PHYS 321 - Theoretical Mechanics

Lecture 7

Instructor: Dr. Andrew James Steinmetz

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Course information

Instructor information:

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Lecture schedule:

- PHYS 321 – 501 AP
  - Mondays at 8:30 AM to 10:05 AM (CST/UTC+8) – Weeks 2-5, 7-14
  - Tuesdays at 10:25 AM to 12:00 PM (CST/UTC+8) – Weeks 2-5, 7-14
- PHYS 321 – 502 ME
  - Mondays at 3:55 PM to 5:30 PM (CST/UTC+8) – Weeks 2-5, 7-14
  - Wednesdays at 3:55 PM to 5:30 PM (CST/UTC+8) – Weeks 2-5, 7-14

In-class tutorial 2:

The damped harmonic oscillator includes the drag force $F_{drag} \propto v$

$$\sum F = ma \Rightarrow -kx - \dot{x} = m\ddot{x}$$

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0$$

Where $\omega \equiv \sqrt{\frac{k}{m}}$ and $\beta \equiv \frac{b}{2m}$ (both have units of frequency 1/seconds)

Let us guess a solution $x = e^{rt}$ just as the case where $\beta = 0$.

Damped harmonic oscillators

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Damped harmonic oscillators

Let us guess a solution $x = e^{rt}$ just as the case where $\beta = 0$.

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0$$

$$r^2 e^{rt} + 2\beta r e^{rt} + \omega^2 e^{rt} = 0$$

$$r^2 + 2\beta r + \omega^2 = 0$$

$$r = \frac{1}{2}\left(-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}\right)$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega^2}$$

We notice that unlike the simple case $(\beta = 0)$, the complex exponential is not pure imaginary anymore: It is complex with real and imaginary parts.

Damped harmonic oscillators

Case (1): $\beta < \omega$ (underdamped oscillator)

Here $r = -\beta \pm i\omega_1$ where $\omega_1 \equiv \sqrt{\omega^2 - \beta^2}$ (real valued). The solution then has two complex roots.

$$\therefore x(t) = c_1 e^{(-\beta + i\omega_1)t} + c_2 e^{(-\beta - i\omega_1)t} \text{ (general complex solution)}$$

$$= e^{-\beta t}\left(c_1 e^{+i\omega_1 t} + c_2 e^{-i\omega_1 t}\right) \text{ (c1 and c2 are still complex)}$$

Let us take the real part only $x(t) \to \mathrm{Re}[x(t)]$

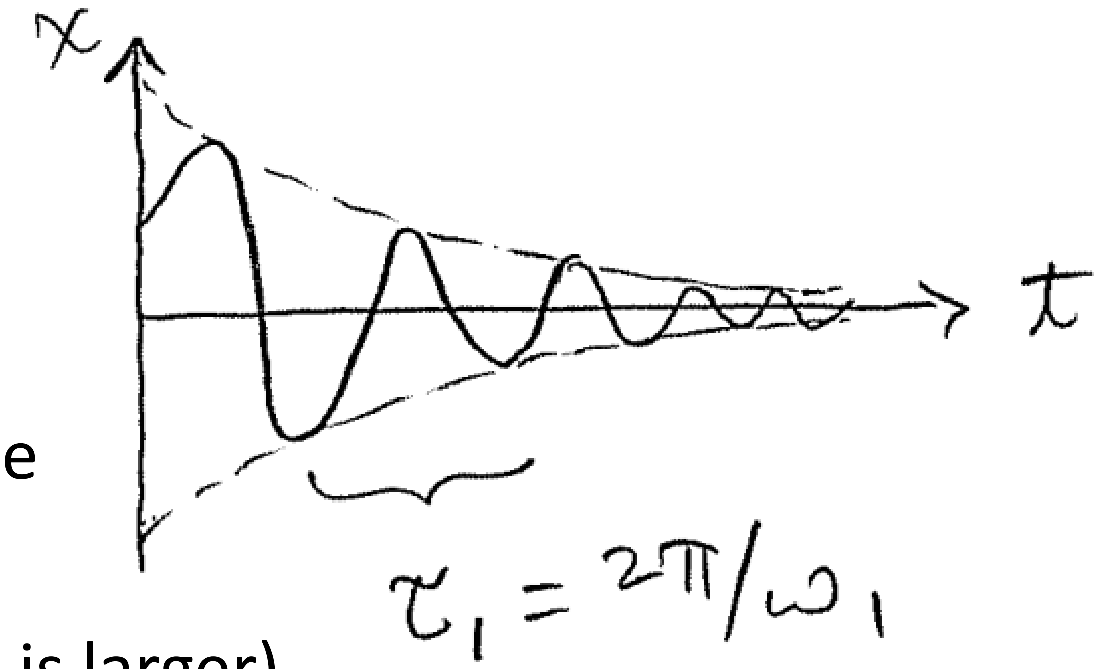
$$x(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t) \text{ (underdamped solution)}$$

Damped harmonic oscillators

Case (1): $\beta < \omega$ (underdamped oscillator)

$$x(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t) \text{ (underdamped solution)}$$

Properties:



(i) Oscillatory motion with a decaying exponential amplitude with time.

(ii) Decay rate depends on damping coefficient β: Large β = fast decay, while small β = slow decay.

(iii) Oscillation frequency is smaller (period is larger) than for the undamped case. $\omega_1 < \omega$

Damped harmonic oscillators

Case (2): $\beta > \omega$ (overdamped oscillator)

Here $r = -\beta \pm \omega_2$ where $\omega_2 \equiv \sqrt{\beta^2 - \omega^2}$ (real valued). The solution then has two real roots.

$$\therefore x(t) = c_1 e^{(-\beta+\omega_2)t} + c_2 e^{(-\beta-\omega_2)t} \text{ (general solution)}$$

$$= e^{-\beta t}\left(c_1 e^{+i\omega_2 t} + c_2 e^{-\omega_2 t}\right) \text{ (c1 and c2 are still complex)}$$

Let us take the real part only $x(t) \to \mathrm{Re}[x(t)]$

$$x(t) = e^{-\beta t}\left(A e^{+i\omega_2 t} + B e^{-\omega_2 t}\right) \text{ (overdamped solution)}$$

Damped harmonic oscillators

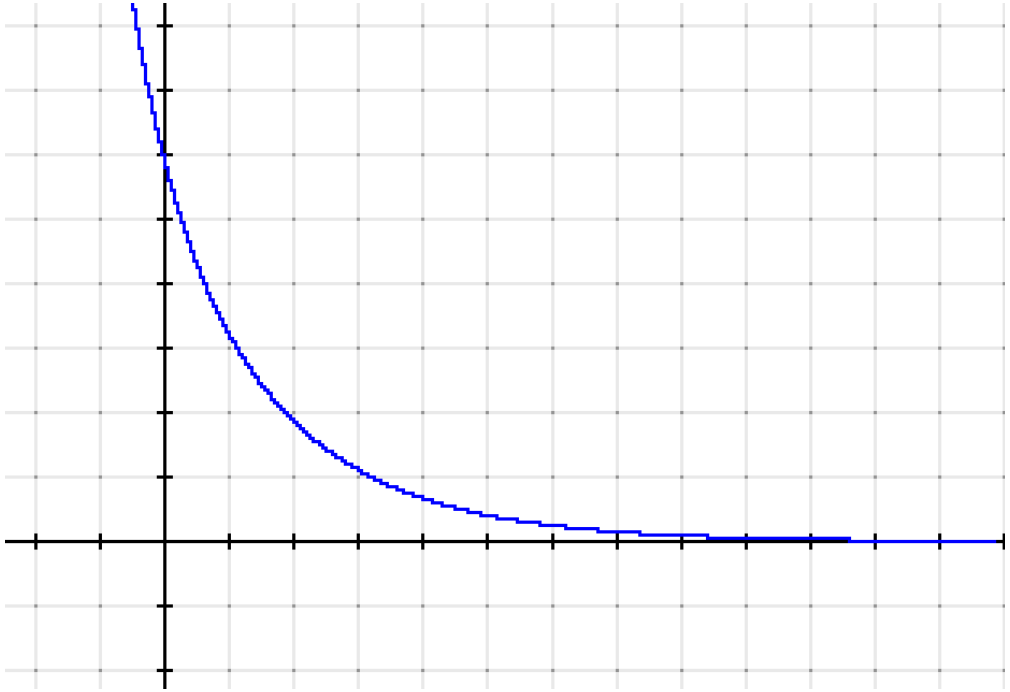
Case (2): $\beta > \omega$ (overdamped oscillator)

$$x(t) = e^{-\beta t}\left(Ae^{+i\omega_2 t} + Be^{-\omega_2 t}\right) \text{ (overdamped solution)}$$

Properties:

(i) Motion is NOT oscillatory. ω_2 is NOT an angular frequency.

(ii) For large time, $x \approx Ae^{(-\beta+\omega_2)t}$. Since $\omega_2 = \sqrt{\beta^2 - \omega^2} < \beta$, the exponential is negative $\therefore x(\infty) = 0$. Motion decays to equilibrium.



Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

Here $r = -\beta \pm 0 = -\beta$ Only ONE root!? Where's the second solution? The single root implies the polynomial is a perfect square. Let's reconsider the differential equations:

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \Rightarrow (\ddot{x} + \beta\dot{x}) + \left(\beta\dot{x} + \beta^2 x\right) = 0$$

$$\frac{d}{dt}\left(\frac{dx}{dt} + \beta x\right) + \beta\left(\frac{dx}{dt} + \beta x\right) = 0$$

Let's define $\gamma(t) = \frac{dx}{dt} + \beta x$

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$\frac{d}{dt}\left(\frac{dx}{dt} + \beta x\right) + \beta\left(\frac{dx}{dt} + \beta x\right) = 0 \quad \Rightarrow \quad \dot{\gamma} + \beta\gamma = 0$$

Note this is a first order equation now.

$$\frac{d\gamma}{dt} + \beta\gamma = 0 \rightarrow \frac{d\gamma}{\gamma} = -\beta dt$$

$$\ln\gamma = -\beta t \rightarrow \gamma = e^{-\beta t}$$

$\therefore \gamma = \frac{dx}{dt} + \beta x = e^{-\beta t}$ (a new first order **inhomogeneous** equation!)

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \quad \Rightarrow \quad \dot{x} + \beta x = e^{-\beta t}$$

We've converted our second order homogeneous ODE into a first order inhomogeneous ODE. To solve it, we use a trick. For any ODE of the form

$$\frac{dx}{dt} + ax = f(t)$$

Let's multiply each side by e^{at} resulting in

$$\left(\frac{dx}{dt} + ax\right)e^{at} = f(t)e^{at}$$

Damped harmonic oscillators

Case (3): $\beta = \omega$ (critically damped oscillator)

$$\left(\frac{dx}{dt} + ax\right)e^{at} = f(t)e^{at} \;\Rightarrow\; \frac{d}{dt}\left(xe^{at}\right) = f(t)e^{at}$$

This rearranges to $xe^{at} = \int f(t)e^{at}dt$. We now solve for x.