IN THE LAND OF MORDOR, IN THE FIRES OF MOUNT DOOM

1 Boosted Schwarzschild Black Strings

The boosted Schwarschild black string is a $5D$ solution to the vacuum Einstein equation,

$$\mathrm{Ric}(g) = 0, \tag{1}$$

which is asymptotically the product of $4D$ Minkowski, Mink_4, and a circle of radius R, $\mathbb{S}^1_R$. It has metric

$$\begin{aligned} g \doteq &-\big(1-[1-D(r)]\cosh^2(\sigma)\big)dt\otimes dt + \frac{1}{D(r)}dr\otimes dr + r^2 d\theta\otimes d\theta + r^2\sin^2\theta d\varphi\otimes d\varphi \\ &+ \big(1+[1-D(r)]\sinh^2(\sigma)\big)dz\otimes dz - \big(1-D(r)\big)\cosh(\sigma)\sinh(\sigma)(dt\otimes dz + dz\otimes dt). \end{aligned} \tag{2}$$

The coordinate ranges are $t\in[0,\infty)$, $r\in(2M,\infty)$ and $z\in[0,2\pi)$. One can 'unboost' the string with the coordinate transformation

$$\tau = \cosh\sigma t + \sinh\sigma z, \tag{3}$$

$$\zeta = \sinh\sigma t + \cosh\sigma z. \tag{4}$$

In this case ζ, τ have infinite image, so the string direction when unboosted looks not compact?

2 Singly Spinning Black Rings

2.1 Metric

The singly spinning black ring solution is a 5D solution to the Einstein vacuum equation,

$$\mathrm{Ric}(g) = 0, \tag{5}$$

which is asymptotically flat. The metric in coordinates $(t, r, \theta, \varphi, \psi)$,

$$\begin{aligned} g \doteq &-\frac{f(r)}{h(\theta)}dt\otimes dt + \frac{h(\theta)}{(1-\frac{r^2}{R^2})(1+\frac{r\cos\theta}{R})^2 p(r)}dr\otimes dr + \frac{r^2 h(\theta)}{(1+\frac{r\cos\theta}{R})^2 q(\theta)}d\theta\otimes d\theta \\ &+ \frac{r^2 q(\theta)\sin^2\theta}{(1+\frac{r\cos\theta}{R})^2}d\varphi\otimes d\varphi + \Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2 f(r)} - \frac{\Xi^2(1-\frac{r}{R})^2}{r^2 f(r)h(\theta)}\Big)d\psi\otimes d\psi \\ &+ \frac{\Xi}{rh(\theta)}\Big(\frac{r}{R}-1\Big)\big(dt\otimes d\psi + d\psi\otimes dt\big) \end{aligned} \tag{6}$$

with the following definitions,

$$f(r) \doteq 1 - \frac{r_+\cosh^2\sigma}{r} \qquad h(\theta) \doteq 1 + \frac{r_+\cosh^2\sigma}{R}\cos\theta \tag{7}$$

$$p(r) \doteq 1 - \frac{r_+}{R}, \qquad q(\theta) \doteq 1 + \frac{r_+}{R}\cos\theta, \tag{8}$$

and

$$\Xi \doteq r_+ R \sinh\sigma \cosh\sigma \sqrt{\frac{R + r_+ \cosh^2\sigma}{R - r_+ \cosh^2\sigma}}. \tag{9}$$

The coordinate ranges are

$$t \in \mathbb{R} \quad r \in (r_+, R) \quad \theta \in [0, \pi) \quad \varphi, \psi \in \Big[0, \frac{2\pi R}{\sqrt{R^2 + r_+^2}}\Big) \tag{10}$$

$r = r_+ < R$ corresponds to the future event horizon, σ is a parameter and R sets the scale of the solution.

The inverse metric is:

$$\begin{aligned} g^{-1} = &-\Big(\frac{h(\theta)}{f(r)} - \frac{\Xi^2 (R-r)(R + r\cos\theta)^2}{r^2 R^4 (r+R) p(r) f(r) h(\theta)}\Big)\partial_t \otimes \partial_t + \frac{(R^2 - r^2)(R + r\cos\theta)^2 p(r)}{R^4 h(\theta)} \partial_r \otimes \partial_r \\ &+ \frac{(R + r\cos\theta)^2 q(\theta)}{r^2 R^2 h(\theta)} \partial_\theta \otimes \partial_\theta + \frac{(R + r\cos\theta)^2}{R^2 r^2 q(\theta) \sin^2\theta} \partial_\varphi \otimes \partial_\varphi + \frac{(R + r\cos\theta)^2 f(r)}{R^2 (R^2 - r^2) h(\theta) p(r)} \partial_\psi \otimes \partial_\psi \\ &- \frac{\Xi (R + r\cos\theta)^2}{r R^3 (r+R) h(\theta) p(r)} (\partial_t \otimes \partial_\psi + \partial_\psi \otimes \partial_t) \end{aligned} \tag{11}$$

For the spacetime to have no conical singularities on the exterior region we require the 'equilibrium' condition:

$$\cosh^2\sigma = \frac{2R^2}{r_+^2 + R^2}, \qquad \sinh^2\sigma = \frac{R^2 - r_+^2}{r_+^2 + R^2}. \tag{12}$$

Thus,

$$f(r) \doteq 1 - \frac{r_e}{r} \qquad h(\theta) = 1 + \frac{2Rr_+}{r_+^2 + R^2}\cos\theta \tag{13}$$

$$p(r) \doteq 1 - \frac{r_+}{R} \qquad q(\theta) \doteq 1 + \frac{r_+}{R}\cos\theta \tag{14}$$

and

$$\Xi \doteq r_e (r_+ + R) \sqrt{\frac{R + r_+}{2(R - r_+)}} \tag{15}$$

2.2 Spacetime Regions

Note that, the vector field ∂_t is null at,

$$g_{tt} = 0 \implies r = r_+ \cosh^2\sigma \tag{16}$$

which corresponds to the ergosurface. So the ergoregion is given by,

$$r \in (r_+, r_e \doteq r_+ \cosh^2\sigma) \tag{17}$$

Note the vector field,

$$k = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial\psi} \qquad \Omega_H^2 = \frac{R - r_+}{2R^2 (R + r_+)} \tag{18}$$

is null future event horizon. Note that ψ is has period $\psi \sim \psi + \Delta\psi$ with $\Delta\psi \doteq \frac{2\pi R}{\sqrt{R^2 + r_+^2}}$ so Ω_H is not the true angular velocity of the horizon. Defining $\tilde{\psi} = \frac{2\pi}{\Delta\psi}\psi$ gives

$$\partial_\psi = \frac{\sqrt{R^2 + r_+^2}}{R} \partial_{\tilde{\psi}}. \tag{19}$$

So the true angular velocity of the horizon is

$$\tilde{\Omega}_H = \frac{1}{R^2}\sqrt{\frac{(R-r_+)(R^2+r_+^2)}{2(R+r_+)}}. \tag{20}$$

Now, for $r \in (r_+, r_+\cosh^2\sigma)$, $f(r) < 0$ so we can make the coordinate transformation,

$$dv = dt + \frac{\Xi}{r(r+R)\sqrt{-f(r)}p(r)}dr \qquad d\psi = d\chi + \frac{R\sqrt{-f(r)}}{(R^2-r^2)p(r)}dr \tag{21}$$

which gives the metric,

$$\begin{aligned} g \doteq & -\frac{f(r)}{h(\theta)}dv\otimes dv + \frac{r^2h(\theta)}{(1+\frac{r\cos\theta}{R})^2q(\theta)}d\theta\otimes d\theta + \frac{r^2q(\theta)\sin^2\theta}{(1+\frac{r\cos\theta}{R})^2}d\varphi\otimes d\varphi \\ & + \Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2f(r)} - \frac{\Xi^2(1-\frac{r}{R})^2}{r^2f(r)h(\theta)}\Big)d\chi\otimes d\chi \\ & + \frac{\Xi}{rh(\theta)}\Big(\frac{r}{R}-1\Big)\big(dv\otimes d\chi + d\chi\otimes dv\big) + \frac{R^3h(\theta)}{(R+r\cos\theta)^2\sqrt{-f(r)}}(dr\otimes d\chi + d\chi\otimes dr) \end{aligned} \tag{22}$$

Then one has the following for k,

$$k = \partial_v + \Omega_H\partial_\chi. \tag{23}$$

Mapping k to a one-form,

$$\begin{aligned} k_\flat = & \Big[\frac{\Xi\Omega_H}{rh(\theta)}\Big(\frac{r}{R}-1\Big) - \frac{f(r)}{h(\theta)}\Big]dv + \Omega_H\frac{R^3h(\theta)}{(R+r\cos\theta)^2\sqrt{-f(r)}}dr \\ & + \Big[\frac{\Xi}{rh(\theta)}\Big(\frac{r}{R}-1\Big) + \Omega_H\Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2f(r)} - \frac{\Xi^2(1-\frac{r}{R})^2}{r^2f(r)h(\theta)}\Big)\Big]d\chi \end{aligned} \tag{24}$$

which at the future event horizon for a balanced ring gives,

$$k_\flat|_{r=r_+} = \frac{R^2}{\sqrt{2(R^2+r_+^2)}(R+r_+)}\frac{(R^2+r_0^2+2Rr_+\cos\theta)}{(R+r_+\cos\theta)^2}dr. \tag{25}$$

Note that,

$$g(k,k) = -\frac{f(r)}{h(\theta)} + 2\Xi\Omega_H\frac{(\frac{r}{R}-1)}{rh(\theta)} + \Omega_H^2\frac{1}{f(r)h(\theta)}\Big(\frac{R^2(R^2-r^2)h(\theta)^2p(r)}{(R+r\cos\theta)^2} - \frac{(\frac{r}{R}-1)^2\Xi^2}{r^2}\Big). \tag{26}$$

To find the surface gravity, κ, of a balanced ring we consider $d(g(k,k))|_{r=r_+} = -2\kappa k|_{r=r_+}$. Evaluating, one finds,

$$\kappa = \frac{(R-r_+)\sqrt{(R^2+r_+^2)}}{2\sqrt{2}R^2r_+}. \tag{27}$$