Complexity of Computing Categorified Invariants

Comparing Khovanov Homology to the Bollobás–Riordan Homology

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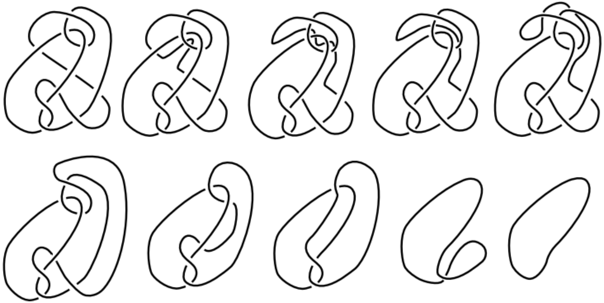
UA System Honors Research Conference

Knots $\rightarrow$ Knot Invariants $\rightarrow$ Jones Polynomial $\rightarrow$ Khovanov Homology

Knots → Knot Invariants → Jones Polynomial → Khovanov Homology

Definition: Knot

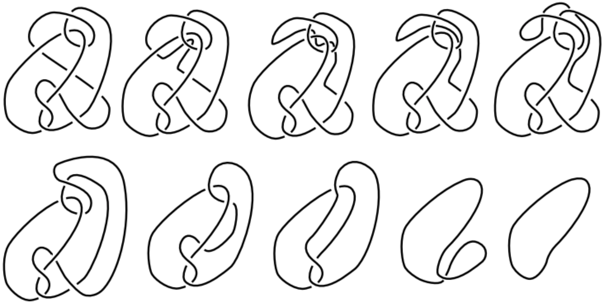
A *knot* is a closed, non-self-intersecting curve in three-dimensional space.
Application: DNA Unknotting



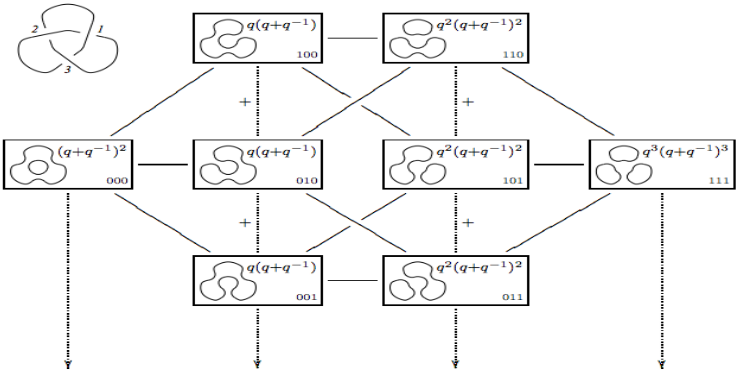
Knots → Knot Invariants → Jones Polynomial → Khovanov Homology

Definition: Knot Invariant

A *knot invariant* is a function that returns the same value for all equivalent knots.
Application: Quantum Money (Fahri et. al, 2010)



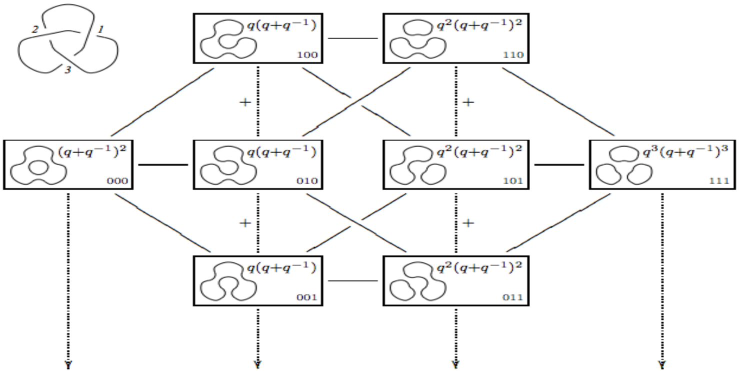
Knots $\to$ Knot Invariants $\to$ Jones Polynomial $\to$ Khovanov Homology



$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3 \tag{1}$$

$$= q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\ ) = q^2+q^6-q^8.$$

Knots $\rightarrow$ Knot Invariants $\rightarrow$ Jones Polynomial $\rightarrow$ Khovanov Homology



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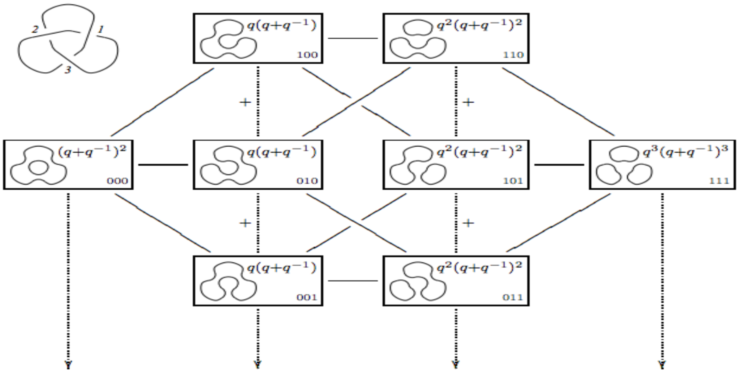

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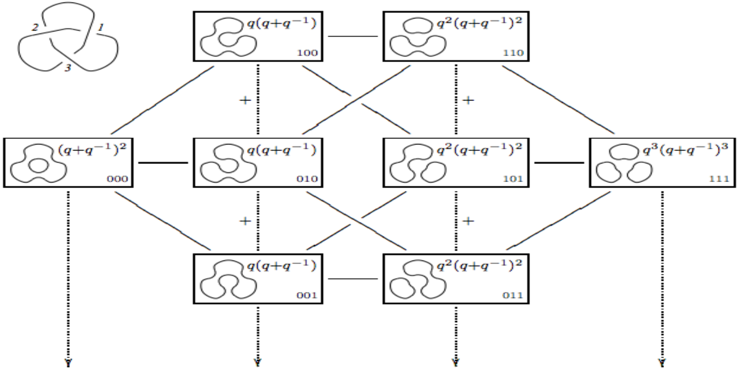
Visualizing the Jones Polynomial (Bar-Natan, 2002)

Knots → Knot Invariants → Jones Polynomial → Khovanov Homology



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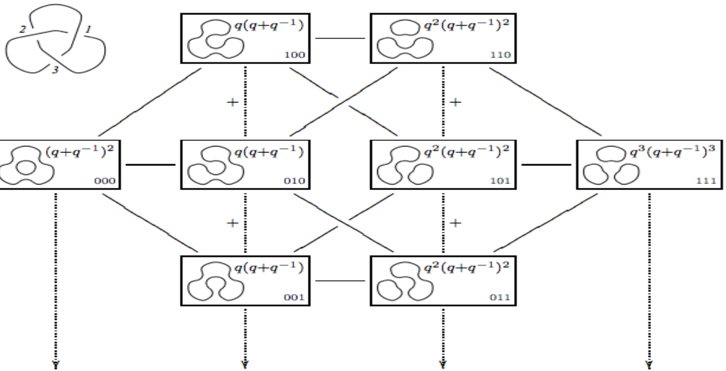

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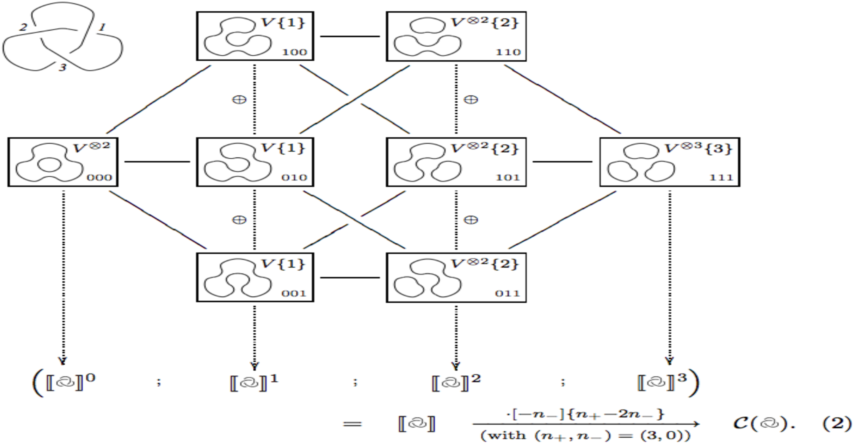
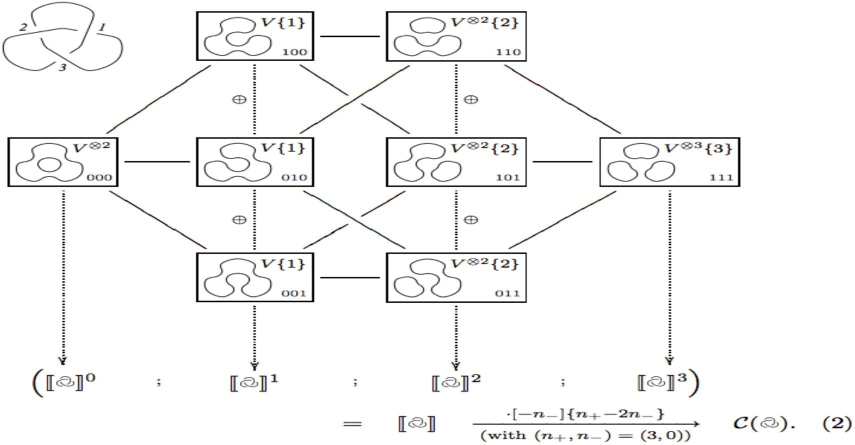
Knots $\to$ Knot Invariants $\to$ Jones Polynomial $\to$ Khovanov Homology



Complexity of Computing Categorified Invariants

Visualizing Khovanov Homology (Bar-Natan, 2002)

Knots → Knot Invariants → Jones Polynomial → Khovanov Homology



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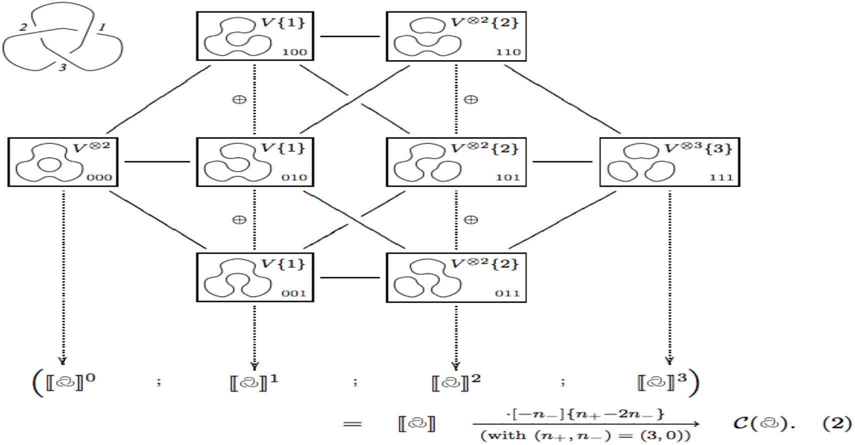

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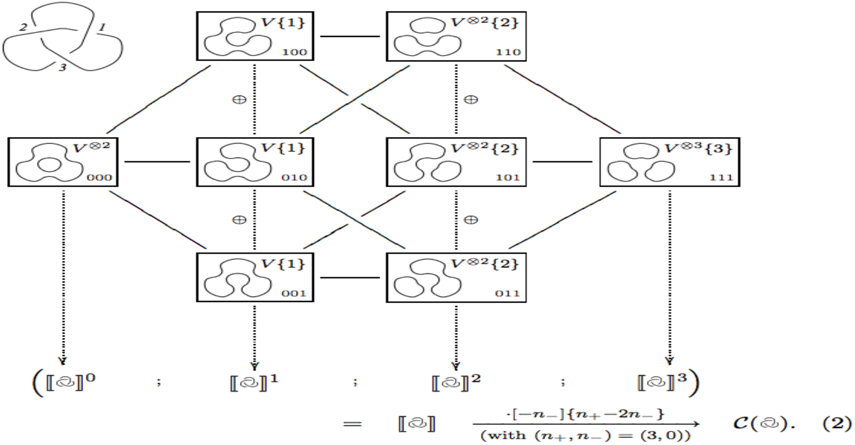


Graphs & Invariants $\rightarrow$ Chromatic Polynomial $\rightarrow$ Chromatic Homology

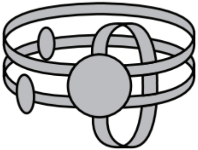
Graphs & Invariants → Chromatic Polynomial → Chromatic Homology

Definition: Graphs & Fatgraphs

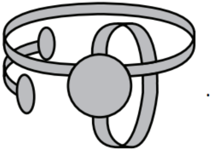
A *graph* is a collection of vertices and edges connecting said vertices. A *fatgraph* is a graph where the vertices and edges are "fattened".

Definition: Graph Invariant

A *graph invariant* is a function that returns the same value for all equivalent graphs.

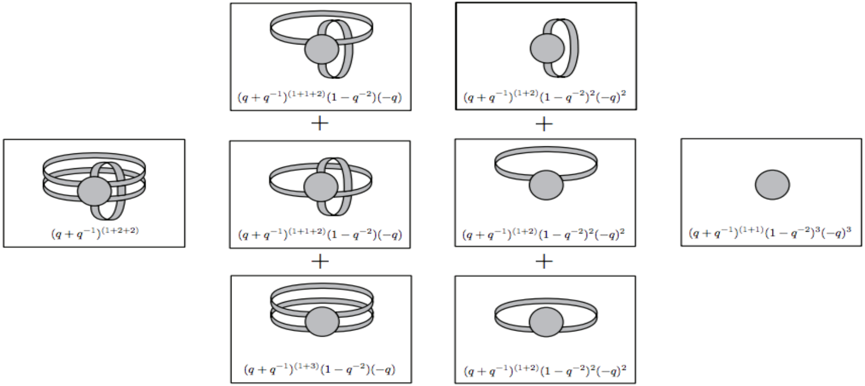


A fatgraph.



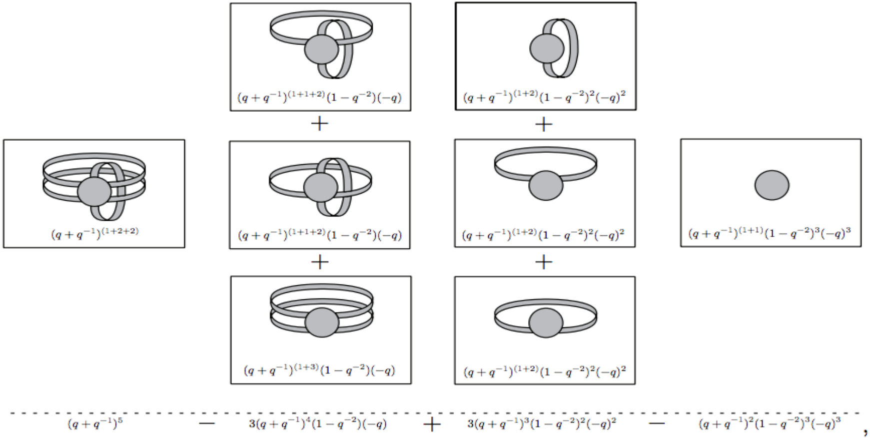
A state.

Graphs & Invariants → Chromatic Polynomial → Chromatic Homology

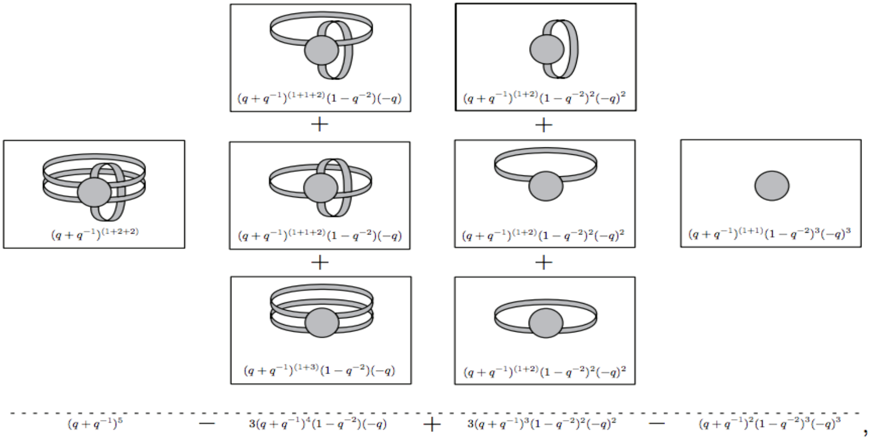


$$(q+q^{-1})^5 \quad - \quad 3(q+q^{-1})^4(1-q^{-2})(-q) \quad + \quad 3(q+q^{-1})^3(1-q^{-2})^2(-q)^2 \quad - \quad (q+q^{-1})^2(1-q^{-2})^3(-q)^3 ,$$

Graphs & Invariants → Chromatic Polynomial → Chromatic Homology



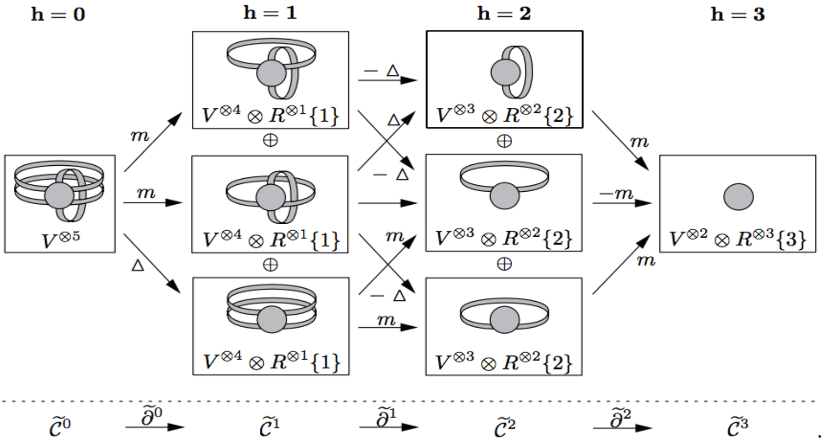
Graphs & Invariants → Chromatic Polynomial → Chromatic Homology



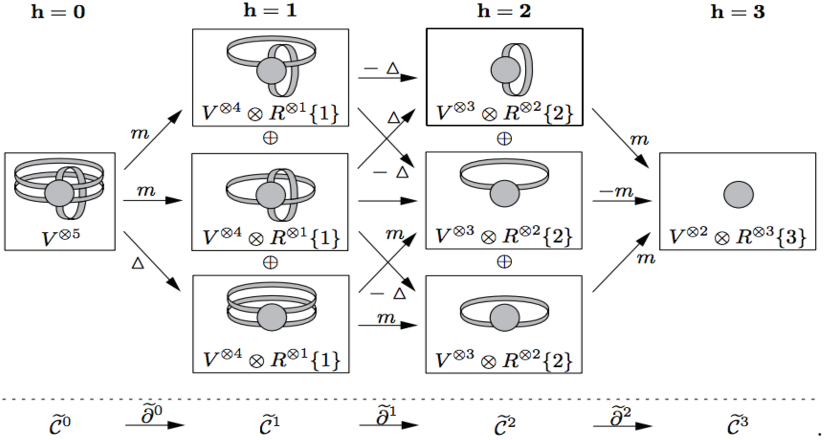
Complexity of Computing Categorified Invariants

Visualizing Chromatic Homology (Loebel & Moffat, 2007)

Graphs & Invariants → Chromatic Polynomial → Chromatic Homology



Graphs & Invariants → Chromatic Polynomial → Chromatic Homology



Bollobás–Riordan Homology

Bollobás–Riordan Homology is the chain complex generated by a three-variable generalization of the chromatic polynomial.

Main Theorem (Loebel & Moffat, 2007):

The Khovanov Homology of some knots can be recovered from the Bollobás–Riordan Homology of related fatgraphs.

Potential Theorem

The Khovanov Homology of any knot can be recovered from a modified Bollobás–Riordan Homology of a related fatgraph.

Thank You!