Electron Positron Partition Function in Early Universe

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I. PARTITION FUNCTION

Considering the partition function of $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, we have

$$\ln \mathcal{Z}_{tot} = eBV \sum_{s=\pm1}\sum_{j=0}^{\infty}\int_{-\infty}^{\infty} dp_z \left[\ln\left(1+e^{-\beta(E_{j,s}-\mu_e)}\right)+\ln\left(1+e^{-\beta(E_{j,s}+\mu_e)}\right)\right], \tag{1}$$

where $\beta = 1/T$, μ_e is the chemical potential of electron, and the electron(positron) energy $E_{j,s}$ can be written as

$$\begin{aligned} E_{j,s} &= \sqrt{m_e^2 + p_z^2 + 2eB\left(j+\frac{1}{2}+\frac{g}{4}s\right)+\frac{g^2}{4}\mu_B^2 B^2} \\ &= \sqrt{\tilde{m}_e^2 + p_z^2 + 2eB\left(j+\frac{1}{2}+\frac{g}{4}s\right)}, \qquad \tilde{m}_e^2 = m_e^2 + \frac{g^2}{4}\mu_B^2 B^2 \qquad s=\pm1, \qquad j = 0,1,2,\ldots \end{aligned} \tag{2}$$

and μ_B is the magnetic moment.

Considering the integral over dp_z we can simplify the integration by integrating by part, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dp_z \left[\ln\left(1+e^{-\beta(E_{j,s}-\mu_e)}\right)+\ln\left(1+e^{-\beta(E_{j,s}+\mu_e)}\right)\right] \\ &= 2\left[\int_0^{\infty} dp_z \ln\left(1+e^{-\beta(E_{j,s}-\mu_e)}\right)+\int_0^{\infty} dp_z \ln\left(1+e^{-\beta(E_{j,s}+\mu_e)}\right)\right] \qquad (3) \\ &= 2\left[\beta\int_0^{z} dp_z p_z \frac{\partial E_{j,s}}{\partial p_z}\frac{1}{e^{\beta(E_{j,s}-\mu_e)}+1}+\beta\int_0^{z} dp_z p_z \frac{\partial E_{j,s}}{\partial p_z}\frac{1}{e^{\beta(E_{j,s}+\mu_e)}+1}\right] \qquad (4) \\ &= 2\beta\int_0^{z} dp_z \frac{p_z^2}{E_{j,s}}\left[\frac{1}{e^{\beta(E_{j,s}-\mu_e)}+1}+\frac{1}{e^{\beta(E_{j,s}+\mu_e)}+1}\right] \qquad (5) \\ &= 2\beta\int_0^{z} dp_z \frac{p_z^2}{E_{j,s}}\frac{e^{-\beta E_{j,s}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,s})} \qquad (6) \end{aligned}$$

where the last step we use the identity

$$\frac{1}{e^{x-y}+1}+\frac{1}{e^{x+y}+1} = \frac{2+(e^{x-y}+e^{x+y})}{(e^{2x}+1)+(e^{x-y}+e^{x+y})} = \frac{e^{-x}+\cosh y}{\cosh x+\cosh y} \tag{7}$$

to simplify the integral.

Then the partition function of $e^{\pm}$ plasma in a uniform magnetic field B can be written as

$$\begin{aligned} \ln \mathcal{Z}_{tot} &= eBV \sum_{s=\pm1}\sum_{j=0}^{\infty} 2\beta\int_0^{z} dp_z \frac{p_z^2}{E_{j,s}}\frac{e^{-\beta E_{j,s}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,s})} \qquad (8) \\ &= V(2eB\beta)\sum_{j=0}^{\infty}\left[\int_0^{z} dp_z \frac{p_z^2}{E_{j,+}}\frac{e^{-\beta E_{j,+}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,+})}+\int_0^{z} dp_z \frac{p_z^2}{E_{j,-}}\frac{e^{-\beta E_{j,-}}+\cosh(\beta\mu_e)}{\cosh(\beta\mu_e)+\cosh(\beta E_{j,-})}\right] \qquad (9) \end{aligned}$$

where the energy $E_{j,\pm}$ are given by

$$E_{j,+} = \sqrt{\tilde{m}_e^2 + p_z^2 + 2eB\left(j+\frac{1}{2}+\frac{g}{4}\right)}, \qquad E_{j,-} = \sqrt{\tilde{m}_e^2 + p_z^2 + 2eB\left(j+\frac{1}{2}-\frac{g}{4}\right)} \tag{10}$$

It is convenient to introduce the dimensionless variables as follow:

$$x = E_{j,+}/T, \qquad x_j = \sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}\left(j + \frac{1}{2} + \frac{g}{4}\right)}, \tag{11}$$

$$y = E_{j,-}/T, \qquad y_j = \sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}\left(j + \frac{1}{2} - \frac{g}{4}\right)}. \tag{12}$$

Then the partition function of $e^{\pm}$ plasma becomes

$$\ln \mathcal{Z}_{tot} = V(2eBT)\sum_{j=0}^{\infty}\left[\int_{x_j}^{\infty} dx\sqrt{x^2 - x_j^2}\,\frac{e^{-x} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(x)} + \int_{y_j}^{\infty} dy\sqrt{y^2 - y_j^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}\right]. \tag{13}$$

Considering the case $g = 2$, then we have

$$x_j = \sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}(j+1)} \longrightarrow x_n = \sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}n}, \qquad n = 1, 2, 3, \ldots \tag{14}$$

$$y_j = \sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}j} \longrightarrow y_n = \sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}n}, \qquad n = 0, 1, 2, 3, \ldots \tag{15}$$

where we change the index from j to n. In this case, the partition function of $e^{\pm}$ plasma can be written as

$$\begin{aligned}
\ln \mathcal{Z}_{tot} &= V(2eBT)\left[\sum_{n=1}^{\infty}\int_{x_n}^{\infty} dx\sqrt{x^2 - x_n^2}\,\frac{e^{-x} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(x)} + \sum_{n=0}^{\infty}\int_{y_n}^{\infty} dy\sqrt{y^2 - y_n^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}\right] \\
&= V(2eBT)\left[\int_{y_0}^{\infty} dy\sqrt{y^2 - y_0^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)} + 2\sum_{n=1}^{\infty}\int_{y_n}^{\infty} dy\sqrt{y^2 - y_n^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}\right] \\
&= V(2eBT)\left[\int_{\tilde{m}_e/T}^{\infty} dy\sqrt{y^2 - \tilde{m}_e^2/T^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)} + 2\sum_{n=1}^{\infty}\int_{y_n}^{\infty} dy\sqrt{y^2 - y_n^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}\right]
\end{aligned} \tag{16}$$

Next we want to use the Euler-Maclaurin formula to approximate the sum by integral. In general, we have

$$\sum_{n=a}^{b} f(n) \approx \int_a^b f(n)dn + \frac{f(b) + f(a)}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right], \tag{17}$$

where B_i is the ith Bernoulli number. In our case, we have

$$f(n) = \int_{\sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}n}}^{\infty} dy\sqrt{y^2 - \left(\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}n\right)}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}. \tag{18}$$

Substituting the function into Euler Maclaurin formula, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_{y_n}^{\infty} dy\sqrt{y^2 - y_n^2}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)} \\
&\approx \left(\int_1^{\infty} dn \int_{\sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}n}}^{\infty} dy\sqrt{y^2 - \left(\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}n\right)}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}\right) \\
&+ \left(\frac{1}{2}\int_{\sqrt{\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}}}^{\infty} dy\sqrt{y^2 - \left(\frac{\tilde{m}_e^2}{T^2} + \frac{2eB}{T^2}\right)}\,\frac{e^{-y} + \cosh(\mu_e/T)}{\cosh(\mu_e/T) + \cosh(y)}\right) + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(\infty) - f^{(2k-1)}(1)\right].
\end{aligned} \tag{19}$$

and the partition function can be written as

$$\ln \mathcal{Z}_{tot} = V(2eBT)\left[\left[\int_{\tilde{m}_e/T}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2}}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} + 2\int_1^{\infty} dn \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}n}}^{\infty} dy \sqrt{y^2 - \left(\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}n\right)}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)}\right.\right.$$

$$\left.+ \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}}}^{\infty} dy \sqrt{y^2 - \left(\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}\right)}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} + 2\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(\infty) - f^{(2k-1)}(1)\right]\right] \tag{20}$$

It is convenient to introduce the dimensionless variable:

$$z = \frac{2eB}{T^2}n \tag{21}$$

then the partition function becomes

$$\ln \mathcal{Z}_{tot} = V(2eBT)\left[\left[\int_{\tilde{m}_e/T}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2}}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} + \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}}}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2} - \frac{2eB}{T^2}}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)}\right.\right.$$

$$\left.\frac{T^2}{eB}\int\limits_{\frac{2eB}{T^2}}^{\infty} dz \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+z}}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2} - z}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} + 2\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(\infty) - f^{(2k-1)}(1)\right]\right] \tag{22}$$

Finally we can change the order of integral between dy and dz and we have

$$\begin{aligned}\int\limits_{\frac{2eB}{T^2}}^{\infty} dz \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+z}}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2} - z}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} &= \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}}}^{\infty} dy \int\limits_{\frac{2eB}{T^2}}^{y^2-\frac{\tilde{m}_e^2}{T^2}} dz \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2} - z}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} \\ &= \frac{2}{3}\int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}}}^{\infty} dy \left[y^2 - \frac{\tilde{m}_e^2}{T^2} - \frac{2eB}{T^2}\right]^{3/2}\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)}\end{aligned} \tag{23}$$

In this case the partition function of $e^{\pm}$ plasma in a uniform magnetic field B can be written as

$$\ln \mathcal{Z}_{tot} = V(2eBT)\left[\left[\int_{\tilde{m}_e/T}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2}}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} + \int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}}}^{\infty} dy \sqrt{y^2 - \frac{\tilde{m}_e^2}{T^2} - \frac{2eB}{T^2}}\,\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)}\right.\right.$$

$$\left.\frac{2T^2}{3eB}\int\limits_{\sqrt{\frac{\tilde{m}_e^2}{T^2}+\frac{2eB}{T^2}}}^{\infty} dy \left[y^2 - \frac{\tilde{m}_e^2}{T^2} - \frac{2eB}{T^2}\right]^{3/2}\frac{e^{-y}+\cosh(\mu_e/T)}{\cosh(\mu_e/T)+\cosh(y)} + 2\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(\infty) - f^{(2k-1)}(1)\right]\right] \tag{24}$$