Solutions to selected exercises from Chapter 1 of *Wasserman — All of Statistics*

(5) The sample space is

$$\Omega = \{HH, THH, HTH, HTTH, THTH, TTHH, HTTTH, THTTH, \ldots\}.$$

I.e. an element of Ω is a sequence in T and H of length k with the last element being H and exactly one of the first $k-1$ elements being H.

There are exactly $k-1$ sequences in Ω of length k, corresponding to the $k-1$ choices for where to put an H in the initial sequence of length $k-1$. Each length k sequence has probability $(1/2)^k$ of occurring. Hence the probability that exactly k tosses are required is $(k-1)(1/2)^k$

(8) We have $\mathbb{P}(\bigcap A_i) = 1 - \mathbb{P}(\bigcup A_i^c) \geq 1 - \sum \mathbb{P}(A_i^c)$ where $\cdot^c$ denotes the complement. Since $\mathbb{P}(A_i^c) = 0$ for each i we have $\mathbb{P}(\bigcap A_i) = 0$, as claimed.

(11) We have $\mathbb{P}(A^c B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(AB)$. Since A and B are independent this is

$$1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) = (1 - \mathbb{P}(A))(1 - \mathbb{P}(B)) = \mathbb{P}(A^c)\mathbb{P}(B^c).$$

(12) The sample space of possible sequences of sides that we can see is $\Omega = \{RR, GG, RG, GR\}$ where R denotes a red side and G denotes a green side. The elements of Ω have probability 1/3, 1/3, 1/6, and 1/6. Hence the probability that the other side is green, given that the first side we see is green is

$$\frac{1/3}{1/3 + 1/6} = \frac{2}{3}.$$

(13) (a) The sample space is

$$\Omega = \{HT, TH, HHT, TTH, HHHT, TTTH, HHHHT, TTTTH, \ldots\}$$

I.e. an element of Ω is a sequence of length n where the first $n-1$ elements are H and the last element is T, or vice versa. In particular, for any $n \geq 2$ there are exactly two sequences in Ω of length n.

(b) The probability of three tosses being required is $2(1/2)^3 = 1/4$ since there are two elements of Ω of length 3, both with probability $(1/2)^3$.

(15) (a) Order the children from youngest to oldest. Thus the sample space consists of sequences of length three in B and N, B indicating that a child has blue eyes and N indicating that they do not have blue eyes:

$$\Omega = \{NNN, BNN, NBN, NNB, BBN, BNB, NBB, BBB\}.$$

If at least one child has blue eyes then the element of the sample space Ω can be anything except for NNN. The probability of this occuring is

$$3(1/4)(3/4)^2 + 3(1/4)^2(3/4) + (1/4)^3.$$

Here the first term is the probability of one of the sequences BNN, NBN, NNB, the second term is the probability of one of the sequences BBN, BNB, NBB, and the third term is the probability of BBB. So the probability that at least two children have blue eyes is

$$3(1/4)^2(3/4) + (1/4)^3.$$

The probability that at least two children have blue eyes given that one child has blue eyes is

$$\frac{3(1/4)^2(3/4) + (1/4)^3}{3(1/4)(3/4)^2 + 3(1/4)^2(3/4) + (1/4)^3} = \frac{10}{37}.$$

(b) If the first child has blue eyes then the sequence is BNN, BBN, BNB, or BBB. The probability for this event is $(1/4)(3/4)^2+2(1/4)^2(3/4)+(1/4)^3$. The probability that the first child has blue eyes and at least two children have blue eyes is $2(1/4)^2(3/4)+(1/4)^3$. So the conditional probability is

$$\frac{2(1/4)^2(3/4) + (1/4)^3}{(1/4)(3/4)^2 + 2(1/4)^2(3/4) + (1/4)^3} = \frac{7}{16}.$$

(19) Let M stand for Mac, W for Windows, L for Linux, and V for virus. By Bayes's Theorem,

$$\mathbb{P}(W|V) = \frac{\mathbb{P}(V|W)\mathbb{P}(W)}{\mathbb{P}(V|M)\mathbb{P}(M) + \mathbb{P}(V|W)\mathbb{P}(W) + \mathbb{P}(V|L)\mathbb{P}(L)}.$$

Substituting yields that $\mathbb{P}(W|V)$ is

$$\frac{(82/100)(50/100)}{(65/100)(30/100) + (82/100)(50/100) + (50/100)(20/100)} = \frac{82}{141} \approx 58.156\%$$

(20) (a) By Bayes's Theorem:

$$\mathbb{P}(C_i|H) = \frac{\mathbb{P}(H|C_i)\mathbb{P}(C_i)}{\sum_j \mathbb{P}(H|C_j)\mathbb{P}(C_j)} = \frac{p_i(1/5)}{\sum_j p_j(1/5)} = \frac{p_i}{1/4+1/2+3/4+1} = \frac{2p_i}{5}.$$

I.e. the probabilities are

$$0,\ 1/10,\ 1/5,\ 3/10,\ 2/5.$$

(b) We have $\mathbb{P}(H_2|H_1) = \mathbb{P}(H_1H_2)/\mathbb{P}(H_1)$ and $\mathbb{P}(H_1) = \sum p_i/5 = 1/2$. We also have

$$\mathbb{P}(H_1H_2) = \sum_{i=1}^{5} \mathbb{P}(H_1H_2C_i) = \sum_i \mathbb{P}(H_1H_2|C_i)\mathbb{P}(C_i) = 1/5\sum_i p_i^2 = 3/8.$$

Therefore

$$\mathbb{P}(H_2|H_1) = \frac{3/8}{1/2} = \frac{3}{4}.$$

(c) By Bayes's Theorem:

$$\mathbb{P}(C_i|B_4) = \frac{\mathbb{P}(B_4|C_i)\mathbb{P}(C_i)}{\sum_j \mathbb{P}(B_4|C_j)\mathbb{P}(C_j)} = \frac{(1-p_i)^3p_i(1/5)}{\sum_j(1-p_j)^3p_j(1/5)} = \frac{128(1-p_i)^3p_i}{23}.$$

The probabilities are

$$0,\ 27/46,\ 8/23,\ 3/46,\ 0.$$

(22) See the Jupyter Notebook 22.ipynb.

(23) See the Jupyter Notebook 23.ipynb.