}_n \in \mathcal{M}$ for $n < \omega$, let $P^{\mathcal{M}} = \{\bar{b}_n \mid n < \omega\}$, and order them by $b_k <^{\mathcal{M}} b_n \Leftrightarrow k < n$. Define functions $F^{\mathcal{M}}_l$ for $l < \ell(\bar{y})$ such that

$$\bar{a}^n_k = \langle F^{\mathcal{M}}_0(b_k, b_n), \ldots, F^{\mathcal{M}}_{\ell(\bar{y})}(b_k, b_n)\rangle := \bar{F}^{\mathcal{M}}(b_k, b_n).$$

The ultrapower $\mathcal{N} = \mathcal{M}^I/D$ inherits these functions and relations with the same properties. So $<^{\mathcal{N}}$ is an infinite discrete linear order on $P^{\mathcal{N}}$ with a first element and no last element to which the $\bar{b}_n$ form an initial segment, and whenever $b, c, d \in P^{\mathcal{N}}$ and $d <^{\mathcal{N}} c$ the type $\{\varphi(x; \bar{F}^{\mathcal{N}}(b, c))^{d <^{\mathcal{N}} b} \mid b <^{\mathcal{N}} c\}$ is consistent.

Since $\kappa = \mathrm{lcf}(\omega, D)$, we can choose a lower cofinal sequence $c_i \in P^{\mathcal{N}}$ for $i < \kappa$ such that $b_n <^{\mathcal{N}} c_j <^{\mathcal{N}} c_i$ for all $n < \omega$ and $i < j < \kappa$ and no $c \in P^N$ has $b_n <^{\mathcal{N}} c <^{\mathcal{N}} c_i$ for all $n < \omega$ and $i < \kappa$. The type

$$\{\neg\varphi(x, \bar{F}^{\mathcal{N}}(b_n, c_0)) \mid n < \omega\} \cup \{(\forall z \in P^{\mathcal{N}}, c_j \leq^{\mathcal{N}} z \leq^{\mathcal{N}} c_i)\varphi(x, \langle \bar{F}^{\mathcal{N}}(z, c_0)) \mid n < \omega, 0 < i < j < \kappa\}$$

is consistent but we claim it is not realized in $\mathcal{N}$. If it was realized, say at a, then the initial b_n segment of $P^{\mathcal{N}}$ would be definable in $\mathcal{N}$ as $\{z \in P^N \mid \neg\varphi(a, \bar{F}^{\mathcal{N}}(z, c_0))\}$. But this is impossible in any ultrapower of an order of type ω. So we conclude $\mathcal{N}$ is not κ^+-compact. $\square$

4 Compactness Results

Here we collect results about when various ultrapowers are compact. Note that we work over a countable theory/language, and so compactness and saturation are the same. But to be consistent with previous work, we will keep saying "compact". The following two lemmas are well known.

This next lemma is specialized to stable theories and requires the characterization of compactness via the extendibility of indiscernible sets.

Lemma 4.1. *Suppose T is countable, $\mathcal{M} \models T$, and D is an $\aleph_1$-incomplete ultrafilter over I. Let $\mathcal{N} = \mathcal{M}^I/D$.*

(i) If T has NFCP, then $\mathcal{N}$ is λ-compact where $\lambda = \aleph^I_0/D$.

(ii) If T is stable with FCP, then $\mathcal{N}$ is λ-compact where λ is the smallest infinite $\prod_i n_i/D$, $n_i < \omega$.

Proof. In both cases we seek to use Corollary 1.7. (Note that the ultrapower is automatically $\aleph_1$-compact by $\aleph_1$-incompleteness of the ultrafilter. In particular we can assume λ is uncountable.) Fix an indiscernible set $\{\bar{c}_i \mid i < \omega\} \subseteq \mathcal{N}$. We show that this extends in $\mathcal{N}$ to an indiscernible set of size λ.

Recursively construct $S \subseteq \mathcal{P}(\mathcal{M})$ such that $|S| = |\mathcal{M}|$, $S \supseteq [\mathcal{M}]^{<\omega}$, and whenever $\Delta \subseteq \mathcal{L}$ is finite and $w \in S$ is a Δ-n-indiscernible set that extends to a Δ-n-indiscernible set in $\mathcal{M}$ of some size μ, S contains one such extension of size μ. Enumerate $\mathcal{M} = \{a_j \mid j < |\mathcal{M}|\}$ and $S = \{w_\alpha \mid \alpha < |\mathcal{M}|\}$. Define a relation $\in^{\mathcal{M}}$ on $\mathcal{M}$ by $a_j \in^{\mathcal{M}} a_\alpha \Leftrightarrow a_j \in w_\alpha$. So with this new relation, there exists a formula $\varphi_{\Delta,n}(x)$ for each finite Δ defining $\{y \mid y \in x\}$ being a Δ-n-indiscernible set.

Moving forward, define $P^{\mathcal{M}} = \{a_\alpha \mid |w_\alpha| \geq \aleph_0\}$ in proving (i). Otherwise in proving (ii) define $P^{\mathcal{M}} = \mathcal{M}$. We claim that in either case, for each finite Δ and $n < \omega$ there is an m such that for all i:

$$(\forall \Delta - n - \text{indiscernible } w_\alpha, |w_\alpha| \geq m)(\exists \Delta - n - \text{indiscernible } w_j) w_\alpha \subseteq w_j \wedge a_j \in P^{\mathcal{M}}).$$

This is trivially true in case (ii). In case (i), we just need to show that there is an m for which every finite Δ-n-indiscernible set of size at least m extends to an infinite Δ-n-indiscernible set. This follows from Lemma 1.3 since T has NFCP in case (i).

The property given above can be expressed in $\mathcal{L} \cup \{\in, P\}$. In this language, the expanded ultrapower is still $\aleph_1$-saturated, which shows that the type

$$\{c_i \in x \mid i < \omega\} \cup \{\varphi_{\Delta,n}(x) \mid \Delta \text{ finite}, n < \omega\} \cup \{P(x)\}$$

is consistent and realized in $\mathcal{N}$, say at b. Then $w = \{a \in \mathcal{N} \mid a \in^{\mathcal{N}} b\}$ will be an indiscernible set containing all c_i. We claim that $|w| \geq \lambda$, which will complete the proof.

In case (i), we have $\lambda = \aleph_0^I/D$ and in each factor model w corresponds to an infinite indiscernible set, and thus has size at least λ in $\mathcal{N}$. In case (ii), w is still infinite in $\mathcal{N}$ since it contains the c_i. If the factors are finite almost everywhere, then $|w| \geq \lambda$. Otherwise the factors are infinite almost everywhere, making $|w| \geq \aleph_0^I/D \geq \lambda$. (Of course, equality must hold in light of Lemma 3.2.) $\square$

5 Defining Keisler's Order

The definition of Keisler's Order gets off the ground with the following lemma, which says that compactness of regular ultrapowers is a property of theories.

Lemma 5.1. *Suppose D is a regular ultrafilter on I and $\mathcal{M} \equiv \mathcal{N}$. Then $\mathcal{M}^I/D$ is $|I|^+$-compact if and only if $\mathcal{N}^I/D$ is $|I|^+$-compact.*

Definition. Given any two theories T_1 and T_2, we say $T_1 \sqsubseteq_\lambda T_2$ if and only if for every regular ultrafilter D on λ and some/all $\mathcal{M}_i \models T_i$,

$$\mathcal{M}_2^\lambda/D \text{ is } \lambda^+\text{-compact} \Rightarrow \mathcal{M}_1^\lambda/D \text{ is } \lambda^+\text{-compact}.$$

We say $T_1 \sqsubseteq T_2$ if and only if $T_1 \sqsubseteq_\lambda T_2$ for all λ.

This defines a preorder, on whose classes Keisler's (Partial) Order is defined. We will later restrict our attention to countable theories, but for now some general results are proven. In what follows, we say T "minimal" or "maximal" in Keisler's Order to mean, respectively, that $T \trianglelefteq T'$ or $T' \trianglelefteq T$ for all T'.

Lemma 5.2. *(i) T is minimal in Keisler's order if and only if for every λ, regular ultrafilter D on λ, and $\mathcal{M} \models T$, $\mathcal{M}^\lambda/D$ is λ^+-compact.*

(ii) T is maximal in Keisler's order if and only if for every λ, regular ultrafilter D on λ, and $\mathcal{M} \models T$,

$$\mathcal{M}^\lambda/D \text{ is } \lambda^+\text{-compact} \Leftrightarrow D \text{ is good.}$$

Proof. (i) If T is minimal, then in particular it lies below the theory of infinite sets T_{inf}. A model $\mathcal{N}$ of T_{inf} is λ^+-compact if and only if $|\mathcal{N}| > \lambda$. Since D is regular, we have $|\mathcal{M}^\lambda/D| = |\mathcal{M}|^{|\lambda|} > \lambda$ by Lemma 2.1. So this ultrapower is λ^+-compact, thus by definition if $\trianglelefteq$, we have that any ultrapower of any model of T modulo D is also λ^+-compact as desired.

The other implication is immediate.

(ii) If T is maximal, then in particular it lies above the theory of dense linear orders. Given a regular ultrafilter D on λ, we already know that if D is good then $\mathcal{M}^\lambda/D$ is λ^+ compact for any $\mathcal{M}$ at all. For the new implication, now using $\mathcal{M} \models T$, if $\mathcal{M}^\lambda/D$ is λ^+-compact then so is $\mathcal{M}'^\lambda/D$ for all dense linear orders $\mathcal{M}'$ by T's maximality in $\trianglelefteq_\lambda$. So by Lemma 2.6 D is λ^+-good and hence good.

Conversely, suppose that models of T are made λ^+ compact in the ultrapower by precisely good ultrafilters. So if $\mathcal{M}' \models T'$ is another model and theory, and $\mathcal{M}^\lambda/D$ is λ^+-compact, then D is good by assumption, so $\mathcal{M}'^\lambda/D$ is λ^+-compact by Lemma 2.6. This holds for all λ so $T' \trianglelefteq T$.

□

The proof of Lemma 5.2 also shows that the two equivalences are nonvoid, i.e. that there actually exists a minimal class and a maximal class.

Corollary 5.3. *The theory of infinite sets is minimal and the theory of dense linear orders is maximal in Keisler's Order.*

The following lemma will come in handy for showing that theories with abstract structural properties lie above concrete theories that exhibit solely those properties.

Lemma 5.4. *Suppose T_i is an $\mathcal{L}_i$-theory for $i = 1, 2$ and λ is infinite. Suppose that for every $\Phi_1 \subseteq \mathcal{L}_1$, $\Phi_1 = \{\varphi_\alpha(x; \bar{y}_\alpha) \mid \alpha < \lambda\}$, there exists $\Phi_2 \subseteq \mathcal{L}_2$, $\Phi_2 = \{\psi_\beta(\bar{v}; \bar{z}_\beta) \mid \beta < \kappa\}$ and $\eta \in \kappa^\lambda$ such that for every $\mathcal{N}_1 \models T_1$ and $\bar{a}_\alpha \in \mathcal{N}_1$, $\alpha < \lambda$, there exists $\mathcal{M}_2 \models T_2$ and $\bar{b}_\beta \in \mathcal{M}_2$, $\beta < \kappa$, such that for every finite $w \subseteq \lambda$, $\{\varphi_\alpha(x; \bar{a}_\alpha) \mid \alpha \in w\}$ is consistent with $\mathcal{N}_1$ if and only if $\{\psi_{\eta(\alpha)}(\bar{v}; \bar{b}_{\eta(\alpha)}) \mid \alpha \in w\}$ is consistent with $\mathcal{M}_2$.*

Then, $T_1 \trianglelefteq_\lambda T_2$.

Proof. Let D be a regular ultrafilter on λ such that every $\mathcal{M}_2 \models T_2$ has that $\mathcal{M}_2^\lambda/D$ is λ^+-compact. Take $\mathcal{M}_1 \models T_1$ and fix a type $p(x)$ of size at most λ consistent with $\mathcal{M}_1^\lambda/D$. Let Φ_1 consist of the formulas in p with the parameters $\bar{a}_\alpha$ replaced by free variables $\bar{y}_\alpha$. We are given Φ_2 and η in the hypothesis, we come in with $\mathcal{N}_1 = \mathcal{M}_1^\lambda/D$ and $\bar{a}_\alpha$, $\alpha < \lambda$, the parameters in p. Now the hypothesis gives us $\mathcal{M}_2 \models T_2$ and $\bar{b}_\beta \in \mathcal{M}_2$. Let $q(\bar{v}) = \{\psi_{\eta(\alpha)}(\bar{v}; \bar{b}_{\eta(\alpha)}) \mid \alpha < \lambda\}$, and this type will be consistent with $\mathcal{M}_2^\lambda/D$, where we embed $\mathcal{M}_2$ and the parameters as usual.

The ultraproduct $\mathcal{M}_2^\lambda/D$ will realize $q(\bar{v})$ at some $\bar{c}$ by hypothesis. Our goal is to pull this back to $\mathcal{M}_1^\lambda/D$. Choose a regularizing family e_α, $\alpha < \lambda$. For each $t < \lambda$, define

$$w(t) = \{\alpha < \lambda \mid \mathcal{M}_2 \models \psi_{\eta(\alpha)}[\bar{c}[t]; \bar{b}_{\eta(\alpha)}[t]] \wedge t \in e_\alpha\}.$$

Note this set is finite and $\{\psi_{\eta(\alpha)}(\bar{v}; \bar{b}_{\eta(\alpha)}[t]) \mid \alpha \in w(t)\}$ is consistent with $\mathcal{M}_2$, hence $\{\varphi_\alpha(x; \bar{a}_\alpha[t]) \mid \alpha \in w(t)\}$ is consistent with $\mathcal{M}_1$, say realized at $d[t]$.

We claim $d = \langle d[t] \rangle_D$ realizes $p(x)$ in $\mathcal{M}_1^\lambda/D$. Given $\varphi_\alpha(x; \bar{a}_\alpha)$, we must show a big set of t have $\alpha \in w(t)$. This is true precisely for those t in e_α and for which $\mathcal{M}_2 \models \psi_{\eta(\alpha)}[\bar{c}[t]; \bar{b}_{\eta(\alpha)}[t]]$, which is the intersection of two big sets and hence big. We conclude that $\mathcal{M}_1^\lambda/D$ is λ^+-compact, and thus $T_1 \sqsubseteq_\lambda T_2$ as desired. □

6 Three + One Classes of Keisler's Order

Here we classify three classes of Keisler's Order for countable theories, with a fourth being consistent. We will henceforth stop mentioning that we are only ranging over countable theories. (Of course as we previously remarked, the fourth outright exists, and many exist between the third and fourth, not linearly ordered, obtainable in ZFC.)

Theorem 6.1. *There exists a first class which is precisely the NFCP theories.*

Proof. We already showed that the theory of infinite sets is minimal, so in particular there exists a first class. We show that this class is precisely the NFCP theories.

If $\mathcal{M} \models T$ has NFCP, any $\mathcal{M}^\lambda/D$ is $\aleph_0^\lambda/D = \aleph_0^\lambda \geq \lambda^+$-compact by Lemma 4.1(i). So by Lemma 5.2, T belongs to the first class.

Conversely, suppose $\mathcal{M} \models T$ has FCP. We show that for some λ and regular ultrafilter D on λ, $\mathcal{M}^\lambda/D$ is not λ^+-compact, hence T does not belong to the first class (again by Lemma 5.2). We take $\lambda = 2^{\aleph_0}$. By Lemma 2.7, there exists a regular ultrafilter D on λ and $n_i < \omega$ such that $\prod_i n_i/D = 2^{\aleph_0}$. If T is stable, then by Lemma 3.2, $\mathcal{M}^\lambda/D$ is not $(2^{\aleph_0})^+$-compact. If T is unstable, then by Lemma 3.1 $\mathcal{M}^\lambda/D$ is also not $(2^{\aleph_0})^+$-compact since $\prod_i n_i/D < 2^{2^{\aleph_0}}$. So either way, T does not belong to the first class. We conclude that the first class consists precisely of the NFCP theories. □

Theorem 6.2. *There exists a second class which is precisely the stable FCP theories.*

Proof. Any theory that is FCP must lie strictly above the first class (of NFCP theories). We first show that the stable FCP theories form a class, and that it must be the next class above the first.

By Lemmas 4.1(i) and 3.2, see that whenever $\mathcal{M} \models T$ a stable FCP theory, $\mathcal{M}^\lambda/D$ is λ^+-compact if and only if λ is strictly less than the least infinite $\prod_i n_i/D$. This is a property of the ultrafilter and not the theory, so the stable FCP theories belong to a single class.

Now suppose $\mathcal{M}' \models T'$ is unstable. To show that T' does not belong to the class containing the stable FCP theories, it suffices by the previous paragraph to find an ultrafilter D on some λ for which λ is strictly less than the least infinite $\prod_i n_i/D$ but $\mathcal{M}'^\lambda/D$ is not λ^+-saturated. By Lemma 2.8, there exists such an ultrafilter on $\lambda = 2^{\aleph_0}$ where $2^{2^{\aleph_0}}$ is the least infinite $\prod_i n_i/D$ with the additional property that $\mathrm{lcf}(\omega, D) = 2^{\aleph_0}$. So by Lemma 3.3, $\mathcal{M}'^\lambda/D$ is not $(2^{\aleph_0})^+$-compact. We conclude that the stable FCP theories form a class.

Finally we we show that any unstable (thus also FCP) theory lies immediately above the class of stable FCP theories, meaning that this is the second class. So, suppose $\mathcal{M}' \models T'$ an unstable theory.

If T' has the SOP, then T' belongs to the maximal class as we will show in Theorem 6.4, so $T \sqsubseteq T'$. Otherwise T' has the IP. If $\mathcal{M}'^\lambda/D$ is λ^+-compact, then $\mathrm{lcf}(\omega, D) > \lambda$ by Lemma 3.3. Thus whenever $\prod_i n_i/D$ is infinite, it must lie above $\mathrm{lcf}(\omega, D)$ which is strictly above λ. So by Lemma 4.1(ii), $\mathcal{M}^\lambda/D$ is λ^+-compact. This shows $T \sqsubseteq T'$ as desired. $\square$

Theorem 6.3. *There exists a third class.*

Proof. We will not classify the third class, but rather argue that there exists a class immediately above the second by exhibiting a theory with this property. We will take T to be the theory of random graphs. This theory has IP and in particular is unstable, so it lies above the second class. Given any other unstable theory T', we must show $T \sqsubseteq T'$.

If T' has the SOP, then as we will show in Theorem 6.4, T' is maximal giving $T \sqsubseteq T'$. Otherwise T' has the IP, say witnessed by $\psi(\bar{v}; \bar{z})$. We seek to use Lemma 5.4 to establish the same. In the notation of that Lemma, take Φ_1 to be an arbitrary collection of at most λ formulas of the form $\varphi_\alpha(x; \bar{y}_\alpha)$ in the language of graphs. By quantifier elimination, each formula is a disjunction of finitely many specified edge and non-edge relations among x and $y \in \bar{y}_\alpha$. Let $\psi_\alpha(\bar{v}; \bar{z}_\alpha)$ be the corresponding formula made by a finite disjunction of conjunctions of instances of ψ and $\neg\psi$, specifying the same pattern of "edge relations" via ψ. Let Φ_2 be the collection of all of such $\psi_\alpha(\bar{v}; \bar{z}_\alpha)$ and take η as the identity map.

Continuing, we are given an arbitrary random graph $\mathcal{N}_1$ with $\bar{a}_\alpha \in \mathcal{N}_1$. Take $\mathcal{M}_2 \models T'$ to be any model of size λ and choose $\bar{b}_\alpha \in \mathcal{M}_2$ with the same equality relations as the $\bar{a}_\alpha$. Then, the IP witnessed by ψ will give that for every finite $w \subseteq \lambda$, $\{\varphi_\alpha(x; \bar{a}_\alpha) \mid \alpha \in w\}$ is consistent with $\mathcal{N}_1$ if and only if $\{\psi_{\eta(\alpha)}(\bar{v}; \bar{b}_{\eta(\alpha)}) \mid \alpha \in w\}$ is consistent with $\mathcal{M}_2$, since a finite consistent set of formulas in $\mathcal{N}_1$ just means that there are no contradictory specified edge relations. So Lemma 5.4 applies, this at every λ, so we get $T \sqsubseteq T'$ as desired. $\square$

Theorem 6.4. *There exists a maximal class containing all SOP theories.*

Proof. We already showed that the theory T of dense linear orders is maximal. So, take any other theory T' with SOP. We show that $T \sqsubseteq T'$ by using Lemma 5.4. The proof is analogous to that given in Theorem 6.3 (where SOP takes the role of IP and we think about dense linear orders as opposed to graphs), and when it comes time to choose $\mathcal{M}_2 \models T'$, we take a model with a definable chain of sets that form a dense linear order isomorphic to $\mathcal{N}_1$. Otherwise the proof goes the same way. $\square$

Theorem 6.5. *It is consistent that the third class is not maximal. (And, one final time, we remark that this is outright provable in ZFC, as is much more about Keisler's Order.)*

Proof. Assume Martin's Axiom and $\aleph_1 < 2^{\aleph_0}$ (this is consistent with ZFC). We will show that the theory of random graphs, shown to lie in the third class in Theorem 6.3, is not maximal. By Lemma 5.2, it suffices to find a regular ultrafilter on some λ that is not good but nonetheless makes $\mathcal{M}^\lambda/D$ λ^+-compact for a random graph $\mathcal{M}$. For this, we appeal to Lemma 2.9 at $\lambda = \aleph_1$ which gives us exactly what we need. So it is consistent that the third class is not maximal, in particular giving us a fourth (maximal) class. $\square$