1 Singly Spinning Black Rings in (r,θ) Coordinates

1.1 Geometry

The singly spinning black ring solution is a 5D solution to the Einstein vacuum equation,

$$\mathrm{Ric}(g) = 0 \tag{1}$$

One can define the singly spinning black ring exterior as the smooth manifold,

$$\mathcal{M} = \mathbb{R} \times (\mathbb{R}^4 \setminus (\mathbb{S}^1 \times \mathbb{B}^3)) \tag{2}$$

with metric in coordinates $(t,r,\theta,\varphi,\psi)$,

$$\begin{aligned} g \ \doteq\ & -\frac{f(r)}{h(\theta)}dt\otimes dt + \frac{h(\theta)}{(1-\frac{r^2}{R^2})(1+\frac{r\cos\theta}{R})^2 p(r)}dr\otimes dr + \frac{r^2 h(\theta)}{(1+\frac{r\cos\theta}{R})^2 q(\theta)}d\theta\otimes d\theta \\ & +\frac{r^2 q(\theta)\sin^2\theta}{(1+\frac{r\cos\theta}{R})^2}d\varphi\otimes d\varphi + \Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2 f(r)} - \frac{K^2(1-\frac{r}{R})^2}{r^2 f(r) h(\theta)}\Big)d\psi\otimes d\psi \\ & +\frac{K}{rh(\theta)}\Big(\frac{r}{R}-1\Big)\big(dt\otimes d\psi + d\psi\otimes dt\big) \end{aligned} \tag{3}$$

with the following definitions,

$$f(r) \doteq 1 - \frac{r_+\cosh^2\sigma}{r} \qquad h(\theta) = 1 + \frac{r_+\cosh^2\sigma}{R}\cos\theta \tag{4}$$

$$p(r) \doteq 1 - \frac{r_+}{R} \qquad q(\theta) \doteq 1 + \frac{r_+}{R}\cos\theta \tag{5}$$

$$K \ \doteq\ r_+ R\sinh\sigma\cosh\sigma\sqrt{\frac{R + r_+\cosh^2\sigma}{R - r_+\cosh^2\sigma}} \tag{6}$$

and coordinate ranges,

$$t\in\mathbb{R} \quad r\in(r_+,R) \quad \theta\in[0,\pi) \quad \varphi,\psi\in\Big[0,\frac{2\pi R}{\sqrt{R^2+r_+^2}}\Big) \tag{7}$$

$r = r_+ < R$ corresponds to the future event horizon, σ is a parameter and R sets the scale of the solution. Note that, the vector field ∂_t is null at,

$$g_{tt} = 0 \implies r = r_+\cosh^2\sigma \tag{8}$$

which corresponds to the ergosurface. So the ergoregion is given by,

$$r\in(r_+, r_e \doteq r_+\cosh^2\sigma) \tag{9}$$

Note that one requires the 'equilibrium' condition,

$$\cosh^2\sigma = \frac{2R^2}{r_+^2 + R^2} \tag{10}$$

to avoid conical singularities. Thus,

$$f(r) \doteq 1 - \frac{r_e}{r} \qquad h(\theta) = 1 + \frac{2Rr_+}{r_+^2 + R^2}\cos\theta \tag{11}$$

$$p(r) \doteq 1 - \frac{r_+}{R} \qquad q(\theta) \doteq 1 + \frac{r_+}{R}\cos\theta \tag{12}$$

$$K \ \doteq\ r_e(r_+ + R)\sqrt{\frac{R + r_+}{2(R - r_+)}} \tag{13}$$

$$r_e \ \doteq\ \frac{2R^2 r_+}{(r_+^2 + R^2)} \tag{14}$$

The inverse metric is the following,

$$\begin{aligned} g^{-1} \;&=\; -\Big(\frac{h(\theta)}{f(r)} - \frac{K^2(R-r)(R+r\cos\theta)^2}{r^2R^4(r+R)p(r)f(r)h(\theta)}\Big)\partial_t\otimes\partial_t + \frac{(R^2-r^2)(R+r\cos\theta)^2p(r)}{R^4h(\theta)}\partial_r\otimes\partial_r \\ &+\frac{(R+r\cos\theta)^2q(\theta)}{r^2R^2h(\theta)}\partial_\theta\otimes\partial_\theta + \frac{(R+r\cos\theta)^2}{R^2r^2q(\theta)\sin^2\theta}\partial_\varphi\otimes\partial_\varphi + \frac{(R+r\cos\theta)^2f(r)}{R^2(R^2-r^2)h(\theta)p(r)}\partial_\psi\otimes\partial_\psi \\ &-\frac{K(R+r\cos\theta)^2}{rR^3(r+R)h(\theta)p(r)}(\partial_t\otimes\partial_\psi + \partial_\psi\otimes\partial_t) \end{aligned} \tag{15}$$

Note the vector field,

$$k = \frac{\partial}{\partial t} + \Omega_H\frac{\partial}{\partial\psi} \qquad \Omega_H^2 = \frac{R-r_+}{2R^2(R+r_+)} \tag{16}$$

is null future event horizon. Note that ψ is has period $\psi\sim\psi+\Delta\psi$ with $\Delta\psi \doteq \frac{2\pi R}{\sqrt{R^2+r_+^2}}$ so Ω_H is not the true angular velocity of the horizon. Defining $\tilde{\psi} = \frac{2\pi}{\Delta\psi}\psi$ gives

$$\partial_\psi = \frac{\sqrt{R^2+r_+^2}}{R}\partial_{\tilde{\psi}}. \tag{17}$$

So the true angular velocity of the horizon is

$$\tilde{\Omega}_H = \frac{1}{R^2}\sqrt{\frac{(R-r_+)(R^2+r_+^2)}{2(R+r_+)}}. \tag{18}$$

Now, for $r\in(r_+, r_+\cosh^2\sigma)$, $f(r)<0$ so we can make the coordinate transformation,

$$dv = dt + \frac{K}{r(r+R)\sqrt{-f(r)}p(r)}dr \qquad d\psi = d\chi + \frac{R\sqrt{-f(r)}}{(R^2-r^2)p(r)}dr \tag{19}$$

which gives the metric,

$$\begin{aligned} g \;&\doteq\; -\frac{f(r)}{h(\theta)}dv\otimes dv + \frac{r^2h(\theta)}{(1+\frac{r\cos\theta}{R})^2q(\theta)}d\theta\otimes d\theta + \frac{r^2q(\theta)\sin^2\theta}{(1+\frac{r\cos\theta}{R})^2}d\varphi\otimes d\varphi \\ &+\Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2f(r)} - \frac{K^2(1-\frac{r}{R})^2}{r^2f(r)h(\theta)}\Big)d\chi\otimes d\chi \\ &+\frac{K}{rh(\theta)}\Big(\frac{r}{R}-1\Big)\big(dv\otimes d\chi + d\chi\otimes dv\big) + \frac{R^3h(\theta)}{(R+r\cos\theta)^2\sqrt{-f(r)}}(dr\otimes d\chi + d\chi\otimes dr) \end{aligned} \tag{20}$$

Then one has the following for k,

$$k = \partial_v + \Omega_H\partial_\chi. \tag{21}$$

Mapping k to a one-form,

$$k_\flat \;=\; \Big[\frac{K\Omega_H}{rh(\theta)}\Big(\frac{r}{R}-1\Big) - \frac{f(r)}{h(\theta)}\Big]dv + \Omega_H\frac{R^3h(\theta)}{(R+r\cos\theta)^2\sqrt{-f(r)}}dr \tag{22}$$

$$+\Big[\frac{K}{rh(\theta)}\Big(\frac{r}{R}-1\Big) + \Omega_H\Big(\frac{R^2(1-\frac{r^2}{R^2})h(\theta)p(r)}{(1+\frac{r\cos\theta}{R})^2f(r)} - \frac{K^2(1-\frac{r}{R})^2}{r^2f(r)h(\theta)}\Big)\Big]d\chi \tag{23}$$

which at the future event horizon for a balanced ring gives,

$$k_\flat|_{r=r_+} = \frac{R^2}{\sqrt{2(R^2+r_+^2)(R+r_+)}}\frac{(R^2+r_0^2+2Rr_+\cos\theta)}{(R+r_+\cos\theta)^2}dr. \tag{24}$$

Note that,

$$g(k,k) = -\frac{f(r)}{h(\theta)} + 2K\Omega_H \frac{\left(\frac{r}{R}-1\right)}{rh(\theta)} + \Omega_H^2 \frac{1}{f(r)h(\theta)}\Big(\frac{R^2(R^2-r^2)h(\theta)^2 p(r)}{(R+r\cos\theta)^2} - \frac{(\frac{r}{R}-1)^2 K^2}{r^2}\Big). \quad (25)$$

To find the surface gravity, κ, of a balanced ring we consider $d(g(k,k))|_{r=r_+} = -2\kappa k|_{r=r_+}$. Evaluating, one finds,

$$\kappa = \frac{(R-r_+)\sqrt{(R^2+r_+^2)}}{2\sqrt{2}R^2 r_+}. \quad (26)$$