Exploring integer partitioning applied to constraints on biodiversity

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Background

The total number of species that can possibly exist is bounded above by the total number of individuals. If there are n individuals there can be at most $s = n$ species. However, the value of s is likely to be much smaller than n. In fact, from a purely combinatorial perspective there are $p_k(n)$ ways of allocating n individuals across $s = k$ species for abitraty $k \leq n$. This is the realm of integer partitioning. From a statistical mechanics perspective, the most likely number of species for a given n is the number that maximizes the microstates (i.e. the value of k that maximizes $p_k(n)$). Because our world is finite, we might expect some reasonable variation about this maximum entropy k. The probabilities for each k_i are equal to $p_{k_i}(n)/p(n)$, where $p(n) = \sum_k p_k(n)$ is the total number of partitions of n across all k.

Set-up

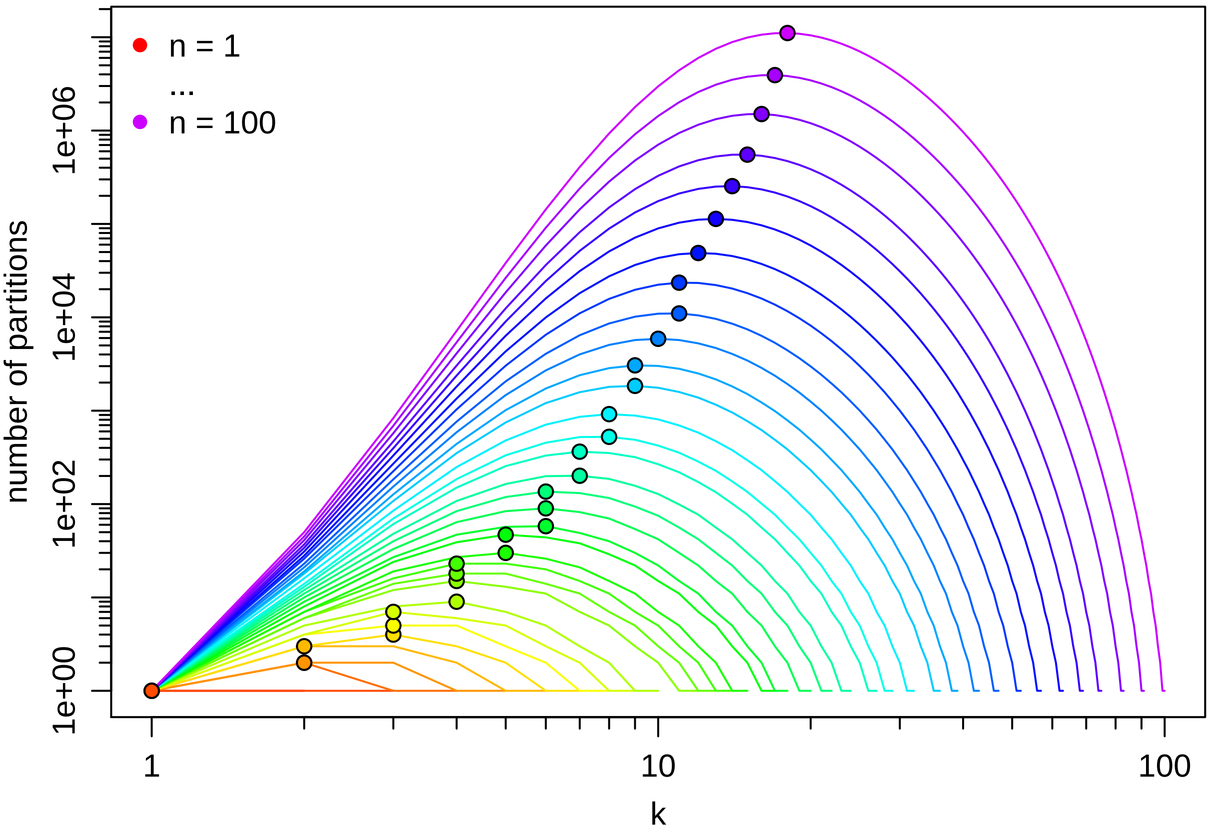
```r
library(partitions)
library(socorro)
```

Initial exploration

Explore how the number of ways to partition an integer n into k parts changes across values of n and k:

```r
## max of n
N <- 100

## loop over n values, for each evaluate k in 1:n
n <- unique(round(10^seq(0, log(N, 10), length = 50)))
npart <- sapply(n, function(ni) {
    out <- sapply(1:ni, function(k) R(k, ni))
    out <- c(out, rep(NA, N - length(out)))
    return(out)
})

## plotting
par(mar = c(3.5, 3.5, 0.5, 0.5), mgp = c(2, 0.75, 0))
matplot(npart, type = 'l', log = 'xy', axes = FALSE, frame.plot = TRUE,
        col = rainbow(length(n), end = 0.8), lty = 1,
        xlab = 'k', ylab = 'number of partitions')
legend('topleft',
       legend = c('n = 1', '...', 'n = 100'), pch = c(16, NA, 16),
       col = rainbow(length(n), end = 0.8)[c(1, 1, length(n))],
       bty = 'n')
logAxis(1)
logAxis(2)
```

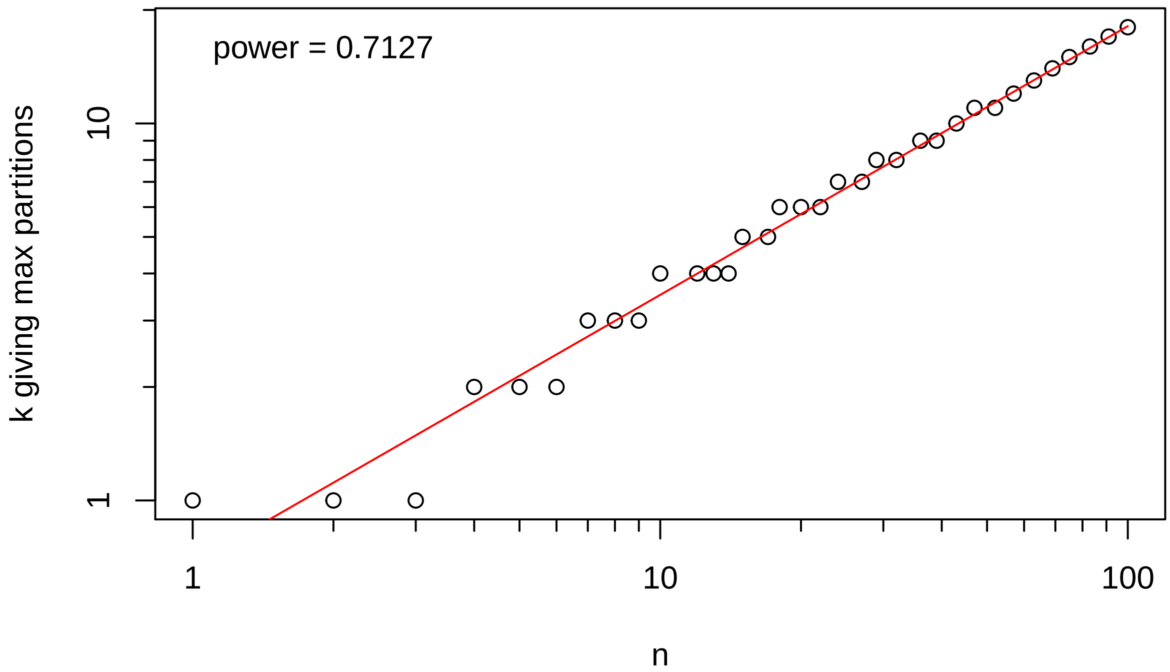
```r
## add the values of k that maximize number of parts
kmax <- apply(npart, 2, which.max)

points(kmax, npart[cbind(kmax, 1:length(kmax))], pch = 21,
       bg = rainbow(length(n), end = 0.8))
```



Note the color gradient starts at $n = 1$ in red, goes through the rainbow colors, and ends at $n = 100$ in purple. Thus each curve represents the number of partitions for a given n across the full range of k values $k \in \{1, \ldots, n\}$.

The filled dots show the maximum number of partitions achieved for each n. We could also look at this in terms of what k value maximizes the number of partitions for each n. Doing so we find a power law:

```r
plot(n, kmax, log = 'xy', axes = FALSE, frame.plot = TRUE,
     xlab = 'n', ylab = 'k giving max partitions')
logAxis(1)
logAxis(2)
mod <- lm(log(kmax) ~ log(n))
curve(exp(mod$coeff[1])*x^mod$coeff[2], add = TRUE, col = 'red')
legend('topleft', legend = paste('power =', round(mod$coeff[2], 4)), bty = 'n')
```


power = 0.7127
k giving max partitions
10
1
1
10
100
n