Solutions to selected exercises from Chapter 5 of *Wasserman — All of Statistics*

(1) (a) We have

$$\mathbb{E}(S_n^2) = \frac{1}{n-1}\sum \mathbb{E}[(X_i - \overline{X}_n)^2] \text{ and } \mathbb{E}(X_i - \overline{X}_n) = 0.$$

So $\mathbb{E}(S_n^2) = \frac{1}{n-1}\sum \mathbb{V}(X_i - \overline{X}_n)$. We have

$$\mathbb{V}(X_i - \overline{X}_n) = \mathbb{V}(X_i) + \mathbb{V}(\overline{X}_n) - 2\operatorname{Cov}(X_i, \overline{X}_n).$$

By independence of the X_i, $\mathbb{V}(\overline{X}_n) = \sigma^2/n$. On the other hand, $\operatorname{Cov}(X_i, \overline{X}_n)$ is equal to

$$\mathbb{E}((X_i - \mu)(\overline{X}_n - \mu)) = \mathbb{E}\left(\frac{1}{n}\sum (X_i - \mu)(X_j - \mu)\right) = \frac{1}{n}\sum \operatorname{Cov}(X_i, X_j).$$

Every term in this sum is zero except for $\operatorname{Cov}(X_i, X_i) = \mathbb{V}(X_i)$. So $\operatorname{Cov}(X_i, \overline{X}_n) = \sigma^2/n$. Thus,

$$\mathbb{V}(X_i - \overline{X}_n) = \sigma^2 + \sigma^2/n - 2\sigma^2/n = \frac{(n-1)\sigma^2}{n}$$

for each i and

$$\mathbb{E}(S_n^2) = \frac{1}{n-1}\sum \mathbb{V}(X_i - \overline{X}_n) = \frac{n}{n-1}\cdot\frac{(n-1)\sigma^2}{n} = \sigma^2.$$

(b) Following the hint,

$$S_n^2 = \frac{1}{n-1}\sum (X_i^2 - 2X_i\overline{X}_n + \overline{X}_n^2) = \frac{1}{n-1}\sum X_i^2 - \frac{2n}{n-1}\overline{X}_n^2 + \frac{n}{n-1}\overline{X}_n^2$$

which is equal to

$$\frac{n}{n-1}\frac{1}{n}\sum X_i^2 - \frac{n}{n-1}\overline{X}_n^2 = \frac{c_n}{n}\sum X_i^2 - d_n\overline{X}_n^2$$

where of course $c_n = d_n \to 1$. By the law of large numbers $\overline{X}_n \xrightarrow{P} \mu$ as $n \to \infty$. Using the LLN plus independence of the X_i^2 we also get $\frac{1}{n}\sum X_i^2 \xrightarrow{P} \mathbb{E}(X_1^2) = \sigma^2 + \mu^2$. Thus by parts (a) and (d) of Theorem 5.5,

$$S_n^2 \xrightarrow{P} \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

(2) The expected value $\mathbb{E}[(X_n - b)^2]$ is equal to

$$\mathbb{E}(X_n^2) - 2b\mathbb{E}(X_n) + b^2 = \mathbb{V}(X_n) + \mathbb{E}(X_n)^2 - 2b\mathbb{E}(X_n) + b^2 = \mathbb{V}(X_n) + (\mathbb{E}(X_n) - b)^2.$$

Since both terms are non-negative, $\mathbb{E}[(X_n - b)^2] \to 0$ if and only if $\mathbb{E}(X_n) \to b$ and $\mathbb{V}(X_n) \to 0$, as claimed.

(3)

$$\mathbb{E}(\overline{X}_n) = \mu \text{ and } \mathbb{V}(\overline{X}_n) = \frac{1}{n}\mathbb{V}(X_1) \to 0 \text{ as } n \to \infty.$$

Hence by Exercise (2), $\overline{X}_n \xrightarrow{\text{qm}} \mu$.

(5)
$$\mathbb{E}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}np = p \text{ and } \mathbb{V}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}np(1-p) \to 0 \text{ as } n \to \infty.$$

Thus by Exercise (2), $\frac{1}{n}\sum X_i \xrightarrow{\text{qm}} p$. Since convergence in quadratic mean implies convergence in probability, the sequence also converges to p in probability.

(6) Let $X_1, \ldots, X_{100}$ be the heights and $\overline{X}$ be the average. Set $\mu = 68$ and $\sigma = 2.6$. By the Central Limit Theorem we may approximate

$$\mathbb{P}(\overline{X} \geq 68) = \mathbb{P}\left(\frac{\sqrt{100}(\overline{X} - 68)}{2.6} \geq 0\right) \approx \mathbb{P}(Z \geq 0) = \frac{1}{2}$$

where Z is a standard normal.

(8) We have $\mathbb{V}(Y) = \sum \mathbb{V}(X_i) = n$ and similarly $\mathbb{E}(Y) = n$. Consider $\overline{X}_n = \frac{1}{n}Y$. We have $\mathbb{E}(\overline{X}_n) = 1$ and $\mathbb{V}(\overline{X}_n) = 1/n$. Consider

$$Z = \frac{\overline{X}_n - \mathbb{E}\overline{X}_n}{\sqrt{\mathbb{V}\overline{X}_n}} = \sqrt{n}(\overline{X}_n - 1).$$

By the central limit theorem we can approximate Z by a standard normal and

$$\Phi(z) \approx \mathbb{P}(Z \leq z) = \mathbb{P}(\overline{X}_n \leq 1 + z/\sqrt{n}) = \mathbb{P}(Y \leq n + \sqrt{n}z).$$

Solving for $n + \sqrt{n}z = 90$ we find $z = (90 - n)/\sqrt{n} = -1$. Thus,

$$\mathbb{P}(Y \leq 90) = \mathbb{P}(Z \leq -1) \approx \Phi(-1) = 0.15865\ldots.$$

(11) It suffices to show that $X_n \xrightarrow{P} X$. We have $\mathbb{P}(X = 0) = 1$. Thus, it suffices to show that $\mathbb{P}(|X_n| > \epsilon) \to 0$. This is clear.

(14) We'll apply the delta method. We have $\mu = \mathbb{E}(X_i) = 1/2$ and $\sigma = \sqrt{\mathbb{V}(X_i)} = \sqrt{1/12}$. By the Central Limit Theorem, $\sqrt{n}(\overline{X}_n - \mu)/\sigma \rightsquigarrow Z$ where Z is a standard normal random variable. Set $g(x) = x^2$ so that $g'(x) = 2x$. By the delta method, we have

$$Y_n \approx N\left(g(\mu), g'(\mu)^2\frac{\sigma^2}{n}\right) = N\left(\frac{1}{4}, \frac{1}{12n}\right)$$

for large n. In particular, $Y_n \rightsquigarrow \frac{1}{4}$.

(15) We will again apply the delta method. By the Central Limit Theorem,

$$\sqrt{n}\left(\begin{pmatrix}\overline{X}_1 \\ \overline{X}_2\end{pmatrix} - \mu\right) \rightsquigarrow N(0, \Sigma)$$

where Σ is the variance matrix for the random vectors. Set $g(x, y) = x/y$. We have

$$\nabla g = \frac{1}{y}\begin{pmatrix}1 \\ -\frac{x}{y}\end{pmatrix} \text{ so } \nabla\mu = \frac{1}{\mu_2}\begin{pmatrix}1 \\ -\frac{\mu_1}{\mu_2}\end{pmatrix}.$$

We may write $\Sigma = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}$. Hence

$$\nabla_\mu^T \Sigma \nabla_\mu = \frac{1}{\mu_2^4} \begin{pmatrix} \mu_2 & -\mu_1 \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} \mu_2 \\ -\mu_1 \end{pmatrix} = \frac{1}{\mu_2^4}(v_{11}\mu_2^2 - 2v_{12}\mu_1\mu_2 + v_{22}\mu_1^2).$$

By the delta method,

$$\sqrt{n}(\overline{X}_1/\overline{X}_2 - \mu_1/\mu_2) \rightsquigarrow N\left(0, \frac{1}{\mu_2^4}(v_{11}\mu_2^2 - 2v_{12}\mu_1\mu_2 + v_{22}\mu_1^2)\right).$$

In particular $\overline{X}_1/\overline{X}_2 \rightsquigarrow \mu_1/\mu_2$.