1 The Toy Problem: The Kerr String/Klein-Gordon on Kerr

1.1 Introduction

The aim is to cast the superradiant mode instability for the Klein-Gordon system on the Kerr spacetime of YSR into a PDE problem, i.e. avoid separation in the (r, θ) variables. Here this will be considered as the wave equation on $\mathrm{Kerr} \times \mathbb{S}^1$. The aim is to produce an open family of frequencies (or masses) for which the wave equation (Klein-Gordon equation) exhibits linear instability. This will result as a superradiant instability. Informally superradiance can occur in a black hole spacetime when there doesn't exist a gloablly defined Killing vector field which is both timelike or null at infinity and timelike or null at $\mathcal{H}^+_A$. The theorem which we wish to prove in the context of PDE analysis is the following:

Theorem 1.1. *Fix a sub-extremal Kerr string $(\mathrm{Kerr} \times \mathbb{S}^1, g_{a,M})$ with $M > 0$. Then $\exists$ a open family of frequencies μ (masses) with ϵ_μ and a non-zero, smooth and finite energy solution Ψ to the corresponding wave equation (KG equation),*

$$\Box_g \Psi = 0 \qquad (\Box_g \Psi - \mu^2 \Psi = 0) \tag{1}$$

such that $\forall (t, r, \theta, \varphi) \in \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$,

$$e^{\epsilon_\mu t} |\partial^\alpha \Psi(0, r, \theta, \varphi)| \leq C_\alpha |\partial^\alpha \Psi(t, r, \theta, \varphi)| \qquad \forall \alpha \tag{2}$$

One can define the following energy momentum tensor,

$$\mathbb{T}[\Psi](X, Y) = \mathrm{Re}\big(\nabla_X \Psi \overline{\nabla_Y \Psi}\big) - \frac{1}{2} g(X, T) \langle \nabla \Psi, \nabla \Psi \rangle_g \tag{3}$$

or,

$$\mathbb{T}[\Psi](X, Y) = \mathrm{Re}\big(\nabla_X \Psi \overline{\nabla_Y \Psi}\big) - \frac{1}{2} g(X, T)\big(\langle \nabla \Psi, \nabla \Psi \rangle_g + \mu^2 |\Psi|^2\big) \tag{4}$$

for Klein-Gordon. One can further define a one form current,

$$J^X = \mathbb{T}[\Psi](\cdot, X). \tag{5}$$

Note,

$$\nabla_a (J^X)^a = \frac{1}{2} \mathbb{T}_{ab} (\pi^X)^{ab} \tag{6}$$

with,

$$(\pi^X)^{ab} = \nabla^a X^b + \nabla^b X^a \tag{7}$$

the deformation tensor. Note this vanishes if X is Killing.

On Kerr/Kerr string there exists a unique, up to normalisation, Killing vector field, $T = \partial_t$, which is future directed and timelike at infinity and spacelike almost everywhere on $\mathcal{H}^+_A$. There is also a unique, up to normalisation, future directed Killing vector field, $\Phi = \partial_\varphi$ which vanishes at $\varphi = 0$. The null generator of the horizon is,

$$k := T + \Omega_H \Phi \tag{8}$$

where $\Omega_H = \frac{a}{2Mr_+}$ is the 'angular velocity' of the horizon. The energy density of the solution to the wave equation (KG equation) is,

$$\mathcal{E}_T(\mathcal{H}^+_A) = J^T(k) = \mathrm{Re}(T(\Psi)\overline{k(\Psi)}) = \mathrm{Re}\Big(T(\Psi)\big(\overline{T(\Psi) + \frac{a}{2Mr_+}\Phi(\Psi)}\big)\Big). \tag{9}$$

Note that if $a \neq 0$ then $\mathcal{E}_T(\mathcal{H}^+_A) < 0$ is possible and therefore energy can radiate out of the black hole in principle. Here the aim is construct exponential growing solutions that are superradiant bound states. Energy flux on $\mathcal{H}^+_A$ will be negative and they will decay to infinity. Therefore energy coming out of the black hole cannot escape.

1.2 Results

In the search for a superradiant instability one can look at mode solutions,

$$\Psi(t,r,\theta,\varphi,z) = e^{-i\omega t + im\varphi + i\mu z} u(r,\theta). \tag{10}$$

Proposition 1. *A mode solution exhibits superradiance iff,*

$$am\mathrm{Re}(\omega) - 2Mr_+|\omega|^2 > 0. \tag{11}$$

Proof. Note $T(\Psi) = -i\omega\Psi$ and $\Phi(\Psi) = im\Psi$. Then,

$$\mathcal{E}_T(\mathcal{H}_A^+) = \mathrm{Re}\Big(-i\omega\Psi\overline{\big(-i\omega\Psi + \frac{a}{2Mr_+}im\Psi\big)}\Big) = \mathrm{Re}\big(|\omega|^2 - \frac{am\omega}{2Mr_+}\big)|\Psi|^2. \tag{12}$$

□

The main idea is to construct exponentially growing modes from real ones. Consider the following theorem,

Theorem 1.2. *Suppose there exists a mode solution with parameters (ω, m, μ) such that $\omega \in \mathbb{R}$ and $\mu^2 > \omega^2$. Then the following statements are true,*

1. $am - 2Mr_+\omega = 0.$
2. $am \neq 0.$

Remark. *Note $am - 2Mr_+\omega = 0$ is equivalent to $k(\Psi) = 0$ so there is no energy flux at the horizon.*

Now, the wave equation on the Kerr string or KG equation on Kerr reduces to,

$$L(u) := \partial_r\big(\Delta\partial_r u\big) + \frac{1}{\sin\theta}\partial_\theta\big(\sin\theta\partial_\theta u\big) - \big(V_a(\theta) + V_r(r)\big)u = 0 \tag{13}$$

with,

$$V_a(\theta) = a^2(\mu^2 - \omega^2)\cos^2\theta + \frac{m^2}{\sin^2\theta} \tag{14}$$

$$V_r(r) = \frac{4aM\omega mr - (r^2+a^2)^2\omega^2 - a^2m^2}{\Delta(r)} + \mu^2r^2 + \omega^2a^2 \tag{15}$$

This is clearly seperable however in as a toy problem for the ring it is instructive not to seperate at this point. Note that the L is elliptic, $\forall(r,\theta) \in (r_+, \infty) \times [0, \pi)$ since,

$$[a^{ij}] = \begin{pmatrix} \Delta(r) & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

so the eigenvalues are positive everywhere except $(r = r_+, \theta)$. Therefore ellipticity holds away from the horizon. The limiting behavior for the PDE as $r \to \infty$ is,

$$\partial_r^2 u - (\mu^2 - \omega^2)u = O\Big(\frac{1}{r}\Big) \tag{17}$$

which has solution,

$$u(r,\theta) = e^{\sqrt{\mu^2-\omega^2}r}u_1(\theta) + e^{-\sqrt{\mu^2-\omega^2}r}u_2(\theta). \tag{18}$$

For sufficently large $r \geq R$,

$$(V_r(r) + V_a(\theta))\frac{1}{\Sigma(r,\theta)} \leq b_{R,a,\mu,\omega,m}\Big(1 + \frac{1}{r^2\sin^2\theta}\Big). \tag{19}$$

Take a cut-off $\chi_R : [r_+, \infty) \to \mathbb{R}$ such that,

$$\chi_R(r) = 0 \qquad \forall r \leq R \tag{20}$$
$$\chi_R(r) = 1 \qquad \forall r \geq 2R \tag{21}$$

with $|\partial_r \chi_R| \leq \frac{C}{R}$ and $|\partial_r^2 \chi_R| \leq \frac{C}{R^2}$. Multiplying through by the cut-off,

$$-\frac{1}{\Sigma(r,\theta)}\partial_r(\Delta\partial_r(\chi_R u)) - \frac{1}{\Sigma(r,\theta)\sin\theta}\partial_\theta(\sin\theta\partial_\theta(\chi_R u)) + b\Big(1+\frac{1}{r^2\sin^2\theta}\Big)\chi_R u \leq C\Big(\frac{1}{R}|\partial_r u| + \frac{1}{R^2}|u|\Big). \tag{22}$$

Multiplying by $\overline{\chi_R u}\Sigma\sin\theta$ and integrating by parts gives,

$$\int_{r\geq R}\int_0^\pi \Big(\Delta|\partial_r(\chi_R u)|^2 + |\partial_\theta(\chi_R u)|^2 + b\Big(1+\frac{1}{r^2\sin^2\theta}\Big)|\chi_R u|^2\Sigma(r,\theta)\Big)\sin\theta dr d\theta \tag{23}$$
$$-\int_0^\pi \overline{\chi_R u}\Delta\partial_r(\chi_R u)\sin\theta\Big|_{r=R}^{r=\infty} d\theta - \int_{r\geq R}\sin\theta\overline{\chi_R u}\partial_\theta(\chi_R u)\Big|_{\theta=0}^{\theta=\pi} dr \leq \frac{C}{R} \tag{24}$$

since the u has finite energy i.e.

$$\int_{r\geq R}\int_0^\pi \Big(|\partial_r u|^2 + \frac{1}{r^2}|\partial_\theta u|^2 + |u|^2\Big) r^2 \sin\theta dr d\theta < \infty. \tag{25}$$

Note that the boundary term at $r = R$ vanishes due to the cut-off and the boundary terms in θ vanish due to the $\sin\theta$. Note the boundary term at infinity can be dealt with by noting,

$$\Big|\int_0^\pi \overline{\chi_R u}\Delta\partial_r(\chi_R u)\sin\theta d\theta\Big| \leq \Delta\Big(\int_0^\pi \sin\theta|\chi_R u|^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_0^\pi \sin\theta|\partial_r(\chi_R u)|^2 d\theta\Big)^{\frac{1}{2}} \tag{26}$$
$$\leq \frac{\Delta}{2}\Big(\int_0^\pi \sin\theta|\chi_R u|^2 d\theta + \int_0^\pi \sin\theta|\partial_r(\chi_R u)|^2 d\theta\Big). \tag{27}$$

Therefore,

$$\int_R^\infty \Delta\Big(\int_0^\pi \sin\theta|\chi_R u|^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_0^\pi \sin\theta|\partial_r(\chi_R u)|^2 d\theta\Big)^{\frac{1}{2}} dr < \infty \tag{28}$$

since u has finite energy. So,

$$\Delta\Big(\int_0^\pi \sin\theta|\chi_R u|^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_0^\pi \sin\theta|\partial_r(\chi_R u)|^2 d\theta\Big)^{\frac{1}{2}} \in L^1((R,\infty)). \tag{29}$$

Hence there exists an increasing sequence of points such that,

$$\Delta\Big(\int_0^\pi \sin\theta|\chi_R u|^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_0^\pi \sin\theta|\partial_r(\chi_R u)|^2 d\theta\Big)^{\frac{1}{2}} \to 0 \implies \lim_{r\to\infty}\Big|\int_0^\pi \overline{\chi_R u}\Delta\partial_r(\chi_R u)\sin\theta d\theta\Big| = 0 \tag{30}$$

So one has the elliptic estimate,

$$\int_{r\geq R}\int_0^\pi \Big(\Delta|\partial_r(\chi_R u)|^2 + |\partial_\theta(\chi_R u)|^2 + b\Big(1+\frac{1}{r^2\sin^2\theta}\Big)|\chi_R u|^2\Sigma(r,\theta)\Big)\sin\theta dr d\theta \leq \frac{C}{R}. \tag{31}$$

Note that this can be iterated to give $\frac{1}{R^k}$ for all $k \in \mathbb{N}$. (Apparently Schauder estimates allow one to extend this to polynomial decay; there is also a stronger statement of exponential decay).

1.3 Boundary Conditions

The solution must extend smoothly to the whole domain of outer communications and have finite energy. For fixed r the angular part we require $e^{im\varphi}u(r,\theta)$ extends smoothly to all of $\mathbb{S}^2$. In terms of boundary conditions in r one needs to examine the horizon with coordinates that do not break down there. Define Kerr coordinates (v, r, θ, χ, z) by,

$$v = t + \bar{t}(r) \qquad \chi = \varphi + \bar{\varphi}(r), \tag{32}$$

with,

$$\frac{d\bar{t}}{dr} = \frac{r^2+a^2}{\Delta(r)} \qquad \frac{d\bar{\varphi}}{dr} = \frac{a}{\Delta(r)}. \tag{33}$$

So,

$$\bar{t}(r) = r + \frac{2M}{r_+ - r_-}\log|r-r_+| - \frac{2M}{r_+ - r_-}\log|r-r_-| \tag{34}$$
$$\bar{\varphi}(r) = \frac{a}{r_+ - r_-}\log|r-r_+| - \frac{a}{r_+ - r_-}\log|r-r_-|. \tag{35}$$

Note $v \in \mathbb{R}$ and $\chi \in [0, 2\pi)$. So that Ψ extends smoothly to $\mathcal{H}_A^+$ one requires,

$$\Psi(v, r, \theta, \chi, z) = e^{-i\omega v + im\chi + i\mu z} e^{i\omega \bar{t}(r) - im\bar{\varphi}(r)} u(r, \theta) \tag{36}$$

to extend smoothly to the horizon. This happens if one can write,

$$u(r,\theta) = e^{-i\omega\bar{t}(r)+im\bar{\varphi}(r)}\tilde{u}(r,\theta) \Leftrightarrow u(r,\theta) = (r-r_+)^{\xi} v(r,\theta) \quad \xi = \frac{i(am - 2Mr_+\omega)}{r_+ - r_-} \tag{37}$$

for $e^{im\chi}v(r,\theta)$ smooth at $r = r_+$ $\forall\theta \in [0,\pi)$.

Remark. *It would be nice to argue that rigorously that solutions 'at infinity' are of the form,*

$$u(r,\theta) = e^{\pm\sqrt{\mu^2-\omega^2}r} c(r,\theta) \tag{38}$$

with $\lim_{r\to\infty} e^{im\varphi} c(r,\theta)$ *smooth and smoothly extending to* $\mathbb{S}^2$*. Maybe one can just impose Dirichlet boundary conditions such that* $\lim_{r\to\infty} e^{im\varphi}u(r,\theta) = 0$.

Note finite energy on spacelike hypersurfaces imposes a restriction of the form,

$$\frac{1}{2}\int_{r\geq R}\int_{\mathbb{S}^2} \Big((|\omega|^2+\mu^2)|u|^2 + |\partial_r u|^2 + \frac{1}{r^2}\big(|\partial_\theta u|^2 + \frac{m^2}{\sin^2\theta}|u|^2\big)\Big)(f(r_*), r, \theta, \varphi) r^2 \sin\theta dr d\theta d\varphi < \infty. \tag{39}$$

This is certainly true if the solution exponentially decays in r.

Suppose $\omega \in \mathbb{R}$ so that $V_r : (r_+, \infty) \times [0, \pi) \to \mathbb{R}$ and consider,

$$Q_T = \mathrm{Im}\Big(\int_{\mathbb{S}^2} \Delta\bar{u}\partial_r u \sin\theta d\theta d\varphi\Big). \tag{40}$$

Then,

$$\partial_r Q_T = \mathrm{Im}\Big(\int_{\mathbb{S}^2} \bar{u}\partial_r(\Delta\partial_r u)\sin\theta d\theta d\varphi\Big) - \mathrm{Im}\Big(\int_{\mathbb{S}^2} \Delta|\partial_r u|^2 \sin\theta d\theta d\varphi\Big) \tag{41}$$
$$= \mathrm{Im}\Big(\int_{\mathbb{S}^2} \Big((V_a(\theta)+V_r(r))|u|^2 - \bar{u}\frac{1}{\sin\theta}\partial_\theta\big(\sin\theta\partial_\theta u\big)\Big)\sin\theta d\theta d\varphi\Big) \tag{42}$$
$$= -\mathrm{Im}\Big(\int_{\mathbb{S}^2} \bar{u}\frac{1}{\sin\theta}\partial_\theta\big(\sin\theta\partial_\theta u\big)\sin\theta d\theta d\varphi\Big) \tag{43}$$
$$= -\mathrm{Im}\Big(\int_{\mathbb{S}^2} \partial_\theta\big(\bar{u}\sin\theta\partial_\theta u\big)d\theta d\varphi\Big) + \mathrm{Im}\Big(\int_{\mathbb{S}^2} \sin\theta|\partial_\theta u|^2 d\theta d\varphi\Big) \tag{44}$$
$$= -\mathrm{Im}\Big(\int_{\mathbb{S}^2} \partial_\theta\big(\bar{u}\sin\theta\partial_\theta u\big)d\theta d\varphi\Big) \tag{45}$$
$$= -2\pi\mathrm{Im}\Big(\sin\theta\bar{u}\partial_\theta u\Big|_{\theta=0}^{\theta=\pi}\Big) \tag{46}$$

Therefore,

$$Q_T(r_+) - Q_T(\infty) = 0. \tag{47}$$

Imposing that $u \sim e^{-\sqrt{\mu^2-\omega^2}r}$ at infinity gives,

$$Q_T(r_+) = 0. \tag{48}$$

Note,

$$\frac{\partial u}{\partial r_*} = \frac{dr}{dr_*}\frac{du}{dr} = \frac{\Delta(r)}{r^2+a^2}\frac{du}{dr} = \frac{\Delta(r)}{r^2+a^2}\frac{\xi}{(r-r_+)}u + \frac{\Delta(r)}{r^2+a^2}(r-r_+)^{\xi}\partial_r v \tag{49}$$

$$= \xi\frac{(r-r_-)}{r^2+a^2}u + \frac{(r-r_-)}{r^2+a^2}(r-r_+)^{\xi+1}\partial_r v. \tag{50}$$

So,

$$\frac{\partial u}{\partial r_*}\Big|_{r=r_+} = \xi\frac{(r_+-r_-)}{2Mr_+}u|_{r=r_+} + O(r-r_+). \tag{51}$$

Now,

$$Q_T = \mathrm{Im}\Big(\int_{\mathbb{S}^2}(r^2+a^2)\bar{u}\frac{\partial u}{\partial r_*}\sin\theta d\theta d\varphi\Big). \tag{52}$$

Therefore,

$$Q_T(r_+) = (r_+-r_-)\mathrm{Im}(\xi)\int_{\mathbb{S}^2}|v|^2(r_+)\sin\theta d\theta d\varphi \tag{53}$$

$$= (am-2Mr_+\omega)\int_{\mathbb{S}^2}|v|^2(r_+,\theta)\sin\theta d\theta d\varphi = 0. \tag{54}$$

Now since $\sin\theta \geq 0$ is positive $\forall\theta\in[0,\pi)$, for $|v|^2(r_+,\theta)\sin\theta = 0$ one requires $v(r_+,\theta)=0$ for almost every θ. Now since v is smooth this extends to $\theta\in[0,\theta)$.

Proposition 2. *Suppose $|v|^2(r_+,\theta)=0$ $\forall\theta\in[0,\pi)$. Then the solution to the PDE is trivial.*

Proof. If one could then argue that $v(r_+,\theta)\neq 0$ (maybe a maximum principle could help) then $(am - 2M\omega r_+) = 0$ would hold. I think maybe from pg.110 in Gilbarg and Trudinger analyticity of v follows. Therefore if $v(r_+,\theta)=0$ then $v\equiv 0$.

□

Now suppose $am - 2Mr_+\omega = 0$ and $am = 0$ so $m = 0 = \omega$ (assuming $a > 0$), then,

$$V_a(\theta) = a^2\mu^2\cos\theta \qquad V_r(r) = \mu^2 r^2 \tag{55}$$

so the potential is strictly positive for $r > r_+ > M > a$. Hence we have an elliptic PDE with positive potential so here one can apply a maximum principle which will give triviality of the solution.

Proposition 3. *Suppose $e^{im\varphi}u\in C^2((r_+,\infty)\times\mathbb{S}^2)$ and $m=0=\omega$ then u cannot attain a positive maxima or negative minima.*

Proof. Suppose that u attains a positive maxima at $r_0\in(r_+,\infty)$, $\theta_0\in[0,\pi)$. Then,

$$\Delta(r_0)\partial_r^2 u|_{r=r_0,\theta=\theta_0} + \partial_\theta^2 u|_{r=r_0,\theta=\theta_0} - V(r_0,\theta_0)u(r_0,\theta_0) = 0 \tag{56}$$

since at a maxima $\partial_r u = 0$ and $\partial_\theta u = 0$. Note further at the maxima $\partial_r^2 u \leq 0$ and $\partial_\theta^2 u \leq 0$. Now,

$$\Delta(r_0)\partial_r^2 u|_{r=r_0,\theta=\theta_0} + \partial_\theta^2 u|_{r=r_0,\theta=\theta_0} = V(r_0,\theta)u(r_0,\theta_0) > 0 \tag{57}$$

which gives a contradiction.

Suppose that u attains a negative minima at $r_0\in(r_+,\infty)$, $\theta_0\in[0,\pi)$. Then,

$$\Delta(r_0)\partial_r^2 u|_{r=r_0,\theta=\theta_0} + \partial_\theta^2 u|_{r=r_0,\theta=\theta_0} - V(r_0,\theta_0)u(r_0,\theta_0) = 0 \tag{58}$$

since at a minima $\partial_r u = 0$ and $\partial_\theta u = 0$. Note further at the minima $\partial_r^2 u \geq 0$ and $\partial_\theta^2 u \geq 0$. Now,

$$\Delta(r_0)\partial_r^2 u|_{r=r_0,\theta=\theta_0} + \partial_\theta^2 u|_{r=r_0,\theta=\theta_0} = V(r_0,\theta)u(r_0,\theta_0) < 0 \tag{59}$$

which gives a contradiction.

□

Remark. *One should note that $u(r_+,\theta)=0$ and $u(\infty,\theta)=0$ therefore the solution must in fact vanish.*

2 Singly Spinning Black Rings in Ring Coordinates

The metric for the balanced singly spinning ring in ring coordinates (t, x, y, φ, ψ) is,

$$
\begin{aligned}
g = &-\frac{F(y)}{F(x)}dt\otimes dt - \frac{1}{F(y)}\Big(\frac{c^2R^2(1+y)^2}{F(x)} + \frac{R^2}{(x-y)^2}F(x)G(y)\Big)d\psi\otimes d\psi \\
&+ \frac{R^2}{(x-y)^2}\Big\{G(x)d\varphi\otimes d\varphi - \frac{F(x)(1+\nu^2)}{G(y)}dy\otimes dy + \frac{F(x)(1+\nu^2)}{G(x)}dx\otimes dx\Big\} \\
&+ \frac{cR(1+y)}{F(x)}\big(dt\otimes d\psi + d\psi\otimes dt\big)
\end{aligned} \tag{60}
$$

with,

$$
F(\xi) = 1 + \frac{2\nu}{1+\nu^2}\xi \qquad G(\xi) = (1-\xi^2)(1+\nu\xi) \qquad c = \sqrt{\frac{2(1+\nu)}{1-\nu}\frac{\nu(1+\nu)}{1+\nu^2}}. \tag{61}
$$

The coordinate/parameter ranges are,

$$
x \in [-1,1], \qquad y \in \big(-\frac{1}{\nu}, -1\big], \qquad \varphi \in [0,2\pi), \qquad \psi \in [0,2\pi), \qquad 0 < \nu < 1. \tag{62}
$$

This spacetime is asymptotically flat with spacelike infinity located at $(x, y) = (-1, -1)$.

Let $\lambda = \frac{2\nu}{1+\nu^2}$ (this is the balancing condition to avoid conical singularities) and note,

$$
\begin{aligned}
G(x) &\geq 0 \quad \forall x \in [-1,1] && (63)\\
F(x) &> 0 \quad \forall x \in [-1,1] && (64)\\
G(y) &\leq 0 \quad \forall y \in \big(-\frac{1}{\nu}, -1\big] && (65)\\
F(y) &> 0 \quad \forall y \in (-\frac{1}{\lambda}, -1] && (66)\\
F(y) &< 0 \quad \forall y \in (-\frac{1}{\nu}, -\frac{1}{\lambda}) && (67)\\
F\Big(-\frac{1}{\lambda}\Big) &= 0 && (68)
\end{aligned}
$$

The solution is determined by two parameters (R, ν), effectively the ring radius and mass. Note that the ergosphere corresponds to $F(y) = 0$. From above, equation (68), this occurs when $y = -\frac{1}{\lambda}$ and the ergoregion is $y \in (-\frac{1}{\nu}, -\frac{1}{\lambda})$. The metric and its inverse are smooth at the ergosurface. The metric has a coordinate singularity at $y = -\frac{1}{\nu}$ which can be shown with,

$$
dt = dv - \frac{c\sqrt{1+\nu^2}R(1+y)}{G(y)\sqrt{-F(y)}}dy \qquad d\psi = d\psi' + \frac{\sqrt{-(1+\nu^2)F(y)}}{G(y)}dy. \tag{69}
$$

The metric in these coordinates is,

$$
\begin{aligned}
g \;=\; &-\frac{F(y)}{F(x)}\Big(dv - \frac{cR(1+y)}{F(y)}d\psi'\Big)\otimes\Big(dv - \frac{cR(1+y)}{F(y)}d\psi'\Big) \\
&+\frac{R^2}{(x-y)^2}F(x)\Big\{\frac{1+\nu^2}{G(x)}dx\otimes dx - \frac{G(y)}{F(y)}d\psi'\otimes d\psi' + \sqrt{\frac{1+\nu^2}{-F(y)}}(d\psi'\otimes dy + dy\otimes d\psi') + \frac{G(x)}{F(x)}d\varphi\otimes d\varphi\Big\}.
\end{aligned} \tag{70}
$$

This allows the metric to be smoothly extended through $y = -\frac{1}{\nu}$. Note this is only valid for $y \in (-\infty, -\frac{1}{\lambda})$. Letting,

$$
k = \partial_t + \Omega_H\partial_\psi = \partial_v + \Omega_H\partial_{\psi'} \tag{71}
$$

and

$$
\Omega_H = \frac{1}{R(1+\nu)}\sqrt{\frac{1-\nu^2}{2}}. \tag{72}
$$

k is null at $y = -\frac{1}{\nu}$ with $k_\flat \propto dy$ so, $y = -\frac{1}{\nu}$ is a Killing horizon with angular velocity Ω_H.

2.1 Scalar Perturbations

First note that,

$$\det(g) = -\frac{R^8 F(x)^2}{(x-y)^8}. \tag{73}$$

The wave equation, with the ansatz,

$$\Psi(t,x,y,\varphi,\psi) = e^{-i\omega t + ik\varphi + im\psi} u(x,y) \tag{74}$$

with $m, k \in \mathbb{Z}$, then reduces to the elliptic problem,

$$\partial_x\Big(\frac{G(x)}{(x-y)^2}\partial_x u\Big) + \partial_y\Big(\frac{-G(y)}{(x-y)^2}\partial_y u\Big) - (1+\nu^2)V(x,y)u = 0 \tag{75}$$

with,

$$V(x,y) = k^2 \frac{F(x)}{(x-y)^2 G(x)} - \omega^2 \frac{R^2 F(x)}{(x-y)^4 F(y)} - \frac{(cR(1+y)\omega - mF(y))^2}{(x-y)^2 F(y) G(y)}$$

Note that the term multiplying ω^2 vanishes sufficently fast at $y = -\frac{1}{\lambda}$ (the ergosurface) so this term is in fact smooth at $y = -\frac{1}{\lambda}$.

Motivated by taking the PDE (75) multiplying by $\bar{u}$ and integrating by parts. Consider the following energy functional

$$E(u) := \int_{-1}^{1}\int_{-\frac{1}{\nu}}^{-1} \Big\{\frac{G(x)}{(x-y)^2}|\partial_x u|^2 + \frac{-G(y)}{(x-y)^2}|\partial_y u| + (1+\nu^2)V(x,y)|u|^2\Big\} dxdy. \tag{76}$$

Note that the minus sign in the second term is deceptive; $-G(y) \geq 0$.

2.2 Energy Flux at the Horizon

The energy flux at the horizon is,

$$\begin{aligned}
\mathcal{E}(\mathcal{H}_A^+) &= J_\alpha^T(k) = \mathrm{Re}(T(\Psi)\overline{k(\Psi)}) - \frac{1}{2}\cancelto{0}{\Big(g(\partial_t,\partial_t) + \Omega_H g(\partial_t,\partial_\psi)\Big)} g(\nabla\Psi, \overline{\nabla\Psi}) && (77)\\
&= \big(|\omega|^2 - \Omega_H m \mathrm{Re}(\omega)\big)|\Psi|^2 && (78)
\end{aligned}$$

Hence a mode is superradiant iff,

$$\Omega_H m \mathrm{Re}(\omega) - |\omega|^2 > 0. \tag{79}$$

2.3 Boundary Conditions

Define the functions $\bar{t}(y)$ and $\bar{\psi}(y)$ by,

$$\begin{aligned}
\frac{d\bar{t}}{dy} &= \frac{c\sqrt{1+\nu^2}R(1+y)}{G(y)\sqrt{-F(y)}} && (80)\\
\frac{d\bar{\psi}}{dy} &= -\frac{\sqrt{-(1+\nu^2)F(y)}}{G(y)} && (81)
\end{aligned}$$

The relevant terms close to $\mathcal{H}^+$ are,

$$\begin{aligned}
\bar{t}(y) &= \frac{cR\sqrt{\nu(1+\nu^2)}}{\sqrt{\lambda-\nu}(1+\nu)} \log\Big(y + \frac{1}{\nu}\Big) + T(y) && (82)\\
\bar{\psi}(y) &= \frac{\sqrt{\nu(1+\nu^2)(\lambda-\nu)}}{(1-\nu)(1+\nu)} \log\Big(y + \frac{1}{\nu}\Big) + \zeta(y). && (83)
\end{aligned}$$

This gives,

$$\xi = i\frac{cR\sqrt{\nu(1+\nu^2)}}{(1+\nu)\sqrt{\lambda-\nu}}\big(\Omega_H m - \omega\big). \tag{84}$$

Hence the boundary behavior of u must be,

$$u(x,y) = \Big(y + \frac{1}{\nu}\Big)^{\xi} v(x,y) \tag{85}$$

For $v(x,y)$ smooth and smoothly extendable to the future event horizon, $\mathcal{H}^+$.

2.4 Conserved Currents and Consequences

Multiplying the equation by $\bar{u}(x,y)$, gives,

$$\begin{aligned} \partial_x\Big(\frac{G(x)}{(x-y)^2}\partial_x u\Big)\bar{u} + \partial_y\Big(\frac{-G(y)}{(x-y)^2}\partial_y u\Big)\bar{u} - (1+\nu^2)V(x,y)|u|^2 &= 0 \\ \partial_x\Big(\bar{u}\frac{G(x)}{(x-y)^2}\partial_x u\Big) + \partial_y\Big(\bar{u}\frac{-G(y)}{(x-y)^2}\partial_y u\Big) - \frac{G(x)}{(x-y)^2}|\partial_x u|^2 + \frac{G(y)}{(x-y)^2}|\partial_y u|^2 - (1+\nu^2)V(x,y) &= 0. \end{aligned}$$

Assuming $\omega \in \mathbb{R}$, taking the imaginary part gives,

$$\mathrm{Im}\Big[\partial_x\Big(\frac{G(x)}{(x-y)^2}\bar{u}\partial_x u\Big) + \partial_y\Big(\frac{-G(y)}{(x-y)^2}\bar{u}\partial_y u\Big)\Big] = 0. \tag{86}$$

Integrating over $x \in [-1,1]$ gives,

$$\partial_y \mathrm{Im}\Big[\int_{-1}^{1}\frac{-G(y)}{(x-y)^2}\bar{u}\partial_y u dx\Big] = \mathrm{Im}\Big[\frac{G(x)}{(x-y)^2}\bar{u}\partial_x u\Big|_{x=-1}^{x=1}\Big] = 0 \tag{87}$$

since $G(\pm 1) \equiv 0$. Hence, integrating now in $y \in [-\frac{1}{\nu}, -1]$,

0 since $G(\pm 1) \equiv 0$

$$\mathrm{Im}\Big[\int_{-1}^{1}\frac{-G(y)}{(x-y)^2}\bar{u}\partial_y u dx\Big]\Big|_{y=-1} - \mathrm{Im}\Big[\int_{-1}^{1}\frac{-G(y)}{(x-y)^2}\bar{u}\partial_y u dx\Big]\Big|_{y=-\frac{1}{\nu}} = 0. \tag{88}$$

Defining y_* via,

$$\frac{dy_*}{dy} = \frac{cR\sqrt{(1+\nu^2)}(1+y)}{G(y)\sqrt{-F(y)}} \tag{89}$$

gives,

$$\mathrm{Im}\Big[\int_{-1}^{1} -\frac{cR\sqrt{(1+\nu^2)}(1+y)}{(x-y)^2\sqrt{-F(y)}}\bar{u}\partial_{y_*} u dx\Big]\Big|_{y=-\frac{1}{\nu}} = 0. \tag{90}$$

Note that,

$$\partial_{y_*} u = \frac{\partial y}{\partial y_*}\partial_y u = \frac{G(y)\sqrt{-F(y)}}{kR\sqrt{(1+\nu^2)}(1+y)}\xi\Big(y+\frac{1}{\nu}\Big)^{-1+\xi} v + \frac{G(y)\sqrt{-F(y)}}{cR\sqrt{(1+\nu^2)}(1+y)}\xi\Big(y+\frac{1}{\nu}\Big)^{\xi}\partial_y v. \tag{91}$$

So,

$$\partial_{y_*} u|_{y=-\frac{1}{\nu}} = \frac{(1+\nu)\sqrt{-F(-\frac{1}{\nu})}}{c\nu R\sqrt{(1+\nu^2)}}\xi\Big(y+\frac{1}{\nu}\Big)^{\xi} v. \tag{92}$$

Hence,

$$-\frac{(1-\nu^2)}{\nu^2}\mathrm{Im}(\xi)\int_{-1}^{1}\frac{1}{(x+\frac{1}{\nu})^2}|v|^2\big(x,-\frac{1}{\nu}\big)dx = 0. \tag{93}$$

So either,

$$\mathrm{Im}(\xi) = 0 \tag{94}$$

or this $|v|^2(x, -\frac{1}{\nu})$ vanishes a.e. (does this nessecarily imply $v \equiv 0$?).

$\xi = 0$ gives,

$$\Omega_H m - \omega = 0 \tag{95}$$

i.e. a real mode has zero flux at the horizon. The potential becomes,

$$V(x,y) = k^2 \frac{(1+2\nu x+\nu^2)}{(1-x^2)(x-y)^2(1+\nu x)} + m^2 \frac{1-\nu}{1+\nu} \frac{p(\nu,x,y)}{(x-y)^4(y^2-1)} \tag{96}$$

with

$$p(\nu,x,y) := 1 - 4x(y-\nu) - (\nu - y(1-\nu))(y+(y-1)\nu) + 2x^2(1-y\nu) \tag{97}$$

Now

$$p(\nu,-1+\epsilon,-1-\epsilon) = -2(1-\nu)^2\epsilon + (7-2\nu-\nu^2)\epsilon^2 + 2\nu\epsilon^3. \tag{98}$$

so $p(\nu,x,y) \to 0^-$ as $(x,y) \to (-1,-1)$ from below at a rate of ϵ. Further,

$$p\Big(\nu,x,-\frac{1}{\nu}\Big) = \frac{(1+2\nu x+\nu^2)^2}{\nu^2} > 0. \tag{99}$$

From this it is clear that this part of the potential has a region of negativity (Initially, I was hoping that it would have the right sign towards infinity, then I hoped that the k term could be use to counter this, like in KG on Kerr). In fact, the roots of $p(\nu,x,y)$ can be written explicitly as

$$x_\pm = -\frac{\nu-y}{1-\nu y} \pm \frac{\sqrt{-2(1-\nu^2)G(y)}}{2(1-\nu y)} \tag{100}$$

Note that $-\frac{\nu-y}{1-\nu y} \leq -1$. So the root in the range of x is always x_+.

However, there is an issue note that the m^2 term in the potential (96) diverges at a rate of $\frac{1}{\epsilon^4}$ whilst the k^2 diverges at a rate $\frac{1}{\epsilon^3}$ as $(x,y) \to (-1,-1)$. Therefore since $p(\nu,x,y) \to 0^-$ as $(x,y) \to (-1,-1)$ the potential diverges to $-\infty$. Whilst this means there is always a region of negativity, proving the existence of a bound state may be futile.

Remark. *Heres the issue. I expected this equation to behave like KG, i.e., for the mass to behave like a mirror and give the possiblity of a bound state. By this I mean that the potential tends to positive infinity, so that one can has reflection of the wave into the potential. The point is, this occurs for AKK spacetimes where the KK momenta acts like an effective mass giving rise to a KG equation. However, when the spacetime is AF the wave equation behaves like the wave equation on Minkowski space far way and therefore disperses. You can see this in the potential where the potential decays to minus infinity. You can see this from the point of loss of compactness also. This means that the analogy Oscar makes in his paper on the doubly spinning ring could be false. It is possible that if one expands the doubly spinning ring metric around the boosted Kerr string the next order perturbation will be large, significantly changing the potential. There is also an analogy with spinning down the KG instability onto Schwarszchild (as $a \to 0$, $\mu \to 0$). The solution of the wave equation is known to be bounded for Schwarszchild but the KG on Kerr is not.*

Remark. *Suppose further that $m = 0$ hence $\omega = 0$ then,*

$$V(x,y) = k^2 \frac{(1+2\nu x+\nu^2)}{(1-x^2)(x-y)^2(1+\nu x)} \geq 0. \tag{101}$$

Suppose u *has a postive maxima* $(x_0, y_0) \in (-1, 1) \times (-\frac{1}{\nu}, -1)$ *then,*

$$\frac{G(x)}{(x-y)^2}\partial_x^2 u + \frac{-G(y)}{(x-y)^2}\partial_y^2 u > 0 \tag{102}$$

which is a contradiction.

Suppose u *has a negative minima* $(x_0, y_0) \in (-1, 1) \times (-\frac{1}{\nu}, -1)$ *then,*

$$\frac{G(x)}{(x-y)^2}\partial_x^2 u + \frac{-G(y)}{(x-y)^2}\partial_y^2 u < 0 \tag{103}$$

which is a contradiction. With the boundary conditions this should give triviality of the solution.

Appendices

A Geometric Analysis

Definition A.1 (Energy-Momentum Tensor)**.** *Consider a Lorentzian manifold (M, g) and a complex-valued smooth scalar field Ψ of mass $\mu \geq 0$. Then, one can define an energy-momentum tensor associated to Ψ:*

$$\mathbf{T}[\Psi](X,Y) = \mathrm{Re}\big(\nabla_X \Psi \overline{\nabla_Y \Psi}\big) - \frac{1}{2} g(X,Y)\big(\langle \nabla\Psi, \nabla\Psi\rangle_g + \mu^2|\Psi|^2\big) \tag{104}$$

where $\langle \nabla\Psi, \nabla\Psi\rangle_g = g(\nabla\Psi, \overline{\nabla\Psi})$. Further one can define a covector field current,

$$J^X[\Psi] = \mathbf{T}[\Psi](X, \cdot) = \mathrm{Re}\big(\nabla_X \Psi \overline{\nabla \Psi_\flat}\big) - \frac{1}{2}\big(\langle \nabla\Psi, \nabla\Psi\rangle_g + \mu^2|\Psi|^2\big) X_\flat \tag{105}$$

Lemma A.1. *Suppose $\Psi \in C^\infty(M, \mathbb{C})$. Then,*

$$\mathrm{div}(J^X[\Psi])^\sharp := \nabla^a J^X_a = \mathrm{Re}\Big((\Box_g - \mu^2)\Psi \overline{X\Psi}\Big) + \frac{1}{2}\langle T, \Pi^X\rangle_g \tag{106}$$

with $\Pi^X = \mathcal{L}_X g$.

Proof.

$$\begin{aligned}
\mathrm{div}(J^X[\Psi])^\sharp \;&=\; \mathrm{Re}\big((\nabla^a\nabla_X\Psi)\overline{\nabla_a\Psi}\big) + \mathrm{Re}\big(\nabla_X\Psi\overline{\Box_g\Psi}\big) && (107)\\
&\quad -\frac{1}{2}\big((\nabla_a\nabla_b\Psi)\overline{\nabla^b\Psi} + \nabla_b\Psi\overline{\nabla_a\nabla^b\Psi} + \mu^2(\nabla_a\Psi)\overline{\Psi} + \mu^2\Psi\overline{\nabla_a\Psi}\big)X^a \\
&\quad -\frac{1}{2}\big(\langle\nabla\Psi,\nabla\Psi\rangle_g + \mu^2|\Psi|^2\big)\nabla^a X_a \\
&= \mathrm{Re}\big((\nabla_{\overline{\nabla\Psi}}\nabla_X\Psi) + \overline{X\Psi}\Box_g\Psi\big) - \mathrm{Re}\big((\nabla_{\overline{\nabla\Psi}}\nabla_a\Psi)\big)X^a + \mathrm{Re}\big(\mu^2(\nabla_X\Psi)\overline{\Psi}\big) && (108)\\
&\quad -\frac{1}{2}\big(\langle\nabla\Psi,\nabla\Psi\rangle_g + \mu^2|\Psi|^2\big)\nabla^a X_a \\
&= \mathrm{Re}\big(\overline{X\Psi}(\Box_g - \mu^2)\Psi\big) + \mathrm{Re}\Big((\nabla_{\overline{\nabla\Psi}}\nabla_X\Psi) - (\nabla_{\overline{\nabla\Psi}}\nabla_a\Psi)X^a\Big) && (109)\\
&\quad -\frac{1}{2}\big(\langle\nabla\Psi,\nabla\Psi\rangle_g + \mu^2|\Psi|^2\big)\nabla^a X_a \\
&= \mathrm{Re}\big(\overline{X\Psi}(\Box_g - \mu^2)\Psi\big) + \mathrm{Re}\big((\nabla_{\overline{\nabla\Psi}}X^a)\nabla_a\Psi\big) - \frac{1}{2}\big(\langle\nabla\Psi,\nabla\Psi\rangle_g + \mu^2|\Psi|^2\big)\nabla^a X_a
\end{aligned}$$

Now,

$$(\mathbf{T}[\Psi])^{ab}\Pi^X_{ab} = 2\mathrm{Re}\big(\nabla^a\Psi(\nabla_{\overline{\nabla\Psi}}X_a)\big) - \nabla_a X^a\big(\langle\nabla\Psi,\nabla\Psi\rangle_g + \mu^2|\Psi|^2\big). \tag{110}$$

Hence result. □

Remark. *If Ψ satisfies the Klein-Gordon equation then and X is Killing then J^X is a conserved 1-form current.*

Lemma A.2. *Let X, Y be two linear independent future-orientated timelike vectors such that,*

$$g(X,X) = -1 = g(Y,Y) \qquad \gamma = -g(X,Y) > 1. \tag{111}$$

Define,

$$\begin{aligned}
W_+ &= \frac{1}{\sqrt{2(\gamma+1)}}(X+Y) && (112)\\
W_- &= \frac{1}{\sqrt{2(\gamma-1)}}(X-Y) && (113)\\
L &= W_+ + W_- && (114)\\
\underline{L} &= W_+ - W_-. && (115)
\end{aligned}$$

Let $\{E_i\}_{i=1}^{n-2}$ be an orthonormal basis for the $(n-2)$-dimensional subspace orthogonal to $\mathrm{span}(X,Y)$. Then,

$$g\big((J^X[\Psi])^\sharp, Y\big) = \frac{1}{4}\Big(|L\Psi|^2 + |\underline{L}\Psi|^2\Big) + \frac{\gamma}{2}\sum_{i=1}^{n-2}|E_i\Psi|^2 - \frac{1}{2}\mu^2|\Psi|^2 \tag{116}$$

Proof. First note that, $g(L,L) = 0 = g(\underline{L},\underline{L})$ and $g(L,\underline{L}) = -2$. Further,

$$\nabla\Psi = \sum_{i=1}^{n-2}(E_i\Psi)E_i + \frac{1}{g(L,\underline{L})}\Big((L\Psi)\underline{L} + (\underline{L}\Psi)L\Big). \tag{117}$$

Next,

$$X = \frac{1}{4}\big(c_+L + c_-\underline{L}\big) \tag{118}$$

$$Y = \frac{1}{4}\big(c_-L + c_+\underline{L}\big) \tag{119}$$

with,

$$c_+ = \sqrt{2(\gamma+1)} + \sqrt{2(\gamma-1)} \qquad c_- = \sqrt{2(\gamma+1)} - \sqrt{2(\gamma-1)}. \tag{120}$$

Note,

$$c_+c_- = 4 \tag{121}$$

$$c_+^2 = 4\big(\gamma + \sqrt{\gamma^2-1}\big) \tag{122}$$

$$c_-^2 = 4\big(\gamma - \sqrt{\gamma^2-1}\big). \tag{123}$$

So,

$$Y(\Psi)\overline{X(\Psi)} = \frac{1}{4}\Big(|L\Psi|^2 + |\underline{L}\Psi|^2\Big) + \frac{1}{2}\gamma\mathrm{Re}\big(L\Psi\underline{L}\overline{\Psi}\big) + \frac{1}{2}i\sqrt{\gamma^2-1}\mathrm{Im}\big(L\Psi\underline{L}\overline{\Psi}\big) \tag{124}$$

$$\mathrm{Re}\big(Y(\Psi)\overline{X(\Psi)}\big) = \frac{1}{4}\Big(|L\Psi|^2 + |\underline{L}\Psi|^2\Big) + \frac{1}{2}\gamma\mathrm{Re}\big(L\Psi\underline{L}\overline{\Psi}\big). \tag{125}$$

Also,

$$\begin{aligned}
\langle\nabla\Psi,\nabla\Psi\rangle_g &= g(\nabla\Psi,\overline{\nabla\Psi}) && (126)\\
&= \sum_{i=1}^{n-2}E_i\Psi\overline{E_j\Psi}g(E_i,E_j) + \frac{1}{g(L,\underline{L})^2}g\big((L\Psi)\underline{L} + (\underline{L}\Psi)L, (\overline{L\Psi})\underline{L} + (\overline{\underline{L}\Psi})L\big) && (127)\\
&= \sum_{i=1}^{n-2}|E_i\Psi|^2 - \mathrm{Re}\big(L\Psi\overline{\underline{L}\Psi}\big) && (128)\\
& && (129)
\end{aligned}$$

Then,

$$\begin{aligned}
g\big((J^X[\Psi])^\sharp, Y\big) &= (\mathbf{T}[\Psi])(X,Y) = \mathrm{Re}\big(Y(\Psi)\overline{X(\Psi)}\big) - \frac{1}{2}g(X,Y)\Big(\langle\nabla\Psi,\nabla\Psi\rangle_g + \mu^2|\Psi|^2\Big) && (130)\\
&= \frac{1}{4}\Big(|L\Psi|^2 + |\underline{L}\Psi|^2\Big) + \frac{\gamma}{2}\mathrm{Re}\big(L\Psi\underline{L}\overline{\Psi}\big) + \frac{\gamma}{2}\Big(\sum_{i=1}^{n-2}|E_i\Psi|^2 - \mathrm{Re}\big(L\Psi\overline{\underline{L}\Psi}\big)\Big) && (131)\\
&= \frac{1}{4}\Big(|L\Psi|^2 + |\underline{L}\Psi|^2\Big) + +\frac{\gamma}{2}\sum_{i=1}^{n-2}|E_i\Psi|^2. && (132)
\end{aligned}$$

□

B Asymptotically Flat Spacelike Hypersurfaces

Definition B.1 (Spacelike Hypersurface). *A hypersurface, Σ, is spacelike if its normal n at each point $p \in \Sigma$ is timelike.*

B.1 Kerr

In Kerr for sufficently large R a spacelike hypersurface satisfies,

$$\Sigma \cap \{r \geq R\} = \{(t, r, \theta, \varphi) : r \geq R \text{ and } t - f(r) = 0\} \tag{133}$$

One requires that Σ intersects the future event horizon and, $f \geq 0$ as $r \to \infty$. To be asymptotically flat one requires $f \sim 1$ as $r \to \infty$. The typical example would be taking $\Sigma = \{t_* = \tau\}$ where τ is constant and,

$$t_* = t + r_+ \log(r - r_+) - r_- \log(r - r_-) \underset{a\to 0}{\longrightarrow} t + 2M \log(r - 2M) \tag{134}$$

Therefore,

$$f(r) = \tau - r_+ \log(r - r_+) + r_- \log(r - r_-). \tag{135}$$

Proposition 4. *Let Σ be an asymptotically flat hypersurface, N be future-orientated time-like vector field which equals ∂_t for large r and Ψ smooth. Then for sufficently large R, the energy of Ψ with respect to N along $\Sigma \cap \{r \geq R\}$ is approximately equal to,*

$$\int_{r\geq R}\int_{\mathbb{S}^2} \frac{1}{2}\Big(|\partial_t\Psi|^2 + |\partial_r\Psi|^2 + \frac{1}{r^2}\big(|\partial_\theta\Psi|^2 + \frac{1}{\sin^2\theta}|\partial_\varphi\Psi|^2\big)\Big)(t = f(r), r, \theta, \varphi)r^2 \sin\theta dr d\theta d\varphi \tag{136}$$

Proof. First note that $n \propto m := -\nabla(t - f)$ and

$$-\nabla t = -g^{tt}\partial_t - g^{t\varphi}\partial_\varphi = \frac{(r^2 + a^2)^2 - a^2\sin^2\theta\Delta}{\Sigma\Delta}\partial_t + \frac{4Mar}{\Sigma\Delta}\partial_\varphi \tag{137}$$

and,

$$g(\nabla t, \nabla t) = \frac{-(r^2 + a^2)^2 + a^2\sin^2\theta\Delta}{\Sigma\Delta}. \tag{138}$$

Now,

$$m = \frac{(r^2 + a^2)^2 - a^2\sin^2\theta\Delta}{\Sigma\Delta}\partial_t + \frac{4Mar}{\Sigma\Delta}\partial_\varphi + \frac{\Delta}{\Sigma}f'\partial_r \tag{139}$$

$$g(m, m) = -\frac{(r^2 + a^2)^2}{\Sigma\Delta} + \frac{a^2\sin^2\theta}{\Sigma} + \frac{\Delta}{\Sigma}f'^2 \to -1 \quad \text{as } r \to \infty \tag{140}$$

Note one assumes that $\lim_{r\to\infty} f(r) = L_1 < \infty$ and $\lim_{r\to\infty} f(r) = L_2 < \infty$ which forces $L_2 = 0$, so $f(r) = 1 + \frac{k}{r} + O(\frac{1}{r^2})$. Hence,

$$n = \frac{m}{\sqrt{-g(m, m)}} = \Big(1 + \frac{M}{r} + O\Big(\frac{1}{r^2}\Big)\Big)\partial_t + \Big(\frac{k}{r} + O\Big(\frac{1}{r^2}\Big)\Big)\partial_r + \Big(\frac{2aM}{r^3} + O\Big(\frac{1}{r^4}\Big)\Big)\partial_\varphi. \tag{141}$$

One can calculate $J^N(n)$ using lemma A.2. $N = \partial_t$ for large $r \geq R$, however this is not normalized,

$$\hat{N} = \sqrt{\frac{\Sigma}{\Delta - a^2\sin^2\theta}}\partial_t = \Big(1 + \frac{M}{r} + O\Big(\frac{1}{r^2}\Big)\Big)\partial_t \quad \text{as } r \to \infty. \tag{142}$$

So,

$$\gamma = -g(\hat{N}, n) = 1 + \frac{k^2}{2r^2} + O\Big(\frac{1}{r^3}\Big) \quad \text{as } r \to \infty. \tag{143}$$

So dropping $O(\frac{1}{r^2})$ and beyond,

$$\begin{aligned} W &= \Big(\frac{1}{2}+O\Big(\frac{1}{r}\Big)\Big)\Big[(2+\frac{2M}{r})\partial_t+\frac{k}{r}\partial_r\Big] = \Big(1+O\Big(\frac{1}{r}\Big)\Big)\partial_t+O\Big(\frac{1}{r}\Big)\partial_r \quad \text{as } r\to\infty & (144)\\ Z &= \Big(\frac{r}{|k|}+O(1)\Big)\frac{k}{r}\partial_r = \partial_r \quad \text{as } r\to\infty & (145)\end{aligned}$$

So,

$$\begin{aligned} L &= \partial_t+\mathrm{sign}(k)\partial_r \qquad r\to\infty & (146)\\ \underline{L} &= \partial_t-\mathrm{sign}(k)\partial_r \qquad r\to\infty & (147)\end{aligned}$$

So,

$$\begin{aligned} |L\Psi|^2 &= |\partial_t\Psi|^2+|\partial_r\Psi|^2+\partial_t\Psi\overline{\partial_r\Psi}+\overline{\partial_t\Psi}\partial_r\Psi & (148)\\ |\underline{L}\Psi|^2 &= |\partial_t\Psi|^2+|\partial_r\Psi|^2-\partial_t\Psi\overline{\partial_r\Psi}-\overline{\partial_t\Psi}\partial_r\Psi & (149)\\ |L\Psi|^2+|\underline{L}\Psi|^2 &= 2|\partial_t\Psi|^2+2|\partial_r\Psi|^2 & (150)\end{aligned}$$

Note further,

$$\begin{aligned} E_1 &= \frac{1}{\sqrt{\Sigma}}\partial_\theta = \frac{1}{r}\partial_\theta \qquad r\to\infty & (151)\\ E_2 &\propto e_2 := f'(r)\partial_t+\partial_r-\frac{f'(r)(\Delta(r)-a^2\sin^2\theta)}{2Mar\sin^2\theta}\partial_\varphi = \partial_r-\frac{k}{2Ma\sin^2\theta}\partial_\varphi \qquad r\to\infty & (152)\end{aligned}$$

and,

$$g(e_2,e_2) = \frac{\Sigma}{\Delta}-\frac{\Sigma\Delta(a^2\sin^2\theta-\Delta)f'^2}{4M^2a^2r^2\sin^2\theta} = \frac{r^2k^2}{4M^2a^2\sin^2\theta} \quad r\to\infty. \tag{153}$$

So,

$$E_2 = \frac{2Ma\sin\theta}{|k|r}\Big(\partial_r-\frac{k}{2Ma\sin^2\theta}\partial_\varphi\Big) = \frac{2Ma\sin\theta}{|k|r}\partial_r-\mathrm{sign}(k)\frac{1}{r\sin\theta}\partial_\varphi \quad r\to\infty \tag{154}$$

$$J^N(n) \approx \frac{1}{2}\Big(|\partial_t\Psi|^2+|\partial_r\Psi|^2+\frac{1}{r^2}\Big[|\partial_\theta\Psi|^2+\frac{1}{\sin^2\theta}|\partial_\varphi\Psi|^2\Big]\Big) \tag{155}$$

Note the ∂_r contribution to the $|\partial_r\Psi|^2$ is negligable as this already goes at $\frac{1}{r^2}$ by dimensional analysis.

The volume form on Kerr is,

$$\mathrm{dvol} = \Sigma\sin\theta dt\wedge dr\wedge d\theta\wedge d\varphi \tag{156}$$

Note that the induced volume form satisfies $i_n\mathrm{dvol}(X_1,...,X_{n-1}) = \mathrm{dvol}(n,X_1,...,X_{n-1})$, so,

$$\begin{aligned} i_n\mathrm{dvol} &= \Big(1+\frac{M}{r}+O\Big(\frac{1}{r^2}\Big)\Big)\Sigma\sin\theta dr\wedge d\theta\wedge d\varphi+f'\Sigma\sin\theta dt\wedge d\theta\wedge d\varphi & (157)\\ &\quad +\frac{2aM}{r^3}\Sigma\sin\theta dt\wedge dr\wedge d\theta+... \\ &= \Big(1+O\Big(\frac{1}{r}\Big)\Big)r^2\sin\theta dr\wedge d\theta\wedge d\varphi+O(r)\sin\theta dt\wedge d\theta\wedge d\varphi & (158)\\ &\quad +O\Big(\frac{1}{r}\Big)\sin\theta dt\wedge dr\wedge d\theta \quad \text{as } r\to\infty \end{aligned}$$

Therefore, after setting $t=f(r)$, the energy is approximately,

$$\int_{r\geq R}\int_{\mathbb{S}^2}\frac{1}{2}\Big(|\partial_t\Psi|^2+|\partial_r\Psi|^2+\frac{1}{r^2}\Big(|\partial_\theta\Psi|^2+\frac{1}{\sin^2\theta}|\partial_\varphi\Psi|^2\Big)\Big)(t=f(r),r,\theta,\varphi)r^2\sin\theta drd\theta d\varphi \tag{159}$$

□

B.2 Rings

For x, y sufficently close to -1 one can define new coordinates,

$$r_1 = R\sqrt{\frac{1-\lambda}{1-\nu}}\frac{\sqrt{2(1+x)}}{(x-y)} \qquad r_2 = R\sqrt{\frac{1-\lambda}{1-\nu}}\frac{\sqrt{-2(1+y)}}{(x-y)} \tag{160}$$

which show that the metric approaches,

$$g = -dt\otimes dt + dr_1\otimes dr_1 + r_1^2 d(c\varphi)\otimes d(c\varphi) + dr_2\otimes dr_2 + r_2^2 d(c\psi)\otimes d(c\psi) \quad c = \frac{1-\nu}{\sqrt{1-\lambda}} \tag{161}$$

so $c\varphi$ and $c\psi$ take values in the range $[0, 2\pi)$. Hence,

$$g = \eta \tag{162}$$

under coordinate transformation,

$$x^1 = r_1\cos c\varphi \qquad x^2 = r_1 \sin c\varphi \qquad x^3 = r_2\cos c\psi \qquad x^4 = r_2\sin c\psi \tag{163}$$

Note one can define,

$$r := \sqrt{r_1^2 + r_2^2} = R\sqrt{\frac{1-\lambda}{1-\nu}}\sqrt{\frac{2}{(x-y)}} \tag{164}$$

For $r \geq R$ with R sufficently large a spacelike hypersurface satisfies,

$$\Sigma_f \cap \{r \geq R\} := \{(t, x, y, \varphi, \psi) : r \geq R \quad t - f(r) = 0\} \tag{165}$$

further it is convenient for Σ_f to intersect $\mathcal{H}^+$ and $f \geq 0$ as $r \to \infty$. To be asymptotically flat one requires $f \sim 1$ for $r \to \infty$.

Proposition 5. *Let Σ be an asymptotically flat hypersurface, N be future-orientated time-like vector field which equals ∂_t for $r \geq R$ with R sufficently large and Ψ smooth. Then for $r \geq R$ the energy of Ψ with respect to N along $\Sigma \cap \{r \geq R\}$ is approximately,*

$$\begin{aligned}\int_{-1}^{-(1-\epsilon)}\int_{-(1+\epsilon)}^{-1}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}\Big(|\partial_t\Psi|^2 \quad + \quad & \frac{(x-y)^2}{R^2\frac{1-\lambda}{1-\nu}}\Big[(y^2-1)|\partial_y\Psi|^2 \\ & +(1-x^2)|\partial_x\Psi|^2 + \frac{1}{(1-x^2)}|\partial_{(c\varphi)}\Psi|^2 \\ & +\frac{1}{y^2-1}|\partial_{(c\psi)}\Psi|^2\Big]\Big)(t = f(r), x, y, \varphi, \psi)\frac{R^4\frac{(1-\lambda)^2}{(1-\nu)^2}}{(x-y)^4}dxdyd(c\varphi)d(c\psi)\end{aligned} \tag{166}$$

Proof. The normal to such a hypersurface is,

$$\begin{aligned} n \quad \propto \quad & -\nabla(t - f(r)) = -g^{tt}\partial_t - g^{t\psi}\partial_\psi + f'\big(g^{xx}\partial_x r\partial_x + g^{yy}\partial_y r\partial_y\big) && (167)\\ = \quad & \frac{k^2(x-y)^2(1+y)^2 + F(x)^2G(y)}{F(x)G(y)F(x)}\partial_t + \frac{k(x-y)^2(1+y)}{RF(x)G(y)}\partial_\psi && (168)\\ & +\frac{(x-y)^2G(x)}{R^2F(x)}(\partial_x r)\partial_x - \frac{(x-y)^2G(y)}{R^2F(x)}(\partial_y r)f'\partial_y && (169)\\ g(\nabla(t-f(y)), \nabla(t-f(y))) \quad = \quad & -\frac{F(x)}{F(y)} - \frac{k^2(x-y)^2(1+y)^2}{F(x)F(y)G(y)} && (170)\\ & +f'^2\Big(\frac{(x-y)^2G(x)(\partial_x r)^2}{R^2F(x)} - \frac{(x-y)^2G(y)(\partial_x r)^2}{R^2F(x)}\Big) && (171)\\ = \quad & -1 - \Big(\frac{\lambda}{1-\lambda} - \frac{f_0^2(1-\nu)}{2R^2(1-\lambda)}\Big)(x-y) \quad \text{for } x\to y\to -1 && (172)\end{aligned}$$

So,

$$\begin{aligned} n &= \Big(1+\frac{f_0^2+2R^2\lambda-f_0^2\nu}{4R^2(1-\lambda)}(x-y)\Big)\partial_t+\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{1-\lambda}}}{2R(1-\lambda)(1-\nu)}(x-y)^2\partial_\psi \\ &\quad -\frac{f_0(1+x)(x-y)(1-\nu)}{R^2(1-\lambda)}\partial_x-\frac{f_0(1+y)(x-y)(1-\nu)}{R^2(1-\lambda)}\partial_y \end{aligned}$$

For $r \geq R$, $N \approx \partial_t$ so,

$$\hat{N}=\frac{1}{\sqrt{(-g_{tt})}}\partial_t=\sqrt{\frac{F(x)}{F(y)}}\partial_t=\Big(1+\frac{\lambda}{2(1-\lambda)}(x-y)-\frac{\lambda^2(x-y)(y+1)}{2(1-\lambda)^2}-\frac{\lambda^2(x-y)^2}{8(1-\lambda)^2}\Big)\partial_t. \tag{173}$$

Therefore,

$$\gamma=-g(\hat{N},n)=1+\frac{f_0^2(1-\nu)}{4R^2(1-\lambda)}(x-y)+.... \tag{174}$$

and,

$$\frac{1}{\sqrt{2(\gamma+1)}} = \frac{1}{2}+O(x-y) \tag{175}$$

$$\frac{1}{\sqrt{2(\gamma-1)}} = \frac{R}{|f_0|}\sqrt{\frac{1-\lambda}{1-\nu}}\sqrt{\frac{2}{(x-y)}} \tag{176}$$

$$\begin{aligned} n+N &= \Big(2+\frac{f_0^2(1-\nu)+4\lambda R^2}{4R^2(1-\lambda)}(x-y)\Big)\partial_t-\frac{f_0(1+x)(x-y)(1-\nu)}{R^2(1-\lambda)}\partial_x \\ &\quad -\frac{f_0(1-\nu)}{R^2(1-\lambda)}(x-y)(1+y)\partial_y+\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{1-\lambda}}}{2R(1-\lambda)(1-\nu)}(x-y)^2\partial_\psi \end{aligned} \tag{177}$$

$$\begin{aligned} n-N &= \frac{f_0^2(1-\nu)}{4R^2(1-\lambda)}(x-y)\partial_t+\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{1-\lambda}}}{2R(1-\lambda)(1-\nu)}(x-y)^2\partial_\psi \\ &\quad -\frac{f_0(1+x)(x-y)(1-\nu)}{R^2(1-\lambda)}\partial_x-\frac{f_0(1-\nu)}{R^2(1-\lambda)}(x-y)(1+y)\partial_y \end{aligned} \tag{178}$$

So,

$$\begin{aligned} W_+ &= \Big(1+\frac{(1-\nu)f_0^2+8R^2\lambda}{16R^2(1+\lambda)}(x-y)\Big)\partial_t-\frac{f_0(1-\nu)}{2R^2(1-\lambda)}(x-y)(x+1)\partial_x \\ &\quad -\frac{f_0(1-\nu)}{2R^2(1-\lambda)}(x-y)(y+1)\partial_y+\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{1-\lambda}}}{4R(1-\lambda)(1-\nu)}(x-y)^2\partial_\psi+... \end{aligned} \tag{179}$$

$$W_- = \frac{|f_0|}{2R\sqrt{\frac{1-\lambda}{1-\nu}}}\sqrt{\frac{(x-y)}{2}}\partial_t+\frac{1}{|f_0|}\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{2(1-\nu)}}}{(1-\lambda)(1-\nu)}(x-y)^{\frac{3}{2}}\partial_\psi \tag{180}$$

$$-\mathrm{sign}(f_0)\frac{(1+x)\sqrt{2(x-y)}}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_x-\mathrm{sign}(f_0)\frac{(1+y)\sqrt{2(x-y)}}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_y. \tag{181}$$

So the approximate null frame is,

$$L=\partial_t+\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{2(1-\nu)}}(x-y)^{\frac{3}{2}}}{|f_0|(1-\lambda)(1-\nu)}\partial_\psi-\sigma(f_0)\frac{(1+x)\sqrt{2(x-y)}}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_x-\sigma(f_0)\frac{(1+y)\sqrt{2(x-y)}}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_y \tag{182}$$

$$\underline{L}=\partial_t-\frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{2(1-\nu)}}(x-y)^{\frac{3}{2}}}{|f_0|(1-\lambda)(1-\nu)}\partial_\psi+\sigma(f_0)\frac{(1+x)\sqrt{2(x-y)}}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_x+\sigma(f_0)\frac{(1+y)\sqrt{2(x-y)}}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_y \tag{183}$$

Note that considering $x=-1+\epsilon$ and $y=-(1+\epsilon)$ gives ,

$$
\begin{aligned}
(1+y)\sqrt{2(x-y)} &\approx -\frac{1}{\sqrt{2}}(x-y)\sqrt{y^2-1} && (184)\\
(1+x)\sqrt{2(x-y)} &\approx \frac{1}{\sqrt{2}}(x-y)\sqrt{1-x^2} && (185)\\
& && (186)
\end{aligned}
$$

So,

$$
L \approx \partial_t + \frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{2(1-\nu)}}(x-y)^{\frac{3}{2}}}{|f_0|(1-\lambda)(1-\nu)}\partial_\psi - \sigma(f_0)\frac{(x-y)\sqrt{1-x^2}}{\sqrt{2}R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_x + \sigma(f_0)\frac{(x-y)\sqrt{y^2-1}}{\sqrt{2}R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_y \quad (187)
$$

$$
\underline{L} \approx \partial_t - \frac{\sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{2(1-\nu)}}(x-y)^{\frac{3}{2}}}{|f_0|(1-\lambda)(1-\nu)}\partial_\psi + \sigma(f_0)\frac{(x-y)\sqrt{1-x^2}}{\sqrt{2}R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_x - \sigma(f_0)\frac{(x-y)\sqrt{y^2-1}}{\sqrt{2}R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_y \quad (188)
$$

Take $E_2 = \frac{(x-y)}{R}\frac{1}{\sqrt{G(x)}}\partial_\varphi$. Therefore, as $x \to y \to -1$,

$$
E_2 = \frac{(x-y)}{R\sqrt{1-\nu}}\frac{1}{\sqrt{1-x^2}}\partial_\varphi = \frac{(x-y)}{R\sqrt{\frac{(1-\lambda)}{1-\nu}}}\frac{1}{\sqrt{1-x^2}}\partial_{(c\varphi)} \quad (189)
$$

E_1 and E_3 are more tricky to construct; they need to be orthogonal to E_2, $N=\partial_t$, n and one another. Therefore,

$$
\begin{aligned}
E_1 &\propto e_1 := \partial_x - \frac{\partial_x r}{\partial_y r}\partial_y && (190)\\
E_3 &\propto e_3 := \frac{f'}{G(x)\partial_x r}\Big(G(x)(\partial_x r)^2 - G(y)(\partial_y r)^2\Big)\partial_t + \partial_x - \frac{G(y)\partial_y r}{G(x)\partial_x r}\partial_y && (191)\\
&\quad + \frac{F(y)f'}{cR(1+y)G(x)\partial_x r}\Big(G(x)(\partial_x r)^2 - G(y)(\partial_y r)^2\Big)\partial_\psi
\end{aligned}
$$

The norm of e_1 and e_3 is,

$$
\begin{aligned}
g(e_1,e_1) &= \frac{-R^2F(x)(G(x)(\partial_x r)^2 - G(y)(\partial_y r)^2)}{(x-y)^2G(x)G(y)(\partial_y r)^2} && (192)\\
g(e_3,e_3) &= \frac{R^2F(x)(G(x)(\partial_x r)^2 - G(y)(\partial_y r)^2)[c^2R^2(1+y)^2 - F(y)G(y)f'^2(G(x)(\partial_x r)^2 - G(y)(\partial_y r)^2))]}{c^2(x-y)^2(1+y)^2G(x)^2(\partial_x r)^2}
\end{aligned}
$$

For $x=-(1-\epsilon)$, $y=-(1+\epsilon)$ one has,

$$
\begin{aligned}
E_3 &\approx O(\epsilon^{\frac{3}{2}})\partial_t + O(\epsilon^{\frac{5}{2}})(\partial_x+\partial_y) + \frac{(x-y)}{R\sqrt{\frac{1-\lambda}{1-\nu}}}\frac{1}{\sqrt{y^2-1}}\partial_{(c\psi)} && (193)\\
E_1 &\approx \frac{(x-y)\sqrt{1-x^2}}{\sqrt{2}R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_x + \frac{(x-y)\sqrt{y^2-1}}{\sqrt{2}R\sqrt{\frac{1-\lambda}{1-\nu}}}\partial_y && (194)
\end{aligned}
$$

Setting $x=-(1-\epsilon)$ and $y=-(1+\epsilon)$ gives,

$$
\begin{aligned}
L &= \partial_t + O(\epsilon^{\frac{3}{2}})\partial_{c\psi} - O(\epsilon^{\frac{3}{2}})\partial_x - O(\epsilon^{\frac{3}{2}})\partial_y && (195)\\
\underline{L} &= \partial_t - O(\epsilon^{\frac{3}{2}})\partial_{c\psi} + O(\epsilon^{\frac{3}{2}})\partial_x + O(\epsilon^{\frac{3}{2}})\partial_y && (196)\\
E_1 &= O(\epsilon^{\frac{3}{2}})\partial_x + O(\epsilon^{\frac{3}{2}})\partial_y && (197)\\
E_2 &= O(\epsilon^{\frac{1}{2}})\partial_{(c\varphi)} && (198)\\
E_3 &= O(\epsilon^{\frac{3}{2}})\partial_t + O(\epsilon^{\frac{5}{2}})(\partial_x+\partial_y) + O(\sqrt{\epsilon})\partial_{(c\psi)} && (199)
\end{aligned}
$$

Note that the dominant terms for ∂_t, ∂_y come from L and $\underline{L}$ not from E_3. So,

$$
\begin{aligned}
|L\Psi|^2 + |\underline{L}\Psi|^2 &= 2|\partial_t\Psi|^2 + \frac{(x-y)^2(1-x^2)}{R^2\frac{1-\lambda}{1-\nu}}|\partial_x\Psi|^2 + \frac{(x-y)^2(y^2-1)}{R^2\frac{1-\lambda}{1-\nu}}|\partial_y\Psi|^2 && (200)\\
&\quad -\frac{(x-y)^2\sqrt{1-x^2}\sqrt{y^2-1}}{R^2\frac{1-\lambda}{1-\nu}}(\partial_x\Psi\overline{\partial_y\Psi} + \partial_y\Psi\overline{\partial_x\Psi}) && (201)\\
&\quad +\frac{\frac{\lambda(1+\lambda)(\lambda-\nu)}{2(1-\nu)}(x-y)^3}{|f_0|^2(1-\lambda)^2(1-\nu)^2}|\partial_\psi\Psi|^2 && (202)\\
|E_1\Psi|^2 + |E_2\Psi|^2 + |E_3\Psi|^2 &= \frac{(x-y)^2}{R^2\frac{(1-\lambda)}{1-\nu}}\frac{1}{1-x^2}|\partial_{(c\varphi)}\Psi|^2 + \frac{(x-y)^2}{R^2\frac{1-\lambda}{1-\nu}}\frac{1}{(y^2-1)}|\partial_{(c\Psi)}\Psi|^2 && (203)\\
&\quad +\frac{(x-y)^2(1-x^2)}{2R^2\frac{1-\lambda}{1-\nu}}|\partial_x\Psi| + \frac{(x-y)^2(y^2-1)}{2R\frac{1-\lambda}{1-\nu}}|\partial_y\Psi|^2\\
&\quad +\frac{(x-y)^2\sqrt{1-x^2}\sqrt{y^2-1}}{2R^2\frac{1-\lambda}{1-\nu}}\big(\partial_x\Psi\overline{\partial_y\Psi} + \partial_x\Psi\overline{\partial_y\Psi}\big)
\end{aligned}
$$

So,

$$
\begin{aligned}
g\big((J^{\hat{N}})^\sharp, n\big) &= \frac{1}{4}\big(|L\Psi|^2 + |\underline{L}\Psi|^2\big) + \frac{1}{4}\big(|E_1\Psi|^2 + |E_2\Psi|^2 + |E_3\Psi|^2\big) && (204)\\
&= \frac{1}{2}\Big(|\partial_t\Psi|^2 + \frac{(x-y)^2(1-x^2)}{R^2\frac{1-\lambda}{1-\nu}}|\partial_x\Psi|^2 + \frac{(x-y)^2(y^2-1)}{R^2\frac{1-\lambda}{1-\nu}}|\partial_y\Psi|^2 && (205)\\
&\quad +\frac{(x-y)^2}{R^2\frac{1-\lambda}{1-\nu}}\frac{1}{y^2-1}|\partial_{(c\psi)}\Psi|^2 + \frac{(x-y)^2}{R^2\frac{(1-\lambda)}{1-\nu}}\frac{1}{1-x^2}|\partial_{(c\varphi)}\Psi|^2\Big). && (206)
\end{aligned}
$$

The volume form for the black ring is,

$$
\mathrm{dvol} = \frac{R^4F(x)}{(x-y)^4}dt\wedge dx\wedge dy\wedge d\varphi\wedge d\psi. \tag{207}
$$

Then one can calculate the induced volume form as,

$$
\begin{aligned}
i_n\mathrm{dvol} &\approx \frac{R^4(1-\lambda)}{(x-y)^4}\Big(dx\wedge dy\wedge d\varphi\wedge d\psi + O((x-y)^2)\big[dt\wedge dy\wedge d\varphi\wedge d\psi && (208)\\
&\quad +dt\wedge dx\wedge d\varphi\wedge d\psi dt\wedge dx\wedge dy\wedge d\varphi\big]\Big) && (209)\\
&\approx \frac{R^4\frac{(1-\lambda)^2}{(1-\nu)^2}}{(x-y)^4}dx\wedge dy\wedge d(c\varphi)\wedge d(c\psi) + O((x-y)^{-2}) && (210)
\end{aligned}
$$

□

C Null Frames and Ring Coordinates Minkowski Spacetime

Take $\mathbb{R}^{4+1}$ with the flat Minkowski metric,

$$
\eta = -dt\otimes dt + \delta_{ij}dx^i\otimes dx^j \tag{211}
$$

with $i, j \in \{1, ..., 4\}$. In ring coordinates the metric is,

$$
g = -dt\otimes dt + \frac{(x-y)^2}{R^2}\Big[\frac{1}{1-x^2}dx\otimes dx + \frac{1}{y^2-1}dy\otimes dy + (1-x^2)d\varphi\otimes d\varphi + (y^2-1)d\psi\otimes d\psi\Big] \tag{212}
$$

So a null frame is,

$$
\begin{aligned}
L &= \partial_t - \frac{(x-y)\sqrt{y^2-1}}{R}\partial_y && (213)\\
\underline{L} &= \partial_t + \frac{(x-y)\sqrt{y^2-1}}{R}\partial_y && (214)\\
E_1 &= -\frac{(x-y)}{R}\sqrt{1-x^2}\partial_x && (215)\\
E_2 &= -\frac{(x-y)}{R}\frac{1}{\sqrt{1-x^2}}\partial_\varphi && (216)\\
E_3 &= -\frac{(x-y)}{R}\frac{1}{\sqrt{y^2-1}}\partial_\psi && (217)
\end{aligned}
$$

D Singly Spinning Black Rings in (r,θ) Coordinates

D.1 Geometry

The singly spinning black ring solution is a 5D solution to the Einstein vacuum equation,

$$\mathrm{Ric}_g = 0 \tag{218}$$

One can define the singly spinning black ring exterior as the smooth manifold,

$$\mathcal{M} = \mathbb{R} \times (\mathbb{R}^4 \setminus (\mathbb{S}^1 \times \mathbb{B}^3)) \tag{219}$$

with metric in coordinates $(t, r, \theta, \varphi, \psi)$,

$$\begin{aligned} g \;:=\; & -\frac{f(r)}{h(\theta)}dt\otimes dt + \frac{h(\theta)}{(1-\frac{r^2}{R^2})(1+\frac{r\cos\theta}{R})^2 p(r)}dr\otimes dr + \frac{r^2 h(\theta)}{(1+\frac{r\cos\theta}{R})^2 q(\theta)}d\theta\otimes d\theta \\ & + \frac{r^2 q(\theta)\sin^2\theta}{(1+\frac{r\cos\theta}{R})^2}d\varphi\otimes d\varphi + \Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2 f(r)} - \frac{K^2(1-\frac{r}{R})^2}{r^2 f(r)h(\theta)}\Big)d\psi\otimes d\psi \\ & + \frac{K}{rh(\theta)}\Big(\frac{r}{R}-1\Big)\big(dt\otimes d\psi + d\psi\otimes dt\big) \end{aligned} \tag{220}$$

with the following definitions,

$$f(r) := 1 - \frac{r_h\cosh^2\sigma}{r} \qquad h(\theta) = 1 + \frac{r_h\cosh^2\sigma}{R}\cos\theta \tag{221}$$

$$p(r) := 1 - \frac{r_h}{R} \qquad q(\theta) := 1 + \frac{r_h}{R}\cos\theta \tag{222}$$

$$K \;:=\; r_h R\sinh\sigma\cosh\sigma\sqrt{\frac{R + r_h\cosh^2\sigma}{R - r_h\cosh^2\sigma}} \tag{223}$$

and coordinate ranges,

$$t\in\mathbb{R} \quad r\in(r_h, R) \quad \theta\in[0,\pi) \quad \varphi,\psi\in\Big[0, \frac{2\pi R}{\sqrt{R^2 + r_h^2}}\Big) \tag{224}$$

$r = r_h < R$ corresponds to the future event horizon, σ is a parameter and R sets the scale of the solution. Note that, the vector field ∂_t is null at,

$$g_{tt} = 0 \implies r = r_h\cosh^2\sigma \tag{225}$$

which corresponds to the ergosurface. So the ergoregion is given by,

$$r \in (r_h, r_e := r_h\cosh^2\sigma) \tag{226}$$

Note that one requires the 'equilibrium' condition,

$$\cosh^2\sigma = \frac{2R^2}{r_h^2 + R^2} \tag{227}$$

to avoid conical singularities. Thus,

$$f(r) := 1 - \frac{r_e}{r} \qquad h(\theta) = 1 + \frac{2Rr_h}{r_h^2 + R^2}\cos\theta \tag{228}$$

$$p(r) := 1 - \frac{r_h}{R} \qquad q(\theta) := 1 + \frac{r_h}{R}\cos\theta \tag{229}$$

$$K \;:=\; r_e(r_h + R)\sqrt{\frac{R + r_h}{2(R - r_h)}} \tag{230}$$

$$r_e \;:=\; \frac{2R^2 r_h}{(r_h^2 + R^2)} \tag{231}$$

The inverse metric is the following,

$$\begin{aligned} g^{-1} &= -\Big(\frac{h(\theta)}{f(r)} - \frac{K^2(R-r)(R+r\cos\theta)^2}{r^2R^4(r+R)p(r)f(r)h(\theta)}\Big)\partial_t\otimes\partial_t + \frac{(R^2-r^2)(R+r\cos\theta)^2p(r)}{R^4h(\theta)}\partial_r\otimes\partial_r \\ &+\frac{(R+r\cos\theta)^2q(\theta)}{r^2R^2h(\theta)}\partial_\theta\otimes\partial_\theta + \frac{(R+r\cos\theta)^2}{R^2r^2q(\theta)\sin^2\theta}\partial_\varphi\otimes\partial_\varphi + \frac{(R+r\cos\theta)^2f(r)}{R^2(R^2-r^2)h(\theta)p(r)}\partial_\psi\otimes\partial_\psi \\ &-\frac{K(R+r\cos\theta)^2}{rR^3(r+R)h(\theta)p(r)}(\partial_t\otimes\partial_\psi+\partial_\psi\otimes\partial_t) \end{aligned} \tag{232}$$

Note the vector field,

$$k = \frac{\partial}{\partial t} + \Omega_H\frac{\partial}{\partial\psi} \qquad \Omega_H^2 = \frac{R-r_h}{2R^2(R+r_h)} \tag{233}$$

is null future event horizon. Note that ψ is has period $\psi\sim\psi+\Delta\psi$ with $\Delta\psi := \frac{2\pi R}{\sqrt{R^2+r_h^2}}$ so Ω_H is not the true angular velocity of the horizon. Defining $\tilde\psi = \frac{2\pi}{\Delta\psi}\psi$ gives

$$\partial_\psi = \frac{\sqrt{R^2+r_h^2}}{R}\partial_{\tilde\psi}. \tag{234}$$

So the true angular velocity of the horizon is

$$\tilde\Omega_H = \frac{1}{R^2}\sqrt{\frac{(R-r_h)(R^2+r_h^2)}{2(R+r_h)}}. \tag{235}$$

Now, for $r\in(r_h, r_h\cosh^2\sigma)$, $f(r)<0$ so we can make the coordinate transformation,

$$dv = dt + \frac{K}{r(r+R)\sqrt{-f(r)}p(r)}dr \qquad d\psi = d\chi + \frac{R\sqrt{-f(r)}}{(R^2-r^2)p(r)}dr \tag{236}$$

which gives the metric,

$$\begin{aligned} g &:= -\frac{f(r)}{h(\theta)}dv\otimes dv + \frac{r^2h(\theta)}{(1+\frac{r\cos\theta}{R})^2q(\theta)}d\theta\otimes d\theta + \frac{r^2q(\theta)\sin^2\theta}{(1+\frac{r\cos\theta}{R})^2}d\varphi\otimes d\varphi \\ &+\Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2f(r)} - \frac{K^2(1-\frac{r}{R})^2}{r^2f(r)h(\theta)}\Big)d\chi\otimes d\chi \\ &+\frac{K}{rh(\theta)}\Big(\frac{r}{R}-1\Big)\big(dv\otimes d\chi+d\chi\otimes dv\big) + \frac{R^3h(\theta)}{(R+r\cos\theta)^2\sqrt{-f(r)}}(dr\otimes d\chi+d\chi\otimes dr) \end{aligned} \tag{237}$$

Then one has the following for k,

$$k = \partial_v + \Omega_H\partial_\chi. \tag{238}$$

Mapping k to a one-form,

$$k_\flat = \Big[\frac{K\Omega_H}{rh(\theta)}\Big(\frac{r}{R}-1\Big) - \frac{f(r)}{h(\theta)}\Big]dv + \Omega_H\frac{R^3h(\theta)}{(R+r\cos\theta)^2\sqrt{-f(r)}}dr \tag{239}$$

$$+\Big[\frac{K}{rh(\theta)}\Big(\frac{r}{R}-1\Big) + \Omega_H\Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2f(r)} - \frac{K^2(1-\frac{r}{R})^2}{r^2f(r)h(\theta)}\Big)\Big]d\chi \tag{240}$$

which at the future event horizon for a balanced ring gives,

$$k_\flat|_{r=r_h} = \frac{R^2}{\sqrt{2(R^2+r_h^2)(R+r_h)}}\frac{(R^2+r_0^2+2Rr_h\cos\theta)}{(R+r_h\cos\theta)^2}dr. \tag{241}$$

Note that,

$$g(k,k) = -\frac{f(r)}{h(\theta)} + 2K\Omega_H \frac{\left(\frac{r}{R}-1\right)}{rh(\theta)} + \Omega_H^2 \frac{1}{f(r)h(\theta)}\left(\frac{R^2(R^2-r^2)h(\theta)^2 p(r)}{(R+r\cos\theta)^2} - \frac{(\frac{r}{R}-1)^2 K^2}{r^2}\right). \quad (242)$$

To find the surface gravity, κ, of a balanced ring we consider $d(g(k,k))|_{r=r_h} = -2\kappa k|_{r=r_h}$. Evaluating, one finds,

$$\kappa = \frac{(R-r_h)\sqrt{(R^2+r_h^2)}}{2\sqrt{2}R^2 r_h}. \quad (243)$$

D.2 Scalar Perturbations

Consider the wave equation on the singly spinning black ring background,

$$\Box_g \Psi = 0. \quad (244)$$

Using the Laplace-Beltrami expansion one has,

$$\frac{1}{\sqrt{|\det(g)|}}\partial_\alpha\left(\sqrt{|\det(g)|}g^{\alpha\beta}\partial_\beta\Psi\right) = 0 \quad (245)$$

with,

$$\det(g) = -\frac{r^4 R^{10} h(\theta)^2 \sin^2\theta}{(R+r\cos\theta)^8} \quad (246)$$

D.3 Mode Solutions and Reduction to an Elliptic Problem

Consider the ansatz,

$$\Psi(t,r,\theta,\varphi,\psi) := e^{-i\omega t + ik\varphi + im\psi} u^{(\omega)}_{m,k}(r,\theta). \quad (247)$$

This can be written as,

$$\frac{1}{\sqrt{(-\det(g))}}\Big(\partial_r\big(f_r(r,\theta)\partial_r u\big) + \partial_\theta\big(f_\theta(r,\theta)\partial_\theta u\big) - V_{(\omega,m,k)}(r,\theta)u\Big) = 0 \quad (248)$$

with,

$$f_1(r,\theta) = \frac{r^2 p(r)(R^2-r^2)\sin\theta}{(R+r\cos\theta)^2} \quad (249)$$

$$f_2(r,\theta) = \frac{R^2 q(\theta)\sin\theta}{(R+r\cos\theta)^2} \quad (250)$$

$$V_{(\omega,m,k)}(r,\theta) = k^2 \frac{R^2 h(\theta)}{q(\theta)\sin\theta(R+r\cos\theta)^2} + V_{(m,\omega)}(r,\theta) \quad (251)$$

$$V_{(m,\omega)}(r,\theta) = \frac{\big(K\omega(R-r) + mrRf(r)\big)^2\sin\theta}{(R^2-r^2)(R+r\cos\theta)^2 f(r)p(r)} - \omega^2\frac{r^2R^4h(\theta)^2\sin\theta}{(R+r\cos\theta)^4 f(r)}. \quad (252)$$

Note that whilst the potential $V_{(\omega,m,k)}$ looks singular at the ergosurface, it is in fact an artifact of the decomposition.

One can assign the following energy functional

$$E_{(\omega,m)}(u) := \int_{r_h}^{R}\int_0^{\pi}\Big(f_r(r,\theta)(\partial_r u)^2 + f_\theta(r,\theta)(\partial_\theta u)^2 + V_{(\omega,m)}u^2\Big)dr d\theta. \quad (253)$$

This is motivated by taking the elliptic equation (248) and multiplying by $\sqrt{-\det(g)}u$ and integrating by parts assuming u satisfies appropriate boundary conditions such that the boundary terms vanish. Note that this would give

$$E(u) = -k^2 \int_{r_h}^{R} \int_0^{\pi} \frac{R^2 h(\theta)}{q(\theta)\sin\theta(R + r\cos\theta)^2} u^2 dr d\theta. \tag{254}$$

The hope is one can find a test function u such that $E(u) < 0$ and therefore argue the minimiser is a weak solution to the problem. Note that at this point the contribution in $V_{(\omega,m,k)}$ from the term multiplied by k is positive therefore this will always shift the potential more positive and therefore making $E(u)$ more positive.

D.3.1 Energy Flux at the Horizon

The energy flux at the horizon is,

$$\begin{aligned} \mathcal{E}(\mathcal{H}_A^+) &= J^T(k) = \mathrm{Re}(T(\Psi)\overline{k(\Psi)}) - \frac{1}{2}\Big(\cancelto{0}{g(\partial_t,\partial_t) + \Omega_H g(\partial_t,\partial_\psi)}\Big) g(\nabla\Psi, \overline{\nabla\Psi}) && (255) \\ &= \big(|\omega|^2 - \Omega_H m \mathrm{Re}(\omega)\big)|\Psi|^2 && (256) \end{aligned}$$

Hence a mode is superradiant iff,

$$\Omega_H m \mathrm{Re}(\omega) - |\omega|^2 > 0. \tag{257}$$

D.3.2 The Potential For a Real Mode with Zero Flux at the Horizon

Consider a mode with $\omega \in \mathbb{R}$ satisfying $\Omega_H m = \omega$, i.e., it has no flux at the horizon. The mixed contribution to the potential gives

$$V_{(m,\omega=\Omega_H m)}(r,\theta) = m^2 R^2 \frac{\big((R-r)r_h(R+r_h) + r(R^2+r_h^2)f(r)\big)^2 \sin\theta}{(R^2-r^2)(R^2+r_h^2)^2(R+r\cos\theta)^2 f(r)p(r)}. \tag{258}$$

This potential now has the property that $\lim_{r\to r_h} V_{m,\Omega_H m} = 0$. Therefore, the full potential now has a finite limit as $r \to r_h$. One has a energy functional

$$E_m(u) := \int_{r_h}^{R} \int_0^{\pi} \Big(f_r(r,\theta)(\partial_r u)^2 + f_\theta(r,\theta)(\partial_\theta u)^2 + m^2 V_0(r,\theta)u^2\Big) dr d\theta. \tag{259}$$

Therefore, if one can find a u such that the integral

$$\int_{r_h}^{R} \int_0^{\pi} V_0(r,\theta)u^2 dr d\theta < 0 \tag{260}$$

and the weighted $\dot{H}^1$-norm in $E(u)$ is finite then by taking m to be large (note that there will be a condition for the periodicity), gives $E(u) < 0$ and therefore,

$$\inf_{\substack{u\in H^1 \\ ||u||_{L^2}=1}} E(u) < 0 \tag{261}$$

provided one can show there is no loss of mass.

Remark. *(Something about boundary conditions.) If in H^1 one automatically has finite energy, so presumably the right boundary conditions are satisfied.*

Scaling $r = Rx$ and $r_h = \lambda R$ and setting $y = \cos\theta$ so that $x \in (\lambda, 1)$, $0 < \lambda < 1$ and $y \in [-1, 1]$ gives

$$V_0(x,y) = \frac{x^2(1-y^2)(1-\lambda)}{2(1+\lambda)(1+\lambda^2)} \frac{P(\lambda,x,y)}{Q(x,y)} \tag{262}$$

with

$$Q(x,y) = (1-x^2)(1+xy)^4\sqrt{1-y^2} \tag{263}$$

$$P(\lambda,x,y) = (1-\lambda^2) + 2\lambda x + (1+\lambda^2)x^2 + 2\lambda xy(2x+y) + 2xy(2+xy). \tag{264}$$

Note that $P(\lambda,\lambda,1) = (1+\lambda)^4 > 0$, for $P(\lambda, 1-\epsilon, -1+\epsilon) < -C_\epsilon$ and $P(\lambda,1,-1) = 0$. Hence, $P(\lambda,x,y)$ has regions of negativity.

Consider the test function

$$u(x,y) = \sqrt{Q(x,y)}\tilde{u}(x,y) \tag{265}$$

where $\tilde{u}(x,y)$ is a smooth function such that $\mathrm{supp}(\tilde{u}) \subset \{P(\lambda,x,y) \leq 0\}$ (maybe some smooth mollifier which has support is contained in the set $\{P(\lambda,x,y) \leq 0\}$ and $\mathbb{1}_{\{P(\lambda,x,y)\leq 0\}}$ is the indicator function on the set where $P(\lambda,x,y) \leq 0$ would do). Therefore,

$$\int_{r_h}^{R}\int_0^{\pi} V_0(r,\theta)drd\theta = R\int_\lambda^1\int_{-1}^1 \frac{x^2(1-y^2)(1-\lambda)}{2(1+\lambda)(1+\lambda^2)}P(\lambda,x,y)(\tilde{u}(x,y))^2\frac{dxdy}{\sqrt{1-y^2}} < 0. \tag{266}$$

Now,

$$f_r(r,\theta)(\partial_r u)^2 = f_x(\partial_x u)^2 \tag{267}$$

where

$$f_x(x,y) := \frac{x(1-x^2)\sqrt{1-y^2}(x-\lambda)}{(1+xy)^2}. \tag{268}$$

Therefore,

$$f_r(r,\theta)(\partial_r u)^2 = f_r(x,y)Q(x,y)(\partial_x\tilde{u})^2 + \tilde{u}^2x(x-\lambda)(1-y^2)(x-2y+3x^2y)^2 \tag{269}$$

which will have finite L^2-norm. Similarly,

$$f_\theta(r,\theta)(\partial_\theta u)^2 = f_y(x,y)(\partial_y u)^2 \tag{270}$$

with

$$f_y(x,y) = \frac{(1+\lambda y)(1-y^2)^{\frac{3}{2}}}{(1+xy)^2}. \tag{271}$$

So,

$$f_\theta(r,\theta)(\partial_\theta u)^2 = f_y(x,y)Q(x,y)(\partial_y\tilde{u})^2 + \tilde{u}^2\frac{1}{4}(1-x^2)(y+x(5y^2-4))^2(1+\lambda y) \tag{272}$$

which again has finite L^2-norm since $(1-y^2)^{-\frac{1}{2}}$ is integrable. Hence, schematically one has

$$||\nabla u||^2_{L^2_{w_1}} + m^2\langle Vu, u\rangle_{L^2} = -k^2||u||^2_{L^2_{w_2}}. \tag{273}$$

D.4 Boundary Conditions

Note one requires that $e^{im\varphi}u(r,\theta)$ extends smoothly to the $\mathbb{S}^2$. To consider the appropriate boundary conditions at $\mathcal{H}_A^+$ one must transform to $(v,r,\theta,\varphi,\chi)$ coordinates. The requirement at $\mathcal{H}_A^+$ is that,

$$e^{-i\omega v+i\omega\bar{t}(r)}e^{i\mu\chi-i\mu\bar{\psi}(r)}e^{im\varphi}u(r,\theta) \tag{274}$$

extends smoothly to $\mathcal{H}_A^+$. This requires that one can write,

$$u(r,\theta)=e^{-i\omega\bar{t}(r)}e^{i\mu\bar{\varphi}(r)}\tilde{u}(r,\theta) \tag{275}$$

for $\tilde{u}(r,\theta)$ smooth. Note that,

$$\frac{d\bar{t}}{dr} = \frac{K}{r(r+R)\sqrt{-f(r)}p(r)} \tag{276}$$

$$\frac{d\bar{\psi}}{dr} = \frac{R\sqrt{-f(r)}}{(R^2-r^2)p(r)} \tag{277}$$

$$\tag{278}$$

These can be integrated explicitly (note the solutions are only real valued for $r\in(r_h,r_e)$). The relavant terms for the boundary conditions are,

$$\bar{t}(r) = \frac{K\sqrt{s}}{\sqrt{r_e-r_h}(R+r_h)}\log(r-r_h)+T(r) \tag{279}$$

$$\bar{\psi}(r) = R\frac{\sqrt{r_h(r_e-r_h)}}{R^2-r_h^2}\log(r-r_h)+\zeta(r) \tag{280}$$

So the regularity condition (275) is equivalent to,

$$u(r,\theta)=(r-r_h)^{\xi}v(r,\theta) \tag{281}$$

for v smooth and,

$$\xi = i\frac{K\sqrt{r_h}}{\sqrt{r_e-r_h}(R+r_h)}\Big(\frac{R(r_e-r_h)}{K(R-r_h)}\mu-\omega\Big) \tag{282}$$

$$= i\frac{K\sqrt{r_h}}{\sqrt{r_e-r_h}(R+r_h)}\Big(\Omega_H\mu-\omega\Big). \tag{283}$$

For $r\to R$ one can maybe find a transformation. However, it will be a necessary condition that these modes have finite energy on a spacelike slice. Note that this means it will be required some H^1-norm finite, something like,

$$\int_{r\geq r_s+1}\int_{\mathbb{S}^2\times\mathbb{S}^1}\Big(|\partial_\theta u|^2+|\partial_r u|^2+|u|^2\Big)f(r,\theta,\varphi,\psi)drd\theta d\varphi d\psi<\infty. \tag{284}$$

For sufficently large $s\leq R$ a spacelike hypersurface satisfies,

$$\Sigma_f\cap\{r\geq s\}:=\{(t,r,\theta,\varphi,\psi):r\geq s\quad t-f(r)=0\} \tag{285}$$

further it is convenient for Σ_f to intersect $\mathcal{H}^+$ and $f\geq 0$ as $r\to R$. The normal to such a hypersurface is,

$$n \propto -\nabla(t-f(r))=-g^{tt}\partial_t-g^{t\psi}\partial_\psi+g^{rr}f'\partial_r \tag{286}$$

$$g(\nabla(t-f(r)),\nabla(t-f(r))) = \tag{287}$$

Further note from the asymptotic form at the horizon,

$$\partial_{r_*}u = \frac{r(r+R)\sqrt{-f(r)}p(r)}{K}\partial_r u \tag{288}$$

$$= \xi\frac{r(r+R)\sqrt{-f(r)}p(r)}{K}(r-r_h)^{-1}u+\frac{r(r+R)\sqrt{-f(r)}p(r)}{K}(r-r_h)^{\xi}\partial_r v \tag{289}$$

$$\partial_{r_*}u|_{r=r_h} = \xi\frac{r_h(r_h+R)\sqrt{-f(r_h)}}{Kr_h}u|_{r=r_h} \tag{290}$$

D.5 Real Modes

Proposition 6 ($\Omega_H\mu - \omega = 0$)**.** *Suppose there exists a mode solution with parameters (ω, m, μ) such that $\omega \in \mathbb{R}$. Then the following statements are true,*

1. $\Omega_H\mu - \omega = 0$.
2. $\Omega_H\mu \neq 0$.

Proof. The charge,

$$Q_T := \mathrm{Im}\Big(\int_0^\pi r^2 p(r)(R^2 - r^2)\Sigma(r,\theta)\bar{u}\partial_r u d\theta\Big) \tag{291}$$

satisfies,

$$\partial_r Q_T = -\mathrm{Im}\Big(R^2 \frac{\left(1 + \frac{r_h}{R}\cos\theta\right)\sin\theta}{(R + r\cos\theta)^2}\bar{u}\partial_\theta u\Big|_0^\pi\Big) \tag{292}$$

via the PDE equation for u. This vanishes due to the boundary conditions ($\lim_{r\to R} u(r,\theta) = 0$ sufficently fast so singularity at (r,π) isn't an issue)? Therefore,

$$Q_T(\infty) - Q_T(r_h) = 0 \implies Q_T(r_h) = 0 \tag{293}$$

by the boundary conditions. Since,

$$Q_T = \mathrm{Im}\Big(\int_0^\pi \frac{Kr(R-r)}{\sqrt{-f(r)}}\Sigma(r,\theta)\bar{u}\partial_{r_*} u d\theta\Big) \tag{294}$$

$$Q_T(r_h) = \mathrm{Im}(\xi)\int_0^\pi r_h(R^2 - r_h^2)\Sigma(r_h,\theta)|v|^2(r_h,\theta)d\theta \tag{295}$$

Hence if the integral doesn't vanish then $\Omega_H\mu - \omega = 0$. □