Solutions to selected exercises from Chapter 11 of *Wasserman — All of Statistics*

(1) We have $f(\theta|X^n) \propto f(\theta)\mathcal{L}(\theta)$. Thus,

$$f(\theta|X^n) \propto e^{-(\theta-a)^2/2b^2} \prod_i e^{-(X_i-\theta)^2/2\sigma^2} = \exp\left(-(\theta-a)^2/2b^2 - \sum(\theta-X_i)^2/2\sigma^2\right).$$

Consider $\frac{(\theta-a)^2}{2b^2} + \sum \frac{(\theta-X_i)^2}{2\sigma^2}$. Up to adding a constant independent of $\theta$, this is equal to

$$\frac{(\theta-a)^2}{2b^2} + \frac{n(\theta-\bar{X})^2}{2\sigma^2},$$

since $\sum(\theta-X_i)^2 = \theta^2 - 2n\bar{X}\theta + \sum X_i^2$. This is in turn equal to

$$\frac{\sigma^2(\theta-a)^2 + nb^2(\theta-\bar{X})^2}{2b^2\sigma^2}.$$

Expanding this out yields

$$\frac{(\sigma^2+nb^2)\theta^2 - 2(\sigma^2 a + b^2 n\bar{X})\theta + C}{2b^2\sigma^2}$$

where $C$ is some constant independent of $\theta$. Up to adding another constant not depending on $\theta$ (by completing the square), this is equal to

$$\frac{(\sigma^2+nb^2)\left(\theta - \frac{\sigma^2 a + b^2 n\bar{X}}{\sigma^2+nb^2}\right)^2}{2b^2\sigma^2}.$$

Thus, up to multiplying by a constant not depending on $\theta$, $f(\theta|X^n)$ is equal to

$$\exp\left(-\frac{\sigma^2+nb^2}{2b^2\sigma^2}\left(\theta - \frac{\sigma^2 a + b^2 n\bar{X}}{\sigma^2+nb^2}\right)^2\right).$$

Finally, compute that $\frac{1}{\tau^2} = \frac{\sigma^2+nb^2}{b^2\sigma^2}$ and $\bar{\theta} = w\bar{X} + (1-w)a = \frac{\sigma^2 a + b^2 n\bar{X}}{\sigma^2+nb^2}$ to conclude that $\theta|X^n \sim N(\bar{\theta}, \tau^2)$, as claimed.

(2) See the Jupyter Notebook 2.ipynb.

(3) Set $M = \max\{x_1, \ldots, x_n\}$. Then

$$\mathcal{L}(\theta) = \begin{cases} 0 & \text{if } \theta < M \\ \theta^{-n} & \text{if } \theta \geq M \end{cases}.$$

Thus,

$$f(\theta|x^n) \propto f(\theta)\mathcal{L}(\theta) = \begin{cases} 0 & \text{if } \theta < M \\ \theta^{-n-1} & \text{if } \theta \geq M \end{cases}.$$

The integral is $\int_M^\infty \theta^{-n-1} = \frac{1}{n}M^{-n}$. Hence

$$f(\theta|x^n) = \begin{cases} \frac{n\max\{x_1,\ldots,x_n\}^n}{\theta^{n+1}} & \text{if } \theta \geq \max\{x_1,\ldots,x_n\} \\ 0 & \text{else} \end{cases}.$$

(4) See the Jupyter Notebook 4.ipynb.

(5) See the Jupyter Notebook 5.ipynb.

(6) (a) We have

$$f(\lambda|x^n) \propto f(\lambda)\mathcal{L}_n(\lambda) \propto \lambda^{\alpha-1}e^{-\lambda/\beta}\prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{x_i}}{x_i!} \propto \lambda^{\alpha+\sum x_i-1}e^{-\lambda/\beta-n\lambda}.$$

Thus,

$$\lambda|x^n \sim \Gamma\left(\alpha+\sum x_i, \frac{1}{n+\frac{1}{\beta}}\right).$$

The posterior mean is the product of the two parameters:

$$\bar{\lambda} = \frac{\alpha+\sum x_i}{n+\frac{1}{\beta}}.$$

(b) We have

$$\log(f(x;\lambda)) = -\lambda + x\log\lambda - \log(x!).$$

Therefore

$$\frac{\partial}{\partial\lambda}\log(f(x;\lambda)) = -1+\frac{x}{\lambda} \text{ and } \frac{\partial^2}{\partial\lambda^2}\log(f(x;\lambda)) = -\frac{x}{\lambda^2}.$$

By Theorem 9.17, we have

$$I(\theta) = -E\left(-\frac{X}{\lambda^2}\right) = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x x}{x!\lambda^2} = \frac{1}{\lambda}\sum_{x=1}^{\infty} e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!} = \frac{1}{\lambda}.$$

Therefore the Jeffreys' prior is $\frac{1}{\sqrt{\lambda}}$. This is an improper prior since the integral of $\lambda^{-1/2}$ is infinite.

The posterior with respect to the Jeffreys' prior is

$$f(\lambda|x^n) \propto \lambda^{-1/2}\prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{x_i}}{x_i!} \propto \lambda^{-1/2}e^{-n\lambda}\lambda^{\sum x_i} = \lambda^{\sum x_i+1/2-1}e^{-n\lambda}.$$

Therefore

$$\lambda|x^n \sim \Gamma\left(\sum x_i+\frac{1}{2}, \frac{1}{n}\right).$$

(8) See the Jupyter Notebook 8.ipynb.