Solutions to selected exercises from Chapter 4 of *Wasserman — All of Statistics*

(2) We have $\mathbb{E}(X) = \lambda = \mathbb{V}(X)$. Thus

$$\mathbb{P}(X \geq 2\lambda) = \mathbb{P}(X - \lambda \geq \lambda) \leq \mathbb{P}(|X - \lambda| \geq \lambda) \leq \lambda/\lambda^2 = 1/\lambda.$$

(3) We have $\mathbb{E}(\overline{X}_n) = p$ and (assuming the X_i are independent)

$$\mathbb{V}(\overline{X}_n) = \frac{1}{n^2} \sum \mathbb{V}(X_i) = \frac{1}{n^2} \sum p(1-p) = \frac{p(1-p)}{n}.$$

Chebyshev's inequality gives

$$\mathbb{P}(|\overline{X}_n - p| > \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}.$$

Hoeffding's inequality gives

$$\mathbb{P}(|\overline{X}_n - p| > \epsilon) \leq 2e^{-2n\epsilon^2}.$$

The bound from Hoeffding's inequality divided by the bound from Chebyshev's inequality approaches 0 as $\epsilon \to \infty$, as can be shown easily using e.g. L'Hôpital's rule.

(4) See the Jupyter Notebook 4.ipynb.

(6) See the Jupyter Notebook 6.ipynb.

(7) We have $\overline{X}_n \sim N(0, 1/n)$ and hence $\sqrt{n}\overline{X}_n \sim N(0,1)$. By Mill's inequality,

$$\mathbb{P}(|\overline{X}_n| > t) = \mathbb{P}(\sqrt{n}|\overline{X}_n| > \sqrt{n}t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-nt^2/2}}{\sqrt{n}t}.$$

By Chebyshev's inequality,

$$\mathbb{P}(|\overline{X}_n| \geq t) \leq \frac{1}{nt^2}.$$

For $t \gg 0$, Mill's inequality gives a much better (i.e. smaller) upper bound than Chebyshev's inequality. On ther other hand for $t \approx 0$ neither inequality tells us anything useful.