ELECTRON POSITRON MAGNETIZATION IN THE EARLY UNIVERSE

Johann Rafelski[ID]**, Cheng Tao Yang**[ID]**, Andrew Steinmetz**[ID]**, and Jeremiah Birrell**

Department of Physics, The University of Arizona, Tucson, Arizona 85721, USA

* Correspondence: JohannR@arizona.edu

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1   **Abstract:** We will write abstract here

2   **Keywords:** Magnetization; Electron-Positron plasma; Cosmology.

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3   **1. Introduction**

4   The electron-positron epoch of the early universe was home to several significant events which have
5   greatly shaped our contemporary universe including neutrino decoupling, Big Bang Nucleosynthesis
6   (BBN), the annihilation of most electrons and positrons partially re-ionizing the universe, as well as
7   setting the stage for the eventual recombination period which would generate the cosmic microwave
8   background (CMB). Therefore, correctly describing the dynamics of this $e^{\pm}$ plasma is of interest when
9   considering modern cosmic mysteries such as the origin of extra-galactic magnetic fields (EGMF). While
10   most approaches tackle magnetized plasmas from the perspective of magneto-hydrodynamics (MHD),
11   a primarily classical or semi-classical approach, our perspective is to demonstrate that fundamental
12   quantum statistical analysis can lead to further insights on the behavior of magnetized plasmas. The
13   universe is filled with magnetic fields at various scales and strengths both within galaxies and in
14   deep extra-galactic space far and away from matter sources. Extra-galactic magnetic fields are not
15   well constrained today, but are required by observation to be non-zero with a magnitude between
16   10^{-12} T $> B_{EGMF} > 10^{-20}$ T over Mpc coherent length scales. The upper bound is constrained from the
17   characteristics of the CMB while the lower bound is constrained by non-observation of ultra-energetic
18   photons from blazars. There are generally considered two possible origins for extra-galactic magnetic
19   fields: (a) matter-induced dynamo processes involving Amperian currents and (b) primordial (or relic)
20   seed magnetic fields whose origins may go as far back as the Big Bang itself. It is currently unknown
21   which origin accounts for extra-galactic magnetic fields today or if it some combination of the two
22   models. Even if magnetic fields in the universe today are primarily driven via amplification through
23   Amperian matter currents, such models still require primordial seed fields at some point to act as
24   catalyst.

25   We then connect back to the plasmas of the early universe as such primordial magnetic fields
26   would be lensed through the various plasmas that existed when the universe was far hotter and
27   denser. Of particular interest to us is the electron-positron plasma which existed in the early universe
28   at temperature $T > 35$ keV which was the last matter/anti-matter plasma in the universe where the
29   medium density exceeded that of the proton/neutron baryon density. This dense plasma environment
30   is where BBN occurred and where similar plasmas can still be found within exotic stars such as
31   magnetars. The contemporary relic magnetic fields may then be an artifact of this time of last
32   universe-scale magnetization in a manner similar to how the CMB is a relic of the time of charge
33   recombination.

34   (CHENG TAO: I added some material here that can be helpful in introduction, feel free to use
35   and edit it) At an early time in the standard cosmology model, the universe began as a fireball with
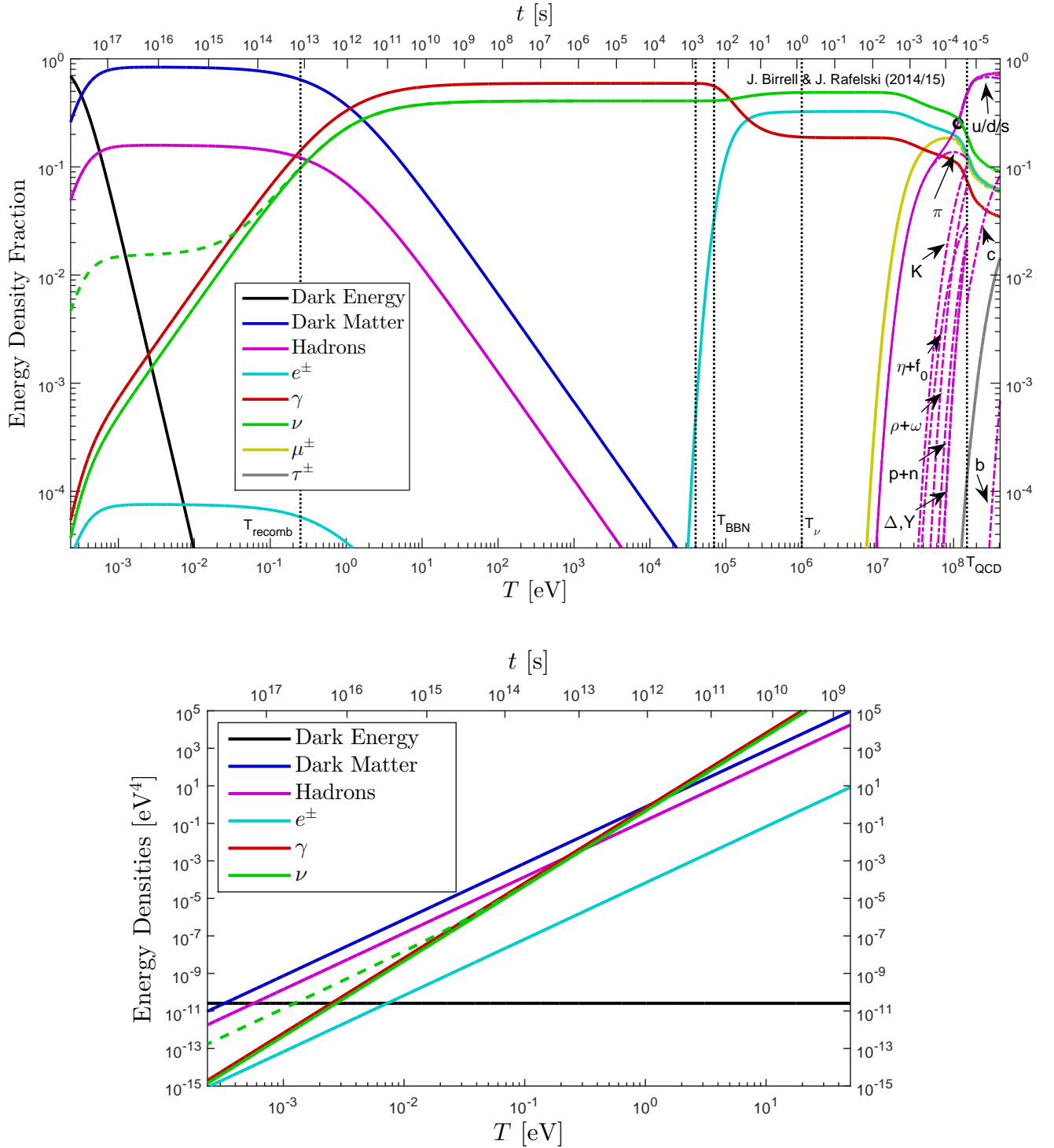
Figure 1. Current era: 69% dark energy, 26% dark matter, 5% baryons, $< 1\%$ photons and neutrinos. Solid neutrino line shows massless neutrinos while the dashed line shows 1 massless and 2×0.1 eV neutrinos (Neutrino mass choice is just for illustration. Other values are possible).
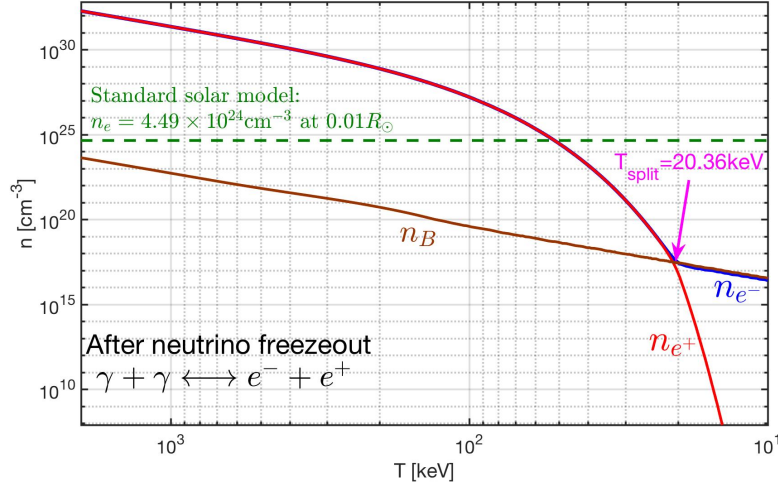
Figure 2. The electron(positron) number density as a function of temperature $2\,\text{MeV} > T > 10\,\text{keV}$. The blue solid line is the electron density, the red solid line is the positron density, and brown solid line is the baryon density. For comparison, we also show the green dotted line as the solar electron density. It shows that we have electron-positron rich plasma in early universe until $T_{\text{split}} = 20.36$ keV, for $T < T_{\text{split}}$ the positron density decreases because of the annihilation.

extremely high temperature and high energy density. In this case, the ultrarelativistic plasma can be produced in the early Universe and transition to different types of properties as the universe expanded and cooled. The comic plasma occurs in the early universe in the following sequence:

- Primordial Quark-gluon plasma: At the early time when $130\text{GeV} > T > 150\text{MeV}$, we have leptons, gauge mesons, and quarks were deconfined and propagated freely; all hadrons are dissolved into their constituents. At that time, strongly interacting nearly massless particles u, d, s, G controlled the fate of the Universe.
- Hadronic plasma: Around the hadronization temperature $T_h \approx 150$ MeV, a phase transformation to the confining state occurs, and matter as we know it today forms, and the Universe turns into hadron matter dominated. In the temperature range $20 < T < 60$ MeV the Universe is rich in physics phenomena involving strange mesons, (anti)baryons including (anti)hyperon abundances [1,2].
- Electroweak plasma: For $10\,\text{MeV} > T > 1\,\text{MeV}$, the universe contains relativistic electron, positron, photon, and 3 species of neutrinos/antineutrinos. In this range, neutrinos still couple to the electron/positron via the weak interaction, and the property of the universe is controlled by this electroweak plasma [3].
- Electron-positron plasma: After neutrino freezeout from the cosmic plasma at $T = 1$ MeV, we have electron-positron-photon plasma in the universe. We show that the electron-positron-photon plasma can exist until $T \approx 0.02\,\text{MeV}$, and the BBN happened in the electron-positron-rich plasma.

The properties of the electron-positron $e\bar{e}$-pair plasma in the early Universe has not received the appropriate attention in the days of precision BBN studies [4]. The presence of $e\bar{e}$-pairs before and during BBN has been acknowledged by Wang, Bertulani and Balantekin [5] nearly a decade ago. This was before necessary tools were developed to explore the connection between electron and neutrino plasma [3,6,7].

In Fig. 2 it shows the dense $e\bar{e}$ plasma in early universe under the hypothesis charge neutrality and entropy conservation as a function of temperature $2\,\text{MeV} > T > 10\,\text{keV}$ [8]. It shows that we have electron-positron rich plasma,i.e, $n_\pm \gg n_B$ in early universe until $T_{\text{split}} = 20.36$ keV. For $T < T_{\text{split}}$ the positron density n_{e^+} decreases because of the annihilation and electron density equal to the proton density because of the charge neutrality.
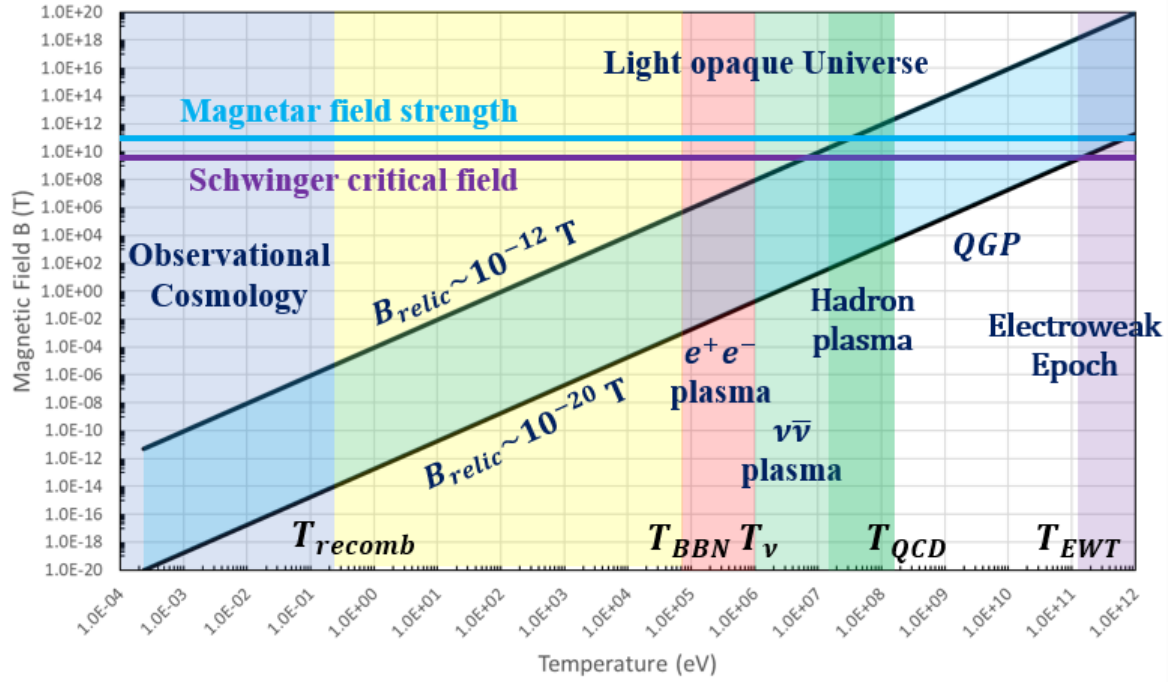
Figure 3. Qualitative value of the primordial magnetic field over the evolutionary lifespan of the universe. As magnetic flux is conserved over co-moving surfaces, the primordial relic field is expected to dilute as $B \propto 1/a(t)^2$ meaning the contemporary small bounded values of 5×10^{-12} T $> B_{relic} >$ 10^{-20} T may have once represented large magnetic fields in the early universe. This is relic magnetic field would then be generated by the last phase of significant magnetization in the early universe. This figure is meant to be illustrative and it is unlikely the magnetization of the universe would proceed unhindered and unaltered into the ultra-dense plasma phases of the early universe. The values of the Schwinger critical field and the upper bound of surface magnetar field strength are included for scale.

2. Energy Eigenvalues

As a starting point, we consider the energy eigenvalues of charged fermions within a homogeneous magnetic field. Here, we have several choices: We could assume the typical Dirac energy eigenvalues with gyro-magnetic g-factor set to $g = 2$. But as electrons, positrons and most plasma species have anomalous magnetic moments (AMM), we require a more complete model. Another option would be to modify the Dirac equation with a Pauli term, often called the Dirac-Pauli (DP) approach, via

$$\hat{H}_{\text{AMM}} = -a \frac{e}{2m_e} \frac{\sigma_{\mu\nu} F^{\mu\nu}}{2} \, , \tag{1}$$

where $\sigma_{\mu\nu}$ is the spin tensor proportional to the commutator of the gamma matrices and $F^{\mu\nu}$ is the EM field tensor. For the duration of this work, we will remain in natural units ($\hbar = c = k_B = 1$) unless explicitly stated otherwise. The AMM is defined via g-factor as

$$\frac{g}{2} = 1 + a \, . \tag{2}$$

This approach, while straightforward, would complicate the energies making analytic understanding and clarity difficult without a clear benefit. Our preferred model for the AMM is through the Klein-Gordon-Pauli (KGP) equation which is given by

$$\left((i\partial_\mu - eA_\mu)^2 - m_e^2 - e\frac{g}{2} \frac{\sigma_{\mu\nu} F^{\mu\mu}}{2} \right) \Psi = 0 \, . \tag{3}$$

We wish to emphasize, that each of the three above models are distinct and have differing physical consequences and are not interchangeable. One benefit of the KGP approach is that the energies take eigenvalues which are mathematically similar to the Dirac energies. Considering the $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, the energy of $e^{\pm}$ fermions can be written as

$$E_n^s = \sqrt{p_z^2 + \tilde{m}^2 + 2eBn}, \qquad \tilde{m}^2 = m_e^2 + eB\left(1 - gs\right), \qquad s = \pm\frac{1}{2}, \qquad n = 0, 1, 2, 3, \ldots \qquad (4)$$

where n is the principle quantum number for the Landau levels and s is the spin quantum number. Here we introduce a notion of "magnetic mass" which inherits the spin-specific part of the energy adding them to the mass. This convention is also generalizable to further non-minimal electromagnetic models with more exotic energy contributions such that we write a general replacement as

$$m_e^2 \to \tilde{m}^2(B). \qquad (5)$$

This definition also pulls out the ground state Landau energy separating it from the remainder of the Landau tower of states. One restriction is that the magnetic-mass must remain positive definite in our analysis thus we require

$$\tilde{m}^2(B) = m_e^2 + eB\left(1 - gs\right) > 0. \qquad (6)$$

This condition fails under ultra-strong magnetic fields of order

$$B_{\text{crit}} = \frac{m_e^2}{ea} = \frac{\mathcal{B}_S}{a} \approx 3.8 \times 10^{12} \text{ T}, \qquad (7)$$

where $\mathcal{B}_S$ is the Schwinger critical field strength. For electrons, this field strength is well above the window of magnetic field strengths of interest during the late $e^{\pm}$ epoch.

3. Landau eigen-energies in cosmology

The standard model of cosmology with a cosmological constant plus cold dark matter (ΛCDM) is described by the Friedman-Lemaître-Robertson-Walker (FLRW) metric. The spatially flat (Gaussian curvature $k = 0$) FLRW metric with metric signature $(+1, -1, -1, -1)$ in Cartesian coordinates is

$$ds^2 = dt^2 - a^2(t)\left[dx^2 + dy^2 + dz^2\right]. \qquad (8)$$

The scale factor $a(t)$ denotes the change of proper distances in an expanding (or contracting) universe which is both homogeneous and isotropic. The evolutionary expansion of the universe is then traditionally defined in terms of the Hubble parameter H as follows

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho_{tot} \qquad (9)$$

where G_N is the Newtonian constant of gravitation and ρ_{tot} is the total energy density of the universe. Eq. (9) is also known as the first Friedmann equation.

There is another natural scale for the magnetic field besides Eq. (7) when considering the consequences of FLRW expansion on the $e^{\pm}$ fluid. As the universe expands, different terms in the energies and thus partition function evolve as a function of the scale factor $a(t)$ which arises in the FLRW metric. We can consider the expansion to be an adiabatic process which results in a

smooth shifting of the relevant dynamical quantities. From the conservation of magnetic flux through a co-moving surface, the magnetic field under expansion starting at some initial time t_0 is given by

$$B(t) = B(t_0) \frac{a(t_0)^2}{a(t)^2} . \tag{10}$$

As the universe expands, the temperature also cools as the cosmological redshift reduces the kinetic energies of particles in the universe lowering their contribution to the energy content of the universe. This cosmological redshift is written as

$$T(t) = T(t_0) \frac{a(t_0)}{a(t)} . \tag{11}$$

The momenta scale with the same factor as temperature, which is the origin of cosmological redshift

$$p_i(t) = p_i(t_0) \frac{a(t_0)}{a(t)} . \tag{12}$$

The energy of massive free particles in the universe scales differently based on their momentum (and thus temperature). When hot and relativistic, particle energy scales with inverse scale factor like radiation. However as particles transition to non-relativistic momenta, their energies scale with the inverse square of the scale factor like magnetic flux.

$$E(t) = E(t_0) \frac{a(t_0)}{a(t)} \xrightarrow{\text{NR}} E(t_0) \frac{a(t_0)^2}{a(t)^2} . \tag{13}$$

This occurs because of the functional dependence of energy on momentum in the relativistic versus non-relativistic cases. The argument in the Boltzmann statistical factor is given by

$$X_n^s \equiv \frac{E_n^s}{T} . \tag{14}$$

We can explore this relationship for the magnetized system explicitly by writing out Eq. (14) using the KGP eigen-energies as

$$X_n^s = \sqrt{\frac{m_e^2}{T^2} + \frac{p_z^2}{T^2} + \frac{2eB}{T^2} \left(n + \frac{1}{2} - \frac{gs}{2} \right)} , \tag{15}$$

where we now introduce the expansion scale factor via Eq. (10) - Eq. (12). The Boltzmann factor can then be written as

$$X_n^s[a(t)] = \sqrt{\frac{m_e^2}{T^2(t_0)} \frac{a(t)^2}{a(t_0)^2} + \frac{p_z^2(t_0)}{T^2(t_0)} + \frac{2eB(t_0)}{T^2(t_0)} \left(n + \frac{1}{2} - \frac{gs}{2} \right)} . \tag{16}$$

This reveals that only the mass contribution is dynamical over cosmological time. For any given eigen-state, the mass term increases driving the state into the non-relativistic limit while the momenta and magnetic contributions are frozen by initial conditions. As a point of comparison, the Boltzmann factor for the DP eigen-energies becomes

$$X_n^s|_{DP} = \sqrt{\left(\sqrt{\frac{m_e^2}{T^2} + \frac{2eB}{T^2} \left(n + \frac{1}{2} - s \right)} - \frac{eB}{2mT} (g - 2)s \right)^2 + \frac{p_z^2}{T^2}} , \tag{17}$$

which over cosmological time under expansion scales as

$$X_n^s[a(t)]|_{DP} = \sqrt{\left(\sqrt{\frac{m_e^2}{T^2(t_0)}\frac{a(t)^2}{a(t_0)^2} + \frac{2eB(t_0)}{T^2(t_0)}\left(n + \frac{1}{2} - s\right)} - \frac{eB(t_0)}{2mT(t_0)}\frac{a(t_0)}{a(t)}(g-2)s\right)^2 + \frac{p_z^2(t_0)}{T^2(t_0)}} .$$

(18)

We note here one important difference between KGP and DP eigen-energies in the context of cosmology: The anomalous magnetic moment portion of the DP statistics is suppressed by $1/a(t)$ over cosmological time while the AMM contribution is preserved in the KGP model. That the universe's expansion makes a distinction between $g = 2$ magnetic moment and AMM for DP fermions appears as a rather artificial and nonphysical trait. While the suppression of AMM may often be small for particles such as electrons, this suppression is non-trivial for particles with large AMM values such as the proton. That cosmological redshift would push DP protons to be described by $g = 2$ eign-energies in the non-relativistic limit counts as a malaise for the model and further strengthens our thinking that the KGP model is more appropriate for cosmological studies. Motivated by Eq. (16), we can introduce a dimensionless cosmic magnetic scale which is frozen in the homogeneous case as

$$b_0 \equiv \frac{eB}{T^2} = \frac{eB\hbar c^2}{(k_B T)^2} \text{ (S.I)},$$

(19)

where we've included the expression explicitly in full SI units. We can estimate the value of b_0 from the bounds on the extra-galactic magnetic field strength and the temperature of the universe today. If the origin of deep space extra-galactic magnetic fields are relic fields from the early universe, which today are expected to exist between 5×10^{-12} T $> B_{relic} > 10^{-20}$ T, then at temperature $T = 2.7$ K, the value of the cosmic magnetic scale is between

$$5.5 \times 10^{-5} > b_0 > 1.1 \times 10^{-11} .$$

(20)

This should remain constant in the universe at-large up to the last epoch the universe was sufficiently magnetized to disturb this value. As the electron-proton plasma which generated the CMB was relatively dilute over its duration, it was unlikely sufficiently magnetized to significantly alter this value over extra-galactic scales. Rather, the best candidate plasma to have been sufficiently magnetized and dense to have set the relic field magnetic scale would have been the electron-positron plasma which existed during the duration of Big Bang Nucleosynthesis (BBN) and beforehand.

Higher order non-minimal magnetic contributions which can be introduced via Eq. (5) to the eigen-energies like $\approx \mu_B^2 B^2/T^2$ are even more surpressed over cosmological time which drives the system into minimal electromagnetic coupling with the exception of the anomalous magnetic moment in the KGP case. It is interesting to note that cosmological expansion serves to "smooth out" the characteristics of more complex BSM electrodynmaics erasing them from a statistical perspective in favor of the minimal or minimal-like dynamics. As b_0 is a constant of expansion, assuming the electron-proton plasma between the CMB and electron-positron annihilation did not greatly disturbed it, we can calculate the remnant values at the temperature $T = 50$ keV with the expression

$$B^{\max}(T = 50\text{keV}) = \frac{B_0^{\max}(k_B T)^2}{(|q|\hbar c^2)} = 2.3 \times 10^5 \text{Tesla},$$

(21)

$$B^{\min}(T = 50\text{keV}) = 4.6 \times 10^{-4} \text{Tesla}$$

(22)

4. Partition function

We now turn our attention now to the statistical behavior of the $e^{\pm}$ system. We can utilize the general fermion partition function given by

$$\ln \mathcal{Z} = \sum_{\alpha} \ln \left(1 + e^{-\beta(E-\eta)}\right), \tag{23}$$

where $\beta = 1/T$, α is the set of all quantum numbers in the system, and η is the generalized chemical potential. The magnetized $e^{\pm}$ system should be considered a system of four quantum species: Particles and anti-particles, and spin aligned and anti-aligned. Taken together we consider a system where all electrons and positrons are spin aligned or anti-aligned with the magnetic field B and the partition function of the system is written as

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \left[\ln \left(1 + \lambda_s e^{-\beta(E_n^+ - \eta_e)}\right) + \ln \left(1 + \lambda_s e^{-\beta(E_n^+ + \eta_e)}\right) \right.$$
$$\left. + \ln \left(1 + \lambda_s^{-1} e^{-\beta(E_n^- - \eta_e)}\right) + \ln \left(1 + \lambda_s^{-1} e^{-\beta(E_n^- + \eta_e)}\right) \right], \tag{24}$$

where η_e is the electron chemical potential, $\lambda_s = e^{\eta_s/T}$ is the spin chemical potentialand energy $E_n^{\pm}$ can be written as

$$E_n^{\pm} = \sqrt{p_z^2 + \tilde{m}_{\pm}^2 + 2eBn}, \qquad \tilde{m}_{\pm}^2 = m_e^2 + eB\left(1 \mp \frac{g}{2}\right), \tag{25}$$

where the $\pm$ script refers to spin aligned and anti-aligned eigenvalues. To simplify the partition function we consider the case $\eta_s/T = 0$ for the first approximation and use the expansion of the logarithmic function into a power series yielding

$$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad \text{for } |x| < 1. \tag{26}$$

We emphasize that this expansion is only value where the Boltzmann factor $\exp(-\beta(E \pm \eta_e))$ remains less than unity, otherwise the partition function becomes divergent and ill-behaved. The partition function of $e^{\pm}$ system can be further expanded to isolate the contribution of the chemical potential η_e as

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[e^{k\beta\eta_e} + e^{-k\beta\eta_e} \right] \left(\int_0^{\infty} dp_z \, e^{-k\beta E_n^+} + \int_0^{\infty} dp_z \, e^{-k\beta E_n^-} \right)$$
$$= \frac{2eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2\cosh(k\beta\eta_e) \right] \sum_{n=0}^{\infty} \left(W_1^+(n,k) + W_1^-(n,k) \right), \tag{27}$$

where we introduce the function $W_1^{\pm}(n,k)$ as follows

$$W_1^{\pm}(n,k) \equiv \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T} K_1\left(\frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\right), \tag{28}$$

defined in terms of the modified (hyperbolic) Bessel function of the second kind $K_i(x)$. We then utilize the Euler-Maclaurin formula to replace the sum over Landau levels with an integration yielding

$$\sum_{n=0}^{\infty} W_1^{\pm}(n,k) = \int_0^{\infty} dn \, W_1^{\pm}(n,k) + \frac{1}{2}\left[W_1^{\pm}(\infty,k) + W_1^{\pm}(0,k) \right] + \frac{1}{12}\left[\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{\infty} - \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_0 \right] + R, \tag{29}$$

where R is the error remainder which is defined by integrals over Bernoulli polynomials. The error remainder has, for well behaved functioned, a finite size and therefore for a more extensive analysis is required to demonstrate to what order in Bessel functions is required to minimize the error remainder for good results. For Eq. (29), we've chosen to include up to the first order in derivatives. Using the asymptotic properties of Bessel function $K_i(x)$ as x becomes large, we can simplify the above expression to obtain

$$\sum_{n=0}^{\infty} W_1^{\pm}(n,k) = \int_0^{\infty} dn \, W_1^{\pm}(n,k) + \frac{1}{2} W_1^{\pm}(0,k) - \frac{1}{12} \left. \frac{\partial W_1^{\pm}}{\partial n} \right|_0 + R \tag{30}$$

$$= \left(\frac{T^2}{k^2 eB} \right) x_{\pm}^2 K_2(x_{\pm}) + \frac{1}{2} x_{\pm} K_1(x_{\pm}) + \frac{1}{12} \left(k^2 b_0 \right) K_0(x_{\pm}) + R, \qquad x_{\pm} = \frac{k \tilde{m}_{\pm}}{T}. \tag{31}$$

Here we see mathematically that the ground state $n = 0$, which has no analogue in the classical continuum limit, contributes to a difference between the sum and integral formulations. Replacing the sum over Landau levels by the integral, the partition function can separated into three distinct parts

$$\ln \mathcal{Z}_{tot} = \ln \mathcal{Z}_{free} + \ln \mathcal{Z}_B + \ln \mathcal{Z}_R \tag{32}$$

where we define

$$\ln \mathcal{Z}_{free} = \frac{T^3 V}{2\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \left[2 \cosh \left(\frac{k \eta_e}{T} \right) \right] \left[x_+^2 K_2 \left(x_+ \right) + x_-^2 K_2 \left(x_- \right) \right], \tag{33}$$

$$\ln \mathcal{Z}_B = \frac{eBTV}{2\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2 \cosh \left(\frac{k \eta_e}{T} \right) \right] \sum_{i=\pm} \left[\frac{x_i}{2} K_1 \left(x_i \right) + \frac{k^2 b_0}{12} K_0 \left(x_i \right) \right], \tag{34}$$

$$\ln \mathcal{Z}_R = \frac{eBTV}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2 \cosh \left(\frac{k \eta_e}{T} \right) \right] R. \tag{35}$$

While this would require further derivation to demonstrate explicitly, the benefit of the Euler-Maclaurin approach is if the error contribution remains finite or bound for the magnetized partition function, then a correspondence between the free Fermi partition function (with noticeably modified magnetic-mass $\tilde{m}_{\pm}$) and the magnetized Fermi partition function can be established. The mismatch between the summation and integral in the Euler-Maclaurin formula would then encapsulate the immediate magnetic response and deviation from the free particle phase space.

4.1. Non-relativistic limit of the partition function

While we label $\ln(\mathcal{Z}_{free})$ in Eq. (33) as the "free" partition function, this is not strictly true as this contribution to the overall partition function is a function of the magnetic-mass we defined earlier in Eq. (5). When determining the magnetization of the quantum Fermi gas, derivatives of the magnetic field B will not fully vanish on this first term which will resulting in an intrinsic magnetization which is distinct from the contribution from the ground state and mismatch between the quantized Landau levels and the continuum of the free momentum. Specifically, this free Fermi contribution represents the magnetization that arises from the spin magnetic energy rather than orbital contributions. To demonstrate this, we will briefly consider the weak field limit for $g = 2$. The magnetic-mass, for $g = 2$, have the following values

$$\tilde{m}_+^2 = m_e^2, \tag{36}$$
$$\tilde{m}_-^2 = m_e^2 + 2eB. \tag{37}$$

The energy eigenvalues are then

$$E_n^+ = \sqrt{p_z^2 + m_e^2 + 2eBn}, \tag{38}$$

$$E_n^- = \sqrt{\left(E_n^+\right)^2 + 2eB}, \tag{39}$$

The spin anti-aligned energies specifically in the non-relativistic limit reduce to

$$E_n^-|_{NR} \approx E_n^+|_{NR} + \frac{eB}{m_e}. \tag{40}$$

This shift in energies is otherwise not influenced by summation over principle quantum number n or summation parameter k, therefore we can interpret this energy shift as a shift in the chemical potential of the species. Introducing a non-relativistic spin-dependant chemical potential

$$\eta_e^\pm = \eta_e \pm \frac{eB}{2m_e}, \tag{41}$$

we can rewrite the total partition function in Eq. (27) as

$$\ln \mathcal{Z}_{tot}|_{NR} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^\infty \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} \left[2\cosh(k\beta\eta_e^+) + 2\cosh(k\beta\eta_e^-)\right] \int_0^\infty dp_z e^{-k\beta\epsilon_n}, \tag{42}$$

$$\epsilon_n = m_e + \frac{p_z^2}{2m_e} + \frac{eB}{2m_e}(n+1). \tag{43}$$

85 Eq. (42) is then the traditional NR quantum harmonic oscillator partition function with a spin
86 dependant chemical potential shift differentiating the aligned and anti-aligned states. We note that
87 in this formulation, the spin contribution is entirely excised from the orbital contribution to the
88 partition function merely acting as a constant coefficient. Under Euler-Maclaurin integration, the now
89 spin-independant Boltzmann factor can be further separated into "free" and Landau quantum parts as
90 was done in Eq. (33) - Eq. (35) for the relativistic case. The non-relativistic "free" partition function
91 is then identical to the free fermion partition function with a simple modification from the intrinsic
92 spin energies. We note however that the inclusion of anomalous magnetic moment (AMM) spoils this
93 clean separation therefore, we take the above as only useful as a limit or approximation of the physical
94 system. ANDREW: I would like to demonstrate this above paragraph mathematically, though this
95 might just be considered a corollary point more appropriate for a subsequent paper.

Another approach to the non-relativistic electron-positron plasma is to consider the partition function obtained in Eq. (33) - Eq. (35) in the Boltzmann approximation $k = 1$ and assuming the error remainder R is small and can be neglected. Here we will not limit ourselves to $g = 2$ and allow the g-factor to be arbitrary in value. In this case, we have

$$\begin{aligned} \ln \mathcal{Z}_{tot} = {} &\frac{T^3 V}{(2\pi)^2} \left[2\cosh\left(\frac{\eta_e}{T}\right)\right] \left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2\left(\frac{\tilde{m}_\pm}{T}\right) \\ &+ \frac{2eBTV}{(2\pi)^2} \left[2\cosh\left(\frac{\eta_e}{T}\right)\right] \left[\frac{1}{2}\left(\frac{\tilde{m}_\pm}{T}\right) K_1\left(\frac{\tilde{m}_\pm}{T}\right) + \frac{1}{12}\left(\frac{eB}{T^2}\right) K_0\left(\frac{\tilde{m}_\pm}{T}\right)\right]. \end{aligned} \tag{44}$$

ANDREW: I don't agree with the above and below equations because this represents the total partition function which means it needs to be a sum of both polarizations. This inflicts a lot of your further analysis where you use charge neutrality because you're using quantities which functions of $x_\pm$ which is ambiguous as these are different. Rather we need to write sums. I see you already starting writing

the explicit sums in the beginning of the Partition Function section, so that correction needs to continue throughout the paper. The above equation Eq. (44) can be simplified into the expression

$$\ln \mathcal{Z}_{tot} = \frac{T^3 V}{(2\pi)^2} \left[2\cosh\left(\frac{\eta_e}{T}\right) \right] \left\{ \left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2\left(\frac{\tilde{m}_\pm}{T}\right) + \frac{eB}{T^2} \left[\left(\frac{\tilde{m}_\pm}{T}\right) K_1\left(\frac{\tilde{m}_\pm}{T}\right) + \frac{1}{6}\left(\frac{eB}{T^2}\right) K_0\left(\frac{\tilde{m}_\pm}{T}\right) \right] \right\}.$$

$$(45)$$

96 ANDREW: We should have more to comment here. This is essentially the "working equation" for the
97 remainder of the paper. So this is a nice key result we should emphasize.

98 **5. Chemical potential and Magnetization**

99 *5.1. Charge neutrality and chemical potential*

Given the partition function in Boltzmann limit Eq.(45) the net number density of electron can be written as

$$(n_e - n_{\bar{e}}) = \frac{T}{V}\frac{\partial}{\partial \eta_e} \ln \mathcal{Z}_{tot}$$

$$= \frac{2T^3}{(2\pi)^2}\sinh\left(\eta_e/T\right)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2\left(\frac{\tilde{m}_\pm}{T}\right)\right] + \frac{4eBT}{(2\pi)^2}\sinh\left(\eta_e/T\right)\left[\frac{\tilde{m}_\pm}{2T}K_1\left(\frac{\tilde{m}_\pm}{T}\right) + \frac{eB}{12T^2}K_0\left(\frac{\tilde{m}_\pm}{T}\right)\right]$$

$$(46)$$

Using the charge neutrality, we have

$$n_p = (n_e - n_{\bar{e}})$$

$$= \frac{2T^3}{(2\pi)^2}\sinh\left(\eta_e/T\right)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2\left(\frac{\tilde{m}_\pm}{T}\right)\right] + \frac{4eBT}{(2\pi)^2}\sinh\left(\eta_e/T\right)\left[\frac{\tilde{m}_\pm}{2T}K_1\left(\frac{\tilde{m}_\pm}{T}\right) + \frac{eB}{12T^2}K_0\left(\frac{\tilde{m}_\pm}{T}\right)\right]$$

$$(47)$$

where n_p is the number density of proton. It is also convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad b_0 = \frac{eB}{T^2}$$

$$(48)$$

and we obtain

$$n_p = \frac{T^3}{(2\pi)^2}\left[2\sinh\left(\eta_e/T\right)\right]\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right]$$

$$(49)$$

and the chemical potential of electron and positron can be written as

$$\sinh\left(\eta_e/T\right) = \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right]}$$

$$(50)$$

$$\longrightarrow \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{x_\pm^2 K_2(x_\pm)}, \qquad \text{for } b_0 = 0$$

$$(51)$$

100 It shows that for the case $b_0 = 0$ the chemical potential agrees with our earlier results.

Considering the homogeneous proton number density in the early universe, it can be written as

$$n_p = \frac{n_p}{n_B}\left(\frac{n_B}{s_{\gamma,\nu,e}}\right) s_{\gamma,\nu,e} = X_p\left(\frac{n_B}{s_{\gamma,\nu}}\right) s_{\gamma,\nu}, \qquad X_p = \frac{n_p}{n_B}$$

$$(52)$$

101 where n_B is the number density of baryon, and the second equality is obtained by considering $e^{\pm}$
102 entropy density is negligible compared to the photon and neutrino entropy density at post-BBN
103 temperature $20 < T < 50$keV.

- The proton fraction X_p can be written as follow:

$$X_p = \frac{n_p}{n_B} = \frac{n_p}{n_p + n_n} = \frac{1}{1 + n_n/n_p} \tag{53}$$

Since all neutrons end up bound in to the ^4He after BBN, then the mass fraction of ^4He can be estimated as

$$X_\alpha = \frac{2(n_n/n_p)}{1 + n_n/n_p} = 0.245 \pm 0.03 \tag{54}$$

where we use $X_\alpha = 0.245 \pm 0.03$ from particle data group [9]. Solving the ratio n_n/n_p and substituting into the X_p, we obtain

$$X_p = 0.878 \pm 0.015 \tag{55}$$

- Since the comoving baryon number and entropy are conserved, hence the ratio $s_{\gamma,\nu}/n_B$ is conserved, then the entropy per baryon ratio can be written as

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right) = \left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \left(\frac{n_\gamma}{n_B}\right)_{t_0} \left(\frac{s_\gamma}{n_\gamma} + \frac{n_\nu}{n_\gamma}\frac{s_\nu}{n_\nu}\right) = \left(\frac{n_\gamma}{n_B}\right)_{t_0} \left[3.601 + \frac{9}{4}\left(\frac{T_\nu}{T_\gamma}\right)^3 4.202\right], \tag{56}$$

where the subscript t_0 denotes the present-day value and considers all neutrinos are relativistic particles today. The entropy per particle for a boson is $(s/n)_{\text{boson}} = 3.601$ and for a fermion is $(s/n)_{\text{fermion}} = 4.202$. From particle data group and standard big bang model [9,10], we have

$$5.8 \times 10^{-10} < \frac{n_B}{n_\gamma} < 6.5 \times 10^{-10}, \quad \frac{n_B}{n_\gamma} = 6.14 \times 10^{10}(\text{BBN observation}), \quad \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}. \tag{57}$$

In this case, the entropy per baryon ratio becomes

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \frac{1}{6.14 \times 10^{-10}}\left[3.601 + \frac{9}{4}\left(\frac{4}{11}\right) 4.202\right] = 1.14 \times 10^{10} \tag{58}$$

- On the other hand, the entropy density at temperature can be written as [10]

$$s = \frac{2\pi^2}{45} g_s T_\gamma^3, \quad g_s = \sum_{i=boson} g_i \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8} \sum_{i=fermion} g_i \left(\frac{T_i}{T_\gamma}\right)^3 \tag{59}$$

where g_s is the effective degree of freedom that contribute to the entropy density. In the temperature we are interested in $50 > T > 20$keV, the entropy density reads

$$s_{\gamma,\nu} = \frac{2\pi^2}{45} g_s T_\gamma^3, \quad g_s = g_\gamma + \frac{7}{8} g_\nu \left(\frac{T_\nu}{T_\gamma}\right)^3 = 2 + \frac{7}{8} 2 \times 3 \left(\frac{4}{11}\right) = 3.91 \tag{60}$$

Substituting above numerical results into Eq.(52), the proton number density is given by

$$n_p = X_p \left(\frac{n_B}{s_{\gamma,\nu}}\right) s_{\gamma,\nu} = \frac{0.878 \times 3.91}{1.14 \times 10^{10}} \left(\frac{2\pi^2}{45}\right) T_\gamma^3 = 3.011 \times 10^{-10} \left(\frac{2\pi^2}{45}\right) T_\gamma^3 \tag{61}$$

In this case, given the magnetic field B we can solve the chemical potential η_e as a function of temperature numerically. In Fig. 4, we plot the chemical potential as a function of temperature T. It
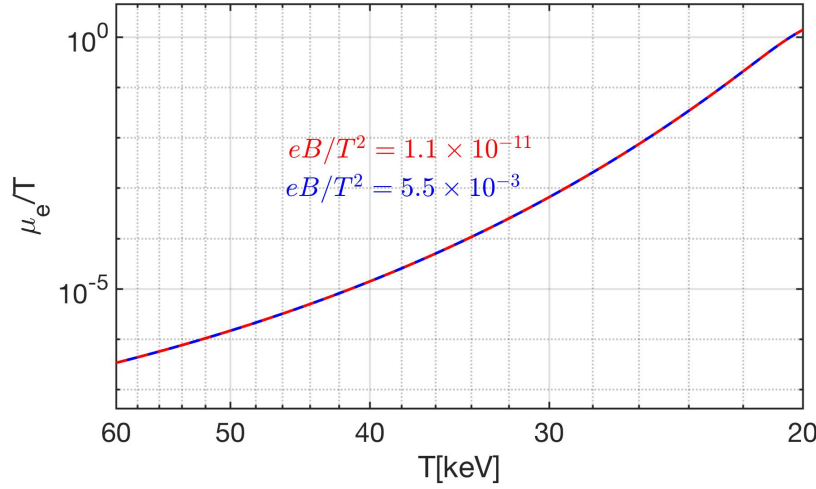

Figure 4. The chemical potential η_e/T a function of temperature $20 < T < 60$keV. It shows that the chemical potential is not sensitive to the magnetic field b_0.

shows that the chemical potential is not sensitive to the magnetic field because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq. (50).

5.2. Magnetization

On the other hand, giving the partition function Eq.(45) the magnetization can be obtained via the definition

$$M = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_{tot}}{\partial B} = \frac{T}{V}\left(\frac{\partial \tilde{m}_\pm}{\partial B}\right)\frac{\partial \ln \mathcal{Z}_{tot}}{\partial \tilde{m}_\pm} \tag{62}$$

Using the the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad b_0 = \frac{eB}{T^2} \tag{63}$$

then the magnetization can be written as

$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 b_0}\left[2\cosh\left(\frac{\eta_e}{T}\right)\right]\left\{\left[1 - \frac{(1\pm g/2)}{2}\left(1 + \frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1\pm g/2)}{2}\right]b_0 K_0(x_\pm)\right\}. \tag{64}$$

Substituting the chemical potential Eq.(50) into Eq.(64) we obtain

$$\frac{M}{B} = \frac{8\pi\alpha}{(2\pi)^2 b_0}\left(\left[1 - \frac{(1\pm g/2)}{2}\left(1 + \frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1\pm g/2)}{2}\right]b_0 K_0(x_\pm)\right)$$
$$\times \sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)}\right]^2} \tag{65}$$

In this case, given the magnetic field B and chemical potential we can solve the magnetization M as a function of temperature numerically.
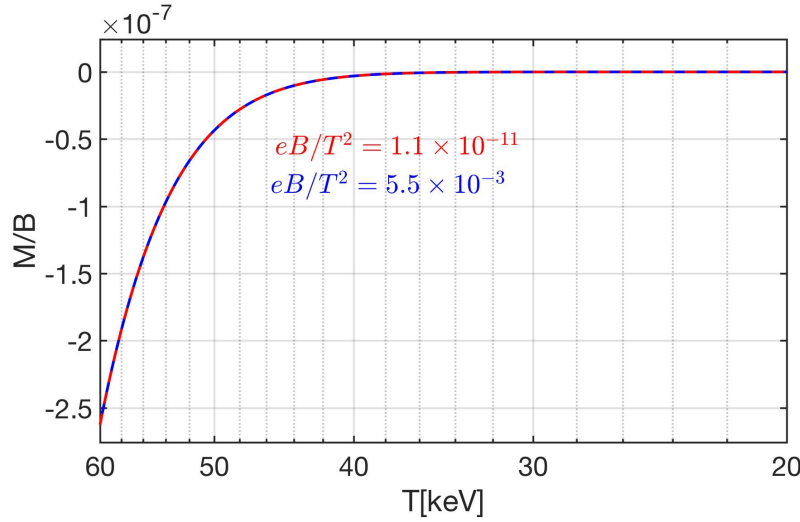
Figure 5. The chemical potential η_e/T (upper) and magnetization M/B (lower) as a function of temperature $20 < T < 60$keV for the case 1. The chemical potential and magnetization are not sensitive to the magnetic field, because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ can be neglected in Eq. (**??**) and Eq. (67).

In our calculation we introduce an effective mass term which incorporates the ground state component of the Landau energy as follow

$$x_{\pm} = \frac{\tilde{m}_{\pm}}{T} = \frac{1}{T}\sqrt{m_e^2 + eB\left(1 \pm \frac{g}{2}\right)} \tag{66}$$

Considering the case $g = 2$ we have following two cases:

- Case1: $\tilde{m}_+ = \sqrt{m_e^2 + 2eB}$, and $x = \tilde{m}_+/T$. The magnetization is given by

$$\left(\frac{M}{B}\right) = -\frac{8\pi\alpha}{(2\pi)^2}\left(\frac{b_0}{6x}K_1(x_{\pm}) + \frac{2}{3}K_0(x_{\pm})\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_{\pm}^2 K_2(x_{\pm}) + b_0 x_{\pm} K_1(x_{\pm}) + \frac{b_0^2}{6}K_0(x_{\pm})\right]}\right]^2} \tag{67}$$

Substituting the magnetic field b_0 and proton density n_p/T^3 we can solve the magnetization and chemical potential numerically. In Fig. 5, we plot the magnetization M/B as a function of temperature T. It shows for the case1 the magnetization is not sensitive to the magnetic field, this is because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq. (**??**) and Eq. (67).

- Case 2: $\tilde{m}_- = m_e$ and $x = \tilde{m}_-/T$, then the magnetization of electron/ positron becomes

$$\left(\frac{M}{B}\right) = \frac{8\pi\alpha}{(2\pi)^2}\left(\frac{1}{b_0}xK_1(x) + \frac{1}{3}K_0(x)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6}K_0(x)\right]}\right]^2} \tag{68}$$

Using the magnetic field b_0 and proton density n_p/T^3 we solve the magnetization for case 2 numerically. In Fig 6, we plot the magnetization M/B as a function of temperature T. It shows that the magnetization depends on the magnetic field b_0 strongly. This is because for a small magnetic field b_0 the dominant term in Eq(68) is $xK_1(x)/b_0$. For given b_0, the value of

magnetization can be larger than the magnetic field, i.e. $M/B > 1$ which shows the possibility that magnetic domains can be formed in the early universe.
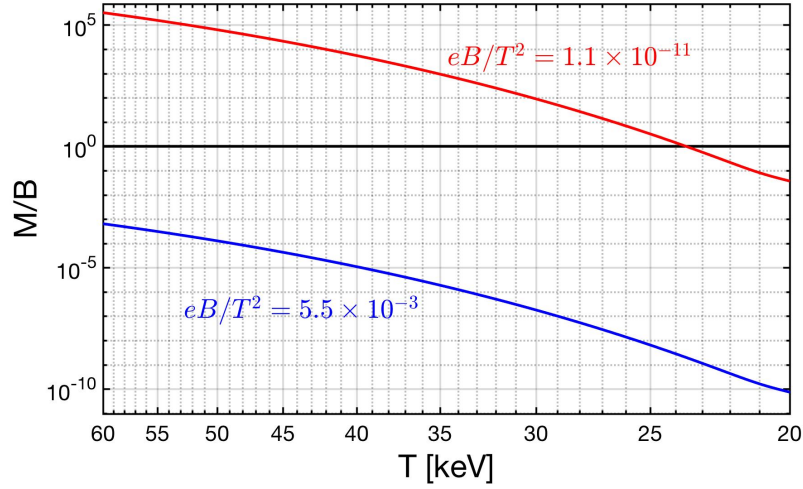

Figure 6. The chemical potential η_e/T(upper) and magnetization M/B(lower) as a function of temperature $20 < T < 60$keV for the case 2. It shows that for giving b_0 we can find the temperature that $M/B > 1$ in early universe.

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