The One Ring

1. The first thing would be to set up the spectral analysis. I suspect that this should be similar to what can be done for Schwarzschild, although I'm not sure whether we would need Gevray spaces. Ideally, the only tools you would need are redshift estimates and the ∂_t estimate. (I don't think this is necessary since we are only interested in studying growing modes.)
2. With the spectral theory, we want to see ideally that the black string mode is isolated (it is part of the point spectrum). This could be seen by proving some Fredholm property of the resolvent (or by basic spectral theory if the problem allows for that).
3. The next step is to study the perturbation theory, here, I think the main difficulty will center around the fact that we need to make sure that perturbing a true mode generates a true mode. We know that the harmonic gauge constraint is propagated by a wave-type operator, so the modes there can be perturbed in a good way. For the pure gauge modes, we should see if theres a way to use the W_{tztz} condition.
4. Also need the expansion of the black ring around the black string as a true perturbation.

1 Setup for spectral analysis

1.1 Estimates

1.1.1 Redshift

1.1.2 Killing energy estimate

1.2 Proving a Fredholm alternative

2 Perturbation theory

2.1 Perturbation theory for principally scalar wave operators

2.2 Mode categorization

There are three classes of modes we are interested in.

Definition 2.1. *We call a mode solution $h = e^{\mu t}u(x)$ a* pure gauge mode solution[1] *if there exists a smooth vectorfield ϑ such that*

$$h = \mathcal{L}_\vartheta g.$$

Definition 2.2. *We call a mode solution $h = e^{\mu t}u(x)$ a* pure constraint mode solution *if h does not satisfy the linearized harmonic coordinate constraints.*[2]

Definition 2.3. *We call a mode solutions a* true mode solution *if it is neither a pure gauge mode nor a pure constraint mode.*

[1] This differs from the definition we use in stability of Kerr-de Sitter where we also consider modes that are linearized changes of the black hole parameters. We should definitely check if we also need to consider these.

[2] Probably also need to consider the spherical gauge that is used.

For the sake of this paper, we are mainly interested in the true mode solutions. The perturbation argument should proceed roughly as follows:

1. We will perform mode perturbation on two separate operators, those being

$$\mathbf{L} = D_g\mathrm{Ric} - \widetilde{\nabla_g \otimes} D_g\mathcal{C}(g, g^0), \qquad \widetilde{\Box^{CP}},$$

   the constraint-damped linearized Einstein operator in harmonic gauge and the constraint-damped constraint propagation operator for harmonic gauge respectively. Both of these operators are principally the scalar wave operator and as such, should be amenable to mode perturbation *in the upper half plane* in a similar way to Kerr-de Sitter. The main general result that we will use from the general perturbation theory is that for small perturbations, the total number of modes is equal (at an eigenvalue).

2. There is an exact correspondence between the pure gauge mode solutions to the linearized Einstein operator and the constraint propagation operator. This gives us a way of counting the pure gauge modes in the perturbation since the constraint propagation operator is principally hyperbolic.

3. If we use constraint damping, then we eliminate any growing modes associated to constraint violations. What we would need to verify is that any mode solution to the constraint-damped operator is a mode of the original operator.

3 Linearized Einstein vacuum equations

In this section, we introduce the linearized Einstein equations in the notation that we will use. Directly linearizing (**??**) around g yields the ungauged linearized Einstein equation

$$D_g\mathrm{Ric}(h) = 0. \tag{1}$$

Given admissible initial data $(\Sigma_0, \underline{g}_0, k_0)$ for g, we define the *linearized constraint equation* as the linearization of (**??**) around $(\underline{g}_0, k_0)$ in terms of the linearized metric $\underline{g}'$ and the linearized second fundamental form k'. An initial data triplet $(\Sigma_0, \underline{g}', k')$ linearized around $(\underline{g}_b, k_b)$ is an *admissible* initial data triplet for Einstein equations linearized around g if $(\underline{g}', k')$ satisfy the linearized constraint equations. Linearizing the gauged Einstein equations in (**??**), we have the linearized gauged Einstein equations

$$D_g(\mathrm{Ric} - \Lambda)(h) - \nabla_g \otimes D_g\mathcal{C}(g, g^0)(h) = 0. \tag{2}$$

Definition 3.1. *Define the* linearized gauge constraint

$$\begin{aligned}\mathcal{C}_g h &:= D_g\mathcal{C}(g + h, g)(h)\\ &= -\nabla_g \cdot G_g h.\end{aligned}$$

We have the following linearized equivalent of Lemma **??**.

Lemma 3.2. *Let h solve (2). Then h also satisfies*

$$\Box_g^{CP}(\mathcal{C}_g h) = 0, \qquad \Box_g^{CP}\psi = \Box_g^{(1)}\psi,$$

where $\Box_g^{(1)} = \nabla^\alpha\nabla_\alpha$ denotes the wave operator acting on 1-tensors.

Proof. The lemma follows directly by applying the twice-contracted linearized second Bianchi identity to the gauged linearized Einstein equation. ∎

Remark 3.3. *From Lemma 3.2, it is clear that if $(\mathcal{C}_g h|_{\Sigma_0}, \mathcal{L}_{\mathbf{T}}\mathcal{C}_g h|_{\Sigma_0}) = (0, 0)$, then $\mathcal{C}_g h = 0$ uniformly.*

Finally, we remark that any solution h to the ungauged linearized Einstein's equations (1) can be put into the linearized gauge $\mathcal{C}_g(h) = 0$ by finding some infinitesimal diffeomorphism $\nabla_g \otimes \omega$[3] such that

$$\mathcal{C}_g(h + \nabla_g \otimes \omega) = 0, \tag{3}$$

as general covariance implies that

$$D_g \mathrm{Ric}(\nabla_g \otimes \omega) = 0$$

for any one-form $\omega \in C^\infty(\mathcal{M}, T^*\mathcal{M})$. This is equivalent to finding some ω such that

$$\Box_g^\Upsilon \omega = 2\mathcal{C}_g(h), \qquad \Box_g^\Upsilon = -2\mathcal{C}_g \circ \nabla_g \otimes, \tag{4}$$

which is principally $\Box_g^{(1)}$, and in fact, in our case we can calculate that

$$\Box_g^\Upsilon = \Box_g^{(1)}.$$

Solving for ω satisfying (4) with Cauchy data $(\omega, \mathcal{L}_{\mathbf{T}}\omega)|_{\Sigma_0} = 0$ then ensures that $h + \nabla_g \otimes \omega$ has the same initial data as h.

3.1 Constraint damping

The goal of constraint damping is to modify the operator

$$\nabla_g \otimes$$

by some lower-order terms so that the quasinormal modes of the constraint propagation operator $\Box_g^{CP}$ lie strictly in the exponentially decaying half-space. This will force any exponentially growing modes of the constraint-damped linearized Einstein operator to satisfy the harmonic gauge constraints.

We remark that since this is a lower-order modification, this will not affect the principal hyperbolic nature of the linearized Einstein operator in harmonic gauge.

[3]Observe that in terms of the Lie derivative, we have that

$$\nabla_g \otimes \omega = \frac{1}{2}\mathcal{L}_{\omega^\sharp} g.$$