Solutions to selected exercises from Chapter 6 of *Wasserman — All of Statistics*

(1) The bias is

$$\mathbb{E}_\lambda(\hat{\lambda}) - \lambda = n^{-1}\sum \mathbb{E}(X_i) - \lambda = \lambda - \lambda = 0.$$

Since $X_1, \ldots, X_n$ are assumed independent,

$$\mathbb{V}_\lambda(\hat{\lambda}) = n^{-2}\sum \mathbb{V}_\lambda(X_i) = n^{-2}(n\lambda) = n^{-1}\lambda$$

and $\mathrm{se}(\hat{\lambda}) = \sqrt{\lambda}/\sqrt{n}$. Since MSE is the sum of se^2 and bias^2, we have

$$\mathrm{MSE}(\hat{\lambda}) = \lambda/n + 0 = \lambda/n.$$

(2) We have

$$\mathbb{E}_\theta(\hat{\theta}) = \frac{1}{\theta^n}\int_0^\theta \cdots \int_0^\theta \max(x_1, \ldots, x_n)dx_1 \cdots dx_n.$$

The region $[0,\theta]^n$ in $\mathbb{R}^n$ breaks up into n regions (with measure 0 overlap) defined by $\max\{x_1, \ldots, x_n\} = x_i$. The integral of $\max(x_1, \ldots, x_n)$ over each region will be the same. So we can integrate over one such region (for instance where x_n is the max) and multiply by n:

$$\mathbb{E}_\theta(\hat{\theta}) = \frac{n}{\theta^n}\int_0^\theta \int_0^{x_n} \cdots \int_0^{x_n} x_n dx_1 \cdots dx_n = \frac{n}{\theta^n}\int_0^\theta x_n^n dx_n = \frac{n}{n+1}\theta.$$

Thus

$$\mathsf{bias}^2(\hat{\theta}) = \mathbb{E}_\theta(\hat{\theta}) - \theta = -\frac{1}{n+1}\theta.$$

To compute standard error we find $\mathbb{E}_\theta(\hat{\theta}^2)$. This integral is handled in essentially the same way as the last one:

$$\mathbb{E}_\theta(\hat{\theta}^2) = \frac{n}{\theta^n}\int_0^\theta \int_0^{x_n} \cdots \int_0^{x_n} x_n^2 dx_1 \cdots dx_n = \frac{n}{\theta^n}\int_0^\theta x_n^{n+1} dx_n = \frac{n}{n+2}\theta^2.$$

Thus,

$$\mathbb{V}_\theta(\hat{\theta}) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+2)(n+1)^2}\theta^2 \text{ and } \mathrm{se}(\hat{\theta}) = \sqrt{\frac{n}{n+2}}\frac{\theta}{n+1}.$$

Finally,

$$\mathrm{MSE}(\hat{\theta}) = \frac{1}{(n+1)^2}\theta^2 + \frac{n}{(n+2)(n+1)^2}\theta^2 = \frac{2n+2}{(n+2)(n+1)^2}\theta^2.$$

(3)

$$\mathbb{E}_\theta(\hat{\theta}) = 2\mathbb{E}_\theta(\overline{X_n}) = \theta \text{ so } \mathsf{bias}(\hat{\theta}) = 0$$

and

$$\mathbb{V}_\theta(\hat{\theta}) = 4\mathbb{V}(\overline{X_n}) = \frac{4}{n^2}\cdot n\frac{\theta^2}{12} = \frac{\theta^2}{3n} \text{ so } \mathrm{se}(\hat{\theta}) = \frac{\theta}{\sqrt{3n}} \text{ and } \mathrm{MSE}(\theta) = \mathbb{V}(\hat{\theta}) = \frac{\theta^2}{3n}.$$