Electron Positron Partition Function in Early Universe

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I. PARTITION FUNCTION

Considering the $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, the energy of $e^{\pm}$ can be written as

$$E_{n,s} = \sqrt{p_z^2 + \tilde{m}^2 + 2eBn}, \qquad \tilde{m}^2 = m_e^2 + eB\left(1 - gs\right), \qquad s = \pm\frac{1}{2}, \qquad n = 0, 1, 2, 3, \ldots \tag{1}$$

$$\tag{2}$$

If we consider a system that all electrons and positrons are spin aligned and antialigned with the magnetic field B, then the partition function of the system can be written as

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \left[\ln\left(1 + e^{-\beta(E_n^{\pm} - \mu_e)}\right) + \ln\left(1 + e^{-\beta(E_n^{\pm} + \mu_e)}\right) \right], \tag{3}$$

where $\beta = 1/T$, μ_e is the chemical potential of electron, and energy $E_n^{\pm}$ can be written as

$$E_n^{\pm} = \sqrt{p_z^2 + \tilde{m}_{\pm}^2 + 2eBn}, \qquad \tilde{m}_{\pm}^2 = m_e^2 + eB\left(1 \pm \frac{g}{2}\right). \tag{4}$$

To simplify the partition function we can consider the expansion of the logarithmic function, we have

$$\ln\left(1 + x\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \qquad \text{for } |x| < 1. \tag{5}$$

Then the partition function of electron/positron system can be written as

$$\begin{aligned} \ln \mathcal{Z}_{tot} =& \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[e^{k\beta\mu_e} + e^{-k\beta\mu_e} \right] e^{-k\beta E_n^{\pm}} \\ =& \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[2\cosh\left(k\beta\mu_e\right) \right] \int_0^{\infty} dp_z\, e^{-k\beta E_n^{\pm}}. \end{aligned} \tag{6}$$

Using the general definition of Bessel function:

$$K_\nu(\beta m) = \frac{\sqrt{\pi}}{\Gamma(\nu - 1/2)} \frac{1}{m} \left(\frac{\beta}{2m}\right)^{\nu - 1} \int_0^{\infty} dp\, p^{2\nu - 2} e^{-2\beta E} \qquad \text{for } \nu > 1/2. \tag{7}$$

the integral over dp_z can be written as

$$\int_0^{\infty} dp_z\, e^{-k\beta E_n^{\pm}} = \frac{\Gamma(1/2)}{\sqrt{\pi}} \sqrt{\tilde{m}_{\pm}^2 + 2eBn}\; K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right) \tag{8}$$

$$= \sqrt{\tilde{m}_{\pm}^2 + 2eBn}\; K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right). \tag{9}$$

In this case, the partition function becomes

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[2\cosh\left(k\beta\mu_e\right) \right] \sqrt{\tilde{m}_{\pm}^2 + 2eBn}\; K_1(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T) \tag{10}$$

$$= \frac{2eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2\cosh\left(k\beta\mu_e\right) \right] \sum_{n=0}^{\infty} W_1^{\pm}(n), \tag{11}$$

where we introduce the function $W_1^{\pm}(n)$ as follows

$$W_1^{\pm}(n) \equiv \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\, K_1\Big(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\Big)\,. \tag{12}$$

Considering the Euler-Maclaurin formula to replace the sum over Landau levels, we have

$$\sum_{n=0}^{\infty} W_1^{\pm}(n) = \int_0^{\infty} dn\, W_1^{\pm}(n) + \frac{1}{2}\Big[W_1^{\pm}(\infty) + W_1^{\pm}(0)\Big] + \frac{1}{12}\Big[\frac{\partial W_1^{\pm}}{\partial n}\Big|_{\infty} - \frac{\partial W_1^{\pm}}{\partial n}\Big|_{0}\Big] + R \tag{13}$$

where R is the error remainder which is defined by integrals over Bernoulli polynomials. Using the properties of Bessel function we have

$$\frac{\partial W_1^{\pm}}{\partial n} = -\frac{k^2 eB}{T^2} K_0\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right), \qquad , W_1^{\pm}(\infty) = 0, \qquad \int_a^{\infty} dx\, x^2 K_1(x) = a^2 K_2(a) \tag{14}$$

then we obtain

$$\sum_{n=0}^{\infty} W_1^{\pm}(n) = \int_0^{\infty} dn\, W_1^{\pm}(n) + \frac{1}{2} W_1^{\pm}(0) - \frac{1}{12}\,\frac{\partial W_1^{\pm}}{\partial n}\Big|_{0} + R \tag{15}$$

$$\begin{aligned}
&= \int_0^{\infty} dn\, \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\, K_1\Big(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\Big) + \frac{1}{2}\left[\frac{k\tilde{m}_{\pm}}{T} K_1(k\tilde{m}_{\pm}/T)\right] + \frac{1}{12}\left[\frac{k^2 eB}{T^2} K_0(k\tilde{m}_{\pm}/T)\right] + R \\
&= \left(\frac{T^2}{k^2 eB}\right)\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] + \frac{1}{2}\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right) K_1(k\tilde{m}_{\pm}/T)\right] + \frac{1}{12}\left[\left(\frac{k^2 eB}{T^2}\right) K_0(k\tilde{m}_{\pm}/T)\right] + R.
\end{aligned} \tag{16}$$

Replacing the sum over Landau levels by the integral, the partition function becomes

$$\ln \mathcal{Z}_{tot} = \ln \mathcal{Z}_{free} + \ln \mathcal{Z}_B + \ln \mathcal{Z}_R \tag{17}$$

where we defined

$$\ln \mathcal{Z}_{free} = \frac{2T^3 V}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cosh(k\mu_e/T) \left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] \tag{18}$$

$$\ln \mathcal{Z}_B = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh(k\mu_e/T) \left[\frac{k\tilde{m}_{\pm}}{2T} K_1(k\tilde{m}_{\pm}/T) + \frac{k^2 eB}{12T^2} K_0(k\tilde{m}_{\pm}/T)\right] \tag{19}$$

$$\ln \mathcal{Z}_R = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh(k\mu_e/T) R \tag{20}$$

Giving the partition function, we can calculate the net number density of electron as follow:

$$\begin{aligned}
(n_e - n_{\bar{e}}) &= \frac{T}{V}\frac{\partial}{\partial \mu_e} \ln \mathcal{Z}_{tot} = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_{free}}{\partial \mu_e} + \frac{T}{V}\frac{\partial \ln \mathcal{Z}_B}{\partial \mu_e} + \frac{T}{V}\frac{\partial \ln \mathcal{Z}_R}{\partial \mu_e} \\
&= (n_e - n_{\bar{e}})_{free} + (n_e - n_{\bar{e}})_B + (n_e - n_{\bar{e}})_R
\end{aligned} \tag{21}$$

we have

$$(n_e - n_{\bar{e}})_{free} = \frac{2T^3}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \sinh(k\mu_e/T) \left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] \tag{22}$$

$$(n_e - n_{\bar{e}})_B = \frac{4eBT}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sinh(k\mu_e/T) \left[\frac{k\tilde{m}_{\pm}}{2T} K_1(k\tilde{m}_{\pm}/T) + s\frac{k^2 eB}{12T^2} K_0(k\tilde{m}_{\pm}/T)\right] \tag{23}$$

$$(n_e - n_{\bar{e}})_R = \frac{4eBT}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sinh(k\mu_e/T) R \tag{24}$$

Considering the Boltzmann approximation and assuming the error remainder R is small and can be neglected, then the net number density of electron can be written as

$$\begin{aligned}(n_e - n_{\bar{e}}) &\approx (n_e - n_{\bar{e}})_{free} + (n_e - n_{\bar{e}})_B \\ &= \frac{2T^3}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\left(\frac{\tilde{m}_\pm}{T} \right)^2 K_2(\tilde{m}_\pm/T) \right] + \frac{4eBT}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\frac{\tilde{m}_\pm}{2T} K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2} K_0(\tilde{m}_\pm/T) \right]\end{aligned} \tag{25}$$

Using the charge neutrality, we have

$$\begin{aligned}n_p &= (n_e - n_{\bar{e}}) \\ &= \frac{2T^3}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\left(\frac{\tilde{m}_\pm}{T} \right)^2 K_2(k\tilde{m}_\pm/T) \right] + \frac{4eBT}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\frac{\tilde{m}_\pm}{2T} K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2} K_0(\tilde{m}_\pm/T) \right]\end{aligned} \tag{26}$$

In this case, given the magnetic field B we can solve the chemical potential μ_e as a function of temperature numerically.