Electron Positron Partition Function in Early Universe

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I. PARTITION FUNCTION

Considering the $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, the energy of $e^{\pm}$ can be written as

$$E_{n,s} = \sqrt{p_z^2 + \tilde{m}^2 + 2eBn}, \qquad \tilde{m}^2 = m_e^2 + eB\left(1 - gs\right), \qquad s = \pm\frac{1}{2}, \qquad n = 0, 1, 2, 3, \ldots \tag{1}$$

$$\tag{2}$$

If we consider a system that all electrons and positrons are spin aligned and antialigned with the magnetic field B, then the partition function of the system can be written as

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \left[\ln\left(1 + e^{-\beta(E_n^{\pm} - \mu_e)}\right) + \ln\left(1 + e^{-\beta(E_n^{\pm} + \mu_e)}\right)\right], \tag{3}$$

where $\beta = 1/T$, μ_e is the chemical potential of electron, and energy $E_n^{\pm}$ can be written as

$$E_n^{\pm} = \sqrt{p_z^2 + \tilde{m}_{\pm}^2 + 2eBn}, \qquad \tilde{m}_{\pm}^2 = m_e^2 + eB\left(1 \pm \frac{g}{2}\right). \tag{4}$$

To simplify the partition function we can consider the expansion of the logarithmic function, we have

$$\ln\left(1 + x\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \qquad \text{for } |x| < 1. \tag{5}$$

Then the partition function of electron/positron system can be written as

$$\begin{aligned} \ln \mathcal{Z}_{tot} =& \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[e^{k\beta\mu_e} + e^{-k\beta\mu_e}\right] e^{-k\beta E_n^{\pm}} \\ =& \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[2\cosh\left(k\beta\mu_e\right)\right] \int_0^{\infty} dp_z \, e^{-k\beta E_n^{\pm}}. \end{aligned} \tag{6}$$

Using the general definition of Bessel function:

$$K_{\nu}(\beta m) = \frac{\sqrt{\pi}}{\Gamma(\nu - 1/2)} \frac{1}{m} \left(\frac{\beta}{2m}\right)^{\nu - 1} \int_0^{\infty} dp \, p^{2\nu - 2} e^{-\beta E} \qquad \text{for } \nu > 1/2. \tag{7}$$

the integral over dp_z can be written as

$$\int_0^{\infty} dp_z \, e^{-k\beta E_n^{\pm}} = \frac{\Gamma(1/2)}{\sqrt{\pi}} \sqrt{\tilde{m}_{\pm}^2 + 2eBn} \; K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right) \tag{8}$$

$$= \sqrt{\tilde{m}_{\pm}^2 + 2eBn} \; K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right). \tag{9}$$

In this case, the partition function becomes

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[2\cosh\left(k\beta\mu_e\right)\right] \sqrt{\tilde{m}_{\pm}^2 + 2eBn} \; K_1(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T) \tag{10}$$

$$= \frac{2eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2\cosh\left(k\beta\mu_e\right)\right] \sum_{n=0}^{\infty} W_1^{\pm}(n), \tag{11}$$

where we introduce the function $W_1^{\pm}(n)$ as follows

$$W_1^{\pm}(n) \equiv \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\, K_1\Big(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\Big)\,. \tag{12}$$

Considering the Euler-Maclaurin formula to replace the sum over Landau levels, we have

$$\sum_{n=0}^{\infty} W_1^{\pm}(n) = \int_0^{\infty} dn\, W_1^{\pm}(n) + \frac{1}{2}\Big[W_1^{\pm}(\infty) + W_1^{\pm}(0)\Big] + \frac{1}{12}\left[\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{\infty} - \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{0}\right] + R \tag{13}$$

where R is the error remainder which is defined by integrals over Bernoulli polynomials. Using the properties of Bessel function we have

$$\frac{\partial W_1^{\pm}}{\partial n} = -\frac{k^2 eB}{T^2} K_0\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right), \qquad , W_1^{\pm}(\infty) = 0, \qquad \int_a^{\infty} dx\, x^2 K_1(x) = a^2 K_2(a) \tag{14}$$

then we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} W_1^{\pm}(n) &= \int_0^{\infty} dn\, W_1^{\pm}(n) + \frac{1}{2} W_1^{\pm}(0) - \frac{1}{12}\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{0} + R \qquad (15)\\
&= \int_0^{\infty} dn\, \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\, K_1\Big(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\Big) + \frac{1}{2}\left[\frac{k\tilde{m}_{\pm}}{T} K_1(k\tilde{m}_{\pm}/T)\right] + \frac{1}{12}\left[\frac{k^2 eB}{T^2} K_0(k\tilde{m}_{\pm}/T)\right] + R\\
&= \left(\frac{T^2}{k^2 eB}\right)\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] + \frac{1}{2}\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right) K_1(k\tilde{m}_{\pm}/T)\right] + \frac{1}{12}\left[\left(\frac{k^2 eB}{T^2}\right) K_0(k\tilde{m}_{\pm}/T)\right] + R.
\end{aligned} \tag{16}$$

Replacing the sum over Landau levels by the integral, the partition function becomes

$$\ln \mathcal{Z}_{tot} = \ln \mathcal{Z}_{free} + \ln \mathcal{Z}_B + \ln \mathcal{Z}_R \tag{17}$$

where we defined

$$\ln \mathcal{Z}_{free} = \frac{2T^3 V}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cosh(k\mu_e/T)\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] \tag{18}$$

$$\ln \mathcal{Z}_B = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh(k\mu_e/T)\left[\frac{k\tilde{m}_{\pm}}{2T} K_1(k\tilde{m}_{\pm}/T) + \frac{k^2 eB}{12T^2} K_0(k\tilde{m}_{\pm}/T)\right] \tag{19}$$

$$\ln \mathcal{Z}_R = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh(k\mu_e/T) R \tag{20}$$

II. CHEMICAL POTENTIAL AND MAGNETIZATION

A. Charge neutrality

Giving the partition function, we can calculate the net number density of electron as follow:

$$\begin{aligned}
(n_e - n_{\bar{e}}) &= \frac{T}{V}\frac{\partial}{\partial \mu_e} \ln \mathcal{Z}_{tot} = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_{free}}{\partial \mu_e} + \frac{T}{V}\frac{\partial \ln \mathcal{Z}_B}{\partial \mu_e} + \frac{T}{V}\frac{\partial \ln \mathcal{Z}_R}{\partial \mu_e}\\
&= (n_e - n_{\bar{e}})_{free} + (n_e - n_{\bar{e}})_B + (n_e - n_{\bar{e}})_R
\end{aligned} \tag{21}$$

we have

$$(n_e - n_{\bar{e}})_{free} = \frac{2T^3}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\sinh\left(k\mu_e/T\right)\left[\left(\frac{k\tilde{m}_\pm}{T}\right)^2 K_2(k\tilde{m}_\pm/T)\right] \tag{22}$$

$$(n_e - n_{\bar{e}})_B = \frac{4eBT}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sinh\left(k\mu_e/T\right)\left[\frac{k\tilde{m}_\pm}{2T}K_1(k\tilde{m}_\pm/T) + s\frac{k^2 eB}{12T^2}K_0(k\tilde{m}_\pm/T)\right] \tag{23}$$

$$(n_e - n_{\bar{e}})_R = \frac{4eBT}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sinh\left(k\mu_e/T\right)R \tag{24}$$

Considering the Boltzmann approximation and assuming the error remainder R is small and can be neglected, then the net number density of electron can be written as

$$\begin{aligned}(n_e - n_{\bar{e}}) &\approx (n_e - n_{\bar{e}})_{free} + (n_e - n_{\bar{e}})_B \\ &= \frac{2T^3}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] + \frac{4eBT}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\frac{\tilde{m}_\pm}{2T}K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2}K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{25}$$

Using the charge neutrality, we have

$$\begin{aligned}n_p &= (n_e - n_{\bar{e}}) \\ &= \frac{2T^3}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] + \frac{4eBT}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\frac{\tilde{m}_\pm}{2T}K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2}K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{26}$$

where n_p is the number density of proton. It is also convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad B_0 = \frac{eB}{T^2} \tag{27}$$

and we obtain

$$n_p = \frac{T^3}{(2\pi)^2}\left[2\sinh\left(\mu_e/T\right)\right]\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right] \tag{28}$$

In this case, given the magnetic field B we can solve the chemical potential μ_e as a function of temperature numerically.

B. Magneticzation

On the other hand, considering the partition function in Boltzmann approximation and neglecting the error remainder R, we have

$$\begin{aligned}\ln\mathcal{Z}_{tot} =& \frac{T^3 V}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T) \\ &+ \frac{2eBTV}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left[\frac{1}{2}\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{12}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{29}$$

It can be written as

$$\ln\mathcal{Z}_{tot} = \frac{T^3 V}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T) + \frac{eB}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right\} \tag{30}$$

In this case, the magnetization can be obtained via the definition

$$M = \frac{T}{V}\frac{\partial\ln\mathcal{Z}_{tot}}{\partial B} = \frac{T}{V}\left(\frac{\partial\tilde{m}_\pm}{\partial B}\right)\frac{\partial\ln\mathcal{Z}_{tot}}{\partial\tilde{m}_\pm} \tag{31}$$

then it can be written as

$$M = \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{\frac{\partial\tilde{m}_\pm}{\partial B}\frac{\partial}{\partial\tilde{m}_\pm}\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] + \frac{e}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right.$$
$$\left. + \frac{eB}{T^2}\left[\frac{\partial\tilde{m}_\pm}{\partial B}\frac{\partial}{\partial\tilde{m}_\pm}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T)\right] + \frac{e}{6T^2}K_0(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)\frac{\partial\tilde{m}_\pm}{\partial B}\frac{\partial}{\partial\tilde{m}_\pm}K_0(\tilde{m}_\pm/T)\right]\right\} \tag{32}$$

we have

$$\frac{\partial\tilde{m}_\pm}{\partial B} = \frac{e(1\pm g/2)}{2\tilde{m}_\pm}, \tag{33}$$
$$\frac{\partial}{\partial\tilde{m}_\pm}K_0(\tilde{m}_\pm/T) = -\frac{1}{T}K_1(\tilde{m}_\pm/T), \tag{34}$$
$$\frac{\partial}{\partial\tilde{m}_\pm}\left(\frac{\tilde{m}_\pm}{T}K_1(\tilde{m}_\pm/T)\right) = -\frac{\tilde{m}_\pm}{T^2}K_0(\tilde{m}_\pm/T), \tag{35}$$
$$\frac{\partial}{\partial\tilde{m}_\pm}\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] = -\frac{\tilde{m}_\pm^2}{T^3}K_1(\tilde{m}_\pm/T). \tag{36}$$

Substituting the above equations into the magnetization we obtain

$$M = \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{-\left[\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm^2}{T^3}K_1(\tilde{m}_\pm/T)\right] + \frac{e}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right.$$
$$\left. + \frac{eB}{T^2}\left[-\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm}{T^2}K_0(\tilde{m}_\pm/T) + \frac{e}{6T^2}K_0(\tilde{m}_\pm/T) - \frac{1}{6}\left(\frac{eB}{T^2}\right)\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{1}{T}K_1(\tilde{m}_\pm/T)\right]\right\} \tag{37}$$

It is convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad B_0 = \frac{eB}{T^2} \tag{38}$$

then the magnetization can be written as

$$\begin{aligned} M &= \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{-\frac{e(1\pm g/2)}{2\tilde{m}_\pm^2}\left[x_\pm^3 K_1(x_\pm) + B_0 x_\pm^2 K_0(x_\pm) + \frac{B_0^2}{6}x_\pm K_1(x_\pm)\right] + \frac{e}{T^2}\left[x_\pm K_1(x_\pm) + \frac{B_0}{3}K_0(x_\pm)\right]\right\} \\ &= \frac{eT^2}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{-\frac{(1\pm g/2)}{2x_\pm^2}\left[x_\pm^3 K_1(x_\pm) + B_0 x_\pm^2 K_0(x_\pm) + \frac{B_0^2}{6}x_\pm K_1(x_\pm)\right] + \left[x_\pm K_1(x_\pm) + \frac{B_0}{3}K_0(x_\pm)\right]\right\} \\ &= \frac{e^2 B}{(2\pi)^2 B_0}\left[2\cosh(\mu_e/T)\right]\left\{-\frac{(1\pm g/2)}{2}\left[\left(x_\pm + \frac{B_0^2}{6x_\pm}\right)K_1(x_\pm) + B_0 K_0(x_\pm)\right] + \left[x_\pm K_1(x_\pm) + \frac{B_0}{3}K_0(x_\pm)\right]\right\} \\ &= \frac{4\pi\alpha B}{(2\pi)^2 B_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1 - \frac{(1\pm g/2)}{2}\left(1 + \frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right\}. \end{aligned} \tag{39}$$

In this case, given the magnetic field B and chemical potential we can solve the magnetization M as a function of temperature numerically.

C. Chemical potential and magnetization

Giving the condition of charge neutrality and magnetization we have

$$\left(\frac{n_p}{T^3}\right) = \frac{1}{(2\pi)^2}\left[2\sinh(\mu_e/T)\right]\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right] \tag{40}$$
$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 B_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1 - \frac{(1\pm g/2)}{2}\left(1 + \frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right\}. \tag{41}$$

The chemical potential of electron and positron can be written as

$$\sinh\left(\mu_e/T\right) = \frac{(2\pi)^2 n_p}{2T^3}\,\frac{1}{\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right]} \tag{42}$$

$$\longrightarrow \frac{(2\pi)^2 n_p}{2T^3}\,\frac{1}{x_\pm^2 K_2(x_\pm)}, \qquad \text{for } B_0 = 0 \tag{43}$$

it shows that for the case $B_0 = 0$ the chemical potential agrees with our earlier results. For magnetization we can use the properties of hyperbolic function $\cosh^2(\mu_e/T) - \sinh^2(\mu_e/T) = 1$,then we have

$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 B_0}\left[2\sqrt{1+(\sinh(\mu_e/T))^2}\right]\left\{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right\}. \tag{44}$$

and the magnetization can be written as

$$\frac{M}{B} = \frac{8\pi\alpha}{(2\pi)^2 B_0}\left(\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right) \times\sqrt{1+\left[\frac{(2\pi)^2 n_p/(2T^3)}{x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)}\right]^2} \tag{45}$$

In this case giving the magnetic field B_0 and proton density n_p/T^3 we can solve the magnetization and chemical potential numerically.

D. Proton number density and magnetic field in early universe

Considering the homogeneous proton number density in early universe, it can be written as

$$n_p = \frac{n_p}{n_B}\left(\frac{n_B}{s_{\gamma,\nu,e}}\right)s_{\gamma,\nu,e} = X_p\left(\frac{n_B}{s_{\gamma,\nu}}\right)s_{\gamma,\nu}, \qquad X_p = \frac{n_p}{n_B} \tag{46}$$

where n_B is the number density of baryon, and the second equality is obtained by considering $e^\pm$ entropy density is negligible compared to the photon and neutrino entropy density at post BBN temperature $20 < T < 50$keV.

- The proton fraction X_p can be written as follow:

$$X_p = \frac{n_p}{n_B} = \frac{n_p}{n_p + n_n} = \frac{1}{1 + n_n/n_p} \tag{47}$$

Since all neutrons end up bound in to the ^4H_e after BBN, then the mass fraction of ^4H_e can be estimated as

$$X_\alpha = \frac{2(n_n/n_p)}{1 + n_n/n_p} = 0.245 \pm 0.03 \tag{48}$$

where we use $X_\alpha = 0.245 \pm 0.03$ from particle data group. Solving the ratio n_n/n_p and substituting into the X_p, we obtain

$$X_p = 0.878 \pm 0.015 \tag{49}$$

- Since the comoving baryon number and entropy are conserved, hence the ratio $s_{\gamma,\nu}/n_B$ is conserved, then the entropy per baryon ratio can be written as

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right) = \left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \left(\frac{n_\gamma}{n_B}\right)_{t_0}\left(\frac{s_\gamma}{n_\gamma} + \frac{n_\nu}{n_\gamma}\frac{s_\nu}{n_\nu}\right) = \left(\frac{n_\gamma}{n_B}\right)_{t_0}\left[3.601 + \frac{9}{4}\left(\frac{T_\nu}{T_\gamma}\right)^3 4.202\right], \tag{50}$$

where the subscript t_0 denotes the present day value. We use the condition $(s/n)_{\rm boson} = 3.601$ and $(s/n)_{\rm fermion} = 4.202$, and three massless neutrino in our calculation. From particle data group and standard bing ban model, we have

$$5.8 \times 10^{-10} \leqslant \frac{n_B}{n_\gamma} \leqslant 6.5 \times 10^{-10}, \qquad \frac{n_B}{n_\gamma} = 6.14 \times 10^{10} (\text{BBN observation}), \qquad \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}. \tag{51}$$

In this case, the entropy per baryon ratio becomes

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \frac{1}{6.14 \times 10^{-10}} \left[3.601 + \frac{9}{4}\left(\frac{4}{11}\right) 4.202\right] = 1.14 \times 10^{10} \tag{52}$$

- On the other hand, the entropy density at temperature can be written as

$$s = \frac{2\pi^2}{45} g_s T_\gamma^3, \qquad g_s = \sum_{i=boson} g_i \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8} \sum_{i=fermion} g_i \left(\frac{T_i}{T_\gamma}\right)^3 \tag{53}$$

where g_s is the effective degree of freedom that contribute to the entropy density. In the temperature we are interested in $50 > T > 20$keV, the entropy density reads

$$s_{\gamma,\nu} = \frac{2\pi^2}{45} g_s T_\gamma^3, \qquad g_s = g_\gamma + \frac{7}{8} g_\nu \left(\frac{T_\nu}{T_\gamma}\right)^3 = 2 + \frac{7}{8} 2 \times 3 \left(\frac{4}{11}\right) = 3.91 \tag{54}$$

Substituting above numerical results into Eq.(46), the proton number density is given by

$$n_p = X_p \left(\frac{n_B}{s_{\gamma,\nu}}\right) s_{\gamma,\nu} = \frac{0.878 \times 3.91}{1.14 \times 10^{10}} \left(\frac{2\pi^2}{45}\right) T_\gamma^3 = 3.011 \times 10^{-10} \left(\frac{2\pi^2}{45}\right) T_\gamma^3 \tag{55}$$

On the other hand, the relic magnetic field in deep intergalactic space today is given by

$$B_{t_0} = 10^{-20} \sim 10^{-12}\,\text{Tesla} \tag{56}$$

where the subscript t_0 denotes the present day value. In this case, we have

$$eB_{t_0} = 5.92 \times \left(10^{-19} \sim 10^{-11}\right)\,\text{eV} \tag{57}$$

On the other hand the photon temperature today is given by

$$T_{t_0} = 2.75\,\text{K} = 2.35 \times 10^{-4}\,\text{eV} \tag{58}$$

If we considering the following case in the universe:

$$B \propto \frac{1}{a^2}, \qquad T \propto \frac{1}{a} \tag{59}$$

where a is the scale factor, then the ratio eB/T^2 is a conserved quantity. We have

$$\frac{eB}{T} = \left(\frac{eB_{t_0}}{T_{t_0}}\right) = 1.07 \times \left(10^{-3} \sim 10^{-8}\right) \tag{60}$$

E. Example: g-factor $g = 2$

In our calculation we introduce an effective mass term which incorporates the ground state component of the Landau energy as follow

$$x_\pm = \frac{\tilde{m}_\pm}{T} = \frac{1}{T}\sqrt{m_e^2 + eB\left(1 \pm \frac{g}{2}\right)} \tag{61}$$

Considering the case $g = 2$ we have following two cases:

- Case1: $\tilde{m}_+ = \sqrt{m_e^2 + 2eB}$, and $x = \tilde{m}_+/T$. The equations for chemical potential and magnetization are given by

$$\sinh(\mu_e/T) = \frac{(2\pi)^2 n_p}{2T^3} \frac{1}{\left[x^2 K_2(x) + B_0 x K_1(x) + \frac{B_0^2}{6} K_0(x)\right]} \tag{62}$$

and

$$\left(\frac{M}{B}\right) = -\frac{8\pi\alpha}{(2\pi)^2}\left(\frac{B_0}{6x}K_1(x_\pm) + \frac{2}{3}K_0(x_\pm)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right]}\right]^2} \tag{63}$$

Substituting the magnetic field B_0 and proton density n_p/T^3 from pervious section, we can solve the magnetization and chemical potential numerically.

In Fig 1, we plot the chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature T. It shows for the case1 the chemical potential and magnetization are not sensitive to the magnetic field, this is because the small value of $B_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq.(62) and Eq.(63).

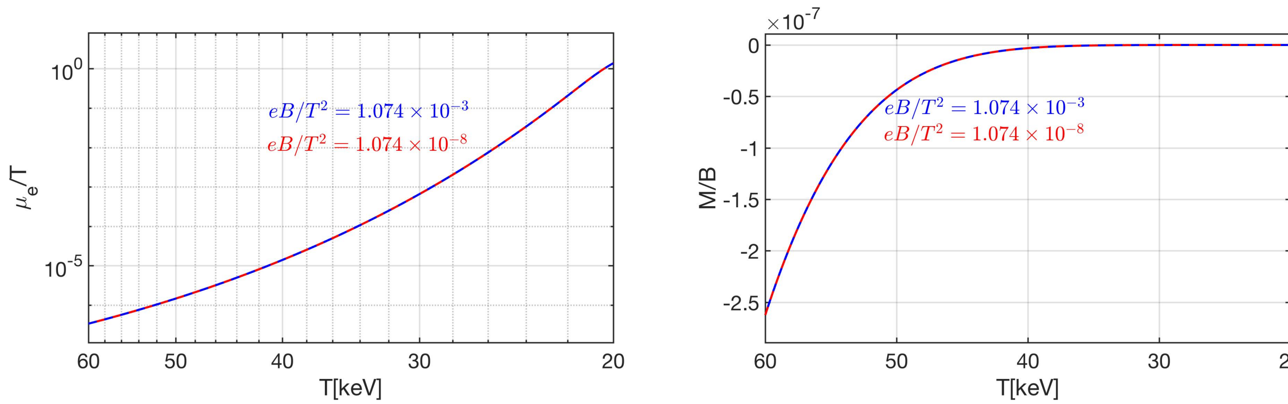

FIG. 1: The chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature $20 \leqslant T \leqslant 60$keV for the case 1. The chemical potential and magnetization are not sensitive to the magnetic field, because the small value of $B_0 = 10^{-3} \sim 10^{-8}$ can be neglected in Eq.(62) and Eq.(63).

- Case2: $\tilde{m}_- = m_e$ and $x = \tilde{m}_-/T$,then the equations for chemical potential and magnetization can be written as

$$\sinh(\mu_e/T) = \frac{(2\pi)^2 n_p}{2T^3} \frac{1}{\left[x^2 K_2(x) + B_0 x K_1(x) + \frac{B_0^2}{6} K_0(x)\right]} \tag{64}$$

and

$$\left(\frac{M}{B}\right) = \frac{8\pi\alpha}{(2\pi)^2}\left(\frac{1}{B_0}xK_1(x) + \frac{1}{3}K_0(x)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x^2 K_2(x) + B_0 x K_1(x) + \frac{B_0^2}{6}K_0(x)\right]}\right]^2} \tag{65}$$

Using the magnetic field B_0 and proton density n_p/T^3 from pervious section, we can solve the magnetization and chemical potential for case 2 numerically.

In Fig 2, we plot the chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature T. It shows that the chemical potential is not sensitive to the magnetic field because the small value of $B_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq.(64). However, the magnetization does depend on the magnetic field B_0 strongly. This is because for small magnetic field B_0 the dominant term in Eq(65) is $xK_1(x)/B_0$. For given B_0, the value of magnetization can be larger than the magnetic filed, i.e. $M/B > 1$ which shows the possibility that magnetic domains can be formed in early universe.

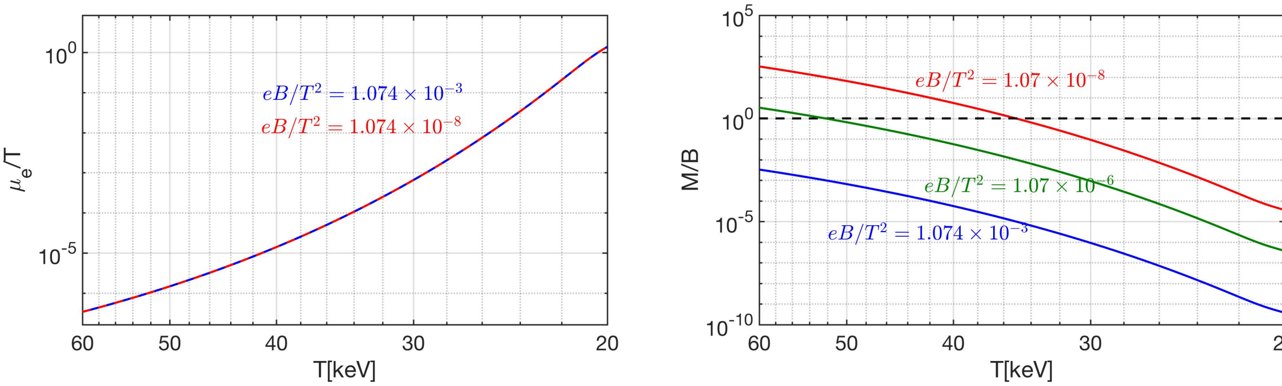

FIG. 2: The chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature $20 \leqslant T \leqslant 60$keV for the case 2. It shows that the chemical potential is not sensitive to the magnetic field B_0. However, the magnetization does depend on the magnetic field B_0 strongly. This is because for small magnetic field B_0 the dominant term in Eq(65) is $xK_1(x)/B_0$. For giving B_0 we can have $M/B > 1$ in early universe.