The partition function of $e^{\pm}$ plasma under magnetic field can be written as:

$$\ln Z = \frac{eBV}{2\pi^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \times \left[\ln\left(1+\lambda_e \lambda_s e^{-\beta E_n^+}\right) + \ln\left(1+\lambda_e \lambda_s^{-1} e^{-\beta E_n^-}\right) + \ln\left(1+\lambda_e^{-1} \lambda_s e^{-\beta E_n^+}\right) + \ln\left(1+\lambda_e^{-1} \lambda_s^{-1} e^{-\beta E_n^-}\right) \right]$$

Using the expansion of ln function. We have

$$\ln Z_0 = \frac{eBV}{2\pi^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp_z \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\left(\lambda_e^k \lambda_s^k e^{-k\beta E_n^+}\right) + \left(\lambda_e^k \lambda_s^{-k} e^{-k\beta E_n^-}\right) + \left(\lambda_e^{-k} \lambda_s^k e^{-k\beta E_n^+}\right) + \left(\lambda_e^{-k} \lambda_s^{-k} e^{-k\beta E_n^-}\right) \right]$$

It can be written as:

$$\ln Z = \frac{eBV}{2\pi^2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot$$

$$\times \Big[(\lambda_e^k + \lambda_e^{-k}) \cdot \lambda_s^k \cdot \int_0^\infty dP_Z \, e^{-k\beta E_n^+}$$

$$+ (\lambda_e^k + \lambda_e^{-k}) \, \lambda_s^{-k} \int_0^\infty dP_Z \, e^{-k\beta E_n^-} \Big]$$

Considering the Boltzmann approximation and Euler Maclaurin formula, we have

$$\ln Z = \frac{T^3 V}{2\pi^2} [\lambda_e + \lambda_e^{-1}] \times \sum_{j=\pm} \lambda_s^j \cdot \Big[x_j^2 K_2(x_j) + \frac{b_0}{2} x_j K_1(x_j) + \frac{b_0^2}{12} K_0(x_j) \Big]$$

Where $x_\pm = \frac{m_\pm}{T} = \sqrt{\left(\frac{m_c}{T}\right)^2 + b_0 \left(1 \pm \frac{g}{2}\right)}$

$b_0 = \frac{eB}{T^2}$

The Partition function becomes:

$$\ln Z = \frac{T^3 V}{2\pi^2}\left[\lambda_e + \lambda_e^{-1}\right] \cdot \sum_{j=\pm} \lambda_s^{\mp}\left[x_j^2 K_{2(x_j)} + \frac{b_0}{2} x_j K_{1}\right.$$
$$\left. + \frac{b_0^2}{12} K_0(x_j)\right]$$

Condition ①. Charge neutrality

$$(n_e - n_{\bar{e}}) = \lambda_e \frac{\partial \ln Z}{\partial \lambda_e}$$
$$= \frac{T^3}{2\pi^2}\left[\lambda_e - \lambda_e^{-1}\right] \sum_{j=\pm} \lambda_s^{\mp}\left[x_j^2 K_{2(x_j)} + \frac{b_0}{2} x_j K_{1(x_j)}\right.$$
$$\left. + \frac{b_0^2}{12} K_{0(x_j)}\right]$$
$$= n_p$$

Condition ② Magnetization:

$$M = \frac{T}{V}\frac{\partial \ln Z}{\partial B}$$
$$= \frac{eT^2}{2\pi^2}\left[\lambda_e + \lambda_e^{-1}\right] \cdot \sum_{j=\pm} \lambda_s^{\mp}\left[C_{1(x_j)} K_{1(x_j)} + C_0 K_{0(x_j)}\right]$$
$$= \delta H_c.$$

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We have

$$\left(\frac{M_p}{T^3}\right) = \frac{1}{2\pi^2}\left[\lambda_e - \lambda_e^{-1}\right]\left(\lambda_s A_{(x_+)} + \lambda_s^{-1} A_{(x_-)}\right)$$

where
$$A_{(x_\pm)} \equiv x_\pm^2 K_{2(x_\pm)} + \frac{b_0}{2} x_\pm K_{1(x_\pm)} + \frac{b_0^2}{12} K_{0(x_\pm)}$$

and

$$S = \frac{eT^2}{2\pi^2 H_c}\left[\lambda_e + \lambda_e^{-1}\right]\left(\lambda_s D_{(x_+)} + \lambda_s^{-1} D_{(x_-)}\right)$$

where
$$D_{(x_+)} = -\frac{1}{2} x_+ K_{1(x_+)} - \frac{b_0^2}{12 x_+^2} x_+ K_{1(x_+)} - \frac{b_0}{3} K_{0(x_+)}$$

$$D_{(x_-)} = \frac{1}{2} x_- K_{1(x_-)} + \frac{1}{6} b_0 K_{0(x_-)}.$$

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Considering the range for $10^{-3} > b_0 > 10^{-}$

we have

$$A_{(x+)} \simeq A_{(x-)} = A_* = x^2 K_2(x)$$

$$-D_{(x+)} \simeq D_{(x-)} = D_* = \frac{1}{2} x K_1(x)$$

and introduce the chemical potentials as follows

$$\lambda_s = e^{\eta_s/T}, \quad \lambda_e = e^{\eta_e/T}.$$

Then we obtain

$$\left(\frac{n_p}{T^3}\right) = \frac{1}{2\pi^2}\left[4\sinh\left(\frac{\eta_e}{T}\right)\cosh\left(\frac{\eta_s}{T}\right)\right] A_*$$

$$\delta = \frac{-eT^2}{2\pi^2 H_c}\left[4\cosh\left(\frac{\eta_e}{T}\right)\sinh\left(\frac{\eta_s}{T}\right)\right] D_*$$

$$= -\left(\frac{e^2 T^2}{2\pi^2 m_e^2}\right)\cdot\left[4\cosh\left(\frac{\eta_e}{T}\right)\sinh\left(\frac{\eta_s}{T}\right)\right]\cdot D_*$$

It can be written as.

$$\sinh\left(\frac{\eta_e}{T}\right)\cosh\left(\frac{\eta_s}{T}\right) = \frac{\pi^2 \cdot \left(\frac{n_p}{T^3}\right)}{2A_*}$$

$$\cosh\left(\frac{\eta_e}{T}\right)\sinh\left(\frac{\eta_s}{T}\right) = \frac{-\pi^2 \delta \cdot \left(\frac{m_e}{T}\right)^2}{2D_*(4\pi\alpha)}$$

Using the property of sinh function, we have

$$\sinh\left[\frac{\eta_e}{T} \pm \frac{\eta_s}{T}\right] = \sinh\left(\frac{\eta_e}{T}\right)\cosh\left(\frac{\eta_s}{T}\right) \pm \cosh\left(\frac{\eta_e}{T}\right)\sinh\left(\frac{\eta_s}{T}\right)$$

Then we obtain.

$$\sinh\left(\frac{\eta_e+\eta_s}{T}\right) = \left[\frac{\pi^2\left(\frac{n_p}{T^3}\right)}{2A_*} - \frac{\pi^2 \delta\left(\frac{m_e}{T}\right)^2}{2D_* 4\pi\alpha}\right]$$

$$\sinh\left(\frac{\eta_e-\eta_s}{T}\right) = \left[\frac{\pi^2\left(\frac{n_p}{T^3}\right)}{2A_*} + \frac{\pi^2 \delta\left(\frac{m_e}{T}\right)^2}{2D_* 4\pi\alpha}\right]$$