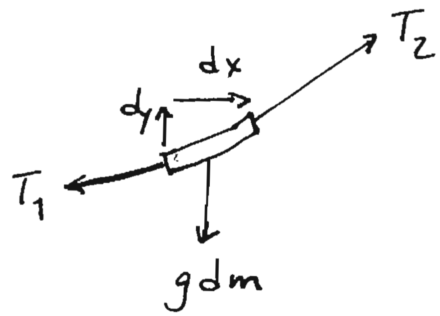
Equilibrium Shape of a Uniform Free-Hanging Chain

1. uniform chain

$$dm = \lambda\sqrt{(dx)^2 + (dy^2)}$$



2. static equilibrium

a) $T_{1x} = T_{2x}$

b) $T_{2y} = T_{1y} + g\,dm$

3. chain can only feel tension forces from other parts of chain

$$\frac{T_{1y}}{T_{1x}} = \frac{dy}{dx}$$

From (2) the x-component of the tension is the same everywhere, call it $T_0 \equiv T_{1x} = T_{2x}$

$$\frac{dT_y}{dx} = g\frac{dm}{dx} = g\lambda\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{and} \quad \frac{dT_y}{dx} = \frac{d}{dx}\left(T_0\frac{dy}{dx}\right)$$

$$\therefore \frac{dT_y}{dx} = T_0\frac{d^2y}{dx^2} = g\lambda\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad : \quad \text{let } y' \equiv \frac{dy}{dx}$$

$$\frac{dy'}{dx} = \frac{g\lambda}{T_0}\sqrt{1+y'^2} \quad : \quad \text{define } a \equiv \frac{g\lambda}{T_0}$$

Solution is $y' = \sinh(a[x - x_0])$, x_0 a constant of integration

$$y = y_0 + \frac{1}{a}\left(\cosh[a(x-x_0)] - 1\right)$$

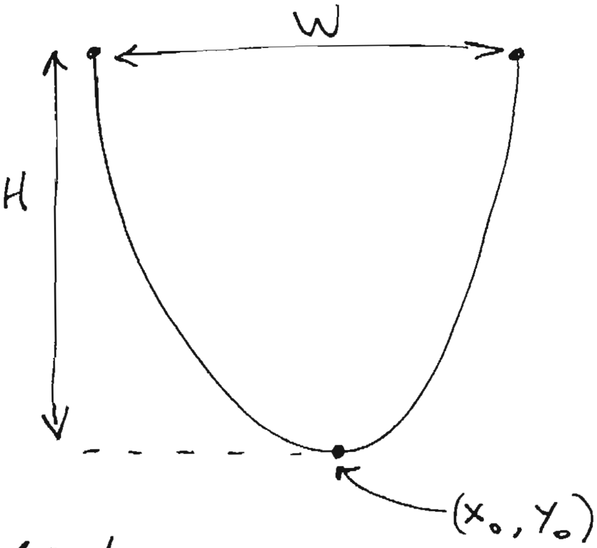
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

x_0, y_0 constants of integration

$$\sinh^2 x + 1 = \cosh^2 x$$

I put in the -1 term so that (X_0, Y_0) are simple to visualize, as shown in the figure.

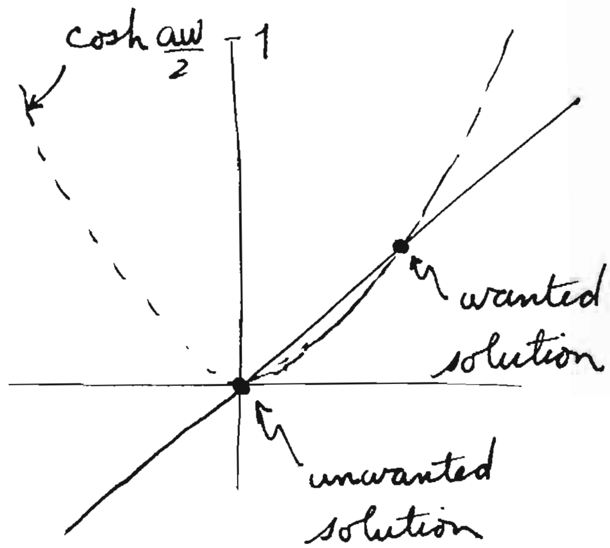
From now on I chose my origin to place it at the apex: $X_0 = 0$, $Y_0 = 0$



$$H = \frac{1}{a}\left\{\cosh\left(\frac{aW}{2}\right) - 1\right\}$$

Measuring H, W lets us find $\underline{\underline{a}}$

$$aH = \cosh\frac{aW}{2} - 1$$



and its error.

$$a\,dH + H\,da = \sinh\frac{aW}{2}\left(\frac{W}{2}da + \frac{a}{2}dW\right)$$

$$\left(H - \frac{W}{2}\sinh\frac{aW}{2}\right)da = \left(\frac{a}{2}\sinh\frac{aW}{2}\right)dW - a\,dH$$

Treat these differentials as random variables, compute errors:

$$(\Delta a)^2 = \left(H - \frac{W}{2}\sinh\frac{aW}{2}\right)^{-2}\left[\left(\frac{a}{2}\sinh\frac{aW}{2}\right)^2(\Delta W)^2 + a^2(\Delta H)^2\right]$$

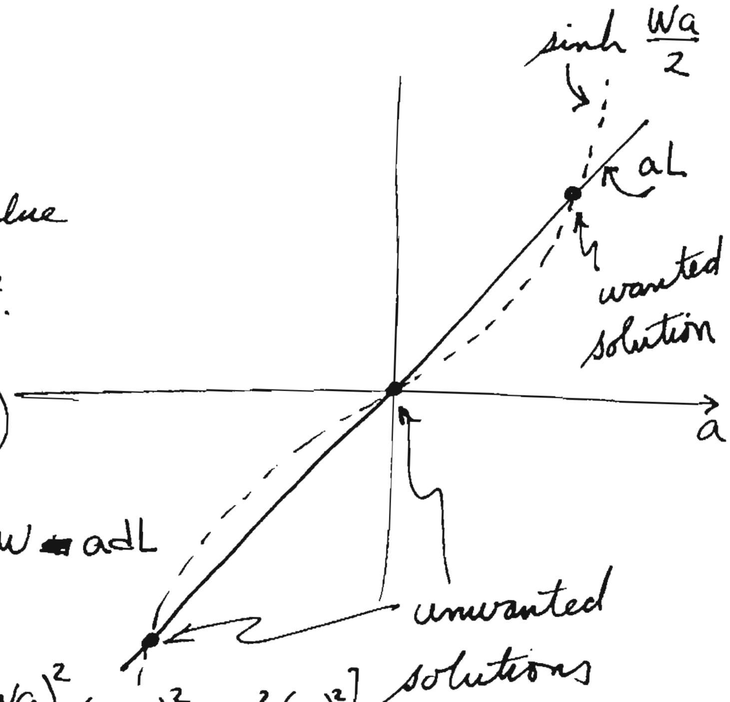
There is another way to find a: compute the length L of the chain and invert to find a.

$$L = \int_0^{\frac{W}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\frac{W}{2}} \cosh ax \, dx$$

$$= \frac{1}{a} \sinh \frac{Wa}{2}$$

$$aL = \sinh \frac{Wa}{2}$$

Again, find error on <u>this</u> value of a, which will be different.



$$L\,da + a\,dL = \cosh \frac{Wa}{2} \left(\frac{a}{2} dW + \frac{W}{2} da\right)$$

$$\left(L - \frac{W}{2} \cosh \frac{Wa}{2}\right) da = \left(\frac{a}{2} \cosh \frac{Wa}{2}\right) dW - a\,dL$$

$$(\Delta a)^2 = \left(L - \frac{W}{2} \cosh \frac{Wa}{2}\right)^{-2} \left[\left(\frac{a}{2} \cosh \frac{Wa}{2}\right)^2 (\Delta W)^2 + a^2 (\Delta L)^2\right]$$

A third way to find <u>a</u> is to fit the measured points (x_i, y_i) along the chain, with errors Δy_i, to a solution with free parameters x_0, y_0, a, or simply one parameter a (if you are careful to put the origin of the measured coordinates at the apex of the chain. Errors come from the fit.