Equation of state for Electron Positron plasma

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(Dated: April 4, 2023)

I. PARTITION FUNCTION

Considering the partition function of $e^{\pm}$ plasma in Boltzmann approximation, we have

$$\ln \mathcal{Z}_{tot} = \frac{g_e}{(2\pi)^2} T^3 V \left[2\cosh\left(\mu_e(T)/T\right)\right] \left(\frac{m_e}{T}\right)^2 K_2(m_e/T). \tag{1}$$

where $g_e = 2$ is the spin degeneracy, K_2 is the 2nd Bessel function and $\mu_e(T)$ is the chemical potential of $e^{\pm}$ which is also a function of temperature.

- Chemical potential:
  Giving the partition function, we canuse charge neutrality to calculate the chemical potential as follow:

$$(n_e - n_{\bar{e}}) = \frac{T}{V}\frac{\partial}{\partial \mu_e} \ln \mathcal{Z}_{tot} = n_p \tag{2}$$

where n_p is the number density of proton. and the net number density of electron can be written as

$$(n_e - n_{\bar{e}}) = \frac{g_e T^3}{(2\pi)^2} \left[2\sinh\left(\mu_e/T\right)\right] \left(\frac{m_e}{T}\right)^2 K_2(m_e/T). \tag{3}$$

Then the charge neutrality becomes:

$$n_p = \frac{g_e T^3}{(2\pi)^2} \left[2\sinh\left(\mu_e/T\right)\right] \left(\frac{m_e}{T}\right)^2 K_2(m_e/T). \tag{4}$$

In Fig.(**??**) we plot the chemical potential μ_e/T and derivative of chemical potential $d\mu_e/dT$ of electron as a function of temperature $200 > T > 20$ keV.

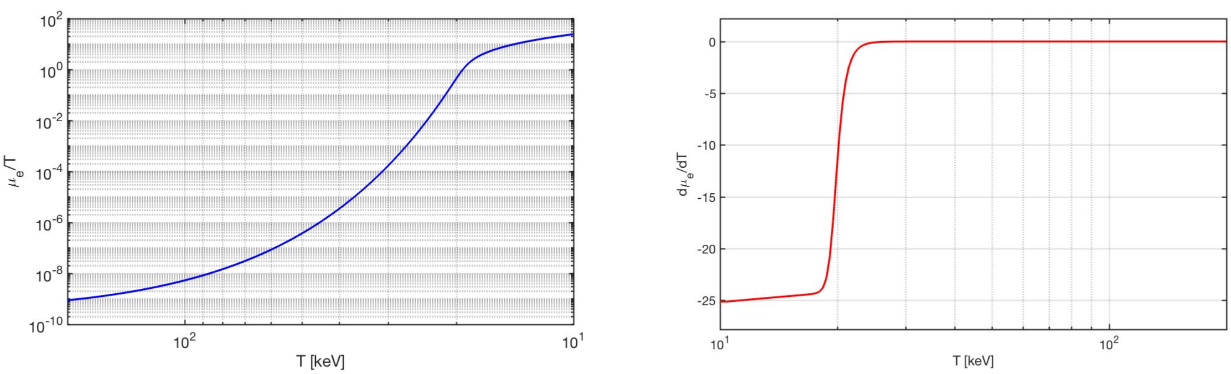

FIG. 1: On left: chemical potential μ_e/T of electron as a function of temperature $200 > T > 20$ keV. On right: the derivative of chemical potential with respect to temperature $d\mu_e/dT$ as a function of temperature.

- Pressure:
  By definition, the pressure of electron-positron is given by

$$P = \frac{1}{\beta}\frac{\partial \ln \mathcal{Z}_{tot}}{\partial V} = \frac{g_e}{2\pi^2} T^4 \left[2\cosh\left(\mu_e/T\right)\right] \left(\frac{m_e}{T}\right)^2 K_2(m_e/T), \tag{5}$$

where $\beta = 1/T$. In Fig.(**??**) we plot the pressure of electron-positron as a function of temperature(blue solid line) $200 > T > 20$ keV. For comparison we also show the pressure for the case $\mu_e = 0$ (blue dashed line).

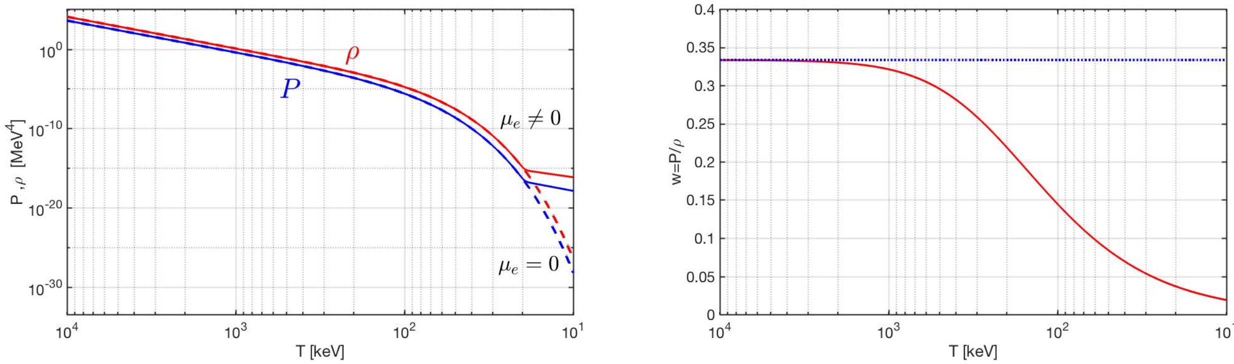

FIG. 2: On left: the energy density (red solid line) and pressure (blue solid line) as a function of temperature $10^4 > T > 10$ keV. For comparison, the dashed lines represent the case $\mu_e = 0$. On right: the equation of state $w = P/\rho$ as a function of temperature $10^4 > T > 10$ keV where the blue dotted line represent the $w = 1/3$ for relativistic limit.

- Energy Density:
  Giving the partition function, energy density can be obtained via the following definition:

$$\rho = \frac{E}{V} = \frac{1}{V}\left(-\frac{\partial \ln \mathcal{Z}_{tot}}{\partial \beta}\right) = \frac{-1}{V}\frac{\partial}{\partial \beta}\left[\frac{g_e}{2\pi^2} V\left(\lambda + \lambda^{-1}\right)\frac{(\beta m_e)^2}{\beta^3} K_2(\beta m_e)\right] \tag{6}$$

It can be written as

$$\rho = -\frac{g_e}{2\pi^2}\left(\lambda + \lambda^{-1}\right)\frac{\partial}{\partial \beta}\left[\frac{(\beta m_e)^2}{\beta^3} K_2(\beta m_e)\right] \tag{7}$$

$$= \frac{g_e}{2\pi^2}\left(\lambda + \lambda^{-1}\right)\left(\frac{m_e}{\beta}\right)^2\left[(\beta m_e) K_1(\beta m_e) + 3K_2(\beta m_e)\right] \tag{8}$$

$$= \frac{g_e}{2\pi^2} T^4 \left[2\cosh\left(\mu_e/T\right)\right]\left(\frac{m_e}{T}\right)^2 K_2(m_e/T)\left[3 + \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)}\right] \tag{9}$$

In Fig.(**??**) we plot the energy density of electron-positron as a function of temperature(red solid line) $10^4 > T > 10$ keV, we also plot the energy denisty for the case $\mu_e = 0$ (red dashed line).

Substituting the Eq.(**??**) we obtain

$$\rho = P\left[3 + \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)}\right] \tag{10}$$

then the equation of state w is given by

$$w = \frac{P}{\rho} = \frac{1}{3 + \left(\frac{m_e}{T}\right)\frac{K_1(m_e/T)}{K_2(m_e/T)}} \tag{11}$$

In Fig.(**??**) on the right, we show the equation of state $w = P/\rho$ as a function of temperature(red solid line) and for the blue dotted line represent the $w = 1/3$ for relativistic limit.