Equation of state for Electron Positron plasma

Cheng Tao Yang[a], Andrew Steinmetz[a], Johann Rafelski[a]
[a]*Department of Physics, The University of Arizona, Tucson, Arizona 85721, USA*
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I. PARTITION FUNCTION

Considering the partition function of $e^{\pm}$ plasma in Boltzmann approximation, we have

$$\ln \mathcal{Z}_{tot} =\frac{g_e}{2\pi^2}T^3V\left[2\cosh\left(\mu_e(T)/T\right)\right]\left(\frac{m_e}{T}\right)^2K_2(m_e/T). \tag{1}$$

where $g_e=2$ is the spin degeneracy, K_2 is the 2nd Bessel function and $\mu_e(T)$ is the chemical potential of $e^{\pm}$ which is also a function of temperature.

- Chemical potential:
  Giving the partition function, we canuse charge neutrality to calculate the chemical potential as follow:

$$(n_e-n_{\bar{e}})=\frac{T}{V}\frac{\partial}{\partial\mu_e}\ln\mathcal{Z}_{tot}=n_p \tag{2}$$

  where n_p is the number density of proton. and the net number density of electron can be written as

$$(n_e-n_{\bar{e}})=\frac{g_eT^3}{2\pi^2}\left[2\sinh\left(\mu_e/T\right)\right]\left(\frac{m_e}{T}\right)^2K_2(m_e/T). \tag{3}$$

  Then the charge neutrality becomes:

$$n_p=\frac{g_eT^3}{2\pi^2}\left[2\sinh\left(\mu_e/T\right)\right]\left(\frac{m_e}{T}\right)^2K_2(m_e/T). \tag{4}$$

  In Fig.(1) we plot the chemical potential μ_e/T and derivative of chemical potential $d\mu_e/dT$ of electron as a function of temperature $200>T>20$ keV.

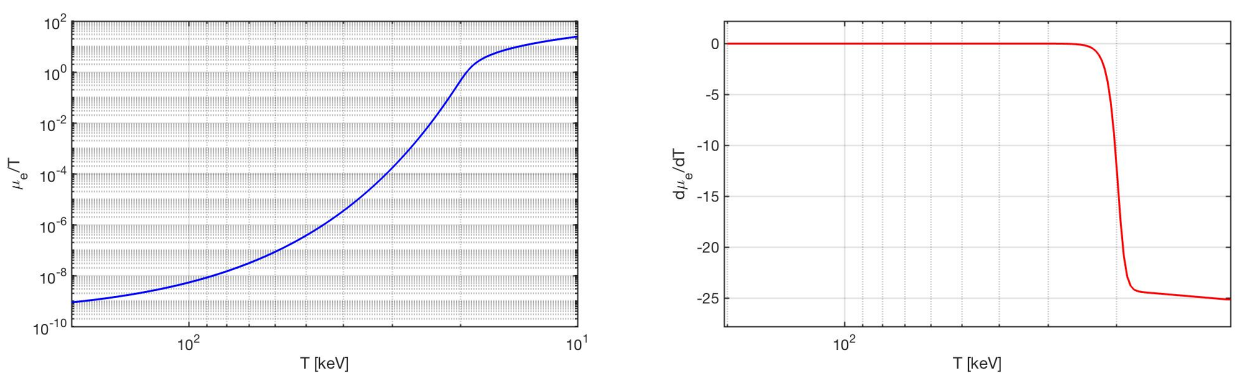

FIG. 1: On left: chemical potential μ_e/T of electron as a function of temperature $200>T>20$ keV. On right: the derivative of chemical potential with respect to temperature $d\mu_e/dT$ as a function of temperature.

- Pressure:
  By definition, the pressure of electron-positron is given by

$$P=\frac{1}{\beta}\frac{\partial\ln\mathcal{Z}_{tot}}{\partial V}=\frac{g_e}{2\pi^2}T^4\left[2\cosh\left(\mu_e/T\right)\right]\left(\frac{m_e}{T}\right)^2K_2(m_e/T), \tag{5}$$

  where $\beta=1/T$. In Fig.(3) we plot the pressure of electron-positron as a function of temperature(blue solid line) $200>T>20$ keV. For comparison we also show the pressure for the case $\mu_e=0$ (blue dashed line).

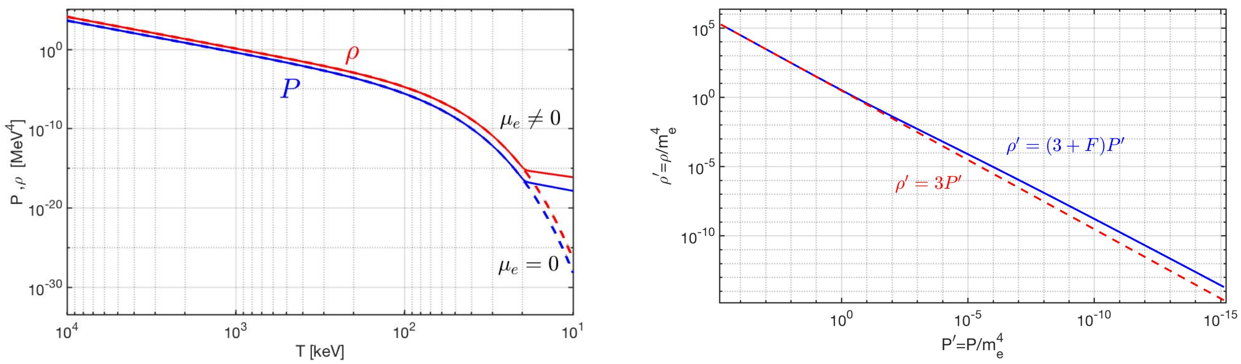

FIG. 2: On left: the energy density (red solid line) and pressure (blue solid line) as a function of temperature $10^4 > T > 10$ keV. For comparison, the dashed lines represent the case $\mu_e = 0$. On right: we plot the dimensionless variables $rho' = \rho/m_e^4$ versus $P' = P/m_e^4$ (blue solid line) and red dashed line represent the relativistic limit.

- Energy Density:
  Giving the partition function, energy density can be obtained via the following definition:

$$\rho = \frac{E}{V} = \frac{1}{V}\left(-\frac{\partial \ln \mathcal{Z}_{tot}}{\partial \beta}\right) = \frac{-1}{V}\frac{\partial}{\partial \beta}\left[\frac{g_e}{2\pi^2} V\left(\lambda + \lambda^{-1}\right)\frac{(\beta m_e)^2}{\beta^3} K_2(\beta m_e)\right] \tag{6}$$

It can be written as

$$\rho = -\frac{g_e}{2\pi^2}\left(\lambda + \lambda^{-1}\right)\frac{\partial}{\partial \beta}\left[\frac{(\beta m_e)^2}{\beta^3} K_2(\beta m_e)\right] \tag{7}$$

$$= \frac{g_e}{2\pi^2}\left(\lambda + \lambda^{-1}\right)\left(\frac{m_e}{\beta}\right)^2\left[(\beta m_e) K_1(\beta m_e) + 3K_2(\beta m_e)\right] \tag{8}$$

$$= \frac{g_e}{2\pi^2} T^4 \left[2\cosh\left(\mu_e/T\right)\right]\left(\frac{m_e}{T}\right)^2 K_2(m_e/T)\left[3 + \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)}\right] \tag{9}$$

In Fig.(3) we plot the energy density of electron-positron as a function of temperature(red solid line) $10^4 > T >$ 10 keV, we also plot the energy density for the case $\mu_e = 0$ (red dashed line).

II. TOLMAN-OPPENHEIMER-VOLKOFF (TOV) EQUATIONS

Combining the Eq.(5) and Eq.(9) we obtain

$$\rho = P\left[3 + \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)}\right] \tag{10}$$

then the equation of state w is given by

$$w = \frac{P}{\rho} = \frac{1}{3 + \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)}} \tag{11}$$

In Fig.(3) on the right, we show the equation of state $w = P/\rho$ as a function of temperature(red solid line) and for the blue dotted line represent the $w = 1/3$ for relativistic limit. On the other hand, the Tolman-Oppenheimer-Volkoff (TOV) equations, we have

$$\frac{dP}{dr} = -\frac{G_N M}{r^2}\rho\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi r^2}{M}P\right)\left(1 - \frac{2GM}{r}\right)^{-1} \tag{12}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \tag{13}$$

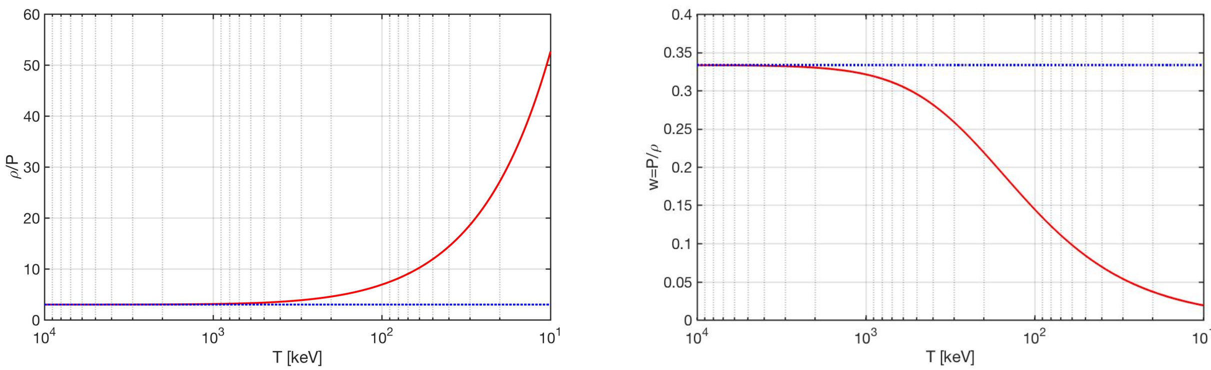

FIG. 3: On left: the ratio ρ/P as a function of temperature $10^4 > T > 10$ keV; On right: the equation of state $w = P/\rho$ as a function of temperature $10^4 > T > 10$ keV, where the blue dotted line represent the $w = 1/3$ for relativistic limit.

where G_N is the Newtonian constant of gravitation. It is convenient to introduce the dimensionless variables as :

$$\rho' = \rho/m_e^4, \qquad P' = P/m_e^4, M' = M\frac{m_e^2}{M_p^3}, \qquad r' = r\frac{m_e^2}{M_p} \tag{14}$$

where $M_P^2 = G_N$ is the Planck mass and m_e is the electron mass. Then the Eq.(10) can be written as

$$\rho' = P' \left[3 + F\right], \qquad F = \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)} \tag{15}$$

We can approximate the function $F(m_e/T)$ as a function of P' as follow:

$$F(P') = a_1 \exp\left[-\left(\frac{\ln P' - b_1}{c_1}\right)^2\right] + a_2 \exp\left[-\left(\frac{\ln P' - b_2}{c_2}\right)^2\right] \tag{16}$$

where the parameters are given by

$$a_1 = 41.72, \qquad b_1 = -40.38, \qquad c_1 = 2.987; \tag{17}$$
$$a_2 = 22.01, \qquad b_2 = -35.94, \qquad c_2 = 18.94. \tag{18}$$

In Fig.(4) we plot the function $F(m_e/T)$ versus P' (blue solid line) and fit the value of F as a function of P' (red dotted line). For the approximation, it shows that function $F(P')$ captures the general behavior of $F(m_e/T)$.

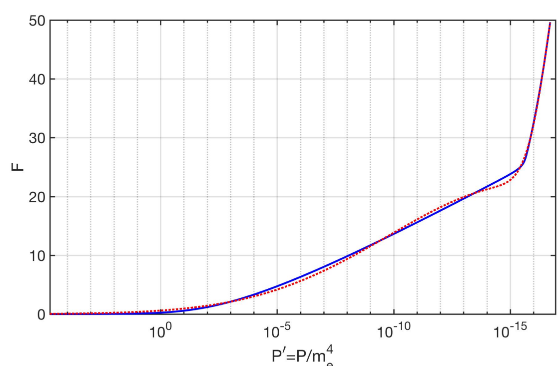

FIG. 4: we plot the function F versus P' (blue solid line) and fit the value of F as a function of P' (red dotted line).

In this case, using the dimensionless variables and function $F(P')$ the TOV equations become

$$\frac{dP'}{dr'} = -\frac{M'}{r'^2}P'\left[4+F(P')\right]\left(1+\frac{4\pi r'^2}{M'}P'\right)\left(1-\frac{2M'}{r'}\right)^{-1} \tag{19}$$

$$\frac{dM'}{dr'} = 4\pi r'^2 P'\left[3+F(P')\right] \tag{20}$$

Using the equation of state to relate P and ρ, the only unknown functions of TOV equations are $P(r)$ and $M(r)$. Given the initial conditions for $P'(r=0)$ and $M'(r=0)$ we can solve Eq.(19) and Eq.(20). Form Eq.(20) we have the initial condition

$$M'(r=0) = 0 \tag{21}$$

However we need to give the initial condition for the central pressure $P'(r=0)$ by hand(need reference), then we can solve the TOV equations from the center of the star $r=0$ to the surface R where the pressure becomes zero $P'(R)=0$ and determine the size of star.

- Example: Considering the initial conditions as follow:

$$M'(r=0) = 0, \qquad , P'(r=0) = 10^{-3},\, 5\times 10^{-3},\, 10^{-2} \tag{22}$$

  here I just try different initial conditions to show that the results depend on the given conditions. Then we can solve the TOV equations. In Fig.(5) we plot the dimensionless mass of star M' as a function of radius r'(on left) and the dimensionless pressure as a function of radius r'(on right). We can determine the radius of stat R'_{star} by the condition $P'(R'_{star})=0$, in Table.(I) we show the radius and total mass of star with different initial conditions.

TABLE I: The radius and total mass of star.

$P'(r=0)$	R'_{star}	M'_{tot}
10^{-3}	2.95×10^2	3.90×10
5×10^{-3}	2.84×10^2	3.78×10
10^{-2}	2.76×10^2	3.64×10

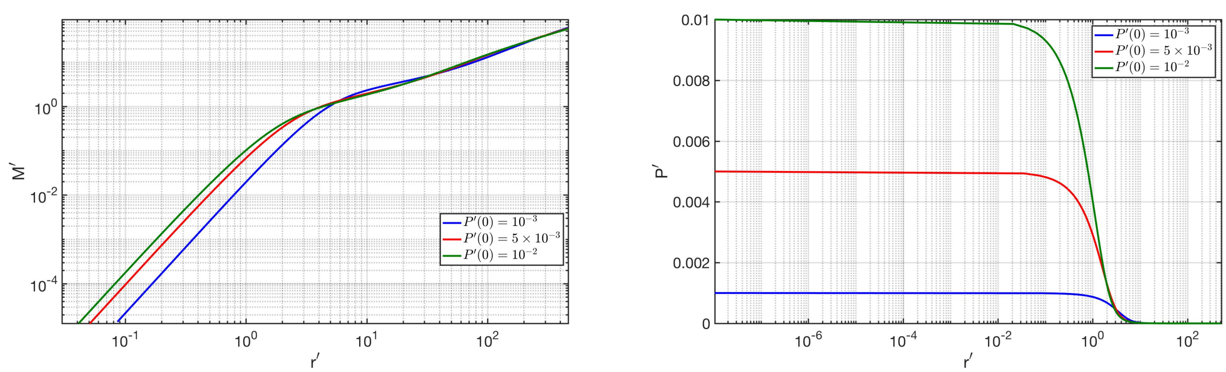

FIG. 5: We solve the TOV equations with the initial condition $M'(r=0)=0, P'(r=0)=10^{-3},\, 5\times 10^{-3},\, 10^{-2}$. On left: the dimensionless mass of star M' as a function of radius r'; On right: the dimensionless pressure as a function of radius r'.