

IN THE LAND OF MORDOR, IN THE FIRES OF MOUNT DOOM

1 Boosted Schwarzschild Black Strings

The boosted Schwarzschild black string is a 5D solution to the vacuum Einstein equation,

$$\text{Ric}(g) = 0, \quad (1)$$

which is asymptotically the product of 4D Minkowski, Mink_4 , and a circle of radius R , \mathbb{S}_R^1 . It has metric

$$g \doteq -(1 - [1 - D(r)] \cosh^2(\sigma)) dt \otimes dt + \frac{1}{D(r)} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\varphi \otimes d\varphi \quad (2)$$

$$+ (1 + [1 - D(r)] \sinh^2(\sigma)) dz \otimes dz - (1 - D(r)) \cosh(\sigma) \sinh(\sigma) (dt \otimes dz + dz \otimes dt).$$

The coordinate ranges are $t \in [0, \infty)$, $r \in (2M, \infty)$ and $z \in [0, 2\pi)$. One can ‘unboost’ the string with the coordinate transformation

$$\tau = \cosh \sigma t + \sinh \sigma z, \quad (3)$$

$$\zeta = \sinh \sigma t + \cosh \sigma z. \quad (4)$$

In this case ζ, τ have infinite image, so the string direction when unboosted is not compact?

2 Singly Spinning Black Rings in (r, θ) Coordinates

2.1 Geometry

The singly spinning black ring solution is a 5D solution to the Einstein vacuum equation,

$$\text{Ric}(g) = 0 \quad (5)$$

One can define the singly spinning black ring exterior as the smooth manifold,

$$\mathcal{M} = \mathbb{R} \times (\mathbb{R}^4 \setminus (\mathbb{S}^1 \times \mathbb{B}^3)) \quad (6)$$

with metric in coordinates $(t, r, \theta, \varphi, \psi)$,

$$g \doteq -\frac{f(r)}{h(\theta)} dt \otimes dt + \frac{h(\theta)}{(1 - \frac{r^2}{R^2})(1 + \frac{r \cos \theta}{R})^2 p(r)} dr \otimes dr + \frac{r^2 h(\theta)}{(1 + \frac{r \cos \theta}{R})^2 q(\theta)} d\theta \otimes d\theta \quad (7)$$

$$+ \frac{r^2 q(\theta) \sin^2 \theta}{(1 + \frac{r \cos \theta}{R})^2} d\varphi \otimes d\varphi + \left(\frac{R^2 (1 - \frac{r^2}{R^2}) h(\theta) p(r)}{(1 + \frac{r \cos \theta}{R})^2 f(r)} - \frac{K^2 (1 - \frac{r}{R})^2}{r^2 f(r) h(\theta)} \right) d\psi \otimes d\psi$$

$$+ \frac{K}{r h(\theta)} \left(\frac{r}{R} - 1 \right) (dt \otimes d\psi + d\psi \otimes dt)$$

with the following definitions,

$$f(r) \doteq 1 - \frac{r_+ \cosh^2 \sigma}{r} \quad h(\theta) \doteq 1 + \frac{r_+ \cosh^2 \sigma}{R} \cos \theta \quad (8)$$

$$p(r) \doteq 1 - \frac{r_+}{R} \quad q(\theta) \doteq 1 + \frac{r_+}{R} \cos \theta \quad (9)$$

$$K \doteq r_+ R \sinh \sigma \cosh \sigma \sqrt{\frac{R + r_+ \cosh^2 \sigma}{R - r_+ \cosh^2 \sigma}} \quad (10)$$

and coordinate ranges,

$$t \in \mathbb{R} \quad r \in (r_+, R) \quad \theta \in [0, \pi) \quad \varphi, \psi \in \left[0, \frac{2\pi R}{\sqrt{R^2 + r_+^2}}\right) \quad (11)$$

$r = r_+ < R$ corresponds to the future event horizon, σ is a parameter and R sets the scale of the solution. Note that, the vector field ∂_t is null at,

$$g_{tt} = 0 \implies r = r_+ \cosh^2 \sigma \quad (12)$$

which corresponds to the ergosurface. So the ergoregion is given by,

$$r \in (r_+, r_e \doteq r_+ \cosh^2 \sigma) \quad (13)$$

Note that one requires the ‘equilibrium’ condition,

$$\cosh^2 \sigma = \frac{2R^2}{r_+^2 + R^2} \quad (14)$$

to avoid conical singularities. Thus,

$$f(r) \doteq 1 - \frac{r_e}{r} \quad h(\theta) = 1 + \frac{2Rr_+}{r_+^2 + R^2} \cos \theta \quad (15)$$

$$p(r) \doteq 1 - \frac{r_+}{R} \quad q(\theta) \doteq 1 + \frac{r_+}{R} \cos \theta \quad (16)$$

$$K \doteq r_e(r_+ + R) \sqrt{\frac{R + r_+}{2(R - r_+)}} \quad (17)$$

$$r_e \doteq \frac{2R^2 r_+}{(r_+^2 + R^2)} \quad (18)$$

The inverse metric is the following,

$$\begin{aligned} g^{-1} = & -\left(\frac{h(\theta)}{f(r)} - \frac{K^2(R-r)(R+r \cos \theta)^2}{r^2 R^4(r+R)p(r)f(r)h(\theta)}\right) \partial_t \otimes \partial_t + \frac{(R^2 - r^2)(R + r \cos \theta)^2 p(r)}{R^4 h(\theta)} \partial_r \otimes \partial_r \\ & + \frac{(R + r \cos \theta)^2 q(\theta)}{r^2 R^2 h(\theta)} \partial_\theta \otimes \partial_\theta + \frac{(R + r \cos \theta)^2}{R^2 r^2 q(\theta) \sin^2 \theta} \partial_\varphi \otimes \partial_\varphi + \frac{(R + r \cos \theta)^2 f(r)}{R^2 (R^2 - r^2) h(\theta) p(r)} \partial_\psi \otimes \partial_\psi \\ & - \frac{K(R + r \cos \theta)^2}{r R^3 (r + R) h(\theta) p(r)} (\partial_t \otimes \partial_\psi + \partial_\psi \otimes \partial_t) \end{aligned} \quad (19)$$

Note the vector field,

$$k = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \psi} \quad \Omega_H^2 = \frac{R - r_+}{2R^2(R + r_+)} \quad (20)$$

is null future event horizon. Note that ψ has period $\psi \sim \psi + \Delta\psi$ with $\Delta\psi \doteq \frac{2\pi R}{\sqrt{R^2 + r_+^2}}$ so Ω_H is not the true angular velocity of the horizon. Defining $\tilde{\psi} = \frac{2\pi}{\Delta\psi} \psi$ gives

$$\partial_\psi = \frac{\sqrt{R^2 + r_+^2}}{R} \partial_{\tilde{\psi}}. \quad (21)$$

So the true angular velocity of the horizon is

$$\tilde{\Omega}_H = \frac{1}{R^2} \sqrt{\frac{(R - r_+)(R^2 + r_+^2)}{2(R + r_+)}}. \quad (22)$$

Now, for $r \in (r_+, r_+ \cosh^2 \sigma)$, $f(r) < 0$ so we can make the coordinate transformation,

$$dv = dt + \frac{K}{r(r+R)\sqrt{-f(r)p(r)}} dr \quad d\psi = d\chi + \frac{R\sqrt{-f(r)}}{(R^2 - r^2)p(r)} dr \quad (23)$$

which gives the metric,

$$\begin{aligned} g \doteq & -\frac{f(r)}{h(\theta)} dv \otimes dv + \frac{r^2 h(\theta)}{(1 + \frac{r \cos \theta}{R})^2 q(\theta)} d\theta \otimes d\theta + \frac{r^2 q(\theta) \sin^2 \theta}{(1 + \frac{r \cos \theta}{R})^2} d\varphi \otimes d\varphi \\ & + \left(\frac{R^2(1 - \frac{r^2}{R^2})h(\theta)p(r)}{(1 + \frac{r \cos \theta}{R})^2 f(r)} - \frac{K^2(1 - \frac{r}{R})^2}{r^2 f(r)h(\theta)} \right) d\chi \otimes d\chi \\ & + \frac{K}{rh(\theta)} \left(\frac{r}{R} - 1 \right) (dv \otimes d\chi + d\chi \otimes dv) + \frac{R^3 h(\theta)}{(R + r \cos \theta)^2 \sqrt{-f(r)}} (dr \otimes d\chi + d\chi \otimes dr) \end{aligned} \quad (24)$$

Then one has the following for k ,

$$k = \partial_v + \Omega_H \partial_\chi. \quad (25)$$

Mapping k to a one-form,

$$k_b = \left[\frac{K\Omega_H}{rh(\theta)} \left(\frac{r}{R} - 1 \right) - \frac{f(r)}{h(\theta)} \right] dv + \Omega_H \frac{R^3 h(\theta)}{(R + r \cos \theta)^2 \sqrt{-f(r)}} dr \quad (26)$$

$$+ \left[\frac{K}{rh(\theta)} \left(\frac{r}{R} - 1 \right) + \Omega_H \left(\frac{R^2(1 - \frac{r^2}{R^2})h(\theta)p(r)}{(1 + \frac{r \cos \theta}{R})^2 f(r)} - \frac{K^2(1 - \frac{r}{R})^2}{r^2 f(r)h(\theta)} \right) \right] d\chi \quad (27)$$

which at the future event horizon for a balanced ring gives,

$$k_b|_{r=r_+} = \frac{R^2}{\sqrt{2(R^2 + r_+^2)}(R + r_+)} \frac{(R^2 + r_0^2 + 2Rr_+ \cos \theta)}{(R + r_+ \cos \theta)^2} dr. \quad (28)$$

Note that,

$$g(k, k) = -\frac{f(r)}{h(\theta)} + 2K\Omega_H \frac{(\frac{r}{R} - 1)}{rh(\theta)} + \Omega_H^2 \frac{1}{f(r)h(\theta)} \left(\frac{R^2(R^2 - r^2)h(\theta)^2 p(r)}{(R + r \cos \theta)^2} - \frac{(\frac{r}{R} - 1)^2 K^2}{r^2} \right). \quad (29)$$

To find the surface gravity, κ , of a balanced ring we consider $d(g(k, k))|_{r=r_+} = -2\kappa k|_{r=r_+}$. Evaluating, one finds,

$$\kappa = \frac{(R - r_+) \sqrt{(R^2 + r_+^2)}}{2\sqrt{2}R^2 r_+}. \quad (30)$$