A Quick Guide to Linear Algebra

Anne Jorstad, Math 246 Fall 2007

1. Linear Independence

The vectors $\vec{v_1}, ..., \vec{v_k}$ are linearly independent if and only if the only solution to

$$\vec{0} = a_1\vec{v_1} + ... + a_k\vec{v_k}$$

is $a_1 = ... = a_k = 0$.
In other words, when no vector $\vec{v_i}$ is a linear combination of any of the other vectors.

If you have n linearly independent vectors $\vec{v_i}$ each with n components, then they form a *basis* for all vectors of length n. This means that any vector $\vec{u}$ of n components can be represented as a weighted sum of these n vectors:

$$\vec{u} = a_1\vec{v_1} + ... + a_n\vec{v_n}, \text{ for some set of constants } a_1, ..., a_n.$$

We also say that these vectors *span* the space of vectors with n components.

2. Matrices

A matrix is an $m \times n$ set of numbers, where m is the number of rows and n is the number of columns. They are useful.

3. Matrix Multiplication

You can multiply two matrices when the number of columns in the first matrix is equal to the number of rows in the second matrix. If matrix A is $m \times n$ and matrix B is $n \times p$, then $A * B$ is a new matrix C of size $m \times p$, where each element C_{ij} of C is found by taking the dot product of row i of A with column j of B (i = # rows down, j = # rows across):

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \end{pmatrix}$$

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matrix dimensions: $(2 \times 4) * (4 \times 3) = (2 \times 3)$

In general, $AB \neq BA$.

3a. I**, The Identity Matrix**

The matrix I is the *identity matrix*, which is the square matrix of any size with 1's on its main diagonal, and 0's everywhere else, and $AI = IA = A$ for all matrices A.

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Matrix Inversion

Given a square matrix A, if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, then A is invertible, and its inverse is A^{-1}.

A matrix is invertible exactly when all its rows (equivalently, all its columns) are linearly independent. The inverse of an invertible matrix is unique.

Formula for the inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To find the inverse of a larger matrix by hand, use row reduction to convert $[A|I]$ to $[I|A^{-1}]$. In practice, use a calculator.

A matrix is invertible if its determinant $\neq 0$.
A matrix is invertible if all its eigenvalues $\neq 0$.

5. Solving a Linear System of Equations

The easiest way to solve a system of equations (n equations, n unknowns) is with matrices. The following are equivalent:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \quad \Rightarrow \quad \vec{x} = A^{-1}\vec{b}$$

This can be easily solved with any graphing calculator!

6. Determinants

The determinant is a number that helps describe a matrix. Written det(A) or $|A|$.

$$2 \times 2: \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$3 \times 3: \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

and so on, recursively finding determinites of submatrices of the larger matrix.

Matrix A is invertible if and only if det(A) $\neq 0$.

7. The Wronskian

Given two functions $y_1(x)$ and $y_2(x)$, the Wronskian is defined as:

$$W(x) = det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2 \ .$$

In general,

$$W(x) = det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}.$$

The functions $y_1(x), ..., y_n(x)$ are linearly independent when $W(x) \neq 0$.

8. Eigenvalues and Eigenvectors

Matrix operations are often described with the help of eigenvalues and eigenvectors. For square matrix A, scalar λ, and nonzero vector $\vec{v}$, if

$$A\vec{v} = \lambda\vec{v},$$

then λ is an eigenvalue of A, and $\vec{v}$ is its corresponding eigenvector. An eigenvector is unique only up to a scalar multiple.

In general, there will be n not necessarily distinct λs for an $n \times n$ matrix.

Matrix A is invertible if and only if all these $\lambda \neq 0$.