Chapter 7: Interpolation

- Topics:
  - Examples
  - Polynomial Interpolation – bases, error, Chebyshev, piecewise
  - Orthogonal Polynomials
  - Splines – error, end conditions
  - Parametric interpolation
  - Multivariate interpolation: f(x,y)

Interpolation

- Basic interpolation problem: for given data

  $$(t_1, y_1),\ (t_2, y_2),\ \ldots\ (t_m, y_m) \quad \text{with} \quad t_1 < t_2 < \cdots < t_m$$

  determine function $f : \mathbb{R} \to \mathbb{R}$ such that

  $$f(t_i) = y_i, \quad i = 1, \ldots, m$$

- f is *interpolating function*, or *interpolant*, for given data
- Additional data might be prescribed, such as slope of interpolant at given points
- Additional constraints might be imposed, such as smoothness, monotonicity, or convexity of interpolant
- f could be function of more than one variable

Note: We might look at multi-dimensional case, even though the text does not.

Purposes for Interpolation

- Plotting smooth curve through discrete data points
- Reading between lines of table
- Differentiating or integrating tabular data
- Quick and easy evaluation of mathematical function
- Replacing complicated function by simple one

  - Basis functions for function approximation in numerical solution of ordinary and partial differential equations (ODEs and PDEs).
  - Basis functions for developing integration rules.
  - Basis functions for developing differentiation techniques.
    (Not just tabular data…)

Interpolation vs Approximation

- By definition, interpolating function fits given data points exactly
- Interpolation is inappropriate if data points subject to significant errors
- It is usually preferable to smooth noisy data, for example by least squares approximation
- Approximation is also more appropriate for special function libraries

Issues in Interpolation

Arbitrarily many functions interpolate given set of data points

- What form should interpolating function have?
- How should interpolant behave between data points?
- Should interpolant inherit properties of data, such as monotonicity, convexity, or periodicity?
- Are parameters that define interpolating function meaningful? ***For example, function values, slopes, etc. ?***
- If function and data are plotted, should results be visually pleasing?

Choosing Interpolant

Choice of function for interpolation based on

- How easy interpolating function is to work with
  - determining its parameters ← ***Conditioning? !!***
  - evaluating interpolant
  - differentiating or integrating interpolant
- How well properties of interpolant match properties of data to be fit (smoothness, monotonicity, convexity, periodicity, etc.)

Functions for Interpolation

- Families of functions commonly used for interpolation include
  - Polynomials
  - Piecewise polynomials
  - Trigonometric functions
  - Exponential functions
  - Rational functions
- For now we will focus on interpolation by polynomials and piecewise polynomials
- We will consider trigonometric interpolation (DFT) later

A Classic Polynomial Interpolation Problem

- Suppose you're asked to tabulate data such that linear interpolation between tabulated values is correct to 4 digits.
- How many entries are required on, say, [0,1]?
- How many digits should you have in the tabulated data?

x	$Si(x) = \int_0^x \frac{\sin t}{t} dt$	δ^2	$Ci(x) = \int_\infty^x \frac{\cos t}{t} dt$	δ^2
41.00	1.59494 33514	-23986	-0.00327 89946	+ 4456
.01	9490 34645	23936	0351 95823	4695
.02	9486 11840	23881	0375 97005	4932
.03	9481 65154	23826	0399 93255	5170
.04	9476 94642	23766	0423 84335	5405
41.05	1.59472 00364	-23705	-0.00447 70010	+ 5642
.06	9466 82381	23642	0471 50043	5878
.07	9461 40756	23577	0495 24198	6110
.08	9455 75554	23508	0518 92243	6347
.09	9449 86844	23439	0542 53941	6577
41.10	1.59443 74695	-23365	-0.00566 09062	+ 6811
.11	9437 39181	23290	0589 57372	7042
.12	9430 80377	23214	0612 98640	7273
.13	9423 98359	23134	0636 32635	7502
.14	9416 93207	23053	0659 59128	7732
41.15	1.59409 65002	-22967	-0.00682 77889	+ 7959
.16	9402 13830	22883	0705 88691	8187
.17	9394 39775	22793	0728 91306	8412
.18	9386 42927	22703	0751 85509	8639
.19	9378 23376	22609	0774 71073	8862

A Classic Polynomial Interpolation Problem

An important polynomial interpolation result for $f(x) \in C^n$:

If $p(x) \in \mathbb{P}_{n-1}$ and $p(x_j) = f(x_j)$, $j = 1, \ldots, n$, then there exists a $\theta \in [x_1, x_2, \ldots, x_n, x]$ such that

$$f(x) - p(x) = \frac{f^n(\theta)}{n!}(x - x_1)(x - x_2) \cdots (x - x_n).$$

In particular, for *linear interpolation*, we have

$$f(x) - p(x) = \frac{f''(\theta)}{2}(x - x_1)(x - x_2)$$

$$|f(x) - p(x)| \leq \max_{[x_1:x_2]} \frac{|f''|}{2}\frac{h^2}{4} = \max_{[x_1:x_2]} \frac{h^2|f''|}{8}$$

where the latter result pertains to $x \in [x_1, x_2]$.

A Classic Polynomial Interpolation Problem

Example: $f(x) = \cos(x)$

We know that $|f''| \leq 1$ and thus, for linear interpolation

$$|f(x) - p(x)| \leq \frac{h^2}{8}.$$

If we want 4 decimal places of accuracy, accounting for rounding, we need

$$\begin{aligned} |f(x) - p(x)| &\leq \frac{h^2}{8} \leq \frac{1}{2} \times 10^{-4} \\ h^2 &\leq 4 \times 10^{-4} \\ h &\leq 0.02 \end{aligned}$$

x	cos x
0.00	1.00000
0.02	0.99980
0.04	0.99920
0.06	0.99820
0.08	0.99680

Basis Functions

- Family of functions for interpolating given data points is spanned by set of *basis functions* $\phi_1(t), \ldots, \phi_n(t)$
- Interpolating function f is chosen as linear combination of basis functions,

$$f(t) = \sum_{j=1}^{n} x_j \phi_j(t)$$

- Requiring f to interpolate data (t_i, y_i) means

$$f(t_i) = \sum_{j=1}^{n} x_j \phi_j(t_i) = y_i, \quad i = 1, \ldots, m$$

which is system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ for n-vector $\boldsymbol{x}$ of parameters x_j, where entries of $m \times n$ matrix $\boldsymbol{A}$ are given by $a_{ij} = \phi_j(t_i)$

Existence, Uniqueness, and Conditioning

- Existence and uniqueness of interpolant depend on number of data points m and number of basis functions n
- If $m > n$, interpolant usually doesn't exist
- If $m < n$, interpolant is not unique
- If $m = n$, then basis matrix $\boldsymbol{A}$ is nonsingular provided data points t_i are distinct, so data can be fit exactly
- Sensitivity of parameters $\boldsymbol{x}$ to perturbations in data depends on $\mathrm{cond}(\boldsymbol{A})$, which depends in turn on choice of basis functions

Polynomial Interpolation

- Simplest and most common type of interpolation uses polynomials
- Unique polynomial of degree at most $n-1$ passes through n data points (t_i, y_i), $i = 1, \ldots, n$, where t_i are distinct
- There are many ways to represent or compute interpolating polynomial, but in theory all must give same result

(i.e., in infinite precision arithmetic)

Monomial Basis

- *Monomial basis functions*

$$\phi_j(t) = t^{j-1}, \quad j = 1, \ldots, n$$

give interpolating polynomial of form

$$p_{n-1}(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$$

with coefficients $\boldsymbol{x}$ given by $n \times n$ linear system

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

- Matrix of this form is called *Vandermonde matrix*

Example: Monomial Basis

- Determine polynomial of degree two interpolating three data points $(-2, -27)$, $(0, -1)$, $(1, 0)$
- Using monomial basis, linear system is

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{y}$$

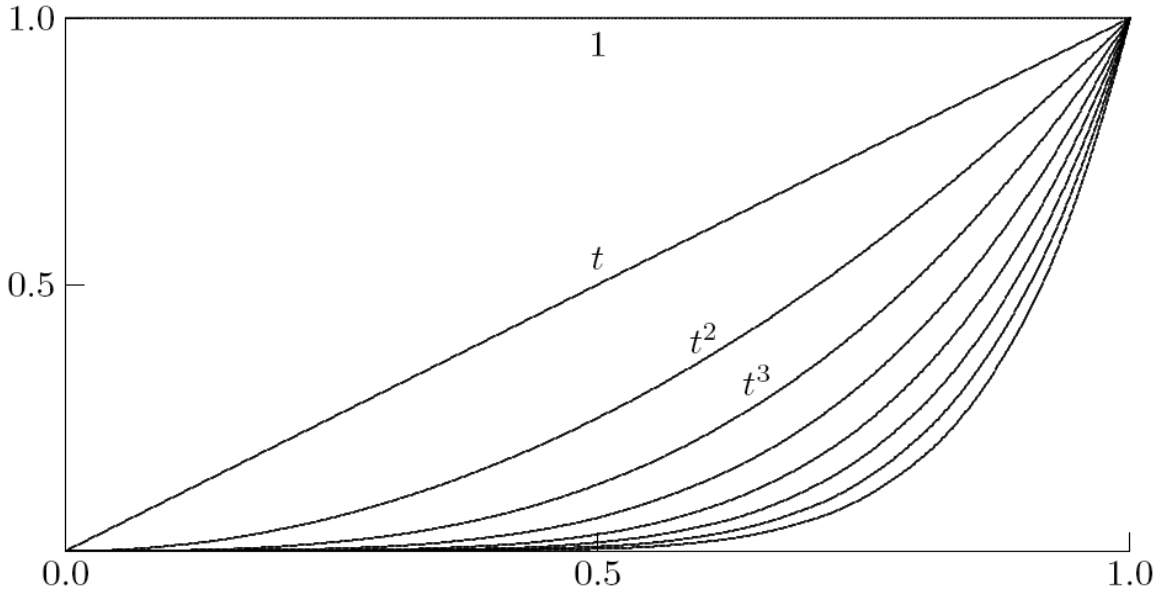
- For these particular data, system is

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix}$$

whose solution is $x = \begin{bmatrix} -1 & 5 & -4 \end{bmatrix}^T$, so interpolating polynomial is

$$p_2(t) = -1 + 5t - 4t^2$$

Monomial Basis, continued



- Solving system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ using standard linear equation solver to determine coefficients $\boldsymbol{x}$ of interpolating polynomial requires $\mathcal{O}(n^3)$ work

Monomial Basis, continued

- For monomial basis, matrix A is increasingly ill-conditioned as degree increases
- Ill-conditioning does not prevent fitting data points well, since residual for linear system solution will be small
- But it does mean that values of coefficients are poorly determined
- Both conditioning of linear system and amount of computational work required to solve it can be improved by using different basis
- Change of basis still gives same interpolating polynomial for given data, but *representation* of polynomial will be different

Monomial Basis, continued

- Conditioning with monomial basis can be improved by shifting and scaling independent variable t

$$\phi_j(t) = \left(\frac{t-c}{d}\right)^{j-1}$$

where, $c = (t_1 + t_n)/2$ is midpoint and $d = (t_n - t_1)/2$ is half of range of data

- New independent variable lies in interval $[-1, 1]$, which also helps avoid overflow or harmful underflow
- Even with optimal shifting and scaling, monomial basis usually is still poorly conditioned, and we must seek better alternatives

< interactive example >

Polynomial Interpolation

- Two types: *Global or Piecewise*
- Choices:
  - A: points are given to you
  - B: you choose the points
- Case A: piecewise polynomials are most common – ***STABLE***.
  - Piecewise linear
  - Splines
  - Hermite (matlab "pchip" – piecewise cubic Hermite int. polynomial)
- Case B: high-order polynomials are OK if points chosen wisely
  - Roots of orthogonal polynomials
  - Convergence is exponential: err ~ $Ce^{-\sigma n}$, instead of algebraic: err~Cn^{-k}

Piecewise Polynomial Interpolation

- Example – Given the table below,

x_j	f_j
0.6	1.2
0.8	2.0
1.0	2.4

- Q: What is f(x=0.75) ?

Polynomial Interpolation

- Example – Given the table below,

x_j	f_j
0.6	1.2
0.8	2.0
1.0	2.4

- Q: What is f(x=0.75) ?

- A: 1.8 --- You've just done (piecewise) linear interpolation.

- Moreover, you know the error is $\leq (0.2)^2$ f'' / 8.

General Polynomial Interpolation

- Whether interpolating on segments or globally, error formula applies over the interval.
  If $p(t) \in \mathbb{P}_{n-1}$ and $p(t_j) = f(t_j)$, $j = 1, \ldots, n$, then there exists a $\theta \in [t_1, t_2, \ldots, t_n, t]$ such that

$$\begin{aligned} f(t) - p(t) &= \frac{f^n(\theta)}{n!}(t - t_1)(t - t_2) \cdots (t - t_n) \\ &= \frac{f^n(\theta)}{n!} q_n(t), \qquad q_n(t) \in \mathbb{P}_n. \end{aligned}$$

- We generally have no control over $f^n(\theta)$, so instead seek to optimize choice of the t_j in order to minimize

$$\max_{t \in [t_1, t_n]} |q_n(t)| .$$

- Such a problem is called a *minimax* problem and the solution is given by the t_js being the roots of a Chebyshev polynomial, as we will discuss shortly.

- First, however, we turn to the problem of constructing $p(t) \in \mathbb{P}_{n-1}(t)$.

Constructing High-Order Polynomial Interpolants

- *Lagrange Polynomials*

$$
\begin{aligned}
p(t) &= \sum_{j=1}^{n} f_j \, l_j(t) \\
l_j(t) &= 1 \qquad t = t_j \\
l_j(t_i) &= 0 \qquad t = t_i, \; i \neq j \\
l_j(t) &\in \mathbb{P}_{n-1}(t)
\end{aligned}
$$

The $l_j(t)$ polynomials are chosen so that $p(t_j) = f(t_j) := f_j$

The $l_j(t)$s are sometimes called the Lagrange cardinal functions.

Constructing High-Order Polynomial Interpolants

- *Lagrange Polynomials*

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p(t) &= \sum_{j=1}^{n} f_j \, l_j(t) \\
l_j(t) &= 1 \qquad t = t_j \\
l_j(t) &= 0 \qquad t = t_i, \; i \neq j \\
l_j(t) &\in \mathbb{P}_{n-1}(t)
\end{aligned}
$$

$$
l_j(t) = \frac{1}{C}(t - t_1)(t - t_2) \cdots (t - t_{j-1})(t - t_{j+1}) \cdots (t - t_n)
$$

- $l_j(t)$ is a polynomial of degree $n - 1$
- It is zero at $t = t_i$, $i \neq j$.
- Choose C so that it is 1 at $t = t_j$.

Constructing High-Order Polynomial Interpolants

- $l_j(t)$ is a polynomial of degree $n-1$
- It is zero at $t = t_i$, $i \neq j$.
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$$
\begin{aligned}
l_j(t) &= \frac{1}{C}(t-t_1)(t-t_2)\cdots(t-t_{j-1})(t-t_{j+1})\cdots(t-t_n) \\
C &= (t_j-t_1)(t_j-t_2)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_n)
\end{aligned}
$$

Constructing High-Order Polynomial Interpolants

- $l_j(t)$ is a polynomial of degree $n-1$
- It is zero at $t = t_i$, $i \neq j$.
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$$
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l_j(t) &= \frac{1}{C}(t-t_1)(t-t_2)\cdots(t-t_{j-1})(t-t_{j+1})\cdots(t-t_n) \\
C &= (t_j-t_1)(t_j-t_2)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_n)
\end{aligned}
$$

$$
l_j(t) = \left(\frac{t-t_1}{t_j-t_1}\right)\left(\frac{t-t_2}{t_j-t_2}\right)\cdots\left(\frac{t-t_{j-1}}{t_j-t_{j-1}}\right)\left(\frac{t-t_{j+1}}{t_j-t_{j+1}}\right)\cdots\left(\frac{t-t_n}{t_j-t_n}\right).
$$

Constructing High-Order Polynomial Interpolants

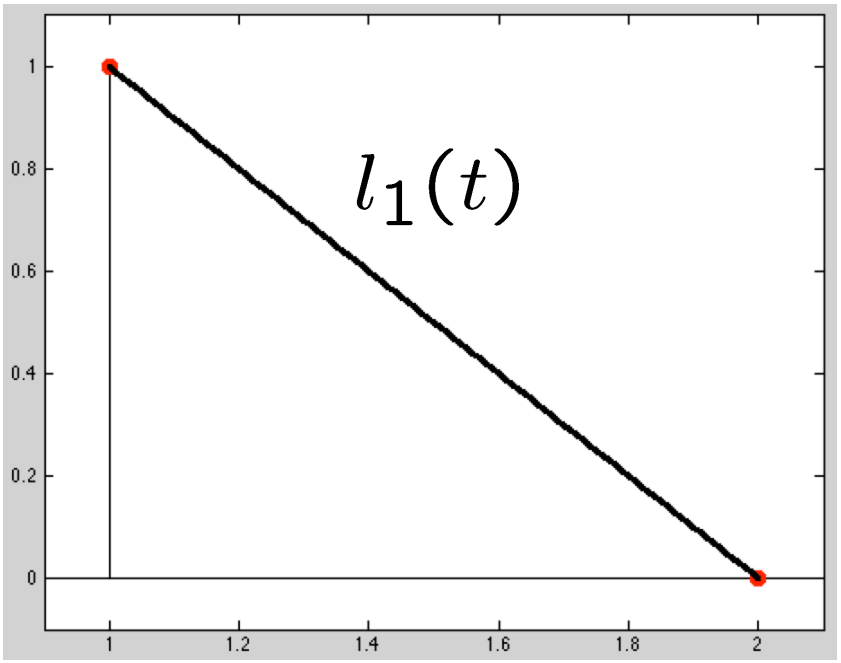
$$l_j(t) = \left(\frac{t-t_1}{t_j-t_1}\right)\left(\frac{t-t_2}{t_j-t_2}\right)\cdots\left(\frac{t-t_{j-1}}{t_j-t_{j-1}}\right)\left(\frac{t-t_{j+1}}{t_j-t_{j+1}}\right)\cdots\left(\frac{t-t_n}{t_j-t_n}\right).$$

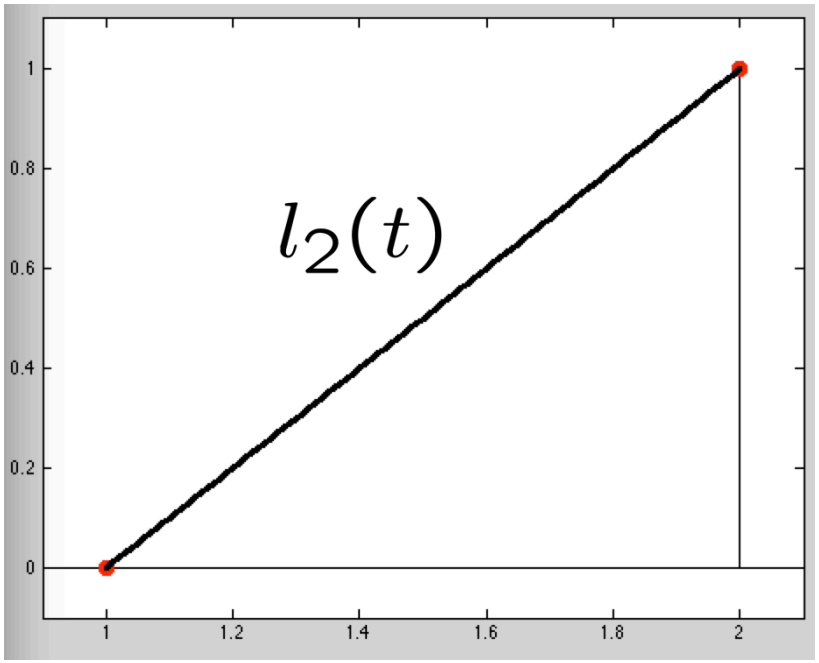
- Although a bit tedious to do by hand, these formulas are relatively easy to evaluate with a computer.

- So, to recap – Lagrange polynomial interpolation:

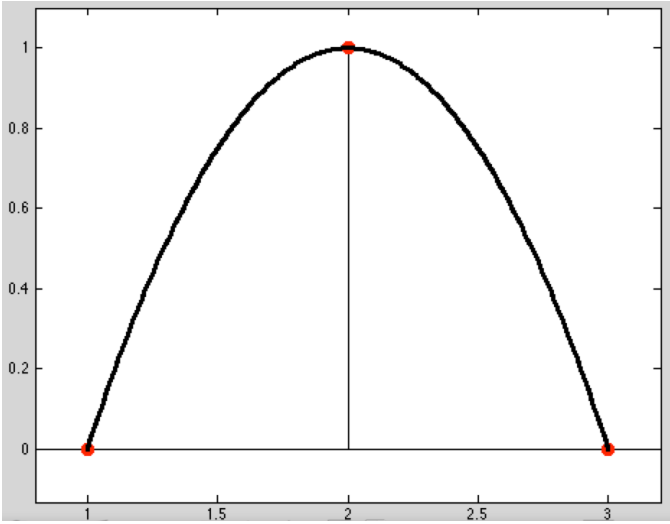
  - Construct $p(t) = \sum_j f_j l_j(t)$.
  - $l_j(t)$ given by above.
  - Error formula $f(t) - p(t)$ given as before.
  - Can choose t_j's to minimize error polynomial $q_n(t)$.

Lagrange Basis Functions, n=2 (linear)

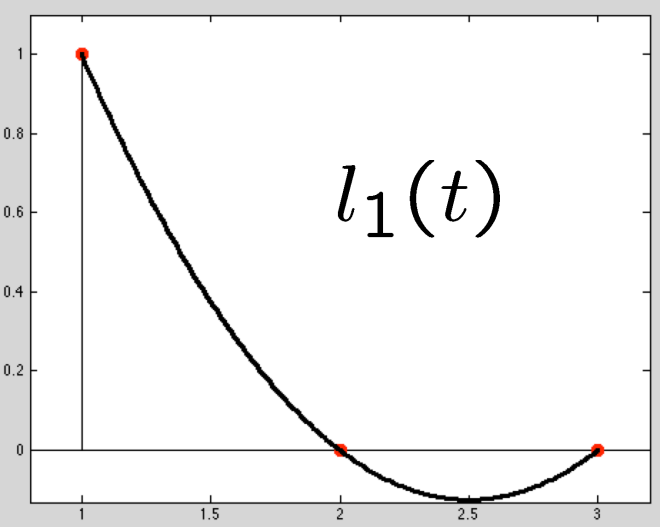




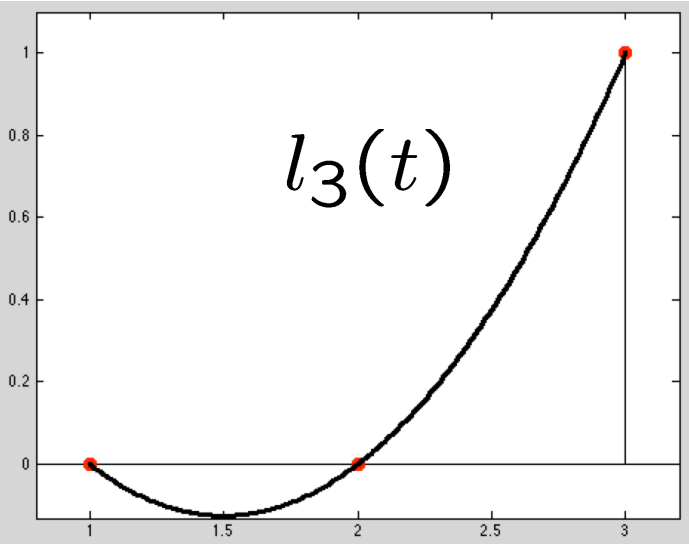
Lagrange Basis Functions, n=3 (quadratic)



$l_2(t)$



$l_1(t)$



$l_3(t)$

Also have the Newton Basis

- Given $p_n(t_j) = f_j$, $j = 1, \ldots, n$ and $p_n \in \mathbb{P}_{n-1}$.
- Let

$$p_{n+1}(t) \;:=\; p_n(t) \,+\, C(t-t_1)(t-t_2)\cdots(t-t_n)$$

such that $p_{n+1}(t_{n+1}) = f_{n+1}$

- Set

$$C \;=\; \frac{f_{n+1} - p_n(t_{n+1})}{q_n(t_{n+1})},$$

with $q_n(t) := (t-t_1)(t-t_2)\cdots(t-t_n) \in \mathbb{P}_n$

- These formulas are interesting because they are adaptive.

(*More details are in the text, but this is the essence of the method.*)

Lagrange Interpolation

- For given set of data points (t_i, y_i), $i = 1, \ldots, n$, *Lagrange basis functions* are defined by

$$\ell_j(t) = \prod_{k=1, k \neq j}^{n} (t - t_k) \; / \; \prod_{k=1, k \neq j}^{n} (t_j - t_k), \quad j = 1, \ldots, n$$

- For Lagrange basis,

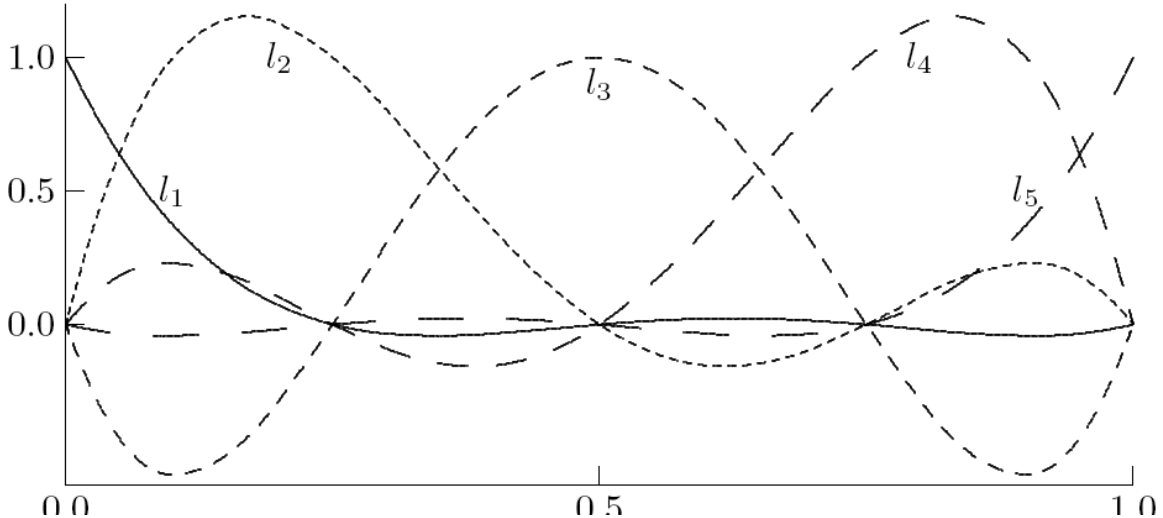
$$\ell_j(t_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i, j = 1, \ldots, n$$

  so matrix of linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ is identity matrix

- Thus, Lagrange polynomial interpolating data points (t_i, y_i) is given by

$$p_{n-1}(t) = y_1 \ell_1(t) + y_2 \ell_2(t) + \cdots + y_n \ell_n(t)$$

Lagrange Basis Functions



- Lagrange interpolant is easy to determine but more expensive to evaluate for given argument, compared with monomial basis representation
- Lagrangian form is also more difficult to differentiate, integrate, etc.

These concerns are important when computing by hand, but not important when using a computer.

Example: Lagrange Interpolation

- Use Lagrange interpolation to determine interpolating polynomial for three data points $(-2, -27)$, $(0, -1)$, $(1, 0)$
- Lagrange polynomial of degree two interpolating three points (t_1, y_1), (t_2, y_2), (t_3, y_3) is given by $p_2(t) =$

$$y_1 \frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)} + y_2 \frac{(t - t_1)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)} + y_3 \frac{(t - t_1)(t - t_2)}{(t_3 - t_1)(t_3 - t_2)}$$

- For these particular data, this becomes

$$p_2(t) = -27 \frac{t(t - 1)}{(-2)(-2 - 1)} + (-1) \frac{(t + 2)(t - 1)}{(2)(-1)}$$

Interpolating Continuous Functions

- If data points are discrete sample of continuous function, how well does interpolant approximate that function between sample points?
- If f is smooth function, and p_{n-1} is polynomial of degree at most $n-1$ interpolating f at n points $t_1, \ldots, t_n$, then

$$f(t) - p_{n-1}(t) = \frac{f^{(n)}(\theta)}{n!}(t-t_1)(t-t_2)\cdots(t-t_n)$$

  where θ is some (unknown) point in interval $[t_1, t_n]$
- Since point θ is unknown, this result is not particularly useful unless bound on appropriate derivative of f is known

Interpolating Continuous Functions, continued

- If $|f^{(n)}(t)| \leq M$ for all $t \in [t_1, t_n]$, and $h = \max\{t_{i+1} - t_i : \ i = 1, \ldots, n-1\}$, then

$$\max_{t \in [t_1, t_n]} |f(t) - p_{n-1}(t)| \leq \frac{Mh^n}{4n}$$

- Error diminishes with increasing n and decreasing h, but only if $|f^{(n)}(t)|$ does not grow too rapidly with n

Convergence

- Polynomial interpolating continuous function may not converge to function as number of data points and polynomial degree increases
- Equally spaced interpolation points often yield unsatisfactory results near ends of interval
- If points are bunched near ends of interval, more satisfactory results are likely to be obtained with polynomial interpolation
- Use of Chebyshev points distributes error evenly and yields convergence throughout interval for any sufficiently smooth function

Unstable and Stable Interpolating Basis Sets

- *Examples of unstable bases are:*
  - Monomials (modal): $\phi_i = x^i$
  - High-order Lagrange interpolants (nodal) on *uniformly-spaced* points.

- Examples of *stable* bases are:
  - Orthogonal polynomials (modal), e.g.,
    - Legendre polynomials: $L_k(x)$, or
    - bubble functions: $\phi_k(x) := L_{k+1}(x) - L_{k-1}(x)$.
  - Lagrange (nodal) polynomials based on Gauss quadrature points (e.g., Gauss-Legendre, Gauss-Chebyshev, Gauss-Lobatto-Legendre, etc.)

- Can map back and forth between stable nodal bases and Legendre or bubble function modal bases, *with minimal information loss*.

Unstable and Stable Interpolating Basis Sets

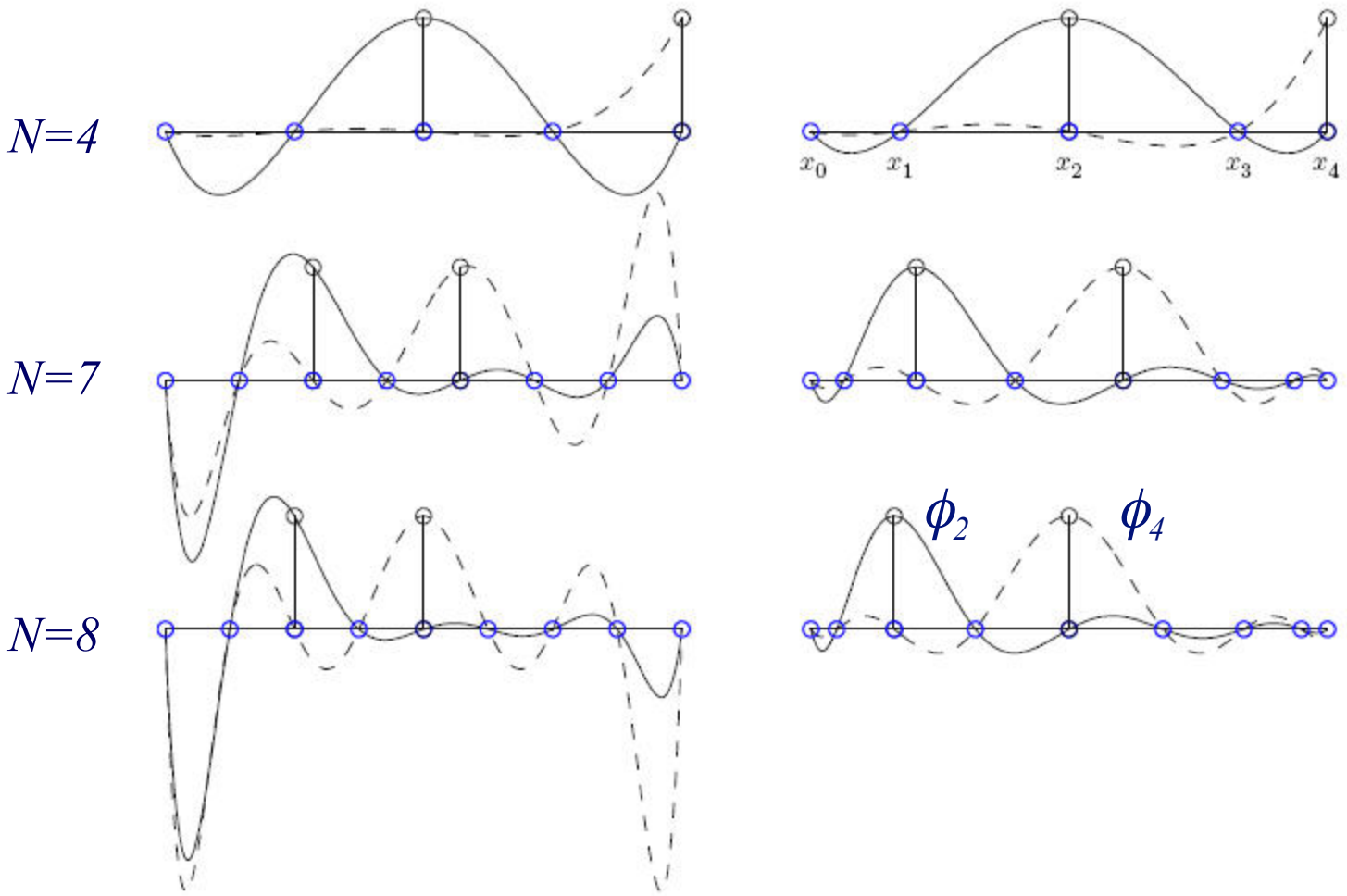
- Key idea for Chebyshev interpolation is to choose points that minimize $\max |q_{n+1}(x)|$ on interval $\mathcal{I} := [-1, 1]$.

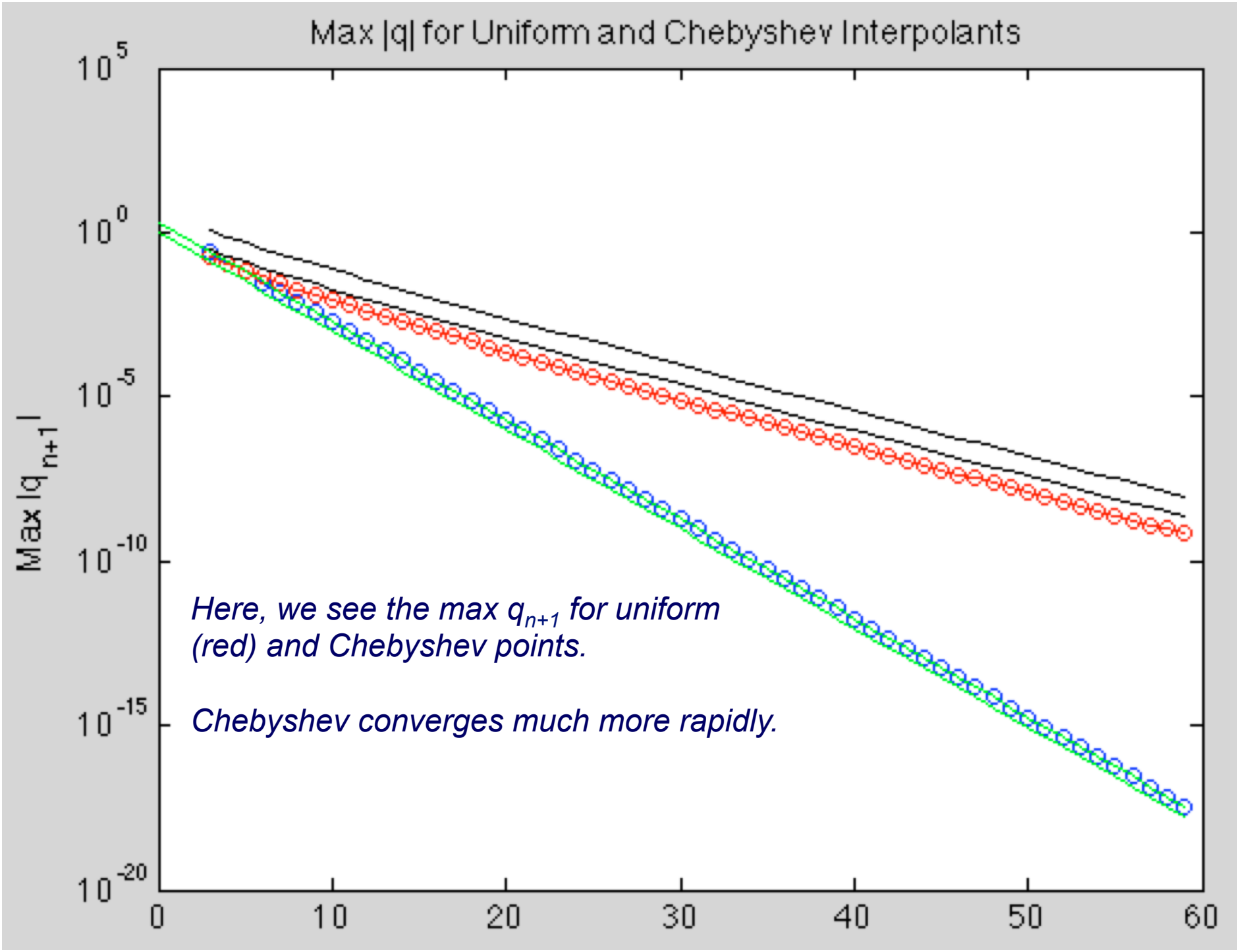
$$\begin{aligned} q_{n+1}(x) &:= (x - x_0)(x - x_1) \ldots (x - x_n) \\ &:= x^n + c_{n-1}x^{n-1} + \ldots + c_0 \end{aligned}$$

  which is a *monic polynomial* of degree $n + 1$.

- The roots of the Chebyshev polynomial $T_{n+1}(x)$ yield such a set of points by clustering near the endpoints.

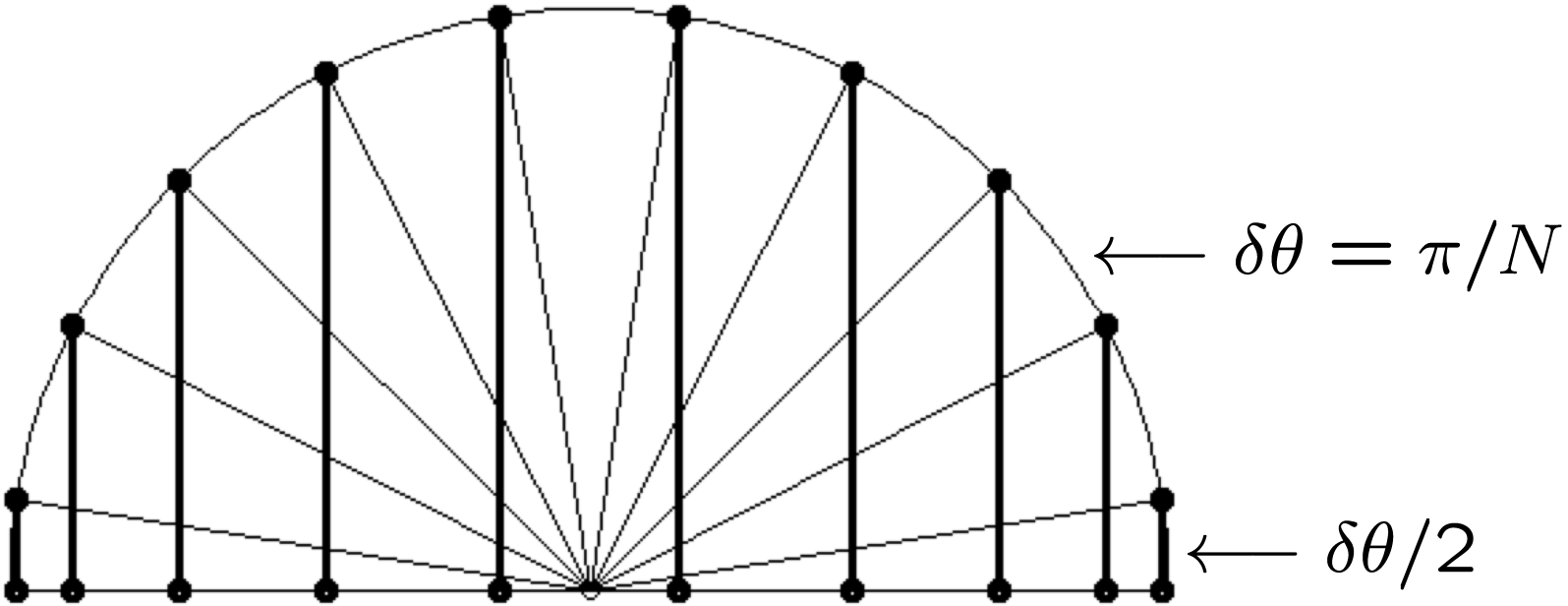
Lagrange Polynomials: Good and Bad Point Distributions




Max |q| for Uniform and Chebyshev Interpolants
Max |q_{n+1}|
10^5
10^0
10^-5
10^-10
10^-15
10^-20
0
10
20
30
40
50
60
Here, we see the max q_{n+1} for uniform (red) and Chebyshev points.
Chebyshev converges much more rapidly.

Nth-order Gauss-Chebyshev Points

- Roots of Nth-order Chebyshev polynomial are projections of equispaced points on the circle, starting with $\theta = \delta\theta/2$, then $\theta = 3\delta\theta/2,\ldots,\pi-\delta\theta/2$.



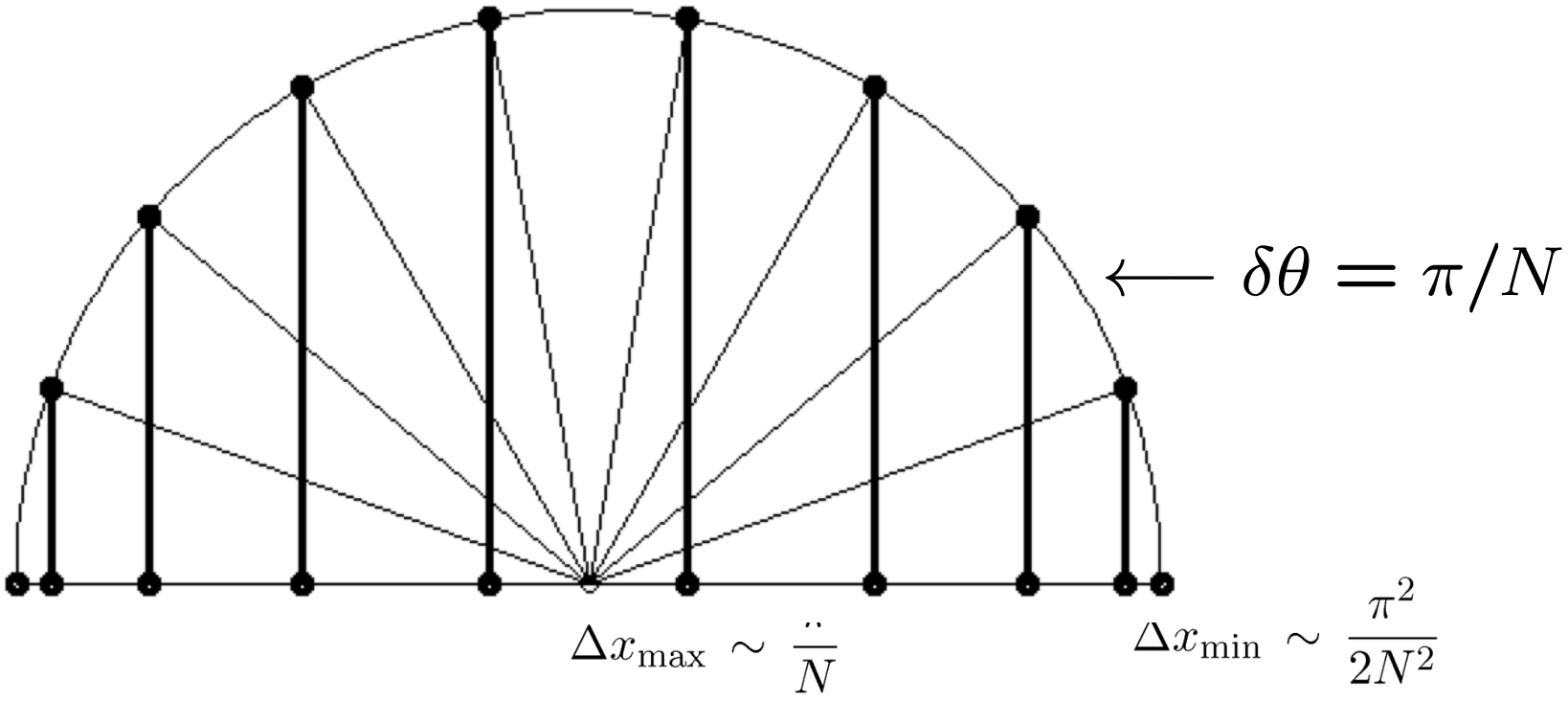

```
ti=(0:N)/(N); ti=pi*ti'; %%% Gauss Lobatto Chebyshev Point generator
ti=(0:N); ti=(ti+.5)/(N+1); ti=pi*ti'; %%% Gauss Chebyshev Point generator
xi=cos(ti); yi=sin(ti);

close all; figure('Color',[1.0 1.0 1.0]);
for i=1:N+1;
    plot([xi(i) xi(i)],[0 yi(i)],'ko-','LineWidth',2); hold on;
    plot([0 xi(i)],[0 yi(i)],'ko-');
end;

N=100;  %% Draw Circle and x-axis:
ti=(0:N)/(N);  ti=pi*ti';  % theta in [0,pi]
xi=cos(ti); yi=sin(ti); plot(xi,0*xi,'k-',xi,yi,'k-');
```

N+1 Gauss-Lobatto Chebyshev Points

- N+1 GLC points are projections of equispaced points on the circle, starting with $\theta = 0$, then $\theta = \pi/N, 2\pi/N, \ldots, k\pi/N, \ldots, \pi$.



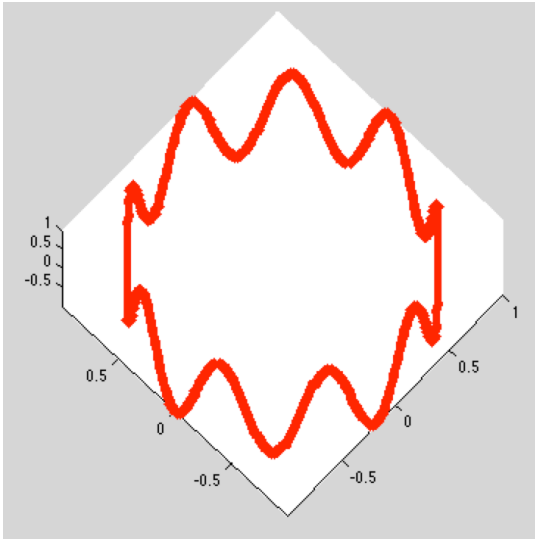
Nth-Order Gauss Chebyshev Points

- Matlab Demo

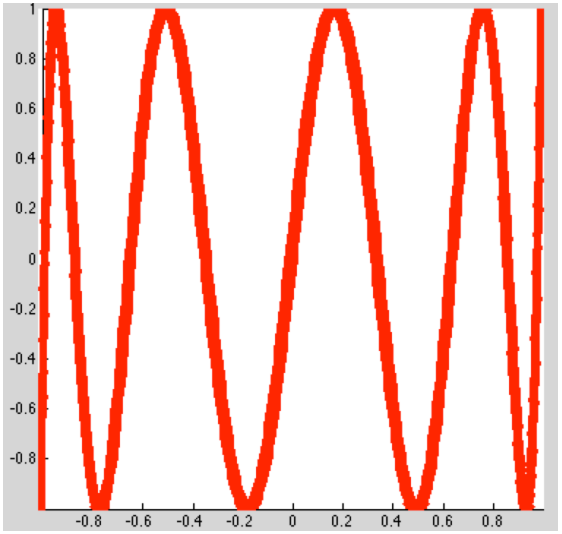
```
t=0:.01:(2*pi); t=t'; x=cos(t); y=sin(t);

n=9; z=cos(n*t);

plot3(x,y,z,'r','LineWidth',5); axis equal
```



$\cos(N\theta)$



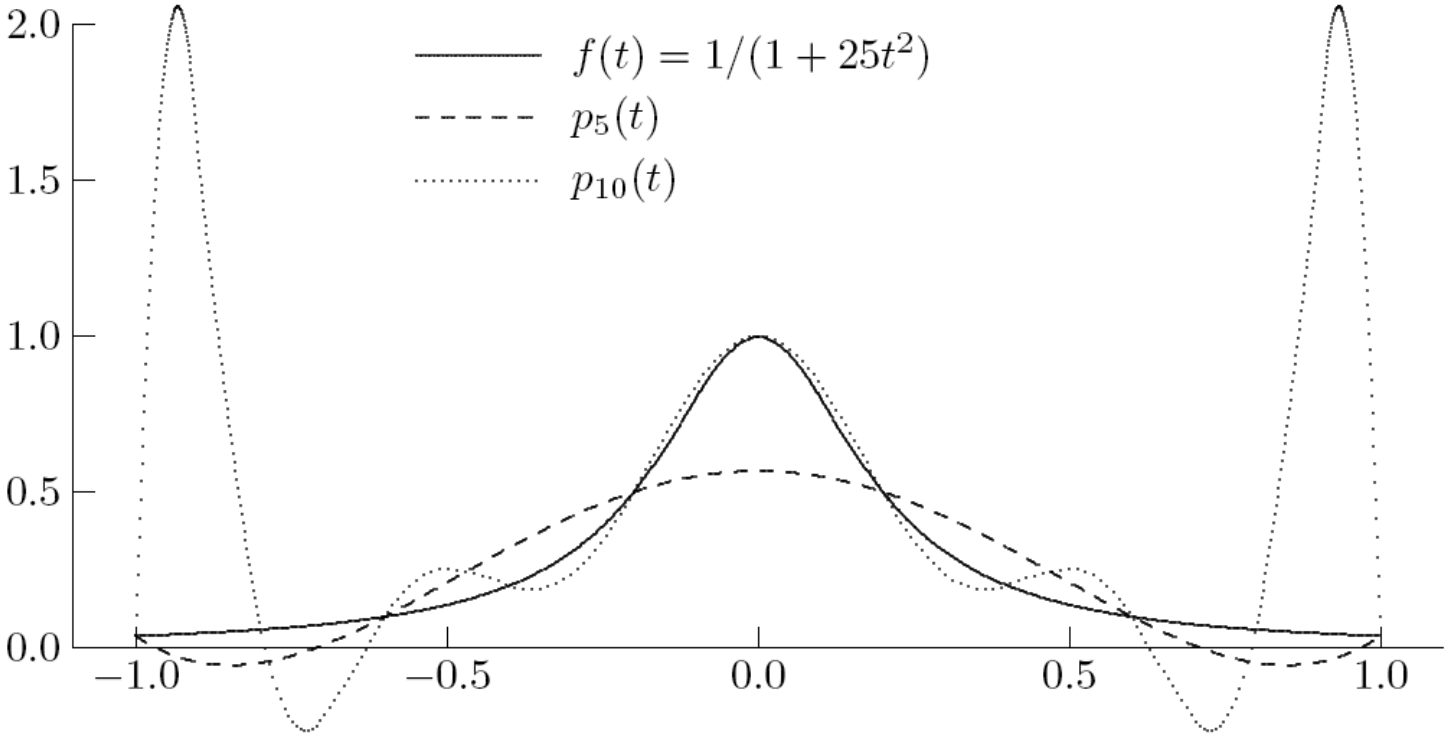
$T_N(x)$

$$T_N(x) = \cos(N\theta)$$

$$x = \cos(\theta)$$

Example: Runge's Function

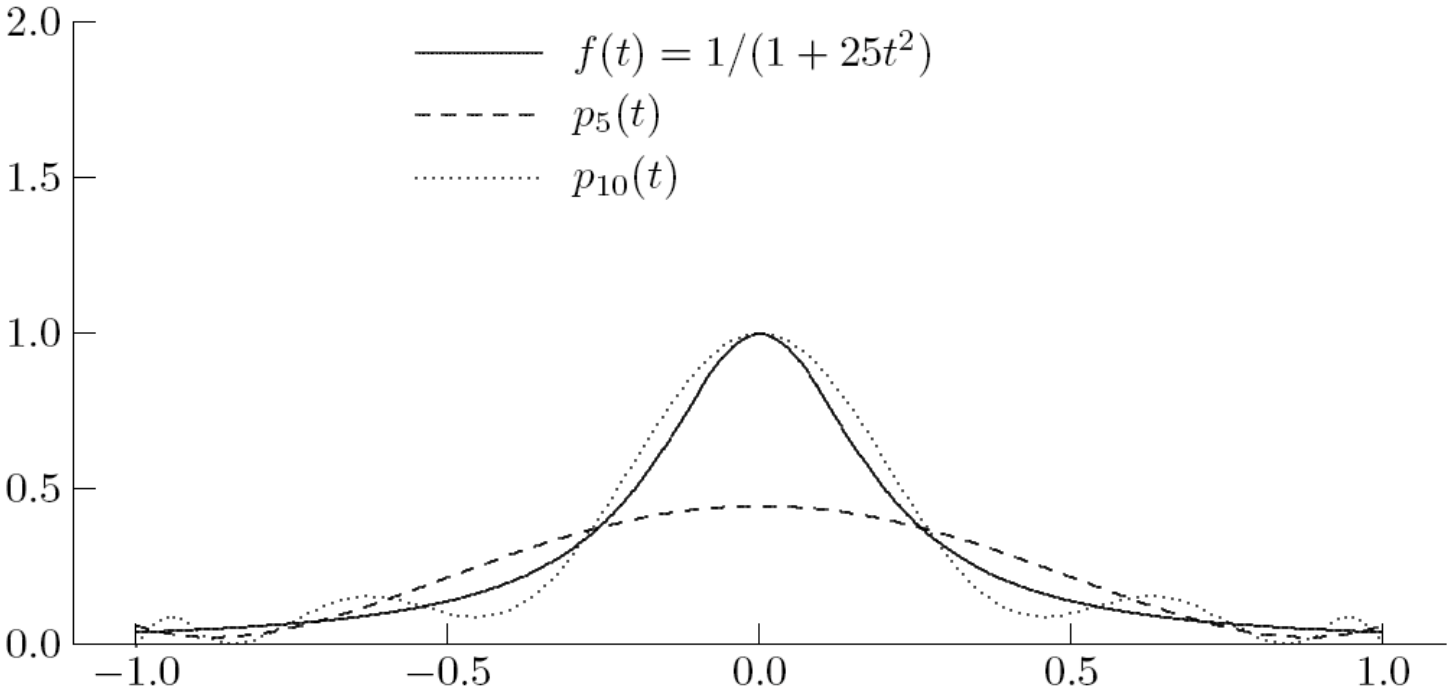
- Polynomial interpolants of Runge's function at *equally spaced* points *do not* converge



< interactive example >

Example: Runge's Function

- Polynomial interpolants of Runge's function at *Chebyshev* points *do* converge



Chebyshev Convergence is exponential for smooth f(t).

Interpolation Testing

- Try a variety of *methods* for a variety of *functions.*
- Inspect by plotting the function and the interpolant.
- Compare with theoretical bounds. (Which *are* accurate!)

Typical Interpolation Experiment

- Given $f(t)$, evaluate $f_j := f(t_j)$, $j = 1, \ldots, n$.
- Construct interpolant:

$$p(t) = \sum_{j=1}^{n} \hat{p}_j \, \phi_j(t).$$

- Evaluate $p(t)$ at $\tilde{t}_i$, $i = 1, \ldots, m$, $m \gg n$. (Fine mesh, for plotting, say.)
- To check error, compare with original function on fine mesh, $\tilde{t}_i$.

$$\begin{aligned} e_i &:= p(\tilde{t}_i) - f(\tilde{t}_i) \\ e_{\max} &:= \frac{\max_i |e_i|}{\max_i |f_i|} \\ &\approx \frac{\max |p - f|}{\max |f|}. \end{aligned}$$

(Remember, it's an *experiment*.)

- Preceding description is for one *trial*.
- Repeat for increasing n and plot $e_{\max}(n)$ on a log-log or semilog plot.
- Compare with other methods *and* with theory:
  - **methods** – identify best method for given function / requirements
  - **theory** – verify that experiment is correctly implemented
- Repeat with a different function.

Summary of Key Theoretical Results

- **Piecewise linear interpolation:**

$$\max_{t\in[a,b]} |\, p \,-\, f \,| \;\;\leq\;\; \frac{h^2}{8}\, M, \qquad \begin{cases} M := \max_{\theta\in[a,b]} \;|f''(\theta)\,| \\ h := \max_{j\in[2,\ldots,n]}(t_j \,-\, t_{j-1}), \;\; t_{j-1} < t_j \end{cases}$$

- **Polynomial interpolation through n points:**

$$\begin{aligned} \max_{t\in[a,b]} |\, p \,-\, f \,| \quad &\leq \quad \frac{q_n(\theta)}{n!}\, M, \\ &\leq \quad \frac{h^n}{4n}\, M, \;\; (\text{for } t \,\in\, [a,b]), \\ \text{with} \quad M &:= \max_{\theta\in[a,b,t]} \;|f^n(\theta)\,|. \end{aligned}$$

  – Here, $q_n(\theta) \;:=\; (\theta - t_1)(\theta - t_2)\cdots(\theta - t_n)$.

  – The first result also holds true for *extrapolation*, i.e., $t \notin [a,b]$.

- **Natural cubic spline ($s''(a) = s''(b) = 0$):**

$$\max_{t\in[a,b]} |\, p - f \,| \;\leq\; C\, h^2\, M, \quad M = \max_{\theta\in[a,b]} |f''(\theta)\,|,$$

  unless $f''(a) = f''(b) = 0$, or other lucky circumstances.

- **Clamped cubic spline ($s'(a) = f'(a)$, $s'(b) = f'(b)$):**

$$\max_{t\in[a,b]} |\, p - f \,| \;\leq\; C\, h^4\, M, \quad M = \max_{\theta\in[a,b]} |f^{iv}(\theta)\,|.$$

- **Nyquist sampling theorem:**

  Roughly: *The maximum frequency that can be resolved with n points is $N = n/2$.*

  There are other conditions, such as limits on the spacing of the sampling.

- **Methods:**
  - piecewise linear
  - polynomial on uniform points
  - polynomial on Chebyshev points
  - natural cubic spline

- **Tests:**
  - e^t
  - $e^{\cos t}$
  - $\sin t$ on $[0, \pi]$
  - $\sin t$ on $[0, \frac{\pi}{2}]$
  - $\sin 15t$ on $[0, 2\pi]$
  - $e^{\cos 11t}$on $[0, 2\pi]$
  - Runge function: $\frac{1}{1+25t^2}$ on $[0, 1]$
  - Runge function: $\frac{1}{1+25t^2}$ on $[-1, 1]$
  - Semi-circle: $\sqrt{1-t^2}$ on $[-1, 1]$
  - Polynomial: t^n
  - Extrapolation
  - Other

interp_test.m *interp_test_runge.m*

- **Methods:**
  - piecewise linear
  - polynomial on uniform points
  - polynomial on Chebyshev points
  - natural cubic spline

- **Tests:**
  - e^t
  - $e^{\cos t}$
  - $\sin t$ on $[0, \pi]$
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  - $\sin 15t$ on $[0, 2\pi]$
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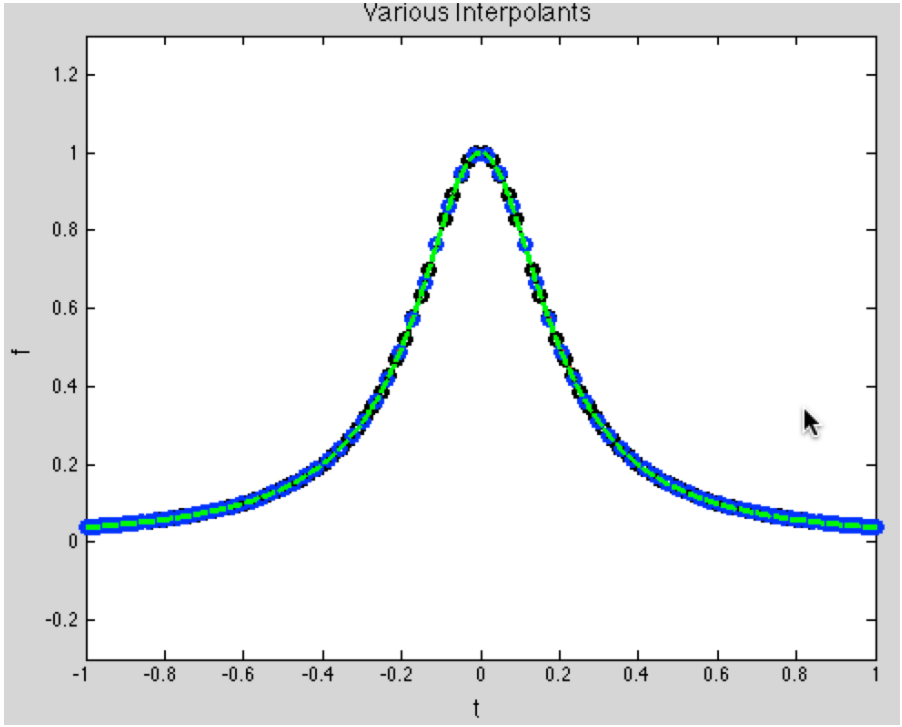
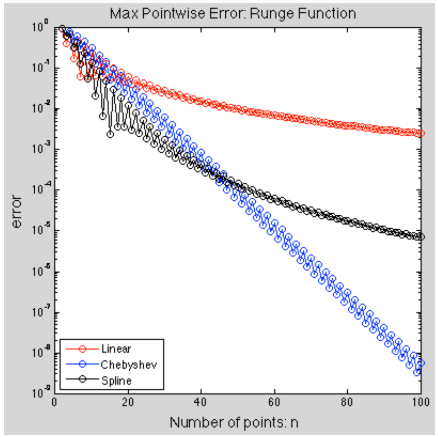
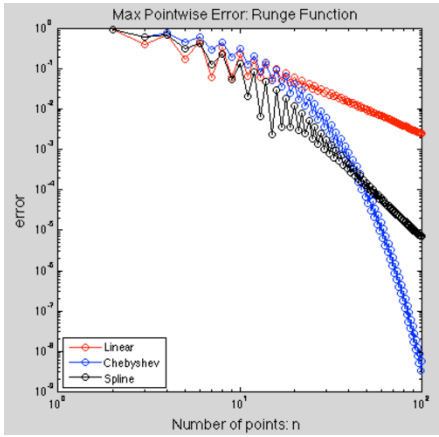
  - Semi-circle: $\sqrt{1-t^2}$ on $[-1, 1]$
  - Polynomial: t^n
  - Extrapolation
  - Other







interp_test.m *interp_test_runge.m*

Piecewise Polynomial Interpolation

- Fitting single polynomial to large number of data points is likely to yield unsatisfactory oscillating behavior in interpolant
- Piecewise polynomials provide alternative to practical and theoretical difficulties with high-degree polynomial interpolation
- Main advantage of piecewise polynomial interpolation is that large number of data points can be fit with low-degree polynomials
- In piecewise interpolation of given data points (t_i, y_i), *different* function is used in each subinterval $[t_i, t_{i+1}]$
- Abscissas t_i are called *knots* or *breakpoints*, at which interpolant changes from one function to another

Piecewise Interpolation, continued

- Simplest example is piecewise linear interpolation, in which successive pairs of data points are connected by straight lines
- Although piecewise interpolation eliminates excessive oscillation and nonconvergence, it appears to sacrifice smoothness of interpolating function
- We have many degrees of freedom in choosing piecewise polynomial interpolant, however, which can be exploited to obtain smooth interpolating function despite its piecewise nature

Piecewise Polynomial Bases: Linear and Quadratic

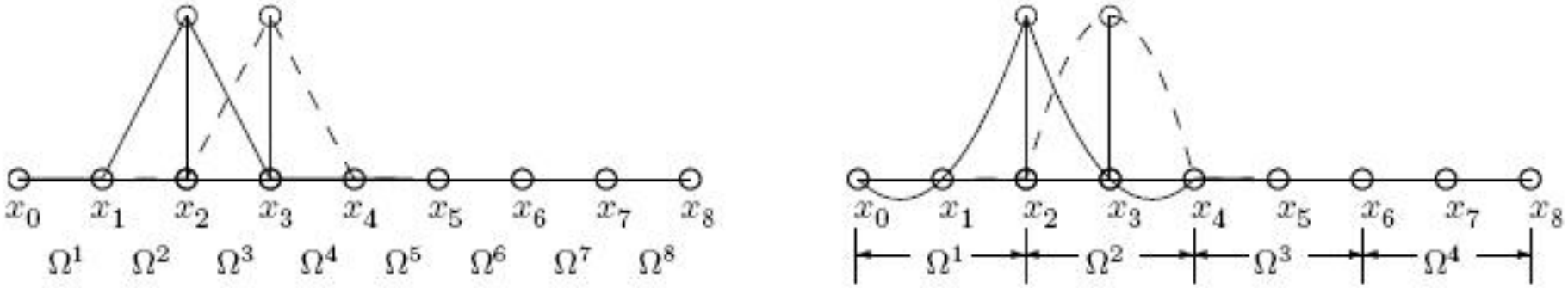


Figure 2: Examples of one-dimensional piecewise linear (left) and piecewise quadratic (right) Lagrangian basis functions, $\phi_2(x)$ and $\phi_3(x)$, with associated element support, Ω^e, $e = 1, \ldots, E$.

Cubic Spline Interpolation

- *Spline* is piecewise polynomial of degree k that is $k-1$ times continuously differentiable
- For example, linear spline is of degree 1 and has 0 continuous derivatives, i.e., it is continuous, but not smooth, and could be described as "broken line"
- *Cubic spline* is piecewise cubic polynomial that is twice continuously differentiable
- As with Hermite cubic, interpolating given data and requiring one continuous derivative imposes $3n-4$ constraints on cubic spline
- Requiring continuous second derivative imposes $n-2$ additional constraints, leaving 2 remaining free parameters

Piecewise cubics:

- Interval $\mathcal{I}_j \;=\; [x_{j-1}, x_j],\; j = 1, \ldots, n$

  $p_j(x) \in \mathbb{P}_3(x)$ on $\mathcal{I}_j$

  $p_j(x) \;=\; a_j \;+\; b_j x \;+\; c_j x^2 \;+\; d_j x^3$

- $4n$ unknowns

  $p_j(x_{j-1}) \;=\; f_{j-1},\; j = 1, \ldots, n$

  $p_j(x_j) \;=\; f_j,\; j = 1, \ldots, n$

  $p_j'(x_j) \;=\; p_{j+1}'(x_j),\; j = 1, \ldots, n-1$

  $p_j''(x_j) \;=\; p_{j+1}''(x_j),\; j = 1, \ldots, n-1$

  } *Spline conditions*

- $4n - 2$ equations

Cubic Splines, continued

Final two parameters can be fixed in various ways

- Specify first derivative at endpoints t_1 and t_n
- Force second derivative to be zero at endpoints, which gives *natural spline*
- Enforce "not-a-knot" condition, which forces two consecutive cubic pieces to be same
- Force first derivatives, as well as second derivatives, to match at endpoints t_1 and t_n (if spline is to be periodic)
- Force first derivatives at endpoints to match y'(x) – *clamped spline.*

Example: Cubic Spline Interpolation

- Determine natural cubic spline interpolating three data points (t_i, y_i), $i = 1, 2, 3$
- Required interpolant is piecewise cubic function defined by separate cubic polynomials in each of two intervals $[t_1, t_2]$ and $[t_2, t_3]$
- Denote these two polynomials by

$$p_1(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3$$

$$p_2(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3$$

- Eight parameters are to be determined, so we need eight equations

Cubic Spline Formulation – 2 Segments

8 Unknowns

$$p_1(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3$$

$$p_2(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3$$

8 Equations

Interpolatory

$$p_1(t_1) = y_1$$

$$p_1(t_2) = y_2$$

$$p_2(t_2) = y_2$$

$$p_2(t_3) = y_3$$

Continuity of Derivatives

$$p_1'(t_2) = p_2'(t_2)$$

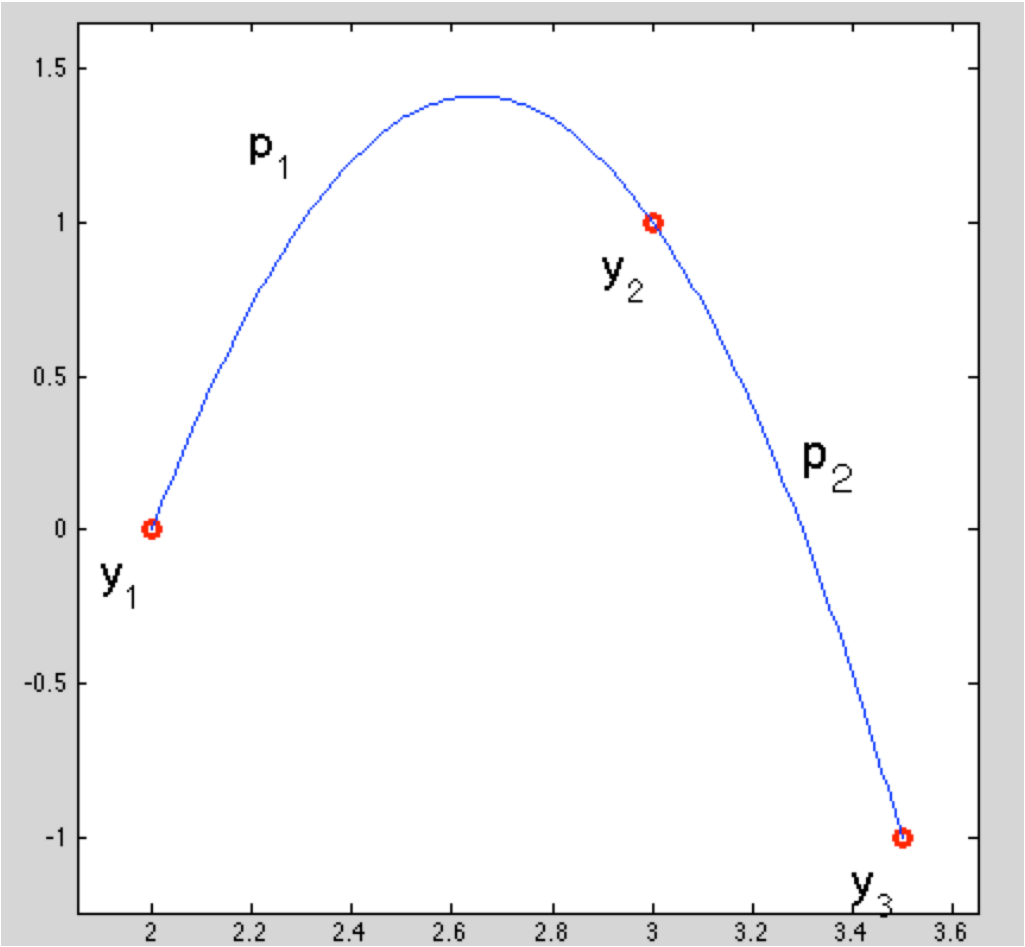
$$p_1''(t_2) = p_2''(t_2)$$

End Conditions

$$p_1''(t_1) = 0$$

$$p_2''(t_3) = 0$$

(*Natural Spline*)



Some Cubic Spline Properties

- Continuous
- 1st derivative: continuous
- 2nd derivative: continuous

- "Natural Spline" minimizes integrated curvature:
  over all twice-differentiable f(x)
  passing through (x_j,f_j), j=1,…,n.

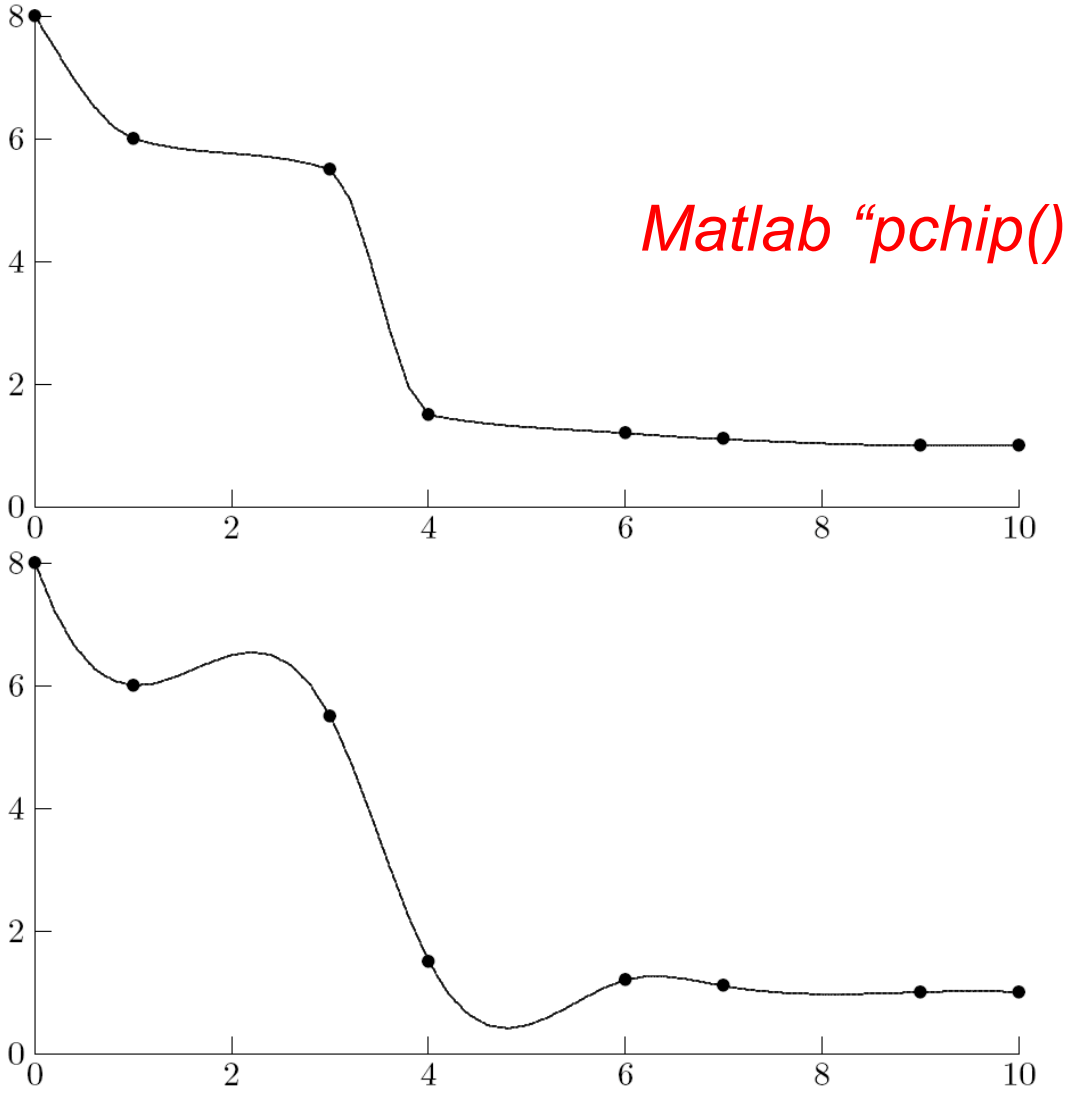
$$\int_{x_1}^{x_n} |S''(x)|^2\,dx \le \int_{x_1}^{x_n} |f''(x)|^2\,dx$$

- Robust / Stable (unlike high-order polynomial interpolation)
- Commonly used in computer graphics, CAD software, etc.
- Usually used in parametric form (DEMO)
- There are other forms, e.g., tension-splines, that are also useful.
- For clamped boundary conditions, convergence is $O(h^4)$
- For small displacements, natural spline is like a ***physical spline.*** ***(DEMO)***

Hermite Cubic vs Spline Interpolation

- Choice between Hermite cubic and spline interpolation depends on data to be fit and on purpose for doing interpolation
- If smoothness is of paramount importance, then spline interpolation may be most appropriate
- But Hermite cubic interpolant may have more pleasing visual appearance and allows flexibility to preserve monotonicity if original data are monotonic
- In any case, it is advisable to plot interpolant and data to help assess how well interpolating function captures behavior of original data

Hermite Cubic vs Spline Interpolation

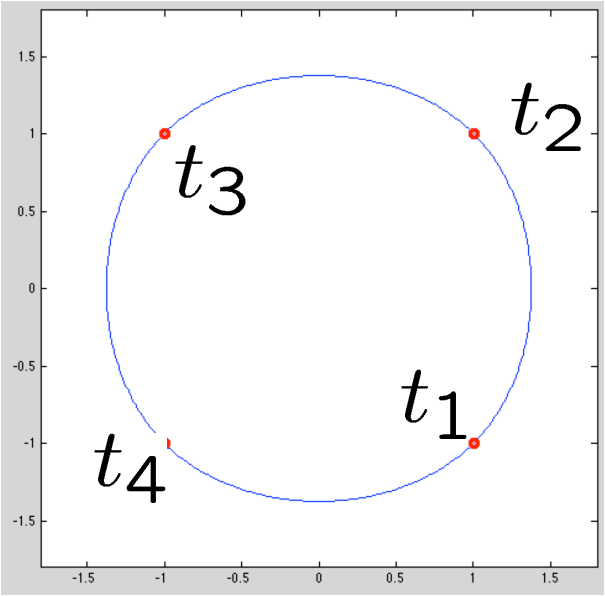


Matlab "pchip()" function

Parametric Interpolation

- Important when y(x) is not a function of x; then, define [x(t), y(t)] such that both are (preferably smooth) functions of t.

- Example 1: a circle.



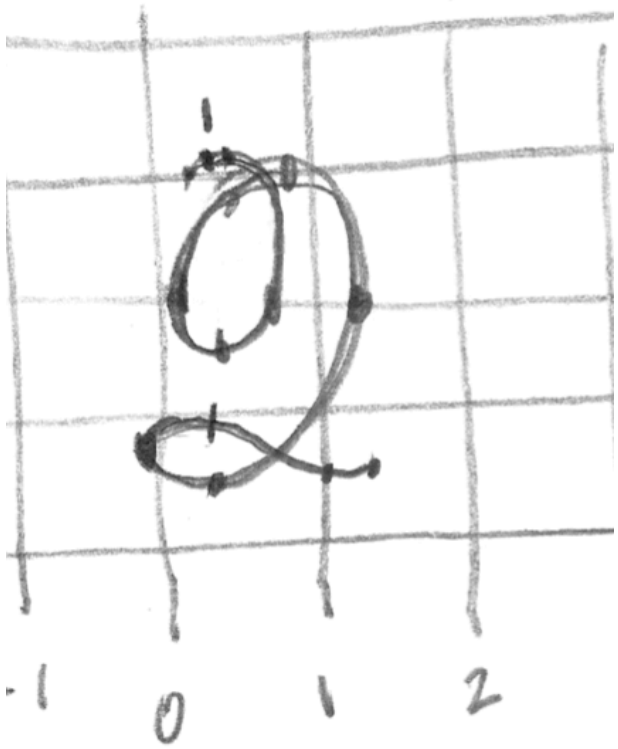

```
%% LAZY WAY TO APPROXIMATE
%% PERIODIC SPLINE

t = -7:8; t=t';

x = [ 1 1 -1 -1 1 1 -1 -1 ]; x=[ x x ];
y = [ -1 1 1 -1 -1 1 1 -1 ]; y=[ y y ];

tt=-2:.01:2;
xx=spline(t,x,tt);
yy=spline(t,y,tt);


hold off;
plot(xx,yy,'b-','LineWidth',1.0); hold on;
plot(x,y,'ro','LineWidth',2.0);
axis equal
axis ([-1.8  1.8  -1.8  1.8 ])
```

Parametric Interpolation: Example 2

- Suppose we want to approximate a cursive letter.
- Use (minimally curvy) splines, parameterized.

Parametric Interpolation: Example 2

i	x_i	y_i
1	.2	3
2	.5	3.2
3	.75	2
	.4	1.6
	.15	2
	.4	2.8
	.9	3
	1.3	2
	.4	.5
	-.15	.8
	.35	.9
	1	.6
	1.4	.6

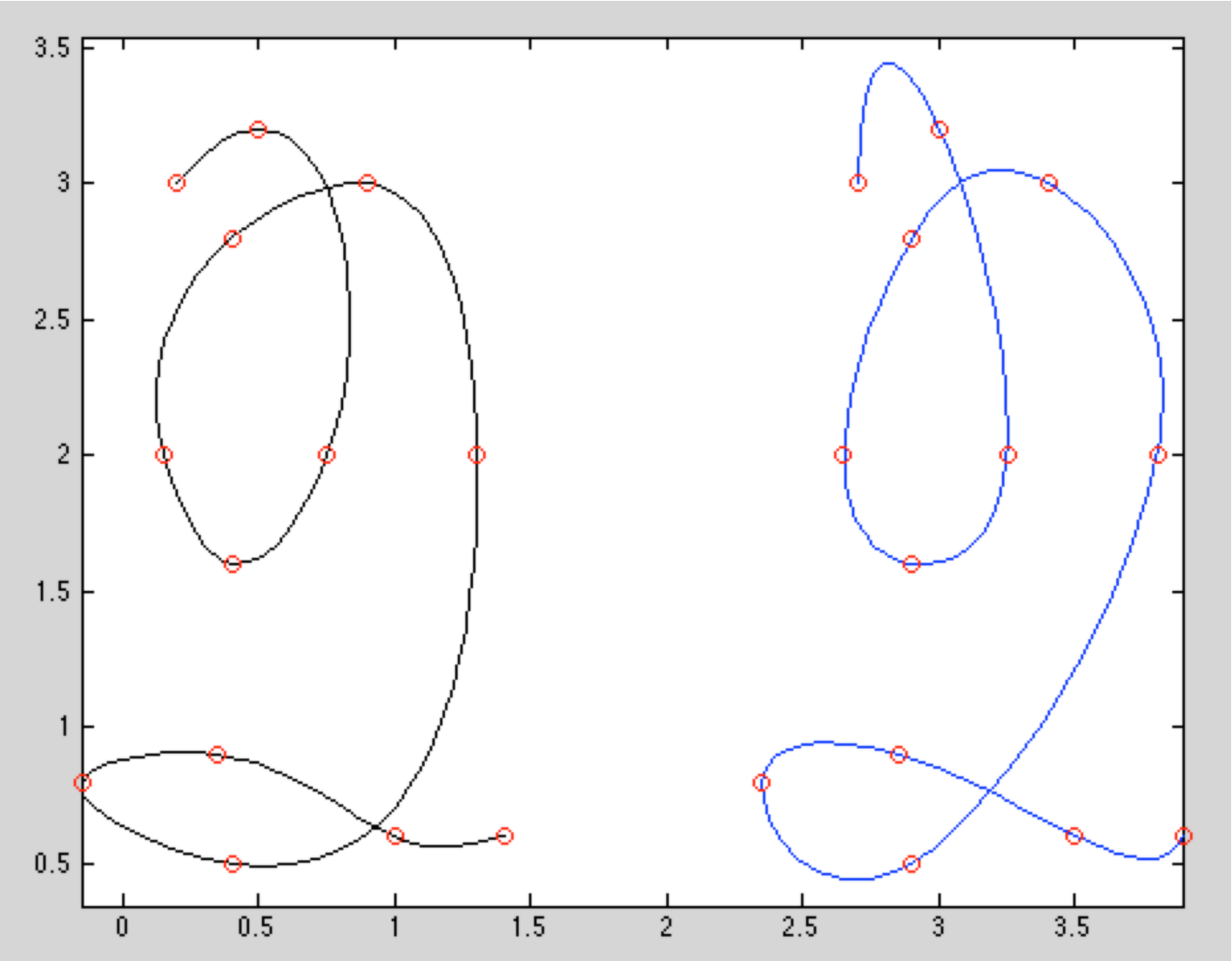


- *Once we have our (x_i , y_i) pairs, we still need to pick t_i .*
- *One possibility: $t_i = i$, but usually it's better to parameterize by arclength, if x and y have the same units.*
- *An approximate arclength is:*

$$s_i = \sum_{j=0}^{i} ds_j, \quad ds_i := ||\mathbf{x}_i - \mathbf{x}_{i-1}||_2$$

- *Note – can also have Lagrange parametric interpolation…*

Parametric Interpolation: Example 2

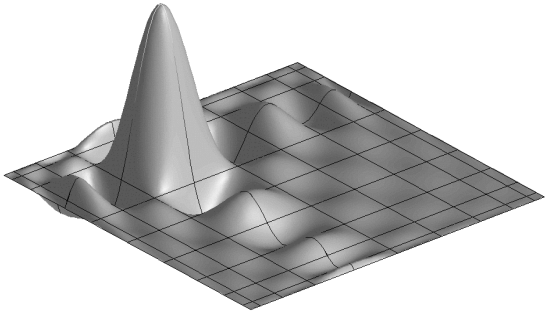


Pseudo-Arclength-Based *Index-Based*

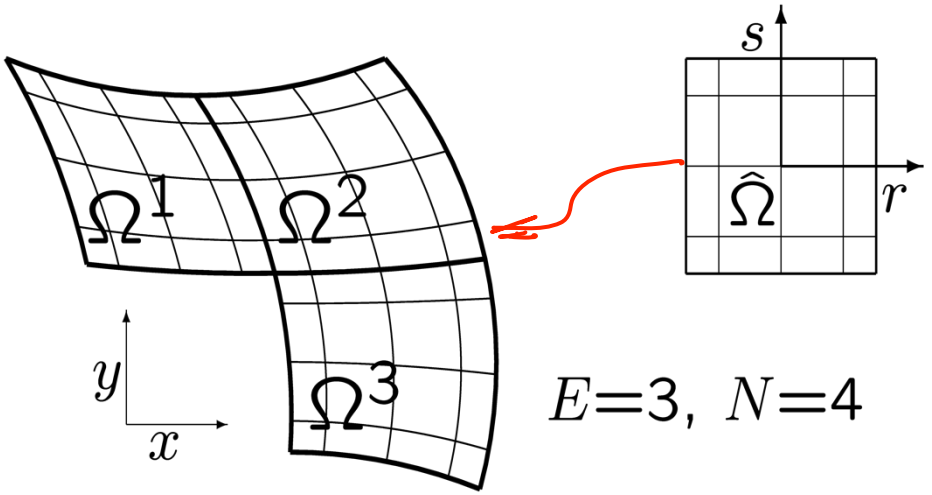
Multidimensional Interpolation

- Multidimensional interpolation has many applications in computer aided design (CAD), partial differential equations, high-parameter data fitting/assimilation.

- Costs considerations can be dramatically different (and of course, higher) than in the 1D case.



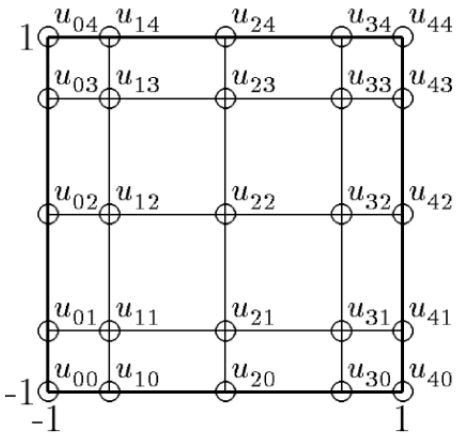
2D basis function, N=10

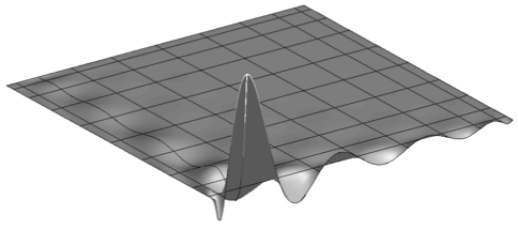


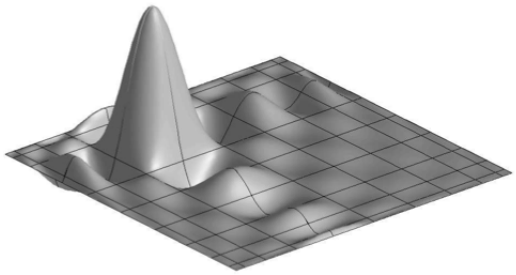
Multidimensional Interpolation

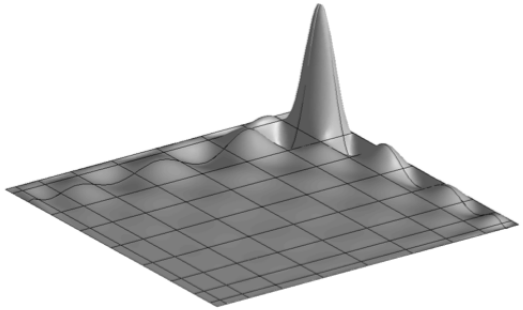
- There are many strategies for interpolating f(x,y) [or f(x,y,z), etc.].
- One easy one is to use tensor products of one-dimensional interpolants, such as bicubic splines or tensor-product Lagrange polynomials.

$$p_n(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} l_i(s)\, l_j(t)\, f_{ij}$$









2D Example: n=2

Consider 1D Interpolation

$$\tilde{f}(s) = \sum_{j=1}^{n} l_j(s)\, f_j$$

$$\tilde{f}(\underline{s}) = \sum_{j=1}^{n} l_j(\underline{s})\, f_j$$

$$\tilde{f}_i = \sum_{j=1}^{n} l_{ij}\, f_j, \qquad l_{ij} := l_j(s_i)$$

$$\underline{\tilde{f}} = L\, \underline{f}$$

- s_i – *fine mesh* (i.e., target gridpoints)
- L is the matrix of Lagrange cardinal polynomials (or, say, spline bases) evaluated at the target points, s_i, $i = 1, \ldots, m$.

Two-Dimensional Case (say, n x n → m x m)

$$\tilde{f}(s,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} l_i(s)\, l_j(t)\, f_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} l_i(s)\, f_{ij}\, l_j(t)$$

$$\tilde{f}_{pq} := \tilde{f}(s_p,t_q) := \sum_{i=1}^{n}\sum_{j=1}^{n} l_{pi}\, f_{ij}\, l_{qj}$$

$$:= \sum_{i=1}^{n}\sum_{j=1}^{n} l_{pi}\, f_{ij}\, l^T_{jq}$$

$$\tilde{F} = LFL^T \qquad \leftarrow \textit{matrix-matrix product, } \textbf{\textit{fast}}$$

- Note that the storage of L is $mn < m^2+n^2$, which is the storage of $\tilde{F}$ and F combined.

Two-Dimensional Case (say, n x n → m x m)

- Note that the storage of L is $mn < m^2 + n^2$, which is the storage of $\tilde{F}$ and F combined.
- That is, in higher space dimensions, the operator cost (L) is less than the data cost ($\tilde{F}$, F).
- This is even more dramatic in 3D, where the relative cost is mn to $m^3 + n^3$.
- Observation: *It is difficult to assess relevant operator costs based on 1D model problems.*

Aside: GLL Points and Legendre Polynomials

The GLL points are the zeros of $(1 - x^2)\, L'_N(x)$.

The Legendre polynomials are orthogonal with respect to the L^2 inner product,

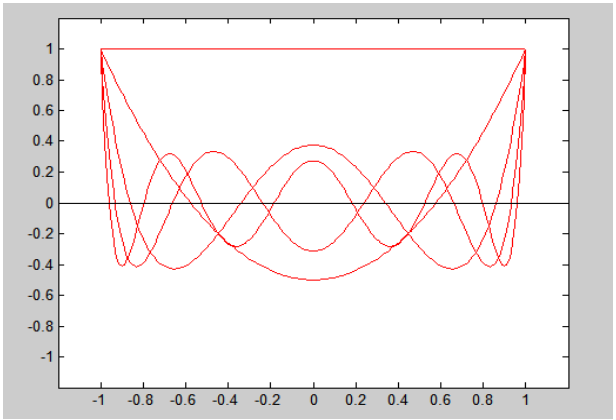
$$\int_{-1}^{1} L_i(x)\, L_j(x)\, dx \;=\; \delta_{ij}, \qquad L_i(x) \in \mathbb{P}_i.$$

They can be efficiently and stably computed using the 3-term recurrence,

$$L_0(x) := 1, \quad L_1(x) = x,$$

$$L_k(x) = \frac{1}{k}\left[(2k-1)\, x\, L_{k-1}(x) \;-\; (k-1)\, L_{k-2}(x)\right].$$

Even Legendre Polynomials, L_0–L_8



Odd Legendre Polynomials, L_1–L_9