Computational and Applied Mathematics III
Mathematical Modelling

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Consultation: Wednesday from 13:15 -14:00, Friday from 08:00-08:45.

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These notes are incomplete. They are meant to be filled in during the lectures. They were developed by:

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1 Introduction to mathematical modelling

Before attempting to define mathematical modelling, let us first try to answer the burning question on our minds, having enrolled for this course, which is:

What exactly is the point of Mathematical Modelling?

A simple answer to that is, mathematical modelling is necessary in order to understand our world, and hopefully to make it better.

Our world is filled with physical processes, governed by physical laws. Some of these processes are man-made, while some are not man-made but natural processes.

Examples of processes that are man-made are industrial processes like mining, fracking, petroleum and gas processing, factory processes, e,g, sugar extraction from sugarcane, glass moulding/ blow moulding/extrusion; environmental processes, e.g., pollution, carbon emission, etc ; spread of diseases (HIV, Corona virus, HPV, STI's); financial processes such as amortisation, annuity, etc.

Examples of natural processes are climatic processes such as hurricanes, flooding, tsunamis, seismic processes such as earthquakes; physiological processes such as blood flow in arteries, tumour growths, cell proliferation and mutation, etc; ecological processes, e.g., wildlife conservation, etc.

What is mathematical modelling and why is it so important?
Mathematical modelling is simply the use of mathematics to understand and explain a real world problem.

What are the basic steps in the mathematical modelling process?

STEP 1: Identification

- *Identify a problem. Industry may give you a problem to solve.*
- *Identify all relevant variables and parameters.*
- *Make clear, simple and reasonable assumptions. Simple models are often easier to understand and solve. For example, the continuum hypothesis - where all variables are treated*

as continuous - can be used.

STEP 2: Model formulation

- *Select a technique to describe the modelling process. For example: dIfferential equations, difference equations, simulations,etc.*

- *Define mathematical relationships between all relevant variables. This is usually the most difficult step. Sometimes conservation laws are used to obtain these relationships.*

STEP 3: Solving

- *Solve equations analytically or numerically. Simulations can also be used.*
- *Plot the solutions.*
- *Find trends.*

STEP 4: Conclusions

- *Discuss the solutions. Do they make sense?*
- *Does our model improve our understanding?*
- *Are the results comparable to experimental data?*

Examples will be done in class.

1.1 Population Models

1.2 Spread of a disease in a community

I ask you to model the spread of a disease in a community. This is clearly very vague. What important questions must be asked? Here are some of the importsnt questions to ask in order to model the process:

- How many people are in the community? say N
- Is the community isolated?
- How many people are currently sick? Say 1 person is sick, so that we have the initial number of susceptible as $N - 1$.
- Is the disease deadly? suppose you answer YES
- Is there a vaccination? suppose you answer NO
- How do people get sick? suppose you answer–via contact/ interaction
- Are there people immuned to the infection? suppose you answer NO, and assume that once infected, you get sick.
- Is the community on lockdown to restrict move and avoid people interaction——NO, people are allowed to interract and there is homogeneous mixing.

Of course, you may have many more questions.

Now we develop the model:

Here we will consider only two distinct population, the healthy (susceptible) and the sick (infective). Call this model the SI epidemic model.

We will model the spread of the disease using Compartmental modelling. In this model, no birth and no death.

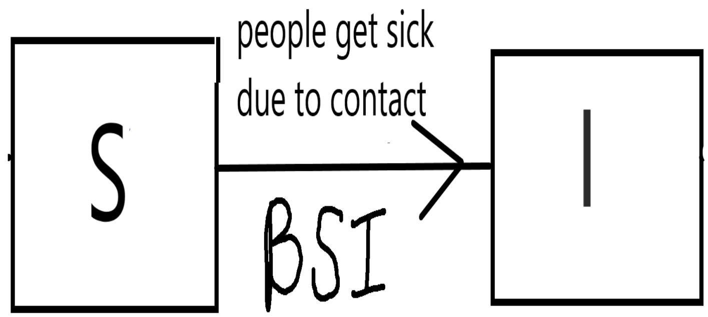


Figure 1: susceptible individuals become infected

We know that

$$I(0) = 1, \quad S(0) = N - 1. \tag{1.1}$$

The community is therefore isolated. The infection spread when healthy people catch the disease from sick people. When a healthy person comes into contact with a sick person, sometimes the healthy person gets sick. Notice that when two healthy people or two sick people come into contact, there will be no new infection since healthy people cannot infect and sick people cannot become sick again.

Let $S(t)$ be the number of healthy people at time t. Let $I(t)$ be the number of sick people at time t. Denoting the total population by N, we see that $N = S + I$. Each person can only move from the susceptibles to the infectives compartment. Thus, as the number of susceptibles decreases, the number of infectives will increase with time.

In the SI model, once a susceptible person becomes infected, he or she will remain infective forever. This applies to epidemics with no immunity, e.g., AIDS.

How fast does a disease spread? This is a question that depends not only on how many sick people there are, but also on how many healthy people there are to get infected. It is easy to think that, no matter how many people are healthy, if there is a large amount of infectives, then there are more people to infect the susceptibles. What is being overlooked here is that, due to the closed population N, as more people become sick, fewer people can become sick. Thus, with a diminishing healthy population, it will be harder for the infectives to find a susceptible whom they can infect. Therefore, the rate of infection is not only proportional to the number of infectives but also to the number of susceptibles. Infact, the rate of infection is proportional to the product of the two groups $S(t)$ and $I(t)$. That is,

$$\text{Infection rate} = \beta(N)S(t)I(t), \tag{1.2}$$

where $\beta(N)$ is the proportionality constant, known as the effective contact rate per susceptible, and depends on the population size.

The effective contact rate, $\beta(N)$, is a product of two separate constants.

Let $c(N)$ be the number of contacts made by a susceptible person per unit time (e.g. per day). Note that these contacts are not necessarily sufficient for transmission of the disease.

Let the probability of a susceptible getting infected for every contact made with an infective be p. Thus, the number of contacts which is sufficient for transmission per susceptible per time is

Then,

$$\beta(N) = cp$$

From the flow diagram, it is clearly seen that the governing equations are given by

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta(N)S(t)I(t)$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta(N)S(t)I(t)$$

Let $\beta(N) = \beta N, \quad \beta \in \mathbb{R}$, then

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta SI$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI$$

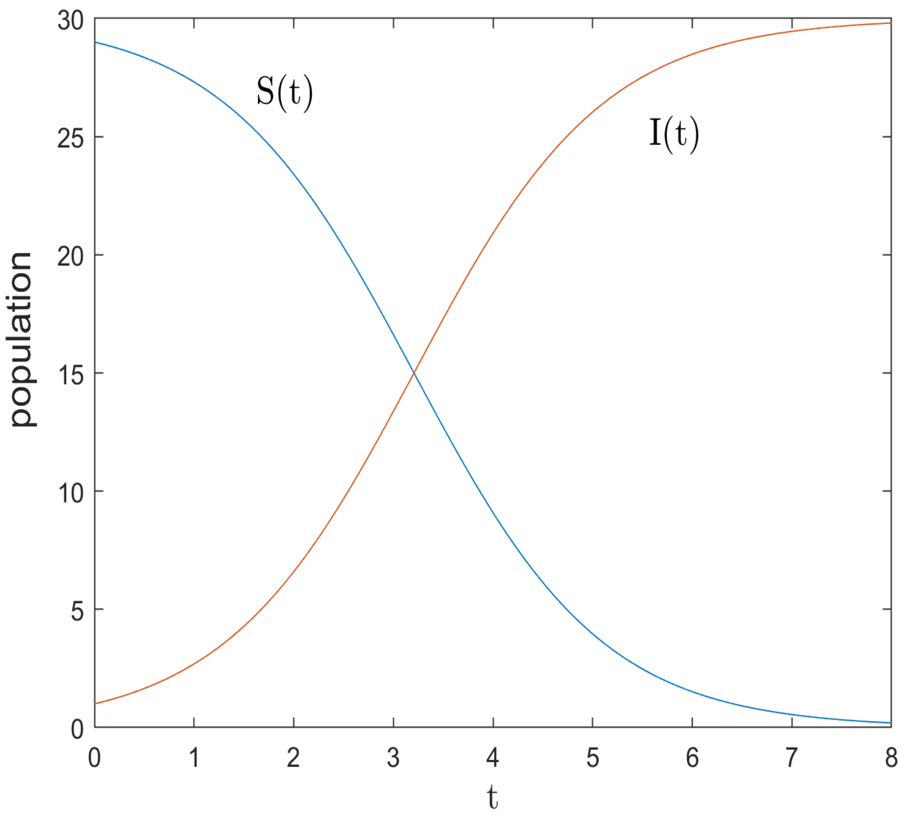


Figure 2: SI model with $\beta = 0.035$

If you look where each curves starts at time zero, we see there are 29 susceptibles and one infective initially. This is because the spread of a disease can not start unless at least one person is spreading it. Figure 2 also tell us when people are becoming sick the fastest: this happens when there is an equal amount of sick and healthy people. On the graph, the maximum rate of infection is where the slope is the steepest, at the inflection point, at t = 4.

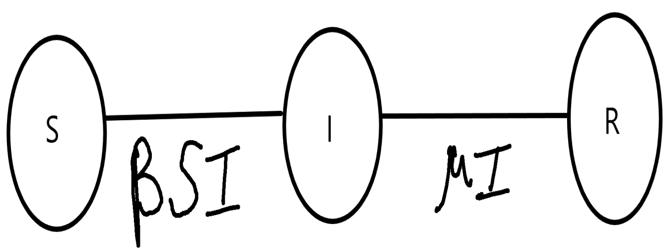
Another thing we can see from the graph of the SI model is that the infectives curve asymptotically approaches the total population, N, from below. Similarly, the susceptibles curve decays to zero. Notice that, in this model, the epidemic never ends.

A more realistic model is the SIR model also called the Kermack-Mckendrick SIR model. This model accounts for three distinct populations: the healthy (Susceptibles), the sick (Infectives), and the recovered (Removeds). In 1927, Kermack and McKendrick proposed a system of differential equations for the SIR model.

Model subdivides total population at time t, denoted $N(t)$ into susceptible $S(t)$, infected $I(t)$ and recovered $R(t)$ compartments.

Thus

$$N(t) = S(t) + I(t) + R(t)$$



$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S(t)I(t)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta S(t)I(t) - \mu I(t)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \mu I(t)$$

β is the average number of contacts an infective makes per unit time. $1/\mu$ is the mean duration of infectivity.

In the SIR model, recovereds can no longer infect susceptibles nor can they become infected themselves. The initial conditions are the same as in the SI model, with zero recovereds at the beginning. The total population is the sum of the three distinct populations: N = S + I + R. Here the infection rate is still proportional to t he product of the susceptible and infective populations, but the removal rate is proportional to the infective population.

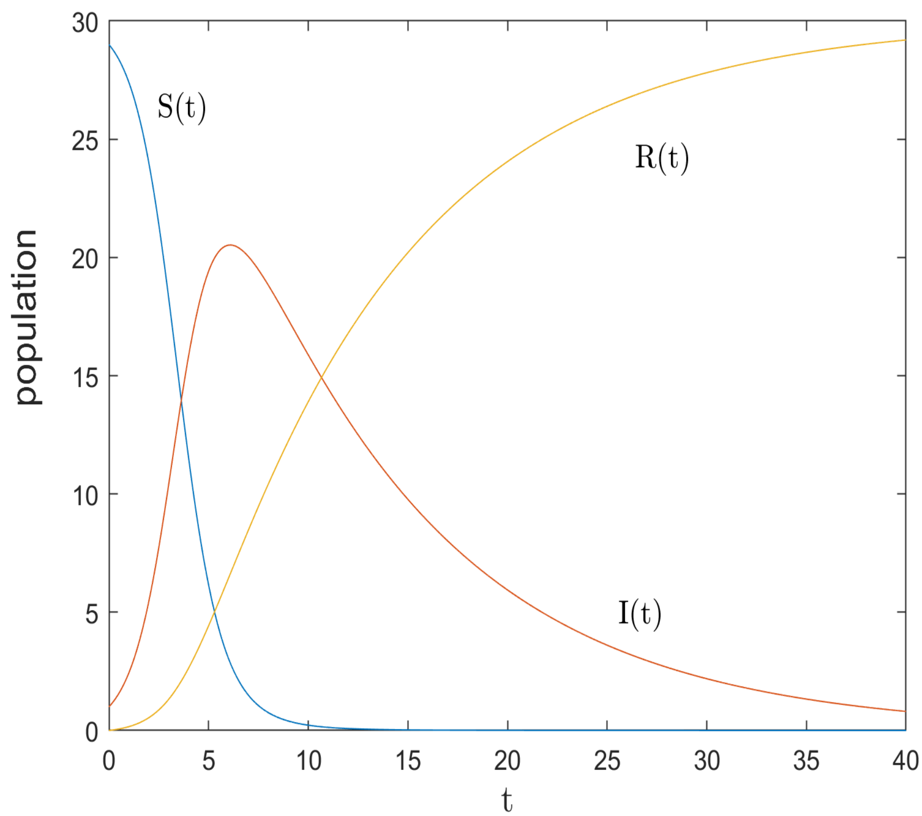


Figure 3: SIR model with $\beta = 0.035$ and $\mu = 0.1$

In Figure 3, the graph of $S(t)$, $I(t)$ and $R(t)$ are shown. Initially, there are 29 susceptibles, one infective and zero recovereds. And as time evolves, we see how the number of susceptibles goes to zero while the number of infectives rises up to a peak and then begins to drop.

Summary of results from SIR model:

- The SIR model explains the rapid rise and fall in the number of infected patients observed during an epidemic of a pandemic

- Epidemics (or pandemic) comes and go without affecting every member of the community. Thus, $S_0 - S_\infty > 0$

Basic reproductive number

Suppose that at time $t \leq 0$, all individuals were susceptible. That is, $S(0) = N$.

Hence, at $t = 0$, one infected individual will infect $\beta S(0) = \beta N$ susceptible individuals per unit time.

Since an infected individual remains infectious for an average period of $1/\mu$, then

$$\mathfrak{R}_0 = \beta S(0) \frac{1}{\mu}.$$

Note that β measures the effective contact rate, and $1/\mu$ is how long people remain infectious.

The values of β and $1/\mu$ can be obtained by fitting a given data of infected people to the mathematical model.

Note that

- $\mathrm{d}S/\mathrm{d}t < 0$ for all t
- $\mathrm{d}I/\mathrm{d}t > 0$ if and only if $S(t) > \mu/\beta$, for which $\mathfrak{R}_0 > 1$
- That is, $I(t)$ increases so long as $S(t) > \mu/\beta$. But since $S(t)$ decreases for all time, $I(t)$ ultimately decreases and approaches zero.
- If $S(0) < \mu/\beta$ (for which $\mathfrak{R}_0 < 1$), then the population $I(t)$ decreases to zero (no epidemic).
- If $S(0) > \mu/\beta$ (for which $\mathfrak{R}_0 > 1$), then the population $I(t)$ first increases to a maximum (attained when $S(t) = \mu/\beta$), and thereafter, decreases to zero (outbreak of epidemic).

Using rigorous mathematical analysis, it is found that

$$\frac{\beta}{\mu} = \frac{\log\left(S(0)/S_\infty\right)}{K - S_\infty}$$

where

$$0 < S_\infty < K$$

and K is the total population at time $t = 0$.

meaning that a part of the population escapes infection.

Conclusion: Thus, $\mathfrak{R}_0$ can be estimated from the above relation.

In any equation, every term must have the same units. In other words, we cannot add something measured in metres with another quantity measured in seconds, for example.

In this example,

- $[P]$ = people
- $[I]$ = people
- $[t]$ = time

Now, find the units of α, β, ρ and k.

Exercise 1

1. In the SIR model, solve the equations for I and S subject to $I(0) = 1$, $S(0) = N - 1$. Provide plots in Mathematica. Experiment with different parameter values for α, β, ρ, and N.
NB: I have used the continuum approximation. Do your foresee any issues with this?

2. Suppose a vaccination is now available. People in S get vaccinated at a rate of νS. Rewrite the governing equations for this scenario and plot your results in Mathematica. Finally, compare the two models.

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2 Buckingham-Pi

2.1 Dimensional analysis

Dimensional analysis is the analysis of a relationship between physical quantities by considering their units of measure. For example, it will be meaningless to construct an equation in which the dimension of the terms in the equation is of the form

$$L + M = T, \tag{2.1}$$

where L is measured in meters, M in kilograms and T in seconds. Simply put, if an equation models a physical process, then, all the terms in it that are separated by +, - or = must have the same physical dimension. If they did not, we would be saying something ridiculous like:

apples + eggs = light bulbs + whisky,

Even though, it is only reasonable to compare apples with apples.

This is, perhaps, the most basic of the many consistency checks that youshould build into your mathematics.

An equation like (2.1) will be called dimensionally inconsistent, or dimensionally non-homogeneous. Therefore, an equation that has the correct and same dimension termwise, will be called dimensionally homogeneous (or dimensionally consistent).

Dimensional analysis is a method for reducing the number (and complexity) of measureable quantities which affect a given physical process by using a compacting technique. That is, if a phenomena depends upon n dimensional variables, dimensional analysis will reduce the problem to k dimensionless variables, where $k < n$.

The standard notation for the dimensions of a quantity is a square bracket [.] around the quantity. All the scientific units that we will be using can be written in terms of the 7 primary dimensions or fundamental dimensions which are

- Mass m: $[m] = M$ measured in kilograms
- Time t: $[t] = T$ measured in seconds
- Length l: $[l] = L$ measured in meters

- Electric current i: $[i] = I$ measured in ampere
- Temperature θ: $[\theta] = \Theta$ measured in kelvin (or $°C$)
- amount of substance with dimension [mol], measured in mole
- luminoscity with dimension [cd], measured in candela

All well-posed models must have consistent units.

2.2 Examples

2.2.1 Newton's second law of motion: $F = ma$

Check that the dimensions of Newton's 2nd law are consistent:

$$[F] = Newtons = \frac{ML}{T^2}$$
$$[m] = M$$
$$[a] = \frac{L}{T^2}$$

$$[ma] = [F] = \frac{ML}{T^2}$$

2.2.2 The Logistic model

Let the dimension for population numbers be such that $[P] = N$, where N means an amount or quantity . Consider the Logistic equation:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{2.2}$$

where r is the per capita growth rate of the population measured in s^{-1}, K is the carrying capacity and P is the population size.

LHS:

$$\left[\frac{\mathrm{d}P}{\mathrm{d}t}\right] = \frac{N}{T}$$

RHS: Since $\left[\frac{P}{K}\right] = 1$

$$\left[rP\left(1 - \frac{P}{K}\right)\right] = [rP] = N[r] = \frac{N}{T}$$

2.2.3 Integrals and derivatives

Integrals and derivatives are defined by limits. For example,

$$\left[\frac{dP}{dt}\right] = \left[\lim_{\delta t\to 0}\frac{\delta P}{\delta t}\right] = \lim_{\delta t\to 0}\left[\frac{\delta P}{\delta t}\right] = \lim_{\delta t\to 0}\frac{N}{T} = \frac{N}{T}. \tag{2.3}$$

Suppose that $v(t)$ is a velocity. Then $[v] = \frac{L}{T}$. What are the units of

$$\int_a^b v(t)dt? \tag{2.4}$$

$$\left[\int_a^b v(t)dt\right] = \left[\lim_{n\to\infty}\sum_{i=1}^{n} v(t_i)\Delta t_i\right] = \frac{L}{T}\times T = L$$

We can find the units of certain parameters in equations by matching the dimensions. Consider the linear partial differential equation governing the diffusion of a substance, for which $u(x,t)$, is the concentration of the substance in units of amount of a substance per unit volume .

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}. \tag{2.5}$$

Given that $[u] = N/L^3$, $[t] = T$ and $[x] = L$, what are the dimensions of α?

LHS:

$$\left[\frac{\mathrm{d}u}{\mathrm{d}t}\right] = \frac{N}{L^3} * \frac{1}{T} = \frac{N}{L^3T}$$

RHS:

$$\left[\alpha^2\frac{\partial^2 u}{\partial x^2}\right] = [\alpha^2]\frac{N}{L^3} * \frac{1}{L^2} = [\alpha^2]\frac{N}{L^5}.$$

Comparing both sides gives

$$\frac{N}{L^3T} = [\alpha^2]\frac{N}{L^5}$$

for which

$$[\alpha^2] = \frac{L^2}{T}$$

and

$$[\alpha] = \frac{L}{\sqrt{T}}$$

What are the units of β in the following expression:

1. $\exp[\beta t]$
2. $\sin(\beta t)$
3. $\cos(\beta t)$

solution

The argument in each of the three function must be dimensionless. In particular, the cos and sin function are defined in terms of angle θ which is dimensionless since $\theta = \frac{s}{r}$ where s is some arc length and r , a radius.

Also, these three functions are written as sums of different powers by taking Taylors expansion about the origin, e.g.,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

A dimensional argument will mean comparing length with area with volume, etc.

Thus,

$$[\beta t] = [\beta]T = 1$$

$$[\beta] = \frac{1}{T}$$

2.3 The Principle of Dimensional Homogeneity

If an equation truly expresses a proper relationship between dimensional variables in a physical process, it will be dimensionally homogeneous. This means that each of its additive terms will have same dimensions.

2.4 Buckingham-Pi theorem

Let $W_1, W_2, W_3, \ldots, W_n$ be n dimensional variables that are physically relevant in a given problem and that are inter-related by a dimensionally homogeneous equation. This dimensionally homogeneous equation has the form

$$F(W_1, W_2, W_3, \ldots, W_n) = 0, \quad \text{or equivalently} \quad W_1 = f(W_2, \ldots, W_n). \tag{2.6}$$

If k is the number of fundamental dimensions rquired to describe the n variables, then there will be $j = (n - k)$ dimensionless and independent quantities or **Pi groups** $\Pi_1, \Pi_2, ..., \Pi_{n-k}$. The functional relationship can thus be reduced to the much more compact form

$$H(\Pi_1, \Pi_2, ..., \Pi_n) = 0, \tag{2.7}$$

or equivalently,

$$\Pi_1 = h(\Pi_2, ..., \Pi_n). \tag{2.8}$$

Note that the set of dimensionless quantities $\Pi_1, \Pi_2, ..., \Pi_{n-k}$ is not unique. They are, however, independent and form a complete set.

It is important to differentiate between the following:

Dimensional variables: These are quantities which actually vary during a given experiment, and thee quantities are often plotted against each other to show the data.

Dimensional constants: These are quantities that may vary between experiments, biut they are often held constant during a particular experiment. Examples are density ρ, gravitational acceleration g, pressure P, viscosity μ, etc.

Pure constants: These quantities have no dimensions, and arise from mathematical manipulations. For example, π and e are pure constants.

Note that dimensionally homogeneous means that the units balance on either side of the equation.

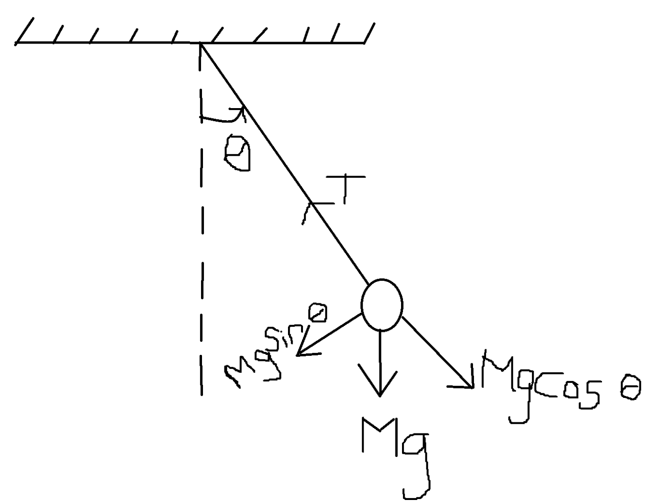
When building a model, initial understanding can be obtained from dimensional analysis and Buckingham-Pi theory.

2.4.1 The simple pendulum

Dimensional analysis is a powerful method for analysing problems without much knowledge of:

- The underlying physics
- Differential equations

We will now illustrate this by considering the period of a pendulum.



Measurable quantities (variables and parameters):

- M - mass of the pendulum.
- l - length of the pendulum

- $x(t)$ - arc length from the vertical or equilibrium configuration
- $\theta(t)$ - angle of deflection from the vertical in radians

The problem is to determine the period of the pendulum.

Method 1: Differential equations

We formulate and solve the ordinary differential equation (ODE) that models the dynamics.

By Newton's second law of motion

$$M\frac{d^2x}{dt^2} = -Mg\sin\theta(t). \tag{2.9}$$

But

$$x(t) = l\theta(t), \tag{2.10}$$

which gives

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{2.11}$$

Now consider small amplitude oscillations so that θ is small. Then

$$\sin\theta = \theta + O(\theta^3). \tag{2.12}$$

Neglecting terms of order θ^3 leads to

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{2.13}$$

The general solution is of the form

$$\theta(t) = A\cos(\omega t + \alpha), \tag{2.14}$$

where

$$\omega = \left(\frac{g}{l}\right)^{1/2}, \tag{2.15}$$

and A and α are constants. Note that the solution does not depend on the mass of the pendulum M.

Exercise 1

Suppose that you are given the initial conditions

$$\theta(0) = \theta_0, \qquad \frac{d\theta}{dt}(0) = 0. \tag{2.16}$$

Find A and α in (2.14).

Now, let T_0 be the period of the simple pendulum. This means that

$$\theta(t) = \theta(t + T_0). \tag{2.17}$$

That is,

$$A\cos(\omega t + \alpha) = A\cos(\omega t + \alpha + \omega T_0)$$

Thus,

$$\omega T_0 = 2\pi$$

Thus

$$T_0 = 2\pi/\omega = 2\pi \left(\frac{l}{g}\right)^{1/2}, \tag{2.18}$$

where T_0 is the period of oscillation of the pendulum, when terms of order θ^3 are neglected.

Method 2: Dimensional analysis

In this method, an alternative derivation of the period T_0 is provided.

Physical quantities in which T_0 could depend:

- l - length of the pendulum
- g - acceleration due to gravity
- m - mass of the pendulum
- θ - angle of deflection

Assume that there exists a function f such that

$$T_0 = f(l, g, m, \theta). \tag{2.19}$$

Denote the dimensions of a physical quantity z by square brackets $[z]$. So

$$[z] = dimensions\ of\ z. \tag{2.20}$$

Fundamental dimensions:

- M - Mass

- L - length
- T - time

M, L, T are the fundamental dimensions of quantities in mechanics. In this problem we will only need the mechanical fundamental dimensions. There are other fundamental dimensions such as Temperature etc.

We will now express the quantities in terms of the fundamental dimensions. The variable θ is dimensionless.

- $[T_0] = T$
- $[l] = L$
- $[g] = L/T^2$
- $[m] = M$
- $[\theta] = M^0 L^0 T^0 = 1$

We consider a product of powers of l, g, m, θ such that we can form dimensionless products. Therefore we write

$$[T_0^a \theta^b l^c m^d g^e] = M^0 L^0 T^0. \tag{2.21}$$

We want to find values for a-e so that this product is dimensionless. Substituting in the dimensions leads to

$$T^a (M^0 L^0 T^0)^b L^c M^d (LT^{-2})^e = M^0 L^0 T^0. \tag{2.22}$$

Equating exponents gives

$$d = 0, \quad c + e = 0, \quad a - 2e = 0. \tag{2.23}$$

The solution to this system is

$$a = 2e, \quad c = -e, \quad d = 0, \tag{2.24}$$

with b arbitrary. There are infinitely many solutions which can be represented by

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = e \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.25}$$

A basis is given by the two vectors $(2\ 0\ \ -1\ 0\ 1)^T$ and $(0\ 1\ 0\ 0\ 0)^T$. One product is thus (by choosing the basis elements)

$$\Pi_1 = T_0^2 l^{-1} g, \tag{2.26}$$

and the other is

$$\Pi_2 = \theta. \tag{2.27}$$

These Π_i form a complete set of dimensionless products as any other dimensionless product can be formed by taking products $\Pi_1^e \Pi_2^b$ for any b and e values.

The Buckingham-Pi theorem gives

$$H(T_0^2 l^{-1} g, \theta) = 0, \tag{2.28}$$

which implies that

$$T_0 = \sqrt{\frac{l}{g}} h(\theta), \tag{2.29}$$

where h is an arbitrary function of θ. One requires more knowledge to find h. We can either conduct experiments to determine h or use a differential equation. From using the differential equation, we see that h is a constant. We can also show that h is a constant by using the following argument: Since there is no damping, $h(\theta)$ is an even function of θ. Expanding in a Taylor series gives

$$h(\theta) = h(0) + \theta h'(0) + \frac{\theta^2}{2} h''(0) + O(\theta^3). \tag{2.30}$$

Now, since h is an even function of θ, $h'(0) = 0$ and $h'''(0) = 0$ and so on. Therefore,

$$h(\theta) = h(0) + \frac{\theta^2}{2} h''(0) + O(\theta^4). \tag{2.31}$$

Thus, correct to first order in θ

$$T_0 = h(0)\sqrt{\frac{l}{g}}. \tag{2.32}$$

Comparison:

Differential equation: $T_0 = 2\pi\sqrt{\dfrac{l}{g}}$

Dimensional analysis: $T_0 = h(0)\sqrt{\dfrac{l}{g}}$.

From this we can conclude that dimensional analysis gives correctly the functional form of the solution in terms of the physical quantities. It does not, however, solve for constant factors. These can be obtained experimentally.

Note that for this example the number of variables and parameters is 5 and the number of equations is 3. If there were more dimensions we would get more equations. Now 5-3 is 2 which is the number of dimensionless products.

2.5 General procedure

Measurable quantities:

A quantity u is to be determined in terms of n measurable quantities (variables and parameters) $W_1, W_2, ..., W_n$ with dimensions $[u]$ and $[W_i]$.

We assume that there exists an unknown function f such that

$$u = f(W_1, W_2, ..., W_n). \tag{2.33}$$

We analyse this function using fundamental dimensions $(L_1, L_2, ..., L_m)$ where $m \leq n$.

Fundamental dimensions:

We postulate that all descriptive quantities in mathematical models have dimensions that are products of powers of fundamental dimensions, $L_1, L_2, ..., L_m$.

In the SI system there are seven fundamental dimensions:

- $L_1 = M$ (mass)
- $L_2 = L$ (length)
- $L_3 = T$ (time)
- $L_4 = A$ (electric current)
- $L_5 = k$ (Temperature)
- $L_6 = mol$ (amount of substance)
- $L_7 = cd$ (luminoscity)

Note that these are different to units. For example the SI units of mass is kg, length is metre and time is second.

Let z be some quantity. The dimensions of z, denoted by $[z]$, is a product of powers of the fundamental dimensions

$$[z] = L_1^{\alpha_1} L_2^{\alpha_2} ... L_m^{\alpha_m}, \tag{2.34}$$

for some real numbers α_i.

The dimension vector of z is the column vector

$$\alpha = (\alpha_1 \alpha_2 ... \alpha_m)^t. \tag{2.35}$$

A quantity z is dimensionless provided

$$[z] = L_1^0 L_2^0 ... L_m^0. \tag{2.36}$$

Exercise 2

Find the dimensions and SI units for the following:

- velocity
- density
- force
- pressure
- energy
- power
- heat flux
- specific heat
- kinematic viscosity

2.6 Further examples

2.6.1 Energy released by the first nuclear bomb 1945

In 1947 a picture of the explosion was published. The energy released by the explosion was strictly classified. G I Taylor determined the energy released during the explosion using only the radius of the expanding blast wave at time t from the published picture, and dimensional analysis. The energy released was still classified but his calculations proved to be quite accurate.

A nuclear explosion is approximated by the release of a large amount of energy E from a point. This results in an expanding fireball.

Let $r =$ the radius of the expanding fireball. We want to find a relationship between r and the other relevant parameters. These other relevant parameter are:

- $t-$ time elapsed after the explosion takes place
- $E-$ energy released by the explosion
- ρ_0- initial or ambient air density
- P_0- initial or ambient air pressure

Thus, we write

$$r = f(t, E, \rho_0, P_0)$$

assuming f exists.

Express measurable quantities in terms of Fundamental dimensions:

- $[r] = L \qquad [t] = T \qquad [E] = \dfrac{ML^2}{T^2}$
- $[\rho_0] = \dfrac{M}{L^3} \qquad [P_0] = \dfrac{M}{T^2 L}$

Consider the products of powers of r, t, E, ρ_0 and P_0 that is dimensionless:

$$[(r)^a (t)^b (E)^c (\rho_0)^d (P_0)^e] = 1$$

$$L^a T^b (ML^2T^{-2})^c (ML^{-3})^d (MT^{-2}L^{-1})^e = 1$$

This gives the three equations

$$L: \quad a + 2c - 3d - e = 0, \tag{2.37}$$

$$M: \quad c + d + e = 0, \tag{2.38}$$

$$T: \quad b - 2c - 2e = 0. \tag{2.39}$$

We have 3 equations and 5 unknowns. There are , therefore, 5-3 = 2 dimensionless products Let's solve in terms of c and d:

$$e = -c - d, \tag{2.40}$$

$$b = 2c + 2e = 2c - 2c - 2d = -2d, \tag{2.41}$$

$$a = -2c + 3d + e = -2c + 3d - c - d = -3c + 2d. \tag{2.42}$$

Thus we can write

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = c \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + d \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \tag{2.43}$$

Using these basis vectors we get

$$\Pi_1 = r^{-3} E P_0^{-1}, \qquad \Pi_2 = r^2 t^{-2} \rho_0 P_0^{-1}. \tag{2.44}$$

But we want to solve explicitly for r so we form a new dimensionless product:

$$\Pi_1 = \Pi_1 \Pi^{\frac{3}{2}} = E t^{-3} \rho_0^{\frac{3}{2}} P_0^{\frac{-5}{2}}$$

and

$$\Pi_2 = \Pi_1^{-1} \Pi^2 = r^5 E^{-1} t^{-2} \rho_0$$

Buckingham-Pi theorem gives

$$H(\Pi_1, \Pi_2) = 0, \qquad \text{or} \quad \Pi_2 = G(\Pi_1)$$

$$r = \left(\frac{Et^2}{\rho_0}\right)^{\frac{1}{5}} F\left(P_0^{\frac{5}{2}} \rho_0^{\frac{3}{2}} \frac{E}{t^3}\right)$$

2.6.2 Heat conduction

Consider a one-dimensional heat conduction in an infinite bar. The bar is heated by a point source of heat. Let

$$u = \text{temperature at any point on the bar} \tag{2.45}$$

We assume that

$$u = f(x, t, \rho, c, k, Q), \tag{2.46}$$

where

- $x-$ distance along the bar from the point source of heat
- $t-$ elapsed time after the initial heating
- $\rho-$ mass density of the bar
- $c-$ the specific heat of the bar
- $\kappa-$ thermal conductivity of the bar.
- $Q-$ strength of the heat source measured in energy per length squared.

Fundamental dimensions:

- $L_1 = M$
- $L_2 = L$
- $L_3 = T$
- $L_4 = K$

Dimensions of the quantities:

- $[u] = K \qquad [t] = T \qquad [x] = L \qquad [Q] = \dfrac{M}{T^2}$
- $[\rho] = \dfrac{M}{L^3} \qquad [\kappa] = \dfrac{ML}{T^3 K} \qquad [c] = \dfrac{L^2}{KT^2}$

There are 4 fundamental dimensions for this problem, and 7 measurable quantities. we therefore have $7 - 4 = 3$ dimensionless products. To obtain them let

$$[u^a x^b t^c \rho^d c^e \kappa^f Q^g] = 1. \tag{2.47}$$

This gives

$$L: \quad b - 3d + 2e + f = 0, \tag{2.48}$$

$$M: \quad d + f + g = 0, \tag{2.49}$$

$$T: \quad c - 2e - 3f - 2g = 0, \tag{2.50}$$

$$K: \quad a - e - f = 0. \tag{2.51}$$

Writing these equations in terms of e, f and d gives

$$a = e + f, \quad g = -d - f \quad c = 2e + 3f + 2g = 2e - 2d + f, \quad b = 3d - 2e - f. \tag{2.52}$$

Therefore we get

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{pmatrix} = e \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + f \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + d \begin{pmatrix} 0 \\ 3 \\ -2 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{2.53}$$

From these basis vectors we get

$$\Pi_1 = ux^{-2}t^2c, \quad \Pi_2 = ux^{-1}tkQ^{-1}, \quad \Pi_3 = x^3t^{-2}\rho Q^{-1}. \tag{2.54}$$

We need only one of the dimensional product Π_i to depend on u. We now eliminate u in one of the Π's and complete the problem.

Take the product $\Pi_1\Pi_2^{-1}$ (or $\Pi_1^{-1}\Pi_2$) to obtain a new dimensionless product which we call $\bar{\Pi}_2$

$$\bar{\Pi}_2 = \Pi_1\Pi_2^{-1} = x^{-1}tck^{-1}Q$$

Thus,

$$\Pi_1 = H(\bar{\Pi}_2, \Pi_3)$$

That is,

$$u = \frac{x^2}{t^2c}H(x^{-1}tck^{-1}Q, x^3t^{-2}\rho Q^{-1})$$

Exercise 3

2.6.3 The damped pendulum

For the damped pendulum, assume that the drag force F is proportional to v and use dimensional analysis to find the period.
Hint: let the force F be given by

$$\underline{F} = -\alpha \underline{v}, \tag{2.55}$$

where $[\alpha] = M/T$. Now look for products of the form

$$[T_0^a \theta^b l^c m^d g^e \alpha^f] = M^0 L^0 T^0. \tag{2.56}$$

Many of you probably got the answer

$$\Pi_1 = T_0^2 l^{-1} g, \quad \Pi_2 = \theta, \quad \Pi_3 = T_0^{-1} m \alpha^{-1}. \tag{2.57}$$

Using (Π_1, Π_2, Π_3) we obtain

$$\Pi_1 = F(\Pi_2, \Pi_3), \tag{2.58}$$

which gives

$$T_0^2 l^{-1} g = F(\theta, T_0^{-1} m \alpha^{-1}). \tag{2.59}$$

Solving for T_0 leads to

$$T_0 = \sqrt{\frac{l}{g}} G(\theta, T_0^{-1} m \alpha^{-1}). \tag{2.60}$$

Recall that the period for the simple pendulum is given by

$$T_0 = \sqrt{\frac{l}{g}} F(\theta). \tag{2.61}$$

Clearly, these two expressions show the difference between the period of the simple and the damped pendulum. However, the expression for the damped pendulum does not solve explicitly for T_0.

If I ask you to solve explicitly for T_0, then multiply

$$\Pi_1 \Pi_3^2 = l^{-1} g m^2 \alpha^{-2} = \Pi_3^*. \tag{2.62}$$

Now using the set (Π_1, Π_2, Π_3^*) we get

$$T_0^2 l^{-1} g = F(\theta, l^{-1} g m^2 \alpha^{-2}), \tag{2.63}$$

which gives

$$T_0 = \sqrt{\frac{l}{g}} G(\theta, l^{-1} g m^2 \alpha^{-2}). \tag{2.64}$$

In this case, we have an explicit expression for T_0 and we can compare the result to the simple pendulum.

Suppose we use the set (Π_3, Π_2, Π_3^*). Then we get

$$T_0 = \frac{m}{\alpha} F(\theta, l^{-1} g m^2 \alpha^{-2}). \tag{2.65}$$

In this case, we have an explicit expression for T_0 but we cannot compare it easily to the simple pendulum.

Which case is best?

2.6.4 Terminal velocity of a falling raindrop (students to complete)

Determine the terminal velocity of a raindrop failing from a motionless cloud using Buckingham Pi theory. Let

- $v-$ terminal velocity of the raindrop
- $r-$ radius of raindrop
- $\rho-$ density of air
- $\mu-$ viscosity of air
- $g-$ gravity

You are given that

$$[\mu] = ML^{-1}T^{-1}, \tag{2.66}$$

where:
M represents the dimensions of mass.
T represents the dimensions of time.
L represents the dimensions of length.

Use products of the form

$$v^a r^b \rho^c \mu^d g^e. \tag{2.67}$$

2.6.5 Wind force on a moving van (students to complete)

Find the wind force F of a van moving down the freeway.
What affects the force F of the wind on a van?

Thus we have the relationship

Consider dimensionless products of the form

Equating coefficients gives

which yields an infinitude of solutions

The dimensionless product is given by

and using Buckingham-Pi we get

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3 A brief introduction to partial differential equations

3.1 Partial differential equations (PDES)

Partial derivatives

Recall that the ordinary derivative of a function $u = f(x)$, denoted by $u'(x)$, is given by

$$u'(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}. \tag{3.1}$$

Now, let u be a function of 2 or more variables, say, $u = u(t, x, y, z)$. Then at the point (t, x, y, z), the partial derivative of u with respect to t is defined as

$$\frac{\partial u}{\partial t} = \lim_{h\to 0} \frac{u(t+h, x, y, z) - u(t, x, y, z)}{h} \tag{3.2}$$

provided the limit exists. Note that x, y, z are regarded as constants when evaluating the partial derivative of u with respect to t. We do frequently employ the notation:

$$u_t = \frac{\partial u}{\partial t},\ u_{tt} = \frac{\partial^2 u}{\partial t^2},,\ u_{tx} = \frac{\partial^2 u}{\partial t \partial x}, etc.$$

Similarly, the partial derivative of u with respect to x is given by

$$u_x = \frac{\partial u}{\partial x} = \lim_{k\to 0} \frac{u(t, x+k, y, z) - u(t, x, y, z)}{k} \tag{3.3}$$

The partial derivative of u with respect to the other independent variables can be obtained in similar fashion.

If all the derivatives of u up to some order r are continuous in some region Ω of the independent variable, it is said that u is in the class $C^r(\Omega)$ or u is C^r in Ω. Also, we say that u is r times continuously differentiable in Ω.

Ordinary differential equation

Recall that an ordinary differential equation is an equation that involves the derivatives of a function that depends on one variable. For example, let u be a dependent variable that is a function of the single independent variable x. The equation

$$\frac{d^5 u}{dx^5} + u\frac{d^4 u}{dx^4} + \cos(x) = 0, \tag{3.4}$$

is an ODE for $u(x)$.

Partial differential equation

Definition: Any equation that involves one or more partial derivatives of an unknown function, called dependent variable, of 2 or more independent variables is called a partial differential equation (pde).

Definition: The order of a partial differential equation is the order of the highest derivative in the partial differential equation.

Examples

(i) $u_{tt} - c^2 u_{xx} = 0$ (one dimensional wave equation), where c is a constant is a PDE in 1 dependent and 2 independent variables. It is second order.

(ii) $u_{tt} - u_{xx} = \sin u$ (Sine-Gordon equation) is a PDE in 1 dependent and 2 independent variables. It is second order.

(iii) $u(u_x^2) + xy(u_y)^3 = u$ is a 1st order PDE in 2 independent variables.

(iv) $u_{xxx} + a(x,y)u_{xy} + b(y)u_y + c(x)u = d(x)$ where a, b, c, d are arbitrary functions of x and y is a third order PDE.

(v) $u_t - \alpha u_{xx} = 0$ (heat equation) is a2nd order PDE in 2 independent variables.

Definition: A partial differential equation is linear if the dependent variable and all the partial derivatives appear linearly.Otherwise, it is nonlinear.

Students to determine which of the above examples is linear.

Definition: A solution to a partial differential equation is any continuosly differentiable (as many times as required by the pde) function $u = \phi(\mathbf{x})$, defined in a neighbourhood of a point $\mathbf{x}_0$ in a region Ω of the independent variables, which upon substitution into the system yields an identity in a neighbourhood of the point $\mathbf{x}_0$.

3.2 Simple examples

1. Let $u = u(x, y)$. The general solution to the PDE

$$\frac{\partial u}{\partial y} = 0. \tag{3.5}$$

is $u = C(x)$.

2. Verify that a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{3.6}$$

where c is a constant is

$$u = f(x + ct). \tag{3.7}$$

Proof: Let $\xi = x + ct$.

Then

$$\frac{\partial}{\partial t} = c\frac{\mathrm{d}}{\mathrm{d}\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}$$

$$\frac{\partial}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}$$

Thus,

$$c^2\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} = c^2\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}, \tag{3.8}$$

3.3 Linear and Quasilinear pdes

The standard form for the scalar general linear partial differential equation of the first order is

$$P^1(\mathbf{x})\frac{\partial u}{\partial x^1} + P^2(\mathbf{x})\frac{\partial u}{\partial x^2} + \ldots P^n(\mathbf{x})\frac{\partial u}{\partial x^n} + f(\mathbf{x})u = g(\mathbf{x}) \tag{3.9}$$

where $P^i(\mathbf{x})$, $i = 1, 2, \ldots, n$, $f(\mathbf{x})$ and $g(\mathbf{x})$ are given functions of the independent variables $\mathbf{x} = (x^1, \ldots, x^n)$ and u is the dependent variables.

Exercise 4

1. Show that $u = g(x - ct)$ is also a solution to the wave equation. Given that

$$u(x, 0) = x^2, \tag{3.10}$$

find g.

2. Let

$$v(x, y) = yA'(x). \tag{3.11}$$

Given the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.12}$$

solve for u.

3. Consider the 1-D heat equation. Show that by letting $u = A(t)B(x)$, the heat equation can be written as two ODES.

4. Find the general solution for u and v where

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \tag{3.13}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.14}$$

5. Let $u = u(x, y)$. Find the general solution to

i)

$$\frac{\partial u}{\partial x} = 0, \tag{3.15}$$

ii)

$$\frac{\partial u}{\partial y} = 0, \tag{3.16}$$

iii)

$$\frac{\partial^2 u}{\partial x^2} = 0, \tag{3.17}$$

iv)

$$\frac{\partial^2 u}{\partial y^2} = 0, \tag{3.18}$$

v)

$$\frac{\partial^2 u}{\partial x \partial y} = 0, \tag{3.19}$$

vi)

$$\frac{\partial^3 u}{\partial y^3} = 0. \tag{3.20}$$

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4 Modelling in fluid mechanics

4.1 Introduction to fluid mechanics

In this section we will focus on very simple fluid flows.

What is a fluid?
A fluid is a substance that cannot support shear stress whilst at rest. Instead, it deforms continuosly under the application of a shear stress. Viscosity is the immediate resistance produced by a fluid to such a rate of deformation. Common examples include: Water, air, syrup, oil, toothpaste, shampoo, honey, blood, etc. For certain fluids, the rate of deformation that they experience has no effects on their viscosity. Such fluids are called Newtonian fluids, and the relationship between the shear stress, τ, and the deformation rate, ϵ, is

$$\tau = \mu\epsilon, \tag{4.1}$$

where μ, called the dynamic viscosity, is constant. Example of fluids in this category include water, air, certain motor oils, honey, gasoline, kerosene (liquid paraffin), etc. The Newtonian fluid is the basics for classical fluid mechanics.

On the other hand, some fluids have a viscosity which changes as they are being deformed. This class of fluids is called non-Newtonian fluids. The relationship between the shear stress and shear deformation rate for such fluids cannot be described by the simple relationship given in (5.25), since μ, their dynamic viscosity, depends on the magnitude of the rate of shear. Examples of non-Newtonian fluids are paints, toothpaste, shampoo, etc.

Newtonian fluids satisfy a linear relationship between stress τ, which is a force per unit area, and rate of strain ϵ, which is extension per unit length per unit time.

From Figure 4, strain is defined as

$$\text{strain} = \frac{L(t) - L_0}{L_0}$$

$$\text{rate of strain} = \frac{\mathrm{d}}{\mathrm{d}t}(\text{strain}) = \frac{1}{L_0}\frac{\mathrm{d}L(t)}{\mathrm{d}t}$$

$$[\text{strain}] = 1$$

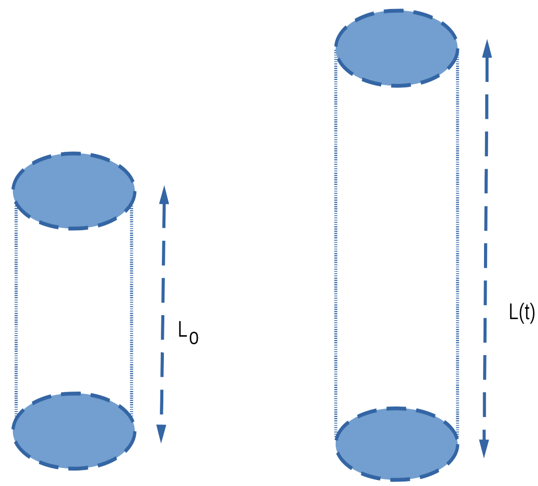


Figure 4: Strain calculation

$$[\text{rate of strain}] = \frac{L}{LT} = \frac{1}{T}$$

$$[\text{velocity gradient}] = \frac{L}{T}\frac{1}{L} = \frac{1}{T}$$

Thus, the extension per unit length per time is a gradient of velocity.

4.1.1 Viscous fluids and stresses

Viscous stress

Consider a steel ball falling under gravity in a viscous liquid such as syrup, shown in Figure 5. The forces acting on the ball are: the weight of the ball $\rho_s V g$ and bouyancy due to the syrup, $\rho_f V g$.

Thus, the net downward force on the ball is

$$F = \rho_s V g - \rho_f V g = (\rho_s - \rho_f)V g \neq 0$$

The net downward force is non-zero, provided that the densities of the ball and the fluid differ. This non-zero net downward force would cause the ball to accelerate if no other forces are acting. When this experiment is performed, what is noticed is that the ball falls at constant speed. What then is the additional unseen force that enabled the ball bearing to fall at a constant speed? This additional force is called the viscous shear stress.

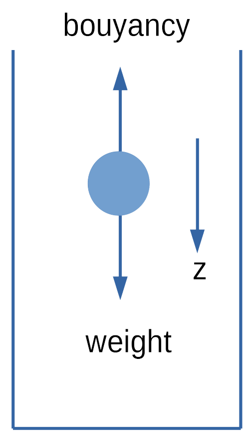


Figure 5: Steel ball bearing falling through a golden syrup

The syrup exerts a tangential viscous stress on the ball. This is similar to the air resistance of an object in free-fall. This stress is called the viscous shear stress and it opposes the motion of the falling object.

Stress

Surface stress τ, which for brevity will be called stress, is force per unit area acting on a surface element. It is a vector quantity because it has direction as well as magnitude.

The stress exerted by a fluid on a surface can be considered in two parts: the normal stress which is perpendicular to the surface and the tangential stress which acts tangentially to the surface.

Normal stress:

These stresses act normally to the surface of the object.

We do know as a matter of fact that the normal force per unit area exerted by a liquid or a gas on a stationary object is the pressure. This means that pressure p is a part of the normal stress exerted by a fluid on the surface of a body. If there is no fluid motion, i.e., when the fluid is stationary, then the stress on a surface with unit outward normal $\mathbf{n}$ is given by the normal

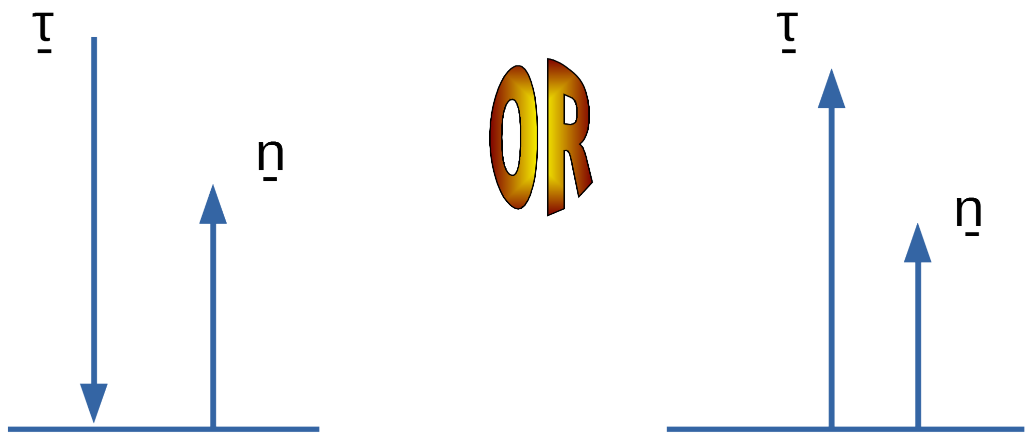


Figure 6: Unit normal to a surface and Normal stress

stress, shown in Figure 7:

$$\tau = -p\mathbf{n}. \tag{4.2}$$

with the convention that $\mathbf{n}$ points into the fluid.

The no-slip condition

Consider again the steel ball falling under gravity in syrup. The fluid around the ball is affected by its movement. The fluid in contact with the steel ball sticks to it and as a result this fluid moves with the same speed as the steel ball. As we move away from the steel ball, the fluid will start to return to its ambient state. As a result, we get gradients in the velocity. These velocity gradients are called shear. Some of the kinetic energy of the steel ball is converted to kinetic energy in the fluid. This kinetic energy is used to overcome the frictional forces between layers of the fluid. Eventually this energy dissipates and the fluid will return to rest. The shear stress of the fluid on the object acts tangentially to the surface of the object.

We will now use an example to show that the shear stress is proportional to the shear in the fluid adjacent to the surface of the object. We will then show that the constant of proportionality is called the dynamic viscosity.

Tangential shear stress

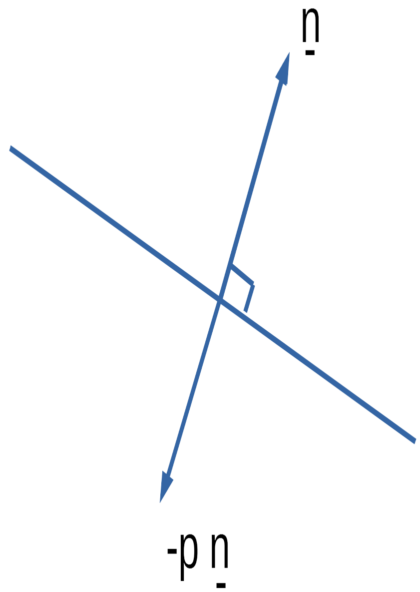


Figure 7: Unit normal to a surface and Normal stress

Consider the flow between two rigid parallel plates separated by a distance h as shown in Figure 8. The lower plate is held stationary while the upper plate is forced to moved in its own plane at a fixed speed U_0 Experimental observations show that the force per unit area (stress) that must be exerted on the upper plate is proportional to speed U_0 and inversely proportional to h. That is,

$$\text{force/area} \propto U_0/h. \tag{4.3}$$

Because the plate is moving at a constant speed, we know that the forces acting on it balance (external force and force due to the fluid), and so the force exerted by the fluid on the plate per unit area is

$$\tau_s = -\mu \frac{U_0}{h} \tag{4.4}$$

The proportionality constant μ is the dynamic viscosity of the fluid. The word "dynamic" means that having to do with the forces. The negative sign in (4.4) indicates that the tangential viscous shear stress τ_s is in the opposite direction to the direction of motion of the upper plate.

Now for this problem, the velocity across the channel is a linear function of y, varying from $u = 0$ at the lower plate $y = 0$ to $u = U$ at the upper plate $y = h$, and is given by

$$u(y) = U_0 \frac{y}{h}. \tag{4.5}$$

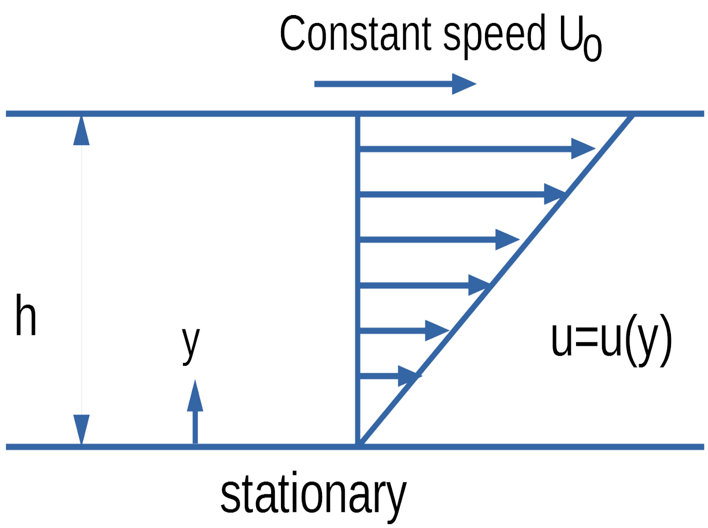


Figure 8: A thin fluid layer sheared between horizontal plates distance h apart

Noting that

$$\frac{\mathrm{d}u}{\mathrm{d}y}(y) = \frac{U_0}{h}, \tag{4.6}$$

we obtain

$$\tau_s = -\mu \frac{du}{dy}, \tag{4.7}$$

which shows that the shear stress is proportional to the velocity gradient adjacent to the surface of the plate.

In general, the tangential shear stress exerted by the fluid on a rigid surface with outward normal $\mathbf{n}$ is

$$\tau_s = \mu \frac{\partial u}{\partial n}. \tag{4.8}$$

or

$$\tau_s = \mu(\mathbf{n}.\nabla)u \tag{4.9}$$

where u is the component of the velocity field tangential to the surface and $\mathbf{n}$ is the normal to the surface pointing into the fluid.

4.2 Parallel viscous flows

In very simple viscous parallel flow, pressure forces drive the flow and viscous stresses are responsible for retarding or slowing down the flow. In what comes next, we will consider a steady fluid flow for which $\underline{u} = (u(y), 0, 0)$. Since only the $x-$component of the velocity is non-zero, this flow is also called parallel flow.

Note that if $\underline{u} = (u_x, u_y, u_z)$, the component u_x will be non-zero if there is an indication that there is motion or movement along x. Similary, the component u_y will be non-zero if there is an indication that there is movement along y. The same applies to the third component u_z.

We will first discuss some important terminologies needed for this section.

Steady flow: A steady flow is a flow that does not change in time. In other words, all forces balance, and there is no acceleration.

Parallel flow: A parallel flow is a flow where all fluid particles are moving in one direction, (say the $x-$direction). For such flows, only one velocity component is different from zero. So the velocity can be written as $\underline{u} = (u(y), 0, 0)$.

Viscous flow: A viscous flow is a flow in which viscous effect are significant and cannot be ignored.

In order to see why $u = u(y)$ and not $u = u(x, y, z)$, consider a fixed region in space D with boundary ∂D and outward normal $\underline{\text{n}}$. Mass is not created or destroyed, so mass inside D can only change by a net flow of mass across ∂D. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{D}} \rho \mathrm{d}V = -\int_{\partial \mathrm{D}} \rho \mathbf{u}\cdot\mathbf{n}\mathrm{d}S \tag{4.10}$$

but

$$-\int_{\partial \mathrm{D}} \rho \mathbf{u}\cdot\mathbf{n}\mathrm{d}S = -\int_{\mathrm{D}} \nabla\cdot(\rho\mathbf{u})\mathrm{d}V \tag{4.11}$$

$$\int_{\mathrm{D}} \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right)\mathrm{d}V = 0 \tag{4.12}$$

and since D is arbitrary,

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\underline{u}) = 0. \tag{4.13}$$

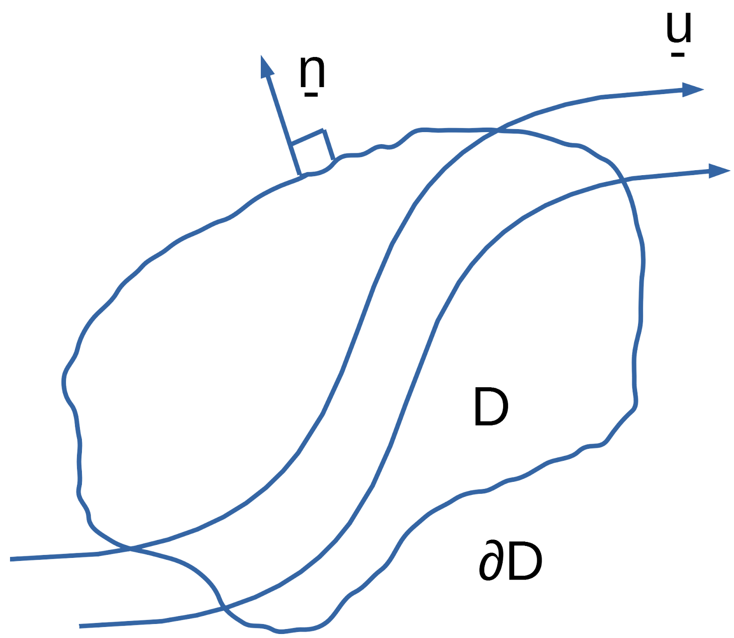


Figure 9: A thin fluid layer sheared between horizontal plates distance h apart

This equation can be written as

$$\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho + \rho \nabla . \underline{u} = 0. \tag{4.14}$$

If $\rho =$ constant, then

$$\nabla . \underline{u} = 0. \tag{4.15}$$

That is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.16}$$

For this particular problem, where there is only one velocity component, i.e., $\bar{u} = (u, 0, 0)$,

$$\frac{\partial u}{\partial x} = 0, \tag{4.17}$$

and thus $u = u(y, z)$. For a 2-D flow, that is, flow is the $x - y$ plane, $u = u(y)$.

4.3 Derivation of Momentum equation

Consider a steady parallel flow of the form $\underline{u} = (u(y), 0, 0)$ in Cartesian coordinate (x, y, z) and the forces acting on a slab of fluid parallel to the x-axis as shown in Figure 10.

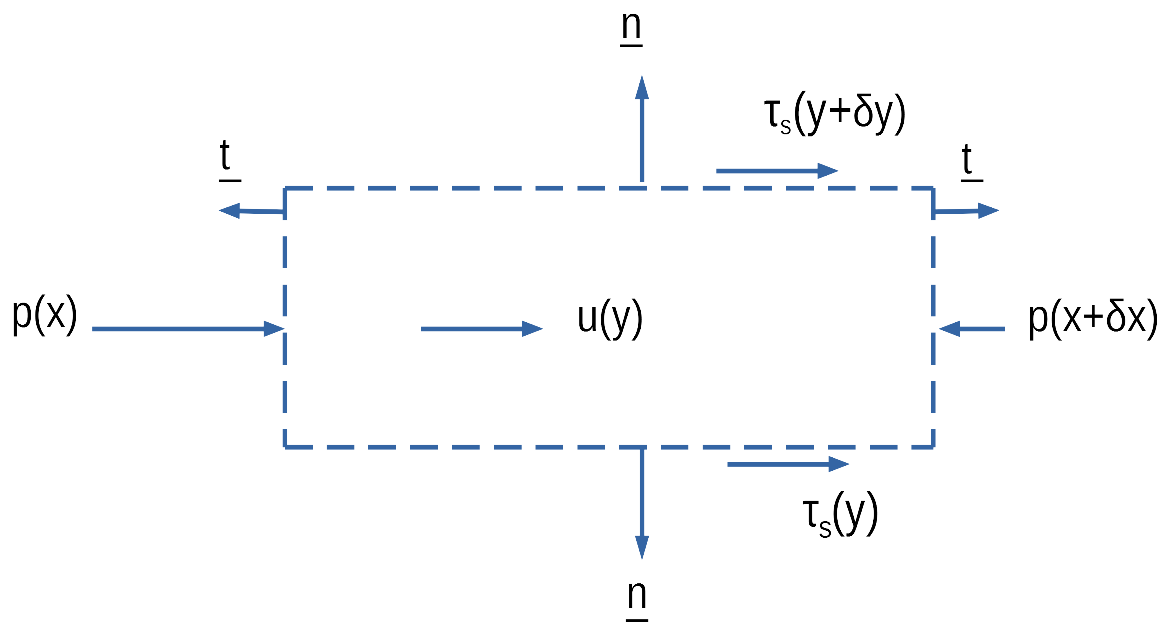


Figure 10: Components of fluid stresses in the flow direction x exerted on a small rectangular slab of length δx and height δy in a parallel shear flow. Note that fluid flows in the positive x direction since $p(x) > p(x+\delta x)$ indicated by the long arrow for $p(x)$.

Explanation of the forces: The vertical sides of the slab experience pressure forces in the x direction that act normally to the surface. These pressure forces are due to adjacent fluid slabs. These forces arise due to the pressure gradient that drives the flow.

In viscous fluids, frictional forces exist between fluid layers. Therefore, the horizontal sides with unit normal **n** experience viscous shear stresses that act tangentially to the surface. These stresses arise due to fluid layers 'rubbing' against one another.

Since the flow is steady, the forces on each fluid slab must balance. Therefore, along the x direction,

$$p(x)\delta y\delta z - p(x+\delta x)\delta y\delta z + \tau_s(y+\delta y)\delta x\delta z + \tau_s(y)\delta x\delta z = 0 \tag{4.18}$$

Note that the normal to the upper surface of the slab points into the surrounding fluid in the positive $y$direction, while the normal to the lower surface of the slab points into the surrounding fluid in the negative $y$direction. This gives, using (4.9)

$$\tau_(y+\delta y) = \mu\left((0,1)\cdot(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\right)u(y+\delta y) = \mu\frac{\partial u}{\partial y}(y+\delta y) \tag{4.19}$$

and

$$\tau_(y) = \mu \left((0, -1) \cdot (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \right) u(y) = -\mu \frac{\partial u}{\partial y}(y) \tag{4.20}$$

If we divide through equation (4.18) by $\delta x \delta y$, we obtain

$$\frac{-(p(x+\delta x, y) - p(x, y))}{\delta x} + \frac{\mu}{\partial y} \left(\frac{\partial u}{\partial y}(x, y+\delta y) - \frac{\partial u}{\partial y}(x, y) \right) = 0. \tag{4.21}$$

Taking the limits $\delta x \to 0$, $\delta y \to 0$ gives

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}. \tag{4.22}$$

Resolving the forces in the y-direction using similar approach leads to

$$\frac{\partial p}{\partial y} = 0. \tag{4.23}$$

Equations (4.22) and (4.23) are partial differential equations. They are called momentum equations along x and y directions respectively. They are momentum equations because they are a statement of Newton's second law relating the net forces (viscous force and pressure force) acting on a fluid element depicted in Figure 10 to the acceleration or rate of change of momentum of the slab. Because the considered flow is steady, the rate of change of momentum is zero, that is, net force is zero.

Students are to determine the dimension of each of the terms in (4.22) and to check that equation (4.22) is dimensionally consistent. If it is not, then it will be a useless piece of equation.

Exercise 5

If a parallel flow $\bar{u} = (u(y, t), 0, 0)$ is unsteady (changing with time) and there is body force (force per unit volume) given by $\mathbf{f} = (f_x, f_y, 0)$, find the governing equations for the fluid flow.

Note that if body force is due to gravitationl force, such that x points horizontally and y vertically upwards, then $\mathbf{f} = (0, -\rho g, 0)$.

4.3.1 Boundary conditions

Equation (4.23) is a statement depicting that pressure p does not vary along the y direction, for which net force is zero along y. Equation (4.22) is a 2nd order pde in two dependent variables u and p and two independent variables x and y.

Before being able to solve a flow problem for the velocity and pressure field, we need boundary conditions for the momentum equation governing the flow.

In fluid mechanics, the task has always been to integrate the given momentum equation, subject to some specified boundary conditions, in order to determine the velocity field u and pressure field p for a given flow problem.

Two types of boundaries can be identified in fluid mechanics. The one boundary is at the interface between a fluid and a rigid surface, while the other is a boundary at a fluid-fluid interface .

Conditions at a rigid boundary

At a fluid-solid interface, we will require that the tangential component of the velocity of the fluid be the same as the tangential component of the velocity of the rigid surface. This is known as the no-slip condition. Similarly, the normal component of the velocity of the fluid be the same as the normal component of the velocity of the rigid boundary.

No-slip condition Let $\mathbf{v_f}$ be the velocity of the fluid and $\mathbf{v_s}$ the velocity of the rigid surface. Let $\hat{\mathbf{n}}$ be a unit normal vector to the rigid surface pointing into the fluid, and $\hat{\mathbf{t}}$, the unit tangential as shown in Figure 11, the no slip boundary condition can be stated as

$$\mathbf{t} \cdot \mathbf{v_f} = \mathbf{t}. \cdot \mathbf{v_s} \quad \text{on a rigid surface} \tag{4.24}$$

Let $v_f = (u, v)$, then equation (4.24) is a statement of the fact that the tangential component of the velocity of the fluid on the boundary is the tangential component of the velocity of the rigid surface itself.

That is,

$$u = 0 \quad \text{at a stationary, rigid boundary} \tag{4.25}$$

$$u = U \quad \text{at a rigid boundary moving in x direction with velocity } U \tag{4.26}$$

Kinematic condition When there is no mass transfer across the boundary, a purely kinematical consequence is that the normal component of the fluid velocity at the boundary must equal the normal component of the velocity of the rigid surface. That is,

$$\hat{\mathbf{n}} \cdot \mathbf{v_f} = \hat{\mathbf{n}} \cdot \mathbf{v_s} \quad \text{on a rigid surface} \tag{4.27}$$

That is, at the fluid-solid boundary,

$$v\Big|_{\text{fluid}} = \text{component of the velocity of rigid surface along } \hat{\mathbf{n}}$$

Figure 11: The unit normal $\hat{\mathbf{n}}$ points into the fluid on top of the rigid/solid boundary. $\mathbf{v_s}$ is the velocity of the fluid-solid boundary and $\mathbf{v_s}$ is the velocity of the fluid above the rigid boundary.

Conditions at a fluid-fluid boundary Sometimes, we encounter a boundary between two fluids. A common example occurs when a liquid film (thin fluid) flows down an inclined plane. The upper surface of the liquid film in contact with the surrounding gas (air) is a fluid-fluid interface. Another example is the interface between two liquid layers.

At the fluid-fluid interface shown in Figure 12, that is, the boundary where two immiscible fluids meet, the x-component and y component of the velocity of each fluid are continuous. Thus (4.24) and (4.27) are also satisfied at the fluid-fluid interface. Both components of viscous stress (normal and tangential) are also continuous at a fluid-fluid interface.

Thus, for parallel viscous flow, for which $\mathbf{u} = (u(y), 0)$, this implies that the velocity, pressure and the tangential viscous stresses are all continuous. Thus, at the fluid-fluid interface,

$$u_{\mathrm{A}} = u_{\mathrm{B}} \tag{4.28}$$

$$p_A = p_B \qquad \text{Normal stress balance} \tag{4.29}$$

$$\mu_A \frac{\partial u_A}{\partial n} = \mu_B \frac{\partial u_B}{\partial n} \qquad \text{tangential stress balance} \tag{4.30}$$

In many cases, such as at a liquid-gas interface, one of the fluids (the gas) has a much smaller dynamic viscosity than the other, and therefore exerts negligible stress on the flow of the liquid.

Suppose that in Figure 12, fluid A is a gas with symbol g, and fluid B is liquid with symbol l, then we have at the interface (or free surface) that

$$\mu_l \frac{\partial u_l}{\partial n} = \mu_g \frac{\partial u_g}{\partial n}$$

Dividing through by μ_l gives

$$\frac{\partial u_l}{\partial n} = \frac{\mu_g}{\mu_l} \frac{\partial u_g}{\partial n} \tag{4.31}$$

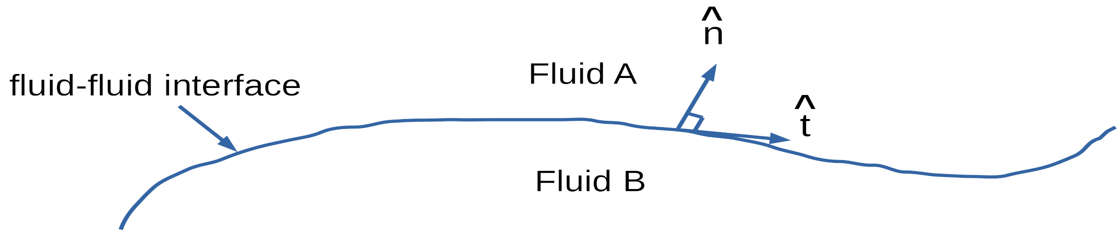


Figure 12: The unit normal $\hat{\mathbf{n}}$ points into the fluid on top of the fluid-fluid interface.

Since the dynamic viscosity of a gas is small compared to the dynamic viscosity for a liquid, we have that $\frac{\mu_g}{\mu_l} << 1$. The right hand side of (4.31) is therefore small and considered negligible. The boundary is thus approximated as being stress-free and we write

$$\frac{\partial u_l}{\partial n} = 0. \qquad \text{at stress-free boundary.}$$

Note that this approximation of the tangential stress condition can be used only when the driving force for the motion of the liquid is not the motion of the gas. When a gas drags a liquid along, as is the case on a windy day when the wind causes motion in a puddle of liquid, the correct boundary condition equating the tangential stresses must be used.

4.4 2-D Poiseuille flow

As a first example, we can use the momentum equations and boundary conditions to determine the parallel flow of fluid in a horizontal channel between two parallel rigid walls, a distance h apart as shown in Figure 19: The flow is driven by the pressure difference between the two ends of the channel. This external pressure gradient drives the flow and viscous stresses due to the solid boundaries (the upper and lower walls) retards the flow. If the flow is steady, the governing momentum equations are given by

$$0 = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}, \tag{4.32}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g. \tag{4.33}$$

Since ρ and g are constant, the second equation is integrated to obtain

$$p(x, y) = -\rho g y + C(x), \tag{4.34}$$

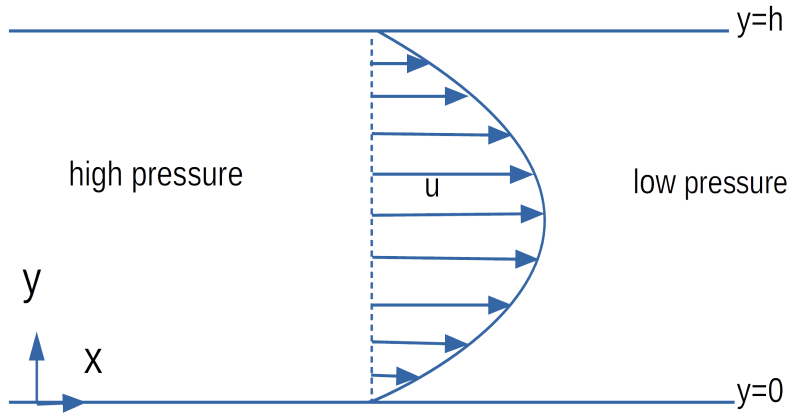


Figure 13: A fluid flow in a two-dimensional channel driven by pressure gradient across x

where $c(x)$ is an arbitrary function of x only. The term $\rho g y$ in (4.34) is simply the hydrostatic pressure which does not cause any flow and only varies along y.

Hydrostatics relates to fluid at rest, and hydrostatic pressure is the pressure that is exerted by a fluid at rest at a given point within the fluid, due to the force of gravity. It is essentially the pressure at a point due to the weight of the fluid above it. Hydrostatic pressure increases in proportion to depth measured from the surface because of the increasing weight of fluid exerting a downward force from above.

When differentiated, (4.34) yields

$$\frac{\partial p}{\partial x} = C'(x) \tag{4.35}$$

Since u depends on y only and $\partial p/\partial x$ on x only, we must have

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = -G, \tag{4.36}$$

where $G > 0$ is a constant. Thus, pressure $p(x, y)$ is linear in x and the problem reduces to solving

$$\mu \frac{\partial^2 u}{\partial y^2} = -G, \qquad 0 < y < h \tag{4.37}$$

subject to the no-slip boundary conditions at the rigid walls of the two-dimensional channel:

$$u = 0, \quad \text{on} \quad y = 0, \tag{4.38}$$

$$u = 0, \quad \text{on} \quad y = h. \tag{4.39}$$

Integrating twice yields

$$u = -\frac{G}{2\mu}y^2 + \frac{A}{\mu}y + \frac{B}{\mu}. \tag{4.40}$$

Now, $u(0) = 0$ yields $B = 0$ and $u(h) = 0$ yields $A = \frac{G}{2}h$. Thus,

$$u = \frac{G}{2\mu}y(h-y) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}y(h-y). \tag{4.41}$$

We see that the velocity profile is parabolic as shown in Figure 14. This parabolic velocity profile is known as Poiseuille flow.

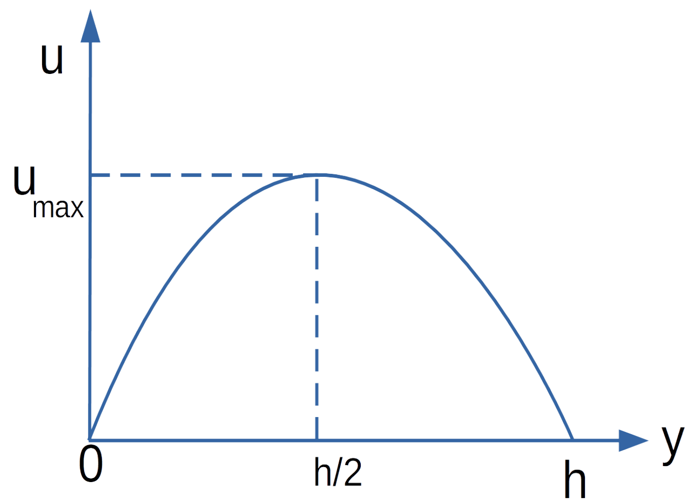


Figure 14: Parabolic velocity profile for a poiseuille flow

We can also derive some important properties for this flow, since we have a mathematical expression for it.

Shear stress at lower and upper boundaries of the channel

$$\tau_s = \mu(\hat{\mathbf{n}} \cdot \underline{\nabla})u$$

Therefore,

$$\tau_s(0) = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{h}{2}\frac{\partial p}{\partial x} \tag{4.42}$$

$$\tau_s(h) = -\mu\frac{\partial u}{\partial y}\bigg|_{y=h} = \frac{h}{2}\frac{\partial p}{\partial x} \tag{4.43}$$

The results in (4.42) and (4.43) are measured in force per unit area. If the channel is of length L, the total viscous force exerted by the fluid on the rigid walls per unit width in the $z-$direction is

$$\int_0^L (\tau_s(0) + \tau_s(h))\mathrm{d}x = hL\frac{\partial p}{\partial x} \sim h\Delta p$$

where $\Delta p = L\frac{\partial p}{\partial x}$ and $h\Delta p$ is equal to the difference in the pressure forces acting at both ends of the channel. This means that the net pressure force exerted on the fluid at the two ends balances the viscous shear stress exerted by the channel walls.

Volume flux Volume flux is the volume of fluid passing any cross section of the channel per unit time. The volume flux per unit width in the z direction for this flow, denoted q, is

$$q = \int_0^h u(y)\mathrm{d}y = \int_0^h -\frac{1}{2\mu}\frac{\partial p}{\partial x}y(h-y)\,\mathrm{d}y = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} \tag{4.44}$$

Students to verity the reason for the minus sign, given that flux is a vector quantity.

Note that the volume flux is proportional to h^3. This means that if the channel is twice as wide (i.e., channel is from y=0 to y=2h), the volume of fluid per unit time transported through a cross section will be eight times as much, for a given pressure gradient driving the flow. The volume flux is also inversely proportional to viscosity μ. This means that the more viscous a fluid is, the less interested it is in flowing, which then means that less volume of fluid is transported across the interface in a unit time. Lastly, the pressure gradient drives this flow. If it is increased (by increasing the fluid pressure at $x = 0$, or by decreasing the pressure at $x = L$), then there will be more volume of fluid transported across the cross section in a unit time.

4.5 2-D Couette flow

Consider the steady flow between two infinite parallel rigid walls $y = 0$ which is at rest and $y = h$ which is moving parallel to itself with velocity $u = U_0$ in the x direction as shown in Figure **??**. This flow is driven only by shear, produced by the motion of the upper plate travelling in the x direction at a constant speed U_0. There is no external pressure gradient applied.

The governing equation and boundary conditions are

$$\frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < y < h, \tag{4.45}$$

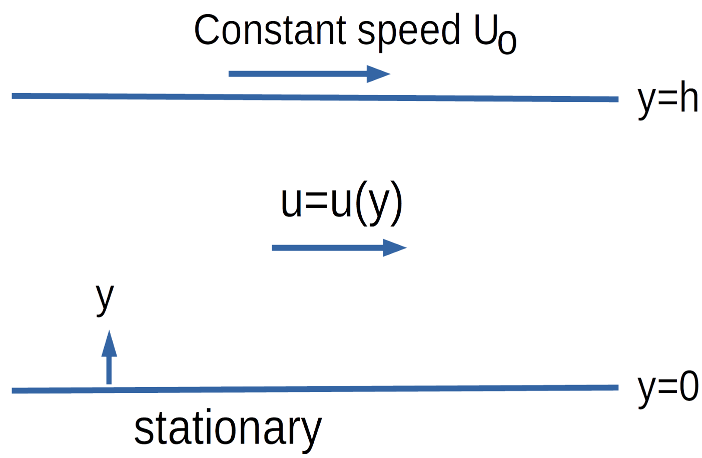


Figure 15: A fluid flow in a two-dimensional channel driven by motion of upper plate

$$u = 0, \qquad y = 0, \tag{4.46}$$

$$u = U_0, \qquad y = h. \tag{4.47}$$

Integrating (4.45) twice with respect to y yields

$$u = A(x)y + B(x).$$

Applying (4.46) yields $B(x) = 0$ and (4.47) yields $A(x) = \frac{U}{h}$.

The solution is

$$u = U_0 \frac{y}{h}. \tag{4.48}$$

There are a number of interesting properties that we can derive for this flow.

Shear stress at boundaries

At the upper boundary, for which $\hat{\mathbf{n}} = (0, -1)$,

$$\tau_s = \mu(\hat{\mathbf{n}} \cdot \nabla)u = \mu \left((0, -1) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right) u = -\mu \frac{\partial u}{\partial y}\bigg|_{y=h} = -\mu \frac{U_0}{h}$$

At the lower boundary, for which $\hat{\mathbf{n}} = (0, 1)$,

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{U_0}{h}$$

For a channel of length L, the total viscous force exerted by the fluid on the walls per unit width in the z direction is

$$\int_0^L (\tau_s(0) + \tau_s(h))\mathrm{d}x = 0$$

Volume flux The volume flux, that is, the volume of fluid transported through any cross section of the channel per unit time , per unit width in the z-direction is

$$q = \int_0^h u(y)\mathrm{d}y = U_0 \frac{h}{2}.$$

Exercise 6

1) Flow down a slope.

Consider the flow of a uniform layer of fluid of constant height h down a slope which has an angle of α. Assume that the atmospheric pressure is uniform and that it exerts no tangential stress on the fluid. Find the governing equations and the boundary conditions. Now show that the solution is given by

$$u = \frac{\rho g}{2\mu} \sin\alpha(y(2h - y)). \tag{4.49}$$

Finally, find the volume flux.

4.6 Unsteady parallel flows

Consider a semi-infinite (used to described an unbounded domain) fluid domain $y > 0$. At $y = 0$, we have a rigid boundary which is initially at rest. At $t = 0$, the boundary is set into motion with constant speed U_0. Assume the fluid flows parallel to the boundary. Using Figure

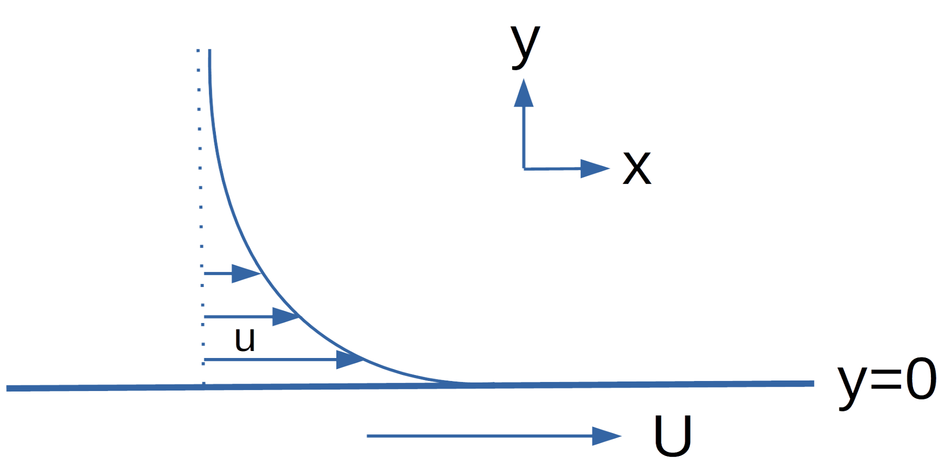


Figure 16: Velocity field generated due to motion of rigid plane $y = 0$

10, the force balance for zero external pressure gradient is, using Newtons second law, where the acceleration is $\frac{\partial u}{\partial t}$, is

$$\tau_s(y+\delta y)\delta x\delta z + \tau_s(y)\delta x\delta z = m\frac{\partial u}{\partial t}, \tag{4.50}$$

where m is the mass of the rectangular slab in Figure 10. If we divide through equation (4.50) by $\delta x\delta y\delta z$, we obtain

$$\frac{\mu}{\delta y}\left(\frac{\partial u}{\partial y}(x, y+\delta y) - \frac{\partial u}{\partial y}(x,y)\right) = \frac{m}{\delta x\delta y\delta z}\frac{\partial u}{\partial t}. \tag{4.51}$$

By writing $\rho = \frac{m}{\delta x\delta y\delta z}$ and taking the limits $\delta x \to 0$, $\delta y \to 0$, we obtain the governing equation given by

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{4.52}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid. Equation (4.52) is a linear partial differential equation that is second order is space x and first order in time t. It has the same form at the one dimensional heat conduction equation which we will study later. It must be solved subject to

$$u = U_0, \quad y = 0, \tag{4.53}$$

$$u \to 0, \quad y \to \infty, \tag{4.54}$$

$$u = 0, \quad t = 0, \quad y > 0. \tag{4.55}$$

We can actually use scaling arguments to find a similarity solution! What happens here is that we can reduce the PDE to an ODE which is easier to solve. This is an incredibly powerful method.

First, let us try to find characteristic length and time scales:

Fluid motion is initially confined to a region close to the moving boundary, and the apparently infinite domain in which the flow occurs has no scale to it. The flow will ultimately establish its own intrinsic length scale of variation, and the velocity profile will remain self similar as the flow evolves with time.

The governing equation is linear and the flow is forced only by the boundary condition at $y = 0$. This then indicates that we can write

$$u = U_0 F(y,t), \tag{4.56}$$

where F is a dimensionless function. F should therefore have dimensionless arguments. But F depends on y and t which are dimensional, and having dimensions of length and time. We can measure length in metres or feet, and time in seconds, hours or years, but the mathematical function F should be independent of such a choice.

Therefore, we must find characteristic length and time scales L and T such that

$$u = U_0 F\left(\frac{y}{L}, \frac{t}{T}\right). \tag{4.57}$$

When (4.57) is substituted into the partial differential equation (4.52), the function F will be a function of dimensionless variables provided

$$\frac{U_0}{T} \sim \nu \frac{U_0}{L^2} \tag{4.58}$$

for which

$$L \sim \sqrt{\nu T} \tag{4.59}$$

Thus we let

$$u = U_0 F(\eta) \quad \text{where} \quad \eta = \frac{y}{\sqrt{\nu T}} \tag{4.60}$$

The quantity $\eta = \frac{y}{\sqrt{\nu T}}$ is a dimensionless product and is called a similarity variable and F is called the similarity solution.

Substituting (4.60) into the govening pde (4.52) transforms the partial differential equation to an ordinary differential equation given by

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\eta^2} + \frac{1}{2}\eta \frac{\mathrm{d}F}{\mathrm{d}\eta} = 0 \tag{4.61}$$

The two boundary conditions and one initial condition reduces to

$$F(0) = 1, \qquad F(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{4.62a-b}$$

Equation (4.61) is linear, first order in $\frac{\mathrm{d}F}{\mathrm{d}\eta}$ and in separable form. It can be integrated to obtain

$$\frac{\mathrm{d}F}{\mathrm{d}\eta} = A e^{-\frac{\eta^2}{4}},$$

where A is an integration constant. Integrating once more yields

$$F(\eta) = A \int_0^{\eta} e^{-\xi^2/4} \mathrm{d}\xi + B = 2A \int_0^{\eta/2} e^{-s^2} \mathrm{d}s + B, \tag{4.63}$$

using the simple change of variable $\xi = 2s$. The last integral term in (4.63) cannot be evaluated in terms of elementary or simple functions, but can be expressed in terms of the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-s^2} \mathrm{d}s, \tag{4.64}$$

which can be easily sketched in Matlab or Mathematica. Thus, the general solution is

$$F(\eta) = 2A\,\mathrm{erf}\left(\frac{\eta}{2}\right) + B, \tag{4.65}$$

The error function has the properties that

$$\mathrm{erf}(0) = 0, \qquad \mathrm{erf}\,(z) \to 1 \text{ as } z \to \infty.$$

and the complimentary error function is related to the error function via

$$\mathrm{erfc}\,(z) = 1 - \mathrm{erf}\,(z).$$

Programming softwares like Matlab and Mathematica has the error function and the complimentary error function as inbuilt functions, and a plot of $\mathrm{erf}(z)$ and $\mathrm{erfc}(z)$ against z using Matlab is shown in Figure 17.

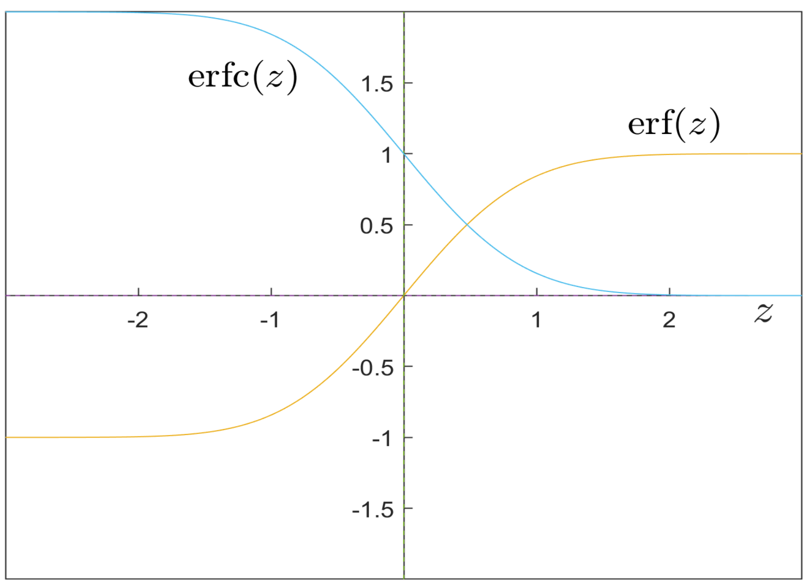
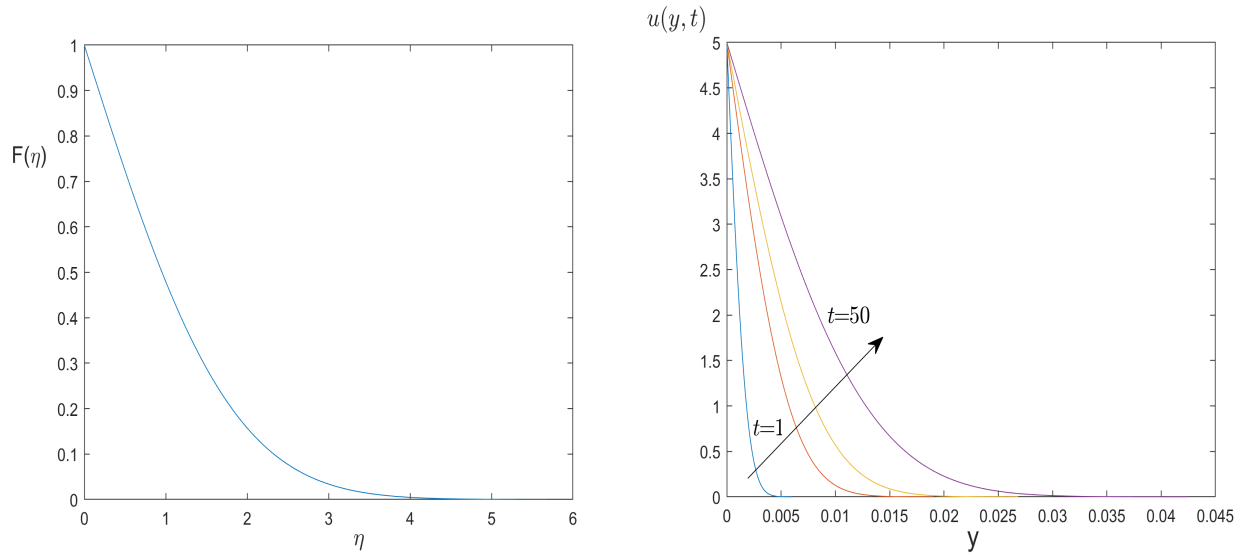


Figure 17: The error function $\mathrm{erf}(z)$ and the complementary error function $\mathrm{erfc}(z)$ plotted against z.

By applying the boundary condition (4.62a) on (4.65), we obtain $B = 1$, and applying (4.62b) gives $A = -1/2$. The solution satisfying the boundary conditions, plotted in Figure (18a) is

$$F(\eta) = 1 - \mathrm{erf}\left(\frac{\eta}{2}\right) = \mathrm{erfc}\left(\frac{\eta}{2}\right) \tag{4.66}$$



(a) (b)

Figure 18: Figure (a) shows plot of similarity function $F(\eta)$ against η. Figure (b) shows time evolution of self similar solutions for the velocity profile of unsteady parallel viscous flow at time $t = 1, 10, 20$ and 50. For water, $\mu = 10^{-3}\,\mathrm{Ns/m}$, $\rho = 10^3\mathrm{kg/m}^3$ and therefore $\nu = \mu/\rho = 10^{-6}\,\mathrm{m}^2/s$.

Using (4.60), the solution $u(y,t)$ is

$$u(y,t) = U_0 \mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right). \tag{4.67}$$

The velocity $u(y,t)$ is a decaying function of y, and the flow problem has a characteristic length scale which is the length through which the velocity decays. This length scale is proportional to $\sqrt{\nu t}$. The velocity profile at different times, shown in Figure (18 b) have the same mathematical form. We say the solutions are self simiilar.

4.7 Poiseuille flow in a pipe

Consider the parallel steady flow $u(r)$ of viscous fluid in a circular pipe of radius a, depicted in Figure 19. The picture also shows a small concentric cylinder of fluid of radius $r < a$ and length δx. Because the flow is parallel, the longitudinal forces on this cylinder consist of viscous shear stresses on the curved surface and pressure forces on the two ends. In steady flow, there is no net force on this fluid cylinder along the x direction and so

$$2\pi r \delta x\, \tau_s(r) + \pi r^2 \left[p(x) - p(x+\delta x)\right] = 0. \tag{4.68}$$

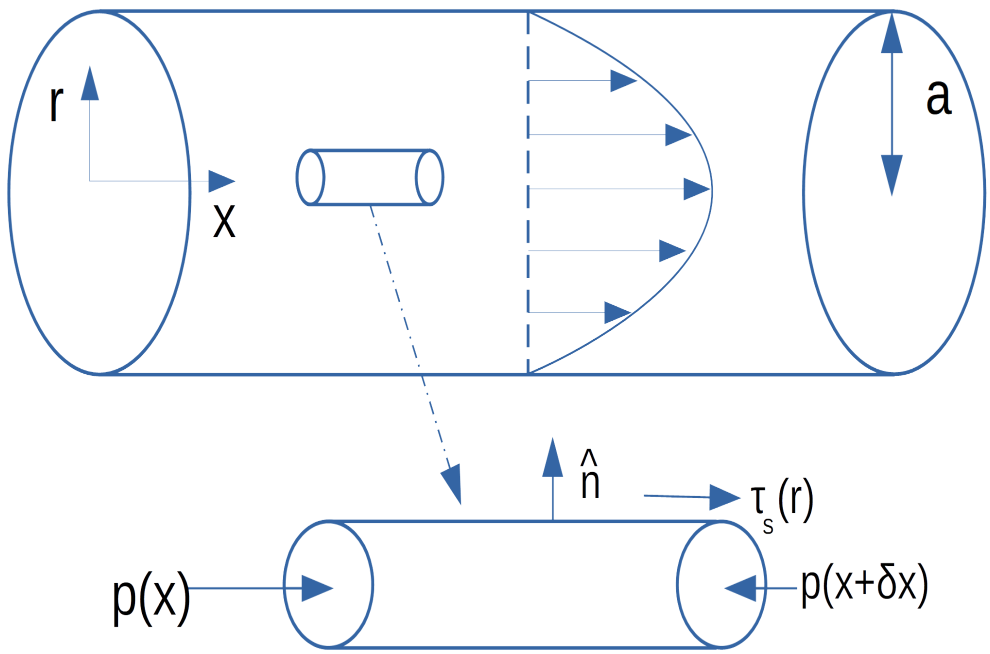


Figure 19: Pressure driven flow in a circular pipe of radius a. The inset shows the components of the stresses acting on a fluid cylinder of radius r and length δx

But

$$\tau_s(r) = \mu \hat{n} \cdot \nabla u,$$

where in cylindrical polar coordinate (r, θ, z), the unit vector $\hat{n} = \hat{r}$, since $\hat{n}$ points in the radial direction. In cartesian coordinate,

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

However, in cylindrical polar coordinate (r, θ, z), it is

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}$$

Thus, $\hat{n} \cdot \nabla = \frac{\partial}{\partial r}$ and

$$\tau_s(r) = \mu\frac{\partial u}{\partial r}.$$

Equation (4.68) becomes

$$2\pi r \delta x \left(\mu \frac{\partial u}{\partial r}\right) + \pi r^2 \left[p(x) - p(x + \delta x)\right] = 0. \tag{4.69}$$

Divide through by $\pi r \delta x$, equation (4.69) becomes, in the limit $\delta x \to 0$

$$\frac{\partial u}{\partial r} = \frac{r}{2\mu}\frac{\partial p}{\partial x}. \tag{4.70}$$

Resolving in the radial direction $\hat{r}$ yields (students to verify)

$$\frac{\partial p}{\partial r} = 0.$$

There is no flow along the $\hat{\theta}$ direction (flow is axisymmetric).

Since $u = u(r)$, we must have that $\partial p/\partial x$ is a constant. Equation (4.70) can therefore be integrated, subject to the boundary condition that at the wall, there is no slip. This yields

$$u(r) = -\frac{1}{4\mu}\frac{\partial p}{\partial x}(a^2 - r^2). \tag{4.71}$$

The volume flux of fluid is given by

$$q = \int_0^a 2\pi r u dr = -\frac{\pi a^4}{8\mu}\frac{\partial p}{\partial x}. \tag{4.72}$$

It is important to see that $q > 0$ since $\partial p/\partial x < 0$. q is a vector quantity which indicates that the volume flow is in the positive x direction.

It is also important to note that $q \propto a^4$. As a result, a pipe twice as wide will transport sixteen times as much fluid per unit time for a given pressure gradient driving the flow.

4.8 The Navier-Stokes momentum equation

Thus far, we have considered very basic parallel flows and explored some of the physical forces acting on the interior of fluids. It is now time for you to meet the general equations for three dimensional fluid flow. More general flows are governed by the Navier-Stokes equations. For 3-D flows, where the velocity vector is $\bar{u} = (u, v, w)$, these are given in vector form by:

$$\rho\frac{D\bar{u}}{Dt} = -\bar{\nabla}p + \mu\nabla^2\bar{\mathbf{u}} + \rho\bar{\mathbf{g}}, \tag{4.73}$$

where $\rho\bar{\mathbf{g}} = (0, 0, -\rho g)$ and where the nabla operator ∇ is defined as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}. \tag{4.74}$$

If a scaler field $\phi = \phi(t, x(t), y(t), z(t))$ is a function of time t and space x and y, then the material derivative $\frac{D}{Dt}$ of ϕ is

$$\begin{aligned} \frac{D\phi}{Dt} &= \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z} \\ &= \frac{\partial \phi}{\partial t} + (\bar{u}\cdot\nabla)\,\phi. \end{aligned} \tag{4.75}$$

Equation (4.73) can be written in component form, that is, along the x, y and z directions.

Along the x direction, it is

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{4.76}$$

Along the y direction, it is

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{4.77}$$

Along the z direction, it is

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g \tag{4.78}$$

Here, u, v and w are the x, y and z components of the velocity respectively, p is the fluid pressure, μ is the dynamic viscosity of the fluid and ρ is the density of the fluid.

4.9 Mass conservation

Consider a fixed region D in space with boundary ∂D and outward unit normal $\underline{\mathrm{n}}$. Mass is not created or destroyed, so mass inside D can only change by a net flow of mass across ∂D. The mass of fluid entering region D per unit time is

$$-\int_{\partial \mathrm{D}} \rho\mathbf{u}\cdot\mathbf{n}\mathrm{d}S. \tag{4.79}$$

This causes a rate of increase of the mass inside region D given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{D}} \rho\mathrm{d}V \tag{4.80}$$

Therefore, equating the two expressions give,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{D}} \rho\mathrm{d}V=-\int_{\partial \mathrm{D}} \rho\mathbf{u}\cdot\mathbf{n}\mathrm{d}S \tag{4.81}$$

but, using the divergence theorem(also known as Gauss's theorem), we have that

$$-\int_{\partial \mathrm{D}} \rho\mathbf{u}\cdot\mathbf{n}\mathrm{d}S=-\int_{\mathrm{D}} \nabla\cdot(\rho\mathbf{u})\mathrm{d}V \tag{4.82}$$

Thus, by putting the time derivative under the integral sign in (4.81), equation (4.81) becomes

$$\int_{\mathrm{D}} \frac{\partial \rho}{\partial t}\mathrm{d}V=-\int_{\mathrm{D}} \nabla\cdot(\rho\mathbf{u})\mathrm{d}V \tag{4.83}$$

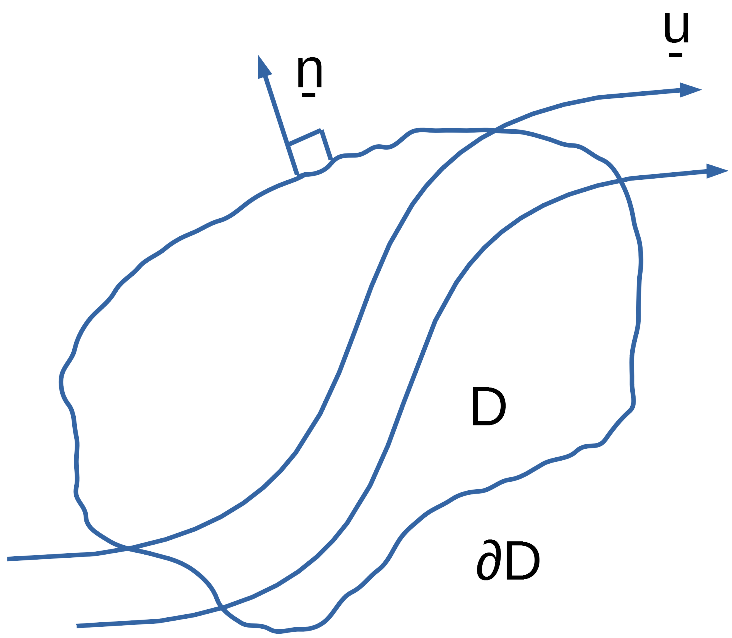


Figure 20: A thin fluid layer sheared between horizontal plates distance h apart

Since (5.40) is true for arbitrary region D, the integrands on both sides must be equal and therefore,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0. \tag{4.84}$$

This is a general mass conservation equation relating the time rate of change of the mass density to the divergence of the mass flux $\rho \mathbf{u}$.

Using the vector identity

$$\nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p,$$

(5.41) becomes

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla.\mathbf{u} = 0. \tag{4.85}$$

If the fluid is incompressible, for which $\rho =$ constant, then

$$\nabla \cdot \mathbf{u} = 0. \tag{4.86}$$

That is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4.87}$$

The equation of constraint states that the velocity field is solenoidal, that is, the velocity field has zero divergence. Equation (4.86) is known as the CONTINUITY EQUATION for incompressible fluid flow.

Complete Navier Stokes equation

The complete set of Navier-Stokes equations, describing three-dimensional fluid flow of an incompressible fluid, is therefore

$$\rho \frac{D\mathbf{u}}{Dt} = -\bar{\nabla} p + \mu \nabla^2 \bar{\mathbf{u}} + \rho \bar{\mathbf{g}}, \tag{4.88}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{4.89}$$

Exercise

Verify that, in Cartesian coordinates (x, y, z), an incompressible parallel flow $\mathbf{u} = (u, 0, 0)$ has the property that $\partial u/\partial x = 0$, and that the Navier-Stokes momentum equation reduces to the parallel flow equations we explored earlier .

4.10 Scaling the Navier Stokes equation

The characteristics of a fluid flow are greatly affected by its speed, the distance between the boundaries of the fluid, and the viscosity of the fluid. For example, the flow of water through a narrow needle is smooth and parallel, like the Poiseuille flow we derived earlier, while the flow of water through a fire hose is likely to be turbulent and full of eddies.

Also, there is no general solution to the complete Navier-Stokes equation. However, we can make mathematical progress by deciding, on the basis of the scale of the flow, that certain terms in the governing Navier-Stokes equation can be reasonably neglected in a particular application or problem. This principle leads to the simplification of the Navier-Stokes equation.

We first want to non-dimensionlize these equations. We start by identifying sensible scales U for the magnitude of the fluid velocity and L for the length over which the velocity varies. For example, for flow through a pipe, we may choose U to be an estimate of the maximum velocity and L as the diameterof the pipe. For an object falling through a fluid, we may choose U to be the fall speed and L to be the diameter of the smallest sphere enclosing the object.

Define dimensionless variables as follows:

$$u' = \frac{u}{U_0}, \quad v' = \frac{v}{U_0}, \quad p' = \frac{p}{P_0}, \tag{4.90}$$

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L}.. \tag{4.91}$$

The scaled Navier-Stokes equations(with zero body force) are

$$\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+w'\frac{\partial u'}{\partial y'}\right)=-\frac{P_0}{\rho U_0^2}\frac{\partial p'}{\partial x'}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}+\frac{\partial^2 u'}{\partial z'^2}\right) \tag{4.92}$$

$$\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}+w'\frac{\partial v'}{\partial z'}\right)=-\frac{P_0}{\rho U_0^2}\frac{\partial p'}{\partial y'}+\mu\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}+\frac{\partial^2 v'}{\partial z'^2}\right) \tag{4.93}$$

$$\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial w'}{\partial x'}+v'\frac{\partial w'}{\partial y'}+w'\frac{\partial w'}{\partial z'}\right)=-\frac{P_0}{\rho U_0^2}\frac{\partial p'}{\partial z'}+\mu\left(\frac{\partial^2 w'}{\partial x'^2}+\frac{\partial^2 w'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}\right). \tag{4.94}$$

where $\mathrm{Re}=\frac{U_0 L}{\nu}$. The continuity equation is

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0. \tag{4.95}$$

There are two extreme flow regimes.

4.10.1 High Reynolds number flows

For $Re \gg 1$, the viscous terms are negligible when compared to the inertial terms. The pressure is scaled by

$$P_0=\rho U_0^2 \tag{4.96}$$

and the governing equations become

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p \tag{4.97}$$

$$\nabla\cdot\mathbf{u}=0 \tag{4.98}$$

The pressure gradient now provides an acceleration of fluid particles.

4.10.2 Low Reynolds number flows

For $Re \ll 1$, the viscous terms will be much larger than the inertial terms. The inertial terms are negligible. The pressure scale P_0 is then chosen as:

$$P_0=\frac{\rho U_0^2}{\mathrm{Re}}=\frac{\mu U_0}{L} \tag{4.99}$$

From this approximation, we get the Stokes equations

$$\mu\nabla^2\mathbf{u} - \nabla p = 0 \tag{4.100}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{4.101}$$

In any approximation we make, we must ensure that the pressure gradient balances the remaining terms and that the continuity equation is not approximated in any way.

4.11 Fluid mechanics and Buckingham-Pi

Although the Navier-Stokes equation describes any fluid flow, it is incredibly difficult to solve. Often, approximations are made. Two such approximations are the Stokes' equation that is obtained for small Reynolds numbers, and the Euler equation which is a good approximation for large Reynolds numbers.

Other types of approximations exist that leads to boundary layer flows and thin film flows. As interesting as these flows are, we will not discuss them in this course.

In order to obtain a rough idea about the nature of a fluid flow, Buckingham-Pi theory is used as a starting point.

Exercise

Show that the velocity vector $\mathbf{u} = (\mathrm{E}x, -\mathrm{E}y)$ satifies the Euler equations in $y > 0$ where E is a constant.

(i) What is the pressure field $p(x, y)$

(ii) What is the velocity at $y = 0$? Does this make sense physically if the plane at $y = 0$ is a rigid boundary

5 Perturbation methods

The majority of equations arising in mathematical modelling do not have exact solutions and one may then be forced to turn to numerical solutions. However, another approach to solving mathematical models is to look for an approximate analytical solution. Of course, we do not want the solution to be too approximate, so in some sense, we are looking for a precise approximation. By this, we mean an approximation with an error that is understood and controllable.

In this section, we will look at perturbation methods. These methods are useful and are based on approximations where some parameter of the differential equation is small and we may look for a solution close to a known solution.

For the low Reynolds flow example in the previous section, we can form a small parameter by setting $Re = \epsilon$. Perturbation methods look at successive approximations to the equation.

5.1 Regular and singular perturbation problems

Perturbation solutions can be loosely separated into two classes, regular or singular perturbations. If the solution is based on a small parameter, say ϵ, then a singular perturbation occurs when the solution differs in the limit $\epsilon = 0$ to the solution obtained along the approach $\epsilon \to 0$. Problems that are not singular are regular.

When a small parameter ϵ occurs naturally in a differential equation as in the example for low Reynolds number flows, a solution in the form of an expansion in powers of ϵ can be considered.

5.1.1 Regular perturbation

An algebraic example

We begin with the simple example of a quadratic equation

$$x^2 + \epsilon x - 1 = 0, \tag{5.1}$$

where ϵ is small. The exact solution to (5.1) is

$$x = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2} \tag{5.2}$$

Using the binomial theorem, we can expand this solution for small ϵ to give

$$x \approx \begin{cases} 1 - \epsilon/2 + \epsilon^2/8 - \epsilon^4/128 + \dots \\ -1 - \epsilon/2 - \epsilon^2/8 + \epsilon^4/128 + \dots \end{cases} \tag{5.3}$$

Now, since ϵ is small, it suggests that the term ϵx does not have a great effect on the solution. Neglecting ϵx in (5.1), we find $x \approx \pm 1$, with expected errors of $\mathcal{O}(\epsilon)$. We can improve on this by looking for a series solution of the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{5.4}$$

where x_0, the zeroth-order term, represents the leading order solution (dominant behaviour), x_1 is the first order term, x_2 the second order term, etc. Substituting the series (5.4) into equation (5.1) gives

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \dots)^2 + \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \dots) - 1 = 0. \tag{5.5}$$

Equating powers of ϵ leads to

$$\epsilon^0 : x_0^2 = 1 \Rightarrow x_0 = \pm 1 \tag{5.6}$$

$$\epsilon^1 : 2x_0 x_1 + x_0 = 0 \Rightarrow x_1 = -\frac{1}{2} \tag{5.7}$$

$$\epsilon^2 : 2x_0 x_2 + x_1^2 + x_1 = 0 \Rightarrow x_2 = \pm\frac{1}{8} \tag{5.8}$$

$$\epsilon^3 : 2x_0 x_3 + 2x_1 x_2 + x_2 = 0 \Rightarrow x_3 = 0 \tag{5.9}$$

That is,

$$x = \pm 1 - \epsilon/2 \pm \epsilon^2/8 + \dots \tag{5.10}$$

which as expected is the series expansion of (5.2).

So, we can reproduce the approximate solution by looking for power series (or perturbation series).

5.2 Ordinary differential equations

In general we would not use perturbation methods to solve quadratic equations, so we now consider a differential equation with a small parameter

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \epsilon u = x, \quad u(0) = 1. \tag{5.11}$$

This is a linear differential equation with constant coefficients. The exact solution is

$$u = \left(1 + \frac{1}{\epsilon^2}\right) e^{-\epsilon x} + \frac{x}{\epsilon} - \frac{1}{\epsilon^2} \tag{5.12}$$

If we expand the exponential for small ϵx, we can find an approximate solution

$$u \quad \approx \left(1 + \frac{1}{\epsilon^2}\right)\left(1 - \epsilon x + \epsilon^2 \frac{x^2}{2} - \epsilon^3 \frac{x^3}{6} \dots\right) + \frac{x}{\epsilon} - \frac{1}{\epsilon^2} \tag{5.13}$$

$$= 1 + \frac{x^2}{2} - \epsilon\left(x + \frac{x^3}{6}\right) + \mathcal{O}\left(\epsilon^2\right) \tag{5.14}$$

Alternatively, we could look for a power series solution from the start. Say

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 \dots .$$

Then, (5.11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u_0 + \epsilon u_1 + \epsilon^2 u_2 \dots\right) + \epsilon(u_0 + \epsilon u_1 + \epsilon^2 u_2 \dots) = x \tag{5.15}$$

We must also always look at how the series acts on the boundary conditions. The boundary condition becomes

$$u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) \dots = 1 \tag{5.16}$$

Equating coefficients of ϵ in the boundary condition gives

$$u_0(0) = 1 \qquad u_1(0) = u_2(0) = \dots = u_n(0) = 0 \tag{5.17}$$

Equating coefficients of ϵ in the govening equation and applying the boundary conditions gives

$$\epsilon^0 : \frac{\mathrm{d}u_0}{\mathrm{d}x} = x \quad \Rightarrow u_0 = \frac{x^2}{2} + 1 \tag{5.18}$$

$$\epsilon^1 : \frac{\mathrm{d}u_1}{\mathrm{d}x} + u_0 = 0 \Rightarrow u_1 = -x - \frac{x^3}{6} \tag{5.19}$$

and so on. The asymptotic solutionis therefore

$$u = 1 + \frac{x^2}{2} - \epsilon\left(x + \frac{x^3}{6}\right) + \mathcal{O}(\epsilon^2), \tag{5.20}$$

which is the small ϵ expansion of the exact solution. Figure 21 shows a comparison of the exact solution with the perturbation solution up to $\mathcal{O}(\epsilon)$. In this case $\epsilon = 0.1$ and we expect errors of 10% at leading order. At first order, we neglect the terms of $\mathcal{O}(\epsilon^2)$ and therefore expect errors of 1%. From the figure, it is clear that the first order solution is more accurate than the leading order, and the accuracy could be improved by looking for higher order corrections. The accuracy will also improve as ϵ decreases, as shown in Figure 22. Infact, we see that the

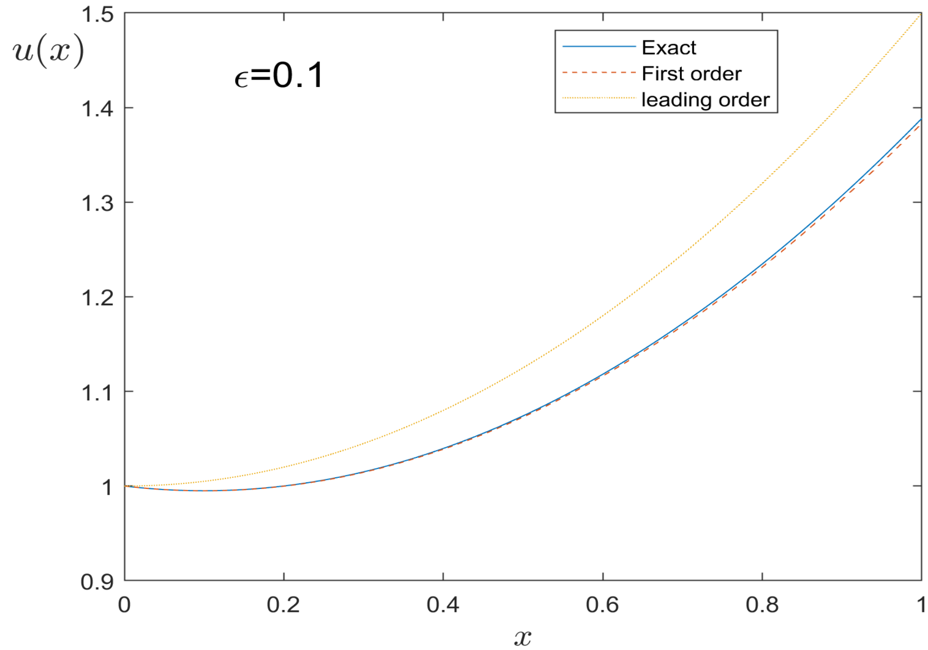


Figure 21: comparison of exact , leading order and first order solutions for $\epsilon = 0.1$

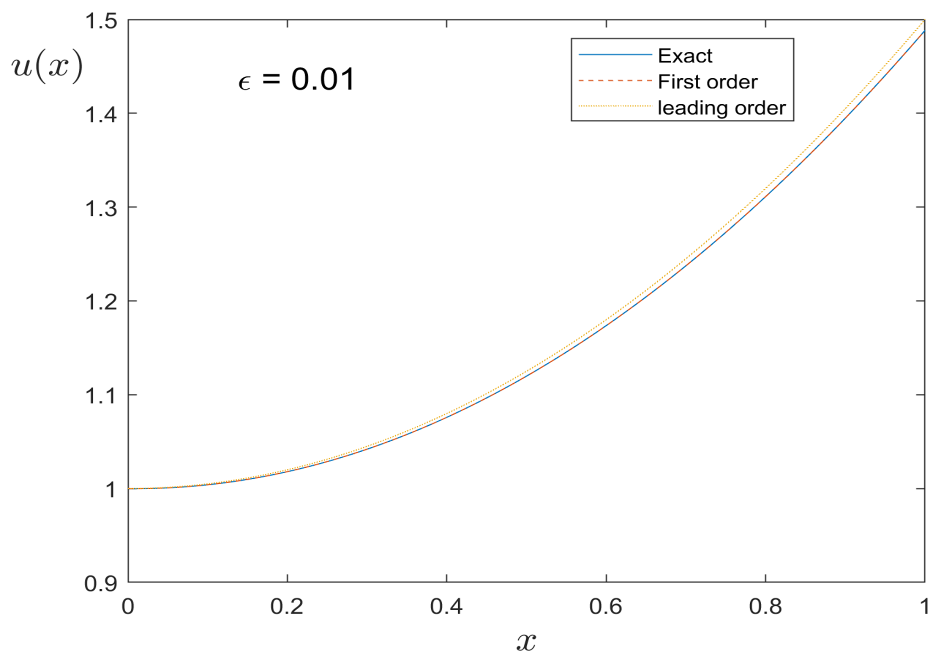


Figure 22: comparison of exact , leading order and first order solutions for $\epsilon = 0.01$

leading order term given by the quadratic function $1 + x^2/2$ will be a good approximation to the exact when ϵ is very small, that is, $\epsilon \ll 1$.Note, when $\epsilon = 0$ the equation is $\frac{du}{dx} = x$ and so $u = x2/2 + 1$. This coincides with the perturbation solution as ϵ and so this is a regular perturbation.

As another example, we consider an initial value problem for second order ordinary differential equations of the form

$$u''(t) + \omega_0^2 u(t) = \epsilon f(u(t), u'(t)), \quad 0 < \epsilon \ll 1, \tag{5.21}$$

$$u(0) = a, \quad u'(0) = b, \tag{5.22}$$

where ω_0, a and b are given constants.

We look for a solution of the form

$$u(t;\epsilon) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + ... + \epsilon^n u_n(t) + O(\epsilon^{n+1}). \tag{5.23}$$

or simply

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + ... + \epsilon^n u_n + O(\epsilon^{n+1}). \tag{5.24}$$

As $\epsilon \to 0$, equation (5.24) is called a straightforward perturbation expansion and ϵ is called the perturbation parameter.

Note: $u_0(t)$ is the solution to (5.21) and (5.22) when $\epsilon = 0$. In other words, $u_0(t)$ solves

$$u'' + \omega_0^2 u = 0, \tag{5.25}$$

$$u(0) = a, \quad u'(0) = b. \tag{5.26a-b}$$

Equation (5.25) is simply the equation of simple harmonic motion. It's general solution is

$$u_0(t) = A\cos(\omega_0 t + \phi), \quad A, \phi \in \mathbb{R} \tag{5.27}$$

or

$$u_0(t) = P\cos(\omega_0 t) + Q\sin(\omega_0 t), \quad P, Q \in \mathbb{R} \tag{5.28}$$

We will use the form (5.28). Applying boundary condition (5.26a) on (5.27) gives:

$$P = a \tag{5.29}$$

Differentiate (5.27) and apply boundary condition (5.26b) yields

$$Q = \frac{b}{\omega_0} \tag{5.30}$$

Thus, the solution which satisfies the boundary condition (5.26) is

$$u_0(t) = a\cos(\omega_0 t) + (b/\omega_0)\sin(\omega_0 t).$$

5.2.1 Example 1

Consider the initial value problem (IVP)

$$u''(t) + u(t) = -2\epsilon u(t) - \epsilon^2 u(t), \qquad 0 < \epsilon \ll 1, \tag{5.31}$$

$$u(0) = a, \quad u'(0) = 0. \tag{5.32}$$

i) Find the exact solution.
ii) Show that

$$u(t;\epsilon) = a\cos(t) - \epsilon a t \sin(t) - \frac{1}{2}\epsilon^2 a t^2 \cos(t) + O(\epsilon^3). \tag{5.33}$$

Solution
i) Rewrite the equation

$$u''(t) + u + 2\epsilon u + \epsilon^2 u = 0, \tag{5.34}$$

as

$$u''(t) + (1+\epsilon)^2 u = 0. \tag{5.35}$$

The general solution is

$$u(t) = A\cos[(1+\epsilon)t] + B\sin[(1+\epsilon)t].$$

The initial conditions give $A = a$ and $B = 0$. Therefore,

$$u(t;\epsilon) = a\cos[(1+\epsilon)t]. \tag{5.36}$$

ii) To determine the perturbation solution, we write

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots. \tag{5.37}$$

Substituting into the governing equation (5.31) gives

$$[u_0'' + \epsilon u_1'' + \epsilon^2 u_2'' + \dots] + (1 + 2\epsilon + \epsilon^2)(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots) = 0 \tag{5.38}$$

Substituting the perturbation expansion (5.37) into the initial conditions (5.32) leads to

$$\begin{aligned} u(0) &= u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + \dots = a, \\ u'(0) &= u_0'(0) + \epsilon u_1'(0) + \epsilon^2 u_2'(0) + \dots = 0. \end{aligned} \tag{5.39}$$

Thus,

$$u_0(0) = a \text{ and } u_1(0) = u_2(0) = \dots = 0,$$

$$u_0'(0) = 0 \text{ and } u_1'(0) = u_2'(0) = \ldots = 0.$$

Now, equating the coefficients of ϵ in equation (5.38) and in the initial condition (5.39) yield

$$\epsilon^0 : \qquad \frac{\mathrm{d}^2 u_0}{\mathrm{d}t^2} + u_0 = 0, \quad u_0(0) = a, \quad \frac{\mathrm{d}u_0}{\mathrm{d}t}(0) = 0. \tag{5.40}$$

$$\epsilon^1 : \qquad \frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + u_1 = -2u_0, \quad u_1(0) = 0, \quad \frac{\mathrm{d}u_1}{\mathrm{d}t}(0) = 0. \tag{5.41}$$

$$\epsilon^2 : \qquad \frac{\mathrm{d}^2 u_2}{\mathrm{d}t^2} + u_2 = -2u_1 - u_0, \quad u_2(0) = 0, \quad \frac{\mathrm{d}u_2}{\mathrm{d}t}(0) = 0 \tag{5.42}$$

and so on for higher powers of ϵ.

Solving for u_0 in (5.40) yields

$$u_0(t) = a\cos(t).$$

This expression for u_0 is substituted into the second order ordinary differential equation in (5.41). We now solve for u_1 in the equation

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + u_1 = -2a\cos(t), \quad u_1(0) = 0, \quad \frac{\mathrm{d}u_1}{\mathrm{d}t}(0) = 0 \tag{5.43}$$

We break this up into finding a homogeneous solution and a particular solution.

Homogeneous solution:

$$u_{1_h} = A\cos(t) + B\sin(t).$$

At this stage we DO NOT use the initial conditions to solve for the unknown constants. Why?

Particular solution:
Since the right hand side of the inhomogeneous equation in (5.48) is a special case of the homogeneous solution, the particular solution must be of the form

$$u_{1_p} = t\left(C\cos(t) + D\sin(t)\right). \tag{5.44}$$

By substituting (5.44) into the non-homogeneous equation in (5.48), show that a particular solution is given by

$$u_{1p} = -at\sin(t). \tag{5.45}$$

The general solution is then

$$u_1 = A\cos(t) + B\sin(t) - at\sin(t). \tag{5.46}$$

Using the initial conditions in (5.48), we obtain $A = 0$ and $B = 0$.

Therefore,

$$u_1 = -at\sin(t). \tag{5.47}$$

Solving for u_2: We also break this up into finding a homogeneous solution and a particular solution. Notice that the form of the homogeneous solution is the same for all the cases.

This expressions for u_0 and u_1 are substituted into the second order ordinary differential equation in (5.42) and we have to solve for u_2 in the initial value problem

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}t^2} + u_2 = 2at\sin t - a\cos(t), \quad u_2(0) = 0, \quad \frac{\mathrm{d}u_2}{\mathrm{d}t}(0) = 0 \tag{5.48}$$

The homogeneous solution is

$$u_{2_h} = A\cos t + B\sin t.$$

For particular solution, try a particular solution of the form

$$u_{2_p} = t^2\left(A\cos t + B\sin t\right) + t\left(P\cos t + Q\sin t\right) \tag{5.49}$$

Exercise: Substitute into the governing equation to show that $B = P = Q = 0$ and $A = -a/2$ so that

$$u_{2_p} = -\frac{at^2}{2}\cos(t). \tag{5.50}$$

Alternatively, rewrite the ode in (5.48) as

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}t^2} + u_2 = 2ate^{it} - aie^{it}, \tag{5.51}$$

where it should be easily noted that the imaginary part of equation (5.51) is the ode in (5.48). Therefore, to get the particular solution to (5.48), we will take the imaginary part of the solution to (5.51).

A particular solution of (5.51) is of the form

$$u_{2_p} = At^2e^{it} + Bte^{it}. \tag{5.52}$$

When substituted into (5.51), we obtain $A = a/2i$ and $B = 0$, so that the particular solution to (5.48) is

$$u_{2_p} = \mathrm{Im}\left(\frac{a}{2i}t^2e^{it}\right) = \mathrm{Im}\left(-\frac{ai}{2}t^2\left(\cos t + i\sin t\right)\right) = -\frac{a}{2}t^2\cos(t) \tag{5.53}$$

The general solution is therefore

$$u_2 = A\cos(t) + B\sin(t) - \frac{a}{2}t^2\cos(t). \tag{5.54}$$

From the initial conditions we get

$$u_2 = -\frac{at^2}{2}\cos(t). \tag{5.55}$$

From the solutions to u_0, u_1 and u_2, the perturbation expansion becomes

$$u(t;\epsilon) = a\cos(t) + \epsilon(-at\sin(t)) + \epsilon^2(-\frac{at^2}{2}\cos(t)) + O(\epsilon^3). \tag{5.56}$$

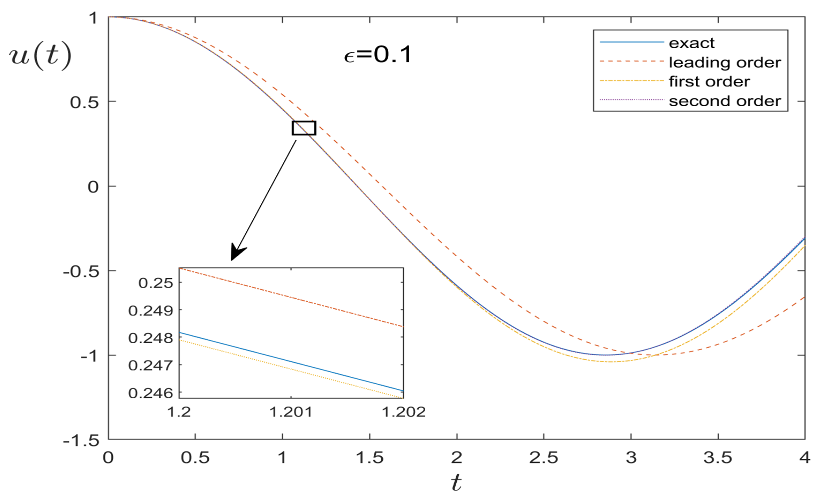


Figure 23: Comparison of exact, leading order, first order and second order solutions for $\epsilon = 0.1$ and $a = 1$.

Figures 23 and 24 show a comparison of the exact solution with the perturbation solution up to order $\mathcal{O}(\epsilon^2)$.

In the case $\epsilon = 0.1$, we expect error of 10% at leading order, 1% at first order and 0.1% at second order since we neglect the $\mathcal{O}(\epsilon)$, $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ terms repectively.

When $\epsilon = 0.01$, however, we expect error of 1% at leading order, 0.1% at first order and 0.01% at second order. In Figure 23, we see that accuracy was improved when higher order terms were plotted, for example, the second order perturbation solution agrees with the exact solution better than the first order, and the first order performed better than the leading order. It is important to note that similar levels of accuracy can be realised when ϵ is decreased. In Figure 24, where $\epsilon = 0.01$, even the leading order term given by $u_0 = a\cos t$, which was far off in Figure 23, performed well enough to qualify as an approximate solution.

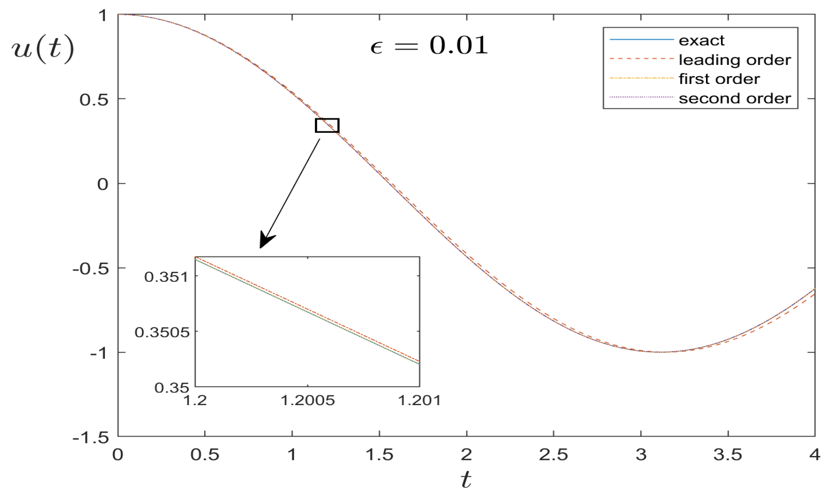


Figure 24: Comparison of exact, leading order, first order and second order solutions for $\epsilon = 0.01$ and $a = 1$

The problem just discussed is a regular perturbation problem since the solution of the unperturbed problem (exact solution at $\epsilon = 0$) agrees with the leading order solution of the perturbed problem (what is left as $\epsilon \to 0$ in the perturbation solution).

Exercise 8

Re-work the example above and find u_3. You may need to use Mathematica.

Exercise 9

Consider

$$\frac{dy}{dx} + y - \epsilon y^2 = 0, \tag{5.57}$$

$$y(0) = a. \tag{5.58}$$

Let

$$y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + O(\epsilon^3). \tag{5.59}$$

Substituting into the governing equation and initial condition gives

Separating by powers of ϵ leads to:
Solving for y_0:

$$\frac{dy_0}{dx} + y_0 = 0, \tag{5.60}$$
$$y_0(0) = a. \tag{5.61}$$

The solution is

Solving for y_1:

$$\frac{dy_1}{dx} + y_1 - y_0^2 = 0, \tag{5.62}$$
$$y_1(0) = 0. \tag{5.63}$$

Using the solution obtained for y_0 gives

$$\frac{dy_1}{dx} + y_1 = a^2 e^{-2x}. \tag{5.64}$$

Homogeneous solution:

Particular solution:

From this we get the general solution to be

$$y_1(x) = Be^{-x} - a^2 e^{-2x}. \tag{5.65}$$

The initial condition gives

$$B = a^2, \tag{5.66}$$

and therefore,

$$y_1 = a^2 e^{-x}(1 - e^{-x}). \tag{5.67}$$

Re-work the above example and solve for y_2.

5.3 The break down of a perturbation expansion

It is important to know when a perturbation expansion breaks down. In a previous example we obtained the expansion

$$u(t;\epsilon) = a\cos(t) + \epsilon(-at\sin(t)) + \epsilon^2(-\frac{at^2}{2}\cos(t)) + O(\epsilon^3). \tag{5.68}$$

Now let us compare this expansion to the exact solution

$$u(t;\epsilon) = a\cos[(1+\epsilon)t], \tag{5.69}$$

to see how well the expansion approximates this solution. Note that the exact solution is periodic and finite. However, the expansion contains secular terms of the form

$$t^n\cos(t), \quad t^m\sin(t) \quad m > 0, \quad n > 0. \tag{5.70}$$

Secular terms are clearly not periodic and they are certainly not finite. What this means is that when we reach larger values for t, the expansion, which is an approximation to the exact solution, will diverge away from it and it will no longer be a good representation of the exact solution.

Perturbation expansions work only if successive terms are significantly smaller than their predecessors. In other words, expansions break down when two consecutive terms have the same order. We use this breaking criteria to find the time t for which the expansion no longer approximates the exact solution. In the example above, consider the two terms

$$a\cos(t), \tag{5.71}$$

and

$$\epsilon(at\sin(t)). \tag{5.72}$$

For each term, we have that

$$|a\cos(t)| \leq |a|, \tag{5.73}$$

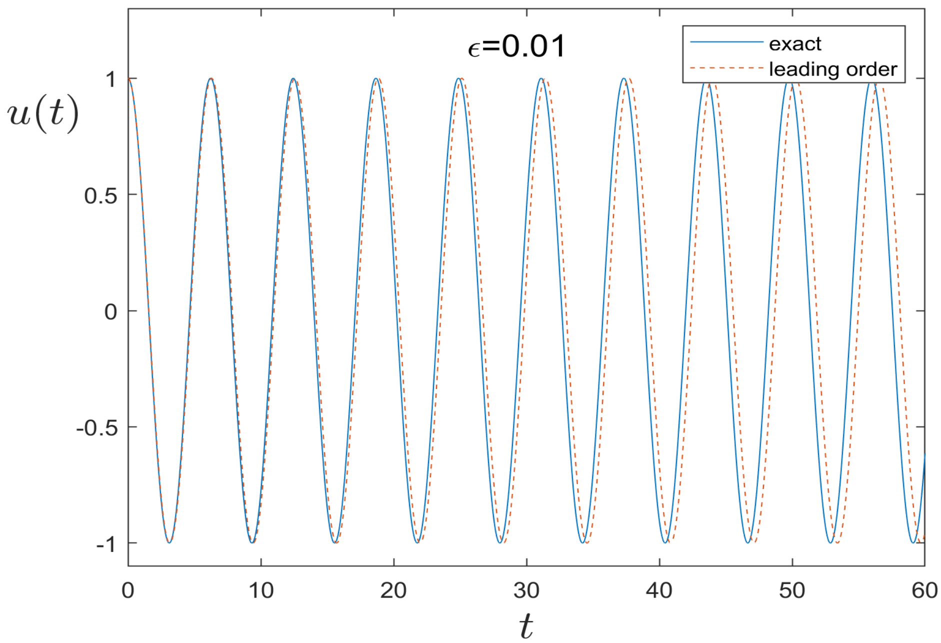


Figure 25: Comparison of the exact and leading order pertubation solution

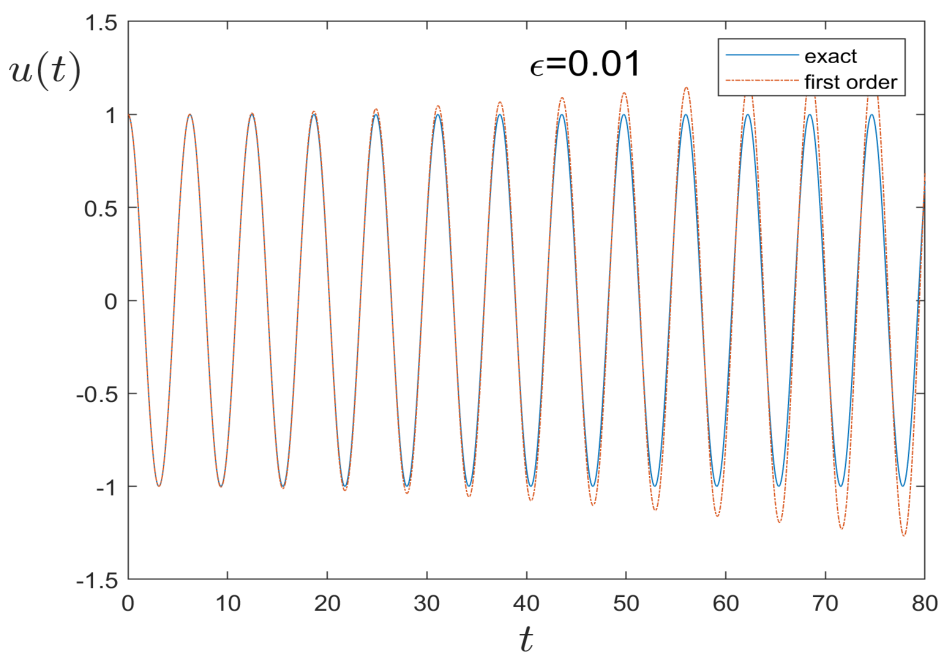


Figure 26: Comparison of the exact and first order pertubation solution

$$|\epsilon(at\sin(t))| \leq \epsilon|a|t. \tag{5.74}$$

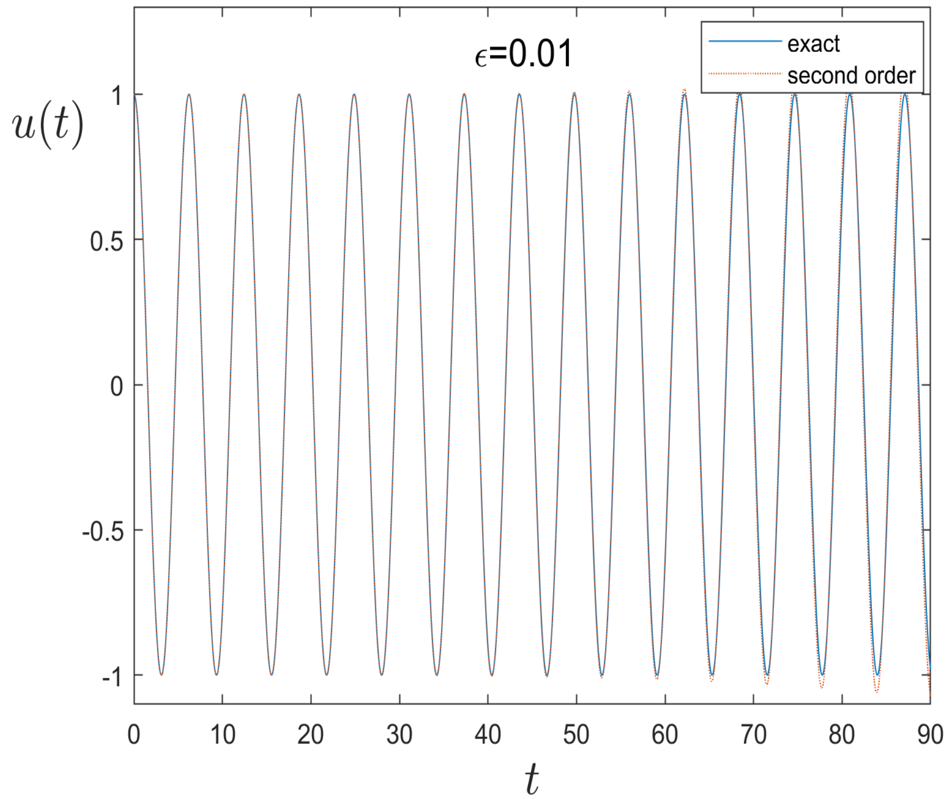


Figure 27: Comparison of the exact and second order pertubation solution

These terms have the same order of magnitude when

$$|a| \approx \epsilon |a| t. \tag{5.75}$$

Solving for t gives

$$t = O(1/\epsilon). \tag{5.76}$$

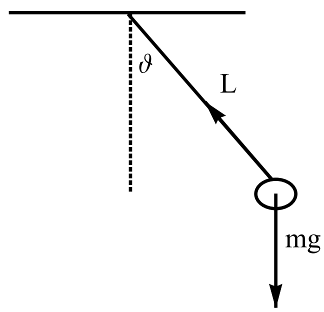
We say that t here is of approximately the same order of magnitude as $1/\epsilon$. Remember ϵ is quite small so $1/\epsilon$ is large. This means that provided t is much less that $1/\epsilon$, the expansion well approximates the exact solution. However, for $t = O(\epsilon)$ and larger, the exact and approximate solutions will diverge. In Figures 25 to 27, a plot is made comparing the exact solution given in (5.69) to the leading order, first order and second order approximate solution given in (5.68). The results on the graphs shows how the aproximate solutions starts to deviate from being good approximations to the exact.

As an exercise, you must plot both the expansion and exact solution on the same set of axes using $\epsilon = 0.001$. Take note of the value on the time axis as to where the two solutions start to diverge.

6 Perturbation theory applications

The damped pendulum

Let T be the tension in the string. For this example, we suppose that $T = mg\cos\theta$ so there is no movement in the radial $\hat{r}$ direction. The only movement will be in the $\hat{\theta}$ direction. Assume that the damping force F_d is proportional to the velocity, where the proportionality constant



is denoted by b.

Let the variable a describe the arc-length. Then

$$F_d = -b\frac{da}{dt} = -bl\frac{d\theta}{dt}, \tag{6.1}$$

$$\begin{aligned} F_{net} &= -bl\frac{d\theta}{dt} - mg\sin\theta & (6.2)\\ &= m\frac{d^2a}{dt^2} \qquad \text{(by NII)} & (6.3)\\ &= ml\frac{d^2\theta}{dt^2}. & (6.4) \end{aligned}$$

Thus, we have

$$ml\frac{d^2\theta}{dt^2} + bl\frac{d\theta}{dt} + mg\sin\theta = 0.$$

For small angles θ, $\sin\theta \approx \theta$ which gives

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0. \tag{6.5}$$

The initial conditions are

$$\theta(0) = \theta_0, \qquad \theta'(0) = 0. \tag{6.6}$$

We are now going to non-dimensionalise. Note that θ is dimensionless, but we can simplify our initial condition, $\theta(0) = \theta_0$.

Exercise

Write equation (6.5) as a system of two first order differential equations.

Let

$$\bar{t} = \frac{t}{A} \qquad \bar{\theta} = \frac{\theta}{B} \tag{6.7}$$

Substituting into the equation and simplifying gives

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{b}{m}A\frac{d\bar{\theta}}{d\bar{t}} + \frac{g}{l}A^2\bar{\theta} = 0, \tag{6.8}$$

and the initial conditions become

$$\bar{\theta}(0) = \frac{\theta_0}{B} \qquad \bar{\theta}'(0) = 0. \tag{6.9}$$

Two obvious choices for B are θ_0 and 1. Take $B = \theta_0$ so the initial condition on $\bar{\theta}$ becomes

$$\bar{\theta}(0) = 1, \tag{6.10}$$

which is simple to deal with.

What are our choices for A? Let's first find the units of all parameters.

- $\left[\dfrac{d^2\theta}{dt^2}\right] = \dfrac{1}{T^2}$
- $\left[\dfrac{b}{m}\dfrac{d\theta}{dt}\right] = [b]\dfrac{1}{MT} \implies [b] = \dfrac{M}{T}$
- $\left[\dfrac{g}{l}\theta\right] = \dfrac{L}{T^2}\dfrac{1}{L} = \dfrac{1}{T^2}$

We can use $\dfrac{m}{b}$ or $\sqrt{\dfrac{l}{g}}$ which also simplify the governing equation. Now let

$$T_1 = \sqrt{\frac{l}{g}}, \tag{6.11}$$

$$T_2 = \frac{m}{b}. \tag{6.12}$$

6.1 Option 1 : $T_1 = \sqrt{\frac{l}{g}}$

Firstly, what is T_1? Consider the simple pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \qquad \text{subject to} \qquad \theta(0) = \theta_0, \quad \theta'(0) = 0. \tag{6.13}$$

The solution is

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right) \tag{6.14}$$

The period ($T \neq 0$) satisfies

$$\theta(t+T) = \theta(t)$$

which implies,

$$\theta_0 \cos\left(\sqrt{\frac{g}{l}}(t+T)\right) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right) \quad \Longrightarrow \quad T = 2\pi\sqrt{\frac{l}{g}} \tag{6.15}$$

Therefore, T_1 is related to the time it takes a simple pendulum to complete one oscillation. Remember, the amplitude of the oscillations change in the damped pendulum.

Now, setting $A = T_1 = \sqrt{\frac{l}{g}}$, the equation for the **damped pendulum** becomes

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{b}{m}\sqrt{\frac{l}{g}}\frac{d\bar{\theta}}{d\bar{t}} + \bar{\theta} = 0,$$

subject to

$$\bar{\theta}(0) = 1 \qquad \bar{\theta}'(0) = 0. \tag{6.16}$$

In terms of T_1 and T_2, we have

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{T_1}{T_2}\frac{d\bar{\theta}}{d\bar{t}} + \bar{\theta} = 0. \tag{6.17}$$

6.1.1 Case A: $T_2 \gg T_1$

For $T_2 \gg T_1$, then $1 \gg \frac{T_1}{T_2}$. Let $\frac{T_1}{T_2} = \epsilon \ll 1$. To zeroth order in ϵ, we have the simple pendulum.

So, we have a perturbation problem

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \epsilon\frac{d\bar{\theta}}{d\bar{t}} + \bar{\theta} = 0 \qquad \text{with} \qquad \bar{\theta}(0) = 1, \quad \bar{\theta}'(0) = 0. \tag{6.18}$$

Consider an expansion of the form,

$$\bar{\theta} = \bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2 + \dots \tag{6.19}$$

Then

$$\left[\frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \epsilon\frac{d^2\bar{\theta}_1}{d\bar{t}^2} + \epsilon^2\frac{d^2\bar{\theta}_2}{d\bar{t}^2}\right] + \epsilon\left[\frac{d\bar{\theta}_0}{d\bar{t}} + \epsilon\frac{d\bar{\theta}_1}{d\bar{t}} + \epsilon^2\frac{d\bar{\theta}_2}{d\bar{t}}\right] + \left[\bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2\right] = 0, \tag{6.20}$$

subject to

$$\left[\bar{\theta}_0(0) + \epsilon\bar{\theta}_1(0) + \epsilon^2\bar{\theta}_2(0)\right] = 1 \qquad \left[\bar{\theta}_0{}'(0) + \epsilon\bar{\theta}_1{}'(0) + \epsilon^2\bar{\theta}_2{}'(0)\right] = 0. \tag{6.21}$$

Now,

$$\epsilon^0 : \qquad \frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \bar{\theta}_0 = 0 \qquad \text{with} \qquad \bar{\theta}_0(0) = 1, \quad \bar{\theta}_0{}'(0) = 0 \tag{6.22}$$

The solution is

$$\bar{\theta}_0(t) = \cos\bar{t}. \tag{6.23}$$

Exercises

1. Find $\bar{\theta}_1$ and $\bar{\theta}_2$ in the expansion (6.19).

2. Find the exact solution to the problem in **Case A**. In other words, find the exact solution to equations (6.18).

3. On the same set of axes, plot:

- the exact solution found in your answer to question 2
- $\bar{\theta}_0$
- $\bar{\theta}_0 + \epsilon\bar{\theta}_1$
- $\bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2$

Use $\epsilon = 0.01$. You will need to play around with the time axis.

4. From your answers to question 3, find $O(\bar{t})$ for which the expansion breaks down. Do your plots reflect this result?

6.1.2 Case B: $T_1 \gg T_2$

In order to obtain a small parameter, take

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{T_1}{T_2}\frac{d\bar{\theta}}{d\bar{t}} + \bar{\theta} = 0,$$

and multiply by $\dfrac{T_2}{T_1}$:

$$\frac{T_2}{T_1}\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \frac{T_2}{T_1}\bar{\theta} = 0.$$

Since $T_1 \gg T_2$, $1 \gg \dfrac{T_2}{T_1}$. Let $\epsilon = \dfrac{T_2}{T_1}$. Then

$$\epsilon\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \epsilon\bar{\theta} = 0. \tag{6.24}$$

Consider an expansion of the form

$$\bar{\theta} = \bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2 + \ldots$$

Then

$$\epsilon\left[\frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \epsilon\frac{d^2\bar{\theta}_1}{d\bar{t}^2}\right] + \frac{d\bar{\theta}_0}{d\bar{t}} + \epsilon\frac{d\bar{\theta}_1}{d\bar{t}} + \epsilon\bar{\theta}_0 + \epsilon^2\bar{\theta}_1 = 0, \tag{6.25}$$

and initial conditions become

$$\bar{\theta}_0(0) + \epsilon\bar{\theta}_1(0) = 1 \qquad \bar{\theta}_0{}'(0) + \epsilon\bar{\theta}_1{}'(0) = 0. \tag{6.26}$$

Now, we have

$$\epsilon^0 : \qquad \frac{d\bar{\theta}}{d\bar{t}} = 0 \quad \text{with} \qquad \bar{\theta}_0(0) = 1, \quad \bar{\theta}_0{}'(0) = 0 \tag{6.27}$$

Solution:

$$\begin{aligned} \bar{\theta}_0(\bar{t}) &= C \\ \bar{\theta}_0(0) &= 1 \implies C = 1 \\ \therefore \bar{\theta}_0(\bar{t}) &= 1 \end{aligned}$$

The condition $\bar{\theta}_0{}'(0) = 0$ is automatically satisfied. Now,

$$\epsilon^1 : \qquad \frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \frac{d\bar{\theta}_1}{d\bar{t}} + \bar{\theta}_0 = 0 \qquad \text{with} \qquad \bar{\theta}_1(0) = 0, \quad \bar{\theta}_1{}'(0) = 0 \tag{6.28}$$

Solution:

$$\frac{d\bar{\theta}_1}{d\bar{t}} + 1 = 0 \implies \bar{\theta}_1 = -\bar{t} + a \tag{6.29}$$

And from $\bar{\theta}_1(0) = 0 \implies a = 0$. But now for $\bar{\theta}_1 = -\bar{t}$, the condition $\bar{\theta}_1{}'(0) = 0$ is not satisfied. The initial conditions are thus violated and we cannot use perturbation methods.

Moral of the Story:

If I tell you the timescale for damping is much greater than the timescale for oscillations, then you need to know that you must use T_1 to non-dimensionalise, if you want to use perturbation methods.

6.2 Option 2: $T_2 = \dfrac{m}{b}$

Firstly, what is T_2? Consider the damped pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0,$$

subject to

$$\theta(0) = \theta_0 \qquad\qquad \theta'(0) = 0.$$

Solution:

$$\lambda^2 + \frac{b}{m}\lambda + \frac{g}{l} = 0$$

$$\begin{aligned}\lambda &= \frac{1}{2}\left[-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{g}{l}}\right] \\ &= -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{g}{l}}\end{aligned}$$

Suppose that $\left(\dfrac{b}{2m}\right)^2 - \dfrac{g}{l} < 0$. Then,

$$\lambda = -\frac{b}{2m} \pm i\sqrt{\frac{g}{l} - \left(\frac{b}{2m}\right)^2} \tag{6.30}$$

We have

$$\theta(t) = e^{-\frac{b}{2m}t}[A\cos\omega t + B\sin\omega t] \tag{6.31}$$

where

$$\omega = \sqrt{\frac{g}{l} - \left(\frac{b}{2m}\right)^2} \tag{6.32}$$

Exercise
Find A and B from the initial conditions.

The factor $\dfrac{b}{m}$ determines how quickly the amplitude $\theta_0 e^{-\frac{b}{2m}t}$, decays. The amplitude is:

$$A(t) = \theta_0 e^{-\frac{b}{2m}t} \tag{6.33}$$

At $t = 0$, $A(0) = \theta_0$. When $t \approx \dfrac{m}{b} = T_2$, then $A(t) \approx \theta_0 e^{-\frac{1}{2}}$ which is a significant decrease in amplitude from the initial value θ_0.

Therefore, T_2 denotes the time it takes for you to notice a significant change in the amplitude of the motion. Now, the equation is

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \left(\frac{T_2}{T_1}\right)^2 \bar{\theta} = 0, \tag{6.34}$$

and initial conditions are

$$\bar{\theta}(0) = 1 \qquad\qquad \bar{\theta}'(0) = 0. \tag{6.35}$$

6.2.1 Case A: $T_1 \gg T_2$

For $T_1 \gg T_2$, then $1 \gg \dfrac{T_2}{T_1}$. Let $\epsilon = \dfrac{T_2}{T_1}$. Then we have a perturbation problem,

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \epsilon^2\bar{\theta} = 0. \tag{6.36}$$

Let $\bar{\theta} = \bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2 + \ldots$ and substituting in gives

$$\frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \epsilon\frac{d^2\bar{\theta}_1}{d\bar{t}^2} + \epsilon^2\frac{d^2\bar{\theta}_2}{d\bar{t}^2} + \frac{d\bar{\theta}_0}{d\bar{t}} + \epsilon\frac{d\bar{\theta}_1}{d\bar{t}} + \epsilon^2\frac{d\bar{\theta}_2}{d\bar{t}} + \epsilon^2(\bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2) = 0 \tag{6.37}$$

So, the *zeroth order*,

$$\epsilon^0 : \qquad \frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \frac{d\bar{\theta}_0}{d\bar{t}} = 0 \qquad \text{with} \qquad \bar{\theta}_0(0) = 1, \quad \bar{\theta}_0{}'(0) = 0 \tag{6.38}$$

Solution:

$$\bar{\theta}_0 = 1 \qquad \forall\ t \tag{6.39}$$

The first solution, $\bar{\theta}_0$, is a pendulum that is situated at its initial position for all times.

Now, *first order*,

$$\epsilon^1: \qquad \frac{d^2\bar{\theta}_1}{d\bar{t}^2} + \frac{d\bar{\theta}_1}{d\bar{t}} = 0 \qquad \text{with} \qquad \bar{\theta}_1(0) = 0, \quad \bar{\theta}_1{}'(0) = 0 \tag{6.40}$$

Solution:

$$\bar{\theta}_1 = 0 \qquad \forall\ t \tag{6.41}$$

Lastly, *second order*,

$$\epsilon^2: \qquad \frac{d^2\bar{\theta}_2}{d\bar{t}^2} + \frac{d\bar{\theta}_2}{d\bar{t}} + \theta_0 = 0 \qquad \text{with} \qquad \bar{\theta}_2(0) = 0, \quad \bar{\theta}_2{}'(0) = 0 \tag{6.42}$$

Solution:

$$\bar{\theta}_2(t) = -t + 1 - e^{-t} \tag{6.43}$$

It is more 'normal' to have a physical pendulum where $T_2 \gg T_1$.

Exercises

1. Find $\bar{\theta}_3$.

2. Find the exact solution to the problem in **Case A**. In other words, find the exact solution to equations (6.36).

3. On the same set of axes, plot:

- the exact solution found in your answer to question 2
- $\bar{\theta}_0$
- $\bar{\theta}_0 + \epsilon\bar{\theta}_1$
- $\bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2$

Use $\epsilon = 0.01$. You will need to play around with the time axis.

4. From your answers to question 3, find $O(\bar{t})$ for which the expansion breaks down. Do your plots reflect this result?

6.2.2 Case B: $T_2 \gg T_1$

Take the equation,

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \left(\frac{T_2}{T_1}\right)^2 \bar{\theta} = 0, \tag{6.44}$$

and multiply by $\frac{T_1^2}{T_2^2}$:

$$\left(\frac{T_1}{T_2}\right)^2 \frac{d^2\bar{\theta}}{d\bar{t}^2} + \left(\frac{T_1}{T_2}\right)^2 \frac{d\bar{\theta}}{d\bar{t}} + \bar{\theta} = 0. \tag{6.45}$$

Let $\epsilon = \frac{T_1}{T_2}$. Then,

$$\epsilon^2 \frac{d^2\bar{\theta}}{d\bar{t}^2} + \epsilon^2 \frac{d\bar{\theta}}{d\bar{t}} + \bar{\theta} = 0. \tag{6.46}$$

Consider the expansion

$$\bar{\theta} = \bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2 + \ldots \tag{6.47}$$

then,

$$\epsilon^2 \left[\frac{d^2\bar{\theta}_0}{d\bar{t}^2} + \epsilon\frac{d^2\bar{\theta}_1}{d\bar{t}^2} + \epsilon^2\frac{d^2\bar{\theta}_2}{d\bar{t}^2}\right] + \epsilon^2 \left[\frac{d\bar{\theta}_0}{d\bar{t}} + \epsilon\frac{d\bar{\theta}_1}{d\bar{t}} + \epsilon^2\frac{d\bar{\theta}_2}{d\bar{t}}\right] + [\bar{\theta}_0 + \epsilon\bar{\theta}_1 + \epsilon^2\bar{\theta}_2] = 0. \tag{6.48}$$

Now,

$$\epsilon^0 : \qquad \bar{\theta}_0 = 0 \tag{6.49}$$

But the initial conditions are $\bar{\theta}_0(0) = 1$ and $\bar{\theta}_0{}'(0) = 0$. The solution, $\bar{\theta}_0(t) = 0$ does not satisfy $\bar{\theta}_0(0) = 1$. Therefore, the initial conditions are violated and we cannot use perturbation methods.

Moral of the Story:

If I tell you that the timescale for oscillations is much greater than the timescale for damping, then use T_2 to non-dimensionalise.

Additional examples

A. Let $w = w(x)$. The function $w(x)$ satisfies the $4th$ order ordinary differential equation

$$\frac{d^4w}{dx^4} + \epsilon\frac{d^2w}{dx^2} = 1, \tag{6.50}$$

subject to

$$w(0) = w(1) = 0, \qquad \frac{d^2w}{dx^2}(0) = \frac{d^2w}{dx^2}(1) = 0, \tag{6.51}$$

where $\epsilon \ll 1$ is a constant.

1. Consider the perturbation expansion

$$w(x) = w_0(x) + \epsilon w_1(x). \tag{6.52}$$

Find the governing equations and boundary conditions for w_0 and w_1.

2. Using your answer in 1, now solve for w_0.

3. Suppose now that $\epsilon \gg 1$. Letting $\epsilon = 1/\sigma$, we can write the equation as

$$\sigma\frac{d^4w}{dx^4} + \frac{d^2w}{dx^2} = \sigma, \tag{6.53}$$

where $\sigma \ll 1$. By considering the perturbation expansion

$$w(x) = w_0(x) + \sigma w_1(x), \tag{6.54}$$

find the solutions for w_0 and w_1 and hence explain why we CANNOT use perturbation methods for this scenario.

B. Repeat the above question with the conditions

$$w(0) = w(1) = \epsilon, \tag{6.55}$$

instead of $w(0) = w(1) = 0$.

6.3 Perturbation methods and the Navier-Stokes equation

Consider the x- and y- components of the steady dimensionless Navier-Stokes equation and the continuity equation:

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right), \tag{6.56}$$

$$u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial y'} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right), \tag{6.57}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0. \tag{6.58}$$

Here, u and v are the dimensionless velocities in the x and y directions respectively, and p is the dimensionless fluid pressure - which is scaled according to the problem at hand. Here we used the scaling for large Reynolds numbers. These equations are obtained when we use the dimensionless variables

$$u' = \frac{u}{U_0}, \quad v' = \frac{v}{U_0}, \quad p' = \frac{p}{\rho U_0^2}, \tag{6.59}$$

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}. \tag{6.60}$$

There are many other scalings that can be used depending on the problem at hand. For example, consider the situation below:

Clearly, it would make more sense to use the scalings

$$x' = \frac{x}{L}, \quad y' = \frac{y}{h}, \tag{6.61}$$

for the independent variables. Now for the velocity u we can still use

$$u' = \frac{u}{U_0}. \tag{6.62}$$

The question is now to find out how we should scale v. Now, the continuity equation must always hold and thus we must never approximate it. So no small parameters must appear in the continuity equation. We can use this to find the scaling for v as follows:

Therefore, we have

$$x' = \frac{x}{L}, \quad y' = \frac{y}{h}, \quad u' = \frac{u}{U_0}, \quad v' = \frac{vL}{hU_0}, \quad p' = \frac{p}{\rho U_0^2}. \tag{6.63}$$

Exercises

1. Suppose that I now tell you that $\frac{h}{L} = \frac{1}{\sqrt{Re}}$. For large Reynolds numbers, find the steady boundary layer equations by approximating the small order terms to zero. Hint: use the scalings in (6.63) and substitute them into the **dimensional** Navier-Stokes and continuity equations.

2. What physical principle does the continuity equation represent?

3. Consider the dimensionless equations (6.56) - (6.58). Let $\frac{1}{Re} = \epsilon^8$, where $\epsilon \ll 1$. Make the further substitutions

$$x' = \epsilon^3 \bar{x}, \quad y' = \epsilon^4 \bar{y}. \tag{6.64}$$

Consider the perturbation expansions

$$u(\bar{x}, \bar{y}) = u_0(\bar{y}) + \epsilon u_1(\bar{x}, \bar{y}) + O(\epsilon^2), \tag{6.65}$$

$$v(\bar{x}, \bar{y}) = \epsilon v_0(\bar{x}, \bar{y}) + \epsilon^2 v_1(\bar{x}, \bar{y}) + O(\epsilon^3) \tag{6.66}$$

$$p(\bar{x}, \bar{y}) = \epsilon p_1(\bar{x}, \bar{y}) + \epsilon^2 p_2(\bar{x}, \bar{y}) + O(\epsilon^3) \tag{6.67}$$

a. By noting that u_0 is a function of $\bar{y}$ only, using the continuity equation, show that $v_0(\bar{x}, \bar{y}) = v_0(\bar{x})$.

b. You are now told that $v_0 = 0$ and $p_1 = 0$. You are also told that u_0 was already solved for previously i.e. it is no longer an arbitrary function. Using (6.64) and by substituting the perturbation expansions into the Navier-Stokes and continuity equations (6.56)-(6.58), show that the governing equations for u_1, v_1 and p_2 are

$$u_0 \frac{\partial u_1}{\partial \bar{x}} + v_1 \frac{du_0}{d\bar{y}} = 0, \tag{6.68}$$

$$\frac{\partial p_2}{\partial \bar{y}} = 0, \tag{6.69}$$

$$\frac{\partial u_1}{\partial \bar{x}} + \frac{\partial v_1}{\partial \bar{y}} = 0. \tag{6.70}$$

c. Find the general solution to the above equations for $\dfrac{du_0}{d\bar{y}} \neq 0$.

Hints:

1) Let

$$u_1(\bar{x}, \bar{y}) = A(\bar{x}) \frac{dB}{d\bar{y}}, \tag{6.71}$$

where A and B are functions to be determined.

2) Find the relationship between B and u_0.

Please email any corrections to A G Fareo at Adewunmi.Fareo@wits.ac.za

I hope you enjoyed the course!