Problem set 1

1a) $\min \; f(x_1, x_2) = \left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^4$

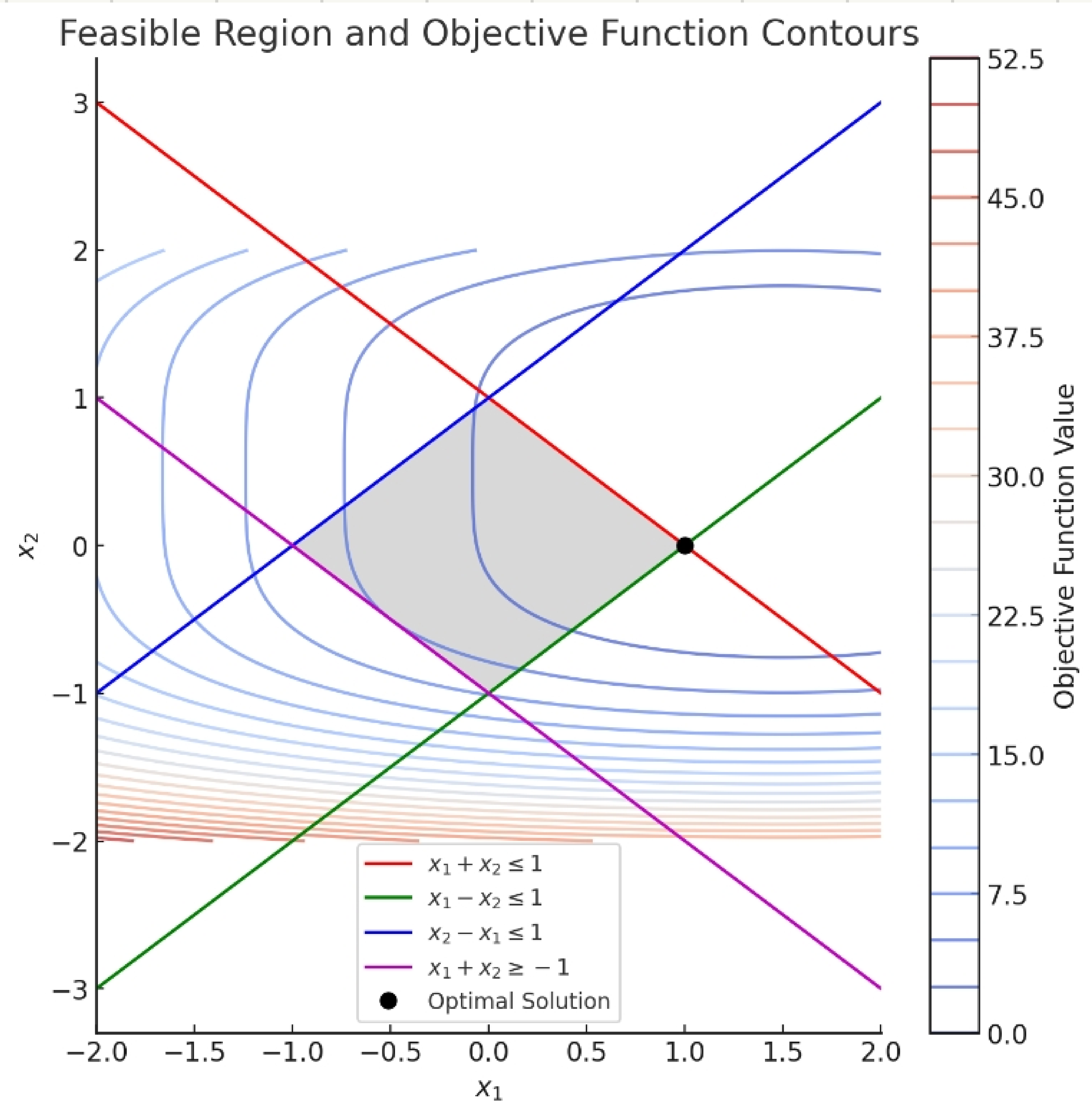
subject to:

$C_1(\bar{x}) = 1 - x_1 - x_2 \geq 0 \Rightarrow x_1 + x_2 \leq 1$

$C_2(\bar{x}) = x_1 - x_2 - 1 \leq 0 \Rightarrow x_1 - x_2 \leq 1$

$C_3(\bar{x}) = x_2 - x_1 - 1 \leq 0 \Rightarrow x_2 - x_1 \leq 1$

$C_4(\bar{x}) = 1 + x_1 + x_2 \geq 0 \Rightarrow x_1 + x_2 \geq -1$



- Very hard to plot such complex contours by hand so I used a computer.

- Optimal solution is $(1;0)$

b) Check LICQ:

$C_1(1,0) = 0 \; \therefore$ Active

$C_2(1,0) = 0 \; \therefore$ Active

$C_3(1,0) = -2 \; \therefore$ Inactive

$C_4(1,0) = 2 \; \therefore$ Inactive

Active constraints:

$C_1(\bar{x}) = 1 - x_1 - x_2 \Rightarrow \nabla C_1(\bar{x}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$C_2(\bar{x}) = x_1 - x_2 - 1 \Rightarrow \nabla C_2(\bar{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\det \left| \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \right| = 2 \; \therefore$ linear independent.

LICQ satisfied.

c) KKT conditions:

1) $\nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = 0$

2) $C_i(\bar{x}^*) = 0 \quad \forall\, i \in E$

3) $C_i(\bar{x}^*) \geq 0 \quad \forall\, i \in I$

4) $\lambda_i^* \geq 0 \quad \forall\, i \in I$

5) $\exists\ \lambda_i^* C_i(\bar{x}^*)\ \forall\, i \in (I \cup E)$

$$\nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = \nabla f(\bar{x}) + \lambda_1 \nabla C_1 + \lambda_2 \nabla C_2 = 0$$

$$\Rightarrow \nabla f = \begin{bmatrix} 2(x_1 - \frac{3}{2}) \\ 4(x_2 - \frac{1}{2})^3 \end{bmatrix} \Rightarrow \nabla f(1,0) = \begin{bmatrix} -1 \\ -0{,}5 \end{bmatrix}$$

$$\therefore \nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = \begin{bmatrix} -1 \\ -0{,}5 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \bar{0}$$

$\Rightarrow \quad -1 - \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 - \lambda_1 = 1$

$\Rightarrow \quad -0{,}5 - \lambda_1 - \lambda_2 = 0 \Rightarrow -\lambda_1 - \lambda_2 = 0{,}5$

$\Rightarrow \quad \underline{\lambda_1^* = -0{,}75 \quad \lambda_2^* = 0{,}25}$

2) $c_1(\bar{x}) = 4x_2^2 + x_1^2 - 4 \leq 0$

$c_2(\bar{x}) = (x_1-2)^2 + x_2^2 - 5 \leq 0$

$c_3(\bar{x}) = x_1 \geq 0$

$c_4(\bar{x}) = x_2 \geq 0$

$x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c_1(x^*) = 0$ ∴ Active

$c_2(x^*) = 0$ ∴ Active

$c_3(x^*) = 0$ ∴ Active

$c_4(x^*) = 1$ ∴ non-active.

$\nabla c_1(\bar{x}) = \begin{bmatrix} 2x_1 \\ 8x_2 \end{bmatrix}$ $\quad \nabla c_3(\bar{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\nabla c_2(\bar{x}) = \begin{bmatrix} 2x_1 - 4 \\ 2x_2 \end{bmatrix}$

$\nabla c_1(x^*) = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad \nabla c_2(x^*) = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \quad \nabla c_3(x^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\alpha_1 \begin{bmatrix} 0 \\ 8 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \bar{0} \Rightarrow$ $-4\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_3 = 4\alpha_2$

$8\alpha_1 + 2\alpha_2 = 0 \Rightarrow \alpha_1 = -\frac{1}{4}\alpha_2$

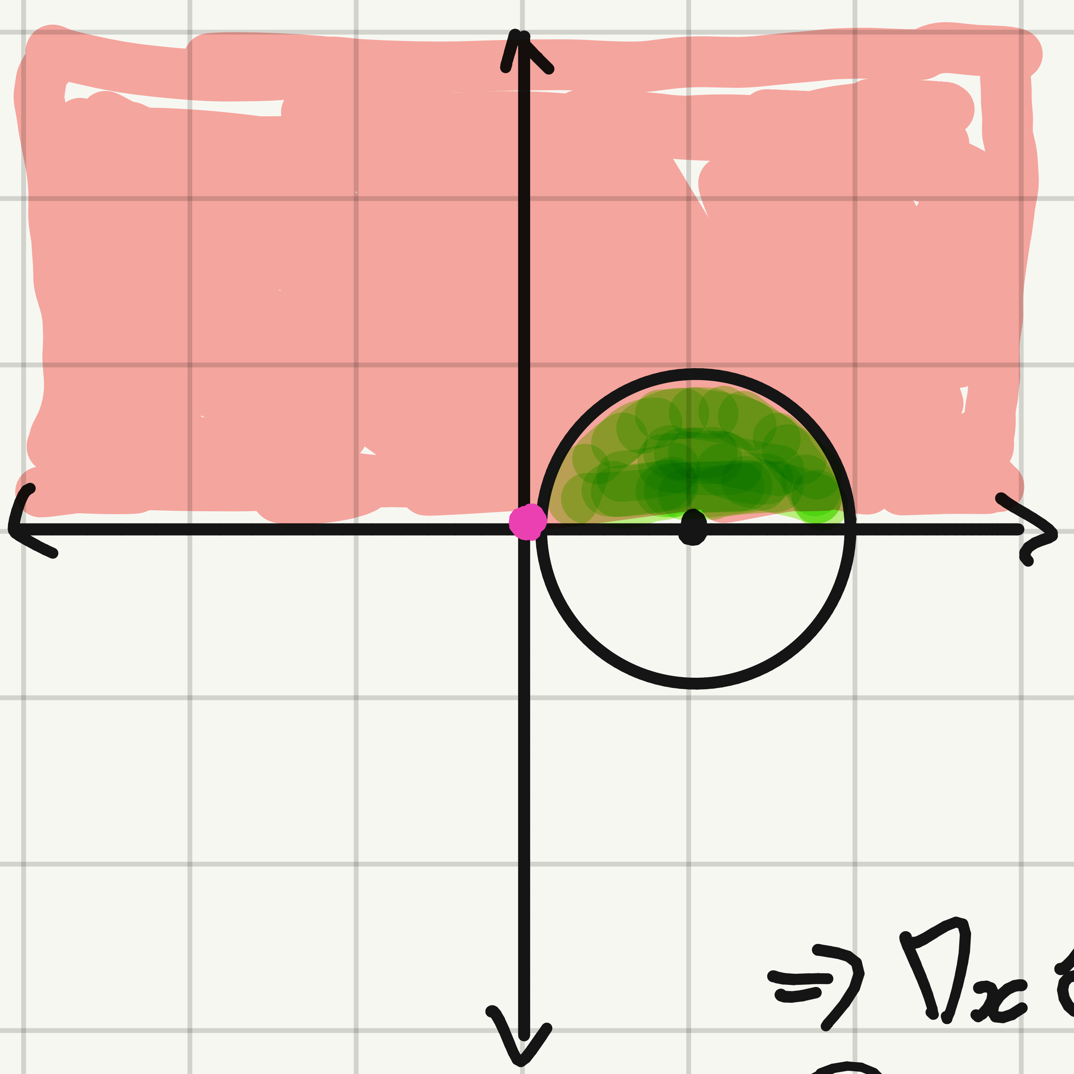
∴ Linearly dependent on each other

LICQ not satisfied.

3)a) min $f(\bar{x}) = x_1$

s.t. $c_1(\bar{x}) = (x_1-1)^2 + x_2^2 - 1 \leq 0$

$c_2(\bar{x}) = x_2 \geq 0$



∴ Optimal point: $(0,0)$

$\nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = \nabla f(\bar{x}) + \lambda_1 \nabla c_1(\bar{x}) + \lambda_2 \nabla c_2(\bar{x}) = 0$

$\Rightarrow \nabla f(\bar{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \nabla c_1(\bar{x}) = \begin{bmatrix} 2x_1 - 2 \\ 2x_2 \end{bmatrix}; \nabla c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \nabla_x \mathcal{L}(\bar{x}^*, \lambda^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

① $c_1(\bar{x}^*) = 0$ ∴ Active

$c_2(\bar{x}^*) = 0$ ∴ Active.

② $\Rightarrow 1 - 2\lambda_1 = 0 \Rightarrow \lambda_1 = \frac{1}{2}$

$\Rightarrow \lambda_2 = 0$

∴ $\lambda_1, \lambda_2 \geq 0$ KKT satsfied.

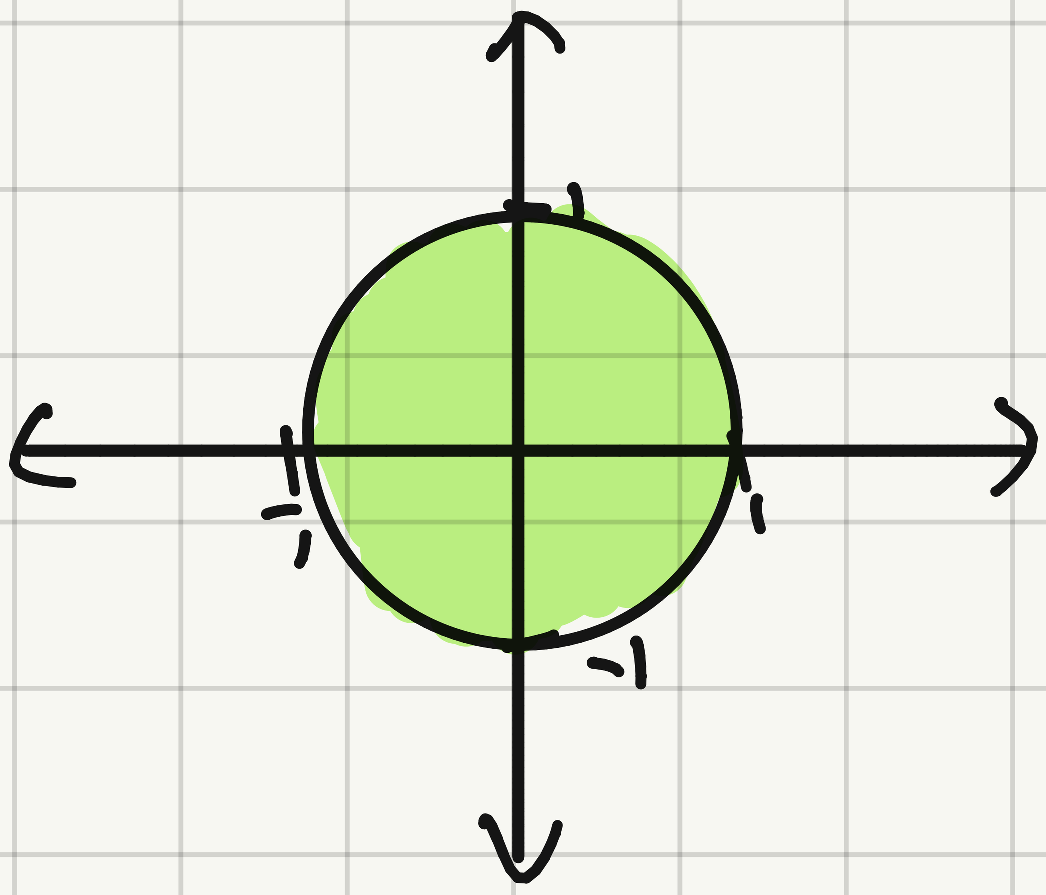
③ $\nabla_x^2 \mathcal{L} = \begin{bmatrix} 2\lambda_1 & 0 \\ 0 & 2\lambda_1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

∴ posite defnate ∴ minum.

3b) $\min f(x) = -0{,}1(x_1-4)^2 + x_2^2$

s.t. $c_1(x) = 1 - x_1^2 - x_2^2 \geq 0$



$$\mathcal{L} = -0{,}1(x_1-4)^2 + x_2^2 + \lambda(1 - x_1^2 - x_2^2)$$

$$\nabla_{x_1}\mathcal{L} = -0{,}2(x_1-4) - 2\lambda x_1 = 0$$
$$\nabla_{x_2}\mathcal{L} = 2x_2 - 2\lambda x_2 = 2x_2(1-\lambda) = 0$$
$$\nabla_\lambda \mathcal{L} = 1 - x_1^2 - x_2^2 = 0$$

① if $x_2 = 0 \Rightarrow x_1 = \pm 1$

if $x_1 = -1 \Rightarrow \lambda = -\frac{1}{2}$ ($\lambda \geq 0$ since inequality constraint)

$\therefore x_1 \neq -1$

if $x_1 = 1 \Rightarrow \lambda = 0{,}3$ (valid)

$\therefore$ Possible solution $(1, 0)$

② if $x_2 \neq 0 \Rightarrow \lambda = 1$

if $\lambda = 1 \Rightarrow x_1 = \frac{4}{11}$

if $x_1 = \frac{4}{11} \Rightarrow x_2 = \pm 0{,}9315$

$\therefore$ Possible solutions $(0{,}3636, \pm 0{,}9315)$

$$H = \begin{bmatrix} -0{,}2-2\lambda & 0 \\ 0 & 2-2\lambda \end{bmatrix}$$

if $\lambda = 1 \Rightarrow H = \begin{bmatrix} -2{,}2 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore$ Not positive definite.

if $\lambda = 0{,}3 \Rightarrow H = \begin{bmatrix} -0{,}8 & 0 \\ 0 & 1{,}4 \end{bmatrix}$ $\therefore$ indefinite.

4) a) min $f(\bar{v}) = (x-1)^2 + (y-2)^2$

s.t. $c_1(\bar{v}) : y - \frac{1}{5}(x-1)^2 = 0$

b) $\mathcal{L} = (x-1)^2 + (y-2)^2 + \lambda\left(y - \frac{1}{5}(x-1)^2\right)$

$\nabla_x \mathcal{L} = 2x - 2 - \frac{2\lambda}{5}x + \frac{2\lambda}{5} = 0 \Rightarrow (2 - \frac{2}{5}\lambda)(x-1) = 0$

$\nabla_y \mathcal{L} = 2y - 2 + \lambda = 0 \Rightarrow$

$\therefore$ if $x = 1 \Rightarrow \lambda = 4$ $\therefore$ ① $(1,0)$ $\lambda = 4$

$\therefore$ if $\lambda = 5 \Rightarrow (x-1)^2 = -\frac{5}{2}$ $\therefore$ Not possible.

$\nabla C_1 = \begin{bmatrix} -\frac{2}{5}(x-1) \\ 1 \end{bmatrix} \Rightarrow \nabla C_1(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\therefore$ non-zero and linearly dependent.

c) $(1,0)$ global minimum.

d) $f(x) = (x-1)^2 + \left(\frac{1}{25}(x-1)^4 - 4\frac{1}{5}(x-1)^2 + 4\right)$

taking the derivative and setting it to 0 will give a different solution.

~~5 and 6 for test prep.~~