ELECTRON POSITRON PARTITION FUNCTION IN EARLY UNIVERSE

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1 **Abstract:** We will write abstract here

2 **Keywords:** Magnetization; Electron-Positron plasma; Cosmology.

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3 **1. Introduction**

4 The electron-positron epoch of the early universe was home to several significant events which have
5 greatly shaped our contemporary universe including neutrino decoupling, Big Bang Nucleosynthesis
6 (BBN), and the annihilation of most electrons and positrons partially reionizing the universe as well as
7 setting the stage for the eventually recombination period which would generate the cosmic microwave
8 background (CMB). Therefore, correctly describing the dynamics of this e^+e^- plasma is of interest
9 when considering modern mysteries such as the origin of cosmic extra-galactic magnetic fields. While
10 most approaches tackle magnetized plasmas from the perspective of magnetohydrodynamics (MHD),
11 a primarily classical or semi-classical approach, our perspective is to demonstrate that fundamental
12 quantum statistical analysis can lead to further insights on the behavior of magnetized plasmas.

13 The universe is filled with magnetic fields at various scales and strengths both within galaxies
14 and in deep inter-galactic space far and away from matter sources. Extragalactic magnetic fields are
15 not well constrained today, but are required by observation to be non-zero with a magnitude between
16 10^{-20} T $< B_{relic} < 10^{-12}$ T over Mpc coherent length scales. The upper bound is contrained from the
17 characteristics of the CMB while the lower bound is contrained by nonobservation of ultra-energetic
18 photons from blazars. There are generally two considered origins for extragalactic magnetic fields:
19 (a) matter-induced dynamo processes involving Amperian currents and (b) primordial (or relic) seed
20 magnetic fields whose origins may go as far back as the Big Bang itself. It is currently unknown
21 which origin accounts for extragalactic magnetic fields today or if it some combination thereof. Even if
22 magnetic fields in the universe today are primarily driven via amplification through Amperian matter
23 currents, such models still require primordial seeds fields at some point to act as catalyst. We then
24 connect back to the plasmas of the early universe as such primordial fields would be lensed through
25 the various plasmas that existed when the universe was far hotter and denser. Of particular interest
26 to us is the electron-positron plasma which existed in the early universe at temperature $T > 20$ KeV
27 which was the last plasma in the universe where the madium density was many orders of magnitude
28 above ordinary lab atomic density. It it in this dense plasma environment that BBN occured and where
29 similar plasmas can still be found within exotic stars such as magnetars.

30 **2. Energy Eigenvalues**

As a starting point, we consider the energy-eigenvalues of charged fermions within a homogenous
magnetic field. Here, we have several choices: We could assume the typical Dirac energy eigen-values
with gyromagnetic g-factor set to $g = 2$. But as electrons, positrons and most plasma species have
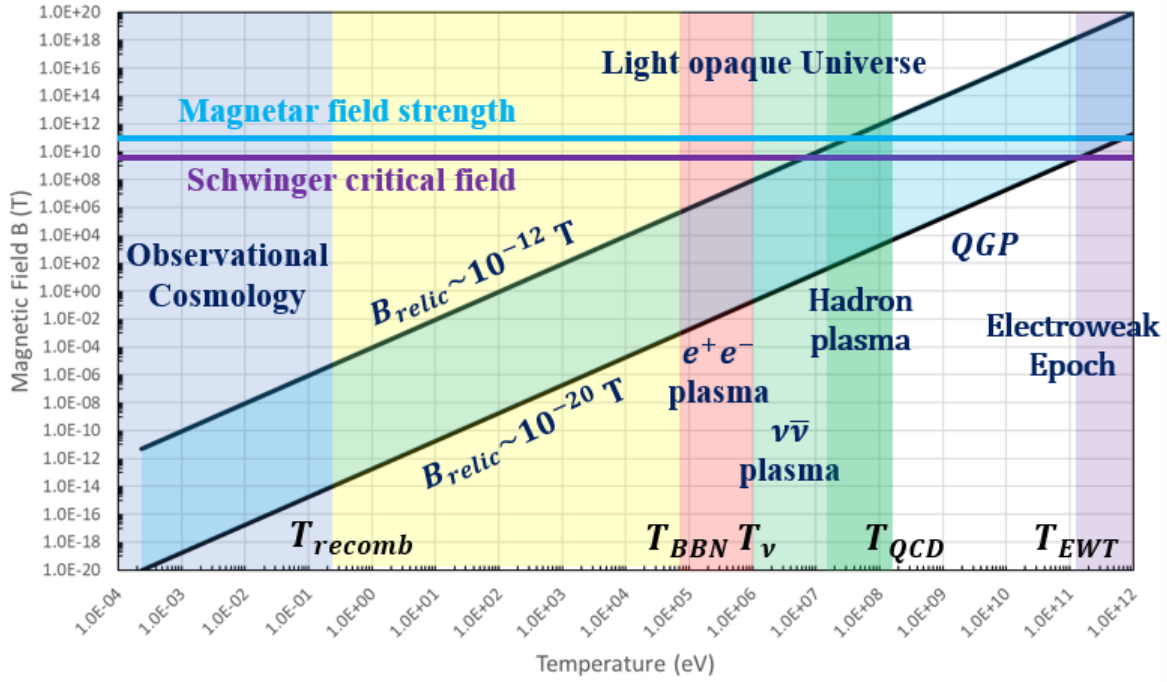
Figure 1. Qualitative value of the primordial magnetic field over the evolutionary lifespan of the universe. As magnetic flux is conserved over comoving surfaces, the primordial relic field is expected to dilute as $\approx 1/a(t)^2$ meaning the contemporary small bound values of 5×10^{-12} T $> B_{relic} > 10^{-20}$ T may have once represented large magnetic fields in the early universe. This is relic magnetic field would then be generated by the last phase of significant magnetization in the early universe. This figure is meant to be illustrative and it is unlikely the magnetization of the universe would proceed unhindered and unaltered into the ultra-dense plasma phases of the early universe. The values of the Schwinger critical field and the upper bound of surface magnetar field strength are included for scale.

anomalous magnetic moments (AMM), we require a more complete model. Another option would be to modify the Dirac equation with a Pauli term, often called the Dirac-Pauli (DP) approach, via

$$\hat{H}_{\text{AMM}} = -a\frac{e}{2m}\frac{\sigma_{\mu\nu}F^{\mu\nu}}{2}, \tag{1}$$

where $\sigma_{\mu\nu}$ is the spin tensor proportional to the commutator of the gamma matrices and $F^{\mu\nu}$ is the EM field tensor. The AMM is defined via g-factor as

$$\frac{g}{2} = 1 + a. \tag{2}$$

This approach, while straightforward, would complicate the energies making analytic understanding and clarity difficult without a clear benefit. Our preferred model for the AMM is through the Klein-Gordon-Pauli (KGP) equation which is given by

$$\left((i\partial_\mu - eA_\mu)^2 - m^2 - e\frac{g}{2}\frac{\sigma_{\mu\nu}F^{\mu\mu}}{2} \right) \Psi = 0. \tag{3}$$

We wish to emphasize, that each of the three above models are distict and have differing physical consequences and are not interchangable. One benefit of the KGP approach is that the energies take

eigenvalues which are mathematically similar to the Dirac energies. Considering the $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, the energy of $e^{\pm}$ fermions can be written as

$$E_n^s = \sqrt{p_z^2 + \tilde{m}^2 + 2eBn}, \qquad \tilde{m}^2 = m_e^2 + eB\left(1 - gs\right), \qquad s = \pm\frac{1}{2}, \qquad n = 0, 1, 2, 3, \dots \qquad (4)$$

Here we introduce a notion of "magnetic mass" which inherents the spin-specific part of the energies adding them to the mass. This convention is also generalizable to further non-minimal electromagnetic models with more exotic energy contributions such that we write

$$m^2 \rightarrow \tilde{m}^2(B). \qquad (5)$$

This definition also pulls out the ground state Landau energy separating it from the remainder of the Landau tower of states. One restriction is that the magnetic-mass must remain positive definite in our analysis thus we require

$$\tilde{m}^2(B) = m_e^2 + eB\left(1 - gs\right) > 0. \qquad (6)$$

This condition fails under ultra-strong magnetic fields of order

$$B_{\text{crit}} = \frac{m^2}{ea} = \frac{\mathcal{B}_S}{a} \approx 3.8 \times 10^{12} \text{ T}, \qquad (7)$$

where $\mathcal{B}_S$ is the Schwinger critical field strength. For electrons, this field strength is well above the window of magnetic field strengths of interest during the late e^+e^- epoch.

3. Partition function

We turn our attention now to the statistical behavior of the e^+e^- system. We can utilize the Fermion partition function given by

$$\ln(\mathcal{Z}) = \sum_{\alpha} \ln\left(1 + e^{-\beta(E-\eta)}\right), \qquad (8)$$

where $\beta = 1/T$, α is the set of quantum numbers in the system, and η is the generalized chemical potential. The magnetized e^+e^- system should be considered a system of four quantum species: Particles and anti-particles, and spign aligned and anti-aligned.

If we consider a system that all electrons and positrons are spin aligned and antialigned with the magnetic field B, then the partition function of the system can be written as

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \left[\ln\left(1 + e^{-\beta(E_n^{\pm} - \mu_e)}\right) + \ln\left(1 + e^{-\beta(E_n^{\pm} + \mu_e)}\right) \right], \qquad (9)$$

where μ_e is the chemical potential of electron, and energy $E_n^{\pm}$ can be written as

$$E_n^{\pm} = \sqrt{p_z^2 + \tilde{m}_{\pm}^2 + 2eBn}, \qquad \tilde{m}_{\pm}^2 = m_e^2 + eB\left(1 \pm \frac{g}{2}\right). \qquad (10)$$

To simplify the partition function we can consider the expansion of the logarithmic function, we have

$$\ln\left(1 + x\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad \text{for } |x| < 1. \qquad (11)$$

Then the partition function of electron/positron system can be written as

$$
\begin{aligned}
\ln \mathcal{Z}_{tot} &= \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[e^{k\beta\mu_e} + e^{-k\beta\mu_e} \right] e^{-k\beta E_n^{\pm}} \\
&= \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[2\cosh\left(k\beta\mu_e\right) \right] \int_0^{\infty} dp_z \, e^{-k\beta E_n^{\pm}} \\
&= \frac{2eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2\cosh\left(k\beta\mu_e\right) \right] \sum_{n=0}^{\infty} W_1^{\pm}(n),
\end{aligned}
\tag{12}
$$

where we introduce the function $W_1^{\pm}(n)$ as follows

$$
W_1^{\pm}(n) \equiv \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T} \, K_1\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T \right).
\tag{13}
$$

Considering the Euler-Maclaurin formula to replace the sum over Landau levels, we have

$$
\sum_{n=0}^{\infty} W_1^{\pm}(n) = \int_0^{\infty} dn \, W_1^{\pm}(n) + \frac{1}{2}\left[W_1^{\pm}(\infty) + W_1^{\pm}(0) \right] + \frac{1}{12}\left[\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{\infty} - \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_0 \right] + R
\tag{14}
$$

where R is the error remainder which is defined by integrals over Bernoulli polynomials. Using the properties of Bessel function we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} W_1^{\pm}(n) &= \int_0^{\infty} dn \, W_1^{\pm}(n) + \frac{1}{2} W_1^{\pm}(0) - \frac{1}{12} \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_0 + R \\
&= \left(\frac{T^2}{k^2 eB}\right) \left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T) \right] + \frac{1}{2}\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right) K_1(k\tilde{m}_{\pm}/T) \right] + \frac{1}{12}\left[\left(\frac{k^2 eB}{T^2}\right) K_0(k\tilde{m}_{\pm}/T) \right] + R.
\end{aligned}
\tag{15}
$$
$$
\tag{16}
$$

Replacing the sum over Landau levels by the integral, the partition function becomes

$$
\ln \mathcal{Z}_{tot} = \ln \mathcal{Z}_{free} + \ln \mathcal{Z}_B + \ln \mathcal{Z}_R
\tag{17}
$$

where we defined

$$
\ln \mathcal{Z}_{free} = \frac{2T^3 V}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cosh\left(k\mu_e/T\right) \left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T) \right]
\tag{18}
$$

$$
\ln \mathcal{Z}_B = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh\left(k\mu_e/T\right) \left[\frac{k\tilde{m}_{\pm}}{2T} K_1(k\tilde{m}_{\pm}/T) + \frac{k^2 eB}{12 T^2} K_0(k\tilde{m}_{\pm}/T) \right]
\tag{19}
$$

$$
\ln \mathcal{Z}_R = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh\left(k\mu_e/T\right) R
\tag{20}
$$

For nonrelativistic electron-positron plasma, we can consider the partition function in Boltzmann approximation and assuming the error remainder R is small and can be neglected. In this case, we have

$$
\begin{aligned}
\ln \mathcal{Z}_{tot} =& \frac{T^3 V}{(2\pi)^2} \left[2\cosh(\mu_e/T) \right] \left(\frac{\tilde{m}_{\pm}}{T}\right)^2 K_2(\tilde{m}_{\pm}/T) \\
&+ \frac{2eBTV}{(2\pi)^2} \left[2\cosh(\mu_e/T) \right] \left[\frac{1}{2}\left(\frac{\tilde{m}_{\pm}}{T}\right) K_1(\tilde{m}_{\pm}/T) + \frac{1}{12}\left(\frac{eB}{T^2}\right) K_0(\tilde{m}_{\pm}/T) \right]
\end{aligned}
\tag{21}
$$

It can be written as

$$\ln \mathcal{Z}_{tot} = \frac{T^3 V}{(2\pi)^2} \left[2\cosh(\mu_e/T) \right] \left\{ \left(\frac{\tilde{m}_\pm}{T} \right)^2 K_2(\tilde{m}_\pm/T) + \frac{eB}{T^2} \left[\left(\frac{\tilde{m}_\pm}{T} \right) K_1(\tilde{m}_\pm/T) + \frac{1}{6} \left(\frac{eB}{T^2} \right) K_0(\tilde{m}_\pm/T) \right] \right\}$$

(22)

4. Chemical potential and Magnetization

4.1. Charge neutrality

Giving the partition function in Boltzmann limit Eq.(22) the net number density of electron can be written as

$$
\begin{aligned}
(n_e - n_{\bar{e}}) &= \frac{T}{V} \frac{\partial}{\partial \mu_e} \ln \mathcal{Z}_{tot} \\
&= \frac{2T^3}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\left(\frac{\tilde{m}_\pm}{T} \right)^2 K_2(\tilde{m}_\pm/T) \right] \\
&\quad + \frac{4eBT}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\frac{\tilde{m}_\pm}{2T} K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2} K_0(\tilde{m}_\pm/T) \right]
\end{aligned}
$$

(23)

Using the charge neutrality, we have

$$
\begin{aligned}
n_p &= (n_e - n_{\bar{e}}) \\
&= \frac{2T^3}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\left(\frac{\tilde{m}_\pm}{T} \right)^2 K_2(\tilde{m}_\pm/T) \right] \\
&\quad + \frac{4eBT}{(2\pi)^2} \sinh\left(\mu_e/T\right) \left[\frac{\tilde{m}_\pm}{2T} K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2} K_0(\tilde{m}_\pm/T) \right]
\end{aligned}
$$

(24)

where n_p is the number density of proton. It is also convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad b_0 = \frac{eB}{T^2}$$

(25)

and we obtain

$$n_p = \frac{T^3}{(2\pi)^2} \left[2\sinh\left(\mu_e/T\right) \right] \left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6} K_0(x_\pm) \right]$$

(26)

In this case, given the magnetic field B we can solve the chemical potential μ_e as a function of temperature numerically.

4.2. Magneticzation

On the other hand, giving the partition function Eq.(22) the magnetization can be obtained via the definition

$$M = \frac{T}{V} \frac{\partial \ln \mathcal{Z}_{tot}}{\partial B} = \frac{T}{V} \left(\frac{\partial \tilde{m}_\pm}{\partial B} \right) \frac{\partial \ln \mathcal{Z}_{tot}}{\partial \tilde{m}_\pm}$$

(27)

then it can be written as

$$M = \frac{T^4}{(2\pi)^2} \left[2\cosh(\mu_e/T)\right] \left\{ -\left[\frac{e(1 \pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm^2}{T^3}K_1(\tilde{m}_\pm/T)\right] + \frac{e}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right] \right.$$
$$\left. + \frac{eB}{T^2}\left[-\frac{e(1 \pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm}{T^2}K_0(\tilde{m}_\pm/T) + \frac{e}{6T^2}K_0(\tilde{m}_\pm/T) - \frac{1}{6}\left(\frac{eB}{T^2}\right)\frac{e(1 \pm g/2)}{2\tilde{m}_\pm}\frac{1}{T}K_1(\tilde{m}_\pm/T)\right] \right\}$$

(28)

It is convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad b_0 = \frac{eB}{T^2}$$

(29)

then the magnetization can be written as

$$M = \frac{4\pi\alpha B}{(2\pi)^2 b_0} \left[2\cosh(\mu_e/T)\right] \left\{ \left[1 - \frac{(1 \pm g/2)}{2}\left(1 + \frac{b_0^2}{6x_\pm^2}\right)\right] x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1 \pm g/2)}{2}\right] b_0 K_0(x_\pm) \right\}.$$

(30)

In this case, given the magnetic field B and chemical potential we can solve the magnetization M as a function of temperature numerically.

4.3. Chemical potential and magnetization

Giving the condition of charge neutrality and magnetization we have

$$\left(\frac{n_p}{T^3}\right) = \frac{1}{(2\pi)^2}\left[2\sinh(\mu_e/T)\right]\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right]$$

(31)

$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 b_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1 - \frac{(1 \pm g/2)}{2}\left(1 + \frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1 \pm g/2)}{2}\right]b_0 K_0(x_\pm)\right\}.$$

(32)

The chemical potential of electron and positron can be written as

$$\sinh(\mu_e/T) = \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right]}$$

(33)

$$\longrightarrow \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{x_\pm^2 K_2(x_\pm)}, \qquad \text{for } b_0 = 0$$

(34)

it shows that for the case $b_0 = 0$ the chemical potential agrees with our earlier results. For magnetization we can use the properties of hyperbolic function $\cosh^2(\mu_e/T) - \sinh^2(\mu_e/T) = 1$, then we have

$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 b_0}\left[2\sqrt{1 + (\sinh(\mu_e/T))^2}\right]\left\{\left[1 - \frac{(1 \pm g/2)}{2}\left(1 + \frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1 \pm g/2)}{2}\right]b_0 K_0(x_\pm)\right\}.$$

(35)

and the magnetization can be written as

$$
\frac{M}{B} = \frac{8\pi\alpha}{(2\pi)^2 b_0} \left(\left[1 - \frac{(1 \pm g/2)}{2} \left(1 + \frac{b_0^2}{6x_\pm^2} \right) \right] x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1 \pm g/2)}{2} \right] b_0 K_0(x_\pm) \right)
$$

$$
\times \sqrt{ 1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6} K_0(x_\pm)} \right]^2 } \tag{36}
$$

In this case giving the magnetic field b_0 and proton density n_p/T^3 we can solve the magnetization and chemical potential numerically.

4.4. Proton number density and magnetic field in early universe

Considering the homogeneous proton number density in early universe, it can be written as

$$
n_p = \frac{n_p}{n_B} \left(\frac{n_B}{s_{\gamma,\nu,e}} \right) s_{\gamma,\nu,e} = X_p \left(\frac{n_B}{s_{\gamma,\nu}} \right) s_{\gamma,\nu}, \qquad X_p = \frac{n_p}{n_B} \tag{37}
$$

where n_B is the number density of baryon, and the second equality is obtained by considering $e^\pm$ entropy density is negligible compared to the photon and neutrino entropy density at post BBN temperature $20 < T < 50$keV.

- The proton fraction X_p can be written as follow:

$$
X_p = \frac{n_p}{n_B} = \frac{n_p}{n_p + n_n} = \frac{1}{1 + n_n/n_p} \tag{38}
$$

Since all neutrons end up bound in to the ^4H_e after BBN, then the mass fraction of ^4H_e can be estimated as

$$
X_\alpha = \frac{2(n_n/n_p)}{1 + n_n/n_p} = 0.245 \pm 0.03 \tag{39}
$$

where we use $X_\alpha = 0.245 \pm 0.03$ from particle data group [1]. Solving the ratio n_n/n_p and substituting into the X_p, we obtain

$$
X_p = 0.878 \pm 0.015 \tag{40}
$$

- Since the comoving baryon number and entropy are conserved, hence the ratio $s_{\gamma,\nu}/n_B$ is conserved, then the entropy per baryon ratio can be written as

$$
\left(\frac{s_{\gamma,\nu}}{n_B} \right) = \left(\frac{s_{\gamma,\nu}}{n_B} \right)_{t_0} = \left(\frac{n_\gamma}{n_B} \right)_{t_0} \left(\frac{s_\gamma}{n_\gamma} + \frac{n_\nu}{n_\gamma} \frac{s_\nu}{n_\nu} \right) = \left(\frac{n_\gamma}{n_B} \right)_{t_0} \left[3.601 + \frac{9}{4} \left(\frac{T_\nu}{T_\gamma} \right)^3 4.202 \right], \tag{41}
$$

where the subscript t_0 denotes the present day value and consider all neutrinos are relativistic particles today. The entropy per particle for a boson is $(s/n)_{\text{boson}} = 3.601$ and for a fermion is $(s/n)_{\text{fermion}} = 4.202$. From particle data group and standard big bang model [1,2], we have

$$
5.8 \times 10^{-10} < \frac{n_B}{n_\gamma} < 6.5 \times 10^{-10}, \quad \frac{n_B}{n_\gamma} = 6.14 \times 10^{10}(\text{BBN observation}), \quad \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11} \right)^{1/3}. \tag{42}
$$

In this case, the entropy per baryon ratio becomes

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \frac{1}{6.14 \times 10^{-10}} \left[3.601 + \frac{9}{4}\left(\frac{4}{11}\right)4.202\right] = 1.14 \times 10^{10} \tag{43}$$

- On the other hand, the entropy density at temperature can be written as [2]

$$s = \frac{2\pi^2}{45}g_s T_\gamma^3, \qquad g_s = \sum_{i=boson} g_i\left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8}\sum_{i=fermion} g_i\left(\frac{T_i}{T_\gamma}\right)^3 \tag{44}$$

where g_s is the effective degree of freedom that contribute to the entropy density. In the temperature we are interested in $50 > T > 20$keV, the entropy density reads

$$s_{\gamma,\nu} = \frac{2\pi^2}{45}g_s T_\gamma^3, \qquad g_s = g_\gamma + \frac{7}{8}g_\nu\left(\frac{T_\nu}{T_\gamma}\right)^3 = 2 + \frac{7}{8}2 \times 3\left(\frac{4}{11}\right) = 3.91 \tag{45}$$

Substituting above numerical results into Eq.(37), the proton number density is given by

$$n_p = X_p\left(\frac{n_B}{s_{\gamma,\nu}}\right)s_{\gamma,\nu} = \frac{0.878 \times 3.91}{1.14 \times 10^{10}}\left(\frac{2\pi^2}{45}\right)T_\gamma^3 = 3.011 \times 10^{-10}\left(\frac{2\pi^2}{45}\right)T_\gamma^3 \tag{46}$$

The dimensionless magnetic constant under expansion in the Boltzmann factor is

$$b_0 = \frac{|e|B(\hbar c^2)}{(k_B T)^2} \tag{47}$$

Magnetic flux drops with $1/a(t)^{(2)}$ where a is the scale factor while temperature drops with $1/a(t)$ making the above a constant. The current modern bounds on the relic magnetic field lies between

$$B_{t_0} = 10^{-20} \sim 10^{-12}\,\text{Tesla} \tag{48}$$

where the subscript t_0 denotes the present day value The current temperature of the universe is

$$T_{t_0} = 2.7\,\text{K} \tag{49}$$

Plugging in those values, b_0 takes the values

$$b_0^{\text{max}} = 5.5 \times 10^{-3}, \tag{50}$$
$$b_0^{\text{min}} = 1.1 \times 10^{-11}. \tag{51}$$

As b_0 is a constant of expansion, assuming the electron-proton plasma between the CMB and electron-positron annihilation did not greatly disturbed it, we can calculate the remnant values at the temperature $k_B T = 50$ keV with the expression

$$B^{\text{max}}(T = 50\text{keV}) = \frac{B_0^{\text{max}}(k_B T)^2}{(|q|\hbar c^2)} = 2.3 \times 10^5\text{Tesla}, \tag{52}$$
$$B^{\text{min}}(T = 50\text{keV}) = 4.6 \times 10^{-4}\text{Tesla} \tag{53}$$

4.5. *Example: g-factor $g = 2$*

In our calculation we introduce an effective mass term which incorporates the ground state component of the Landau energy as follow

$$x_\pm = \frac{\tilde{m}_\pm}{T} = \frac{1}{T}\sqrt{m_e^2 + eB\left(1 \pm \frac{g}{2}\right)} \tag{54}$$

Considering the case $g = 2$ we have following two cases:

- Case1: $\tilde{m}_+ = \sqrt{m_e^2 + 2eB}$, and $x = \tilde{m}_+/T$. The equations for chemical potential and magnetization are given by
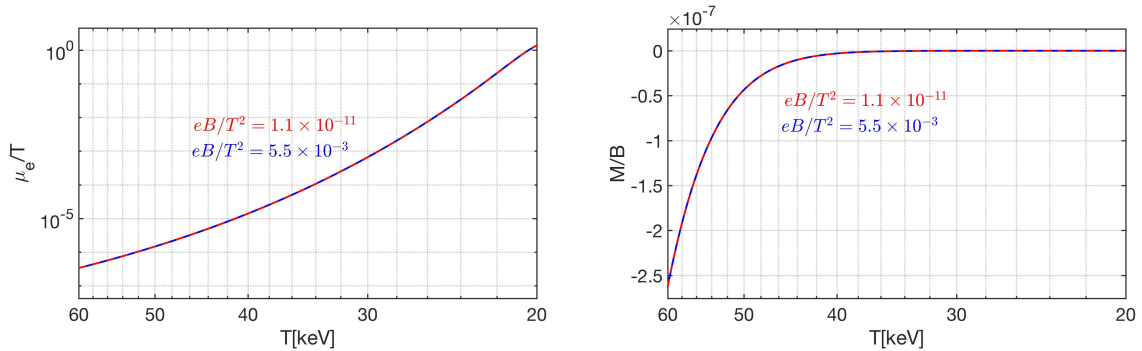
$$\sinh\left(\mu_e/T\right) = \frac{(2\pi)^2 n_p}{2T^3} \frac{1}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6}K_0(x)\right]} \tag{55}$$

and

$$\left(\frac{M}{B}\right) = -\frac{8\pi\alpha}{(2\pi)^2}\left(\frac{b_0}{6x}K_1(x_\pm) + \frac{2}{3}K_0(x_\pm)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right]}\right]^2} \tag{56}$$

Substituting the magnetic field b_0 and proton density n_p/T^3 from pervious section, we can solve the magnetization and chemical potential numerically.

In Fig 2, we plot the chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature T. It shows for the case1 the chemical potential and magnetization are not sensitive to the magnetic field, this is because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq.(55) and Eq.(56).

Figure 2. The chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature $20 < T < 60$keV for the case 1. The chemical potential and magnetization are not sensitive to the magnetic field, because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ can be neglected in Eq.(55) and Eq.(56).

- Case 2: $\tilde{m}_- = m_e$ and $x = \tilde{m}_-/T$,then the equations for chemical potential and magnetization can be written as

$$\sinh\left(\mu_e/T\right) = \frac{(2\pi)^2 n_p}{2T^3} \frac{1}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6}K_0(x)\right]} \tag{57}$$

and

$$\left(\frac{M}{B}\right) = \frac{8\pi\alpha}{(2\pi)^2}\left(\frac{1}{b_0}xK_1(x) + \frac{1}{3}K_0(x)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6}K_0(x)\right]}\right]^2} \quad (58)$$

Using the magnetic field b_0 and proton density n_p/T^3 from pervious section, we can solve the magnetization and chemical potential for case 2 numerically.

In Fig 3, we plot the chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature T. It shows that the chemical potential is not sensitive to the magnetic field because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq.(57). However, the magnetization does depend on the magnetic field b_0 strongly. This is because for small magnetic field b_0 the dominant term in Eq(58) is $xK_1(x)/b_0$. For given b_0, the value of magnetization can be larger than the magnetic filed, i.e. $M/B > 1$ which shows the possibility that magnetic domains can be formed in early universe.
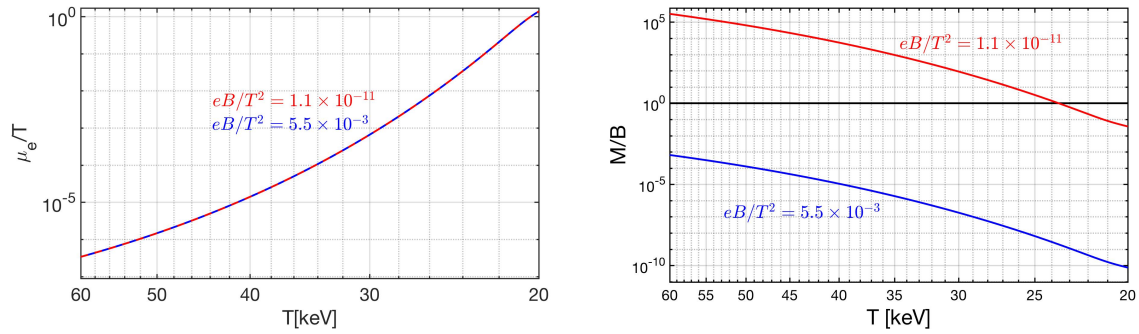
Figure 3. The chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature $20 < T < 60$keV for the case 2. It shows that the chemical potential is not sensitive to the magnetic field b_0. However, the magnetization does depend on the magnetic field b_0 strongly. This is because for small magnetic field b_0 the dominant term in Eq(58) is $xK_1(x)/b_0$. For giving b_0 we can have $M/B > 1$ in early universe.

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