Solutions to selected exercises from Chapter 10 of *Wasserman — All of Statistics*

(5) (a) Choose some $c \geq 0$. The rejection region is

$$R = \{(x_1, \ldots, x_n) : x_i > c \text{ for some } i\}.$$

Hence the power function is $\beta(\theta) = \mathbb{P}_\theta((X_1, \ldots, X_n) \in R)$ and this is

$$\mathbb{P}_\theta(X_i > c \text{ for some } i) = 1 - \mathbb{P}_\theta(X_i < c \text{ for all } i) = 1 - \Pi_{i=1}^n \mathbb{P}_\theta(X_i < c).$$

If $c \geq \theta$ then $\mathbb{P}_\theta(X_i < c) = 1$ for all i. If $c < \theta$ then $\mathbb{P}_\theta(X_i < c) = c/\theta$ for all i. Hence

$$\beta(\theta) = \begin{cases} 1 - (c/\theta)^n & \theta > c \\ 0 & \theta \leq c \end{cases}$$

(b) The size is

$$\alpha = \sup_{\theta = 1/2} \beta(\theta) = \beta(1/2) = 1 - (2c)^n.$$

So we need to solve $1 - (2c)^n = 0.05$, i.e. $(2c)^n = 19/20$. The solution is

$$c = \frac{1}{2}\left(\frac{19}{20}\right)^{1/n}.$$

(c) Choosing the parameter c, the size is $\alpha = 1 - (2c)^n$ with corresponding rejection region $R_\alpha = \{(x_1, \ldots, x_n) : x_i > c \text{ for some } i\}$. The p-value is

$$\inf\{\alpha \in (0,1) : Y \in R_\alpha\} = \inf\{\alpha \in (0,1) : Y > c\}$$

Since α is strictly monotonically decreasing with c, the p-value is exactly the value of α when $Y = c$. In other words, the p-value is

$$\alpha = 1 - (2Y)^n = 1 - (2 \cdot 0.48)^{20} \approx 0.558.$$

We would not reject the null hypothesis H_0 in this case.

(d) Of course we can reject H_0 in this case: θ is definitely not equal to $1/2$. Let's confirm this using the p-value. The p-value in this case is

$$\inf\{\alpha \in (0,1) : Y > c\} = \inf\{\alpha \in (0,1) : Y > 1/2(1-\alpha)^{1/20}\}.$$

This is equal to 0 since $Y > 1/2$ and we reject H_0 since our p-value 0 is < 0.05.

(6) Set $n = 1919$. Then we have Bernoulli random variables $X_1, \ldots, X_n$ with $X_i = 0$ if person i died the week before Passover and $X_i = 1$ otherwise. Let $\theta \in [0,1]$ be the parameter with $X_i \sim \text{Bernoulli}(\theta)$ for each i. So we want to test $H_0 : \theta = \theta_0 = 1/2$ versus $H_1 : \theta \neq 1/2$. We will test this using the Wald test.

Our test statistic is

$$w = \frac{\hat{\theta} - \theta_0}{\widehat{\text{se}}}.$$

Our estimate for $\hat{\theta}$ is the mean

$$\hat{\theta} = \overline{X} = \frac{1}{n}\sum X_i = \frac{997}{1919} \approx 0.5195.$$

The standard error of $\overline{X}$ is $\sigma/\sqrt{n}$ where σ is the standard deviation of the X_i. So we estimate se by $\widehat{\text{se}} = \widehat{\sigma}/\sqrt{n}$ where $\widehat{\sigma}$ is the estimated standard deviation of the X_i:

$$\widehat{\text{se}}^2 = \frac{1}{n}\hat{\theta}(1-\hat{\theta}) = \frac{1}{1919}\frac{997}{1919}\frac{922}{1919} \approx 0.00013, \quad \widehat{\text{se}} \approx 0.0114.$$

Plugging in,

$$w \approx \frac{0.5195 - 0.5}{0.0114} \approx 1.7105.$$

By Theorem 10.13, our estimated p-value is

$$2\Phi(-w) \approx 0.08717.$$

We can view this as weak evidence against H_0 but we don't reject H_0 at the size threshold of 0.05.

A 95% confidence interval for θ is given by

$$(\hat{\theta} - 2\widehat{\text{se}}, \hat{\theta} - 2\widehat{\text{se}}) = (0.4967, 0.5423).$$

(7) See the Jupyter Notebook 7.ipynb.

(8) (a) We have

$$\sum X_i \sim N(n\theta, n), \quad \overline{X_n} = \frac{1}{n}\sum X_i \sim N(\theta, n/n^2) = N(\theta, 1/n).$$

So

$$\beta(\theta) = \mathbb{P}_\theta((X_1, \ldots, X_n) \in R) = \mathbb{P}_\theta(\overline{X_n} > c)$$

and this is equal to

$$\mathbb{P}_\theta(\sqrt{n}(\overline{X_n} - \theta) > \sqrt{n}(c-\theta)) = 1 - \Phi(\sqrt{n}(c-\theta)).$$

The size is

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(0) = 1 - \Phi(c\sqrt{n}).$$

I.e. $\Phi(c\sqrt{n}) = 1 - \alpha$ and solving for c yields $c = \frac{1}{\sqrt{n}} z_\alpha$.

(b) $\beta(1) = 1 - \Phi(\sqrt{n}(c-1))$

(c) As $n \to \infty$, $c - 1 \to -1$ so $\sqrt{n}(c-1) \to -\infty$. Thus $\Phi(\sqrt{n}(c-1)) \to 0$ and $\beta(1) \to 1$.

(11) See the Jupyter Notebook 11.ipynb.

(12) See the Jupyter Notebook 12.ipynb.

(13) We assume σ to be known. The likelihood ratio test statistic is

$$\lambda = 2\log\left(\frac{\mathcal{L}(\hat{\mu})}{\mathcal{L}(\mu_0)}\right)$$

where $\hat{\mu}$ is the maximum likelihood estimator $\hat{\mu} = \overline{X} = \frac{1}{n}\sum X_i$.

We have

$$\mathcal{L}(\hat{\mu}) = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\hat{\mu})^2/2\sigma^2} = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\sum(X_i-\hat{\mu})^2/2\sigma^2}.$$

Expanding $(X_i - \mu_0)^2$ as

$$(X_i - \hat{\mu})^2 + 2(\hat{\mu} - \mu_0)(X_i - \hat{\mu}) + (\hat{\mu} - \mu_0)^2$$

yields

$$\sum (X_i - \mu_0)^2 = \sum (X_i - \hat{\mu})^2 + n(\hat{\mu} - \mu_0)^2.$$

So we have

$$\mathcal{L}(\mu_0) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\left(\sum(X_i-\hat{\mu})^2/2\sigma^2 + n(\hat{\mu}-\mu_0)^2/2\sigma^2\right)}.$$

Thus

$$\frac{\mathcal{L}(\hat{\mu})}{\mathcal{L}(\mu_0)} = e^{n(\hat{\mu}-\mu_0)^2/2\sigma^2}$$

and our statistic is

$$\lambda = 2\log\left(\frac{\mathcal{L}(\hat{\mu})}{\mathcal{L}(\mu_0)}\right) = n(\hat{\mu} - \mu_0)^2/\sigma^2.$$

Say that we reject H_0 if $\lambda > c$ for some fixed c. This is equivalent to

$$\frac{|\hat{\mu} - \mu_0|}{\sigma/\sqrt{n}} > \sqrt{c}.$$

Let's compute the size of this test. Assuming $\mu = \mu_0$, we have that $\frac{\hat{\mu}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$. In this case, the probability of the inequality above is

$$\Phi(-\sqrt{c}) + (1 - \Phi(\sqrt{c})) = 2 - 2\Phi(\sqrt{c})$$

and this is the size of our test. Setting the size to be α yields $c = z_{\alpha/2}^2$. So the size α likelihood ratio test is

$$\frac{|\hat{\mu} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}.$$

The size α Wald test for rejecting H_0 is

$$\frac{|\hat{\mu} - \mu_0|}{\widehat{\text{se}}} > z_{\alpha/2}.$$

So we see that the two tests are the same up to replacing $\widehat{\text{se}}$ (an estimate of the standard error of $\hat{\mu}$) by $\sigma/\sqrt{n}$ (the actual standard error of $\hat{\mu}$).