

MIU EFTERÅR 2021

ALGORITMER OG DATASTRUKTURER

Forelæsning 3: Algoritmeanalyse

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Example: 3-SUM

3-SUM. Given N distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt  
8  
30 -40 -20 -10 40 0 10 5  
  
% java ThreeSum 8ints.txt  
4
```

a[i]	a[j]	a[k]	sum
30	-40	10	0
30	-20	-10	0
-40	40	0	0
-10	0	10	0

3-SUM: brute-force algorithm

```
public class ThreeSum {  
    public static int count(int[] a) {  
        int N = a.length;  
        int count = 0;  
        for (int i = 0; i < N; i++)  
            for (int j = i+1; j < N; j++)  
                for (int k = j+1; k < N; k++)  
                    if (a[i] + a[j] + a[k] == 0) ← check each triple  
                        count++;  
        return count;  
    }  
  
    public static void main(String[] args) {  
        StdIn in = new StdIn(args[0]);  
        int[] a = in.readAllInts();  
        StdOut.println(count(a));  
    }  
}
```

Measuring the running time

Q. How to time a program?

A. Manual.



```
% java ThreeSum 1Kints.txt
```



70

```
% java ThreeSum 2Kints.txt
```



528

```
% java ThreeSum 4Kints.txt
```



Measuring the running time

Q. How to time a program?

A. Automatic.

public class Stopwatch

(part of stdlib.jar)

Stopwatch()

create a new stopwatch

double elapsedTime()

time since creation (in seconds)

```
public static void main(String[] args){  
    StdIn in = new StdIn(args[0]);  
    int[] a = in.readAllInts();  
    Stopwatch stopwatch = new Stopwatch();  
    StdOut.println(ThreeSum.count(a));  
    double time = stopwatch.elapsedTime();  
    StdOut.println("elapsed time " + time);  
}
```

Empirical analysis

Run the program for various input sizes and measure running time.

```
% ]
```

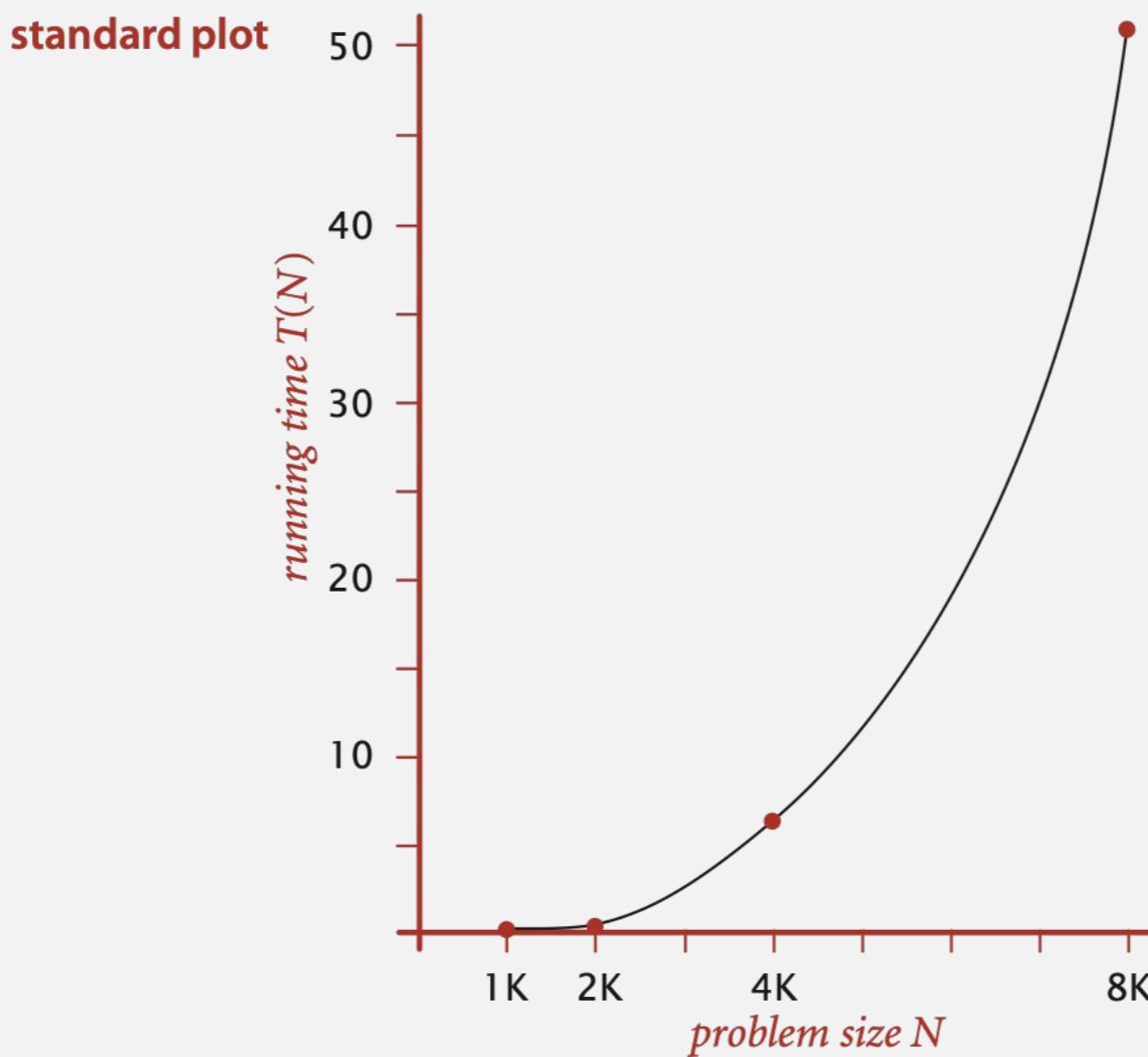
Empirical analysis

Run the program for various input sizes and measure running time.

N	time (seconds) [†]
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

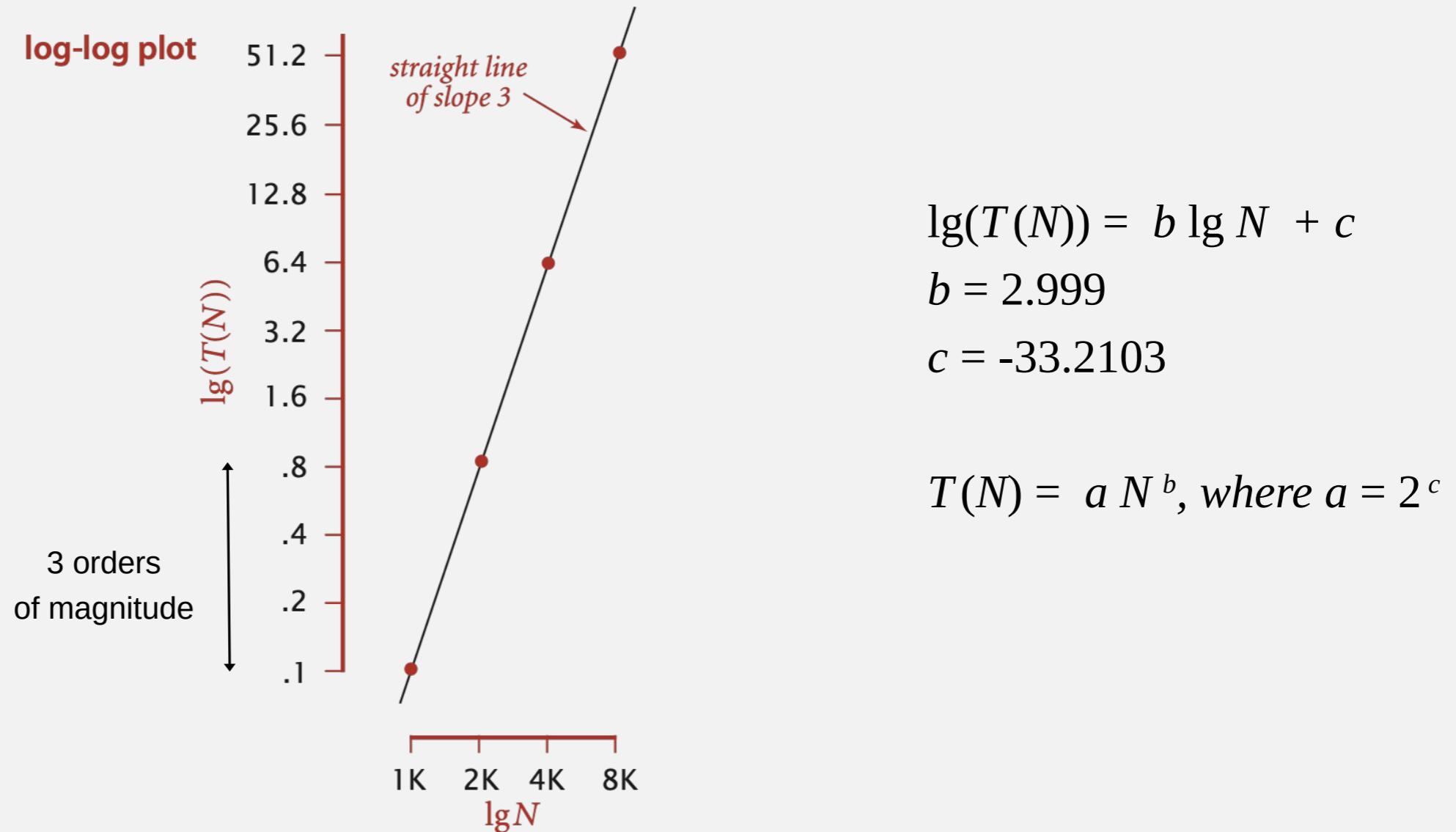
Data analysis

Standard plot. Plot running time $T(N)$ vs. input size N .



Data analysis

Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



Regression. Fit straight line through data points: $a N^b$.

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.



"order of growth" of running
time is about N^3 [stay tuned]

Predictions.

- 51.0 seconds for $N = 8,000$.
- 408.1 seconds for $N = 16,000$.

Observations.

N	time (seconds) [†]
8,000	51.1
8,000	51
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Run program, **doubling** the size of the input.

N	time (seconds) [†]	ratio	lg ratio
250	0		-
500	0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8	3
8,000	51.1	8	3

$$\begin{aligned}\frac{T(2N)}{T(N)} &= \frac{a(2N)^b}{aN^b} \\ &= 2^b\end{aligned}$$

$\lg (6.4 / 0.8) = 3.0$

seems to converge to a constant $b \approx 3$

Hypothesis. Running time is about a N^b with $b = \lg$ ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of N) and solve for a .

N	time (seconds) [†]
8,000	51.1
8,000	51
8,000	51.1

$$51.1 = a \times 8000^3$$

$$\Rightarrow a = 0.998 \times 10^{-10}$$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times N^3$ seconds.



almost identical hypothesis
to one obtained via linear regression

Experimental algorithmics

System independent effects.

- Algorithm.
 - Input data.
- 
- determines exponent
in power law

System dependent effects.

- Hardware: CPU, memory, cache, ...
 - Software: compiler, interpreter, garbage collector, ...
 - System: operating system, network, other apps, ...
- 
- determines constant in
power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments



Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

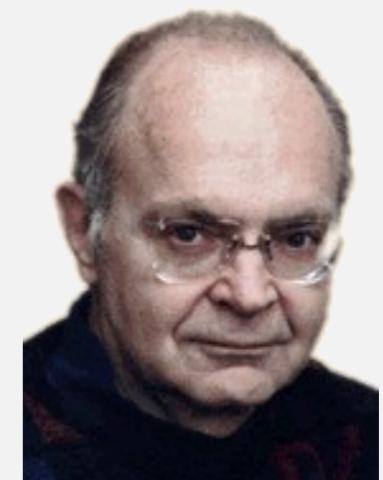
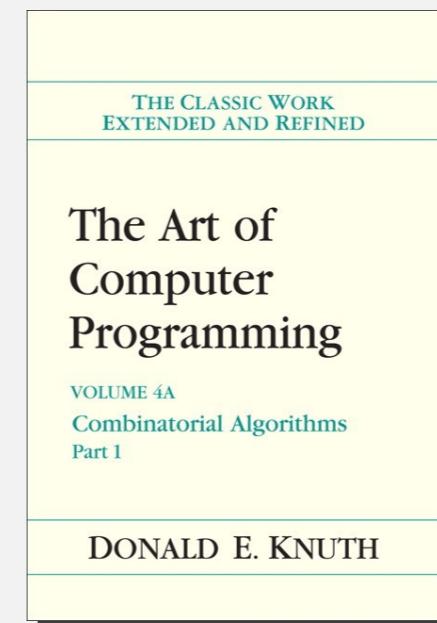
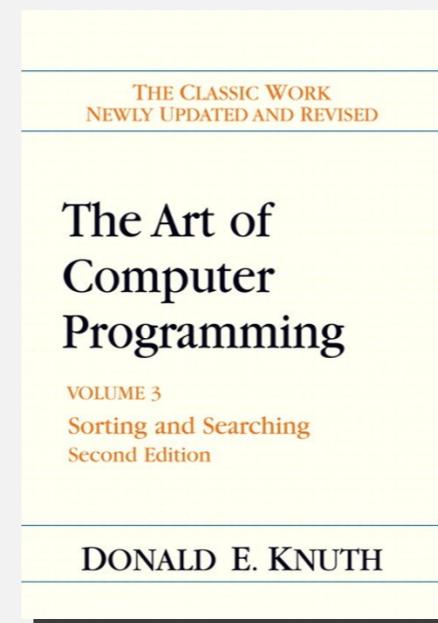
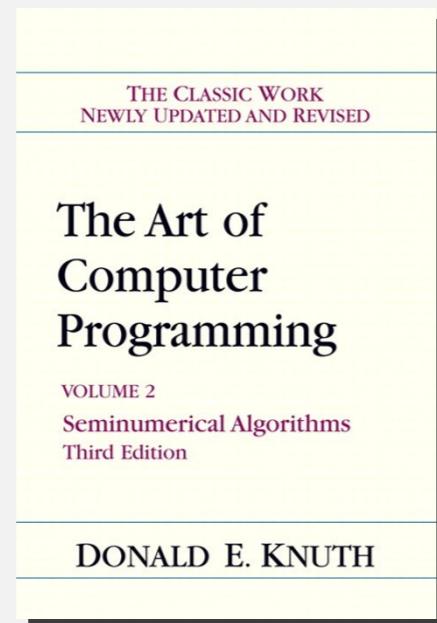
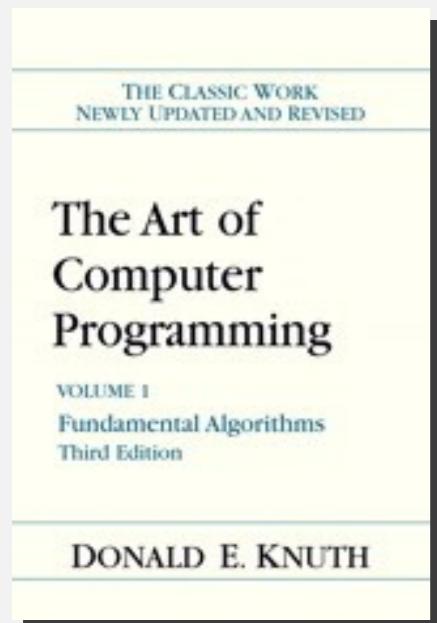
1.4 ANALYSIS OF ALGORITHMS

- ▶ *introduction*
- ▶ *observations*
- ▶ ***mathematical models***
- ▶ *order-of-growth classifications*
- ▶ *theory of algorithms*
- ▶ *memory*

Mathematical models for running time

Total running time: sum of cost \times frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



Donald Knuth
1974 Turing Award

Cost of basic operations

Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	$a + b$	2.1
integer multiply	$a * b$	2.4
integer divide	a / b	5.4
floating-point add	$a + b$	4.6
floating-point multiply	$a * b$	4.2
floating-point divide	a / b	13.5
sine	<code>Math.sin(theta)</code>	91.3
arctangent	<code>Math.atan2(y, x)</code>	129
...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Cost of basic operations

Observation. Most primitive operations take constant time.

operation	example	nanoseconds [†]
variable declaration	int a	c_1
assignment statement	a = b	c_2
integer compare	a < b	c_3
array element access	a[i]	c_4
array length	a.length	c_5
1D array allocation	new int[N]	$c_6 N$
2D array allocation	new int[N][N]	$c_7 N^2$

Caveat. Non-primitive operations often take more than constant time.



novice mistake: abusive string concatenation

Example: 1-SUM

Q. How many instructions as a function of input size N ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    if (a[i] == 0)  
        count++;
```

N array accesses

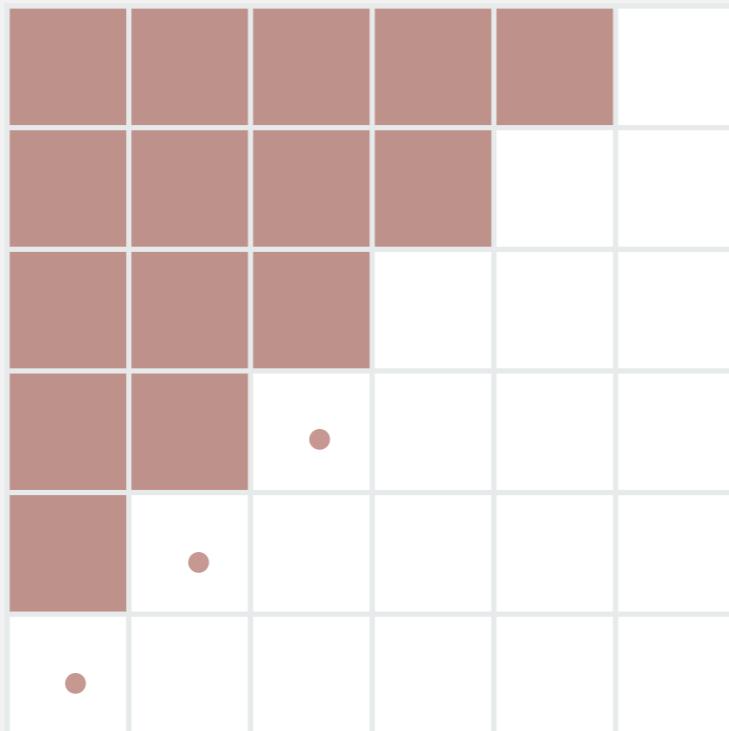
operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	N
array access	N
increment	N to $2N$

Example: 2-SUM

Q. How many instructions as a function of input size N ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

Pf. [n even]



$$\begin{aligned} 0 + 1 + 2 + \dots + (N-1) &= \frac{1}{2}N(N-1) \\ &= \binom{N}{2} \end{aligned}$$

$$0 + 1 + 2 + \dots + (N-1) = \frac{1}{2}N^2 - \frac{1}{2}N$$

half of square half of
diagonal

Example: 2-SUM

Q. How many instructions as a function of input size N ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

$$\begin{aligned}0 + 1 + 2 + \dots + (N - 1) &= \frac{1}{2} N(N - 1) \\&= \binom{N}{2}\end{aligned}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2} (N + 1)(N + 2)$
equal to compare	$\frac{1}{2} N(N - 1)$
array access	$N(N - 1)$
increment	$\frac{1}{2} N(N - 1)$ to $N(N - 1)$

tedious to count exactly

Simplifying the calculations

*“ It is convenient to have a **measure of the amount of work involved in a computing process**, even though it be a very **crude** one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and we shall therefore only attempt to count the number of multiplications and recordings. ” — Alan Turing*

ROUNDING-OFF ERRORS IN MATRIX PROCESSES

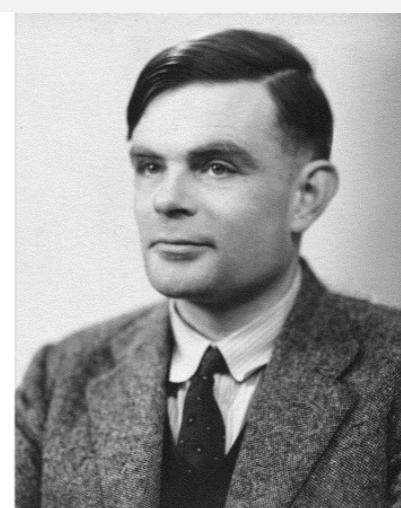
By A. M. TURING

(National Physical Laboratory, Teddington, Middlesex)

[Received 4 November 1947]

SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known ‘Gauss elimination process’, it is found that the errors are normally quite moderate: no exponential build-up need occur.



Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

int count = 0; for (int i = 0; i < N; i++) for (int j = i+1; j < N; j++) if (a[i] + a[j] == 0) count++;		$0 + 1 + 2 + \dots + (N - 1) = \frac{1}{2}N(N - 1)$ $= \binom{N}{2}$
operation	frequency	
variable declaration	$N + 2$	
assignment statement	$N + 2$	
less than compare	$\frac{1}{2}(N + 1)(N + 2)$	
equal to compare	$\frac{1}{2}N(N - 1)$	
array access	$N(N - 1)$	cost model = array accesses
increment	$\frac{1}{2}N(N - 1)$ to $N(N - 1)$	(we assume compiler/JVM do not optimize any array accesses away!)

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N .
- Ignore lower order terms.
 - when N is large, terms are negligible
 - when N is small, we don't care

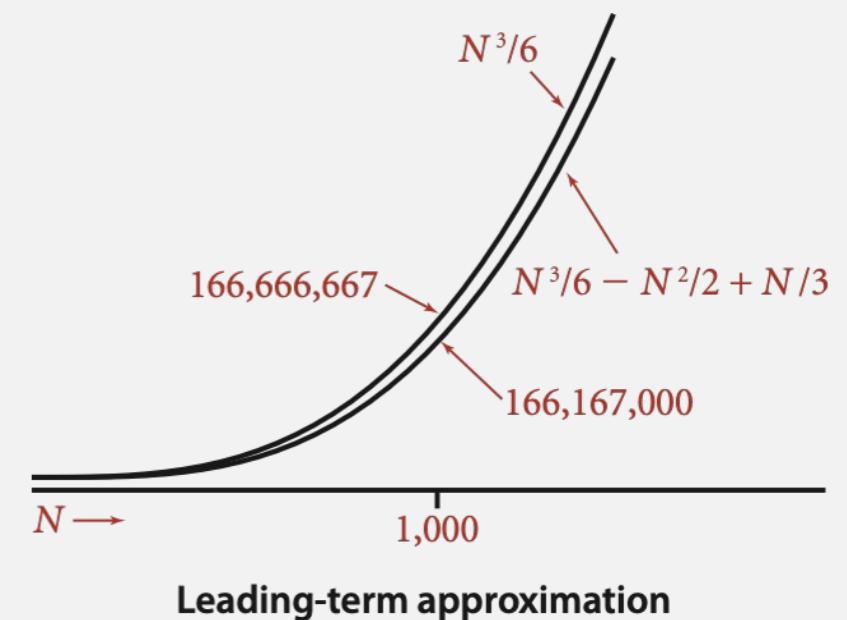
Ex 1. $\frac{1}{6}N^3 + 20N + 16 \sim \frac{1}{6}N^3$

Ex 2. $\frac{1}{6}N^3 + 100N^{4/3} + 56 \sim \frac{1}{6}N^3$

Ex 3. $\frac{1}{6}N^3 - \frac{1}{2}N^2 + \frac{1}{3}N \sim \frac{1}{6}N^3$

discard lower-order terms

(e.g., $N = 1000$: 166.67 million vs. 166.17 million)



Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N .
- Ignore lower order terms.
 - when N is large, terms are negligible
 - when N is small, we don't care

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2} (N + 1) (N + 2)$	$\sim \frac{1}{2} N^2$
equal to compare	$\frac{1}{2} N (N - 1)$	$\sim \frac{1}{2} N^2$
array access	$N (N - 1)$	$\sim N^2$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$	$\sim \frac{1}{2} N^2$ to $\sim N^2$

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size N ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

← "inner loop"

$$\begin{aligned}0 + 1 + 2 + \dots + (N - 1) &= \frac{1}{2} N(N - 1) \\&= \binom{N}{2}\end{aligned}$$

A. $\sim N^2$ array accesses.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size N ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        for (int k = j+1; k < N; k++)  
            if (a[i] + a[j] + a[k] == 0)  
                count++;
```

← "inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!}$$
$$\sim \frac{1}{6}N^3$$

A. $\sim \frac{1}{2} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Estimating a discrete sum

Q. How to estimate a discrete sum?

A. Use Maple or Wolfram Alpha.

The screenshot shows the WolframAlpha PRO interface. The search bar contains the input: `sum(sum(sum(1, k=j+1..N), j = i+1..N), i = 1..N)`. Below the search bar are several small icons. To the right of the search bar are links for "Examples" and "Random". The main result area is titled "Sum:" and displays the mathematical expression:
$$\sum_{i=1}^N \left(\sum_{j=i+1}^N \left(\sum_{k=j+1}^N 1 \right) \right) = \frac{1}{6} N(N^2 - 3N + 2)$$
.

wolframalpha.com

```
[wayne:nobel.princeton.edu] > maple15
|\^/|  Maple 15 (X86 64 LINUX)
\_|\_ |/_\_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2011
\ MAPLE / All rights reserved. Maple is a trademark of
<______> Waterloo Maple Inc.
      | Type ? for help.
> factor(sum(sum(sum(1, k=j+1..N), j = i+1..N), i = 1..N));
```

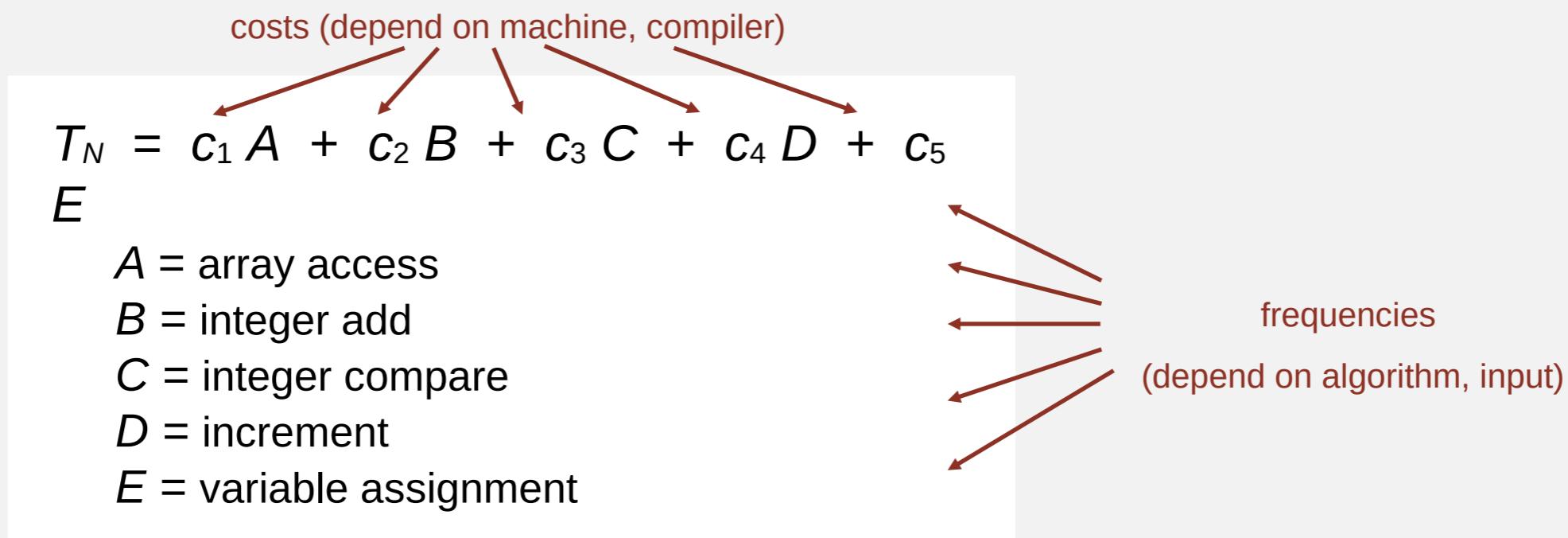
$$\frac{N(N - 1)(N - 2)}{6}$$

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.



Bottom line. We use approximate models in this course: $T(N) \sim c N^3$.

Common order-of-growth classifications

Definition. If $f(N) \sim c g(N)$ for some constant $c > 0$, then the **order of growth** of $f(N)$ is $g(N)$.

- Ignores leading coefficient.
- Ignores lower-order terms.

Ex. The order of growth of the **running time** of this code is N^3 .

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
                count++;
```

Typical usage. With running times.

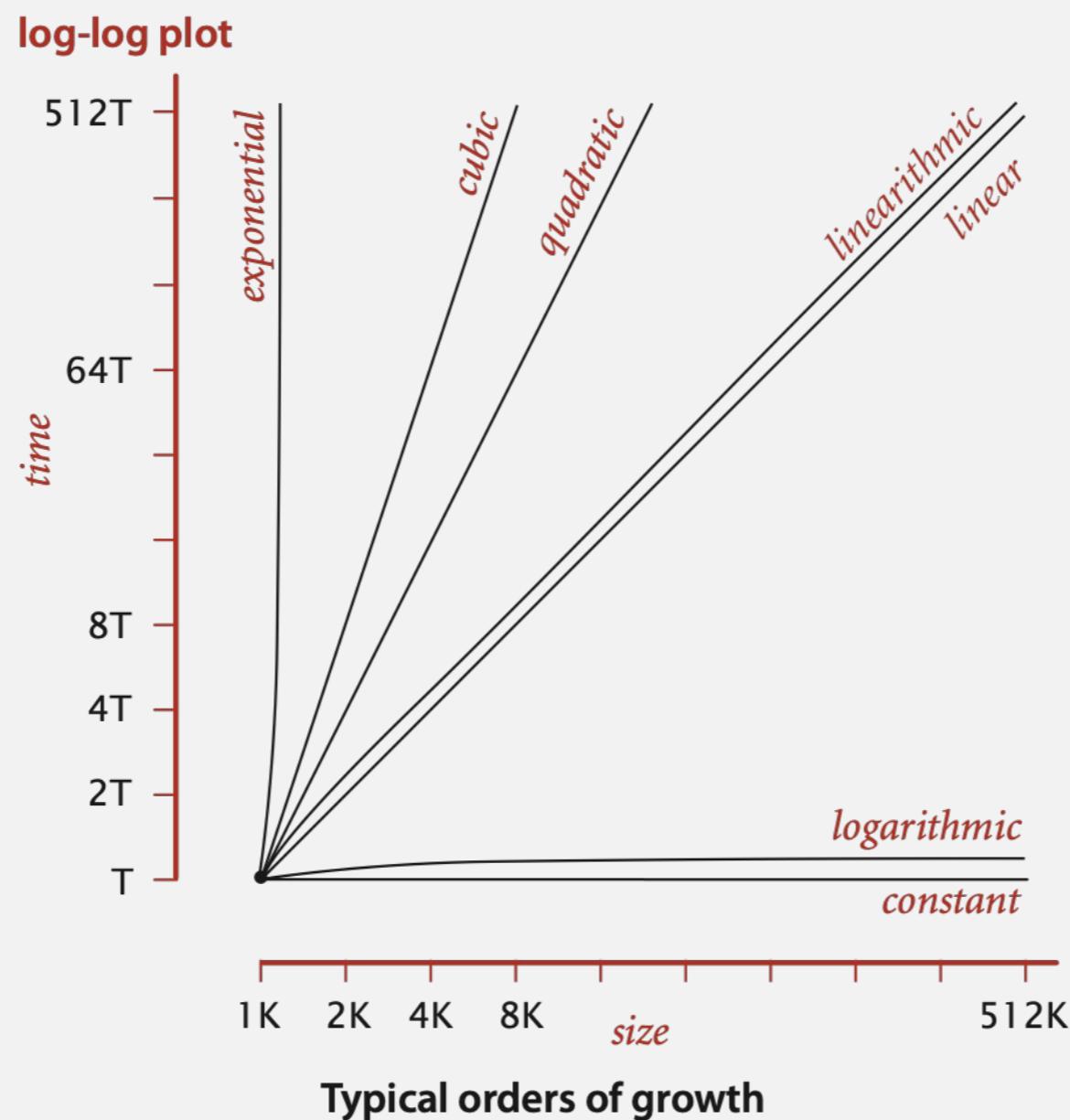
→ where leading coefficient
depends on machine, compiler, JVM, ...

Common order-of-growth classifications

Good news. The set of functions

$1, \log N, N, N \log N, N^2, N^3$, and 2^N

suffices to describe the order of growth of most common algorithms.



Binary search demo

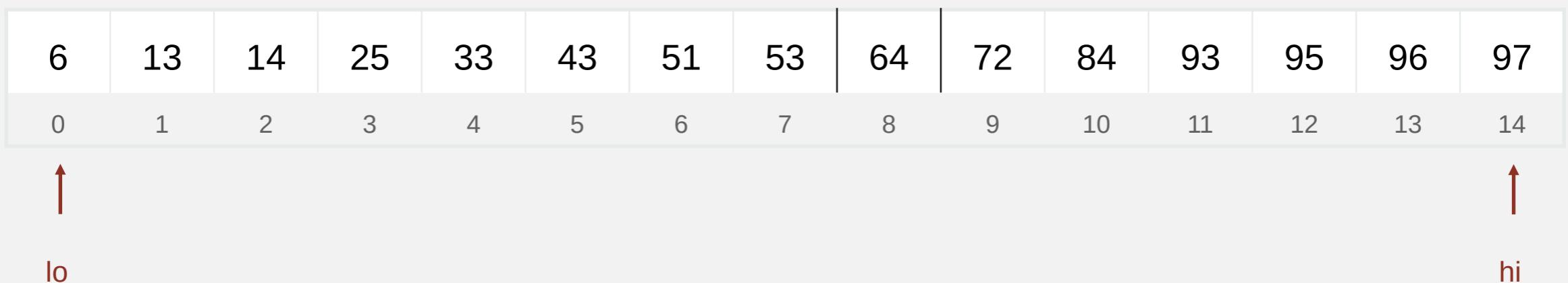
Goal. Given a sorted array and a key, find index of the key in the array?

[Binary search.](#) Compare key against middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.



successful search for 33

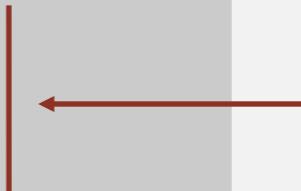


Binary search: Java implementation

Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's Arrays.binarySearch() discovered in 2006.

```
public static int binarySearch(int[] a, int key) {  
    int lo = 0, hi = a.length-1;  
    while (lo <= hi) {  
        int mid = lo + (hi - lo) / 2;  
        if      (key < a[mid]) hi = mid - 1;  
        else if (key > a[mid]) lo = mid + 1;  
        else return mid;  
    }  
    return -1;  
}
```



one "3-way compare"

Invariant. If key appears in the array a[], then $a[lo] \leq key \leq a[hi]$.

Binary search: mathematical analysis

Proposition. Binary search uses at most $1 + \lg N$ key compares to search in a sorted array of size N .

Def. $T(N) = \# \text{ key compares to binary search a sorted subarray of size } \leq N$.

Binary search recurrence. $T(N) \leq T(N/2) + 1$ for $N > 1$, with $T(1) = 1$.

left or right half
(floored division)

possible to implement with one
2-way compare (instead of 3-way)

Pf sketch. [assume N is a power of 2]

$T(N) \leq T(N/2) + 1$	[given]
$\leq T(N/4) + 1 + 1$	[apply recurrence to first term]
$\leq T(N/8) + 1 + 1 + 1$	[apply recurrence to first term]
\vdots	
$\leq T(N/N) + 1 + 1 + \dots + 1$	[stop applying, $T(1) = 1$]
$= 1 + \lg N$	

An $N^2 \log N$ algorithm for 3-SUM

Algorithm.

- Step 1: Sort the N (distinct) numbers.
- Step 2: For each pair of numbers $a[i]$ and $a[j]$, binary search for $-(a[i] + a[j])$.

Analysis. Order of growth is $N^2 \log N$.

- Step 1: N^2 with insertion sort.
- Step 2: $N^2 \log N$ with binary search.

input

30 -40 -20 -10 40 0 10 5

sort

-40 -20 -10 0 5 10 30 40

binary search

(-40, -20) 60

(-40, -10) 50

(-40, 0) 40

(-40, 5) 35

(-40, 10) 30

: :

(-20, -10) 30

: :

(-10, 0) 10

: :

(10, 30) -40

(10, 40) -50

(30, 40) -70

only count if
 $a[i] < a[j] < a[k]$
to avoid
double counting

Comparing programs

Hypothesis. The sorting-based $N^2 \log N$ algorithm for 3-SUM is significantly faster in practice than the brute-force N^3 algorithm.

N	time (seconds)
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1

`ThreeSum.java`

N	time (seconds)
1,000	0.14
2,000	0.18
4,000	0.34
8,000	0.96
16,000	3.67
32,000	14.88
64,000	59.16

`ThreeSumDeluxe.java`

Guiding principle. Typically, better order of growth \Rightarrow faster in practice.

Types of analyses

Best case. Lower bound on cost.

- Determined by “easiest” input.
- Provides a goal for all inputs.

Worst case. Upper bound on cost.

- Determined by “most difficult” input.
- Provides a guarantee for all inputs.

Average case. Expected cost for random input.

- Need a model for “random” input.
- Provides a way to predict performance.

this course

Ex 1. Array accesses for brute-force 3-SUM.

Best: $\sim \frac{1}{2} N^3$

Average: $\sim \frac{1}{2} N^3$

Worst: $\sim \frac{1}{2} N^3$

Ex 2. Comparisons for binary search.

Best: ~ 1

Average: $\sim \lg N$

Worst: $\sim \lg N$

Theory of algorithms

Goals.

- Establish “difficulty” of a problem.
- Develop “optimal” algorithms.

Approach.

- Suppress details in analysis: analyze “to within a constant factor.”
- Eliminate variability in input model: focus on the worst case.

Upper bound. Performance guarantee of algorithm for any input.

Lower bound. Proof that no algorithm can do better.

Optimal algorithm. Lower bound = upper bound (to within a constant factor).

Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N \log N + 3N$ ⋮	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3 N$ ⋮	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ N^5 $N^3 + 22 N \log N + 3 N$ ⋮	develop lower bounds

Theory of algorithms: example 1

Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 1-SUM = “*Is there a 0 in the array?*”

Upper bound.

A specific algorithm.

- Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
- Running time of the optimal algorithm for 1-SUM is $O(N)$.

Lower bound.

Proof that no algorithm can do better.

- Ex. Have to examine all N entries (any unexamined one might be 0).
- Running time of the optimal algorithm for 1-SUM is $\Omega(N)$.

Optimal algorithm.

- Lower bound equals upper bound (to within a constant factor).
- Ex. Brute-force algorithm for 1-SUM is optimal: its running time is $\Theta(N)$.

Theory of algorithms: example 2

Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^3)$.

Theory of algorithms: example 2

Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Improved algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^2 \log N)$.

Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all N entries to solve 3-SUM.
- Running time of the optimal algorithm for solving 3-SUM is $\Omega(N)$.

Open problems.

- Optimal algorithm for 3-SUM?
- Subquadratic algorithm for 3-SUM?
- Quadratic lower bound for 3-SUM?

Algorithm design approach

Start.

- Develop an algorithm.
- Prove a lower bound.

Gap?

- Lower the upper bound (discover a new algorithm).
- Raise the lower bound (more difficult).

Golden Age of Algorithm Design.

- 1970s-.
- Steadily decreasing upper bounds for many important problems.
- Many known optimal algorithms.

Caveats.

- Overly pessimistic to focus on worst case?
- Need better than “to within a constant factor” to predict performance.

Typical memory usage for primitive types and arrays

type	bytes
boolean	1
byte	1
char	2
int	4
float	4
long	8
double	8

primitive types

type	bytes
char[]	$2N + 24$
int[]	$4N + 24$
double[]	$8N + 24$

one-dimensional arrays

type	bytes
char[][]	$\sim 2MN$
int[][]	$\sim 4MN$
double[][]	$\sim 8MN$

two-dimensional arrays

Typical memory usage for objects in Java

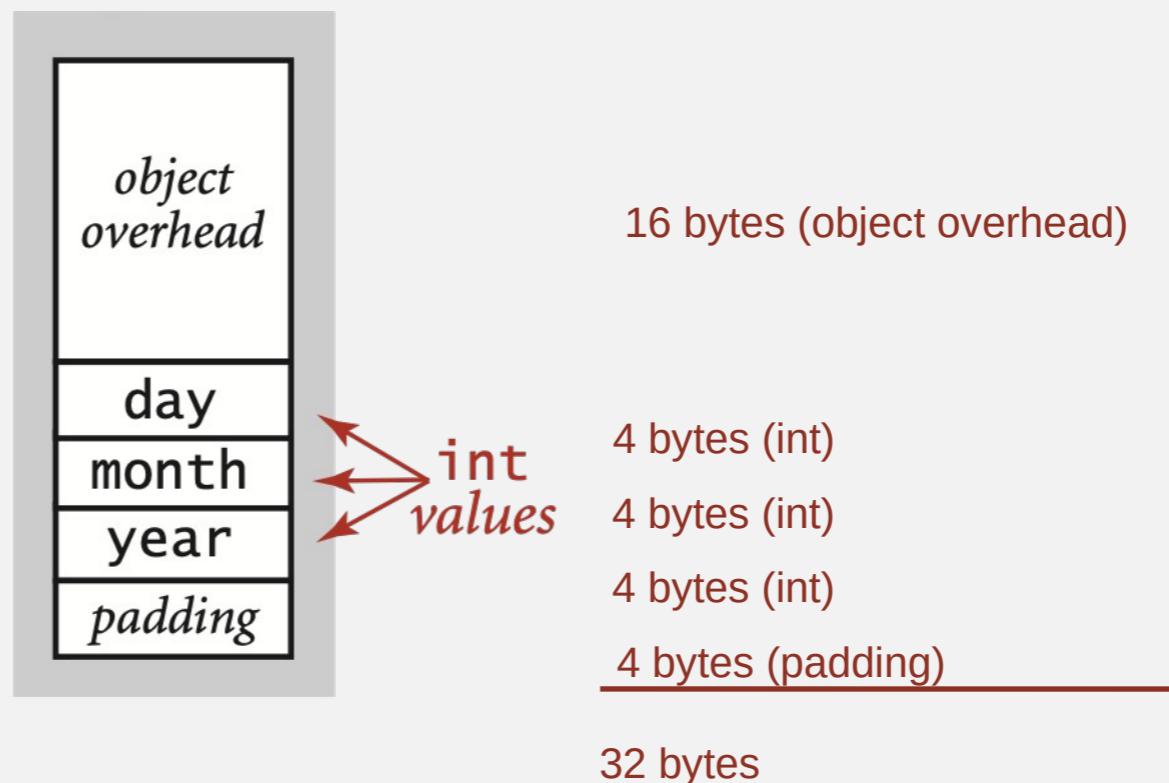
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 1. A Date object uses 32 bytes of memory.

```
public class Date
{
    private int day;
    private int month;
    private int year;
    ...
}
```



Typical memory usage summary

Total memory usage for a data type value:

- Primitive type: 4 bytes for int, 8 bytes for double, ...
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable.
- Padding: round up to multiple of 8 bytes.

+ 8 extra bytes per inner class object
(for reference to enclosing class)

Shallow memory usage: Don't count referenced objects.

Deep memory usage: If array entry or instance variable is a reference, count memory (recursively) for referenced object.

Example

Q. How much memory does WeightedQuickUnionUF use as a function of N ?

Use tilde notation to simplify your answer.

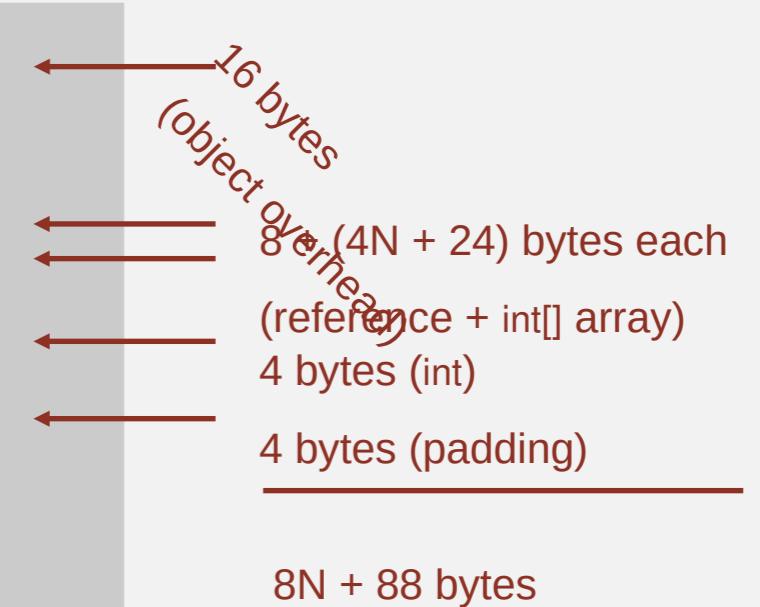
```
public class WeightedQuickUnionUF
{
    private int[] id;
    private int[] sz;
    private int count;
```

```
public WeightedQuickUnionUF(int N)
```

```
{
    id = new int[N];
    sz = new int[N];
    for (int i = 0; i < N; i++) id[i] = i;
    for (int i = 0; i < N; i++) sz[i] = 1;
}
```

...

```
}
```



A. $8N + 88 \sim 8N$ bytes.

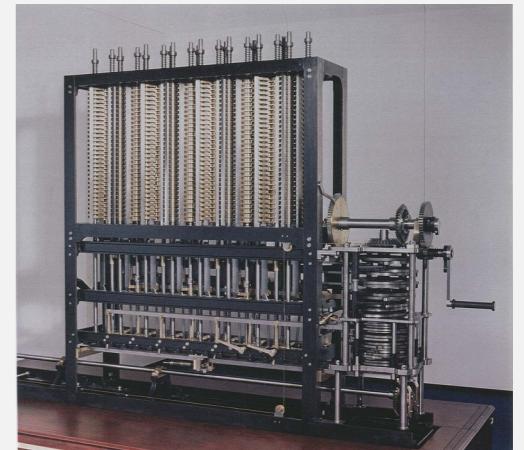
Turning the crank: summary

Empirical analysis.

- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to **make predictions**.

Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to **explain behavior**.



Scientific method.

- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.