COMPUTATION OF GRAPH MINORS

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Abstract. The Graph Minors Project is based on Robertson and Seymour's proof of Wagner's Conjecture. This paper is intended to provide an introduction to the study of graphs, graph minors, and computational problems surrounding graph minors, ending with Fellows and Langston's result stating that the computation of an obstruction set for a generalized minor-closed class of graphs is impossible.

1. Introduction

In the mid-1980's, Neil Robertson and Paul Seymour announced their proof of one of the deepest and most impactful theorems in combinatorics. The full proof was stretched over twenty papers, and this result opened up new areas of study surrounding graphs and their structure.

This paper is structured as follows. In Section 2, graphs, subgraphs, and the edge- and vertex-deletion operations are defined. In Section 3, the concepts of graph minors and minor-closed classes of graphs are introduced, and Robertson and Seymour's titular result is stated. Section 4 introduces the idea of algorithmic complexity and the complexity of testing membership in a minor closed class. Kuratowski's Theorem and the general problem of embedding graphs on surfaces is discussed in Section 5. Finally, the issue of computing obstruction sets for minor-closed classes is discussed in Section 6, utilizing several results in computer science and mathematics.

2. Graphs and Subgraphs

In this section, we give the definitions of graphs, subgraphs, edge deletions, and vertex deletions.

Definition 2.1. A *graph* is a pair $G = (V, E)$ of sets such that:

(1) V (or $V(G)$) is a finite set whose elements are called *vertices*.

Date: April 17, 2019.

1991 *Mathematics Subject Classification.* 05C83.

(2) E (or $E(G)$) is a finite set of 2-element subsets of V whose elements are called *edges*.

Example 2.2. Consider the sets $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{b, d\}\}$, where each element in E is a 2-element subset of V. The pair $G = (V, E)$ is an example of a graph. We can draw this graph by drawing the elements in V (vertices) as points and the elements in E (edges) as lines between the elements in V.

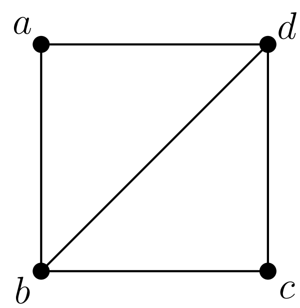

FIGURE 1. The graph G in Example 2.2.

For an edge $e = \{u, v\}$, we say that the vertices u and v are *endpoints* of e. An edge is *incident* to a vertex if the edge has the vertex as an endpoint. Similarly, a vertex is *adjacent* to another vertex if both vertices are endpoints of the same edge. We use these definitions to build the following graph operations.

Let $G = (V, E)$ be a graph, and let $e \in E$ be an edge in G. Deleting e from G results in a graph $G \setminus e = (V', E')$ such that $V' = V$ and $E' = E \setminus \{e\}$. To put it another way, we remove the edge e from the edge set of G. We call this the *edge-deletion* operation.

Similarly, let $H = (V, E)$ be a graph, and let $v \in V$ be a vertex in H. Deleting v from H results in a graph $H - v = (V', E')$ such that $V' = V \setminus \{v\}$ and $E' = E \setminus K$, where K is the set of edges incident to v. To put it another way, we remove both the vertex v from the vertex set of H and the set of all edges incident to v from the edge set of H. We call this the *vertex-deletion* operation.

Example 2.3. Consider the graph G. We build the graph H by deleting the edge e and deleting the vertex v. We say that H is a subgraph of G.

A graph H is a *subgraph* of G if it can be obtained from G by a sequence of edge deletions and vertex deletions. More formally, let $G = (V, E)$ and $H = (V', E')$ be graphs such that $V' \subseteq V$ and $E' \subseteq E$. We say that H is a subgraph of G.

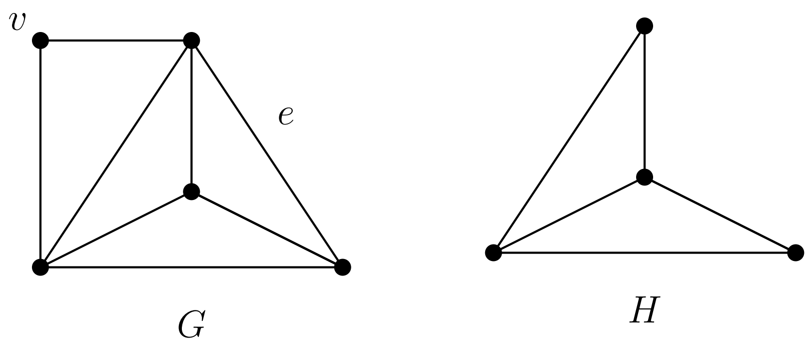

Figure 2. The graphs G and H in Example 2.3.

3. Robertson and Seymour's Graph Minors Theorem

In this section, we define a minor of a graph, introduce the concept of well-quasi-ordering, and state the foundational result of Robertson and Seymour.

Contracting an edge $e = \{u, v\}$ in a graph G is an operation that first introduces a new vertex a_e and new edges such that a_e is adjacent to all the vertices adjacent to u or v and then deletes the vertices u and v.

Example 3.1. Consider the graph A. We build the graph B by contracting the edge e.

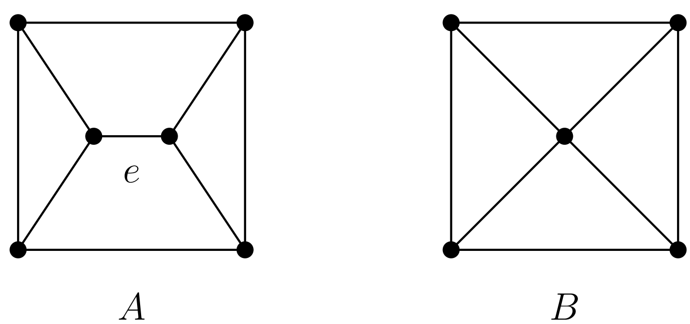

Figure 3. The graphs A and B in Example 3.1.

Definition 3.2. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of edge deletions, vertex deletions, and edge contractions.

Two examples of minors have already appeared in this paper. In Example 2.3, the graph H is a minor of the graph G. Likewise, in Example 3.1, the graph B is a minor of the graph A.

Definition 3.3. A class F of graphs is said to be *minor closed* (closed under the operation of taking minors) if every minor of a graph in F also belongs to F.

A simple example of a class of graphs closed under minors is the class of cycle-free graphs. A cycle is a set of at least three vertices connected by a closed chain of edges, and a cycle-free graph is a graph that does not contain cycles. Deleting edges or vertices from a cycle-free graph cannot create a cycle (since that would mean a cycle was already in the graph); likewise, obtaining a cycle from contracting edges in a cycle-free graph means that the graph was not cycle-free in the first place. Thus, the class of cycle-free graphs is closed under minors.

A significant part of the work done on graph minors is based on classes of graphs that are minor closed. Perhaps equally important is to note what is not included in these classes — forbidden minors and obstruction sets.

Definition 3.4. Let F be a class of graphs closed under minors, and let S be the set of graphs not in F. The set S' is considered to be the *obstruction set* of F if $S' \subseteq S$ and, for every graph G in S', all minors of G are in F. The elements of S' are the *forbidden minors* of F.

Using our previous example of the class of cycle-free graphs, the obstruction set of this class contains exactly one graph: the complete graph on three vertices, K_3. This graph is, by definition, a set of three vertices connected by a closed chain of edges, and is thus not cycle-free. However, every minor of K_3 does not contain a cycle, making every minor of K_3 cycle-free. All other graphs that are not cycle-free have at least one minor that is also not cycle-free. This also means that, since every graph that is not cycle-free has at least one cycle, deleting all vertices and edges not part of the cycle and contracting all but three edges gives K_3 as a minor of the graph.

Classes of graphs are not the only way to use graph minors to denote the structure of graphs. The relation formed by graph minors, denoted as $\preceq$, creates a partial order on the set of all graphs. However, Robertson and Seymour's titular result shows that this order is actually much stronger than just a partial order.

Theorem 3.5 (Robertson and Seymour [6])**.** *The graph minor relation $\preceq$ is reflexive, transitive, and, for every infinite sequence $G_0, G_1, G_2 \cdots$ of graphs, there are indices $i < j$ such that $G_i \preceq G_j$.*

Stated differently, the graph minor relation $\preceq$ is a well-quasi-ordering.

This statement asserts a very important fact, which is expressed in the following corollary.

Corollary 3.6. *For a fixed class of graphs closed under minors, there exists a finite list of forbidden minors.*

This idea is quite significant. If the list of forbidden minors for a minor-closed class was infinite, it would make no sense to use minors to determine if a graph is in a minor closed class. However, since the list is finite, the process of determining if a graph is in a minor-closed class boils down to testing if at least one of the minors in the forbidden minor list is a minor of that graph.

4. Testing Membership in A Minor-Closed Class

In this section, we discuss the complexity of determining if a fixed graph is in a minor closed class. However, we first need the necessary notation to discuss the complexity of problems and algorithms.

Big O notation is a symbolism used to describe the asymptotic behavior of functions. This notation is used to express the growth rate of an algorithm. For example, when analyzing an algorithm, one might find that the time it takes to complete a problem of size n is given by $T(n) = 3n^2 + 4n + 10$. If we ignore constants and slower-growing terms, we could say that $T(n)$ grows at the order of n^2 and write $T(n) = O(n^2)$. Formally, for functions $f(n)$ and $g(n)$ defined on the positive integers, $f(n) = O(g(n))$ if there are constants M and N such that $|f(n)| \leq M \cdot g(n)$ for all $n \geq N$.

The first key result in applying graph minors to complexity theory is the following theorem.

Theorem 4.1 (Robertson and Seymour [7])**.** *For a fixed graph H, there is an algorithm in $O(n^3)$ time to determine whether a given graph of order n has H as a minor.*

Corollary 4.2 follows from Theorems 4.1 and 3.5.

Corollary 4.2. *For a fixed graph of order n and a fixed class of graphs closed under minors, there exists an algorithm in $O(n^3)$ time to determine whether that graph belongs to the class of graphs.*

Proof. By Theorem 4.1, there is an algorithm in $O(n^3)$ time to determine whether a given graph of order n has some fixed graph H as a minor. By Theorem 3.5, the list of forbidden minors for a class of graphs closed under minors is finite. Therefore, if there are k graphs in this list, the amount of time to run the algorithm from Theorem 3.3 on k elements is $k \times O(n^3)$. Since k is a constant not dependent on n, this means that the amount of time to run the algorithm from Theorem 4.1 on k elements is $O(n^3)$. □

Corollary 4.2, while true, does hide an issue with the application of this algorithm: the coefficients of the growth rate are extremely large. This means that, when actually running the algorithm, the amount of time needed to compute the algorithm is also extremely large.

5. Kuratowski's Theorem and Wagner's Conjecture

In this section, we give an example of a type of minor-closed class — classes of graphs embeddable on surfaces, specifically 2-manifolds.

An embedding of a graph G on a surface Σ is a representation of G on Σ in which points of Σ are associated with vertices and simple arcs (homeomorphic images of $[0, 1]$) are associated with edges in such a way that the endpoints of the arc associated with an edge e are the points associated with the end vertices of e, no arcs include points associated with other vertices, and two arcs never intersect at a point that is interior to either of the arcs.

A graph G is *planar* if G is embeddable onto a plane. Likewise, a graph G is *nonplanar* if G is not embeddable onto a plane.

Theorem 5.1. *If G is planar, then every minor of G is also planar.*

Proof. Consider the case where an edge is deleted. Since G is planar, removing an edge will not cause a pair of edges to intersect. Likewise, removing a vertex will not cause a pair of edges to intersect. Finally, note that edge contraction will keep the graph planar because no edges are created; rather, edges are extended to an already existing point and a vertex is deleted. Since a minor is built through edge deletion, vertex deletion, and edge contraction, and none of these operations can create an intersection where one did not already exist, the minor of a planar graph G is also planar. □

Another way to state the result of Theorem 5.1 is that the class of planar graphs is closed under minors. This means that, by Theorem 3.5, there exists a finite list of forbidden minors for the class of planar graphs. However, the question then arises regarding the contents of the obstruction set. Kuratowski's Theorem is a classic result in graph theory regarding planar graphs, which was proved several decades before Robertson and Seymour proved Theorem 3.5. In 1937, Klaus Wagner proved the following refinement of Kuratowski's Theorem, using minors instead of subgraphs.

Theorem 5.2 (Wagner [1])**.** *A finite graph G is planar if and only if neither $K_{3,3}$ nor K_5 is a minor of G.*

According to Robertson and Seymour, Wagner followed this theorem with the following conjecture.

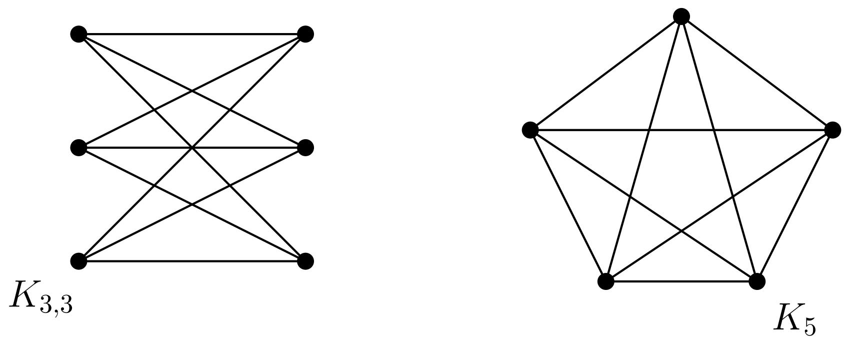

FIGURE 4. The graphs $K_{3,3}$ and K_5 in Theorem 5.2

Conjecture 5.3 (Wagner [8]). *Let C be the class of graphs embeddable on a surface Σ. The obstruction set for C is finite.*

Theorem 3.5 says that the conjecture is true. In fact, this is the conjecture the theorem was attempting to prove. The issue, however, lies in the fact that Theorem 3.5 does not provide a method of finding the set of excluded minors; the theorem only guarantees that the list is finite. In some cases, it is possible to conclude that an obstruction set is actually complete. However, many other cases have yet to reach this point. Two surfaces that showcase this issue are the projective plane and the torus.

The obstruction set of the class of graphs embeddable on the projective plane $\mathbb{R}P^2$ is a set of exactly 35 graphs. [2] In comparison, Chambers and Myrvold have found 16,629 distinct forbidden minors for graphs embeddable on the torus. [3] However, there has been no guarantee that this list is complete.

By Theorem 3.5, we know that the obstruction set for graphs embeddable on the torus is finite. The issue is that figuring out if the current list is complete or not is rather difficult. Simply attempting to compute the obstruction set is also not a valid solution, and this idea is discussed in the next section.

6. Computing Obstruction Sets

In this section, we discuss the problem of computing obstruction sets and several results related to this problem. As mentioned earlier, one of the issues with Theorem 3.5 is that it is nonconstructive. In other words, the theorem only guarantees that the obstruction set is finite but does not give a way to find the obstruction set itself. The question that naturally arises is as follows: is it possible to compute the obstruction set for an arbitrary class of graphs closed under minors?

Adler, Grohe, and Kreutzer were able to show that it is possible to compute the obstruction set for a class of graphs closed under minors in certain situations, particularly using the class of apex graphs. [5] Let F be a class of graphs closed under minors. The graph H is considered to be an *apex graph* over F if removing one vertex from H results in a graph from F.

Theorem 6.1 (Adler, Grohe, Kreutzer [5]). *Let F be a class of graphs closed under minors with a known obstruction set. Then there exists an algorithm to compute the obstruction set of the class of graphs apex over F.*

Theorem 6.2 (Adler, Grohe, Kreutzer [5]). *Let F, F' be classes of graphs closed under minors with known obstruction sets. Then there exists an algorithm to compute the obstruction set of the minor-closed class $F \cup F'$.*

Despite these specific cases having positive solutions, the general case of this problem has a negative answer from Fellows and Langston.

Theorem 6.3 (Fellows and Langston [4]). *There is no algorithm to compute, from a finite description of an arbitrary minor-closed class F of graphs, the obstruction set of F.*

This theorem works similarly to the halting problem. The halting problem consists of being able to determine whether an algorithm will run forever using only the description of the algorithm. Because the set of all finite graphs is infinite, it is impossible to guarantee that the algorithm will halt once it has found the complete obstruction set.

Acknowledgements

The author gives special thanks to Dr. Bogdan Oporowski for his advice and encouragement with this project.

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