Electron Positron Partition Function in Early Universe

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I. PARTITION FUNCTION

Considering the partition function of $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, we have

$$\ln \mathcal{Z}_{tot} = eBV \sum_{s=\pm1}\sum_{j=0}^{\infty}\int_{-\infty}^{\infty} dp_z \left[\ln\left(1+e^{-\beta(E_{j,s}-\mu_e)}\right)+\ln\left(1+e^{-\beta(E_{j,s}+\mu_e)}\right)\right], \tag{1}$$

where $\beta = 1/T$, μ_e is the chemical potential of electron, and the electron(positron) energy $E_{j,s}$ can be written as

$$E_{j,s} = \sqrt{m_e^2 + p_z^2 + 2eB\left(j+\frac{1}{2}+\frac{g}{4}s\right)}, \qquad s=\pm1, \qquad j=0,1,2,\dots \tag{2}$$

and μ_B is the magnetic moment.

Considering the integral over dp_z we can simplify the integration by integrating by part, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} dp_z \left[\ln\left(1+e^{-\beta(E_{j,s}-\mu_e)}\right)+\ln\left(1+e^{-\beta(E_{j,s}+\mu_e)}\right)\right] \\
&= 2\left[\int_0^{\infty} dp_z \ln\left(1+e^{-\beta(E_{j,s}-\mu_e)}\right)+\int_0^{\infty} dp_z \ln\left(1+e^{-\beta(E_{j,s}+\mu_e)}\right)\right] && (3)\\
&= 2\left[\beta\int_0^{\infty} dp_z p_z \frac{\partial E_{j,s}}{\partial p_z}\frac{1}{e^{\beta(E_{j,s}-\mu_e)}+1}+\beta\int_0^{z} dp_z p_z \frac{\partial E_{j,s}}{\partial p_z}\frac{1}{e^{\beta(E_{j,s}+\mu_e)}+1}\right] && (4)\\
&= 2\beta\int_0^{\infty} dp_z \frac{p_z^2}{E_{j,s}}\left[\frac{1}{e^{\beta(E_{j,s}-\mu_e)}+1}+\frac{1}{e^{\beta(E_{j,s}+\mu_e)}+1}\right] && (5)\\
&= 2\beta\int_0^{\infty} dp_z \frac{p_z^2}{E_{j,s}}\frac{e^{-\beta E_{j,s}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,s})} && (6)
\end{aligned}$$

where the last step we use the identity

$$\frac{1}{e^{x-y}+1}+\frac{1}{e^{x+y}+1} = \frac{2+(e^{x-y}+e^{x+y})}{(e^{2x}+1)+(e^{x-y}+e^{x+y})} = \frac{e^{-x}+\cosh y}{\cosh x+\cosh y} \tag{7}$$

to simplify the integral.

Then the partition function of $e^{\pm}$ plasma in a uniform magnetic field B can be written as

$$\begin{aligned}
\ln \mathcal{Z}_{tot} &= eBV\sum_{s=\pm1}\sum_{j=0}^{\infty} 2\beta\int_0^{\infty} dp_z \frac{p_z^2}{E_{j,s}}\frac{e^{-\beta E_{j,s}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,s})} && (8)\\
&= V(2eB\beta)\sum_{j=0}^{\infty}\left[\int_0^{\infty} dp_z \frac{p_z^2}{E_{j,+}}\frac{e^{-\beta E_{j,+}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,+})}+\int_0^{\infty} dp_z \frac{p_z^2}{E_{j,-}}\frac{e^{-\beta E_{j,-}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,-})}\right] && (9)
\end{aligned}$$

where the energy $E_{j,\pm}$ are given by

$$E_{j,+} = \sqrt{m_e^2+p_z^2+2eB\left(j+\frac{1}{2}+\frac{g}{4}\right)}, \qquad E_{j,-} = \sqrt{m_e^2+p_z^2+2eB\left(j+\frac{1}{2}-\frac{g}{4}\right)} \tag{10}$$

Considering the case $g=2$, then we have

$$\begin{aligned}
E_{j,+} &= \sqrt{m_e^2+p_z^2+2eB\,(j+1)} \longrightarrow E_n = \sqrt{m_e^2+p_z^2+2eBn}, \qquad n=1,2,3,\dots && (11)\\
E_{j,-} &= \sqrt{m_e^2+p_z^2+2eB\,(j)} \longrightarrow E_n = \sqrt{m_e^2+p_z^2+2eBn}, \qquad n=0,1,2,3,\dots && (12)
\end{aligned}$$

where we change the index from j to n. In this case, the partition function of $e^{\pm}$ plasma can be written as

$$\ln \mathcal{Z}_{tot} = V(2eB\beta)\left[\sum_{n=1}^{\infty}\int_0^{\infty}dp_z\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}+\sum_{n=0}^{\infty}\int_0^{\infty}dp_z\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right] \tag{13}$$

$$= V(2eB\beta)\left[\int_0^{\infty}dp_z\frac{p_z^2}{\sqrt{m_e^2+p_z^2}}\frac{e^{-\beta\sqrt{m_e^2+p_z^2}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta\sqrt{m_e^2+p_z^2})}+2\sum_{n=1}^{\infty}\int_0^{\infty}dp_z\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right] \tag{14}$$

Giving the partition function, the magnetization can be obtained via the definition

$$M=\frac{1}{V\beta}\frac{\partial \ln Z_{tot}}{\partial B} \tag{15}$$

In the case of electron/positron system, we have

$$M=(2e)\left[\int_0^{\infty}dp_z\frac{p_z^2}{\sqrt{m_e^2+p_z^2}}\frac{e^{-\beta\sqrt{m_e^2+p_z^2}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta\sqrt{m_e^2+p_z^2})}+2\sum_{n=1}^{\infty}\int_0^{\infty}dp_z\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right.$$
$$\left.+2\sum_{n=1}^{\infty}\int_0^{\infty}dp_z B\frac{\partial}{\partial B}\left(\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right)\right] \tag{16}$$

It can be written as

$$M=M_0+M_1+M_2 \tag{17}$$

where

$$M_0=(2e)\int_0^{\infty}dp_z\frac{p_z^2}{\sqrt{\tilde{m}_e^2+p_z^2}}\frac{e^{-\beta\sqrt{\tilde{m}_e^2+p_z^2}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta\sqrt{\tilde{m}_e^2+p_z^2})} \tag{18}$$

$$M_1=(4e)\sum_{n=1}^{\infty}\int_0^{\infty}dp_z\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)} \tag{19}$$

and

$$M_2=(4e)\sum_{n=1}^{\infty}\int_0^{\infty}dp_z B\frac{\partial}{\partial B}\left(\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right)$$
$$=-(4e)\sum_{n=1}^{\infty}\int_0^{\infty}dp_z\left(\frac{p_z^2}{E_n}\frac{e^{-\beta E_n}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right)\left(\frac{eBn}{E_n^2}\right)\left[1+\frac{\beta E_n e^{-\beta E_n}}{e^{-\beta E_n}+\cosh(\beta\mu_e)}+\frac{\beta E_n\sinh(\beta E_n)}{\cosh(\beta\mu_e)+\cosh(\beta E_n)}\right] \tag{20}$$

It is convenient to introduce the dimensionless variables as follow

$$\eta=\frac{p_z}{m_e},\qquad \xi=\frac{2eB}{m_e^2},\qquad \kappa=\frac{m_e}{T} \tag{21}$$

then the magnetization can be written as

$$M_0=2em_e^2\int_0^{\infty}d\eta\frac{\eta^2}{\sqrt{1+\eta^2}}\frac{e^{-\kappa\sqrt{1+\eta^2}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\kappa\sqrt{1+\eta^2})} \tag{22}$$

$$M_1=4em_e^2\sum_{n=1}^{\infty}\int_0^{\infty}d\eta\frac{\eta^2}{\sqrt{1+\eta^2+\xi n}}\frac{e^{-\kappa\sqrt{1+\eta^2+\xi n}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\kappa\sqrt{1+\eta^2+\xi n})} \tag{23}$$

and

$$M_2=-4em_e^2\sum_{n=1}^{\infty}\int_0^{\infty}d\eta\frac{\eta^2}{\sqrt{1+\eta^2+\xi n}}\frac{e^{-\kappa\sqrt{1+\eta^2+\xi n}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\kappa\sqrt{1+\eta^2+\xi n})}\left(\frac{\xi n}{1+\eta^2+\xi n}\right)$$
$$\times\left[1+\frac{\kappa\sqrt{1+\eta^2+\xi n}\;e^{-\kappa\sqrt{1+\eta^2+\xi n}}}{e^{-\kappa\sqrt{1+\eta^2+\xi n}}+\cosh(\beta\mu_e)}+\frac{\kappa\sqrt{1+\eta^2+\xi n}\;\sinh(\kappa\sqrt{1+\eta^2+\xi n})}{\cosh(\beta\mu_e)+\cosh(\kappa\sqrt{1+\eta^2+\xi n})}\right] \tag{24}$$

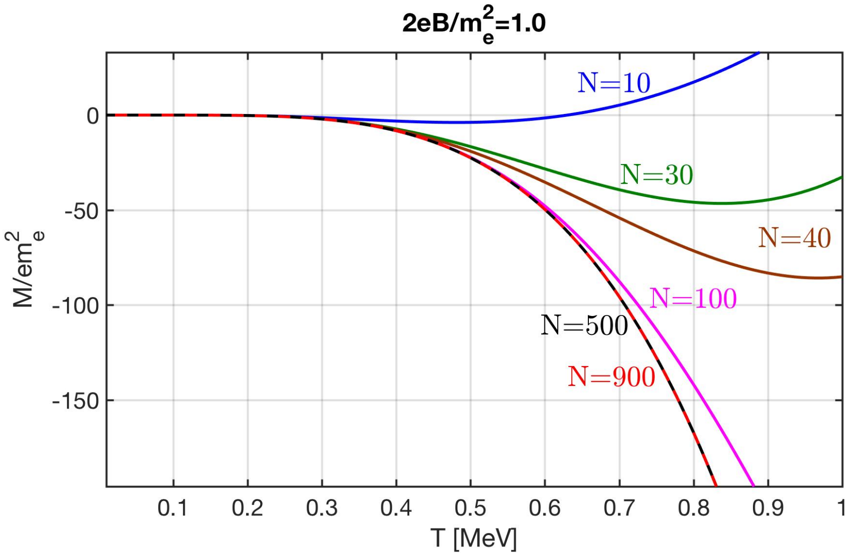

FIG. 1: The magnetization Eq.(25) as a function of temperature T. We use different colors to label the number of terms N in the summation. For the case $2eB/m_e^2 = 1.0$, the value of magnetization becomes stable after $N = 500$.

Substituting Eq.(22), Eq.(23), and Eq.(24) into Eq.(17) we can obtain the magnetization of $e^{\pm}$ plasma in the constant magnetic field B.

Considering the case for strong magnetic field $2eB/m_e^2 = 1.0$, we can the magnetization numerically. Giving the strong field in calculation, the summation for larger number of n would be suppressed. The magnetization can be written as

$$\begin{aligned}\left(\frac{M}{em_e^2}\right) =&2\int_0^\infty d\eta\frac{\eta^2}{\sqrt{1+\eta^2}}\,\frac{e^{-\kappa\sqrt{1+\eta^2}}+\cosh\left(\beta\mu_e\right)}{\cosh\left(\beta\mu_e\right)+\cosh\left(\kappa\sqrt{1+\eta^2}\right)}+4\sum_{n=1}^{N}\int_0^\infty d\eta\frac{\eta^2}{\sqrt{1+\eta^2+\xi n}}\,\frac{e^{-\kappa\sqrt{1+\eta^2+\xi n}}+\cosh\left(\beta\mu_e\right)}{\cosh\left(\beta\mu_e\right)+\cosh\left(\kappa\sqrt{1+\eta^2+\xi n}\right)}\\ &-4\sum_{n=1}^{N}\int_0^\infty d\eta\frac{\eta^2}{\sqrt{1+\eta^2+\xi n}}\,\frac{e^{-\kappa\sqrt{1+\eta^2+\xi n}}+\cosh\left(\beta\mu_e\right)}{\cosh\left(\beta\mu_e\right)+\cosh\left(\kappa\sqrt{1+\eta^2+\xi n}\right)}\left(\frac{\xi n}{1+\eta^2+\xi n}\right)\\ &\times\left[1+\frac{\kappa\sqrt{1+\eta^2+\xi n}\;e^{-\kappa\sqrt{1+\eta^2+\xi n}}}{e^{-\kappa\sqrt{1+\eta^2+\xi n}}+\cosh(\beta\mu_e)}+\frac{\kappa\sqrt{1+\eta^2+\xi n}\;\sinh(\kappa\sqrt{1+\eta^2+\xi n})}{\cosh\left(\beta\mu_e\right)+\cosh\left(\kappa\sqrt{1+\eta^2+\xi n}\right)}\right]\end{aligned} \tag{25}$$

where we include $n = 1 \sim N$ in our summation.

In Fig.1, we plot the magnetization M/em_e^2 Eq.(25) as a function of temperature T. We use different colors to label the number of terms N in summation, it shows that the magnetization depends on the number of terms we include in the summation. For the case $2eB/m_e^2 = 1.0$, the value of magnetization becomes stable after $N = 500$.

In Fig.2 we plot the magnetization M/em_e^2 as a function of temperature with different value of magnetic field $2eB/m_e^2 = 0.5, 1.0, 1.5$. We set $N = 500$ in our calculation.

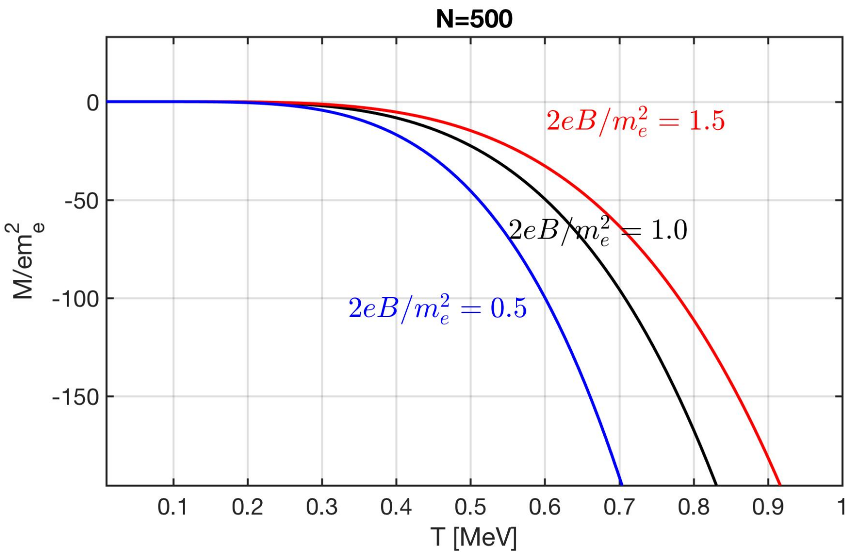

FIG. 2: The magnetization Eq.(25) as a function of temperature T with magnetization $M/em_e^2 = 1.0$(black), $M/em_e^2 = 1.5$(red), and $M/em_e^2 = 0.5$(blue).In our calculation we set $N = 500$ for all three different cases.