Article

ELECTRON POSITRON PARTITION FUNCTION IN EARLY UNIVERSE

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Abstract: We will write abstract here

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1. Introduction

The electron-positron epoch of the early universe was home to several significant events which have greatly shaped our contemporary universe including neutrino decoupling, Big Bang Nucleosynthesis (BBN), the annihilation of most electrons and positrons partially re-ionizing the universe, as well as setting the stage for the eventually recombination period which would generate the cosmic microwave background (CMB). Therefore, correctly describing the dynamics of this $e^{\pm}$ plasma is of interest when considering modern cosmic mysteries such as the origin of extra-galactic magnetic fields. While most approaches tackle magnetized plasmas from the perspective of magnetohydrodynamics (MHD), a primarily classical or semi-classical approach, our perspective is to demonstrate that fundamental quantum statistical analysis can lead to further insights on the behavior of magnetized plasmas.

The universe is filled with magnetic fields at various scales and strengths both within galaxies and in deep extragalactic space far and away from matter sources. Extragalactic magnetic fields are not well constrained today, but are required by observation to be non-zero with a magnitude between 10^{-20} T $< B_{relic} < 10^{-12}$ T over Mpc coherent length scales. The upper bound is constrained from the characteristics of the CMB while the lower bound is constrained by non-observation of ultra-energetic photons from blazars. There are generally considered two possible origins for extragalactic magnetic fields: (a) matter-induced dynamo processes involving Amperian currents and (b) primordial (or relic) seed magnetic fields whose origins may go as far back as the Big Bang itself. It is currently unknown which origin accounts for extragalactic magnetic fields today or if it some combination of the two models. Even if magnetic fields in the universe today are primarily driven via amplification through Amperian matter currents, such models still require primordial seeds fields at some point to act as catalyst.

We then connect back to the plasmas of the early universe as such primordial magnetic fields would be lensed through the various plasmas that existed when the universe was far hotter and denser. Of particular interest to us is the electron-positron plasma which existed in the early universe at temperature $T > 20$ KeV which was the last plasma in the universe where the medium density was many orders of magnitude above ordinary lab atomic density. This dense plasma environment is where BBN occurred and where similar plasmas can still be found within exotic stars such as magnetars. The contemporary relic magnetic fields may then be an artifact of this time of last universe-scale magnetization in a manner similar to how the CMB is a relic of the time of charge recombination.

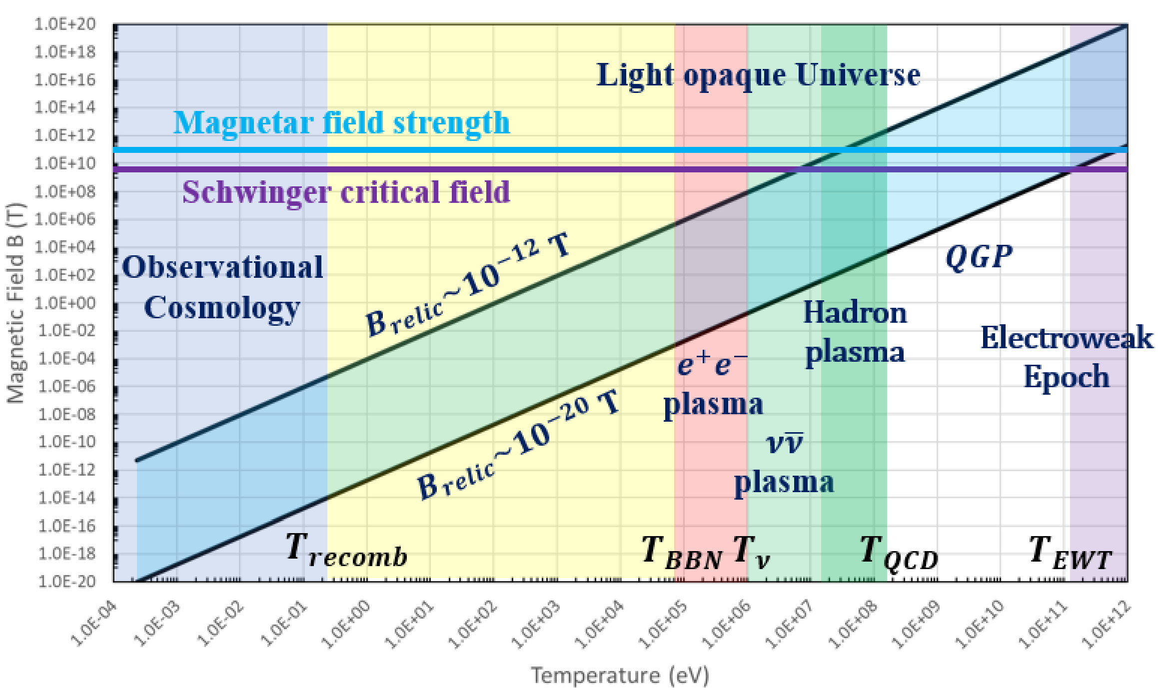

Figure 1. Qualitative value of the primordial magnetic field over the evolutionary lifespan of the universe. As magnetic flux is conserved over comoving surfaces, the primordial relic field is expected to dilute as $\approx 1/a(t)^2$ meaning the contemporary small bound values of 5×10^{-12} T $> B_{relic} > 10^{-20}$ T may have once represented large magnetic fields in the early universe. This is relic magnetic field would then be generated by the last phase of significant magnetization in the early universe. This figure is meant to be illustrative and it is unlikely the magnetization of the universe would proceed unhindered and unaltered into the ultra-dense plasma phases of the early universe. The values of the Schwinger critical field and the upper bound of surface magnetar field strength are included for scale.

2. Energy Eigenvalues

As a starting point, we consider the energy-eigenvalues of charged fermions within a homogenous magnetic field. Here, we have several choices: We could assume the typical Dirac energy eigen-values with gyromagnetic g-factor set to $g = 2$. But as electrons, positrons and most plasma species have anomalous magnetic moments (AMM), we require a more complete model. Another option would be to modify the Dirac equation with a Pauli term, often called the Dirac-Pauli (DP) approach, via

$$\hat{H}_{\mathrm{AMM}} = -a\frac{e}{2m}\frac{\sigma_{\mu\nu}F^{\mu\nu}}{2}\,, \tag{1}$$

where $\sigma_{\mu\nu}$ is the spin tensor proportional to the commutator of the gamma matrices and $F^{\mu\nu}$ is the EM field tensor. The AMM is defined via g-factor as

$$\frac{g}{2} = 1 + a\,. \tag{2}$$

This approach, while straightforward, would complicate the energies making analytic understanding and clarity difficult without a clear benefit. Our preferred model for the AMM is through the Klein-Gordon-Pauli (KGP) equation which is given by

$$\left((i\partial_\mu - eA_\mu)^2 - m^2 - e\frac{g}{2}\frac{\sigma_{\mu\nu}F^{\mu\mu}}{2}\right)\Psi = 0\,. \tag{3}$$

We wish to emphasize, that each of the three above models are distinct and have differing physical consequences and are not interchangeable. One benefit of the KGP approach is that the energies take eigenvalues which are mathematically similar to the Dirac energies. Considering the $e^{\pm}$ plasma in a uniform magnetic field B pointing along the z-axis, the energy of $e^{\pm}$ fermions can be written as

$$E_n^s = \sqrt{p_z^2 + \tilde{m}^2 + 2eBn}, \qquad \tilde{m}^2 = m_e^2 + eB\left(1 - gs\right), \qquad s = \pm\frac{1}{2}, \qquad n = 0, 1, 2, 3, \ldots \tag{4}$$

where n is the principle quantum number for the Landau levels and s is the spin quantum number. Here we introduce a notion of "magnetic mass" which inherits the spin-specific part of the energy adding them to the mass. This convention is also generalizable to further non-minimal electromagnetic models with more exotic energy contributions such that we write a general replacement as

$$m^2 \rightarrow \tilde{m}^2(B)\,. \tag{5}$$

This definition also pulls out the ground state Landau energy separating it from the remainder of the Landau tower of states. One restriction is that the magnetic-mass must remain positive definite in our analysis thus we require

$$\tilde{m}^2(B) = m_e^2 + eB\left(1 - gs\right) > 0\,. \tag{6}$$

This condition fails under ultra-strong magnetic fields of order

$$B_{\mathrm{crit}} = \frac{m^2}{ea} = \frac{\mathcal{B}_S}{a} \approx 3.8 \times 10^{12}\ \mathrm{T}\,, \tag{7}$$

where $\mathcal{B}_S$ is the Schwinger critical field strength. For electrons, this field strength is well above the window of magnetic field stregnths of interest during the late $e^{\pm}$ epoch.

3. Partition function

We now turn our attention now to the statistical behavior of the $e^{\pm}$ system. We can utilize the general Fermion partition function given by

$$\ln(\mathcal{Z}) = \sum_{\alpha} \ln\left(1 + e^{-\beta(E-\eta)}\right)\,, \tag{8}$$

where $\beta = 1/T$, α is the set of all quantum numbers in the system, and η is the generalized chemical potential. The magnetized $e^{\pm}$ system should be considered a system of four quantum species: Particles and anti-particles, and spign aligned and anti-aligned. Taken together we consider a system where all electrons and positrons are spin aligned or anti-aligned with the magnetic field B and the partition function of the system is written as

$$\ln \mathcal{Z}_{tot} = \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \left[\ln\left(1 + e^{-\beta(E_n^{\pm} - \mu_e)}\right) + \ln\left(1 + e^{-\beta(E_n^{\pm} + \mu_e)}\right)\right], \tag{9}$$

where μ_e is the electron chemical potential, the positron chemical potential is the negative of the electron's, and energy $E_n^{\pm}$ can be written as

$$E_n^{\pm} = \sqrt{p_z^2 + \tilde{m}_{\pm}^2 + 2eBn}, \qquad \tilde{m}_{\pm}^2 = m_e^2 + eB\left(1 \mp \frac{g}{2}\right)\,, \tag{10}$$

where the $\pm$ script refers to spin aligned and anti-aligned eigenvalues. To simplify the partition function we can consider the expansion of the logarithmic function into a power series yielding

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad \text{for } |x| < 1. \tag{11}$$

We emphasize that this expansion is only value where the Boltzmann factor $\exp(-\beta(E \pm \mu_e))$ remains less than unity, otherwise the partition function becomes divergent and ill-behaved. The partition function of $e^{\pm}$ system can be further expanded as

$$\begin{aligned}
\ln \mathcal{Z}_{tot} =& \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[e^{k\beta\mu_e} + e^{-k\beta\mu_e} \right] e^{-k\beta E_n^{\pm}} \\
=& \frac{2eBV}{(2\pi)^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[2\cosh(k\beta\mu_e) \right] \int_0^{\infty} dp_z \, e^{-k\beta E_n^{\pm}} \\
=& \frac{2eBTV}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[2\cosh(k\beta\mu_e) \right] \sum_{n=0}^{\infty} W_1^{\pm}(n,k),
\end{aligned} \tag{12}$$

where we introduce the function $W_1^{\pm}(n,k)$ as follows

$$W_1^{\pm}(n,k) \equiv \frac{k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}}{T} \, K_1\!\left(k\sqrt{\tilde{m}_{\pm}^2 + 2eBn}/T \right), \tag{13}$$

defined in terms of the modified (hyperbolic) Bessel function of the second kind $K_1(x)$. We then utilize the Euler-Maclaurin formula to replace the sum over Landau levels with an integration yielding

$$\sum_{n=0}^{\infty} W_1^{\pm}(n,k) = \int_0^{\infty} dn \, W_1^{\pm}(n,k) + \frac{1}{2}\left[W_1^{\pm}(\infty,k) + W_1^{\pm}(0,k) \right] + \frac{1}{12}\left[\left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{\infty} - \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{0} \right] + R\,, \tag{14}$$

where R is the error remainder which is defined by integrals over Bernoulli polynomials. Using the asymptotic properties of Bessel function $K_1(x)$ as x becomes large, we can simplify the above expression to obtain

$$\sum_{n=0}^{\infty} W_1^{\pm}(n,k) = \int_0^{\infty} dn \, W_1^{\pm}(n,k) + \frac{1}{2} W_1^{\pm}(0,k) - \frac{1}{12} \left.\frac{\partial W_1^{\pm}}{\partial n}\right|_{0} + R\,, \tag{15}$$

$$\sum_{n=0}^{\infty} W_1^{\pm}(n,k) = \left(\frac{T^2}{k^2 eB} \right) z^2 K_2(z) + \frac{1}{2} z K_1(z) + \frac{1}{12} \left(\frac{k^2 eB}{T^2} \right) K_0(z) + R\,, \qquad z = \frac{k\tilde{m}_{\pm}}{T}\,. \tag{16}$$

Here we see mathematically, it is the ground state which has no analogue in the classical continuum limit which contributes to a difference between the sum and integral formulations. Replacing the sum over Landau levels by the integral, the partition function can separated into three distinct parts

$$\ln \mathcal{Z}_{tot} = \ln \mathcal{Z}_{free} + \ln \mathcal{Z}_B + \ln \mathcal{Z}_R \tag{17}$$

where we define

$$\ln \mathcal{Z}_{free} = \frac{2T^3V}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4}\cosh\left(\frac{k\mu_e}{T}\right)\left[\left(\frac{k\tilde{m}_\pm}{T}\right)^2 K_2\left(\frac{k\tilde{m}_\pm}{T}\right)\right] \tag{18}$$

$$\ln \mathcal{Z}_{B} = \frac{4eBTV}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\cosh\left(\frac{k\mu_e}{T}\right)\left[\frac{k\tilde{m}_\pm}{2T}K_1\left(\frac{k\tilde{m}_\pm}{T}\right)+\frac{k^2eB}{12T^2}K_0\left(\frac{k\tilde{m}_\pm}{T}\right)\right] \tag{19}$$

$$\ln \mathcal{Z}_{R} = \frac{4eBTV}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\cosh\left(\frac{k\mu_e}{T}\right)R \tag{20}$$

While this would require further derivation to demonstrate explicitly, the benefit of the Euler-Maclaurin approach is if the error contribution remains finite or bound for the magnetized partition function, then a correspondence between the free Fermi partition function (with noticeably modified magnetic-mass $\tilde{m}_\pm$) and the magnetized Fermi partition function can be established. The mismatch between the summation and integral in the Euler-Maclaurin formula would then encapsulate the immediate magnetic response and deviation from the free particle phase space.

While we label $\ln(\mathcal{Z}_{free})$ as the "free" partition function, this is not strictly true as this contribution to the overall partition function is a function of the magnetic-mass we defined earlier in Eq. (5). When determining the magnetization of the magnetized quantum Fermi gas using, derivatives of the magnetic field B will not fully vanish on this first term which will resulting in an intrinsic magnetization which is distinct from the contribution from the ground state and mismatch between the quantized Landau levels and the continuum of the free momentum. Specifically, this free Fermi contribution represents the magnetization that arises from the spin magnetic energy rather than orbital contributions. To demonstrate this, consider the weak field limit for $g = 2$. The magnetic-mass reduces to the following values

$$\tilde{m}_+^2 = m_e^2\,, \tag{21}$$

$$\tilde{m}_-^2 = m_e^2 + 2eB\,. \tag{22}$$

For non-relativistic electron-positron plasma, we can consider the partition function in Boltzmann approximation and assuming the error remainder R is small and can be neglected. In this case, we have

$$\begin{aligned}\ln \mathcal{Z}_{tot} =& \frac{T^3V}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\\ &+\frac{2eBTV}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left[\frac{1}{2}\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T)+\frac{1}{12}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{23}$$

It can be written as

$$\ln \mathcal{Z}_{tot} = \frac{T^3V}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\left\{\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)+\frac{eB}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T)+\frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right]\right\} \tag{24}$$

4. Chemical potential and Magnetization

4.1. Charge neutrality

Giving the partition function in Boltzmann limit Eq.(24) the net number density of electron can be written as

$$\begin{aligned}(n_e - n_{\bar{e}}) &= \frac{T}{V}\frac{\partial}{\partial \mu_e} \ln \mathcal{Z}_{tot} \\ &= \frac{2T^3}{(2\pi)^2}\sinh(\mu_e/T)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] \\ &\quad + \frac{4eBT}{(2\pi)^2}\sinh(\mu_e/T)\left[\frac{\tilde{m}_\pm}{2T}K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2}K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{25}$$

Using the charge neutrality, we have

$$\begin{aligned}n_p &= (n_e - n_{\bar{e}}) \\ &= \frac{2T^3}{(2\pi)^2}\sinh(\mu_e/T)\left[\left(\frac{\tilde{m}_\pm}{T}\right)^2 K_2(\tilde{m}_\pm/T)\right] \\ &\quad + \frac{4eBT}{(2\pi)^2}\sinh(\mu_e/T)\left[\frac{\tilde{m}_\pm}{2T}K_1(\tilde{m}_\pm/T) + \frac{eB}{12T^2}K_0(\tilde{m}_\pm/T)\right]\end{aligned} \tag{26}$$

where n_p is the number density of proton. It is also convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad b_0 = \frac{eB}{T^2} \tag{27}$$

and we obtain

$$n_p = \frac{T^3}{(2\pi)^2}\left[2\sinh(\mu_e/T)\right]\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right] \tag{28}$$

In this case, given the magnetic field B we can solve the chemical potential μ_e as a function of temperature numerically.

4.2. Magneticzation

On the other hand, giving the partition function Eq.(24) the magnetization can be obtained via the definition

$$M = \frac{T}{V}\frac{\partial \ln \mathcal{Z}_{tot}}{\partial B} = \frac{T}{V}\left(\frac{\partial \tilde{m}_\pm}{\partial B}\right)\frac{\partial \ln \mathcal{Z}_{tot}}{\partial \tilde{m}_\pm} \tag{29}$$

then it can be written as

$$\begin{aligned}M = \frac{T^4}{(2\pi)^2}\left[2\cosh(\mu_e/T)\right]\Bigg\{ -\left[\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm^2}{T^3}K_1(\tilde{m}_\pm/T)\right] + \frac{e}{T^2}\left[\left(\frac{\tilde{m}_\pm}{T}\right)K_1(\tilde{m}_\pm/T) + \frac{1}{6}\left(\frac{eB}{T^2}\right)K_0(\tilde{m}_\pm/T)\right] \\ + \frac{eB}{T^2}\left[-\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{\tilde{m}_\pm}{T^2}K_0(\tilde{m}_\pm/T) + \frac{e}{6T^2}K_0(\tilde{m}_\pm/T) - \frac{1}{6}\left(\frac{eB}{T^2}\right)\frac{e(1\pm g/2)}{2\tilde{m}_\pm}\frac{1}{T}K_1(\tilde{m}_\pm/T)\right]\Bigg\}\end{aligned} \tag{30}$$

It is convenient to introduce the dimensionless variables:

$$x_\pm = \frac{\tilde{m}_\pm}{T}, \qquad b_0 = \frac{eB}{T^2} \tag{31}$$

then the magnetization can be written as

$$M = \frac{4\pi\alpha B}{(2\pi)^2 b_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm)+\left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]b_0 K_0(x_\pm)\right\}. \tag{32}$$

In this case, given the magnetic field B and chemical potential we can solve the magnetization M as a function of temperature numerically.

4.3. Chemical potential and magnetization

Giving the condition of charge neutrality and magnetization we have

$$\left(\frac{n_p}{T^3}\right) = \frac{1}{(2\pi)^2}\left[2\sinh(\mu_e/T)\right]\left[x_\pm^2 K_2(x_\pm)+b_0 x_\pm K_1(x_\pm)+\frac{b_0^2}{6}K_0(x_\pm)\right] \tag{33}$$

$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 b_0}\left[2\cosh(\mu_e/T)\right]\left\{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm)+\left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]b_0 K_0(x_\pm)\right\}. \tag{34}$$

The chemical potential of electron and positron can be written as

$$\sinh(\mu_e/T) = \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{\left[x_\pm^2 K_2(x_\pm)+b_0 x_\pm K_1(x_\pm)+\frac{b_0^2}{6}K_0(x_\pm)\right]} \tag{35}$$

$$\longrightarrow \frac{(2\pi)^2 n_p}{2T^3}\frac{1}{x_\pm^2 K_2(x_\pm)}, \qquad \text{for } b_0 = 0 \tag{36}$$

it shows that for the case $b_0 = 0$ the chemical potential agrees with our earlier results. For magnetization we can use the properties of hyperbolic function $\cosh^2(\mu_e/T) - \sinh^2(\mu_e/T) = 1$,then we have

$$\left(\frac{M}{B}\right) = \frac{4\pi\alpha}{(2\pi)^2 b_0}\left[2\sqrt{1+(\sinh(\mu_e/T))^2}\right]\left\{\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm)+\left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]b_0 K_0(x_\pm)\right\}. \tag{37}$$

and the magnetization can be written as

$$\frac{M}{B} = \frac{8\pi\alpha}{(2\pi)^2 b_0}\left(\left[1-\frac{(1\pm g/2)}{2}\left(1+\frac{b_0^2}{6x_\pm^2}\right)\right]x_\pm K_1(x_\pm)+\left[\frac{1}{3}-\frac{(1\pm g/2)}{2}\right]b_0 K_0(x_\pm)\right) \times\sqrt{1+\left[\frac{(2\pi)^2 n_p/(2T^3)}{x_\pm^2 K_2(x_\pm)+b_0 x_\pm K_1(x_\pm)+\frac{b_0^2}{6}K_0(x_\pm)}\right]^2} \tag{38}$$

In this case giving the magnetic field b_0 and proton density n_p/T^3 we can solve the magnetization and chemical potential numerically.

4.4. Proton number density and magnetic field in early universe

Considering the homogeneous proton number density in early universe, it can be written as

$$n_p = \frac{n_p}{n_B}\left(\frac{n_B}{s_{\gamma,\nu,e}}\right) s_{\gamma,\nu,e} = X_p\left(\frac{n_B}{s_{\gamma,\nu}}\right) s_{\gamma,\nu}, \qquad X_p = \frac{n_p}{n_B} \tag{39}$$

where n_B is the number density of baryon, and the second equality is obtained by considering $e^{\pm}$ entropy density is negligible compared to the photon and neutrino entropy density at post BBN temperature $20 < T < 50$keV.

- The proton fraction X_p can be written as follow:

$$X_p = \frac{n_p}{n_B} = \frac{n_p}{n_p + n_n} = \frac{1}{1 + n_n/n_p} \tag{40}$$

  Since all neutrons end up bound in to the ^4H_e after BBN, then the mass fraction of ^4H_e can be estimated as

$$X_\alpha = \frac{2(n_n/n_p)}{1 + n_n/n_p} = 0.245 \pm 0.03 \tag{41}$$

  where we use $X_\alpha = 0.245 \pm 0.03$ from particle data group [1]. Solving the ratio n_n/n_p and substituting into the X_p, we obtain

$$X_p = 0.878 \pm 0.015 \tag{42}$$

- Since the comoving baryon number and entropy are conserved, hence the ratio $s_{\gamma,\nu}/n_B$ is conserved, then the entropy per baryon ratio can be written as

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right) = \left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \left(\frac{n_\gamma}{n_B}\right)_{t_0}\left(\frac{s_\gamma}{n_\gamma} + \frac{n_\nu}{n_\gamma}\frac{s_\nu}{n_\nu}\right) = \left(\frac{n_\gamma}{n_B}\right)_{t_0}\left[3.601 + \frac{9}{4}\left(\frac{T_\nu}{T_\gamma}\right)^3 4.202\right], \tag{43}$$

  where the subscript t_0 denotes the present day value and consider all neutrinos are relativistic particles today. The entropy per particle for a boson is $(s/n)_{\text{boson}} = 3.601$ and for a fermion is $(s/n)_{\text{fermion}} = 4.202$. From particle data group and standard big bang model [1,2], we have

$$5.8 \times 10^{-10} < \frac{n_B}{n_\gamma} < 6.5 \times 10^{-10}, \quad \frac{n_B}{n_\gamma} = 6.14 \times 10^{10}(\text{BBN observation}), \quad \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}. \tag{44}$$

  In this case, the entropy per baryon ratio becomes

$$\left(\frac{s_{\gamma,\nu}}{n_B}\right)_{t_0} = \frac{1}{6.14 \times 10^{-10}}\left[3.601 + \frac{9}{4}\left(\frac{4}{11}\right) 4.202\right] = 1.14 \times 10^{10} \tag{45}$$

- On the other hand, the entropy density at temperature can be written as [2]

$$s = \frac{2\pi^2}{45} g_s T_\gamma^3, \qquad g_s = \sum_{i=boson} g_i\left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8}\sum_{i=fermion} g_i\left(\frac{T_i}{T_\gamma}\right)^3 \tag{46}$$

where g_s is the effective degree of freedom that contribute to the entropy density. In the temperature we are interested in $50 > T > 20$keV, the entropy density reads

$$s_{\gamma,\nu} = \frac{2\pi^2}{45} g_s T_\gamma^3, \qquad g_s = g_\gamma + \frac{7}{8} g_\nu \left(\frac{T_\nu}{T_\gamma}\right)^3 = 2 + \frac{7}{8} 2 \times 3 \left(\frac{4}{11}\right) = 3.91 \tag{47}$$

Substituting above numerical results into Eq.(39), the proton number density is given by

$$n_p = X_p \left(\frac{n_B}{s_{\gamma,\nu}}\right) s_{\gamma,\nu} = \frac{0.878 \times 3.91}{1.14 \times 10^{10}} \left(\frac{2\pi^2}{45}\right) T_\gamma^3 = 3.011 \times 10^{-10} \left(\frac{2\pi^2}{45}\right) T_\gamma^3 \tag{48}$$

The dimensionless magnetic constant under expansion in the Boltzmann factor is

$$b_0 = \frac{|e|B(\hbar c^2)}{(k_B T)^2} \tag{49}$$

Magnetic flux drops with $1/a(t)^(2)$ where a is the scale factor while temperature drops with $1/a(t)$making the above a constant. The current modern bounds on the relic magnetic field lies between

$$B_{t_0} = 10^{-20} \sim 10^{-12}\,\text{Tesla} \tag{50}$$

where the subscript t_0 denotes the present day value The current temperature of the universe is

$$T_{t_0} = 2.7\,\text{K} \tag{51}$$

Plugging in those values, b_0 takes the values

$$b_0^{\text{max}} = 5.5 \times 10^{-3}, \tag{52}$$
$$b_0^{\text{min}} = 1.1 \times 10^{-11}. \tag{53}$$

As b_0 is a constant of expansion, assuming the electron-proton plasma between the CMB and electron-positron annihilation did not greatly disturbed it, we can calculate the remnant values at the temperature $k_B T = 50$ keV with the expression

$$B^{\text{max}}(T = 50\text{keV}) = \frac{B_0^{\text{max}}(k_B T)^2}{(|q|\hbar c^2)} = 2.3 \times 10^5\text{Tesla}, \tag{54}$$
$$B^{\text{min}}(T = 50\text{keV}) = 4.6 \times 10^{-4}\text{Tesla} \tag{55}$$

57 *4.5. Example: g-factor $g = 2$*

In our calculation we introduce an effective mass term which incorporates the ground state component of the Landau energy as follow

$$x_\pm = \frac{\tilde{m}_\pm}{T} = \frac{1}{T}\sqrt{m_e^2 + eB\left(1 \pm \frac{g}{2}\right)} \tag{56}$$

58 Considering the case $g = 2$ we have following two cases:

- Case1: $\tilde{m}_+ = \sqrt{m_e^2 + 2eB}$, and $x = \tilde{m}_+/T$. The equations for chemical potential and magnetization are given by

$$\sinh(\mu_e/T) = \frac{(2\pi)^2 n_p}{2T^3} \frac{1}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6} K_0(x)\right]} \tag{57}$$

and

$$\left(\frac{M}{B}\right) = -\frac{8\pi\alpha}{(2\pi)^2}\left(\frac{b_0}{6x}K_1(x_\pm) + \frac{2}{3}K_0(x_\pm)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x_\pm^2 K_2(x_\pm) + b_0 x_\pm K_1(x_\pm) + \frac{b_0^2}{6}K_0(x_\pm)\right]}\right]^2} \tag{58}$$

Substituting the magnetic field b_0 and proton density n_p/T^3 from pervious section, we can solve the magnetization and chemical potential numerically.

In Fig 2, we plot the chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature T. It shows for the case1 the chemical potential and magnetization are not sensitive to the magnetic field, this is because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq.(57) and Eq.(58).

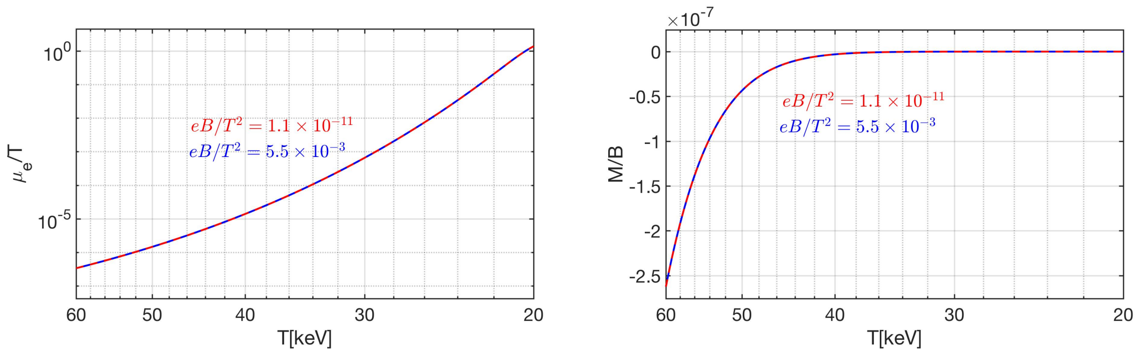

Figure 2. The chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature $20 < T < 60$keV for the case 1. The chemical potential and magnetization are not sensitive to the magnetic field, because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ can be neglected in Eq.(57) and Eq.(58).

- Case 2: $\tilde{m}_- = m_e$ and $x = \tilde{m}_-/T$,then the equations for chemical potential and magnetization can be written as

$$\sinh(\mu_e/T) = \frac{(2\pi)^2 n_p}{2T^3} \frac{1}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6} K_0(x)\right]} \tag{59}$$

and

$$\left(\frac{M}{B}\right) = \frac{8\pi\alpha}{(2\pi)^2}\left(\frac{1}{b_0}xK_1(x) + \frac{1}{3}K_0(x)\right)\sqrt{1 + \left[\frac{(2\pi)^2 n_p/(2T^3)}{\left[x^2 K_2(x) + b_0 x K_1(x) + \frac{b_0^2}{6}K_0(x)\right]}\right]^2} \tag{60}$$

Using the magnetic field b_0 and proton density n_p/T^3 from pervious section, we can solve the magnetization and chemical potential for case 2 numerically.

In Fig 3, we plot the chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature T. It shows that the chemical potential is not sensitive to the magnetic field because the small value of $b_0 = 10^{-3} \sim 10^{-8}$ and can be neglected in Eq.(59). However, the magnetization does depend on the magnetic field b_0 strongly. This is because for small magnetic field b_0 the dominant term in Eq(60) is $xK_1(x)/b_0$. For given b_0, the value of magnetization can be larger than the magnetic filed, i.e. $M/B > 1$ which shows the possibility that magnetic domains can be formed in early universe.

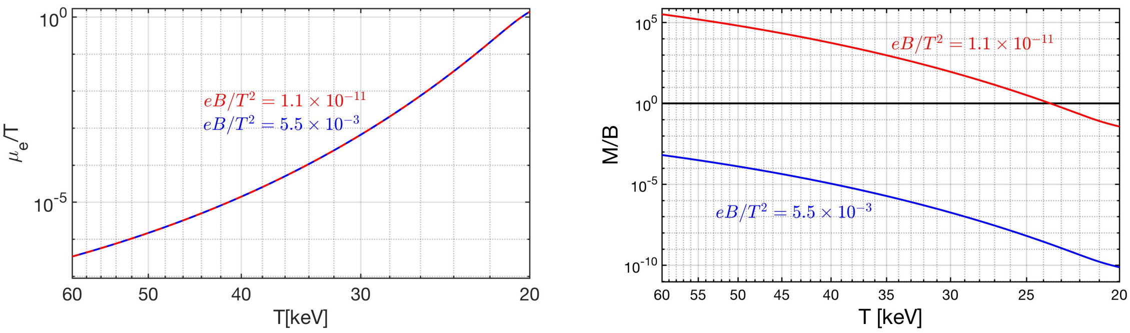

Figure 3. The chemical potential μ_e/T(left) and magnetization M/B(right) as a function of temperature $20 < T < 60$keV for the case 2. It shows that the chemical potential is not sensitive to the magnetic field b_0. However, the magnetization does depend on the magnetic field b_0 strongly. This is because for small magnetic field b_0 the dominant term in Eq(60) is $xK_1(x)/b_0$. For giving b_0 we can have $M/B > 1$ in early universe.

References

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