Solutions to selected exercises from Chapter 12 of *Wasserman — All of Statistics*

(1) (a) Let's calculate the posterior distribution $p|x$. We have

$$f(p|x) \propto f(p)\mathcal{L}(p) \propto p^{\alpha-1}(1-p)^{\beta-1}p^x(1-p)^{n-x}.$$

Thus we recognize $p|x$ as a Binomial$(\alpha+x, \beta+n-x)$ random variable. By Theorem 12.8, the Bayes estimator for p is the posterior mean:

$$\hat{p}(x) = \frac{\alpha+x}{\alpha+\beta+n}.$$

(b) Here we have

$$f(\lambda|x) \propto f(\lambda)\mathcal{L}(\lambda) \propto \lambda^{\alpha-1}e^{-x/\beta}\lambda^x.$$

Thus we recognize $\lambda|x$ as a Gamma$(\alpha+x, \beta)$ random variable. Again, the Bayes estimator for λ is the posterior mean:

$$\hat{\lambda}(x) = (\alpha+x)\beta.$$

(c) Here

$$f(\theta|x) \propto \exp\left(\frac{-(\theta-a)^2}{2b^2}\right)\exp\left(\frac{-(x-\theta)^2}{2\sigma^2}\right) = \exp\left(-\frac{\sigma^2(\theta-a)^2+b^2(\theta-x)^2}{2b^2\sigma^2}\right).$$

Completing the square yields that $f(\theta|x)$ is proportional to

$$\exp\left(-\frac{(\sigma^2+b^2)\theta^2-2(a\sigma^2+xb^2)\theta}{2b^2\sigma^2}\right) \propto \exp\left(-\frac{\sigma^2+b^2}{2b^2\sigma^2}\left(\theta-\frac{a\sigma^2+xb^2}{\sigma^2+b^2}\right)^2\right).$$

Thus we recognize $\theta|x$ as a $N\left(\frac{a\sigma^2+xb^2}{\sigma^2+b^2}, \frac{b^2\sigma^2}{\sigma^2+b^2}\right)$ random variable. The Bayes estimator is the posterior mean:

$$\hat{\theta}(x) = \frac{a\sigma^2+xb^2}{\sigma^2+b^2}.$$