An Accelerated Fixed Point Iteration Scheme: The Nonlinear Krylov Method

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FP Iteration For $f(u) = 0$

Fixed point iteration for $f(u) = 0$, $f : \mathbb{R}^m \to \mathbb{R}^m$:

u_0 *given*
for $n = 0, 1, 2, \ldots$ ***do***
  $u_{n+1} = u_n - f(u_n)$
end for

- $u_n \to u$ if $\|u - u_0\|$ and $\|\mathrm{D}f(u) - I\|$ are sufficiently small.
- Rapid convergence when $\mathrm{D}f \approx I$.

Newton's Method as a Preconditioned FP Iteration

Newton's method for $g(u) = 0$, $g : \mathbb{R}^m \to \mathbb{R}^m$:

> u_0 *given*
> ***for*** $n = 0, 1, 2, \ldots$ ***do***
> $\quad u_{n+1} = u_n - \mathrm{D}g(u_n)^{-1} g(u_n)$
> ***end for***

If we define $f(u) = \mathrm{D}g(u)^{-1} g(u)$, then Newton's method is precisely a FP iteration for the equivalent problem $f(u) = 0$.

- At the solution u, $\mathrm{D}f = I$.

Accelerated FP Correction: Motivation

The fixed point iteration for $f(u) = 0$,

> ***for*** $n = 0, 1, 2, \ldots$ ***do***
> $v_{n+1} = f(u_n)$
> $u_{n+1} = u_n - v_{n+1}$
> ***end for***

generates a sequence of corrections $v_1, v_2, \ldots$
At step $n+1$ how might we choose a better correction v_{n+1}? Perhaps as the solution of the Newton correction equation

$$0 = f(u_n) - \mathrm{D}f(u_n)\, v_{n+1} \approx f(u_{n+1}).$$

- FPI view: Don't know $\mathrm{D}f(u_n)$, so approximate it by I.
  Indeed, FPI converges best when $\mathrm{D}f \approx I$.
- But, if $\mathrm{D}f \approx$ constant, we do know something about $\mathrm{D}f$!
  We have the f-values $f(u_0), f(u_1), \ldots, f(u_n)$ available.

Accelerated FP Correction: Motivation (cont.)

To generate the correction v_{n+1} we have available:

- Corrections: $v_1, \ldots, v_n$,

$$\mathcal{V}_n = \mathrm{span}\{v_1, \ldots, v_n\}, \qquad V_n = [v_1 \cdots v_n]$$

- f-differences: $w_1, \ldots, w_n$, where $w_j = f(u_{j-1}) - f(u_j)$,

$$\mathcal{W}_n = \mathrm{span}\{w_1, \ldots, w_n\} \qquad W_n = [w_1 \cdots w_n]$$

Note that $w_j \approx \mathrm{D}f\, v_j$, so that we know (approximately) the action of $\mathrm{D}f$ on $\mathcal{V}_n$.

The Accelerated FP Correction

Idea: Split the correction $v_{n+1} = v' + v''$, with $v' \in \mathcal{V}_n$, for which we know the action of $\mathrm{D}f$:

$$0 = f(u_n) - \mathrm{D}f(u_n)\,(v' + v'') \quad \rightsquigarrow \quad 0 = f(u_n) - \mathrm{D}f\; v' - I\, v''.$$

For $\mathrm{D}f\; v''$ we invoked the FP iteration viewpoint.

Accelerated correction (Carlson & Miller, SISC '98)
Choose v' such that $\mathrm{D}f\; v'$ is the l_2 projection of $f(u_n)$ into $\mathcal{W}_n$:

$$v_{n+1} = \underbrace{V_n z}_{\in \mathcal{V}_n} + \underbrace{(f(x_n) - W_n z)}_{\in \mathcal{W}_n^{\perp}}$$

where $z = \mathrm{argmin}_{\zeta \in \mathbb{R}^n} \| f(u_n) - W_n \zeta \|$.

- We are not required to use all the available subspace info.

The Nonlinear Reality

Of course $\mathrm{D}f$ isn't constant. In recognition of this fact we consider the most recent corrections and differences to be the most reliable.

- Use the v_j in reverse order.
- Use only a limited number of the most recent v_j.
- Drop any v_j that is nearly in the span of the preceding vectors (in the reverse sense).

The Nonlinear Krylov (NLK) Method

To summarize, the accelerated fixed point iteration is

$u \leftarrow u_0$

repeat

$\quad v \leftarrow \mathrm{NKA}(f(u))$

$\quad u \leftarrow u - v$

until *converged*

- The black-box procedure $\mathrm{NKA}()$ returns the accelerated correction just defined. Formally it is a function of all the previous corrections and previous f values, but it accumulates this info over the course of the iteration.
- $v \leftarrow f(u)$ is the unaccelerated correction.