Solutions to selected exercises from Chapter 16 of *Wasserman — All of Statistics*

(1) Suppose the data are given by the following table

X	Y	C_0	C_1
0	0	0	0^*
0	0	0	0^*
1	0	1^*	0
1	1	1^*	1

Then $\theta = \mathbb{E}(C_1) - \mathbb{E}(C_0) = 1/4 - 1/2 = -1/4$. On the other hand $\alpha = \mathbb{E}(Y|X = 1) - \mathbb{E}(Y|X = 0) = 1/2 - 0 = 1/2$.

(2) Saying that X is randomly assigned means that X is independent of the collection $\{C(x) : x \in \mathbb{R}\}$. Thus in this case

$$\theta(x) = \mathbb{E}(C(x)) = \mathbb{E}(C(x)|X = x) = \mathbb{E}(Y|X = x) = r(x).$$

To see that $\theta \neq r$ in general, suppose that $X \sim N(0, 1)$ and that for each $x \in \mathbb{R}$, $C(x)$ is the random variable X. Then $\theta(x) = \mathbb{E}(X) = 0$. However, $r(x) = \mathbb{E}(Y|X = x) = \mathbb{E}(X|X = x) = x$.

(3) We have

$$\mathbb{E}(C_1) = \mathbb{E}(C_1|X = 1)\mathbb{P}(X = 1) + \mathbb{E}(C_1|X = 0)\mathbb{P}(X = 0).$$

Since $\mathbb{E}(C_1|X = 1) = \mathbb{E}(Y|X = 1)$, we have

$$\mathbb{E}(Y|X = 1)\mathbb{P}(X = 1) \leq \mathbb{E}(C_1) \leq \mathbb{E}(Y|X = 1)\mathbb{P}(X = 1) + \mathbb{P}(X = 0).$$

Similarly,

$$\mathbb{E}(Y|X = 0)\mathbb{P}(X = 0) \leq \mathbb{E}(C_0) \leq \mathbb{E}(Y|X = 0)\mathbb{P}(X = 0) + \mathbb{P}(X = 1).$$

Thus,

$$\mathbb{E}(C_0) \geq \mathbb{E}(Y|X = 1)\mathbb{P}(X = 1) - \mathbb{E}(Y|X = 0)\mathbb{P}(X = 0) - \mathbb{P}(X = 1)$$

and

$$\mathbb{E}(C_0) \leq \mathbb{E}(Y|X = 1)\mathbb{P}(X = 1) + \mathbb{P}(X = 0) - \mathbb{E}(Y|X = 0)\mathbb{P}(X = 0).$$

The conclusion is that $\mathbb{E}(C_1) - \mathbb{E}(C_0)$ lies between

$$\mathbb{E}(Y|X = 1)\mathbb{P}(X = 1) - \mathbb{E}(Y|X = 0)\mathbb{P}(X = 0) - \mathbb{P}(X = 1)$$

and

$$\mathbb{E}(Y|X = 1)\mathbb{P}(X = 1) - \mathbb{E}(Y|X = 0)\mathbb{P}(X = 0) + \mathbb{P}(X = 0)$$

and this interval has diameter 1.