Solutions to selected exercises from Chapter 21 of *Wasserman — All of Statistics*

(1) We have $R(J) = \int b(x)^2 dx + \int v(x) dx$. Furthermore,

$$b(x) = \mathbb{E}(\hat{f}(x)) - f(x) = \sum_{j=1}^{J} \mathbb{E}(\hat{\beta}_j)\phi_j(x) - f(x) = \sum_{j=1}^{J} \beta_j \phi_j(x) - f(x) = \sum_{j=J+1}^{\infty} \beta_j \phi_j(x).$$

By orthogonality of the ϕ_j,

$$\int b^2 = \sum_{j=J+1}^{\infty} \beta_j^2$$

For the other term,

$$v(x) = \mathbb{E}\left(\hat{f}(x) - \mathbb{E}\hat{f}(x)\right)^2 = \mathbb{E}\left(\sum_{j=1}^{J} (\hat{\beta}_j - \beta_j)\phi_j(x)\right)^2 = \sum_{j=1}^{J}\sum_{k=1}^{J} \mathrm{Cov}(\hat{\beta}_j, \hat{\beta}_k)\phi_j(x)\phi_k(x).$$

Again using orthogonality of the ϕ_j yields,

$$\int v(x) dx = \sum_{j=1}^{J} \mathbb{V}(\hat{\beta}_j) = \sum_{j=1}^{J} \frac{\sigma_j^2}{n}.$$

Adding $\int b^2$ to $\int v$ proves the theorem.

(4) Set $f_n = \sum_{j=1}^{n} \beta_j \phi_j$. By hypothesis, $\int (f - f_n)^2 \to 0$ as $n \to \infty$. We have

$$\int (f - f_n)^2 = \int f^2 - 2\int f f_n + \int f_n^2.$$

By the Cauchy-Schwarz inequality (equivalently Hölder's inequality),

$$\left|\int f f_n\right| \leq \sqrt{\int f^2}\sqrt{\int f_n^2}.$$

Hence,

$$\int (f - f_n)^2 \geq \int f^2 - 2\sqrt{\int f^2}\sqrt{\int f_n^2} + \int f_n^2 = \left(\sqrt{\int f^2} - \sqrt{\int f_n^2}\right)^2.$$

Thus $\int f_n^2 \to \int f^2$. On the other hand,

$$\int f_n^2 = \sum_{j=1}^{n} \beta_j^2$$

by the orthogonality of the ϕ_j. The proof of Hölder's inequality follows from Young's inequality for products. See Wikipedia.

(5) See the Jupyter Notebook 5.ipynb.

(6) See the Jupyter Notebook 6.ipynb.

(7) See the Jupyter Notebook 9.ipynb.

(11)