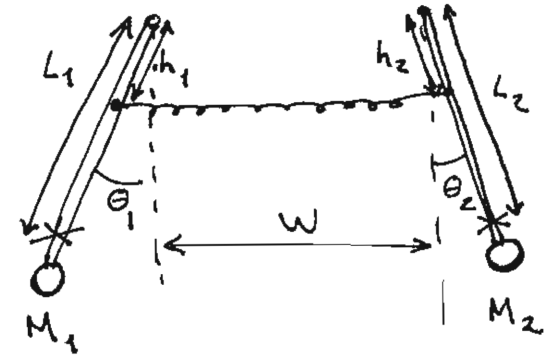
Notes on the Coupled Pendulums Lab



$$x = W - S + h_1\theta_1 + h_2\theta_2$$

Assume $h_1 \approx h_2$ so that torque from the spring is $|\Gamma| = K h x$

$$I_1 \ddot{\theta}_1 = -M_g L_1 \theta_1 - K h_1 x$$

$$I_2 \ddot{\theta}_2 = -M_g L_2 \theta_2 - K h_2 x$$

$$I_1 \ddot{\theta}_1 = -(M_1 L_1 g + K h_1^2)\theta_1 - K h_1 h_2 \theta_2 - K h_1 (W-S)$$

$$I_2 \ddot{\theta}_2 = -(M_2 L_2 g + K h_2^2)\theta_2 - K h_1 h_2 \theta_1 - K h_2 (W-S)$$

* Static equilibrium:
$$\begin{pmatrix} M_1 L_1 g + K h_1^2 & K h_1 h_2 \\ K h_1 h_2 & M_2 L_2 g + K h_2^2 \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = -K(W-S) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = \begin{pmatrix} \dfrac{M_2 L_2 g h_1 + \cancel{K h_2^2 h_1} - \cancel{K h_1 h_2^2}}{(M_1 L_1 g + K h_1^2)(M_2 L_2 g + K h_2^2) - K^2 h_1^2 h_2^2} (S-W) K \\[2ex] \dfrac{-\cancel{K h_1^2 h_2} + (M_1 L_1 g h_2) + \cancel{K h_1^2 h_2}}{(M_1 L_1 g + K h_1^2)(M_2 L_2 g + K h_2^2) - K h_1^2 h_2^2} (S-W) K \end{pmatrix}$$

$$\theta_{i0} = \frac{K(S-W) M_{i'} L_{i'} h_i \cancel{g}}{M_1 M_2 L_1 L_2 g^{\cancel{2}} + K M_1 L_1 \cancel{g} h_2^2 + K M_2 L_2 \cancel{g} h_1^2}, \quad i' = 3 - i$$

$$\theta_0 = \frac{K(S-W)h}{MLg + 2Kh^2}$$ for the case of identical pendulums

Let $\varphi_i = \theta_i - \theta_{i0}$, then

$$I_1 \ddot{\varphi}_1 = -(M_1 L_1 g + K h_1^2)\varphi_1 - K h_1 h_2 \varphi_2$$

$$I_2 \ddot{\varphi}_2 = -(M_2 L_2 g + K h_2^2)\varphi_2 - K h_1 h_2 \varphi_2$$

In matrix form, $\underline{\mathcal{I}}\,\ddot{\Phi} = -\underline{\mathcal{R}}\,\Phi$

where $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$, $\underline{\mathcal{I}} = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$, $\underline{\mathcal{R}} = \begin{pmatrix} M_1L_1g + Kh_1^2 & Kh_1h_2 \\ Kh_1h_2 & M_2L_2g + Kh_2^2 \end{pmatrix}$

or $\ddot{\Phi} = -\underline{\mathcal{M}}\,\Phi$, $\underline{\mathcal{M}} = \underline{\mathcal{I}}^{-1}\underline{\mathcal{R}}$

$$\underline{\mathcal{M}} = \begin{pmatrix} \dfrac{M_1L_1g + Kh_1^2}{I_1} & \dfrac{Kh_1h_2}{I_1} \\[2ex] \dfrac{Kh_1h_2}{I_2} & \dfrac{M_2L_2g + Kh_2^2}{I_2} \end{pmatrix}$$

To solve a matrix-format linear differential equation, break into parts: 1) a matrix eigenvalue problem, and 2) a simple differential equation problem.

Let $\Phi = \begin{pmatrix} a \\ b \end{pmatrix} f(t)$

1) Solve for vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $\underline{\mathcal{M}} \begin{pmatrix} a \\ b \end{pmatrix} = r \begin{pmatrix} a \\ b \end{pmatrix}$

Here r is called an <u>eigenvalue</u>, $\begin{pmatrix} a \\ b \end{pmatrix}$ an <u>eigenvector</u> of $\underline{\mathcal{M}}$

★ Find values of r:

- Recall that uncoupled pendulum 1 has natural frequency $\omega_1^2 = \dfrac{M_1L_1g}{I_1}$ and for #2, $\omega_2^2 = \dfrac{M_2L_2g}{I_2}$
- Define a new "frequency" $\omega_3^2 = \dfrac{Kh_1^2}{I_1}$, $\omega_4^2 = \dfrac{Kh_2^2}{I_2}$

$$\underline{\mathcal{M}} = \begin{pmatrix} \omega_1^2 + \omega_3^2 & \omega_3^2 \dfrac{h_2}{h_1} \\[2ex] \omega_4^2 \dfrac{h_1}{h_2} & \omega_2^2 + \omega_4^2 \end{pmatrix}$$

$$\underline{\mathfrak{M}} \begin{pmatrix} a \\ b \end{pmatrix} = m \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \underbrace{\left\{ \underline{\mathfrak{M}} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right\}}_{\text{must have determinant } = 0} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\therefore \quad (\omega_1^2 + \omega_3^2 - m)(\omega_2^2 + \omega_4^2 - m) - \omega_3^2 \omega_4^2 = 0$$

$$\underbrace{1}_{A} m^2 - \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2)}_{-B} m + \underbrace{(\omega_1^2 + \omega_3^2)(\omega_2^2 + \omega_4^2) - \omega_3^2 \omega_4^2}_{C} = 0$$

$$m = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad : \text{ two solutions } m_+ , m_-$$

$$B^2 - 4AC = \omega_1^4 + \omega_2^4 + \omega_3^4 + \omega_4^4 + 2\omega_1^2\omega_2^2 + 2\omega_1^2\omega_3^2 + 2\omega_1^2\omega_4^2 + 2\omega_2^2\omega_3^2 + 2\omega_2^2\omega_4^2 + 2\omega_3^2\omega_4^2$$
$$- 4(\omega_1^2\omega_2^2 + \omega_1^2\omega_4^2 + \omega_2^2\omega_3^2)$$

A big simplification takes place in special case $\underline{\omega_1 = \omega_2}$

Then $B^2 - 4AC \to (\omega_3^2 + \omega_4^2)^2$

$$m_\pm \to \left\{ \begin{array}{ll} +\omega_0^2 & : - \text{ sign} \\ +(\omega_0^2 + \omega_3^2 + \omega_4^2) & : + \text{ sign} \end{array} \right\} \text{ where } \omega_0 \equiv \omega_1 = \omega_2$$

Direct substitution shows that the − sign eigenvector becomes

$$\begin{pmatrix} h_2 \\ -h_1 \end{pmatrix} : \begin{pmatrix} \omega_0^2 + \omega_3^2 & \omega_3^2 \frac{h_2}{h_1} \\ \omega_4^2 \frac{h_1}{h_2} & \omega_0^2 + \omega_4^2 \end{pmatrix} \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix} = \omega_0^2 \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix}$$

and the + sign solution becomes

$$\begin{pmatrix} \omega_3^2 h_2 \\ \omega_4^2 h_1 \end{pmatrix} : \begin{pmatrix} \omega_0^2 + \omega_3^2 & \omega_3^2 \frac{h_2}{h_1} \\ \omega_4^2 \frac{h_1}{h_2} & \omega_0^2 + \omega_4^2 \end{pmatrix} \begin{pmatrix} \omega_3^2 h_2 \\ \omega_4^2 h_1 \end{pmatrix} = \left[\omega_0^2 + \omega_3^2 + \omega_4^2 \right] \begin{pmatrix} \omega_3^2 h_2 \\ \omega_4^2 h_1 \end{pmatrix}$$

2) Solve the differential equation for these 2 cases

$$\ddot{\Phi} = \begin{pmatrix} a \\ b \end{pmatrix} \ddot{f}(t) = -\underline{m} \begin{pmatrix} a \\ b \end{pmatrix} f(t)$$
$$= -m_{\pm} \begin{pmatrix} a \\ b \end{pmatrix} f(t)$$

$\Rightarrow \quad \ddot{f} = -m_{\pm} f \leftarrow$ harmonic oscillator

$$f(t) = f_0 \sin(\omega_{\pm} t + \phi_0), \quad \omega_{\pm} = \sqrt{m_{\pm}}$$

$$\omega_+ = \sqrt{\omega_0^2 + \omega_3^2 + \omega_4^2} \quad , \quad \omega_- = \sqrt{\omega_0^2} = \omega_0$$

for the special case with $\omega_1 = \omega_2$. For the more general case, the algebra for the values $\omega_{\pm}$ is more complicated, but conceptually the same.

3) Form a general solution out of a sum of these "normal mode" solutions

$$\Phi(t) = f_{0+} \Phi_+ \sin(\omega_+ t + \phi_+) + f_{0-} \Phi_- \sin(\omega_- t + \delta_-)$$

amplitude of mode +   eigenvector of mode +   phase of mode+   same for mode -

$\Phi_{\pm}$ are constant column vectors, eg $\Phi_+ = \begin{pmatrix} \omega_3^2 h_2 \\ \omega_4^2 h_1 \end{pmatrix}$