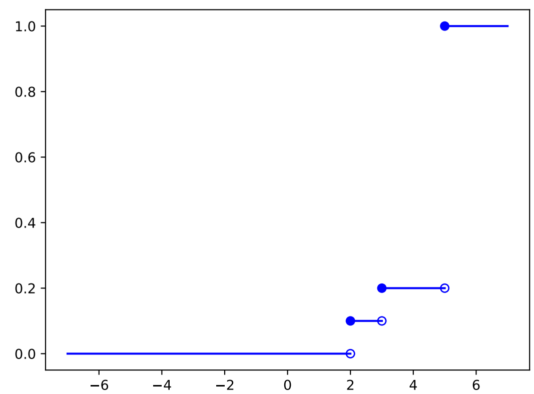
Solutions to selected exercises from Chapter 2 of *Wasserman — All of Statistics*

(2) The CDF is

$$F(x) = \begin{cases} 0 & x < 2 \\ 1/10 & 2 \le x \le 3 \\ 2/10 & 3 \le x \le 5 \\ 1 & x \ge 5 \end{cases}$$

Thus $\mathbb{P}(2 < X \le 4.8) = F(4.8) - F(2) = 2/10 - 1/10 = 1/10$ and $\mathbb{P}(2 \le X \le 4.8) = \mathbb{P}(X = 2) + 1/10 = 2/10$. A plot is shown below.



(4) (a) The CDF $F_X(x)$ is equal to 0 at $x = 0$, 1/4 at $x = 1$, 1/4 at $x = 3$, and 1 at $x = 5$. It interpolates between these values affinely. I.e.

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1/4x & 0 \le x \le 1 \\ 1/4 & 1 \le x \le 3 \\ 1/4 + 3/8(x-3) & 3 \le x \le 5 \\ 1 & x \ge 5 \end{cases}$$

(b) We will compute $F_Y(y)$ and take its derivative to compute $f_Y(y)$. We want to calculate $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(1/X \le y)$. This is 0 if $y \le 0$ so we may assume without loss of generality that $y > 0$. We consider the set $A_y = \{x > 0 : 1/x \le y\}$. This is equal to $[1/y, \infty)$, so $F_Y(y) = \int_{1/y}^{\infty} f_X(x)dx$.

First consider the case $1/y \ge 5$ i.e. $y \le 1/5$. Then we have

$$F_Y(y) = \int_{1/y}^{\infty} f_X(x)dx \le \int_{5}^{\infty} f_X(x)dx = 0.$$

Now consider the case $3 \le 1/y \le 5$ i.e. $1/5 \le y \le 1/3$. Since $1/y \ge 3$ we have

$$F_Y(y) = \int_{1/y}^{5} \frac{3}{8}dx = \frac{3}{8}(5 - 1/y).$$

Now consider the case $1 \le 1/y \le 3$ i.e. $1/3 \le y \le 1$. Since $1/y \ge 1$ we have

$$F_Y(y) = \int_{1/y}^{\infty} f_X(x)dx = \int_3^5 \frac{3}{8}dx = \frac{3}{4}.$$

Finally if $0 < 1/y \le 1$, i.e. $1 \le y$, we have

$$F_Y(y) = \int_{1/y}^{1} f_X(x)dx + \frac{3}{4} = \frac{1}{4}(1 - 1/y) + \frac{3}{4}.$$

Thus,

$$F_Y(y) = \begin{cases} 0 & y \le 1/5 \\ \frac{3}{8}(5 - \frac{1}{y}) & 1/5 \le y \le 1/3 \\ \frac{3}{4} & 1/3 \le y \le 1 \\ \frac{3}{4} + \frac{1}{4}(1 - \frac{1}{y}) & y \ge 1 \end{cases}$$

and

$$f_Y(g) = \begin{cases} 0 & y \le 1/5 \\ \frac{3}{8y^2} & 1/5 \le y \le 1/3 \\ 0 & 1/3 \le y \le 1 \\ \frac{1}{4y^2} & y \ge 1. \end{cases}$$

(7) We have $\mathbb{P}(Z > z) = \mathbb{P}(X > z \text{ and } Y > z) = \mathbb{P}(X > z)\mathbb{P}(Y > z)$ by independence. This is equal to $(1 - F_X(z))(1 - F_Y(z))$ and

$$F_X(z) = F_Y(z) = \begin{cases} 0 & z \le 0 \\ z & 0 \le z \le 1 \\ 1 & z \ge 1. \end{cases}$$

Thus

$$F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z)) = \begin{cases} 0 & z \le 0 \\ 2z - z^2 & 0 \le z \le 1 \\ 1 & z \ge 1 \end{cases}$$

and

$$f_Z(z) = \begin{cases} 2 - 2z & 0 \le z \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(8) If $x \ge 0$ then $X^+ > x$ if and only if $X > x$. Thus $F_{X^+}(x) = F(x)$.
If $x < 0$ then $X^+ > x$ automatically. So $F_X^+(x) = \mathbb{P}(X^+ \le x) = 0$.
Thus

$$F_{X^+}(x) = \begin{cases} 0 & x < 0 \\ F_X(x) & x \ge 0. \end{cases}$$

(9) We have $f_X(x) = \frac{1}{\beta}e^{-x/\beta}$ for $x > 0$. Thus if $x > 0$

$$F_X(x) = \int_{\infty}^{x} f_X(t)dt = 1 - e^{-x/\beta}.$$

In general,

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x/\beta} & x \ge 0. \end{cases}$$

For $0 \le q \le 1$, $F_X^{-1}(q) = \inf\{x : F_X(x) \ge q\}$. Solving $1 - e^{-x/\beta} \ge q$ for x yields $x \ge -\beta \log(1-q)$. So $F_X^{-1}(q) = -\beta \log(1-q)$.

(11) (a) $\mathbb{P}(X = 1, Y = 1) = 0$ but $\mathbb{P}(X = 1) = 1/2$ and $\mathbb{P}(Y = 1) = 1/2$.

(b) Choose $n \ge 0$. Then

$$\mathbb{P}(X = n) = f_X(n) = \sum_{N=n}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!}\binom{N}{n}.$$

I.e. the total number of flips N ranges from n to ∞, there is probability $e^{-\lambda}\lambda^N/N!$ for the number N of flips, and the number of possible ways for n of those flips to be heads is $\binom{N}{n}$. The above infinite sum is also equal to $f_Y(n)$. Simplifying $\binom{N}{n}/N!$ to $\frac{1}{n!(N-n)!}$ and pulling out factors of e^{λ} and $1/n!$ from the sum yields that $f_X(n)$ is

$$\frac{e^{-\lambda}}{n!}\sum_{N=n}^{\infty}\frac{(\lambda/2)^N}{(N-n)!} = \frac{e^{-\lambda}}{n!}(\lambda/2)^n\sum_{m=0}^{\infty}\frac{(\lambda/2)^m}{m!} = \frac{e^{-\lambda}}{n!}(\lambda/2)^n e^{\lambda/2} = \frac{e^{-\lambda/2}}{n!}(\lambda/2)^n$$

On the other hand,

$$\mathbb{P}(X = m, Y = n) = f_{X,Y}(m,n) = \frac{e^{-\lambda}\lambda^{m+n}}{(m+n)!}\binom{m+n}{m}(1/2)^{m+n} = \frac{e^{-\lambda}(\lambda/2)^{m+n}}{m!n!}.$$

I.e. the first term in the product is the probability of $m+n$ total flips and the second term is the probability of seeing m heads and n tails among these flips. By our simplified expression for f_X and f_Y we see that the above expression for $\mathbb{P}(X = m, Y = n)$ is equal to $\mathbb{P}(X = m)\mathbb{P}(Y = n)$ and X and Y are indeed independent.

(13) (a) The PDF for X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x/2}.$$

We have

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = F_X(\log(y)) = \int_{-\infty}^{\log(y)} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

for $y > 0$ and $F_Y(y) = 0$ elsewhere. The derivative is

$$f_Y(y) = \frac{1}{y}\frac{1}{\sqrt{2\pi}}e^{-\log^2 y/2}$$

for $y > 0$ and $F_Y(y) = 0$ elsewhere.

(b) See the Jupyter Notebook: 13.ipynb

(14) We have

$$F_R(r) = \mathbb{P}(\sqrt{x^2+y^2} \le r) = \begin{cases} 0 & r \le 0 \\ r^2 & 0 \le r \le 1 \\ 1 & r \ge 1 \end{cases}$$

i.e. this is the area of the disk of radius r after normalizing so that the unit disk has area 1 by dividing by its usual area, π. Taking the derivative yields

$$f_R(r) = \begin{cases} 2r & 0 \le r \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(15) Since F is continuous and strictly increasing, F^{-1} exists as a function $F^{-1} : (0,1) \to \mathbb{R}$. For $0 \le y \le 1$ we have

$$F_Y(y) = \mathbb{P}(F(X) \le y) = \mathbb{P}(X \le F^{-1}(y)) = F(F^{-1}(y)) = y.$$

So

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 \le y \le 1 \\ 1 & y \ge 1 \end{cases}$$

and $f_Y = \chi_{[0,1]}$, the indicator function of $[0,1]$.

If $U \sim \text{Uniform}(0,1)$ and $X = F^{-1}(U)$ then

$$F_X(x) = \mathbb{P}(F^{-1}(U) \le x) = \mathbb{P}(U \le F(x)) = \begin{cases} 0 & F(x) \le 0 \\ F(x) & 0 \le F(x) \le 1 \\ 1 & F(x) \ge 1 \end{cases}$$

Of course the first and third cases are vacuous, so $F_X(x) = F(x)$.

Now we consider the case $F(x) = -e^{-x/\beta} + 1$, the CDF for the exponential distribution. The inverse of this function is $F^{-1} : (0,1) \to \mathbb{R}$, $F^{-1}(y) = -\beta \log(1-y)$. So if $U \sim \text{Uniform}(0,1)$ then $F^{-1}(U) \sim \text{Exp}(\beta)$.

See the Jupyter Notebook 15.ipynb for a demo.

(16) We consider conditional probability distributions:

$$f_{X|X+Y}(x|n) = \frac{f_{X,X+Y}(x,n)}{f_{X+Y}(n)}.$$

We have $f_{X,X+Y}(x,y) = \mathbb{P}(X = x, X+Y = n) = \mathbb{P}(X = x)\mathbb{P}(Y = n-x) = f_X(x) f_Y(n-x)$. Plugging in for f_X, f_Y, and f_{X+Y} and noting that $X+Y \sim \text{Poisson}(\lambda+\mu)$ yields that $f_{X|X+Y}(x|n)$ is equal to

$$\frac{e^{-\lambda}\frac{\lambda^x}{x!} e^{-\mu}\frac{\mu^{n-x}}{(n-x)!}}{e^{-\lambda-\mu}\frac{(\lambda+\mu)^n}{n!}} = \binom{n}{x} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{n-x} = \binom{n}{x} (\lambda/\pi)^x (1-\lambda/\pi)^{n-x}.$$

This is exactly the expression for the PDF of Binomial(n, π) at x.

(17) We plug in $y = 1/2$ and consider the resulting distribution. We have

$$\mathbb{P}\left(X < \frac{1}{2} | Y = \frac{1}{2}\right) = \int_0^{1/2} \frac{c(x+1/4)}{f_Y(1/2)} dx = \frac{c}{4 f_Y(1/2)}$$

So we need to compute the marginal distribution $f_Y(y)$. We have

$$f_Y(y) = \int_0^1 c(x+y^2) dx = c\left(\frac{1}{2} + y^2\right)$$

To compute c we solve

$$1 = \int_0^1 \int_0^1 c(x+y^2) dx dy = \int_0^1 \left(\frac{1}{2}c + cy^2\right) dy = \frac{1}{2}c + \frac{1}{3}c = \frac{5}{6}c.$$

Thus $c = \frac{6}{5}$. Finally we can plug in to find

$$f_Y\left(\frac{1}{2}\right) = \frac{6}{5}\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{9}{10} \text{ and } \mathbb{P}\left(X < \frac{1}{2} | Y = \frac{1}{2}\right) = \frac{6/5}{4 \cdot 9/10} = \frac{1}{3}.$$

(18) See the Jupyter Notebook: 18.ipynb