Electron Positron Partition Function in Early Universe

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I. PARTITION FUNCTION

Considering the $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, the energy of $e^{\pm}$ can be written as

$$E_{n,s}=\sqrt{p_z^2+\tilde{m}^2+2eBn},\qquad \tilde{m}^2=m_e^2+eB\left(1-gs\right),\qquad s=\pm\frac{1}{2},\qquad n=0,1,2,3,\ldots \tag{1}$$

$$\tag{2}$$

If we consider a system that all electrons and positrons are spin aligned and antialigned with the magnetic field B, then the partition function of the system can be written as

$$\ln\mathcal{Z}_{tot}=\frac{2eBV}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^{\infty}dp_z\left[\ln\left(1+e^{-\beta(E_n^{\pm}-\mu_e)}\right)+\ln\left(1+e^{-\beta(E_n^{\pm}+\mu_e)}\right)\right], \tag{3}$$

where $\beta=1/T$, μ_e is the chemical potential of electron, and energy $E_n^{\pm}$ can be written as

$$E_n^{\pm}=\sqrt{p_z^2+\tilde{m}_{\pm}^2+2eBn},\qquad \tilde{m}_{\pm}^2=m_e^2+eB\left(1\pm\frac{g}{2}\right). \tag{4}$$

To simplify the partition function we can consider the expansion of the logarithmic function, we have

$$\ln\left(1+x\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k,\qquad \text{for } |x|<1. \tag{5}$$

Then the partition function of electron/positron system can be written as

$$\begin{aligned}\ln\mathcal{Z}_{tot}=&\frac{2eBV}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^{\infty}dp_z\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left[e^{k\beta\mu_e}+e^{-k\beta\mu_e}\right]e^{-k\beta E_n^{\pm}}\\ =&\frac{2eBV}{(2\pi)^2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left[2\cosh\left(k\beta\mu_e\right)\right]\int_0^{\infty}dp_z\,e^{-k\beta E_n^{\pm}}.\end{aligned} \tag{6}$$

Using the general definition of Bessel function:

$$K_\nu(\beta m)=\frac{\sqrt{\pi}}{\Gamma(\nu-1/2)}\frac{1}{m}\left(\frac{\beta}{2m}\right)^{\nu-1}\int_0^{\infty}dp\,p^{2\nu-2}e^{-\beta E}\qquad \text{for } \nu>1/2. \tag{7}$$

the integral over dp_z can be written as

$$\int_0^{\infty}dp_z\,e^{-k\beta E_n^{\pm}}=\frac{\Gamma(1/2)}{\sqrt{\pi}}\sqrt{\tilde{m}_{\pm}^2+2eBn}\;K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2+2eBn}/T\right) \tag{8}$$

$$=\sqrt{\tilde{m}_{\pm}^2+2eBn}\;K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2+2eBn}/T\right). \tag{9}$$

In this case, the partition function becomes

$$\ln\mathcal{Z}_{tot}=\frac{2eBV}{(2\pi)^2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left[2\cosh\left(k\beta\mu_e\right)\right]\sqrt{\tilde{m}_{\pm}^2+2eBn}\;K_1(k\sqrt{\tilde{m}_{\pm}^2+2eBn}/T) \tag{10}$$

$$=\frac{2eBTV}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\left[2\cosh\left(k\beta\mu_e\right)\right]\sum_{n=0}^{\infty}W_1^{\pm}(n), \tag{11}$$

where we introduce the function $W_1^{\pm}(n)$ as follows

$$W_1^{\pm}(n) \equiv \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\, K_1\Big(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\Big)\,. \tag{12}$$

Considering the Euler-Maclaurin formula to replace the sum over Landau levels, we have

$$\sum_{n=0}^{\infty} W_1^{\pm}(n) = \int_0^{\infty} dn\, W_1^{\pm}(n) + \frac{1}{2}\Big[W_1^{\pm}(\infty) + W_1^{\pm}(0)\Big] + \frac{1}{12}\Big[\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{\infty} - \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{0}\Big] + R \tag{13}$$

where R is the error remainder which is defined by integrals over Bernoulli polynomials. Using the properties of Bessel function we have

$$\frac{\partial W_1^{\pm}}{\partial n} = -\frac{k^2 eB}{T^2} K_0\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\right), \qquad , W_1^{\pm}(\infty) = 0, \qquad \int_a^{\infty} dx\, x^2 K_1(x) = a^2 K_2(a) \tag{14}$$

then we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} W_1^{\pm}(n) &= \int_0^{\infty} dn\, W_1^{\pm}(n) + \frac{1}{2} W_1^{\pm}(0) - \frac{1}{12}\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{0} + R && (15)\\
&= \int_0^{\infty} dn\, \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T}\, K_1\Big(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T\Big) + \frac{1}{2}\left[\frac{k\tilde{m}_{\pm}}{T} K_1(k\tilde{m}_{\pm}/T)\right] + \frac{1}{12}\left[\frac{k^2 eB}{T^2} K_0(k\tilde{m}_{\pm}/T)\right] + R \\
&= \left(\frac{T^2}{k^2 eB}\right)\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] + \frac{1}{2}\left[\left(\frac{k\tilde{m}_{\pm}}{T}\right) K_1(k\tilde{m}_{\pm}/T)\right] + \frac{1}{12}\left[\left(\frac{k^2 eB}{T^2}\right) K_0(k\tilde{m}_{\pm}/T)\right] + R. && (16)
\end{aligned}$$

Replacing the sum over Landau levels by the integral, the partition function becomes

$$\ln \mathcal{Z}_{tot} = \ln \mathcal{Z}_{free} + \ln \mathcal{Z}_B + \ln \mathcal{Z}_R \tag{17}$$

where we defined

$$\ln \mathcal{Z}_{free} = \frac{2T^3 V}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cosh(k\mu_e/T) \left[\left(\frac{k\tilde{m}_{\pm}}{T}\right)^2 K_2(k\tilde{m}_{\pm}/T)\right] \tag{18}$$

$$\ln \mathcal{Z}_B = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh(k\mu_e/T) \left[\frac{k\tilde{m}_{\pm}}{2T} K_1(k\tilde{m}_{\pm}/T) + \frac{k^2 eB}{12T^2} K_0(k\tilde{m}_{\pm}/T)\right] \tag{19}$$

$$\ln \mathcal{Z}_R = \frac{4eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cosh(k\mu_e/T) R \tag{20}$$

II. CHEMICAL POTENTIAL AND MAGNETIZATION

A. Charge neutrality

Giving the partition function, we can calculate the net number density of electron as follow:

$$\begin{aligned}
(n_e - n_{\bar{e}}) &= \frac{T}{V}\frac{\partial}{\partial \mu_e} \ln \mathcal{Z}_{tot} = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_{free}}{\partial \mu_e} + \frac{T}{V}\frac{\partial \ln \mathcal{Z}_B}{\partial \mu_e} + \frac{T}{V}\frac{\partial \ln \mathcal{Z}_R}{\partial \mu_e} \\
&= (n_e - n_{\bar{e}})_{free} + (n_e - n_{\bar{e}})_B + (n_e - n_{\bar{e}})_R && (21)
\end{aligned}$$

we have

$$(n_e - n_{\bar{e}})_{free} = \frac{2T^3}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\sinh\left(k\mu_e/T\right)\left[\left(\frac{k\tilde{m}_\pm}{T}\right)^2 K_2(k\tilde{m}_\pm/T)\right] \tag{22}$$

$$(n_e - n_{\bar{e}})_B = \frac{4eBT}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sinh\left(k\mu_e/T\right)\left[\frac{k\tilde{m}_\pm}{2T}K_1(k\tilde{m}_\pm/T) + s\frac{k^2 eB}{12T^2}K_0(k\tilde{m}_\pm/T)\right] \tag{23}$$

$$(n_e - n_{\bar{e}})_R = \frac{4eBT}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sinh\left(k\mu_e/T\right)R \tag{24}$$

Considering the Boltzmann approximation and assuming the error remainder R is small and can be neglected, then the net number density of electron can be written as

$$\begin{aligned}(n_e - n_{\bar{e}}) &\approx (n_e - n_{\bar{e}})_{free} + (n_e - n_{\bar{e}})_B \\ &= \frac{2T^3}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] + \frac{4eBT}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\frac{\tilde{m}_\pm}{2T}K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2}K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{25}$$

Using the charge neutrality, we have

$$\begin{aligned}n_p &= (n_e - n_{\bar{e}}) \\ &= \frac{2T^3}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] + \frac{4eBT}{(2\pi)^2}\sinh\left(\mu_e/T\right)\left[\frac{\tilde{m}_\pm}{2T}K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2}K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{26}$$

where n_p is the number density of proton. It is also convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad B_0 = \frac{eB}{T^2} \tag{27}$$

and we obtain

$$n_p = \frac{T^3}{(2\pi)^2}\left[2\sinh\left(\mu_e/T\right)\right]\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right] \tag{28}$$

In this case, given the magnetic field B we can solve the chemical potential μ_e as a function of temperature numerically.

B. Magneticzation

On the other hand, considering the partition function in Boltzmann approximation and neglecting the error remainder R, we have

$$\begin{aligned}\ln \mathcal{Z}_{tot} =& \frac{T^3 V}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T) \\ &+ \frac{2eBTV}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left[\frac{1}{2}\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{12}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{29}$$

It can be written as

$$\ln \mathcal{Z}_{tot} = \frac{T^3 V}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T) + \frac{eB}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right\} \tag{30}$$

In this case, the magnetization can be obtained via the definition

$$M = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_{tot}}{\partial B} = \frac{T}{V}\left(\frac{\partial \tilde{m}_\pm}{\partial B}\right)\frac{\partial \ln \mathcal{Z}_{tot}}{\partial \tilde{m}_\pm} \tag{31}$$

then it can be written as

$$M = \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{\frac{\partial\tilde{m}_\pm}{\partial B}\frac{\partial}{\partial\tilde{m}_\pm}\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] + \frac{e}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right.$$
$$\left. + \frac{eB}{T^2}\left[\frac{\partial\tilde{m}_\pm}{\partial B}\frac{\partial}{\partial\tilde{m}_\pm}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T)\right] + \frac{e}{6T^2}K_0(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)\frac{\partial\tilde{m}_\pm}{\partial B}\frac{\partial}{\partial\tilde{m}_\pm}K_0(\tilde{m}_\pm/T)\right]\right\} \tag{32}$$

we have

$$\frac{\partial\tilde{m}_\pm}{\partial B} = \frac{e(1\pm g/2)}{2\tilde{m}_\pm}, \tag{33}$$
$$\frac{\partial}{\partial\tilde{m}_\pm}K_0(\tilde{m}_\pm/T) = -\frac{1}{T}K_1(\tilde{m}_\pm/T), \tag{34}$$
$$\frac{\partial}{\partial\tilde{m}_\pm}\left(\frac{\tilde{m}_\pm}{T}K_1(\tilde{m}_\pm/T)\right) = -\frac{\tilde{m}_\pm}{T^2}K_0(\tilde{m}_\pm/T), \tag{35}$$
$$\frac{\partial}{\partial\tilde{m}_\pm}\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] = -\frac{\tilde{m}_\pm^2}{T^3}K_1(\tilde{m}_\pm/T). \tag{36}$$

Substituting the above equations into the magnetization we obtain

$$M = \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{-\left[\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm^2}{T^3}K_1(\tilde{m}_\pm/T)\right] + \frac{e}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right.$$
$$\left. + \frac{eB}{T^2}\left[-\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm}{T^2}K_0(\tilde{m}_\pm/T) + \frac{e}{6T^2}K_0(\tilde{m}_\pm/T) - \frac{1}{6}\left(\frac{eB}{T^2}\right)\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{1}{T}K_1(\tilde{m}_\pm/T)\right]\right\} \tag{37}$$

It is convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad B_0 = \frac{eB}{T^2} \tag{38}$$

then the magnetization can be written as

$$\begin{aligned}
M &= \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{-\frac{e(1\pm g/2)}{2\tilde{m}_\pm^2}\left[x_\pm^3K_1(x_\pm) + B_0x_\pm^2K_0(x_\pm) + \frac{B_0^2}{6}x_\pm K_1(x_\pm)\right] + \frac{e}{T^2}\left[x_\pm K_1(x_\pm) + \frac{B_0}{3}K_0(x_\pm)\right]\right\}\\
&= \frac{eT^2}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{-\frac{(1\pm g/2)}{2x_\pm^2}\left[x_\pm^3K_1(x_\pm) + B_0x_\pm^2K_0(x_\pm) + \frac{B_0^2}{6}x_\pm K_1(x_\pm)\right] + \left[x_\pm K_1(x_\pm) + \frac{B_0}{3}K_0(x_\pm)\right]\right\}\\
&= \frac{e^2B}{(2\pi)^2B_0}\left[2\cosh(\mu_e/T)\right]\left\{-\frac{(1\pm g/2)}{2}\left[\left(x_\pm + \frac{B_0^2}{6x_\pm}\right)K_1(x_\pm) + B_0K_0(x_\pm)\right] + \left[x_\pm K_1(x_\pm) + \frac{B_0}{3}K_0(x_\pm)\right]\right\}\\
&= \frac{4\pi\alpha B}{(2\pi)^2B_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1 - \frac{(1\pm g/2)}{2}\left(1 + \frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1\pm g/2)}{2}\right]B_0K_0(x_\pm)\right\}.
\end{aligned} \tag{39}$$

In this case, given the magnetic field B and chemical potential we can solve the magnetization M as a function of temperature numerically.

C. chemical potential and magnetization

Giving the condition of charge neutrality and magnetization we have

$$\left(\frac{n_p}{T^3}\right) = \frac{1}{(2\pi)^2}\left[2\sinh\left(\mu_e/T\right)\right]\left[x_\pm^2K_2(x_\pm) + B_0x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right] \tag{40}$$
$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2B_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1 - \frac{(1\pm g/2)}{2}\left(1 + \frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3} - \frac{(1\pm g/2)}{2}\right]B_0K_0(x_\pm)\right\}. \tag{41}$$

It can be written as

$$\sinh\left(\mu_e/T\right) = \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right]} \tag{42}$$

$$\cosh\left(\mu_e/T\right) = \frac{(2\pi)^2 M B_0}{8\pi\alpha B}\frac{1}{\left\{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right\}} \tag{43}$$

Using the properties of hyperbolic function, we can obtain the relation

$$\tanh\left(\mu_e/T\right) = \left(\frac{n_p}{T^3}\frac{B}{M}\right)\frac{4\pi\alpha}{B_0}\frac{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)}{x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)} \tag{44}$$

and using the relation $\cosh^2(\mu_e/T) - \sinh^2(\mu_e/T) = 1$ we have

$$\begin{aligned}\left(\frac{(2\pi)^2 M B_0}{8\pi\alpha B}\right)^2 &= \left(1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right]}\right]^2\right)\\ &\quad\times\left\{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right\}^2\end{aligned} \tag{45}$$

and the magnetization can be written as

$$\begin{aligned}\frac{M}{B} &= \frac{8\pi\alpha}{(2\pi)^2 B_0}\sqrt{\left(\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{B_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm) + \left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]B_0 K_0(x_\pm)\right)^2}\\ &\quad\times\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right]}\right]^2}\end{aligned} \tag{46}$$

D. Example: g-factor $g = 2$

Considering the case $g = 2$ we have following two cases:

- Case1: $\tilde{m}_+ = \sqrt{m_e^2 + 2eB}$, and $x = \tilde{m}_+/T$. The equations for chemical potential and magnetization are given by

$$\tanh\left(\mu_e/T\right) = \left(\frac{n_p}{T^3}\frac{B}{M}\right)\frac{4\pi\alpha}{B_0}\frac{-B_0^2 K_1(x)/(6x) - 2B_0 K_0(x)/3}{x^2 K_2(x) + B_0 x K_1(x) + \frac{B_0^2}{6}K_0(x)} \tag{47}$$

and

$$\left(\frac{M}{B}\right) = \frac{8\pi\alpha}{(2\pi)^2 B_0}\left(\frac{B_0^2}{6x}K_1(x_\pm) + \frac{2}{3}B_0 K_0(x_\pm)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_\pm^2 K_2(x_\pm) + B_0 x_\pm K_1(x_\pm) + \frac{B_0^2}{6}K_0(x_\pm)\right]}\right]^2} \tag{48}$$

- Case2: $\tilde{m}_- = m_e$ and $x = \tilde{m}_-/T$,then the equations for chemical potential and magnetization can be written as

$$\tanh\left(\mu_e/T\right) = \left(\frac{n_p}{T^3}\frac{B}{M}\right)\frac{4\pi\alpha}{B_0}\frac{[xK_1(x) + B_0 K_0(x)/3]}{x^2 K_2(x) + B_0 x K_1(x) + \frac{B_0^2}{6}K_0(x)} \tag{49}$$

and

$$\left(\frac{M}{B}\right) = \frac{8\pi\alpha}{(2\pi)^2 B_0}\left(xK_1(x) + \frac{B_0}{3}K_0(x)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x^2 K_2(x) + B_0 x K_1(x) + \frac{B_0^2}{6}K_0(x)\right]}\right]^2} \tag{50}$$

In both cases, giving the magnetic field B_0 we can solve the magnetization and chemical potential numerically.