PHYS 321 - Theoretical Mechanics

Lecture 7

Instructor: Dr. Andrew James Steinmetz
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Course information

Instructor information:

- **Instructor:** Dr. Andrew Steinmetz
- **E-mail:** ajsteinmetz@arizona.edu
- **Office:** C315
- **Office Hours:** Monday/Tuesday/Wednesday at 1:00 PM to 2:00PM
- **Webpage:** http://d2l.arizona.edu

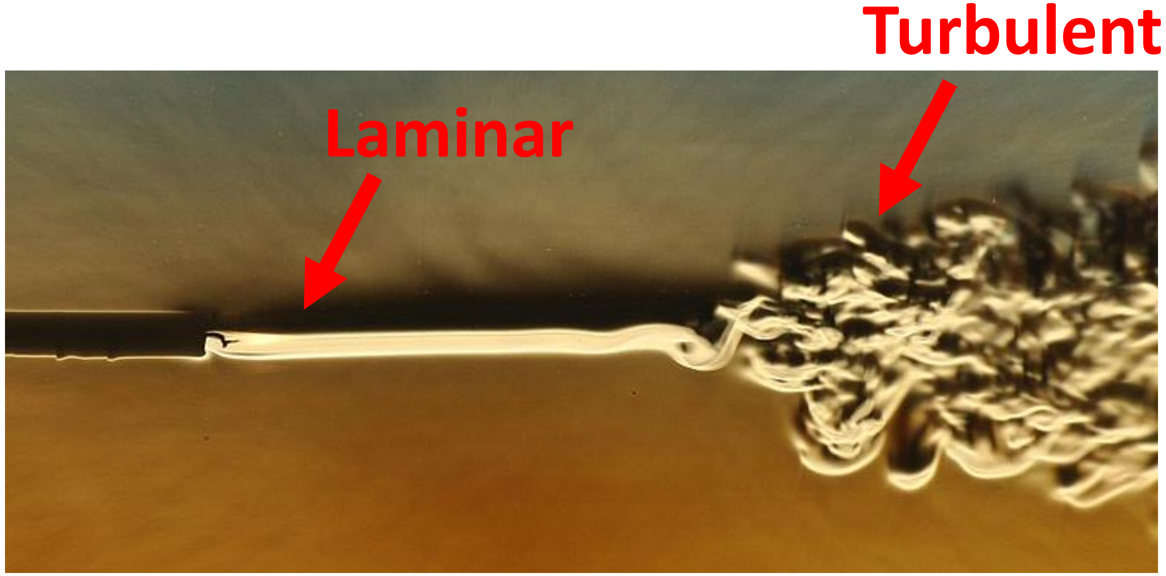
Lecture schedule:

- PHYS 321 – 501 AP
  - Mondays at 8:30 AM to 10:05 AM (CST/UTC+8) – Weeks 2-5, 7-14
  - Tuesdays at 10:25 AM to 12:00 PM (CST/UTC+8) – Weeks 2-5, 7-14
- PHYS 321 – 502 ME
  - Mondays at 3:55 PM to 5:30 PM (CST/UTC+8) – Weeks 2-5, 7-14
  - Wednesdays at 3:55 PM to 5:30 PM (CST/UTC+8) – Weeks 2-5, 7-14

A few more notes on second-order drag:

There are two kinds of fluid flow:

- Laminar (sheet-like) flow
- Turbulent (chaotic/eddy current) flow



A few more notes on second-order drag:

The drag coefficient depends on both the object geometry/shape and the Reynolds number of the fluid. The Re number is the "ratio" of inertial forces to viscous forces

$$Re = \frac{uL}{\nu} = \frac{\rho u L}{\mu}$$

Where u is the relative flow speed, L the characteristic size, $\mu \left[\frac{kg}{m \cdot s}\right]$the dynamic viscosity and $\nu \left[\frac{m^2}{s}\right]$ is the kinematic viscosity. A "low" Re# is Laminar (with 1st order drag) while a "high" Re# is turbulent (with 2nd order drag)

A few more notes on second-order drag:

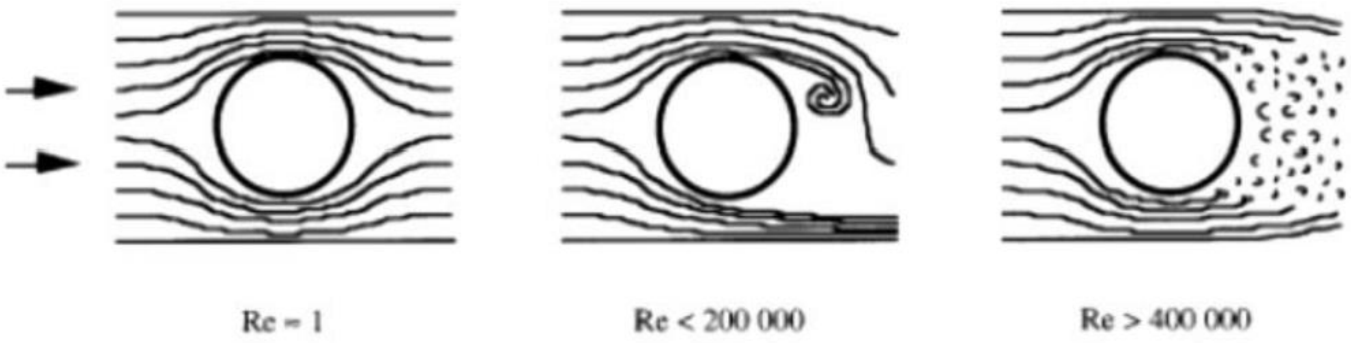
A falling object undergoing 1D second-order drag has the force:

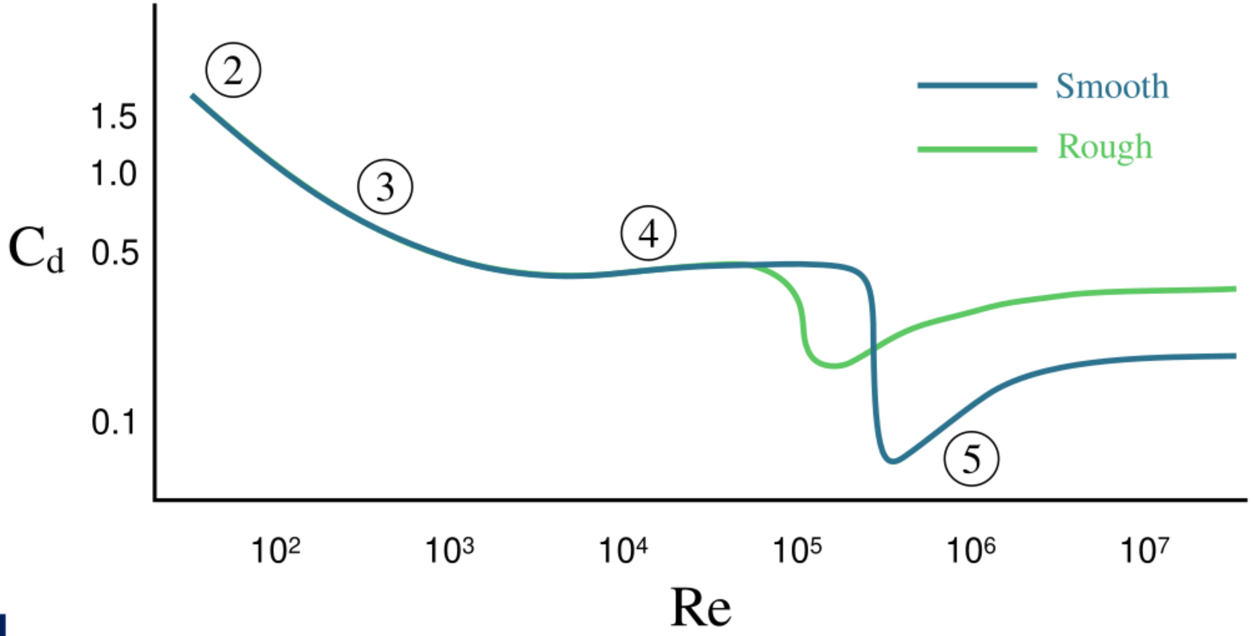
$$\sum F = ma = mg - b\dot{x}$$

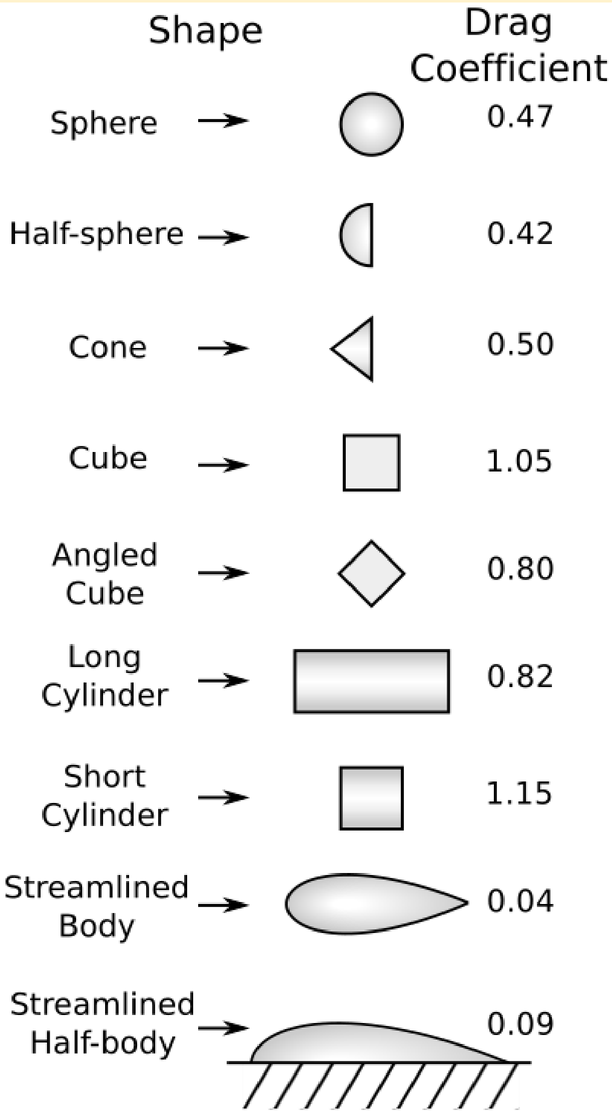
Physically the parameter b can be understood physically as: $b = \frac{1}{2}\rho A C_d$

A is the object cross-sectional area, ρ is the fluid density, and C_d is the drag coefficient which is both fluid and object dependent.

A few more notes on second-order drag:

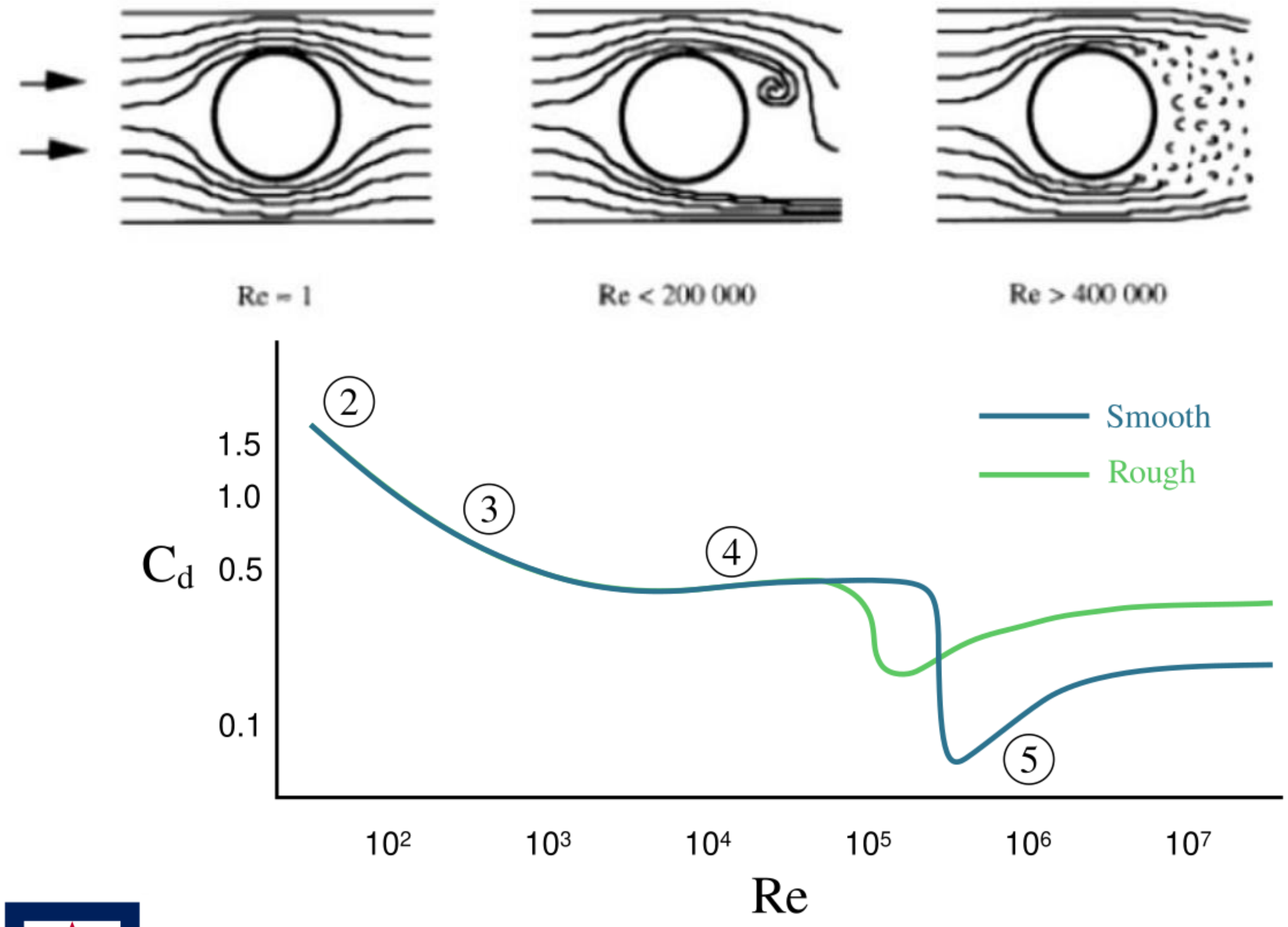






Measured Drag Coefficients

A few more notes on second-order drag:



Drag coefficient of a sphere at different Reynolds numbers:

2. Laminar/Stokes flow
3. Separated unsteady flow (Laminar flow with a turbulent boundary layer)
4. Separated unsteady flow (Laminar flow mixed with chaotic turbulent flow)
5. Chaotic turbulent flow

Damped harmonic oscillators

The damped harmonic oscillator includes the linear drag force $F_{drag} \propto v$

$$\sum F = ma \Rightarrow -kx - \dot{x} = m\ddot{x}$$

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0$$

Where $\omega \equiv \sqrt{\frac{k}{m}}$ and $\beta \equiv \frac{b}{2m}$ (both have units of frequency 1/seconds)

Let us guess a solution $x = e^{rt}$ just as the case where $\beta = 0$.

Damped harmonic oscillators

Let us guess a solution $x = e^{rt}$ just as the case where $\beta = 0$.

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0$$

$$r^2 e^{rt} + 2\beta r e^{rt} + \omega^2 e^{rt} = 0$$

$$r^2 + 2\beta r + \omega^2 = 0$$

$$r = \frac{1}{2}\left(-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}\right)$$

$$\boxed{r = -\beta \pm \sqrt{\beta^2 - \omega^2}}$$

We notice that unlike the simple case $(\beta = 0)$, the complex exponential is not pure imaginary anymore: It is complex with real and imaginary parts.

Damped harmonic oscillators

Case (1): $\beta < \omega$ (underdamped oscillator)

Here $r = -\beta \pm i\omega_1$ where $\omega_1 \equiv \sqrt{\omega^2 - \beta^2}$ (real valued). The solution then has two complex roots.

$$\therefore x(t) = c_1 e^{(-\beta + i\omega_1)t} + c_2 e^{(-\beta - i\omega_1)t} \text{ (general complex solution)}$$

$$= e^{-\beta t}\left(c_1 e^{+i\omega_1 t} + c_2 e^{-i\omega_1 t}\right) \text{ (c1 and c2 are still complex)}$$

Let us take the real part only $x(t) \rightarrow \mathrm{Re}[x(t)]$

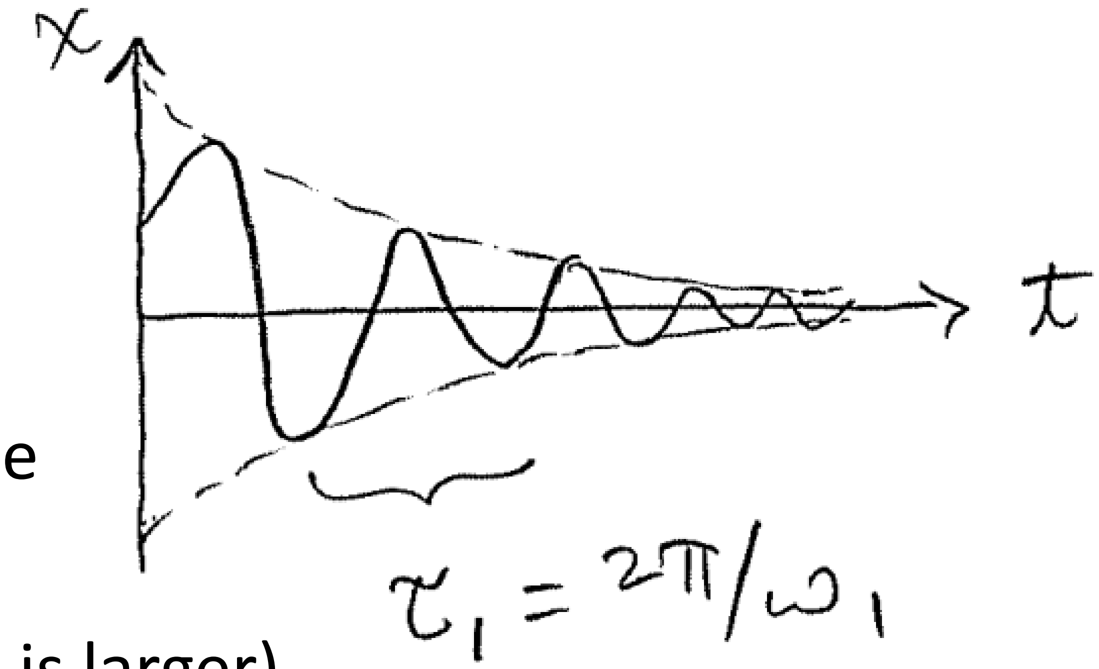
$$x(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t) \text{ (underdamped solution)}$$

Damped harmonic oscillators

Case (1): $\beta < \omega$ (underdamped oscillator)

$$x(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t) \text{ (underdamped solution)}$$

Properties:



(i) Oscillatory motion with a decaying exponential amplitude with time.

(ii) Decay rate depends on damping coefficient β: Large β = fast decay, while small β = slow decay.

(iii) Oscillation frequency is smaller (period is larger) than for the undamped case. $\omega_1 < \omega$

Damped harmonic oscillators

Case (2): $\beta > \omega$ (overdamped oscillator)

Here $r = -\beta \pm \omega_2$ where $\omega_2 \equiv \sqrt{\beta^2 - \omega^2}$ (real valued). The solution then has two real roots.

$$\therefore x(t) = c_1 e^{(-\beta+\omega_2)t} + c_2 e^{(-\beta-\omega_2)t} \text{ (general solution)}$$

$$= e^{-\beta t}\left(c_1 e^{+\omega_2 t} + c_2 e^{-\omega_2 t}\right) \text{ (c1 and c2 are still complex)}$$

Let us take the real part only $x(t) \rightarrow \mathrm{Re}[x(t)]$

$$x(t) = e^{-\beta t}\left(A e^{+\omega_2 t} + B e^{-\omega_2 t}\right) \text{ (overdamped solution)}$$

Damped harmonic oscillators

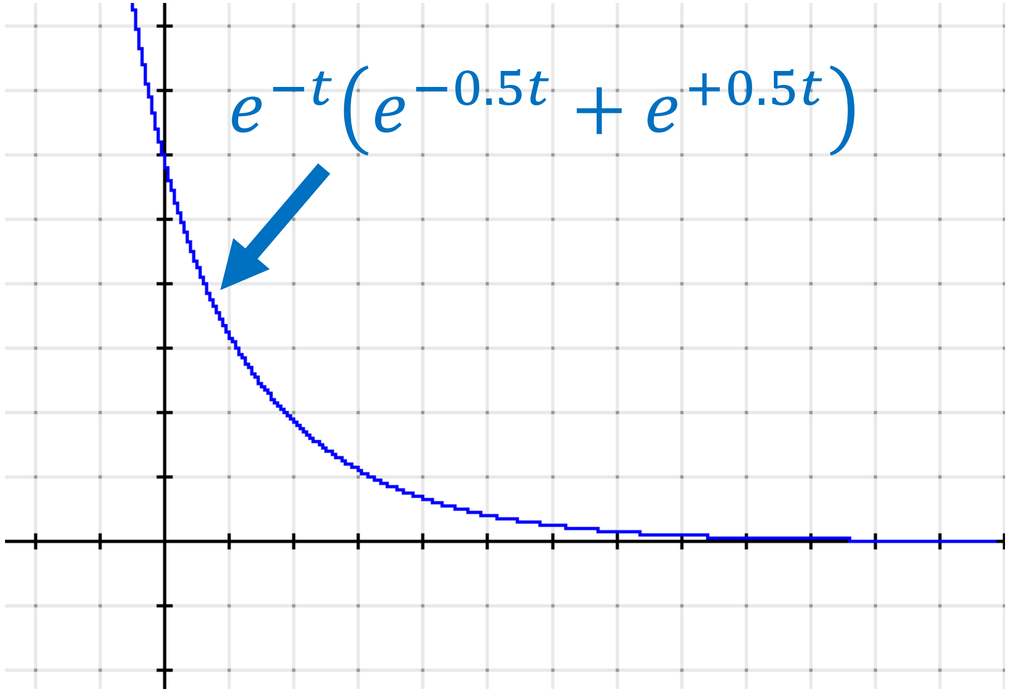
Case (2): $\beta > \omega$ (overdamped oscillator)

$$x(t) = e^{-\beta t}\left(Ae^{+\omega_2 t} + Be^{-\omega_2 t}\right) \text{ (overdamped solution)}$$

Properties:

(i) Motion is NOT oscillatory. ω_2 is NOT an angular frequency.

(ii) For large time, $x \approx Ae^{(-\beta+\omega_2)t}$. Since $\omega_2 = \sqrt{\beta^2 - \omega^2} < \beta$, the exponential is negative $\therefore x(\infty) = 0$. Decays to equilibrium.



Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

Here $r = -\beta \pm 0 = -\beta$ **Only ONE root!? Where's the second solution?** Let us confirm the single root is indeed a solution.

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \Rightarrow \beta^2 e^{-\beta t} - 2\beta^2 e^{-\beta t} + \beta^2 e^{-\beta t} = 0$$

$$2\beta^2 - 2\beta^2 = 0$$

Therefore, we know that the decaying exponential is ONE (of two) solutions:

$$x(t) = Ae^{-\beta t} + x_2(t) \Rightarrow x_2(t) = ?$$

A linear 2-ODE will have 2 linearly independent solutions, so we're not done.

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

The single root implies a polynomial solution which can be found using the method of a perfect square. Let's reconsider the differential equations:

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \Rightarrow (\ddot{x} + \beta\dot{x}) + \left(\beta\dot{x} + \beta^2 x\right) = 0$$

$$\frac{d}{dt}\left(\frac{dx}{dt} + \beta x\right) + \beta\left(\frac{dx}{dt} + \beta x\right) = 0$$

We see we have an auxiliary function whose total time derivative is taken.

Therefore, let's define $\gamma(t) = \frac{dx}{dt} + \beta x$

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$\frac{d}{dt}\left(\frac{dx}{dt} + \beta x\right) + \beta\left(\frac{dx}{dt} + \beta x\right) = 0 \qquad \Rightarrow \quad \dot{\gamma} + \beta\gamma = 0$$

Note this is a first order equation now.

$$\frac{d\gamma}{dt} + \beta\gamma = 0 \rightarrow \frac{d\gamma}{\gamma} = -\beta dt$$

$$\ln \gamma = -\beta t \rightarrow \gamma = e^{-\beta t}$$

$\therefore \gamma = \frac{dx}{dt} + \beta x = e^{-\beta t}$ (a new first order **inhomogeneous** equation!)

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \quad \Rightarrow \quad \dot{x} + \beta x = e^{-\beta t}$$

We've converted our second order homogeneous ODE into a first order inhomogeneous ODE. To solve it, we use a trick. For any ODE of the form

$$\frac{dx}{dt} + \beta x = f(t)$$

Let's multiply each side by $e^{\beta t}$ resulting in

$$\left(\frac{dx}{dt} + \beta x\right) e^{\beta t} = f(t) e^{\beta t}$$

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$\left(\frac{dx}{dt} + \beta x\right) e^{\beta t} = f(t)e^{\beta t} \Rightarrow \frac{d}{dt}\left(xe^{\beta t}\right) = f(t)e^{\beta t}$$

This rearranges to $xe^{\beta t} = \int f(t)e^{\beta t}dt$. We solve for $x(t)$ using $f(t) = e^{-\beta t}$

$$d\left(xe^{\beta t}\right) = e^{-\beta t}e^{\beta t}dt = dt$$

$$xe^{\beta t} = t \Rightarrow x_2(t) = te^{-\beta t}$$

We now put this together writing a general solution of $x_1(t)$ and $x_2(t)$.

$$\boxed{x(t) = Ae^{-\beta t} + Bte^{-\beta t} = (A + Bt)e^{-\beta t}}$$

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

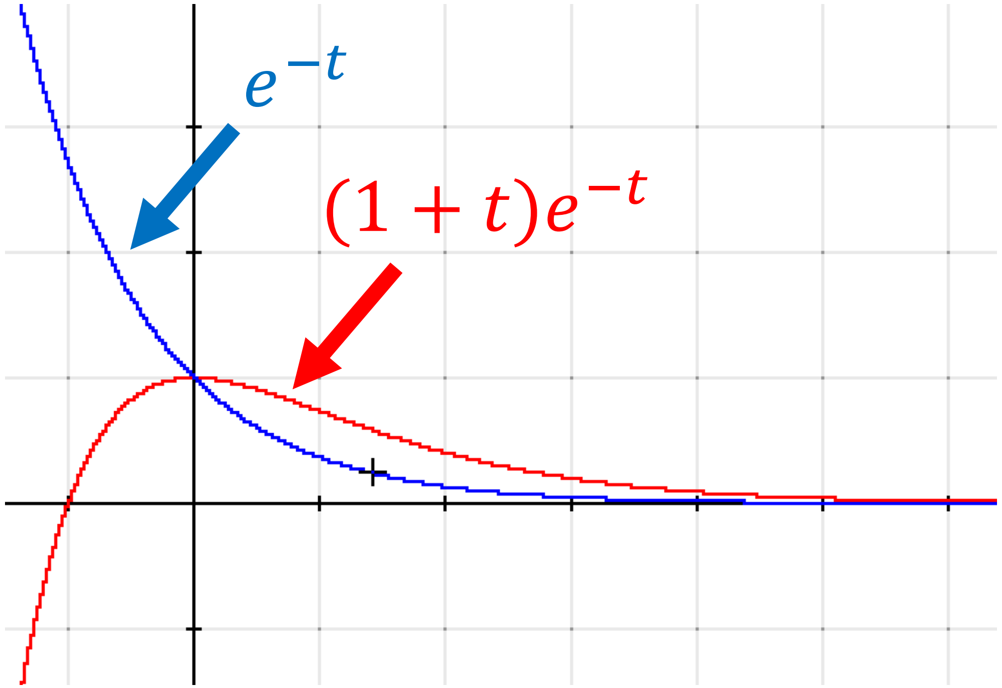
$$x(t) = Ae^{-\beta t} + Bte^{-\beta t} = (A + Bt)e^{-\beta t}$$

Properties of the critically damped solution:

(i) NOT oscillatory

(ii) For $\lim t \to \infty$, $x \approx Bte^{-\beta t} \to 0$.
Decays exponentially to equilibrium.

(iii) Decays "faster" than overdamped oscillator as $\beta > \beta - \omega_2$.



Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$x(t) = Ae^{-\beta t} + Bte^{-\beta t} = (A + Bt)e^{-\beta t}$$

A note on roots:

Generally, when an n-order linear homogeneous equation has degenerate roots (i.e. multiples of the same), its independent solutions corresponding to this root are polynomials constructed from:

$$1, t, t^2, \ldots, t^{n-1}.$$