Short note on spin magnetization in primordial QGP]Short note on spin magnetization in primordial quark-gluon plasma

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Key words. quark-gluon plasma, magnetism, magnetic moment, primordial Universe

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Abstract. We outline the theory of spin magnetization of the primordial quark-gluon plasma (QGP) that existed shortly after the Big Bang, focusing on a magnetized fermion gas of light quarks and electrons within temperatures of 150 MeV to 500 MeV. Using a grand partition function approach and evaluating magnetized Fermi-Dirac integrals, we calculate the magnetization and estimate that a fully polarized up quark gas could generate cosmic magnetic fields of approximately 1.4×10^{14} Tesla. This field strength is significantly larger than the critical Schwinger field and exceeds magnetar surface fields by over three orders of magnitude. This suggest that even a weakly polarized quark gas would have a profound impact on the early Universe.

1 Introduction

Quark-gluon plasma (QGP) is a state of matter that existed microseconds after the Big Bang, where quarks and gluons were not confined within hadrons but existed as a free deconfined plasma [?,?,?,?,?,?]. Understanding the properties of QGP is crucial for determining the conditions of the primordial Universe and the formation of matter as we observe it today [?,?,?,?,?].

Table 1. Properties of selected particles

Particle	**Mass** [$\approx$ MeV]	**Magneton** [μ_f/μ_B]
Up (u)	2.2	1.55×10^{-1}
Down (d)	4.7	3.62×10^{-2}
Strange (s)	96	1.77×10^{-3}
Electron (e)	0.511	1

If the incredibly dense gas of up (u) quarks (the most magnetically relevant particle in QGP) was fully polarized during this era at a temperature of 150 MeV shortly prior to hadronization, this would correspond to an estimated cosmic magnetic strength $M = \mathcal{M}/V$ of

$$M = \frac{\mathcal{M}}{V} = \mu_u n_u \sim 1.4 \times 10^{14}\ \mathrm{T} \tag{1}$$

where $\mathcal{M}$ is the magnetization, μ_u is the up quark magneton, and n_u is the up quark number density. This can also be understood as $\approx 3.2 \times 10^4$ the critical magnetic field strength $\mathcal{B} \equiv m_e^2/e$ and over 10^3 stronger than the upper estimated surface field strengths of magnetars.

In [?,?], we explored the magnetic properties of the magnetized electron-positron plasma around the period of Big Bang Nucleosynthesis. We expand those efforts to now consider the magnetization of QGP where both density of magnetic particles and the external fields are greatly increased as we consider temperatures upward of 500 MeV. We aim to understand how magnetic dipole moments impact the chemical potentials of quarks and evaluate the overall magnetization of the primordial Universe QGP.

In this work, we explore the thermodynamic properties of a magnetized QGP in the primordial Universe, focusing on the interplay between quarks, leptons, and magnetic fields. The presence of strong magnetic fields in the primordial Universe could have significantly affected the equilibrium properties of Standard Model particles in the earliest moments after the Big Bang [?,?]. Such magnetic fields have long been thought to be connected to baryon asymmetry [?,?]. The connection between magnetism QGP chirality has been studied in [?,?,?].

For relativistic species [?], under conditions of thermal and chemical equilibrium—as was the case in the primordial Universe—the chemical potential of each particle is opposite in sign to that of its antiparticle

$$\eta_q = T \ln \lambda_q\,, \qquad \lambda_q = 1/\lambda_{\bar{q}}\,, \qquad \eta_q = -\eta_{\bar{q}}\,. \tag{2}$$

During this period, particle-antiparticle pairs of quarks and antiquarks were freely produced and annihilated through photon-mediated processes, represented by $q+\bar{q} \rightleftharpoons 2\gamma$. Since the Universe expanded very slowly compared to collision reaction times during the QGP epoch [?,?], the expansion can be considered adiabatic, conserving entropy. Therefore, the light-quark gas we consider is in full thermal equilibrium.

2 Magnetization of a polarized gas

We consider a free but magnetized fermion gas in the temperature range 500 MeV $> T >$ 150 MeV composed of light-quark species $q \in u, d, s$. As the quark magneton scales with $\mu_q \sim Q_q/2m_q$, these species are magnetically the most relevant due to their lighter masses m_q and consequently larger magnetic moments.

Here, Q_q denotes the electric charges of the light quarks, taking values $Q_q \in \pm\frac{1}{3}, \pm\frac{2}{3}$. Therefore in accounting for the internal energy E of QGP, the energies of the electrons and neutrinos must also be included to fully determine the QGP state. We take into account the following properties:

[nosep]The energy of adding or removing a baryon $\eta_B B$, the energy of adding or removing a lepton $\eta_\ell(N_\ell - N_\ell)$ with $\ell \in e, \nu$, the magnetic energy $\mathcal{M}\mathcal{B}$ where $\mathcal{M}$ is the net magnetization and $\mathcal{B}$ is the magnetic field strength and the electromagnetic energy density generated by the external magnetic field.

Magnetic field strength $\mathcal{B}$ should not be confused with baryon number B. Additionally, the internal energy and chemical potential of QGP is sensitive to electric fields which cause a statistical Coulomb distortion [?,?]. For the purpose of clarity and to maintain focus on magnetism, electrical fields will be omitted.

The dependency of E on $\mathcal{M}$ reflects that $\mathcal{B}$ is the incremental energy cost to change the magnetization by flipping the spin of a particle [?]. Therefore, this makes magnetization $\mathcal{M}$ an extensive property of the system which changes which particle number. We see this explicitly by writing the magnetization as the sum over all particles $i \in 1, \ldots, k$

$$\mathcal{M} = \sum_{i=1}^{k} (\mu_i N_i^{\uparrow} + \mu_{\bar{i}} N_{\bar{i}}^{\uparrow} - \mu_i N_i^{\downarrow} - \mu_{\bar{i}} N_{\bar{i}}^{\downarrow}) , \qquad N_i = N_i^{\uparrow} + N_i^{\downarrow} , \tag{3}$$

where μ_i is the magnetic dipole moment per particle $\mu_i \propto Q_i/m_i$. The $\uparrow\downarrow$ notation refers to spin-up ($\uparrow$) and spin-down ($\downarrow$) states along the direction of the external field. Therefore, $N_i^{\uparrow\downarrow}$ refers to the i-th constituent population number in either spin-up or spin-down orientation. The signs of each term in Eq. (**??**) arises from the sign of the spin eigenvalue. While Eq. (**??**) presumably includes contributions from each particle with a magnetic dipole, we expect the magnetization to be dominated by electrons, positrons, and the lightest quarks due to their charge and low mass therefore we sum over $i \in u, d, s, e$

$$\begin{aligned}
\mathcal{M} \approx &+ |\mu_u|(N_u^{\uparrow} - N_{\bar{u}}^{\uparrow}) - |\mu_u|(N_u^{\downarrow} - N_{\bar{u}}^{\downarrow}) \\
&- |\mu_d|(N_d^{\uparrow} - N_{\bar{d}}^{\uparrow}) + |\mu_d|(N_d^{\downarrow} - N_{\bar{d}}^{\downarrow}) \\
&+ |\mu_s|(N_s^{\uparrow} - N_{\bar{s}}^{\uparrow}) - |\mu_s|(N_s^{\downarrow} - N_{\bar{s}}^{\downarrow}) \\
&- |\mu_e|(N_e^{\uparrow} - N_{\bar{e}}^{\uparrow}) + |\mu_e|(N_e^{\downarrow} - N_{\bar{e}}^{\downarrow}) .
\end{aligned} \tag{4}$$

The magnetic dipole of a particle is opposite in sign to its antiparticle $\mu_i = -\mu_{\bar{i}}$ as charge is flipped. Any deviation from this condition would represent a violation of CPT [?,?,?]. The magnetic moments of the relevant magnetic species are

$$\begin{aligned}
\mu_u &= +\frac{2}{3}\left|\frac{g_u}{2}\right|\frac{e\hbar}{2m_u} , & \mu_d &= -\frac{1}{3}\left|\frac{g_d}{2}\right|\frac{e\hbar}{2m_d} , \\
\mu_s &= +\frac{2}{3}\left|\frac{g_s}{2}\right|\frac{e\hbar}{2m_s} , & \mu_e &= -1\left|\frac{g_e}{2}\right|\frac{e\hbar}{2m_e} ,
\end{aligned} \tag{5}$$

where $e = -Q_e$ is the value of elementary charge and g_i the g-factor of the particle.

We recognize that Eq. (**??**) contains terms representing asymmetry in the spin alignment though we can organize them in two different ways: (a) We group terms of the same spin alignment or (b) we group terms of matter and antimatter. The second approach may allow definition of spin-asymmetry in terms of conserved quantities. Therefore, we define net spin-asymmetry numbers $\delta_i^{\uparrow\downarrow}$ and write

$$\delta_i^{\uparrow\downarrow} \equiv N_i^{\uparrow\downarrow} - N_{\bar{i}}^{\uparrow\downarrow} , \tag{6}$$

$$\mathcal{M} = +|\mu_u|(\delta_u^{\uparrow} - \delta_u^{\downarrow}) - |\mu_d|(\delta_d^{\uparrow} - \delta_d^{\downarrow}) + |\mu_s|(\delta_s^{\uparrow} - \delta_s^{\downarrow}) - |\mu_e|(\delta_e^{\uparrow} - \delta_e^{\downarrow}) . \tag{7}$$

The net spin-asymmetry warrants some discussion: It is the asymmetry of particles and antiparticles of the same spin. Therefore $\delta_u^{\uparrow} \neq 0$ represents a situation where there are more up quarks than up antiquarks in the spin-up $\uparrow$ state.

3 Magnetized grand partition function

The partition function allows us to calculate various thermodynamic quantities found in Eq. (**??**) by taking appropriate derivatives of $\mathcal{F}$. In the temperature range considered (500 MeV $> T >$ 150 MeV), the lightest quarks act as essentially massless particles with only the strange quark requiring significant mass corrections. It is worth remarking on the uniqueness of the situation: As magnetic moment scales inverse with mass, it is the particles which are most massless in character which contribute most to magnetization.

The relevant contributions to the magnetized primordial plasma arise from the quarks, gluons, leptons, and the vacuum. The grand potential Eq. (**??**) can be recast in terms of the grand partition function $\ln \mathcal{Z}$

$$\mathcal{F} = -T \ln \mathcal{Z} \,, \tag{8}$$

$$\ln \mathcal{Z}_{\text{total}} = \ln \mathcal{Z}_{\text{quarks}} + \ln \mathcal{Z}_{\text{gluons}} + \ln \mathcal{Z}_{\text{vac.}} + \ln \mathcal{Z}_{\text{leptons}} + \ldots \tag{9}$$

We consider a homogeneous magnetic field domain defined along the z-axis with magnetic field magnitude $\mathcal{B}$. The volume $V = L^3$ is not necessarily infinite and is to be considered the size of the homogeneous domain such that $\nabla \cdot \mathcal{B} \approx 0$. For a fermion species f of charge Q, mass m, and g-factor g, the energy eigenvalues of the magnetized particles is given by [**?**]

$$E(p_z, n, s) = \sqrt{m^2 + p_z^2 + 2|Q|\mathcal{B}\left(n + \frac{1}{2} - \frac{g}{2}s\right)} \,, \tag{10}$$

where E are the relativistic Landau energy eigenvalues. The micro-state energies depend on spin $s \in \pm 1/2$ and orbital Landau $n \in 0, 1, 2, 3, \ldots$ quantum number.

4 Magnetized Fermi-Dirac integrals

The power and utility of the partition function in statistical systems is found by examining the Fermi integral in various limits and expansions. We define the Fermi-Dirac distribution in the usual way noting that fugacity λ is related to chemical potential via $\eta = T \ln \lambda$. Thus,

$$F\left(E - \sigma\eta\right) = \frac{1}{e^{(E-\sigma\eta)/T} + 1} \,. \tag{11}$$

We can further simplify Eq. (**??**) by rewriting the partition function in spherical coordinates $d\mathbf{p}^3 = 4\pi p^2 dp$. We substitute coordinates and integrate by parts yielding

$$\ln \mathcal{Z} = \frac{2N_{\text{dof}}V}{(2\pi)^2} \sum_s^{\pm 1/2} \sum_\sigma^{\pm 1} \int_0^\infty \frac{dp}{3T} \frac{p^4}{E} F\left(E - \sigma\eta\right) \,. \tag{12}$$

The form of the partition function expressed by Eq. (**??**) more directly lets us evaluate thermodynamic quantities in terms of Fermi integrals. However, integrating over momentum is not an ideal description as relativistic expansions in momentum yield series that are only semi-convergent; see Sect. **??**.

4.1 Dimensionless change of variables

To simplify the integration process, we introduce dimensionless variables by normalizing relevant physical quantities with the temperature T. This approach renders the equations dimensionless and highlights the thermal contributions explicitly. The dimensionless variables are defined as

$$p_T = \frac{p}{T}, \qquad E_T(p_T, s) = \frac{E(p,s)}{T}, \qquad \eta_T = \frac{\eta}{T}\,, \qquad m_T(s) = \frac{m(s)}{T} \,. \tag{13}$$

This defines momentum-like p_T, energy-like E_T, potential-like η_T and mass-like m_T parameters. Using the relativistic dispersion relation, the dimensionless energy E_T can be expressed in terms of the dimensionless momentum p_T and the dimensionless mass m_T

$$E_T = \frac{E}{T} = \sqrt{p_T^2 + m_T^2} \,. \tag{14}$$

The differential dp_T and dE_T transform as

$$dp = T\, dp_T \,, \qquad p_T dp_T = E_T dE_T \,, \tag{15}$$

and the limits of integration change accordingly

$$p_T = 0 \quad \Rightarrow \quad E_T = m_T, \quad p_T \to \infty \quad \Rightarrow \quad E_T \to \infty. \tag{16}$$

Substituting these dimensionless variables and differentials into the partition function $\ln \mathcal{Z}$, we obtain expressions for both momentum-like p_T integration and energy-like E_T integration

$$\ln \mathcal{Z} = \frac{2N_{\mathrm{dof}}V}{(2\pi)^2}\frac{T^3}{3}\sum_s^{\pm 1/2}\sum_\sigma^{\pm 1}\int_0^\infty dp_T\,\frac{p_T^4}{\sqrt{p_T^2+m_T^2}}\,F\left(\sqrt{p_T^2+m_T^2}-\sigma\eta_T\right), \tag{17}$$

$$= \frac{2N_{\mathrm{dof}}V}{(2\pi)^2}\frac{T^3}{3}\sum_s^{\pm 1/2}\sum_\sigma^{\pm 1}\int_{m_T}^\infty dE_T\,(E_T^2-m_T^2)^{3/2}F\left(E_T-\sigma\eta_T\right). \tag{18}$$

In this dimensionless formulation, it is evident that the logarithm of the partition function scales as $\ln \mathcal{Z} \propto T^3$, consistent with the expected thermodynamic behavior for a relativistic gas in three spatial dimensions.

4.2 Evaluation of magnetization from the dimensionless partition function

Here we evaluate the magnetization using a different approach to Sect. **??** which provides greater clarity especially in high temperature systems. Given the dimensionless form of the partition function Eq. (**??**) from Sect. **??**, we proceed to evaluate the magnetization $\mathcal{M}$, acknowledging that the dimensionless mass m_T depends on the magnetic field $\mathcal{B}$ via Eq. (**??**). The magnetization is defined in Eq. (**??**).

Since $\ln \mathcal{Z}$ depends on $\mathcal{B}$ solely through m_T, we apply the chain rule

$$\frac{\partial \ln \mathcal{Z}}{\partial \mathcal{B}} = \frac{\partial \ln \mathcal{Z}}{\partial m_T}\frac{\partial m_T}{\partial \mathcal{B}}, \qquad 2m_T\frac{\partial m_T}{\partial \mathcal{B}} = -\frac{g|Q|}{T^2}s. \tag{19}$$

Taking the derivative with respect to m_T, we write

$$\frac{\partial \ln \mathcal{Z}}{\partial m_T} = \frac{2N_{\mathrm{dof}}VT^3}{3(2\pi)^2}\sum_s^{\pm 1/2}\sum_\sigma^{\pm 1}\int_0^\infty dp_T\,\frac{\partial}{\partial m_T}\left(\frac{p_T^4}{\sqrt{p_T^2+m_T^2}}F\left(\sqrt{p_T^2+m_T^2}-\sigma\eta_T\right)\right). \tag{20}$$

Given that $E_T = \sqrt{p_T^2+m_T^2}$, then, $\partial E_T/\partial m_T = m_T/E_T$. The derivative of the integrand is computed using the product and chain rules

$$\frac{\partial}{\partial m_T}\left(\frac{p_T^4}{E_T}F(E_T-\sigma\eta_T)\right) = \frac{p_T^4}{E_T}\frac{\partial F}{\partial E_T}\frac{\partial E_T}{\partial m_T} + F(E_T-\sigma\eta_T)\frac{\partial}{\partial m_T}\left(\frac{p_T^4}{E_T}\right). \tag{21}$$

The second term involves

$$\frac{\partial}{\partial m_T}\left(\frac{p_T^4}{E_T}\right) = -\frac{p_T^4 m_T}{E_T^3}. \tag{22}$$

Substituting these results back, the integrand becomes

$$\frac{\partial}{\partial m_T}\left(\frac{p_T^4}{E_T}F(E_T-\sigma\eta_T)\right) = \frac{p_T^4 m_T}{E_T^2}F'(E_T-\sigma\eta_T) - \frac{p_T^4 m_T}{E_T^3}F(E_T-\sigma\eta_T). \tag{23}$$

Replacing E_T with $\sqrt{y^2+m_T^2}$, the derivative of $\ln \mathcal{Z}$ is

$$\frac{\partial \ln \mathcal{Z}}{\partial m_T} = \frac{2N_{\mathrm{dof}}VT^3}{3(2\pi)^2}\sum_s^{\pm 1/2}\sum_\sigma^{\pm 1}\int_0^\infty dp_T\,p_T^4 m_T \times\left(\frac{F'\left(\sqrt{p_T^2+m_T^2}-\sigma\eta_T\right)}{p_T^2+m_T^2} - \frac{F\left(\sqrt{p_T^2+m_T^2}-\sigma\eta_T\right)}{(p_T^2+m_T^2)^{3/2}}\right). \tag{24}$$

This result provides the explicit form of $\partial \ln \mathcal{Z}/\partial m_T$ in terms of F and its derivative F', with all dependencies on m_T and p_T made explicit.

Given that $F(x) = \frac{1}{e^x+1}$ is the Fermi-Dirac distribution, its derivative is

$$F'(x) = \frac{dF}{dx} = -\frac{e^x}{(e^x+1)^2} = -F(x)\left[1-F(x)\right]. \tag{25}$$

Substituting $F'(x)$ into the expression for the derivative of the integrand

$$\int_0^\infty dp_T\, p_T^4 m_T \left(-\frac{F(E_T-\sigma\eta_T)\left[1-F(E_T-\sigma\eta_T)\right]}{E_T^2} - \frac{F(E_T-\sigma\eta_T)}{E_T^3}\right) =$$
$$\int_{m_T}^\infty dE_T\, \left(E_T^2-m_T^2\right)^{3/2} m_T \left(-\frac{F(E_T-\sigma\eta_T)\left[1-F(E_T-\sigma\eta_T)\right]}{E_T} - \frac{F(E_T-\sigma\eta_T)}{E_T^2}\right). \tag{26}$$

Substituting the transformed integral into the derivative of $\ln\mathcal{Z}$, we obtain:

$$\frac{\partial\ln\mathcal{Z}}{\partial m_T} = -\frac{2N_{\rm dof}VT^3}{3(2\pi)^2}\sum_s^{\pm1/2}\sum_\sigma^{\pm1}\int_{m_T}^\infty dE_T\, m_T\left(E_T^2-m_T^2\right)^{3/2} \tag{27}$$
$$\times\left[\frac{F(E_T-\sigma\eta_T)\left(1-F(E_T-\sigma\eta_T)\right)}{E_T}+\frac{F(E_T-\sigma\eta_T)}{E_T^2}\right]. \tag{28}$$

Substituting the expression for $\partial\ln\mathcal{Z}/\partial m_T$ into Eq. (**??**) and Eq. (**??**), we obtain the magnetization $\mathcal{M}$

$$\mathcal{M} = -\frac{g|Q|}{2T}\cdot\frac{2N_{\rm dof}VT^3}{3(2\pi)^2}\sum_s^{\pm1/2}s\sum_\sigma^{\pm1}\int_{m_T(s)}^\infty dE_T\,\left(E_T^2-m_T^2(s)\right)^{3/2} \tag{29}$$
$$\times\left[\frac{F(E_T-\sigma\eta_T)\left(1-F(E_T-\sigma\eta_T)\right)}{E_T}+\frac{F(E_T-\sigma\eta_T)}{E_T^2}\right]. \tag{30}$$

This expression is equivalent to the lowest order term in Eq. (**??**). Much how we expected the free energy to be $\ln\mathcal{Z}\sim T^3$, we see the magnetization is $\mathcal{M}\sim T^2$ via dimensional analysis. This is in agreement to our prior work [**?**,**?**,**?**] where we evaluated the magnetization in the Boltzmann limit. The benefit of expressing the magnetization in the form of Eq. (**??**) is that the integrand within the brackets [...] entirely contains the Fermi-Dirac distribution scaled by energy without mass (or magnetic fields) except as a boundary condition on the integration.