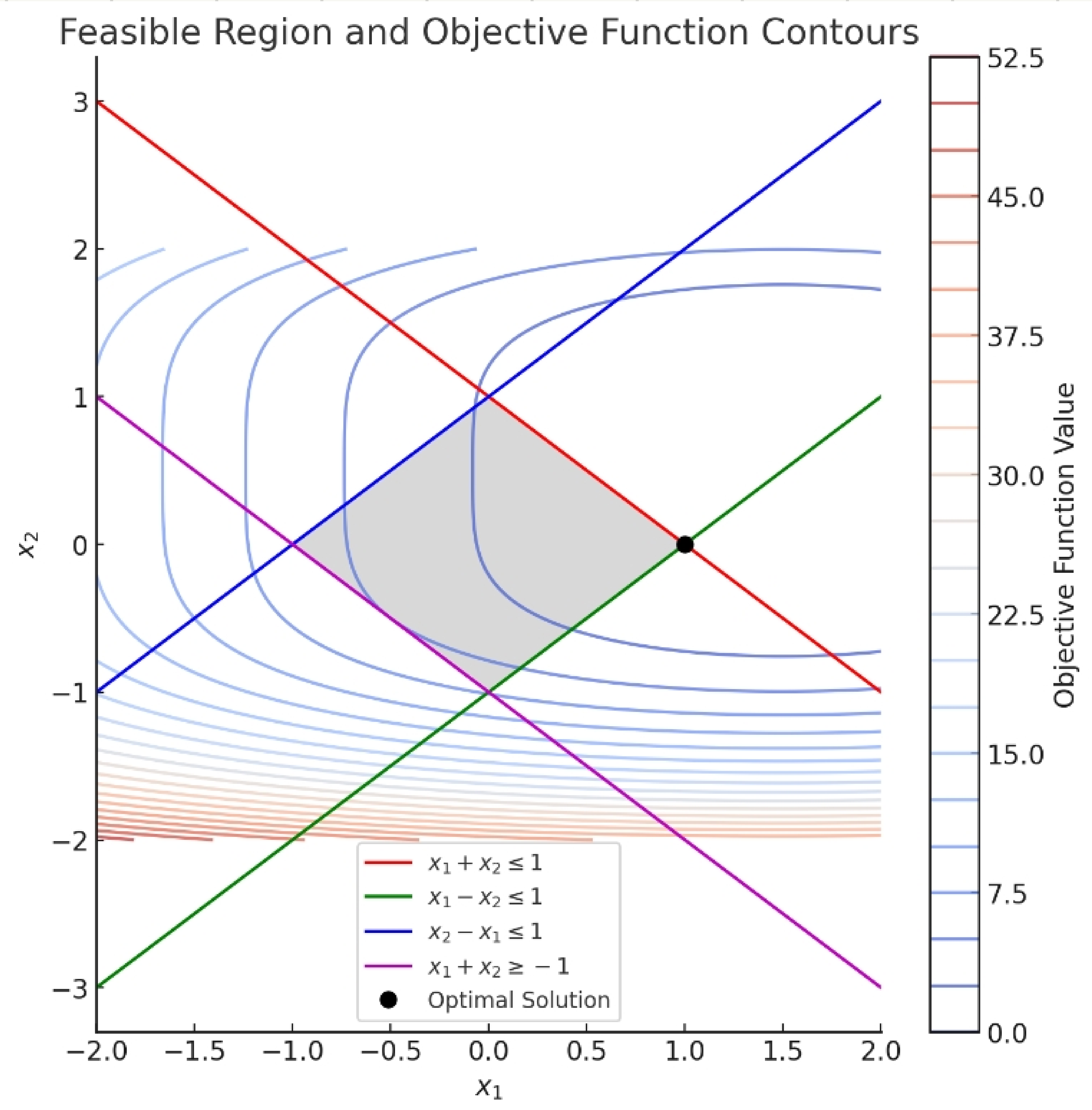
Problem set 1

1a) $\min \; f(x_1, x_2) = \left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^4$

subject to:
$C_1(\bar{x}) = 1 - x_1 - x_2 \geq 0 \Rightarrow \boxed{x_1 + x_2 \leq 1}$
$C_2(\bar{x}) = x_1 - x_2 - 1 \leq 0 \Rightarrow \boxed{x_1 - x_2 \leq 1}$
$C_3(\bar{x}) = x_2 - x_1 - 1 \leq 0 \Rightarrow \boxed{x_2 - x_1 \leq 1}$
$C_4(\bar{x}) = 1 + x_1 + x_2 \geq 0 \Rightarrow \boxed{x_1 + x_2 \geq -1}$



- Very hard to plot such complex contours by hand so I used a computer.

- Optimal solution is $(1;0)$

b) Check LICQ:

$C_1(1,0) = 0 \;\therefore$ Active
$C_2(1,0) = 0 \;\therefore$ Active
$C_3(1,0) = -2 \;\therefore$ Inactive
$C_4(1,0) = 2 \;\therefore$ Inactive

Active constraints:

$C_1(\bar{x}) = 1 - x_1 - x_2 \Rightarrow \nabla C_1(\bar{x}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$C_2(\bar{x}) = x_1 - x_2 - 1 \Rightarrow \nabla C_2(\bar{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\det \left| \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \right| = 2 \;\therefore$ linear independent.

LICQ satisfied.

c) KKT conditions:

1) $\nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = 0$

2) $c_i(\bar{x}^*) = 0 \quad \forall\, i \in E$

3) $c_i(\bar{x}^*) \geq 0 \quad \forall\, i \in I$

4) $\lambda_i^* \geq 0 \quad \forall\, i \in I$

5) $\exists\ \lambda_i^* c_i(\bar{x}^*) \quad \forall\, i \in (I \cup E)$

$$\nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = \nabla f(\bar{x}) + \lambda_1 \nabla c_1 + \lambda_2 \nabla c_2 = 0$$

$$\Rightarrow \nabla f = \begin{bmatrix} 2(x_1 - \frac{3}{2}) \\ 4(x_2 - \frac{1}{2})^3 \end{bmatrix} \Rightarrow \nabla f(1,0) = \begin{bmatrix} -1 \\ -0,5 \end{bmatrix}$$

$$\therefore \nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = \begin{bmatrix} -1 \\ -0,5 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \bar{0}$$

$\Rightarrow \quad -1 - \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 - \lambda_1 = 1$

$\Rightarrow \quad -0,5 - \lambda_1 - \lambda_2 = 0 \Rightarrow -\lambda_1 - \lambda_2 = 0,5$

$\Rightarrow \quad \underline{\lambda_1^* = -0,75 \quad \lambda_2^* = 0,25}$

Saving the Rest for when there is a test to refresh.