Solutions to selected exercises from Chapter 11 of *Wasserman — All of Statistics*

(1) We have $f(\theta|X^n) \propto f(\theta)\mathcal{L}(\theta)$. Thus,

$$f(\theta|X^n) \propto e^{-(\theta-a)^2/2b^2} \prod_i e^{-(X_i-\theta)^2/2\sigma^2} = \exp\left(-(\theta-a)^2/2b^2 - \sum(\theta - X_i)^2/2\sigma^2\right).$$

Consider $\frac{(\theta-a)^2}{2b^2} + \sum \frac{(\theta-X_i)^2}{2\sigma^2}$. Up to adding a constant independent of θ, this is equal to

$$\frac{(\theta-a)^2}{2b^2} + \frac{n(\theta-\bar{X})^2}{2\sigma^2},$$

since $\sum(\theta - X_i)^2 = \theta^2 - 2n\bar{X}\theta + \sum X_i^2$. This is in turn equal to

$$\frac{\sigma^2(\theta-a)^2 + nb^2(\theta-\bar{X})^2}{2b^2\sigma^2}.$$

Expanding this out yields

$$\frac{(\sigma^2+nb^2)\theta^2 - 2(\sigma^2 a + b^2 n\bar{X})\theta + C}{2b^2\sigma^2}$$

where C is some constant independent of θ. Up to adding another constant not depending on θ (by completing the square), this is equal to

$$\frac{(\sigma^2+nb^2)\left(\theta - \frac{\sigma^2 a + b^2 n\bar{X}}{\sigma^2+nb^2}\right)^2}{2b^2\sigma^2}.$$

Thus, up to multiplying by a constant not depending on θ, $f(\theta|X^n)$ is equal to

$$\exp\left(-\frac{\sigma^2+nb^2}{2b^2\sigma^2}\left(\theta - \frac{\sigma^2 a + b^2 n\bar{X}}{\sigma^2+nb^2}\right)^2\right).$$

Finally, compute that $\frac{1}{\tau^2} = \frac{\sigma^2+nb^2}{b^2\sigma^2}$ and $\bar{\theta} = w\bar{X} + (1-w)a = \frac{\sigma^2 a + b^2 n\bar{X}}{\sigma^2+nb^2}$ to conclude that $\theta|X^n \sim N(\bar{\theta}, \tau^2)$, as claimed.

(2) See the Jupyter Notebook 2.ipynb.

(3) Set $M = \max\{x_1, \ldots, x_n\}$. Then

$$\mathcal{L}(\theta) = \begin{cases} 0 & \text{if } \theta < M \\ \theta^{-n} & \text{if } \theta \geq M \end{cases}.$$

Thus,

$$f(\theta|x^n) \propto f(\theta)\mathcal{L}(\theta) = \begin{cases} 0 & \text{if } \theta < M \\ \theta^{-n-1} & \text{if } \theta \geq M \end{cases}.$$

The integral is $\int_M^\infty \theta^{-n-1} = \frac{1}{n}M^{-n}$. Hence

$$f(\theta|x^n) = \begin{cases} \frac{n\max\{x_1,\ldots,x_n\}^n}{\theta^{n+1}} & \text{if } \theta \geq \max\{x_1, \ldots, x_n\} \\ 0 & \text{else} \end{cases}.$$

(4) See the Jupyter Notebook 4.ipynb.

(5) See the Jupyter Notebook 5.ipynb.

(8) See the Jupyter Notebook 8.ipynb.