Solutions to selected exercises from Chapter 10 of *Wasserman — All of Statistics*

(1) If in reality $\theta = \theta_*$, then we have

$$W = \frac{\hat{\theta} - \theta_0}{\hat{\mathrm{se}}} = \frac{\hat{\theta} - \theta_*}{\hat{\mathrm{se}}} + \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}}.$$

Asymptotically, this is a $N\left(\frac{\theta_* - \theta_0}{\hat{\mathrm{se}}}, 1\right)$ random variable. We reject the null hypothesis if $|W| > z_{\alpha/2}$. There are two cases. The probability that $W > z_{\alpha/2}$ is

$$\mathbb{P}(W > z_{\alpha/2}) = \mathbb{P}\left(W - \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}} > z_{\alpha/2} - \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}}\right) = 1 - \Phi\left(z_{\alpha/2} - \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}}\right).$$

On the other hand,

$$\mathbb{P}(W < -z_{\alpha/2}) = \mathbb{P}\left(W - \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}} < -z_{\alpha/2} - \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}}\right) = \Phi\left(-z_{\alpha/2} - \frac{\theta_* - \theta_0}{\hat{\mathrm{se}}}\right).$$

Adding these two expressions gives the correct expression for the probability that we reject the null hypothesis. I.e. the sum of the two expressions is the power.

(5) (a) Choose some $c \geq 0$. The rejection region is

$$R = \{(x_1, \ldots, x_n) : x_i > c \text{ for some } i\}.$$

Hence the power function is $\beta(\theta) = \mathbb{P}_\theta((X_1, \ldots, X_n) \in R)$ and this is

$$\mathbb{P}_\theta(X_i > c \text{ for some } i) = 1 - \mathbb{P}_\theta(X_i < c \text{ for all } i) = 1 - \Pi_{i=1}^n \mathbb{P}_\theta(X_i < c).$$

If $c \geq \theta$ then $\mathbb{P}_\theta(X_i < c) = 1$ for all i. If $c < \theta$ then $\mathbb{P}_\theta(X_i < c) = c/\theta$ for all i. Hence

$$\beta(\theta) = \begin{cases} 1 - (c/\theta)^n & \theta > c \\ 0 & \theta \leq c \end{cases}$$

(b) The size is

$$\alpha = \sup_{\theta = 1/2} \beta(\theta) = \beta(1/2) = 1 - (2c)^n.$$

So we need to solve $1 - (2c)^n = 0.05$, i.e. $(2c)^n = 19/20$. The solution is

$$c = \frac{1}{2}\left(\frac{19}{20}\right)^{1/n}.$$

(c) Choosing the parameter c, the size is $\alpha = 1 - (2c)^n$ with corresponding rejection region $R_\alpha = \{(x_1, \ldots, x_n) : x_i > c \text{ for some } i\}$. The p-value is

$$\inf\{\alpha \in (0,1) : Y \in R_\alpha\} = \inf\{\alpha \in (0,1) : Y > c\}$$

Since α is strictly monotonically decreasing with c, the p-value is exactly the value of α when $Y = c$. In other words, the p-value is

$$\alpha = 1 - (2Y)^n = 1 - (2 \cdot 0.48)^{20} \approx 0.558.$$

We would not reject the null hypothesis H_0 in this case.

(d) Of course we can reject H_0 in this case: θ is definitely not equal to 1/2. Let's confirm this using the p-value. The p-value in this case is

$$\inf\{\alpha \in (0,1) : Y > c\} = \inf\{\alpha \in (0,1) : Y > 1/2(1-\alpha)^{1/20}\}.$$

This is equal to 0 since $Y > 1/2$ and we reject H_0 since our p-value 0 is < 0.05.

(6) Set $n = 1919$. Then we have Bernoulli random variables $X_1, \ldots, X_n$ with $X_i = 0$ if person i died the week before Passover and $X_i = 1$ otherwise. Let $\theta \in [0,1]$ be the parameter with $X_i \sim \text{Bernoulli}(\theta)$ for each i. So we want to test $H_0 : \theta = \theta_0 = 1/2$ versus $H_1 : \theta \neq 1/2$. We will test this using the Wald test.

Our test statistic is

$$w = \frac{\hat{\theta} - \theta_0}{\widehat{\text{se}}}.$$

Our estimate for $\hat{\theta}$ is the mean

$$\hat{\theta} = \overline{X} = \frac{1}{n}\sum X_i = \frac{997}{1919} \approx 0.5195.$$

The standard error of $\overline{X}$ is $\sigma/\sqrt{n}$ where σ is the standard deviation of the X_i. So we estimate se by $\widehat{\text{se}} = \widehat{\sigma}/\sqrt{n}$ where $\widehat{\sigma}$ is the estimated standard deviation of the X_i:

$$\widehat{\text{se}}^2 = \frac{1}{n}\hat{\theta}(1-\hat{\theta}) = \frac{1}{1919}\frac{997}{1919}\frac{922}{1919} \approx 0.00013, \quad \widehat{\text{se}} \approx 0.0114.$$

Plugging in,

$$w \approx \frac{0.5195 - 0.5}{0.0114} \approx 1.7105.$$

By Theorem 10.13, our estimated p-value is

$$2\Phi(-w) \approx 0.08717.$$

We can view this as weak evidence against H_0 but we don't reject H_0 at the size threshold of 0.05.

A 95% confidence interval for θ is given by

$$(\hat{\theta} - 2\widehat{\text{se}}, \hat{\theta} - 2\widehat{\text{se}}) = (0.4967, 0.5423).$$

(7) See the Jupyter Notebook 7.ipynb.

(8) (a) We have

$$\sum X_i \sim N(n\theta, n), \quad \overline{X_n} = \frac{1}{n}\sum X_i \sim N(\theta, n/n^2) = N(\theta, 1/n).$$

So

$$\beta(\theta) = \mathbb{P}_\theta((X_1, \ldots, X_n) \in R) = \mathbb{P}_\theta(\overline{X_n} > c)$$

and this is equal to

$$\mathbb{P}_\theta(\sqrt{n}(\overline{X_n} - \theta) > \sqrt{n}(c-\theta)) = 1 - \Phi(\sqrt{n}(c-\theta)).$$

The size is

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(0) = 1 - \Phi(c\sqrt{n}).$$

I.e. $\Phi(c\sqrt{n}) = 1 - \alpha$ and solving for c yields $c = \frac{1}{\sqrt{n}} z_\alpha$.

(b) $\beta(1) = 1 - \Phi(\sqrt{n}(c-1))$

(c) As $n \to \infty$, $c - 1 \to -1$ so $\sqrt{n}(c-1) \to -\infty$. Thus $\Phi(\sqrt{n}(c-1)) \to 0$ and $\beta(1) \to 1$.

(11) See the Jupyter Notebook 11.ipynb.

(12) See the Jupyter Notebook 12.ipynb.

(13) We assume σ to be known. The likelihood ratio test statistic is

$$\lambda = 2\log\left(\frac{\mathcal{L}(\hat{\mu})}{\mathcal{L}(\mu_0)}\right)$$

where $\hat{\mu}$ is the maximum likelihood estimator $\hat{\mu} = \overline{X} = \frac{1}{n}\sum X_i$.
We have

$$\mathcal{L}(\hat{\mu}) = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\hat{\mu})^2/2\sigma^2} = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\sum(X_i-\hat{\mu})^2/2\sigma^2}.$$

Expanding $(X_i - \mu_0)^2$ as

$$(X_i - \hat{\mu})^2 + 2(\hat{\mu} - \mu_0)(X_i - \hat{\mu}) + (\hat{\mu} - \mu_0)^2$$

yields

$$\sum(X_i - \mu_0)^2 = \sum(X_i - \hat{\mu})^2 + n(\hat{\mu} - \mu_0)^2.$$

So we have

$$\mathcal{L}(\mu_0) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\left(\sum(X_i-\hat{\mu})^2/2\sigma^2 + n(\hat{\mu}-\mu_0)^2/2\sigma^2\right)}.$$

Thus

$$\frac{\mathcal{L}(\hat{\mu})}{\mathcal{L}(\mu_0)} = e^{n(\hat{\mu}-\mu_0)^2/2\sigma^2}$$

and our statistic is

$$\lambda = 2\log\left(\frac{\mathcal{L}(\hat{\mu})}{\mathcal{L}(\mu_0)}\right) = n(\hat{\mu} - \mu_0)^2/\sigma^2.$$

Say that we reject H_0 if $\lambda > c$ for some fixed c. This is equivalent to

$$\frac{|\hat{\mu} - \mu_0|}{\sigma/\sqrt{n}} > \sqrt{c}.$$

Let's compute the size of this test. Assuming $\mu = \mu_0$, we have that $\frac{\hat{\mu}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$. In this case, the probability of the inequality above is

$$\Phi(-\sqrt{c}) + (1 - \Phi(\sqrt{c})) = 2 - 2\Phi(\sqrt{c})$$

and this is the size of our test. Setting the size to be α yields $c = z_{\alpha/2}^2$. So the size α likelihood ratio test is

$$\frac{|\hat{\mu} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}.$$

The size α Wald test for rejecting H_0 is

$$\frac{|\hat{\mu} - \mu_0|}{\widehat{\text{se}}} > z_{\alpha/2}.$$

So we see that the two tests are the same up to replacing $\widehat{\text{se}}$ (an estimate of the standard error of $\hat{\mu}$) by $\sigma/\sqrt{n}$ (the actual standard error of $\hat{\mu}$).