Electron Positron Number Density in Early Universe

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(Dated: January 30, 2023)

I. THE EFFECT OF ELECTRON NUMBER DENSITY

A. Massless Approximation

In order to study the neutrino chemical potential, we need to study the net number density of electron. Let we consider the distribution function of electron(positron), we have

$$f_e = \frac{1}{\exp\left[(\frac{1}{T_\gamma}(E-\sigma\mu_e)\right]+1}, \tag{1}$$

where $\sigma = 1(-1)$ for electron(positron) respectively and we assume that electron/positron and photon are in equilibrium and express the number density in term of the photon temperature T_γ. The excess of electron over positron can is given by

$$(n_e - n_{\overline{e}}) = \frac{g_e}{2\pi^2}\left[\int_0^\infty \frac{p^2\,dp}{\exp\left[\frac{1}{T_\gamma}(E-\mu_e)\right]+1} - \int_0^\infty \frac{p^2\,dp}{\exp\left[\frac{1}{T_\gamma}(E+\mu_e)\right]+1}\right], \tag{2}$$

Let we write the net number density as

$$(n_e - n_{\overline{e}}) = (n_e - n_{\overline{e}})_{m=0} + \Delta_{m\neq 0}, \tag{3}$$

where $(n_e - n_{\overline{e}})_{m=0}$ is the electron number density in the massless limit, and $\Delta_{m\neq 0}$ is the correction of the finite mass of electron. The number density in massless approximation can be written as

$$\begin{aligned}(n_e - n_{\overline{e}})_{m=0} &= \frac{g_e}{2\pi^2}\left[\int_0^\infty \frac{p^2\,dp}{\exp\left[\frac{1}{T_\gamma}(p-\mu_e)\right]+1} - \int_0^\infty \frac{p^2\,dp}{\exp\left[\frac{1}{T_\gamma}(p+\mu_e)\right]+1}\right]\\ &= \frac{g_e}{2\pi^2} I_2\\ &= \frac{g_e}{6\pi^2}T_\gamma^3\left[\pi^2\left(\frac{\mu_e}{T_\gamma}\right)+\left(\frac{\mu_e}{T_\gamma}\right)^3\right].\end{aligned} \tag{4}$$

Then in general the electron number density is given by

$$\begin{aligned}(n_e - n_{\overline{e}}) &= (n_e - n_{\overline{e}})_{m=0} + \Delta_{m\neq 0}\\ &= \frac{g_e}{6\pi^2}T_\gamma^3\left[\pi^2\left(\frac{\mu_e}{T_\gamma}\right)+\left(\frac{\mu_e}{T_\gamma}\right)^3 + \Delta'_{m\neq 0}\right]\\ &\simeq \frac{g_e}{6\pi^2}T_\gamma^3\left[\pi^2\left(\frac{\mu_e}{T_\gamma}\right) + \Delta'_{m\neq 0}\right]\end{aligned} \tag{5}$$

where we neglect the term $(\mu_e/T_\gamma)^3$ in last step. Substituting Eq.(5) into the neutrino chemical potential , then we have

$$\mu_\nu = \frac{2T_f}{3\pi^2}\left(\frac{T_\gamma}{T_\nu}\right)^3\left(\frac{n_n}{n_p}\right)\left[\pi^2\left(\frac{\mu_e}{T_\gamma}\right)+\Delta'_{m\neq 0}\right]. \tag{6}$$

B. Chemical Potential of Electron

To estimate the chemical potential of election μ_e, we use the conservation law for the electric charge. From the cosmological data [?], the ratio between number of baryons and number of photon is given by

$$\eta \equiv \frac{n_B}{n_\gamma} = 6.05 \times 10^{-10}, \tag{7}$$

where n_B is the number of baryons and n_γ is the number of photons. Since the neutrality of the Universe and the value of the baryon-photon ratio η, the number of electron-to-photon ratio is given by

$$\frac{n_e}{n_\gamma} = \frac{n_p}{n_\gamma} \sim \frac{n_B}{n_\gamma} = 6.05 \times 10^{-10}. \tag{8}$$

The number density of photon can be written as

$$n_\gamma = \frac{g_\gamma}{\pi^2}\zeta(3)T_\gamma^3, \tag{9}$$

where the zeta function $\zeta(3) = 1.20206$. Then using Eq.(4), Eq.(9), and Eq.(8), we obtain

$$\frac{n_e - n_{\overline{e}}}{n_\gamma} = \frac{\pi^2}{6\zeta(3)}\left(\frac{\mu_e}{T_\gamma}\right) \simeq 6.05 \times 10^{-10}, \tag{10}$$

then the chemical potential of electron is given by

$$\frac{\mu_e}{T_\gamma} \sim 4.42 \times 10^{-10}. \tag{11}$$

If we consider the freezeout temperature $T_f = 2.0$MeV and massless electrons, then the neutrino chemical potential is given by

$$\mu_\nu = \frac{2T_f}{3}\left(\frac{T_\gamma}{T_\nu}\right)^3\left(\frac{\mu_e}{T_\gamma}\right)\left(\frac{n_n}{n_p}\right) = 3.087 \times 10^{-10}\text{MeV} = 3.087 \times 10^{-4}\text{eV}. \tag{12}$$

C. The Correction of Electron Mass Effect

In order to study the effect of the finite mass of electron on the neutrino chemical potential, we consider the partition function of the electron/positron system first. We have

$$\ln \mathcal{Z} = g_e V \int \frac{d^3p}{(2\pi)^3}\left[\ln\left(1 + e^{-\beta(E-\mu_e)}\right) + \ln\left(1 + e^{-\beta(E+\mu_e)}\right)\right], \tag{13}$$

where we define $\beta = 1/T_\gamma$. If we assume that $0 < \mu_e < m_e$, then we can use a series of the logarithmic function, we have

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n, \qquad \text{for } |x| < 1. \tag{14}$$

Hence the partition function of electron/positron system can be written as

$$\begin{aligned}\ln \mathcal{Z} &= g_e V \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[e^{\beta n\mu_e} + e^{-\beta n\mu_e}\right]\int \frac{d^3p}{(2\pi)^3}e^{-\beta nE} \\ &= \left(\frac{g_e V}{2\pi^2}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}2\cosh(\beta n\mu_e)\int_0^\infty dp\, p^2 e^{-\beta nE}.\end{aligned} \tag{15}$$

Since the Bessel function is given by

$$K_\nu(\beta m) = \frac{\sqrt{\pi}}{\Gamma(\nu - 1/2)}\frac{1}{m}\left(\frac{\beta}{2m}\right)^{\nu-1}\int_0^\infty dp\, p^{2\nu-2}e^{-2\beta E} \qquad \text{for } \nu > 1/2. \tag{16}$$

Using the Bessel function and replace $\beta \to n\beta$ the integral in partition function can be written as

$$\int_0^\infty dp\, p^2 e^{-\beta n E} = \frac{\Gamma(3/2)}{\sqrt{\pi}} \frac{2m_e^2}{n\beta} K_2(n\beta m_e) = \frac{m_e^2}{n\beta} K_2(n\beta m_e). \tag{17}$$

Then the grand partition function of electron/positron system in the early universe can be written as

$$\ln \mathcal{Z} = \left(\frac{g_e V}{2\pi^2}\right) T_\gamma^3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left[2\cosh\left(\frac{n\mu_e}{T_\gamma}\right)\right] \left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma). \tag{18}$$

Let we consider the quantity

$$\begin{aligned}
\frac{\partial^2 \ln \mathcal{Z}}{\partial^2 \mu_e} &= \left(\frac{g_e V}{2\pi^2}\right) T_\gamma^3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \frac{\partial^2}{\partial^2 \mu_e} \left[2\cosh\left(\frac{n\mu_e}{T_\gamma}\right)\right] \left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) \\
&= \left(\frac{g_e V}{2\pi^2}\right) T_\gamma^3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left(\frac{n}{T_\gamma}\right)^2 \left[2\cosh\left(\frac{n\mu_e}{T_\gamma}\right)\right] \left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma).
\end{aligned} \tag{19}$$

Hence, the excess of electron over positron to the first order of (μ_e/T_γ) can be written as

$$\begin{aligned}
(n_e - n_{\overline{e}}) &= \frac{\mu_e}{V} T_\gamma \left(\frac{\partial^2 \ln \mathcal{Z}}{\partial^2 \mu_e}\right)\Bigg|_{\mu_e = 0} \\
&= \frac{g_e}{\pi^2} T_\gamma^3 \left(\frac{\mu_e}{T_\gamma}\right) \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma)\right].
\end{aligned} \tag{20}$$

In order to point out the effect of the electron mass, it is convenient to rewrite Eq.(20) into

$$\begin{aligned}
(n_e - n_{\overline{e}}) &= \frac{g_e}{\pi^2} T_\gamma^3 \left(\frac{\mu_e}{T_\gamma}\right) \left\{2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]\right\} \\
&= \frac{g_e}{\pi^2} T_\gamma^3 \left(\frac{\mu_e}{T_\gamma}\right) \left\{\zeta(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]\right\} \\
&= \frac{g_e}{\pi^2} T_\gamma^3 \left(\frac{\mu_e}{T_\gamma}\right) \left\{\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]\right\} \\
&= \frac{g_e}{6\pi^2} T_\gamma^3 \left\{\pi^2 \left(\frac{\mu_e}{T_\gamma}\right) + 6\left(\frac{\mu_e}{T_\gamma}\right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]\right\}.
\end{aligned} \tag{21}$$

Compare Eq.(21) with Eq.(5) the correction of electron mass to the first order of μ_e/T_γ can be written as

$$\Delta'_{m\neq 0} = 6\left(\frac{\mu_e}{T_\gamma}\right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]. \tag{22}$$

In this case, the chemical potential of neutrino in the early universe can be written as

$$\begin{aligned}
\mu_\nu &= \frac{2T_f}{3\pi^2} \left(\frac{T_\gamma}{T_\nu}\right)^3 \left(\frac{n_n}{n_p}\right) \left(\frac{\mu_e}{T_\gamma}\right) \left\{\pi^2 + 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]\right\} \\
&= \frac{2T_f}{3} \left(\frac{T_\gamma}{T_\nu}\right)^3 \left(\frac{n_n}{n_p}\right) \left(\frac{\mu_e}{T_\gamma}\right) \left\{1 + \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \left[\left(\frac{nm_e}{T_\gamma}\right)^2 K_2(nm_e/T_\gamma) - 2\right]\right\}
\end{aligned} \tag{23}$$

If we consider the temperature around $T_\gamma = 0.4\,\mathrm{MeV}$, then the effect of electron mass is given by

$$\begin{aligned}
\Delta'_{m\neq 0} &= \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[\left(\frac{n\times 0.5}{0.4}\right)^2 K_2(n\times 0.5/0.4) - 2\right] \\
&= \frac{6}{\pi^2}\Bigg[\left(\left(\frac{0.5}{0.4}\right)^2 K_2(0.5/0.4) - 2\right) - \frac{1}{2^2}\left(\left(\frac{2\times 0.5}{0.4}\right)^2 K_2(2\times 0.5/0.4) - 2\right) \\
&+ \frac{1}{3^2}\left(\left(\frac{3\times 0.5}{0.4}\right)^2 K_2(3\times 0.5/0.4) - 2\right) - \frac{1}{4^2}\left(\left(\frac{4\times 0.5}{0.4}\right)^2 K_2(4\times 0.5/0.4) - 2\right) + \cdots\Bigg] \\
&= \frac{1}{\pi^2}\Big(- 1.7282 + \cdots\Big) \\
&= -0.1751 + \cdots\,.
\end{aligned} \tag{24}$$

In this case the finite electron mass will 10% correction to the massless approximation at the temperature around $T_\gamma = 0.4\,\mathrm{MeV}$.