BrainSync in case of exact equality of sample correlations, or white noise

Let $X_{T\times V}$ and $Y_{T\times V}$ be the normalized data sets. In general $V \gg T$.

Let us assume that both X and Y have identical spatial correlation structure. Therefore, $X^TX = Y^TY$. Let us consider SVD decomposition of X and Y.

$X = U_X\Sigma_X V_X^T$ and $Y = U_Y\Sigma_Y V_Y^T$. We can choose U, Σ and V matrices to be real.

$X^TX = Y^TY = V_X\Sigma_X^2 V_X^T = V_Y\Sigma_Y^2 V_Y^T$.

Since this decomposition is unique if there are no repeated singular values, we can say $\Sigma_X = \Sigma_Y$ and $V_X = V_Y$. Even in case of repeated singular values we can choose $V_X = V_Y$. So removing subscripts, we have

$X = U_X\Sigma V^T$ and $Y = U_Y\Sigma V^T$.

What is the cross covarince matrix?

$XY^T = U_X\Sigma V^TV\Sigma U_Y^T = U_X\Sigma^2U_Y^T$.

The optimal rotation given by BrainSync is $O = U_XU_Y^T$.

$X = OY = U_XU_Y^TU_Y\Sigma V = U_X\Sigma V$.

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We have proven: X and Y have same spatial correlation $\implies$ $\Sigma_X = \Sigma_Y$ and $V_X = V_Y$. The proof in other direction is easy by substitution. Therefore we have proven the following claim.

Claim. $XX^T = YY^T \iff \Sigma_X = \Sigma_Y$ *and* $V_X = V_Y$.

(1) Similar analysis can be done by ICA.
(2) All the information about XX^T is in Σ_X and V_X. So for brain network analysis in the resting state, we should just keep those two.
(3) For the task data spatial basis and singular values represent brain networks. Does the temporal basis represent driving processes such as thoughts or motor actions? It would be interesting to check correlation of spatial basis and block design process in task data. The basis that has highest correlation can be kept and rest could be nulled out. This will give us brain data that is driven by the task and remove the one that is driven by physiological processes.
(4) *In noisy case, wlog, we can say that* $XX^T = YY^T + D$ *where* D *is some diagonal matrix. We can see that the orthogonal matrix we get is still the same* $O = U_XU_Y^T$, $\Sigma_X = \Sigma_Y + \sqrt{D}$, $V_X = V_Y$.

Choosing an Atlas. Let's consider 3 subjects, with $XX^T = YY^T = ZZ^T$, and $Z = U_Z\Sigma V$ as above. The orthogonal matrix O_{XY} that synchronizes X to Y is given by $O_{XY} = U_XU_Y^T$. Also, the orthogonal matrix O_{YZ} that synchronizes Y to Z is given by $O_{YZ} = U_YU_Z^T$.

The orthogonal matrix O_{XZ} that synchronizes X to Z is given by $O_{XZ} = U_XU_Z^T = U_XU_Y^TU_YU_Z^T = O_{XY}O_{YZ}$.

The transitivity property $\boxed{O_{XZ} = O_{XY}O_{YZ}}$ implies that, if we synchronize multiple subjects to one single representative subject, then it is equivalent to synchronizing them to each other, and then synchronizing them to the representative subject with a common transform.

This still holds true in case of white noise.

Therefore, for the case of exact equality of sample correlation, or white noise, choosing one subject as 'atlas' and aligning all the subjects to that atlas minimizes the joint cost function of pairwise synchronizations.