1 Linear Perturbation Theory

Consider a Lorentzian manifold (M, g) with metric satisfying the Einstein equation,

$$Ric_g(X,Y) - \frac{1}{2}g(X,Y)R = T(X,Y) \qquad \forall X,Y \in \mathfrak{X}(M). \tag{1}$$

Suppose we perturb the initial data for this spacetime. In general this will result in a change of spacetime manifold and metric. However, here we are concerned with 'small' perturbations so we model this perturbation with just a change to the metric $g(X,Y) \mapsto g(X,Y) + \epsilon h(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$ with $\epsilon > 0$. h here is a symmetric bilinear form on the fibres of TM. In the following we will derive a series of results on how various quantities change to $O(\epsilon)$, i.e. the linear level. This will result in an expression for the Einstein tensor under such a perturbation to linear order. For this to be a good toy model for the full non-linear problem we want ϵ to be small.

Remark. *An important point to note is that we have to be careful that $h^{ab} := g^{ac}g^{bd}h_{cd} \neq (h^{-1})^{ab}$.*

Proposition 1 (Change in the Ricci Tensor)**.** *Consider a Lorentzian manifold (M, g). Suppose the metric $\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric. Then Ricci tensor, $\tilde{R}_{ab}$, of $\tilde{g}_{ab}$ to $O(\epsilon)$ is,*

$$\tilde{R}_{ab} = R_{ab} - \epsilon\frac{1}{2}\Delta_L h_{ab} \tag{2}$$

where, Δ_L denotes the Lichnerowicz operator given by,

$$\Delta_L h_{ab} = \Box_g h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} - 2R^c{}_{(a}h_{b)c} - 2\nabla_{(a}\nabla^c h_{b)c} + \nabla_a\nabla_b h \tag{3}$$

and $h = g^{ab}h_{ab}$.

Proof. Now suppose we take an arbitrary point $p \in M$ and introduce normal coordinates at this point, then recall that the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}(p)$ vanish and therefore so does $g_{\mu\nu,\alpha}(p)$. This means we can calculate the new in the Christoffel symbols, $\tilde{\Gamma}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} + \epsilon\gamma^\alpha{}_{\beta\mu}$,

$$\tilde{\Gamma}^\mu_{\alpha\beta}(p) = \epsilon\frac{1}{2}g^{\mu\gamma}(h_{\alpha\gamma,\beta} + h_{\beta\gamma,\alpha} - h_{\alpha\beta,\gamma})(p) \tag{4}$$

since $g_{\mu\nu,\alpha}(p)$ vanish. We can replace the partial derivatives by the covariant derivative associated the the Levi-Civita connection since the Christoffel symbols vanish. The resulting expression is basis independent, so holds in all frames. The point picked was also arbitrary hence the resulting expression holds at all $p \in M$.

$$\gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\nabla_c h_{bd} + \nabla_b h_{cd} - \nabla_d h_{bc}) \tag{5}$$

To work out the Riemann tensor $\tilde{R}^a_{bcd} = R^a{}_{bcd} + \epsilon S^a{}_{bcd}$ we again work in normal coordinates,

$$\begin{aligned} S^\alpha{}_{\beta\mu\nu}(p) &= \big(\partial_\mu\gamma^\alpha{}_{\beta\nu} - \partial_\nu\gamma^\alpha{}_{\beta\mu}\big)(p) & (6)\\ &= \big(\nabla_\mu\gamma^\alpha{}_{\beta\nu} - \nabla_\nu\gamma^\alpha{}_{\beta\mu}\big)(p). & (7)\end{aligned}$$

Hence,

$$S^a{}_{bcd} = \nabla_c\gamma^a{}_{bd} - \nabla_d\gamma^a{}_{bc}. \tag{8}$$

The Ricci tensor $(\tilde{\mathrm{Ric}}_g)_{ab} = (\mathrm{Ric}_g)_{ab} + \epsilon\mathbf{R}_{ab}$ is then given by,

$$\begin{aligned} \mathbf{R}_{ab} = S^c{}_{acb} &= \nabla_c\gamma^c{}_{ab} - \nabla_b\gamma^c{}_{ac} & (9)\\ &= \frac{1}{2}g^{cd}(\nabla_c\nabla_a h_{bd} + \nabla_c\nabla_b h_{ad} - \nabla_c\nabla_d h_{ab} - \nabla_b\nabla_a h_{cd}) & (10)\\ &= -\frac{1}{2}\Box_g h_{ab} - \frac{1}{2}\nabla_a\nabla_b h + g^{cd}\nabla_c\nabla_{(b}h_{a)d}. & (11)\end{aligned}$$

Using the Ricci Identity on the last term,

$$\begin{aligned}\mathbf{R}_{ab} &= -\frac{1}{2}\Box_g h_{ab} - \frac{1}{2}\nabla_a\nabla_b h + g^{cd}\Big(\nabla_{(b}\nabla_{|c|}h_{a)d} + R^e{}_{(ab)c}h_{ed} - R^e{}_{dc(b}h_{a)e}\Big) && (12)\\ &= -\frac{1}{2}\Box_g h_{ab} - \frac{1}{2}\nabla_a\nabla_b h + \nabla_{(b}\nabla^c h_{a)c} - R_a{}^c{}_b{}^d h_{cd} + R^c{}_{(b}h_{a)c} && (13)\\ &= -\frac{1}{2}\Delta_L h_{ab}. && (14)\end{aligned}$$

□

Corollary 1.0.1 (Change in the Ricci Scalar). *Consider a Lorentzian manifold (M, g). Suppose the metric $\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric. Then the Ricci Scalar $\tilde{R} = R + \epsilon S$ of $\tilde{g}_{ab}$ to $O(\epsilon)$ is,*

$$S = -R^{ab}h_{ab} - \Box_g h + \nabla^a\nabla^b h_{ab}. \tag{15}$$

Proof. First we note that from the invariance of δ^a_b, the inverse metric pertubation $g^{ab} \mapsto g^{ab} + (h^{-1})^{ab}$ is given by,

$$(h^{-1})^{ab} = -g^{ac}g^{bd}h_{cd} \tag{16}$$

Therefore the Ricci scalar change as,

$$\begin{aligned}S &= -g^{ac}g^{bd}h_{cd}R_{ab} + g^{ab}\mathbf{R_{ab}} && (17)\\ &= g^{ab}\Big(-\frac{1}{2}\Box_g h_{ab} - \frac{1}{2}\nabla_a\nabla_b h + \nabla_{(b}\nabla^c h_{a)d} - R_a{}^c{}_b{}^d h_{cd} + R^c{}_{(b}h_{a)c}\Big) - h_{ab}R^{ab} && (18)\\ &= -\Box_g h + \nabla^a\nabla^b h_{ab} - h_{ab}R^{ab}. && (19)\end{aligned}$$

□

Corollary 1.0.2 (Change in the Einstein Tensor). *Consider a Lorentzian manifold (M, g). Suppose the metric $\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$ is a Lorentzian metric. Then Einstein tensor, $\tilde{G}_{ab}$, of $\tilde{g}_{ab}$ to $O(\epsilon)$ is,*

$$\tilde{G}_{ab} = G_{ab} - \frac{1}{2}\Delta_L h_{ab} - \frac{1}{2}g_{ab}\Big(\nabla^c\nabla^d h_{cd} - h_{cd}R^{cd} - \Box_g h\Big) - \frac{1}{2}h_{ab}R. \tag{20}$$

Proof. Direct calculation. □

If we have a vacuum spacetime, we require that the perturbed spacetime satisfies,

$$\Delta_L h_{ab} = 0 \implies -\frac{1}{2}\Box_g h_{ab} - \frac{1}{2}\nabla_a\nabla_b h + \nabla_{(b}\nabla^c h_{a)c} - R_a{}^c{}_b{}^d h_{cd} = 0 \tag{21}$$

This will be the main equation of interest, with g the black string metric,

$$g := -\Big(1 - \frac{2M}{r}\Big)dt\otimes dt + \Big(1 - \frac{2M}{r}\Big)^{-1}dr\otimes dr + r^2\Big(d\theta\otimes d\theta + \sin^2\theta d\varphi\otimes d\varphi\Big) + dz\otimes dz. \tag{22}$$

Remark. *If we take the transverse traceless gauge,*

$$\nabla^a h_{ab} = 0 = h \tag{23}$$

then the equation reduces further to,

$$\Box_g h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} = 0. \tag{24}$$

2 ODE Theory

2.1 Adjoint Operators

Definition 2.1 (Adjoint and Symmetric ODE)**.** *Consider the operator,*

$$L[u] = p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) \tag{25}$$

then the adjoint of the linear operator L, M, is given by,

$$M[v] := (p_0(x)v)'' - (p_1(x)v)' + p_2(x)v \tag{26}$$

The adjoint ODE to $L[u] = 0$ is given by,

$$M[v] = 0 \tag{27}$$

If $M[u] = L[u]$ then the ODE is symmetric.

Remark. *This gives the nessecary and sufficent condition $p_0' = p_1$.*

Proposition 2. *Consider a coordinate transformation $s = s(r)$ and the second order homogeneous linear ODE,*

$$\frac{d^2u}{dr^2} + f(r)\frac{du}{dr} + g(r)u = 0 \qquad f, g \in C^0(I) \tag{28}$$

Then this can be reduce to the form,

$$-\frac{d^2z}{ds^2}(s) + V(s)z(s) = 0 \tag{29}$$

with,

$$V(s) = \frac{1}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{df}{dr} - \frac{3}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{d^2s}{dr^2}\right)^2 + \frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^3s}{dr^3} + \frac{f^2}{2} - 2g\right) \tag{30}$$

Proof. The proof is a straight-forward calculation. Take $u(s) = w(s)z(s)$, then

$$\left(\frac{ds}{dr}\right)^2 w\frac{d^2z}{ds^2} + \left(2\left(\frac{ds}{dr}\right)^2\frac{dw}{ds} + w\frac{d^2s}{dr^2} + fw\frac{ds}{dr}\right)\frac{dz}{ds} + \left(\left(\frac{ds}{dr}\right)^2\frac{d^2w}{ds^2} + \frac{dw}{ds}\frac{d^2s}{dr^2} + f\frac{dw}{ds}\frac{ds}{dr} + gw\right)z = 0$$

To reduce this to symmetric form we set,

$$2\left(\frac{ds}{dr}\right)^2\frac{dw}{ds} + w\frac{d^2s}{dr^2} + fw\frac{ds}{dr} = 0 \tag{31}$$

which is equivalent to $w(r)$ satisfying,

$$\frac{dw}{dr} + \frac{1}{2}\left(\frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^2s}{dr^2} + f\right)w = 0 \tag{32}$$

Hence,

$$\frac{d^2w}{dr^2} = -\frac{1}{2}\left(\frac{df}{dr} - \frac{1}{\left(\frac{ds}{dr}\right)^2}\left(\frac{d^2s}{dr^2}\right)^2 + \frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^3s}{dr^3} - \frac{1}{2}\left(\frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^2s}{dr^2} + f\right)^2\right)w \tag{33}$$

Notice the last term in the ODE for z reduces to,

$$\frac{d^2w}{dr^2} + f\frac{dw}{dr} + gw \tag{34}$$

Reducing this with our expressions for the derivatives of w gives the potential for $-\frac{d^2z}{ds^2} + V(s)z = 0$ as,

$$V(s) \;=\; \frac{1}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{df}{dr} - \frac{3}{2\left(\frac{ds}{dr}\right)^2}\left(\frac{d^2s}{dr^2}\right)^2 + \frac{1}{\left(\frac{ds}{dr}\right)}\frac{d^3s}{dr^3} + \frac{f^2}{2} - 2g\right) \tag{35}$$

□

Remark. *Applying this to $s = r_*(r)$ gives,*

$$V(r(r_*)) = \frac{(r-2M)^2}{2r^2}\left(\frac{df}{dr} + \frac{2M(2r-3M)}{r^2(r-2M)^2} + \frac{f^2}{2} - 2g\right). \tag{36}$$

2.2 Schrödinger Operators

Note that this section is largely based upon Reed and Simon Volume II and IV and Davies Spectral Theory and Analysis of Operators [1, 2, 3].

Definition 2.2 (Symmetric, Self-Adjoint and Densely-Defined Operators)**.** *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be an unbounded operator on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. A is symmetric if,*

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \qquad \forall x, y \in D(A) \tag{37}$$

where A^* denotes the adjoint of A. A is self-adjoint if it is symmetric and $D(A) = D(A^*)$. A is densely-defined if $D(A) \subset \mathcal{H}$ is a dense subspace.

Definition 2.3 (A-Bounded)**.** *Let A and B be densely defined operators on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then B is said A-bounded, if $D(A) \subset D(B)$ and there exist $a, b \in \mathbb{R}$, such that,*

$$||B\Psi||_{\mathcal{H}} \leq a||A\Psi||_{\mathcal{H}} + b||\Psi||_{\mathcal{H}} \qquad \forall \Psi \in D(A) \tag{38}$$

Theorem 2.1 (Kato-Rellich Theorem)**.** *Let A be a densely-defined self adjoint operator on a Hilbert space $\mathcal{H}$ and B a symmetric operator. If B is A-bounded, then $H := A + B$ with $D(H) = D(A)$ is self-adjoint.*

Proof. See Reed and Simon IV pg. 162. □

The main theorem from the spectral theory of Schrödinger operators we want to employ comes from a variational method,

Theorem 2.2 (Rayleigh-RItz)**.** *Let H be a self-adjoint operator on $\mathcal{H}$ that is bounded from below. Then there exists a non-decreasing sequence of numbers defined by,*

$$\lambda_n := \inf_{L^n \subset D(H)} \sup_{\Psi \in L^n} \frac{(\Psi, H\Psi)}{||\Psi||^2} \tag{39}$$

where L^n is any n-dimensional subspace of $D(H)$. Then,

1. $\lim_{n\to\infty} \lambda_n = \inf \sigma_{ess}(H)$*, with* $\lim_{n\to\infty} \lambda_n = \infty$ *implying* $\sigma_{ess}(H) = \emptyset$.
2. $(\lambda_n)_n \cap (-\infty, \lim_{n\to\infty} \lambda_n) = \sigma_d(H) \cap (-\infty, \lim_{n\to\infty} \lambda_n)$*, each* $\lambda_n \in (-\infty, \lim_{n\to\infty} \lambda_n)$ *being an eigenvalue of H repeated a number of times equal to its multiplicity.*

Proof. See Davies Chapter 4, section 4.5 or Reed and Simon Vol. IV. □

Corollary 2.2.1. *Let H be a self-adjoint operator on $\mathcal{H}$ that is bounded from below. Then*

$$\lambda_1 = \inf \sigma(H) = \inf_{\Psi \in D(H)\setminus\{0\}} \frac{(\Psi, H\Psi)}{||\Psi||^2} \tag{40}$$

This corollary will allow us to deduce the existance of a negative eigenvalue which will give rise to the unstable growing mode for the Gregory-Laflamme instability. However, we need some spectral theory machinary to conclude such a result. The following theorem gives us a setting in which to apply the Rayleigh-Ritz quotient for a wide class of Schrödinger operators.

Theorem 2.3. *Let $\Psi \in D(-\Delta) \subset L^2(\mathbb{R})$. If Ψ is bounded and continuous then for any $a > 0$ there is a b, independent of Ψ, so that,*

$$||\Psi||_{L^\infty(\mathbb{R})} \leq a||\Delta\Psi||_{L^2(\mathbb{R})} + b||\Psi||_{L^2(\mathbb{R})} \tag{41}$$

Proof. By the Riemann-Lebesque lemma and the Plancherel theorem the result follows if we can show $\hat{\Psi} \in L^1(\mathbb{R})$ and

$$||\hat{\Psi}||_{L^1(\mathbb{R})} \leq a||\lambda^2\hat{\Psi}||_{L^2(\mathbb{R})} + b||\hat{\Psi}||_{L^2(\mathbb{R})} \tag{42}$$

Finish: Reed/Simon pg.56 □

Theorem 2.4. *Let $V \in (L^2 + L^\infty)(\mathbb{R})$. Then $H = -\Delta + V$ is a self-adjoint operator and $D(H) = H^2(\mathbb{R})$.*

Proof. By the Kato-Rellich Theorem it is sufficent to show, V is $-\Delta$-bounded. If $V \in L^p + L^\infty$ then $V = V_p + V_\infty$ with $V_p \in L^p(\mathbb{R})$ and $V_\infty \in L^\infty(\mathbb{R})$. Let $\Psi \in D(-\Delta) = H^2(\mathbb{R})$, in fact we can work in $C_c^\infty(\mathbb{R})$ then argue by density the conclusion. So for $\Psi \in C_c^\infty(\mathbb{R})$,

$$||V\Psi||_{L^2(\mathbb{R})} \leq ||V_2\Psi||_{L^2(\mathbb{R})} + ||V_\infty\Psi||_{L^2(\mathbb{R})} \leq ||V_2||_{L^2(\mathbb{R})}||\Psi||_{L^\infty(\mathbb{R})} + ||V_\infty||_{L^\infty(\mathbb{R})}||\Psi||_{L^2(\mathbb{R})} \tag{43}$$

By theorem 2.3,

$$||\Psi||_{L^\infty(\mathbb{R})} \leq a||\Delta\Psi||_{L^2(\mathbb{R})} + b||\Psi||_{L^2(\mathbb{R})} \qquad \forall \Psi \in C_c^\infty(\mathbb{R}) \tag{44}$$

Therefore,

$$||V\Psi||_{L^2(\mathbb{R})} \leq a||V_2||_{L^2(\mathbb{R})}||\Delta\Psi||_{L^2(\mathbb{R})} + \Big(b||V_2||_{L^2(\mathbb{R})} + ||V_\infty||_{L^\infty(\mathbb{R})}\Big)||\Psi||_{L^2(\mathbb{R})} \tag{45}$$

This shows that V is $-\Delta$-bounded. □

2.3 A Regularity Result

Theorem 2.5 (Regularity for the Schrödinger Equation)**.** *Let u be a weak solution of the equation $(-\Delta + V)u = \lambda u$ where V is a measurable function and $\lambda \in \mathbb{C}$. Then, if $V \in C^\infty(\Omega)$ with $\Omega \subset \mathbb{R}$ open, not nessecarily bounded, then $u \in C^\infty(\Omega)$ also.*

Proof. Reed and Simon Vol. II pg.55. (I think this can just be argued via elliptic regularity and Sobolev embeddings.) □

3 The Gregory–Laflamme Instability

3.1 Black Strings

We will take the $5D$ black string spacetime $\mathrm{Sch}_4 \times \mathbb{R}$. The metric on the exterior is,

$$g := -\Big(1 - \frac{2M}{r}\Big)dt \otimes dt + \Big(1 - \frac{2M}{r}\Big)^{-1} dr \otimes dr + r^2\Big(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi\Big) + dz \otimes dz. \tag{46}$$

We now want to perform linear perturbation theory around this metric and determine the stability of this metric to linear order. Note, ${R^4}_{\mu\alpha\beta} = {R^\mu}_{4\alpha\beta} = {R^\mu}_{\alpha 4\beta} = {R^\mu}_{\alpha\beta 4} = 0$. So the Riemann tensor components that are relevant are the ones with spacetime indices $\mu \in \{0, ..., 3\}$ which are just the usual Schwarzschild Riemann tensor components; the non-zero ones are listed below for completeness,

$$\begin{aligned}
{R^0}_{101} &= \frac{2M}{r^2(r-2M)} & {R^0}_{202} &= -\frac{M}{r} & {R^0}_{303} &= -\frac{M\sin^2\theta}{r} && (47)\\
{R^1}_{010} &= -\frac{2M(r-2M)}{r^4} & {R^1}_{212} &= -\frac{M}{r} & {R^1}_{313} &= -\frac{M\sin^2\theta}{r} && (48)\\
{R^2}_{020} &= \frac{M(r-2M)}{r^4} & {R^2}_{121} &= -\frac{M}{r^2(r-2M)} & {R^2}_{323} &= \frac{2M\sin^2\theta}{r} && (49)\\
{R^3}_{030} &= \frac{M(r-2M)}{r^4} & {R^3}_{131} &= -\frac{M}{r^2(r-2M)} & {R^3}_{232} &= \frac{2M}{r}. && (50)
\end{aligned}$$

Any others can be found from the ${R^a}_{b(cd)} = 0$ symmetry. To compute $\Box_g h_{ab}$ we require the Christoffel symbols; the non-zero ones are listed below,

$$\begin{aligned}
\Gamma^0_{01} &= \frac{M}{r(r-2M)} && && && (51)\\
\Gamma^1_{00} &= \frac{M(r-2M)}{r^3} & \Gamma^1_{11} &= \frac{-M}{r(r-2M)} & \Gamma^1_{22} &= (2M-r) & \Gamma^1_{33} &= (2M-r)\sin^2\theta \quad (52)\\
\Gamma^2_{12} &= \frac{1}{r} = \Gamma^3_{13} & \Gamma^2_{33} &= -\sin\theta\cos\theta & \Gamma^3_{23} &= \cot\theta. && && (53)
\end{aligned}$$

The others are obtained from symmetry of lower indices.

3.2 Wiseman Gauge

In the paper [4], it is stated that the metric perturbation can be taken to be,

$$h_{\mu\nu} = e^{\mu t + i\omega z}\begin{pmatrix} H_t(r) & \mu H_v(r) & 0 & 0 & 0 \\ \mu H_v(r) & H_r(r) & 0 & 0 & -i\omega H_v(r) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i\omega H_v(r) & 0 & 0 & H_z(r) \end{pmatrix} \tag{54}$$

In this section we show that this can be achieved via a gauge choice, which we will from now on refer to as the Wiseman gauge due to its original apperance in the work of Aharony et al. [5].

Proposition 3 (Consistency of the Wiseman Gauge)**.** *The Wiseman gauge can be consistently imposed.*

Proof. First we start with the standard spherically symmetric metric perturbation,

$$h_{\mu\nu} = e^{\mu t + i\omega z}\begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 & H_{tz}(r) \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 & H_{rz}(r) \\ 0 & 0 & H_{\theta\theta}(r) & 0 & 0 \\ 0 & 0 & 0 & H_{\theta\theta}\sin^2\theta & 0 \\ H_{tz}(r) & H_{rz}(r) & 0 & 0 & H_{zz}(r) \end{pmatrix} \tag{55}$$

Now an infinitesimal diffeomorphism generated by a vector field ξ changes this pertubation by $2\nabla_{(a}\xi_{b)}$. So we have, $h'_{ab} = h_{ab} + 2\nabla_{(a}\xi_{b)}$ as the new metric. Take,

$$\xi_\mu = e^{\mu t + i\omega z}\zeta_\mu(r, \theta, \varphi) \tag{56}$$

We then try to impose Wiseman gauge on h'_{ab}.The following equations must be respected,

$$\begin{aligned}
H'_{zz}(r) &= H_{zz}(r) + 2i\omega\zeta_z(r,\theta,\varphi) \qquad H'_{rr}(r) = H_{rr}(r) + \frac{2M\zeta_r}{r(r-2M)} + 2\partial_r\zeta_r && (57)\\
H'_{tt} &= 2\mu\zeta_t - \frac{2M(r-2M)}{r^3}\zeta_r + H_{tt} && (58)\\
\zeta_t &= \frac{i}{\omega}\Big(H_{tz} + \mu\zeta_z\Big) && (59)\\
\zeta_r &= -\frac{2\cot\theta\zeta_\theta + H_{\theta\theta} + 2\mathrm{cosec}^2\theta\partial_\varphi\zeta_\varphi}{2(r-2M)} \qquad \zeta_r = -\frac{H_{\theta\theta} + 2\partial_\theta\zeta_\theta}{2(r-2M)} && (60)\\
\zeta_\theta &= \frac{i}{\omega}\partial_\theta\zeta_z \qquad \zeta_\theta = \frac{r}{2}\Big(\partial_\theta\zeta_r + \partial_r\zeta_\theta\Big) \qquad \zeta_\theta = -\frac{1}{\mu}\partial_\theta\zeta_t && (61)\\
\zeta_\varphi &= \frac{i}{\omega}\partial_\varphi\zeta_z \qquad \zeta_\varphi = \frac{r}{2}\Big(\partial_\varphi\zeta_r + \partial_r\zeta_\varphi\Big) \qquad \zeta_\varphi = -\frac{1}{\mu}\partial_\varphi\zeta_t && (62)\\
0 &= -2\cot\theta\zeta_\theta + \partial_\varphi\zeta_\theta + \partial_\theta\zeta_\varphi && (63)\\
\partial_r\zeta_z &= -i\omega(\zeta_r - H_{rz} + H'_v) && (64)\\
\partial_r\zeta_t &= \frac{2M\zeta_t}{r(r-2M)} - \mu\Big(\zeta_r + H_{tr} - H'_v\Big) && (65)
\end{aligned}$$

Now under spherical symmetry we see that $\zeta_z = \zeta_z(r)$ which gives, $\zeta_\theta = 0 = \zeta_\varphi$. This then further tells us $\zeta_r = \zeta_r(r)$ and $\zeta_t = \zeta_t(r)$. The equations for ζ_θ and ζ_φ are then consistent. The equations then reduce to,

$$\begin{aligned}
H'_{zz}(r) &= H_{zz}(r) + 2i\omega\zeta_z(r) \qquad H'_{rr}(r) = H_{rr}(r) + \frac{2M\zeta_r}{r(r-2M)} + 2\partial_r\zeta_r && (66)\\
H'_{tt} &= 2\mu\zeta_t - \frac{2M(r-2M)}{r^3}\zeta_r + H_{tt} && (67)\\
\zeta_t &= \frac{i}{\omega}\Big(H_{tz} + \mu\zeta_z\Big) && (68)\\
\zeta_r &= -\frac{H_{\theta\theta}}{2(r-2M)} && (69)\\
\partial_r\zeta_z &= -i\omega(\zeta_r - H_{rz} + H'_v) && (70)\\
\partial_r\zeta_t &= \frac{2M\zeta_t}{r(r-2M)} - \mu\Big(\zeta_r + H_{tr} - H'_v\Big) && (71)
\end{aligned}$$

Substituting the equations for ζ_t and ζ_r gives two first order ODE's for ζ_z.

$$\begin{aligned}
\partial_r\zeta_z &= i\omega\Big(\frac{H_{\theta\theta}}{2(r-2M)} + H_{rz} - H'_v\Big) && (72)\\
\partial_r\zeta_z &= \mu\partial_r H_{tz} + \frac{2M}{\mu r(r-2M)}\Big(H_{04} + \mu\zeta_z\Big) - i\omega\Big(\frac{H_{\theta\theta}}{2(r-2M)} - H_{tr} + H'_v\Big) && (73)
\end{aligned}$$

One can eliminate $\partial_r\zeta_z$ to give,

$$\zeta_z = \frac{i((2M-r)r\mu\omega H_{tr} + 2iMH_{tz} + r((r-2M)\mu\omega H_{rz} + \mu\omega H_{\theta\theta} + i(2M-r)\partial_r H_{tz}))}{2M\mu} \tag{74}$$

□

3.3 Proof of GL Instability

Suppose we take the Wiseman gauge defined in previous section. In this gauge the metric perturbation can be put into the form,

$$h_{\mu\nu} = e^{\mu t + i\omega z} \begin{pmatrix} H_t(r) & \mu H_v(r) & 0 & 0 & 0 \\ \mu H_v(r) & H_r(r) & 0 & 0 & -i\omega H_v(r) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i\omega H_v(r) & 0 & 0 & H_z(r) \end{pmatrix} \tag{75}$$

In this gauge there is no further residual gauge freedom. The equation of study is,

$$\Box_g h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} = 0 \tag{76}$$

Note, it is unclear whether we are considering a well-posed problem at this stage. Later this will be resolved by making a gauge transformation to harmonic gauge and showing the resulting perturbation is non-trivial. This results in the following equations for the metric perturbation components that need to be decoupled,

$$\begin{aligned}
H_r(r) &= \frac{2M}{r(r-2M)} H_v(r) && (77)\\
H_v(r) &= \frac{-2(r-2M)^2 H_r(r) + r(r(r-2M)H_z'(r) - MH_z(r))}{2r^2(r-2M)\omega^2} && (78)\\
H_t'(r) &= \frac{MH_t(r)}{r(r-2M)} - \frac{(r-2M)(2r-3M)H_r(r)}{r^3} + 2\mu^2 H_v(r) && (79)\\
H_t(r) &= \frac{(r-2M)(2r^2(r(\omega^2-\mu^2) - 2M\omega^2)H_v(r) - 2(2M^2 + Mr - r^2)H_r(r))}{2Mr^2} && (80)\\
&\quad + \frac{(r-2M)(r(r^2 H_t'(r) - (r-2M)(rH_z'(r) - (r-2M)H_r'(r))))}{2Mr^2} && (81)\\
H_z''(r) &= \frac{-r^3\omega^2 H_t + r(-2M+r)^2\omega^2 H_r + (8M^2\omega^2 - 12Mr\omega^2 + 4r^2\omega^2)H_v + r^3\mu^2 H_z}{r(r-2M)^2} && (82)\\
&\quad + \frac{(8M^2 r\omega^2 - 8Mr^2\omega^2 + 2r^3\omega^2)H_v' - 2(2M^2 - 3Mr + r^2)H_z'}{r(r-2M)^2} && (83)\\
H_z''(r) &= \frac{2M(2r-3M)}{r(r-2M)^3} H_t(r) - \frac{\left(6M^2 - (\mu^2+\omega^2)r^4 + 2Mr(\omega^2 r^2 - 2)\right)}{r^3(r-2M)} H_r(r) && (84)\\
&\quad - \frac{2M(2M\omega^2 + r(\mu^2 - \omega^2))}{r(r-2M)^2} H_v(r) + \frac{2r-3M}{r^2} H_r'(r) - \frac{2\mu^2 r + 4M\omega^2 - 2\omega^2 r}{r-2M} H_v'(r) \\
&\quad - \frac{M}{r(r-2M)} H_z'(r) - \frac{M}{(r-2M)^2} H_t'(r) + \frac{r}{r-2M} H_t''(r) \\
H_t''(r) &= \frac{\omega^2 r^4 - 2M\omega^2 r^3 - 2M^2}{r^2(r-2M)^2} H_t(r) - \left(\mu^2 + \frac{2M^2}{r^4}\right) H_r(r) - \frac{r\mu^2}{r-2M} H_z(r) && (85)\\
&\quad + \frac{4\mu^2 r^2 + 4M^2\omega^2 - 2Mr(3\mu^2 + \omega^2)}{r^2(r-2M)} H_v(r) - \frac{2r-5M}{r(r-2M)} H_t'(r) - \frac{M(r-2M)}{r^3} H_r'(r) \\
&\quad + 2\mu^2 H_v'(r) + \frac{M}{r^2} H_z'(r)
\end{aligned}$$

Now from the first and second we can find H_v in terms of $\{H_z, H_z'\}$. This can then be used in the third to give and equation for H_t' in terms of $\{H_t, H_z, H_z'\}$. All of these expressions can be used to express H_t

in terms of $\{H_z, H_z', H_z''\}$ via the fourth equation. The resulting equations are,

$$
\begin{aligned}
H_r(r) &= -\frac{M^2 r}{(r-2M)^2(\omega^2 r^2+2M)}H_z(r) + \frac{Mr^2}{(r-2M)(\omega^2 r^2+2M)}H_z'(r) &(86)\\
H_v(r) &= -\frac{Mr^2}{(2(r-2M)(\omega^2 r^2+2M)}H_z(r) + \frac{r^3}{2(\omega^2 r^2+2M)}H_z'(r) &(87)\\
H_t(r) &= \frac{2M^2(r-3M)+M\omega^2 r^3(2r-5M)-\omega^4 r^6(r-2M)}{r(\omega^2 r^3+2M)^2}H_z(r) &(88)\\
&\quad -\frac{2(r-2M)(M(r-4M)+(2r-5M)\omega^2 r^3)}{(\omega^2 r^3+2M)^2}H_z'(r) + \frac{r(r-2M)^2}{\omega^2 r^3+2M}H_z''(r)
\end{aligned}
$$

Then one can use the above expressions to obtain a decoupled ODE for H_z.

$$
\begin{aligned}
H_z''(r) &= \Big(\frac{2(2r-5M)}{r(r-2M)} - \frac{12M}{r(\omega^2 r^3+2M)}\Big)H_z' &(89)\\
&\quad +\Big(\frac{r\omega^2}{r-2M} + \frac{\mu^2 r^4 - 6Mr + 12M^2}{r^2(r-2M)^2} + \frac{12M^2}{r^2(r-2M)(\omega^2 r^3+2M)}\Big)H_z(r)
\end{aligned}
$$

After eliminating the mass parameter we find,

$$
H_z'' = \Big(\frac{4}{x} - \frac{1}{x(x-1)} + \frac{6}{x(\hat\omega^2 x^3+1)}\Big)H_z' + \frac{x(\hat\mu^2 x + \hat\omega^2(\hat\mu^2 x^4 - 2x + 2) + \hat\omega^4 x^3(x-1))}{(\hat\omega^2 x^3+1)(x-1)^2}H_z \tag{90}
$$

Asymptotically,

$$
\begin{aligned}
H_z'' - (\omega^2+\mu^2)H_z &= 0 \qquad x\to\infty &(91)\\
H_z'' + \frac{1}{x-1}H_z' - \frac{\hat\mu^2}{(x-1)^2}H_z &= 0 \qquad x\to 1 &(92)
\end{aligned}
$$

So the asymptotic solutions are,

$$
\begin{aligned}
H_z &= c_1 e^{\sqrt{\hat\omega^2+\hat\mu^2}x} + c_2 e^{-\sqrt{\hat\omega^2+\hat\mu^2}x} \qquad x\to\infty &(93)\\
H_z &= k_1(x-1)^{\hat\mu} + k_2(x-1)^{-\hat\mu} \qquad x\to 1 &(94)
\end{aligned}
$$

For regularity on the future event horizon we require $k_2 = 0$ and at infinity $c_1 = 0$. We will call a solution to the ODE for H_z a mode solution if it satisfies these boundary conditions. It will be called unstable if it has $\hat\mu > 0$.

Proposition 4 (Rough Bounds on Frequency Parameters)**.** *There does not exist an mode solution with $\hat\omega \in \mathbb{R}\setminus(-\sqrt{2},\sqrt{2})$ or $\hat\mu \in \mathbb{R}\setminus(-\frac{1}{2},\frac{1}{2})$. In fact, this can be stengthened to $\hat\mu \in \mathbb{R}\setminus(-\frac{3}{8}\sqrt{\frac{3}{2}},\frac{3}{8}\sqrt{\frac{3}{2}})$.*

Proof. Without loss of generality take $k_1 > 0$ and suppose $k_2 = 0 = c_1$ for regularity at the boundaries. Now, since the solution must decay exponentially towards infinity, we must have a maxima $a \in (1,\infty)$. At such a point we have,

$$
H_z''|_{x=a} = \frac{a(\hat\mu^2 a + \hat\omega^2(\hat\mu^2 a^4 - 2a + 2) + \hat\omega^4 a^3(a-1))}{(\hat\omega^2 a^3+1)(a-1)^2}H_z(a) \tag{95}
$$

with $H_z(a) > 0$. To derive a contradition we must have the coefficent positive. A sufficent condition for the numerator to be positive is,

$$
\hat\mu^2 a^4 - 2a + 2 \geq 0 \tag{96}
$$

This has discriminant,

$$
\Delta = 16\hat\mu^4(128\hat\mu^2 - 27) \qquad D = 128\hat\mu^2 \tag{97}
$$

Hence if $\hat{\mu}^2 > \frac{27}{128}$ then there are no real roots. Thus because the polynomial is positive at say $a = 1$ it's positive everywhere. If $\Delta = 0$ we have a double real root and two complex conjugate roots. The real roots can only occur at a stationary point of the polynomial and therefore the polynomial cannot be negative anywhere. Since all other terms in the numericator are positive the prefactor of H_z also is. Hence there can be no unstable mode solution if $\hat{\mu} \geq \frac{3}{8}\sqrt{\frac{3}{2}}$.

Another sufficent condition for positivity of the numerator is,

$$\hat{\omega}^2 a^3 - 2 \geq 0 \tag{98}$$

This polynomial has a single real root at $a = \left(\frac{2}{\hat{\omega}^2}\right)^{\frac{1}{3}}$. For positivity on $a \in (1, \infty)$ we need $\frac{2}{\hat{\omega}^2} \leq 1$ or $\hat{\omega}^2 \geq 2$. Note that if $\hat{\mu} = 0$ then this is precisely the polynomial that governs positivity. Hence this bound for $\hat{\omega}$ is sharp. □

Remark. *One can make the bound for $\hat{\mu}$ even sharper and bound $\hat{\mu} \in \mathbb{R} \setminus (-\frac{1}{4}, \frac{1}{4})$ and $|\hat{\mu}| \leq \sqrt{2}|\omega|$. (To do).*

We now move the equation into symmetric form with a change of coordinates to $x_* = \frac{r_*}{2M}$. In x_* coordinates with the weight function,

$$w(x) = \frac{C(1 + \omega^2 x^3)}{x} \tag{99}$$

using (36) we find the ODE in formally self-adjoint (symmetric) form is,

$$-\frac{d^2 \tilde{H}_z}{dx_*^2} + \frac{(x-1)(1 + 9x^2\omega^2 - 12\omega^2 x^4 - 9\omega^4 x^6 + 6\omega^4 x^7 + \omega^6 x^9)}{x^4(1 + \omega^2 x^3)^2} \tilde{H}_z = -\mu^2 \tilde{H}_z \tag{100}$$

We will define the following function $q : \mathbb{R} \to \mathbb{R}$ via,

$$(q \circ x_*)(x) := \frac{(x-1)(1 + 9x^2\omega^2 - 12\omega^2 x^4 - 9\omega^4 x^6 + 6\omega^4 x^7 + \omega^6 x^9)}{x^4(1 + \omega^2 x^3)^2} \tag{101}$$

Noting that $x \to 1$ is equivalent to $x_* \to -\infty$, we can examine the asymptotics. We have,

$$w = C(1 + \omega^2) \qquad \tilde{H}_z = k_1 e^{\mu x_*} + k_2 e^{-\mu x_*} = \tilde{k}_1 (x-1)^{\mu} + \tilde{k}_2 (x-1)^{-\mu} \qquad x \to 1 \tag{102}$$

$$\tilde{H}_z = c_1 e^{\sqrt{\hat{\omega}^2 + \hat{\mu}^2} x_*} + c_2 e^{-\sqrt{\hat{\omega}^2 + \hat{\mu}^2} x_*} \qquad x_* \to \infty \tag{103}$$

Relating back to the asymptotics of the ODE (89), we want $k_2 = 0 = c_1$. We are now in a form where we can apply the min-max methods layed out in the section on spectral theory of self-adjoint operators, however we need to know the domain on which to define $-\Delta_{x_*} + q$ so that it is self-adjoint. The following proposition remedies this problem,

Proposition 5. $q \in (L^2 + L^\infty)(\mathbb{R})$

Proof. Note that q can be written in partial fractions as,

$$q(x) = \omega^2 \frac{x-1}{x} + \frac{(6x-11)(x-1)}{x^4} + \frac{18(x-1)^2}{x^4(1 + \omega^2 x^3)^2} - \frac{6(4x-5)(x-1)}{x^4(1 + \omega^2 x^3)} \tag{104}$$

The first term,

$$q_\infty := \omega^2 \frac{x-1}{x} \tag{105}$$

has supremum, $\sup q_\infty = \omega^2$ and therefore is in $L^\infty(\mathbb{R})$. Now, the other terms,

$$q_2 := \frac{(6x-11)(x-1)}{x^4} + \frac{18(x-1)^2}{x^4(1 + \omega^2 x^3)^2} - \frac{6(4x-5)(x-1)}{x^4(1 + \omega^2 x^3)} \tag{106}$$

are square integrable, which can be seen as follows,

$$\int_{\mathbb{R}} q_2(x_*)^2 dx_* = \int_{(1,\infty)} q_2(x)^2 \frac{x}{x-1} dx \tag{107}$$

Note that,

$$1 \geq \frac{1}{1+\omega^2 x^3} \geq \frac{1}{5x^3} \implies q_2 \leq \frac{(x-1)}{5x^6}\Big(24\big(5x^3-1\big)+5x^2\Big) \tag{108}$$

Hence,

$$\int_{\mathbb{R}} q_2(x_*) dx_* \leq \int_{(1,\infty)} \frac{(x-1)}{25x^{11}}\Big(24\big(5x^3-1\big)+5x^2\Big)^2 dx < 46 \tag{109}$$

□

With theorem 2.4, this proposition tells us that we should work in $H^2(\mathbb{R})$ for self-adjointness. To find a negative eigenvalue we apply the Ragleigh-Quotient method,

$$-\mu^2 = \inf_{u\in H^2(\mathbb{R})} \int_{-\infty}^{\infty} \Big(q(x_*)u^2 - \frac{d^2u}{dx_*^2}\Big) dx_* = \inf_{u\in H^2(\mathbb{R})} \int_{1}^{\infty} \Big(\frac{x}{x-1}q(x)u^2 + \frac{x-1}{x}u'^2\Big) dx. \tag{110}$$

where integration by parts, a change of variables have been used and it is assumed that $||u||_{L^2(1,\infty)} = 1.$ If we can find a function such that the integral here is negative then this must hold for the infimum. Note, we can write,

$$\frac{x}{x-1}q(x) = \omega^2 + \frac{6x-11}{x^3} + \frac{18(x-1)}{x^3(1+\omega^2x^3)^2} - \frac{6(4x-5)}{x^3(1+\omega^2x^3)} \tag{111}$$

Inspired by the asymptotic analysis a good guess for u is,

$$u = K(x-1)^{\frac{1}{n}} e^{-\omega x} \tag{112}$$

However, this leads to difficulties in performing the integrals and further difficulties in bounding them. It can be check numerically that for $n \geq 10$ this gives the existance of a negative eigenvalue for a range of ω.

Largely inspired by the difficulties in performing the integration with the above test function, the following guess was decided upon,

$$u = K(1+\omega^2x^3)(x-1)^{\frac{1}{n}} e^{-3\omega(x-1)} \tag{113}$$

with $n \in \mathbb{N} \setminus \{0\}$.

Proposition 6. $u(x_*) \in H^2(\mathbb{R})$

Proof. We will argue that this functions H^2-norm can be bounded as follows:

$$||u||_{H^2(\mathbb{R})} \leq C_\omega ||v||_{L^2(\mathbb{R})} \tag{114}$$

where $v(x) := x^3(x-1)^{\frac{1}{n}} e^{-3\omega(x-1)}$. Note the following,

$$u \leq K(1+\omega^2)v \tag{115}$$

since $x \geq 1$. Now,

$$\begin{aligned} \Delta_{x_*} u &= \frac{x-1}{x}\Delta_x u = \frac{1+\omega^2x^3 - 3n\omega(x-1)(1-\omega x^2+\omega^2x^3)}{nx} K(x-1)^{\frac{1}{n}} e^{-3\omega(x-1)} && (116)\\ &\leq K(1+3\omega+4\omega^2+3\omega^3)v && (117) \end{aligned}$$

since $x, n \geq 1$. Finally,

$$\Delta_{x_*}^2 u = \frac{x-1}{x}\Delta_x(\Delta_{x_*}u) = \frac{P_{\omega,n}(x)}{n^2x^3}K(x-1)^{\frac{1}{n}}e^{-3\omega(x-1)} \tag{118}$$

with $P_{\omega,n}(x) \leq n^2x^6(2+9\omega+21\omega^2+39\omega^3+18\omega^4)$. So,

$$\Delta_{x_*}^2 u \leq K(2+9\omega+21\omega^2+39\omega^3+18\omega^4)v \tag{119}$$

This means $u(x_*) \in H^2(\mathbb{R})$ if $v(x_*) \in L^2(\mathbb{R})$. Now, we can split the integrals of v into two parts,

$$||v||_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} v(x_*)^2 dx_* = \int_{(2,\infty)} v(x)^2\frac{x}{x-1}dx + \int_{(1,2)} v(x)^2\frac{x}{x-1}dx \tag{120}$$

On $(2,\infty)$ we find, $v \leq x^3(x-1)e^{-3\omega(x-1)}$ so,

$$\int_{(2,\infty)} v(x)^2\frac{x}{x-1}dx \leq \frac{\tilde{P}(\omega)}{34992\omega^9}e^{-6\omega} \tag{121}$$

with,

$$\tilde{P}(\omega) := 140+1575\omega+8820\omega^2+32760\omega^3+90720\omega^4+199584\omega^5+362880\omega^6+559872\omega^7+746496\omega^8$$

For the bounded region we find $v \leq 8(x-1)^{\frac{1}{n}}$, which has integral,

$$\int_{(1,2)} v(x)^2\frac{x}{x-1}dx \leq 64n \tag{122}$$

Hence $u(x_*) \in H^2(\mathbb{R})$. □

For the integral in the Ragleigh-Quotient we find,

$$\begin{aligned}
\int_1^\infty \frac{x-1}{x}u'^2dx &= \frac{2^{-3-\frac{2}{n}}3^{-5-\frac{2}{n}}\omega^{-\frac{2+n}{n}}\Gamma\big(\frac{2}{n}\big)K^2}{n^6}\Big(4+n(32+60\omega)+n^2(95+12\omega(23+30\omega)) \\
&+ 12\omega(23+30\omega)n^2+n^3(130+3\omega(223+36\omega(7+10\omega))) \\
&+ 3n^4(27+\omega(60\omega(10+3\omega+9\omega^2)-77))-17496n^6\omega^3 \\
&+ 18n^5(1+2\omega(-7+3\omega(-59+3\omega(6+\omega(-1+3\omega))))) \\
&+ 11664e^{6\omega}n^4\omega^2(1+3n\omega)^2E_{\frac{2}{n}}(6\omega)\Big)
\end{aligned} \tag{123}$$

$$\begin{aligned}
\int_1^\infty \frac{18(x-1)}{x^3(1+\omega^2x^3)^2}u^2dx &= \frac{(2\omega)^{\frac{n-2}{n}}3^{3-\frac{2}{n}}K^2\Gamma\big(2+\frac{2}{n}\big)}{n+2}\Big[-(n+n^2+3\omega n^2) \\
&+ e^{6\omega}(2+n+6n\omega(2+n+3n\omega))E_{\frac{2}{n}}(6\omega)\Big]
\end{aligned} \tag{124}$$

$$\begin{aligned}
\int_1^\infty \frac{6(5-4x)}{x^3(1+\omega^2x^3)}u^2dx &= \frac{3^{-\frac{2+n}{n}}4^{-\frac{1}{n}}\omega^{-\frac{2}{n}}\Gamma\big(\frac{2+n}{n}\big)}{n}\Big(-4+n(-2+3\omega(91+18n(4+15\omega))) \\
&- 54e^{6\omega}\omega(10+3n+6n\omega(10+n(4+15\omega)))E_{\frac{2}{n}}(6\omega)\Big)
\end{aligned} \tag{125}$$

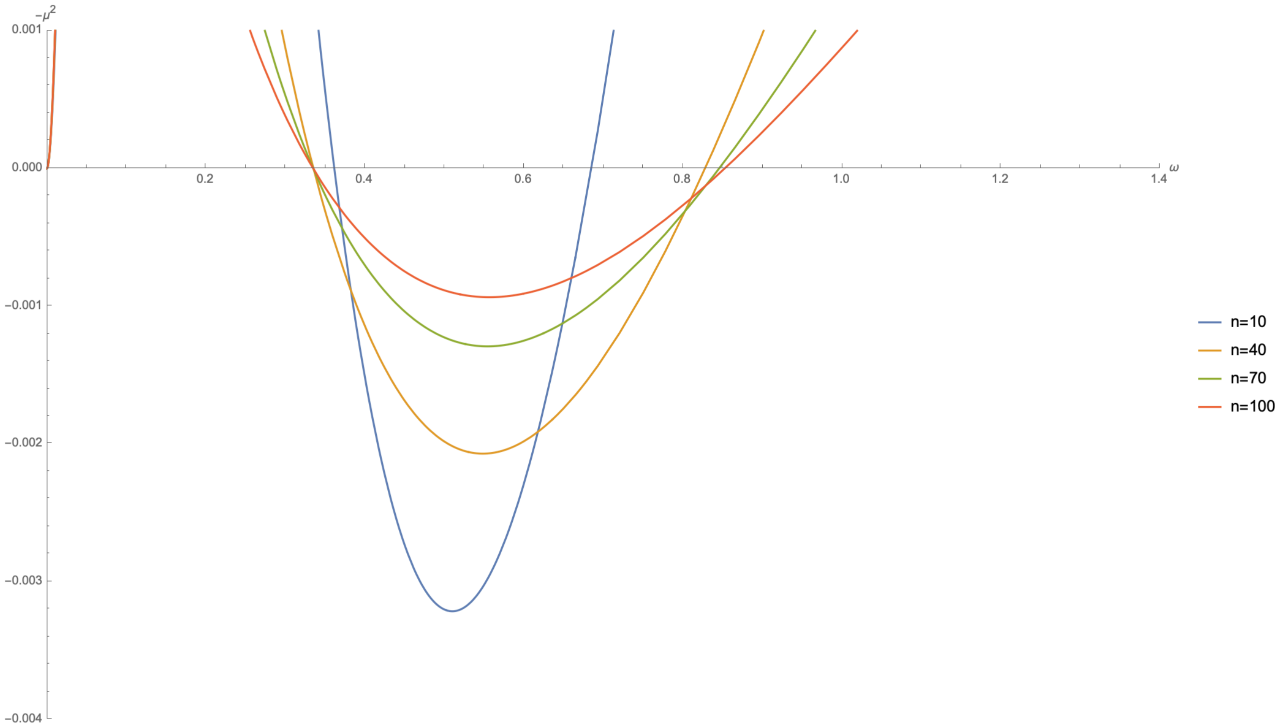
$$\begin{aligned}
\int_1^\infty \frac{6x-11}{x^3}u^2dx &= \frac{2^{-3-\frac{2}{n}}9^{-2-\frac{1}{n}}\omega^{-\frac{2+n}{n}}K^2\Gamma\big(\frac{2+n}{n}\big)}{n^4}\Big(8+n(40+52\omega+n(70+12\omega(13+3\omega) \\
&- 5832n^3\omega^2(2+11\omega)+n(50+\omega(575+54(1-6\omega)\omega)) \\
&- 3n^2(-4+\omega(-85+6\omega(1247+9\omega+30\omega^2))))) \\
&+ 1944e^{6\omega}n^3\omega^2(22+n+6n\omega(22+n(6+33\omega)))E_{\frac{2}{n}}(6\omega)\Big)
\end{aligned} \tag{126}$$

$$\int_1^\infty \omega^2u^2dx = \omega^2||u||_{L^2(1\infty)}^2 \tag{127}$$

Therefore we have a bound,

$$\begin{aligned} -\mu^2(\omega) \;\; &\leq \;\; \frac{K^2}{||u||^2_{L^2(1,\infty)}}\Big[2(1+n)(2+n)(1+2n)(2+3n)(1+5n(2+15n))\omega^2 \\ &- \;\; 9n(2+n)(-8+n(-86+n(-445+n(-877+1218n))))\omega^3 + 5832n^6\omega^8 \\ &+ \;\; 108n^3\omega^5\Big(40+n(210+n(101+3n(55+486n))) - 5832e^{6\omega}n^3E_{\frac{2}{n}}(6\omega)\Big) \\ &- \;\; 1944n^4\omega^6\Big(n(-15+8n)+486e^{6\omega}n^3E_{\frac{2}{n}}(6\omega)-5\Big) + 2916n^5(4+5n)\omega^7 \\ &+ \;\; 54n^2\omega^4\Big(20+n(160+n(409+n(587+642n)))+1944e^{6\omega}(n-1)n^3E_{\frac{2}{n}}(6\omega)\Big)\Big] \end{aligned} \tag{128}$$

This function can be plotted, varying n, as a function of ω as,



As we see from the graph, there exist regions in which $\mu^2 > 0$ and hence there exists a solution to the ODE (89) with $\mu \in \mathbb{R}_+$. This corresponds to an unstable mode solution.

At this point, it is unclear whether the PDE of study 76 is well-posed in Wiseman gauge. However, we can transform to the transverse-traceless gauge which is a sub-gauge of the harmonic gauge, which is known to be well-posed. Hence if we find that the solution to 76 corresponds to a solution of 24, then we know that the instability is physical. Taking the diffeomorphism generating vector field for the change of gauge to be,

$$[\xi_\mu] = e^{\mu t + i\omega z}\Big(-\frac{\mu H_z(r)}{2\omega^2}, -\frac{Mr^2\omega^2 H_z(r) + 2M(r-2M)H_z'(r)}{2\omega^2(r-2M)(\omega^2 r^3+2M)}, 0, 0, \frac{iH_z(r)}{2\omega}\Big) \tag{129}$$

This gives a new metric perturbation,

$$h'_{\mu\nu} \;\; = \;\; h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)} \tag{130}$$

$$= \;\; e^{\mu t + i\omega z}\begin{pmatrix} \tilde{H}_{tt} & \tilde{H}_{tr} & 0 & 0 & 0 \\ \tilde{H}_{tr} & \tilde{H}_{rr} & 0 & 0 & 0 \\ 0 & 0 & \tilde{H}_{\theta\theta} & 0 & 0 \\ 0 & 0 & 0 & \tilde{H}_{\theta\theta}\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{131}$$

with the following expressions for the metric components,

$$\begin{aligned} \tilde{H}_{tt} &= -\frac{2M(2M\mu^2 r + \omega^2(\mu^2 r^4 - Mr + 2M^2) + \omega^4 r^3(r-2M))}{r(\omega^3 r^3 + 2M\omega)^2} H_z(r) \\ &\quad -\frac{2M(r-2M)(\omega^2 r^3(3r-7M) - 2M^2)}{r^3(\omega^3 r^3 + 2M\omega)^2} H_z'(r) \end{aligned} \tag{132}$$

$$\begin{aligned} \tilde{H}_{rr} &= -\frac{2Mr(2M\mu^2 r + \omega^2(6M^2 - 3Mr + \mu^2 r^4))}{\omega^2(r-2M)^2(\omega^2 r^3 + 2M)^2} H_z(r) \\ &\quad -\frac{2M(6M^2 + \omega^2 r^4 - Mr(4 + 3r^2\omega^2))}{r(r-2M)(\omega^3 r^3 + 2M\omega)^2} H_z'(r) \end{aligned} \tag{133}$$

$$\tilde{H}_{tr} = -\frac{2M\mu}{2M\omega^2 + r^3\omega^4} H_z'(r) + \frac{2M^2\mu}{\omega^2 r(r-2M)(\omega^2 r^3 + 2M)} H_z(r) \tag{134}$$

$$\tilde{H}_{\theta\theta} = -\frac{Mr^2}{\omega^2 r^3 + 2M} H_z(r) - \frac{2M(r-2M)}{\omega^4 r^3 + 2M\omega^2} H_z'(r) \tag{135}$$

where the equation 89 has been used. This new metric satisfies the transverse-traceless gauge: $g^{\mu\nu}h'_{\mu\nu} = 0$ and $\nabla^\mu h'_{\mu\nu} = 0$. We need to convince ourselves that this is not trivial. Suppose the components are vansihing then we find the following expressions for the first derivative of H_z,

$$H_z' = \frac{M}{r(r-2M)} H_z \tag{136}$$

$$H_z' = -\frac{\omega^2 r^2}{2(r-2M)} H_z \tag{137}$$

$$H_z' = -\frac{r^2(6M^2\omega^2 + r^4\mu^2\omega^2 + Mr(2\mu^2 - \omega^2))}{(r-2M)(6M^2 + r^4\omega^2 - Mr(4 + 3r^2\omega^2))} H_z \tag{138}$$

$$H_z' = \frac{r^2(2M^2\omega^2 + r^4\omega^2(\mu^2 + \omega^2) - Mr(2\omega^4 r^2 + \omega^2 - 2\mu^2))}{(r-2M)(2M^2 + 7Mr^3\omega^2 - 3r^4\omega^2)} H_z \tag{139}$$

From the first two equations alone we find, $2M + \omega^2 r^3 = 0$, which is a contradiction.

Note this new metric satisfies the Lichneriwicz equation, 24, which can be demonstrated by doing the following gauge transformation explicitly. We know $h_{\mu\nu}$ satisfies 76. Therefore,

$$\begin{aligned} 0 &= \Box_g h'_{ab} - 2[\Box_g, \nabla_{(a}]\xi_{b)} + 2R_a{}^c{}_b{}^d h'_{cd} - 4R_a{}^c{}_b{}^d \nabla_{(c}\xi_{d)} + \nabla_a\nabla_b \cancelto{0}{h'} \qquad (140) \\ &\quad -2\nabla_a\nabla_b\nabla^c\xi_c - \cancelto{0}{2\nabla_{(a}\nabla^c h'_{b)c}} + 2\nabla_{(a}\nabla^c\nabla_{b)}\xi_c \qquad (141) \end{aligned}$$

In vacuum, we have $\nabla^c\nabla_b\xi_c = \nabla_b(\nabla^c\xi_c)$.

$$0 = \Box_g h'_{ab} + 2R_a{}^c{}_b{}^d h'_{cd} - 2[\Box_g, \nabla_{(a}]\xi_{b)} - 4R_a{}^c{}_b{}^d \nabla_{(c}\xi_{d)} \tag{142}$$

Again, in vacuum we find,

$$\nabla_a(\Box_g\xi_b) = \Box_g(\nabla_a\xi_b) - \xi_e\nabla^c R^e{}_{bac} - 2R^e{}_{bac}\nabla^c\xi_e \tag{143}$$

leaving,

$$0 = \Box_g h'_{ab} + 2R_a{}^c{}_b{}^d h'_{cd} - 2\xi_e\nabla^c R^e{}_{bac} \tag{144}$$

From the Bianchi identity,

$$\nabla_{[d}R^e{}_{|b|ac]} = 0 \implies \nabla_{[d}R^d{}_{|b|ac]} = 0 \implies \nabla^d R_{dbac} = 0 \implies \nabla^d R^c{}_{bad} = 0 \tag{145}$$

Hence h'_{ab} satisfies,

$$\Box_g h'_{ab} + 2R_a{}^c{}_b{}^d h'_{cd} = 0. \tag{146}$$

Proposition 7. *A perturbation of the form (131) satisfying (24) cannot be pure gauge unless* $\omega = 0$.

Proof. In the following, indices $i, j, k, l, m \in (0, ..., 3)$ and Greek characters for the full spacetime indices. Note that, since (131) has $h_{z\mu} = 0$ and the Riemann tensor vanishes in this direction, the equation (24) reduces to,

$$g^{km}\nabla_k\nabla_m h_{ij} + 2R_i{}^k{}_j{}^m h_{km} + \partial_z^2 h_{ij} = 0. \tag{147}$$

For the perturbation (131) this further reduces to a massive tensor wave equation,

$$g^{km}\nabla_k\nabla_m h_{ij} + 2R_i{}^k{}_j{}^m h_{km} - \omega^2 h_{ij} = 0. \tag{148}$$

For

$$2\nabla_{(\mu}\xi_{\nu)} = e^{\mu t + i\omega z}\begin{pmatrix} \tilde{H}_{tt} & \tilde{H}_{tr} & 0 & 0 & 0 \\ \tilde{H}_{tr} & \tilde{H}_{rr} & 0 & 0 & 0 \\ 0 & 0 & \tilde{H}_{\theta\theta} & 0 & 0 \\ 0 & 0 & 0 & \tilde{H}_{\theta\theta}\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{149}$$

we require $\partial_z\xi_z = 0$ and $\partial_z\xi_i + \partial_i\xi_z = 0$. Using these results and the symmetry of partial derivatives, the gauge conditions reduce to,

$$\nabla_i\xi^i = 0 \qquad g^{ij}\nabla_i\nabla_j\xi_k = 0. \tag{150}$$

Commuting covariant derivatives and using $R_{ij} = 0$ and the Bianchi Identity gives,

$$g^{km}\nabla_k\nabla_m\nabla_{(i}\xi_{j)} + 2R_i{}^k{}_j{}^m\nabla_{(k}\xi_{m)} = 0 \tag{151}$$

Hence we require,

$$\omega^2\nabla_{(i}\xi_{j)} = 0. \tag{152}$$

So if $\omega \neq 0$ then $\nabla_{(i}\xi_{j)} = 0$ and therefore the perturbation cannot be pure gauge. □

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