Solutions to selected exercises from Chapter 17 of *Wasserman — All of Statistics*

(1) We have

$$f_{X,Y|Z}(x,y|z) = f_{X|Z}(x|z) f_{Y|Z}(y|z)$$

if and only if

$$\frac{f_{X,Y,Z}(x,y,z)}{f_Z(z)} = \frac{f_{Y,Z}(y,z)}{f_Z(z)} f_{X|Z}(x|z)$$

if and only if

$$f_{X|Y,Z}(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} = f_{X|Z}(x|z).$$

(2) The first implication, $X \amalg Y|Z \implies Y \amalg X|Z$, is clear.

For the second implication, if $X \amalg Y|Z$ then

$$\mathbb{P}(U \le u|Y=y, Z=z) = \mathbb{P}(X \le h^{-1}(u)|Y=y, Z=z) = \mathbb{P}(X \le h^{-1}(u)|Z=z)$$

which is finally equal to $\mathbb{P}(U \le u|Z=z)$. Taking derivatives with respect to u yields $f(u|y,z) = f(u|z)$.

For the third implication, Consider x, y, z, and $u = h(x')$. Then

$$\mathbb{P}(X \le x, Y \le y|Z=z, U=u) = \mathbb{P}(X \le x, Y \le y|Z=z, X=x').$$

If $x' \le x$, then $\mathbb{P}(X \le x|X=x') = 1$ and the above conditional probability is

$$\mathbb{P}(Y \le y|Z=z, U=u) = \mathbb{P}(Y \le y|Z=z, U=u)\mathbb{P}(X \le x|Z=z, U=u).$$

If $x' > x$ then $\mathbb{P}(X \le x|X=x') = 0$ and the above conditional probability is

$$0 = \mathbb{P}(Y \le y|Z=z, U=u)\mathbb{P}(X \le x|Z=z, U=u).$$

In any case, taking the mixed second partial derivative yields that

$$f(x,y|z,u) = f(x|z,u)f(y|z,u),$$

as desired.

For the fourth implication,

$$f(x|w,y,z) = \frac{f(x,w,y,z)}{f(w,y,z)} = \frac{f(x,w,y,z)}{f(y,z)}\frac{f(y,z)}{f(w,y,z)} = \frac{f(x,w|y,z)}{f(w|y,z)}.$$

By (17.1), this is equal to

$$\frac{f(x|y,z)f(w|y,z)}{f(w|y,z)} = f(x|y,z).$$

Finally, by (17.2) this is equal to $f(x|z)$. By (17.2), this shows that $X \amalg (W,Y)|Z$, as desired.

For the last implication, we have

$$f(x|z) = f(x|y,z) = f(x|y).$$

This shows that $f(x|z)$ is independent of z, i.e. $X \amalg Z$. By symmetry, $X \amalg Y$. Thus,

$$\frac{f(x,y,z)}{f(y,z)} = f(x|y,z) = f(x|z) = f(x) \text{ so } f(x,y,z) = f(x)f(y,z),$$

as desired.

(3) (a) The probability that $Z = 0$ is 0.5 and the probability that $Z = 1$ is 0.5. Therefore the distribution $f(x,y|0) = f(x,y,0)/0.5$ is given by

	$Y = 0$	$Y = 1$
$X = 0$	0.81	0.09
$X = 1$	0.09	0.01

.

Similarly, the distribution $f(x,y|1)$ is given by

	$Y = 0$	$Y = 1$
$X = 0$	0.25	0.25
$X = 1$	0.25	0.25

.

(b) Let's calculate the distribution $f_{X|Z}(x|z)$. If $Z = 0$ then $X = 0$ with probability 0.45 and $X = 1$ with probability 0.05. The conditional distribution is $f_{X|Z}(0|0) = 0.9$ and $f_{X|Z}(1|0) = 0.1$. The conditional distribution if $Z = 1$ is $f_{X|Z}(0|1) = 0.5$ and $f_{X|Z}(1|1) = 0.5$.

Now let's calculate $f_{Y|Z}(y|z)$. We have $f_{Y|Z}(0|0) = 0.9$, $f_{Y|Z}(1|0) = 0.1$, $f_{Y|Z}(0|1) = 0.5$, $f_{Y|Z}(1|1) = 0.5$.

Finally then, the fact that $X \amalg Y|Z$ follows from the calculations

$$\begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.81 & 0.09 \\ 0.09 & 0.01 \end{pmatrix}$$

for the case $Z = 0$ and

$$\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}$$

for the case $Z = 1$.

(c) The marginal distribution of X and Y is

	$Y = 0$	$Y = 1$
$X = 0$	0.53	0.17
$X = 1$	0.17	0.13

.

(d) Clearly X and Y are not independent, as $f_{X,Y}(x|0) \neq f_{X,Y}(x|1)$.

(4) In the upper left graph we have $f(x,y,z) = f(z|y)f(y|x)f(x)$. Thus,

$$f(x,z|y)f(y) = f(z|y)f(y|x)f(x) \text{ and } f(x,z|y) = f(z|y)\frac{f(y|x)f(x)}{f(y)}.$$

By Bayes's rule for PDF's we have $f(x,z|y) = f(z|y)f(x|y)$, as desired.

The upper right graph is the same as the upper left up to swapping X and Z. Hence by the last paragraph we again have $X \amalg Z|Y$.

In the lower left graph we have $f(x, y, z) = f(x|y)f(z|y)f(y)$. Dividing both sides by $f(y)$ yields $f(x, z|y) = f(x|y)f(z|y)$, as desired.

(6) See the Jupyter Notebook 6.ipynb.

(8) See the Jupyter Notebook 8.ipynb.