MSH7 - APPLIED PROBABILITY AND STOCHASTIC CALCULUS

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1. Lecture 1 - Tuesday 1 March

Definition 1.1 (Finite dimensional distribution)**.** The **finite dimensional distribution** of a stochastic process X is the joint distribution of $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$

Definition 1.2 (Equality in distribution)**.** Two random variables X and Y are **equal in distribution** if $\mathbb{P}(X \leq \alpha) = \mathbb{P}(Y \leq \alpha)$ for all $\alpha \in \mathbb{R}$. We write $X \stackrel{d}{=} Y$.

Definition 1.3 (Strictly stationary)**.** A stochastic process X is **strictly stationary** if

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}) = (X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h})$$

for all t_i, h.

Definition 1.4 (Weakly stationary)**.** A stochastic process X is **weakly stationary** if $\mathbb{E}\left(X_t\right) = \mathbb{E}\left(X_{t+h}\right)$ and $\operatorname{Cov}\left(X_t, X_s\right) = \operatorname{Cov}\left(X_{t+h}, X_{s+h}\right)$ for all t, s, h.

Lemma 1.5. *If $\mathbb{E}\left(X_t^2\right) < \infty$, then strictly stationary implies weakly stationary.*

Example 1.6.

- The stochastic process $\{X_t\}$ with X_t all IID is strictly stationary.
- The stochastic process W_t with W_t $N(0, t)$ and $X_t - X_s$ independent of X_s (for $s < t$) is not strictly or weakly stationary.

Definition 1.7 (Stationary increments)**.** A stochastic process has **stationary increments** if

$$X_t - X_s \stackrel{d}{=} X_{t-h} - X_{s-h}$$

for all s, t, h.

2. Lecture 2 - Thursday 3 March

Example 2.1. Let $X_n, n \geq 1$ be IID random variables. Consider the stochastic process $\{S_n\}$ where $S_n = \sum_{j=1}^{n} X_j$. Then $\{S_n\}$ has stationary increments.

2.1. **Concepts of convergence.** There are three major concepts of convergence of random variables.

- Convergence in distribution
- Convergence in probability
- Almost surely convergence

Definition 2.2 (Convergence in distribution)**.** $X_n \xrightarrow{d} X$ if $\mathbb{P}(X_n \leq x) \to \mathbb{P}(X \leq x)$ for all x.

Definition 2.3 (Convergence in probability)**.** $X_n \xrightarrow{p} X$ if $\mathbb{P}((|X_n - X| \geq \epsilon)) \to 0$ for all $\epsilon > 0$.

Definition 2.4 (Almost surely convergence)**.** $X_n \xrightarrow{a.s.} X$ if except on a null set A, $X_n \to X$, that is, $\lim_{n\to\infty} X_n = X$. And hence $\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$

Definition 2.5 (σ-field generated by X)**.** Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We call $\sigma(X)$ the σ-field generated by X, and we have

$$\sigma(X) = \left\{X^{-1}(B) : B \in \mathcal{B}\right\}$$

where $X^{-1}(B) = \{\omega, X(\omega) \in \mathcal{B}\}$ and $\mathcal{B}$ is the Borel set on $\mathbb{R}$.

Definition 2.6 (Conditional probability)**.** We have $\mathbb{P}(A|B) = \frac{\mathbb{P}(A,B)}{\mathbb{P}(B)}$ if $\mathbb{P}(B) \neq 0$.

Definition 2.7 (Naive conditional expectation)**.** We have $\mathbb{E}\left(X|B\right) = \frac{\mathbb{E}(X\mathbb{I}_B)}{\mathbb{P}(B)}$.

Definition 2.8 (Conditional density)**.** Let $g(x,y)$ be the joint density function for X and Y. Then we have Y $\int_{\mathbb{R}} g(x,y)\,dx \equiv g_Y(y)$. We also have

$$g_{X|Y=y} = \frac{g(x,y)}{g_Y(y)}$$

which defines the conditional density given $Y = y$.

Finally, we define $\mathbb{E}\left(X|Y=y\right) = \int_{\mathbb{R}} x g_{X|Y=y}(x)\,dx$.

Definition 2.9 (Conditional expectation)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{A}$ be a sub σ-field of $\mathcal{F}$. Let X be a random variable such that $\mathbb{E}(|X|) < \infty$. We define $\mathbb{E}\left(X|\mathcal{A}\right)$ to be a random variable Z such that

(i) Z is $\mathcal{A}$-measurable,
(ii) $\mathbb{E}(X\mathbb{I}_A) = \mathbb{E}(Z\mathbb{I}_A)$ for all $A \in \mathcal{A}$.

Proposition 2.10 (Properties of the conditional expectation)**.** *Consider* $Z = \mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y))$

- *If* T *is* $\sigma(Y)$*-measurable, then* $\mathbb{E}(XT|Y) = T\mathbb{E}(X|Y)$ *a.s.*
- *If* T *is independent of* Y*, then* $\mathbb{E}(T|Y) = \mathbb{E}(T)$*.*
- $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|T))$

3. Lecture 3 - Tuesday 8 March

Definition 3.1 (Martingale)**.** Let $\{X_t, t \geq 0\}$ be a **right-continuous** with **left-hand limits**.

$$\lim_{t \uparrow t_0} X_t \quad \text{exists}$$

Let $\{\mathcal{F}_t, t \geq 0\}$ be a filtration.

Then X is called a martingale with respect to $\mathcal{F}_t$ if

(i) X is **adapted to** $\mathcal{F}_t$, i.e. X_t is $\mathcal{F}_t$-measurable
(ii) $\mathbb{E}(|X|) < \infty, t \geq 0$
(iii) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ a.s.

Example 3.2. Let X_n be IID with $\mathbb{E}(X_n) = 0$. Then $\{S_k, k \geq 0\}$, where $S_k = \sum_{i=0}^{k} X_i$, is a martingale.

Example 3.3. An independent increment process $\{X_t, t \geq 0\}$ with $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(|X_t|) \leq \infty$ is a martingale with respect to $\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$

Definition 3.4 (Gaussian process)**.** Let $\{X_t, t \geq 0\}$ be a stochastic process. If the finite dimensional distributions are multivariate normal, that is,

$$(X_{t_1}, \dots, X_{t_m}) \equiv N(\mu, \Sigma)$$

for all $t_1, \dots, t_m$, then we call X_t a **Gaussian process**

Definition 3.5 (Markov process)**.** A continuous time process X is a **Markov process** if for all t, each $A \in \sigma(X_s, s > t)$ and $B \in \sigma(X_s, s < t)$, we have

$$\mathbb{P}(A|X_t, B) = \mathbb{P}(A|X_t)$$

Definition 3.6 (Diffusion process)**.** Consider the stochastic differential equation

$$dX_t = \mu(t, x)dt + \sigma(t, x)dB_t$$

A diffusion process is **path-continuous**, **strong Markov** process such that

$$\lim_{h \to 0} h^{-1}\mathbb{E}(X_{t+h} - X_t|X_t = x) = \mu(t, x)$$
$$\lim_{h \to 0} h^{-1}\mathbb{E}([X_{t+h} - X_t - h\mu(t, X)]^2|X_t = x) = \sigma^2(t, x)$$

Definition 3.7 (Path-continuous)**.** A process is **path-continuous** if $\lim_{t \to t_0} X_t = X_{t_0}$.

Definition 3.8 (Lévy process)**.** Let $\{X_t, t \geq 0$ be a stochastic process. We call X a **Lévy process**

(i) $X_0 = 0$ a.s.
(ii) It has stationary and independent increments

(iii) X is stochastically continuous, that is, for all $s, t, \epsilon > 0$, we have

$$\lim_{s \to t} \mathbb{P}(|X_s - X_t| \geq \epsilon) = 0.$$

Equivalently, $X_s \xrightarrow{p} X_t$ if $s \to t$.

Example 3.9 (Poisson process)**.** Let $(N(t), t \geq 0)$ be a stochastic process. We call $N(t)$ a **Poisson process** if the following all hold:

(i) $N(0) = 0$
(ii) N has independent increments
(iii) For all $s, t \geq 0$,

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, 2, \ldots$$

The Poisson process is stochastically continuous - that is,

$$\begin{aligned}\mathbb{P}(|N(t) - N(s)| \geq \epsilon) &= \mathbb{P}(|N(t-s) - N(0)| \geq \epsilon) \\ &= 1 - \mathbb{P}(|N(t-s)| < \alpha) \\ &= 1 - \mathbb{P}(|N(t-s)| = 0) \\ &= 1 - e^{-\lambda(t-s)} \to 0 \quad \text{as } t \to s\end{aligned}$$

The Poisson process is **not** path-continuous, that is

$$\mathbb{P}(\lim_{t \to s} |N(t) - N(s)| = 0) \neq 1$$

because

$$\mathbb{P}(\cup_{|t-s| \geq \epsilon} |N(t) - N(s)| > \delta) \geq \mathbb{P}(|N(s+1) - N(s)| \geq \delta) > 0$$

4. Lecture 4 - Thursday 10 March

Definition 4.1 (Self-similar process)**.** For any $t_1, t_2, \ldots, t_n \geq 0$, for any $c > 0$, there exists an H such that

$$(X_{ct1}, X_{ct_2}, \ldots, X_{ct_n}) \stackrel{d}{=} (c^H X_{t1}, c^H X_{t_2}, \ldots, c^H X_{t_n}).$$

We call H the **Hurst index**.

Example 4.2 (Fractional process)**.**

$$(1 - B)^d X_t = \epsilon_t, \quad \epsilon_t \text{ martingale difference}$$

$$BX_t = X_{t-1}, \quad 0 < d < 1$$

Definition 4.3 (Brownian motion)**.** Let $\{B_t, t \geq 0\}$ be a stochastic process. We call B_t a **Brownian motion** if the following hold:

(i) $B_0 = 0$ a.s.

(ii) $\{B_t\}$ has stationary, independent increments.
(iii) For any $t > 0$, $B_t \equiv N(0, t)$
(iv) The path $t \mapsto B_t$ is continuous almost surely, i.e.

$$\mathbb{P}(\lim_{t \to t_0} B_t = B_{t_0}) = 1$$

Definition 4.4 (Alternative formulations of Brownian motion)**.** A process $\{B_t, t \geq 0\}$ is a Brownian motion if and only if

- $\{B_t, t \geq 0\}$ is a Gaussian process
- $\mathbb{E}(B_t) = 0, \mathbb{E}(B_s B_t) = \min(s, t)$
- The process $\{B_t\}$ is path-continuous

Proof. ($\Rightarrow$) For all $t_1, \ldots, t_m$, we have

$$(B_{t_m} - B_{t_{m-1}}, \ldots, B_{t_2} - B_{t_1}) \equiv N(0, \Sigma)$$

as B_t has stationary, independent increments. Hence, $(B_{t_1}, B_{t_2,}, \ldots, B_{t_n})$ is normally distributed. Thus B_t is a Gaussian process. We can also show that $\mathbb{E}(B_t) = 0$ and $\mathbb{E}(B_s B_t) = \min(t, s)$.

($\Leftarrow$) TO PROVE $\square$

Corollary. *Let B_t be a Brownian motion. The so are the following:*

- $\{B_{t+t_0} - B_{t_0}, t \geq 0\}$
- $\{-B_t, t \geq 0\}$
- $\{cB_{t/c^2}, t \geq 0, c \neq 0\}$
- $\{tB_{1/t}, t \geq 0\}$

Proof. Here, we prove that $\{X_t\} = \{tB_{1/t}\}$ is a Brownian motion. Consider $\sum^n \alpha_i X_{t_i}$. Then by a telescoping argument, we know that the process is Gaussian (can be written as a sum of $X_{t_1}, X_{t_2} - X_{t_1}$, etc). We can also easily show that $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(X_t X_s) = \min(s, t)$ as required.

We now show that $\lim_{t \to 0} X_t = 0$ a.s. Fix $\epsilon \geq 0$. We must show

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{0<t<\frac{1}{m}} \{|X_t| \leq \frac{1}{n}\} \right) = 1$$

However, as $|X_t|$ has the same distribution as $|B_t|$ (as they are both Gaussian with same mean and covariance), we have that this is equivalent to

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{0<t<\frac{1}{m}} \{|B_t| \leq \frac{1}{n}\} \right)$$

which is clearly one.

$\square$

5. Lecture 5 - Tuesday 15 March

Theorem 5.1 (Properties of the Brownian motion)**.** *We have*

$$\mathbb{P}(B_t \leq x \,|\, B_{t_0} = x_0, \ldots, B_{t_n} = x_n) = \mathbb{P}(B_t \leq x | B_s = x_s) = \Phi(\frac{x - x_s}{\sqrt{t-s}})$$

Theorem 5.2. *The joint density of* $(B_{t_1}, \ldots, B_{t_n}$ *is given by*

$$g(x_1, \ldots, x_n) = \prod_{j=1}^{n} f(x_{t_j} - x_{t_{j-1}}, t_j - t_1)$$

where $f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

Theorem 5.3 (Density and distribution of Brownian bridge)**.** *Let* $t_1 < t < t_2$*. Then*

$$g(B_t | B_{t_1} = a, B_{t_2} = b) \equiv N\left(a + \frac{(b-a)(t-t_1)}{t_2 - t_1}, \frac{(t_2 - t)(t - t_1)}{t_2 - t_1}\right).$$

The density of $B_t | B_{t_1} = a, B_{t_2} = b$ *is given as*

$$\frac{g_{t_1,t,t_2}(a,b,t)}{g_{t_1,t_2}(a,b)}$$

Theorem 5.4 (Joint distribution of B_t and B_s)**.** *We have*

$$\mathbb{P}(B_s \leq x, B_t \leq y) =$$

$$\mathbb{P}(B_s \leq x, B_t - B_s \leq y - B_s)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y-z} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x_1^2}{2s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{-z_2^2}{2(t-s)}} \, dz_1 \, dz_2$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x_1^2}{2s}}$$

5.1. **Properties of paths of Brownian motion.**

Definition 5.5 (Variation)**.** Let g be a real function. Then the variation of g over an interval $[a,b]$ is defined as

$$V([a.b]) = \lim_{\Delta \to 0} \sum_{j=1}^{n} |g(t_j) - g(t_{j-1})|$$

where Δ is the size of the partition $a = t_0, \ldots, t_n = b$ of $[a,b]$.

Definition 5.6 (Quadratic variation)**.** The quadratic variation of a function g over an interval $[a,b]$ is defined by

$$[g,g] = \lim_{\Delta \to 0} \sum_{j=1}^{n} |g(t_j) - g(t_{j-1})|^2$$

Theorem 5.7 (Non-differentiability of Brownian motion)**.** *Paths of Brownian motion are continuous almost everywhere, by definition. Consider now, the differentiability of* B_t*. We claim that a*

Brownian motion is non-differentiable almost surely, that is,

$$\lim_{t \to s} \frac{|B_t - B_s|}{|t-s|} = \infty \quad a.s.$$

We claim that

(i) Brownian motion is not differentiable almost surely for any $t \geq 0$.
(ii) Brownian motion has infinite variation on any interval $[a,b]$, that is,

$$V_B([a,b]) = \lim_{\Delta \to 0} \sum_{j=1}^{n} |B(t_j) - B(t_{j-1})| = \infty \quad a.s.$$

(iii) Brownian motion has quadratic variation t on $[0,t]$, that is

$$[B,B]([0,t]) = \lim_{\Delta \to 0} \sum_{j=1}^{n} |B(t_j) - B(t_{j-1})|^2 = t \quad a.s.$$

Proof. We know that $(B_{ct_1}, \ldots, B_{ct_n}) \stackrel{d}{=} (c^H B_{t_1}, \ldots, c^H B_{t_n})$. This is true as $B_{ct} \equiv N(0,ct) = c^{1/2} N(0,t)$ as required.

Now suppose X_t is H-self-similar with stationary increments for some $0 < H < 1$ with $X_0 = 0$. Then, for any fixed t_0, we have

$$\lim_{t \to t_0} \frac{|X_t - X_{t_0}|}{t - t_0} = \infty \quad a.s.$$

Consider

$$\mathbb{P}(\limsup_{t \to t_0} \frac{|X_t - X_{t_0}|}{t - t_0} \geq M)$$

which by stationary increments, is equal to

$$\begin{aligned}
\mathbb{P}(\limsup_{t \to t_0} \frac{|X_t|}{t} \geq M) &= \lim_{n \to \infty} \mathbb{P}(\bigcup_{k \geq n} \frac{|B_{t_n} - B_{t_0}}{|t_n - t_0|} \geq M) \\
&> \lim_{n \to \infty} \mathbb{P}(\frac{|B_{t_n} - B_{t_0}}{|t_n - t_0|} \geq M) \\
&= \lim_{n \to \infty} \mathbb{P}(|N(0,1)| \geq M \cdot (t_n - t_0)^{1/2})
\end{aligned}$$

and as the RHS goes to zero, we have

$$\mathbb{P}(|N(0,1)| \geq M \cdot (t_n - t_0)^{1/2}) \to 1$$

as required. □

Now, Assume $V_B([0,t]) < \infty$ almost surely. Consider $Q_n = \sum_{j=1}^{n} |B_{t_j} - B_{t_{j-1}}|^2$. Then we have

$$Q_n \leq \max_{0 \leq j \leq n} |B_{t_j} - B_{t_{j-1}}| \cdot \sum_{j=1}^{n} |B_{t_j} - B_{t_{j-1}}|$$

Let $\Delta \to 0$. Then

$$\begin{aligned}
\lim_{\Delta\to 0} Q_n &\le \lim_{\Delta\to 0}\max_{0\le j\le n} |B_{t_j} - B_{t_{j-1}}| \cdot \sum_{j=1}^{n} |B_{t_j} - B_{t_{j-1}}| \\
&\le \lim_{\Delta\to 0}\max_{0\le j\le n} |B_{t_j} - B_{t_{j-1}}| \cdot V_B([0,t]) \\
&\le 0 \cdot V_B([0,t]) &= 0
\end{aligned}$$

because B_s is uniformly continuous on $[0,t]$. This is a contradiction to $V_B([0,t]) < \infty$.

Proof. We now show that $\mathbb{E}((Q_n - t)^2) \to 0$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
\mathbb{E}(Q_n) &= \sum_{i=1}^{n} \mathbb{E}(B_{t_j} - B_{t_{j-1}})^2 \\
&= \sum_{j=1}^{n} (t_j - t_{j-1}) = t
\end{aligned}$$

We now show L^2 convergence. We have

$$\begin{aligned}
\mathbb{E}(Q_n - t)^2 = \mathbb{E}((Q_n - \mathbb{E}(Q_n))^2) \\
&= \mathbb{E}(\sum_{j=1}^{n} Y_j \quad \text{where } Y_j = |B_{t_j} - B_{t_{j-1}}|^2 - \mathbb{E}(|B_{t_j} - B_{t_{j-1}}|^2) \\
&= \sum_{j=1}^{n} \mathbb{E}(Y_j^2)) \\
&\le \sum_{j=1}^{n} |t_j - t_{j-1}|^2 \cdot \mathbb{E}|N(0,1)|^4 \\
&\le C \cdot \Delta \cdot t \to 0
\end{aligned}$$

as $\Delta \to 0$. Thus we have convergence in L^2. □

6. Lecture 6 - Thursday 17 March

Theorem 6.1 (Martingales related to Brownian motion)**.** *Let $B_t, t \ge 0$ be a Brownian motion. Then the following are martingales with respect to $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$.*

(1) $B_t, t \ge 0$.
(2) $B_t^2 - t, t \ge 0$.
(3) For any u, $e^{uB_t - \frac{u^2 t}{2}}$

Proof. (1) is simple.

(2). We know that $\mathbb{E}(|B_t^2|)$ is finite for any t. We can also easily show $\mathbb{E}(B_t^2 - t \,|\, \mathcal{F}_s) = B_s^2 - s$ a.s. □

Theorem 6.2. *Let X_t be a martingale satisfying $X_t^2 - t$ is also a martingale. Then X_t is a Brownian motion.*

Definition 6.3 (Hitting time)**.** Let $T_\alpha = \inf_{\{t\geq 0, B_t=\alpha\}}$

(1) If $\alpha = 0$, $T_0 = 0$.

(2) If $\alpha > 0$, then

$$\mathbb{P}(T_\alpha \leq t) = 2\mathbb{P}(B_t \geq \alpha) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}}\, dx$$

We clearly have $T_\alpha > t \iff \sup_{0\leq s\leq t} < \alpha$

7. Lecture 7 - Tuesday 22 March

Theorem 7.1 (Arcsine law)**.** *Let B_t be a Brownian motion. Then*

$$\mathbb{P}(B_t = 0, \textit{ for at least once, } t \in [a, b]) = \frac{2}{\pi} arcos\sqrt{\frac{a}{b}}$$

Example 7.2. Processes derived from Brownian motion

(1) Brownian bridge

$$X_t = B_t - tB_1 \quad t \in [0, 1]$$

Consider $X \equiv F(x)$, with $X_1, \ldots X_n$ data. Our empirical distribution

$$F_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_j \leq X}$$

We can then prove that

$$\sqrt[n]{F_n(x) - F(x)} \to X_t \quad t \in (0, 1)$$

(2) Diffusion process

$$X_t = \mu t + \sigma B_t$$

This is a Gaussian process with $\mathbb{E}(X_t) = \mu t$, $\mathrm{Cov}(X_t, X_s) = \sigma^2 \min(s, t)$.

(3) Geometric Brownian motion

$$X_t = X_0 e^{\mu t + \sigma B_t}$$

This is not a Gaussian process.

(4) Higher dimensional Brownian motion

$$B_t = (B_t^1, \ldots, B_t^n)$$

where the B^i are independent Brownian motions, then

7.1. Construction of Brownian motion.

Define a stochastic process

$$\hat{B}_t^n = \frac{S_{[nt]} - \mathbb{E}(S_{[nt]})}{\sqrt{n}}$$

With

$$\tilde{B}_t^n = \begin{cases} \hat{B}_t^n & \text{if } t = \frac{i}{n} \\ 0 & \text{otherwise} \end{cases}$$

Then we can prove that

$$\tilde{B}_t^n \Rightarrow B_t \quad \text{on } [0,1]$$

Definition 7.3 (Stochastic integral)**.** We now turn to defining expressions of the form

$$\int_0^A X_t \, dY_t$$

with X_t, Y_t stochastic processes.

Definition 7.4 ($\int_0^t f(B_s)\, ds$)**.** We have

$$\int_0^t f(t)\, dt$$

exists if f is bonded and continuous, except on a set of Lebesgue measure zero. Thus, we can set

$$\int_0^t f(t)\, dt = \lim_{\Delta \to 0} \sum_{j=1}^n f(B_{y_j})(t_j - t_{j-1})$$

if $f(x)$ is bounded.

We now seek to find $\int_0^1 B_s \, ds$. Consider $Q_n = \sum_{j=1}^n B_{y_j}(t_j - t_{j-1})$. As the sum of normal variables, we know that $Q_n \equiv N(\mu_n, \sigma_n^2)$.

Since $\int_0^1 B_s \, ds$ is normally distributed with mean 0, we have

$$\begin{aligned} \mathbb{E}(X^2) &= \mathbb{E} \int_0^1 B_s \, dx \int_0^1 B_t \, dt \\ &= \int_0^1 \mathbb{E}(B_s B_t)\, ds \, dt \\ &= \int_0^1 \min(s,t)\, ds\, dt \\ &= 1/3 \end{aligned}$$

8. Lecture 8 - Thursday 24 March

$$\int_0^1 f(B_s)\, ds = \lim_{\Delta \to 0} \sum_{j=1}^n f(B_{y_j})(t_{j+1} - t_j)$$

Recall

$$\int_0^1 B_s \, ds \sim N(0, \frac{1}{3}) = \lim_{\Delta \to 0} \sum_{i=1}^n B_{y_j}(t_{j+1} - t_j)$$

Consider $I = \int X_s \, dY_s$. We have the following:

Theorem 8.1. *I exists if*

(1) The functions f, g are not discontinuous at the same point x.
(2) f is continuous and g has bounded variation or,
(2)' f has finite p-variation and g has finite q-variation, where

$$1/p + 1/q = 1$$

For any p, we define p-variation by

$$\lim_{\Delta \to 0} \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|^p$$

Theorem 8.2. *If $J = \int_0^1 f(t)\, dg(t)$ exists for any continuous f, then g must have finite variation.*

Theorem 8.3. *B_s has bounded p variation for any $p \geq 2$ and unbounded q-variation for any $q < 2$.*

Proof. We can write (for $p \geq 2$), $p = 2 + (p - 2)$. Then we have

$$\Delta_t = \lim_{\Delta \to 0} \max |B_{t_j} - B_{t_{j-1}}|^{p-2} \sum_{j=1}^{n} |B_{t_j} - B_{t_{j-1}}|^2$$

and hence Δ_t exists. □

Corollary. *Thus $\int_0^1 dB_t$ is well defined if f has finite variation (as setting $q = 1$, $p \geq 2$ gives $1/p + 1/q > 1$).*

Consider $\int_0^1 B_s\, dB_s$ - is this an R-S integral? Consider

$$\Delta_{1n} = \sum_{j=1}^{\infty} B_{t_j}(B_{t_{j+1}} - B_{t_j}), \Delta_{2n} = \sum_{j=1}^{\infty} B_{t_{j+1}}(B_{t_{j+1}} - B_{t_j})$$

We have that

$$\Delta_{2n} - \Delta_{1n} = \sum_{j=1}^{n} (B_{t_{j+1}} - B_{t_j})^2 \to 1$$

and

$$\Delta_{1n} + \Delta_{2n} = \sum_{j=1}^{n} B^2_{t_{j+1}} - B^2_{t_j}) = B_1^2.$$

Thus we know

$$\Delta_{2n} \to \frac{1}{2}(B_1^2 + 1)$$
$$\Delta_{1n} \to \frac{1}{2}(B_1^2 - 1)$$

Definition 8.4 (Itô integral)**.** The Itô integral is defined by evaluating $f(B_{t_j})$, the left-hand end-point at each partition interval $[t_j, t_{j+1})$

9. Lecture 9 - Tuesday 29 March

Definition 9.1 (Itô integral)**.** Consider $\int_0^1 f(s)\,dB_s$. Where $f(s)$ is a real function, B_s a Brownian motion. We define the integral in two steps.

(1) If $f(s)$ is a step function, define

$$\begin{aligned} I(f) = \int_0^1 f(s)\,dB_s &= \sum_{j=1}^{m} \int_{t_j}^{t_{j+1}} f(s)\,dB_s \\ &= \sum_{j=1}^{m} c_j(B_{t_{j+1}} - B_{t_j}) \end{aligned}$$

(2) If $f(s) \in L^2([0,1])$, then let f_n be a sequence in $L^2([0,1])$ such that $f_n \to f$ in $L^2([0,1])$.
Then define $I(f)$ to be the limitation in such situations such that

$$\mathbb{E}(I(f_n) - I(f))^2 \to 0$$

in $L^2([0,1])$ or Let $I(f) = \lim_{n\to 0} I(f_n)$ in probability.

Remark. If $f(x), g(x)$ are given step functions then $\alpha I(f) + \beta I(g) = I(\alpha f + \beta g)$.

Remark. If $f(x)$ is a step function, then $I(f) \sim N(0, \int_0^1 f^2(s)\,ds)$

Proof.

$$I(f) = \sum_{j=1}^{m} c_j(B_{t_{j+1}} - B_{t_j}) \sim N(0, \sigma^2)$$

where $\sigma^2 = \sum_{j=1}^{m} c_j^2(t_{j+1} - t_j)$. □

Theorem 9.2. *$I(f)$ is well defined (independent of the choice of f_n.)*

Proof. Let $f_n, g_n \to f$ in $L^2([0,1])$. We then need to only compute

$$\Delta_{n,m} = \mathbb{E}(I(f_n) - I(g_m))^2 \to 0$$

as $m, n \to \infty$.

In fact,

$$\begin{aligned} \Delta_{n,m} = \mathbb{E}(I(f_n - g_m))^2 = \int_0^1 (f_n - g_m)^2\,dx \\ &\le 2\int_0^1 (f_n - f)^2\,dx + 2\int_0^1 (g_m - f)^2\,dx \\ &\to 0 \end{aligned}$$

as $n, m \to \infty$. □

Remark. If f is continuous bounded variation, then

$$\int_0^1 f(s)\,dB_s = (R.S.)\int)0^1 f(s)\,dB_s$$
$$= \lim_{\delta\to 0}\sum_{j=1}^{n} f(t_j)B(t_{j+1} - B_{t_j})$$

Proof. Case 1).

$$\alpha I(f) + \beta I(g) = \lim_{n\to\infty}[\alpha I(f_n) + \beta I(g_n) = \lim_{n\to\infty}[I(\alpha f_n + \beta g_n)] = I(\alpha f + \beta g)$$

and thus $\alpha f_n + \beta g_n \to \alpha f + \beta g \in L^2([0,1])$. □

Case 2). If $I(f) = \lim_{\delta\to 0} I(f_n)$ in probability. Then

$$I(f_n) \sim N(0, \sigma_n^2)$$
$$\sim N(0, \int_0^1 f_n^2(s)\,ds)$$
$$\to N(0, \int_0^1 f^2(s)\,ds)$$

if $\sigma_n^2 \to \int_0^1 f^2(x)\,dx$. In fact, as

$$\int_0^1 f_n^2\,dx = \int_0^1 (f_n - f + f)^2\,dx$$
$$= \int_0^1 f^2\,dx + \int_0^1 (f_n - f)^2\,dx + \int_0^1 (f_n - f)f\,dx$$
$$\to \int_0^1 f^2\,dx$$

as other terms tend to zero by L^2 convergence and Hölder's inequality.

Remark. $(R.S.)\int_0^1 f(s)\,dB_s$ exists if f is of bounded variation.

Remark. If f is continuous then $\int_0^1 f^2\,dx < \infty$ and

$$f_n(t) = \sum_{j=1}^{n} f(t_j) I_{[t_j, t_{j+1})} \to f(t) \text{ in } L^2([0,1])$$

Thus,

$$I(f) = \lim_{\delta\to 0} I(f_n)$$
$$= \lim_{\delta\to 0}\sum_{j=1}^{m} f(t_j)(B_{t_{j+1}} - B_{t_j})$$

We have to prove

$$\int_0^1 (f_n - f)^2 \, dx \to 0$$

if f is continuous.

10. Lecture 10 - Thursday 31 March

We have

$$(\text{Itô }) \int_0^1 f(s) \, dB_s \sim N(0, \int_0^1 f(t)^2 \, dt$$

if f is continuous, and of finite variation. In this case, we can write

$$(\text{R.S.}) \int_0^1 f(s) \, dB_s = (\text{Itô }) \int_0^1 f(s) \, dB_s = \lim_{\Delta \to 0} \sum_{j=1}^n f(t_j)(B_{t_{j+1}} - B_{t_j})$$

Example 10.1. We have

$$\int_0^1 (1-t) \, dB_t = (\text{Itô }) \int_0^1 (1-t) \, dB_t \sim N(0, \int_0^1 (1-t)^2 \, dt) = N(0, \frac{1}{3})$$

and by integrating by parts, we have

$$\int_0^1 (1-t) \, dB_t = (1-t)B_t|_0^1 - \int_0^1 B_t \, d(1-t) = 0 + \int_0^1 B_t \, dt \sim N(0, \frac{1}{3})$$

Now consider

$$(\text{Itô }) \int_0^1 X_s \, dB_s$$

where we now allow X_s to be a stochastic process.

Write $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$. Let π be the collection of X_s such that

(1) X_s is adapted to $\mathcal{F}_s$, that is, for any s, X_s is $\mathcal{F}_s$ measurable.
(2) $\int_0^1 X_s^2 \, ds < \infty$ almost surely (R.S)

Let $\pi' = \{X_s \mid X_s \in \pi, \int_0^1 \mathbb{E}(X_s^2) < \infty\}$. Then $\pi' \subset \pi$. Let $X_s = e^{B_s^2}$. Then

$$\mathbb{E}(X_s^2) = \mathbb{E}(e^{B_s^2}) = \begin{cases} \frac{1}{\sqrt{1-4s}} & 0 \leq s < \frac{1}{4} \\ \infty & s \geq \frac{1}{4} \end{cases}$$

Definition 10.2 (Itô integral for stochastic integrands)**.** We proceed in two steps.

(1) Let $X_s = \sum_{j=1}^n \zeta_i \mathbf{1}_{[t_j, t_{j+1})}$ where ζ_j is $\mathcal{F}_{t_j}$ measurable. Then

$$I(X) = \sum_{j=1}^n \zeta_j (B_{t_{j+1}} - B_{t_j}).$$

(2) If $X \in \pi$, there exists a sequence $X^n \in \pi'$ such that X^n are step process with

$$\int_{0^1} |X_s^n - X_s|^2 \, ds \to 0$$

as $n \to \infty$ in probability or in $L^2([0,1])$ if $X_s \in \pi'$.

Proof. We show only for X_s continuous, the general case can be found in Hu-Hsing Kuo (p.65). As X_s is continuous, then it is in π. Then choose

$$X_s^n = X_0 + \sum j = 1^n X_{t_j} \mathbf{1}_{[t_j, t_{j+1})}$$

Then $X_s \in pi'$ and

$$\mathbb{E}(|X_s^n - X_s|^2) \to 0$$

for any $s \in (0,1)$.

We can also show that

$$\mathbb{E}(|X_s^n - X_s|^2) < \infty$$

and so by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^1 \mathbb{E}(|X_s^n - X_s|^2)\, ds = 0.$$

□

Finally if $X_s \in \pi$, the Itô integral $\int_0^1 X_s \, dB_s$ is defined as

$$I(X) = \lim_{\delta \to 0} I(X^n)$$

in probability or in L^2 if $X \in \pi'$.

11. Lecture 11 - Tuesday 5 March

Let $I(X) = \int_0^1 X_s \, dB_s$ in the Itô sense. We then require

(1) X_s is $\mathcal{F}_s = \sigma\{B_t, 0 \le t \le s\}$-measurable
(2) $\int_0^1 X_s^2 \, ds < \infty$ almost surely.

Then $I(X) = \lim_{n\to\infty} I(X_n)$ where X_n is a sequence of step functions converging to X in L^2, that is,

$$\int_0^1 (X_s^n - X_s)^n \, ds \to 0$$

We then show that this definition is independent of the sequence of step functions. For any Y_s step process, we have

$$\int_0^1 (Y_s^n - Y_s^m)^2 \, dx \le 2 \int_0^1 (Y_s^m - X_s)^2 \, ds + 2 \int_0^1 (X_s^n - X_s)^2 \, dx \to 0$$

Theorem 11.1 (Properties of the Itô integral)**.**

(1) For any $\alpha, \beta \in \mathbb{R}$, $X, Y \in \pi$,

$$I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y)$$

(2) For any $X_s \in \pi'$, we have

$$\mathbb{E}(I(X)) = 0, \quad \mathbb{E}(I^2(X)) = \int_0^1 E(X_s^2)\, ds$$

If $X' \in \pi'$, $Y_s \in \pi'$, then

$$\mathbb{E}(I(X)I(Y)) = \int_0^1 \mathbb{E}(X_s Y_s)\, ds$$

(3) If X_s continuous then

$$I(X) = \lim_{\delta \to 0} \sum_{j=1}^{n} X_{t_j}(B_{t_{j+1}} - B_{t_j})$$

in probability.

(4) If X is continuous and of finite variation and $\int_0^1 B_s\, dX_s < \infty$ then

$$(R.S.) \int_0^1 = (It\hat{o}\ ) \int_0^1 X_s\, dB_s = X_1 B_1 - X_0 B_0 - \int_0^1 B_s\, dX_s$$

Proposition 11.2. *We now show why we require*

(1) X_s is $\mathcal{F}_s = \sigma\{B_t, 0 \leq t \leq s\}$-measurable
(2) $\int_0^1 X_s^2\, ds < \infty$ almost surely.

Proof. Motivation for (1)

$$\begin{aligned}
X_s &= \sum_{j=1}^{n} \zeta_j \mathbf{1}_{(t_j, t_{j+1})} \\
I(X) &= \sum_{j=1} \zeta_j (B_{t_{j+1}} - B_{t_j}) \\
\mathbb{E}(I(X)) &= \sum_{j=1}^{n} \mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j})) \\
&= \sum_{j=1}^{n} \mathbb{E}(\mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \,|\, \mathcal{F}_{t_j})) \\
&= \sum_{j=1}^{n} \mathbb{E}(\zeta_j \mathbb{E}(B_{t_{j+1} - B_{t_j}} \,|\, \mathcal{F}_{t_j}))
\end{aligned}$$

Motivation for (2)

$$\begin{aligned}
\mathbb{E}(I^2(X)) &= \mathbb{E}(\sum_{j=1}^{n} \zeta_j (B_{t_{j+1}} - B_{t_j}))^2 \\
&= 2\sum_{i<j} \mathbb{E}(\mathbb{E}(\zeta_i \zeta_j (B_{t_{j+1}} - B_{t_j})(B_{t_{i+1}} - B_{t_i})) \,|\, \mathcal{F}_{t_i}) + \sum_{j=1}^{n} \mathbb{E}(\zeta_j^2 (B_{t_{j+1}} - B_{t_j})^2) \\
&= \sum_{j=1}^{n} \mathbb{E}(\zeta_j^2)(t_{j+1} - t_j) \\
&= \int_0^1 \mathbb{E}(X_t^2)\, dt
\end{aligned}$$

We now show that there $X_s^n \in \pi'$. such that $\int_0^1 \mathbb{E}(X_s^n - X_s)^2\, ds \to 0$. We have

$$\begin{aligned}
\mathbb{E}(I^2(X)) &= \mathbb{E}(I(X)) - I(X^n) + \mathbb{E}(I(X_s^n))^2 \\
&= \mathbb{E}(I^2(X^n)) + 2\mathbb{E}(I(X^n)(I(X) - I(X^n))) + \mathbb{E}(I(X) - I(X^n))^2
\end{aligned}$$

By Cauchy-Swartz, the middle term tens do zero, and by definition, the third term tends to zero. Thus we have

$$\lim_{n\to\infty} \mathbb{E}(I^2(X_n)) = \mathbb{E}(I^2(X))$$

Let X_s^n be step processes. We now show

$$\mathbb{E}(X_s^n - X_s)^2 \to 0 \Rightarrow \int_0^1 \mathbb{E}(X_s^n - X_s)^2\, ds \to 0$$

By definition, we have

$$I(X) = \lim_{n\to\infty} I(X^n) = \lim_{\delta\to 0} \sum_{j=1}^{n} X_{t_j}(B_{t_{j+1}} - B_{t_j}).$$

and we proved the required result in lectures.

We now show that the $(R.S.)$ integral exists if X is continuous and of finite variation, and $\int_0^1 B_s\, dX_s < \infty$, the our integration by parts formula holds.

We have

$$\begin{aligned}
(R.S.) \int_0^1 X_s\, dB_s &= \lim_{\delta\to 0} \sum_{j=1}^{n} X_{y_j}(B_{t_{j+1}} - B_{t_j}) \\
&= \lim_{\delta\to 0} \sum_{j=1}^{n} X_{t_j}(B_{t_{j+1}} - B_{t_j})
\end{aligned}$$

which is our Itô integral by definition.

□

Definition 11.3 (Itô process)**.** Suppose $Y_t = \int_0^t X_s\, dB_s$, $t \geq 0$ is well defined, for $X_s \in \pi$. Then Y_t is an *Itô processes*. To show a process Y_t is an Itô process, we need to show that $\int_0^t |X_s|\, ds)M\infty$ a.s. and $\int_0^t |X_s|^2\, ds < \infty$ a.s.

Theorem 11.4. *We have that Y_t is continuous (except on a null set), is of infinite variation, as*

$$\begin{aligned}\sum_{j=1}^{n} |Y_{t_{j+1}} - Y_{t_j}| &= \sum_{j=1}^{n} |\int_{t_j}^{t_{j+1}} X_s\, dB_s| \\ &\geq \sum_{j=1}^{n} \min_s |X_s||B_{t_{j+1}-B_{t_j}} \\ &\geq C \sum_{j=1^n} |B_{t_{j+1}} - B_{t_j}|\end{aligned}$$

12. Lecture 12 - Thursday 7 March

From before, consider the Itô process $Y_t = \int_0^t X_s\, dB_s$.

Lemma 12.1. $\mathbb{E}(\int_s^t X_u dB_u \,|\, \mathcal{F}_s) = 0$.

Proof. Let $X_u = \sum_{j=1}^n \zeta_j \mathbf{1}_{[t_j,t_{j+1})}$. Then

$$\begin{aligned}\mathbb{E}(\int_s^t X_u\, dB_u \,|\, \mathcal{F}_s) &= \sum_{j=1}^{n} \mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \,|\, \mathcal{F}_s) \\ &= \sum_{j=1}^{n} \mathbb{E}(\mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \,|\, \mathcal{F}_s) \,|\, \mathcal{F}_{t_j}) \\ &= \sum_{j=1}^{n} \mathbb{E}(\mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \,|\, \mathcal{F}_{t_j}) \,|\, \mathcal{F}_s) \\ &= 0\end{aligned}$$

where the third equality follows from the fact that $(\mathcal{F}_t)$ is an increasing sequence of σ-fields and the final follows from the fact that $B_{t_{j+1}} - B_{t_j}$ is independent of $\mathcal{F}_{t_j}$. □

Definition 12.2 (Local martingale)**.** A process Y_t is a local martingale if there exists a sequence of stopping times $\tau_n, n \geq 1$ such that $Y_{\min(t,\tau),\mathcal{F}_t}$ is a martingale.

Proposition 12.3. *We have the following.*

(1) If $X_s \in \pi'$, that is, $\int_0^t \mathbb{E}(X_s^2)\, ds < \infty$, then $(Y_t, \mathcal{F}_t)$ is a martingale.
(2) If $X_s \in \pi$, that is $\int_0^t X_s^2\, ds < \infty$ a.s., then $(Y_t, \mathcal{F}_t)$ is a local martingale.
(3) For $f(x)$ satisfying $\int_0^t f^2(z)\, dz < \infty$., we have

$$Y_t = \int_0^t f(s)\, dB_s$$

is a Gaussian process.

Proof. We have Y_t is trivially $\mathcal{F}_t$-measurable.

We have $\mathbb{E}(|Y_t|) < \infty$ a.s. as $\mathbb{E}(Y_t^2) = \int_0^t \mathbb{E}(X_s^2)\, ds < \infty$ by assumption.

We have

$$\begin{aligned}
\mathbb{E}(Y_t \mid \mathcal{F}_s) &= \mathbb{E}(\int_0^s X_u\, dB_u + \int_s^t X_u\, dB_u \mid \mathcal{F}_s) \\
&= \mathbb{E}(Y_s \mid \mathcal{F}_s) + \mathbb{E}(\int_s^t X_u\, dB_u \mid \mathcal{F}_s) \\
&= Y_s
\end{aligned}$$

from the previous lemma.

Now assuming $X_s \in \pi'$, the exists an X_s^n a step process such that

$$\int_0^t \mathbb{E}(X_u^n - X_u)^2\, du \to 0$$

as $n \to \infty$.

Set $Y_t^n = \int_0^t X_u^n\, dB_u$. Then

$$\begin{aligned}
\int_s^t X_u\, dB_u &= Y_t - Y_s \\
&= Y_t - Y_t^n + Y_t^n - Y_s^n + Y_s^N - Y_s
\end{aligned}$$

Then for each n, we have

$$Z = \mathbb{E}(\int_s^t X_u\, dB_u) = \mathbb{E}(Y_t - Y_t^n \mid \mathcal{F}_s) + \mathbb{E}(Y_s^n - Y_s \mid \mathcal{F}_s)$$

We only need to prove $\mathbb{E}(|Z|^2) \to 0$ which implies $\mathbb{E}(Z^2) = 0$ and thus $Z = 0$ almost surely. We have

$$\mathbb{E}(Z^2) \leq 2\mathbb{E}(Y_t - Y_t^n)^2 + 2\mathbb{E}(Y_s - Y_s^n)^2 \to 0$$

by definition of Y_t^n.

□

13. Lecture 13 - Tuesday 12 March

Theorem 13.1. *Let f be continuous. Then*

$$\lim_{\delta \to 0} \sum_{j=1}^n f(\theta_j)(B_{t_{j+1}} - B_{t_j})^2 = \int_0^t f(B_s)\, ds$$

Proof. Let

$$Q_n = \sum_{j=1}^{n} |f(\theta_j) - f(B_{t_j})| B_{t_{j+1}} - B_{t_j}|^2$$

Note that $\sum_{j=1}^{n} f(B_{t_j})(B_{t_{j+1}} - B_{t_j})^2 \to \int_0^t f(B_s)\, ds$ in probability.

It only needs to show that $Q_n \to 0$ in probability. We have

$$Q_n \leq \max_{1\leq j\leq n} |f(\theta_j) - f(B_{t_j}) \sum_{j=1}^{n} |B_{t_{j+1}} - B_{t_j}|^2 \to 0 \cdot t = 0$$

in probability from the quadratic variance of the brownian motion □

Theorem 13.2. *Let f be bounded on $[0,1]$. Then*

$$\lim_{\delta\to 0} \sum_{j=1}^{n} f(X_{t_j})(B_{t_{j+1}} - B_{t_j})^2 = \int_0^t f(X_s\, ds)$$

Proof. We only prove for cases where $\int_0^t \mathbb{E}(X_s^2)\, ds$ is finite. In this case, there exists X_s^n step process such that

$$\int_0^t \mathbb{E}(X_s^n - X_s)^2\, ds \to 0$$

as $n \to \infty$.

In fact, we can let $X_s^n = \sum_{j=1}^{n} X_{t_j} \mathbf{1}_{[t_j, t_{j+1}]}$. Let

$$Y_t^n = \int_0^t X_s^n\, dB_s$$

Then $Y_{t_{j+1}}^n - Y_{t_j}^n = \int_{t_j}^{t_{j+1}} X_s^n\, dB_s$.

Therefore,

$$\sum_{j=1}^{n} (Y_{t_{j+1}} - Y_{t_j})^2 = \sum_{j=1}^{n} (Y_{t_{j+1}} - Y_{t_j}^n + Z_{1j} + Z_{2j})$$

where $Z_{1j} = Y_{t_{j+1}} - Y_{t_{j+1}}^n$ and $Z_{2j} = Y_{t_j} = Y_{t_j}^n$. Continuing, we have

$$\begin{aligned}
&= \sum_{j=1}^{n} (Y_{t_{j+1}} - Y_{t_j}^n)^2 + \text{error} \\
&= \sum_{j=1}^{n} X_{t_j}^2 (B_{t_{j+1}} - B_{t_j})^2 + \text{error} \\
&\to \int_0^t X_s^2\, ds
\end{aligned}$$

if the error term goes to zero. We have

$$\begin{aligned}
R_n &\leq 2\sum Z_{1j}^2 + 2\sum Z_{2j}^2 + 2\sum |Z_{1j}| \cdot |Y_{t_{j+1}}^n - Y_{t_j}^n| + 2\sum |Z_{2j}| \cdot |Y_{t_{j+1}}^n - Y_{t_j}^n| + 2\sum |Z_{1j}| \cdot |Z_{2j}| \\
&\leq \cdots + 2(\sum Z_{1j}^2)^{1/2} A_n^{1/2} + 2(\sum Z_{2j}^2)^{1/2} A_n^{1/2} + 2(\sum Z_{1j}^2)^{1/2} \cdot (\sum Z_{2j}^2)^{1/2}
\end{aligned}$$

and as $\sum {Z_{ij}}^2 \to 0$ in probability, we have our result. □

Theorem 13.3 (Itô's first formula)**.** *If $f(x)$ is a twice-differentiable function then for any t,*

$$f(B_t) - f(B_s) = \int_s^t f'(B_u)\,dB_u + \frac{1}{2}\int_s^t f''(B_u)\,du$$

Proof. Let $s = t_1, \ldots, t_n = t$. We have

$$f(B_t) - f(B_s) = \sum_{j=1}^{n} \left[f(B_{t_{j+1}}) - f(B_{t_j})\right]$$

Applying Taylor's expansion to $f(B_{t_{j+1}}) - f(B_{t_j})$ we get

$$f(B_{t_{j+1}}) - f(B_{t_j} = f'(B_{t_j})(B_{t_{j+1}} - B_{t_j}) + \frac{1}{2}f''(\theta_j)(B_{t_{j+1}} - B_{t_j})^2$$

and so

$$\begin{aligned} f(B_t) - f(B_s) &= \lim_{\delta\to 0}\sum_{j=1}^{n} f'(B_{t_j})(B_{t_{j+1}} - B_{t_j}) + \lim_{\delta\to 0}\sum_{j=1}\frac{1}{2}f''(\theta_j)(B_{t_{j+1}} - B_{t_j})^2 \\ &= \int_s^t f'(B_u)\,dB_u + \frac{1}{2}\int_s^t f''(B_u)\,du \end{aligned}$$

□

Example 13.4.

14. Lecture 14 - Thursday 14 March

Definition 14.1 (Covariation)**.** The covariation of two stochastic processes X_t, Y_t is defined as

$$[X,Y](t) = \frac{1}{4}([X+Y, X+Y](t) - [X-Y, X-Y](t))$$

where $[\cdot,\cdot]$ is the quadratic variation previously defined.

Definition 14.2 (Stochastic differential equation)**.** Let X_t be an Itô process. Then

$$X_t = X_a + \int_a^t \mu(s)\,ds + \int_a^b \sigma(s)\,dB_s$$

We write

$$dX_t = \mu(t)\,dt + \sigma(t)\,dB_s$$

By convention, we write $dX_t \cdot dY_t = d[X,Y](t)$. In particular, $(dY_t)^2 = d[Y,Y](t)$.

Theorem 14.3. *Let Y_t be path continuous, and let X_t have finite variation. Then*

$$[X,Y](t) = 0.$$

Proof. We have

$$\begin{aligned}[X,Y](t) &= \frac{1}{4}([X+Y,X+Y]-[X-Y,X-Y]) \\ &= \lim_{\delta\to 0}\sum_{j=1}^{n}(X_{t_{j+1}}-X_{t_j})(Y_{t_{j+1}}-Y_{t_j}) \\ &\leq \lim_{\delta\to 0}\max|Y_{t_{j+1}}-Y_{t_j}|\sum_{j=1}^{n}|X_{t_{j+1}}-X_{t_j}| \\ &\to 0\end{aligned}$$

as Y_t is path continuous and X_t has finite variation. □

Corollary. *From this theorem, we then have*

$$dB_t\cdot dt = 0,\quad (dt)^2 = 0,\quad (dB_t)^2 = dt$$

Corollary. *For an Itô process X_t given above, we then have*

$$d[X,X](t) = dX_t\cdot dX_t = (\mu(t)dt+\sigma(t)\,dB_t)^2 = \sigma^2(t)\,dt$$

Corollary. *If $f(x)$ has a twice continuous derivative, then*

$$[f(B_t),B_t](t) = \int_0^t f'(B_s)\,ds$$

Proof. From Itô's formula, we have

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2}f''(B_t)\,dt$$

Then from above, we have

$$d[f(B_t),B_t](t) = df(B_t)\cdot dB_t = f'(B_t)dt$$

□

Theorem 14.4 (Itô's lemma)**.** *By Taylor's theorem, we have*

$$\begin{aligned}df(t,X_t) &= \frac{\partial f(t,X_t)}{\partial t}dt + \left.\frac{\partial f(t,x)}{\partial x}\right|_{x=X_t}\cdot dX_t + \frac{1}{2}\left.\frac{\partial^2 f(t,x)}{\partial x^2}\right|_{x=X_t}\cdot (dX_t)^2 \\ &\quad + \frac{1}{2}\left.\frac{\partial^2 f(t,x)}{\partial x\partial t}\right|_{x=X_t}\cdot dX_t\cdot dt \\ &= \frac{\partial f(t,X_t)}{\partial t}dt + \left.\frac{\partial f(t,x)}{\partial x}\right|_{x=X_t}\cdot(\mu(t)\,dt+\sigma(t)\,dB_t) + \frac{1}{2}\left.\frac{\partial^2 f(t,x)}{\partial x^2}\right|_{x=X_t}\cdot\sigma^2\,dt \\ &= \left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\mu(t)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(t)\right)dt + \frac{\partial f}{\partial x}\sigma(t)\,dB_t\end{aligned}$$

Example 14.5. Let $f(B_t) = e^{B_t}$. Then

$$df(B_t) = e^{B_t}\, dB_t + \frac{1}{2} e^{B_t}\, dt$$

Theorem 14.6. *Assume* $\int_0^T f^2(t)\, dt < \infty$*. Let* $X_t = \int_0^t f(s)\, dB_s - \frac{1}{2}\int_0^t f^2(s)\, ds$*. Then*

$$Y_t = e^{X_t}$$

is a martingale with respect to $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$*.*

Proof. We have

$$\begin{aligned} dY_t &= e^{X_t}\, dX_t + \frac{1}{2} e^{X_t} (dX_t)^2 \\ &= e^{X_t}[f(t)\, dB_t - \frac{1}{2} f^2(s)\, dt] + \frac{1}{2} e^{X_t} f^2\, dt \\ &= e^{X_t} f(t)\, dB_t \end{aligned}$$

and thus $Y_t = \int_0^t e^{X_s} f(s)\, dB_s$ which is a martingale from previous work. □

15. Lecture 15 - Tuesday 19 March

Theorem 15.1 (Multivariate Itô's formula)**.** *Let* $B_j(t)$ *be a sequence of independent Brownian motions. Consider the Itô processes* X_t^i*, with*

$$dX_t^i = b_i(t)\, dt + \sum_{k=1}^m \sigma_{ki}(t)\, dB_k(t)$$

Suppose $f(t, x_1, \ldots, x_n)$ *is a continuous function of its components and has continuous partial derivatives* $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial^2 x_i \partial x_j}$*. Then*

$$df(t, X_t^1, \ldots, X_t^m) = \frac{\partial f}{\partial t}\, dt + \sum_{i=1}^m \frac{\partial f}{\partial x_i}\, dX_t^i + \frac{1}{2} \sum_{i,j=1} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dX_t^i\, dX_t^j$$

We have

$$\begin{aligned} dX_t^i\, dX_t^j &= \sum_{k,s=1}^m \sigma_{ki}\sigma_{sj}\, dB_k(t)\, dB_s(t) \\ &= \sum_{k=1}^n \sigma_{ki}\sigma_{kj}\, dt + \end{aligned}$$

as $d[B_i, B_j](t) = 0$ *when* B_i, B_j *are independent*

Example 15.2. Let

$$\begin{aligned} dX_t &= \mu_1(t)dt + \sigma_1(t)\, dB_1(t) \\ dY_t &= \mu_2(t)dt + \sigma_2(t)\, dB_2(t) \end{aligned}$$

with B_1, B_2 independent. Then

$$d(X_t \cdot Y_t) = Y_t\, dX_t + X_t\, dY_t + d[X,Y](t)$$

and by independence, $d[X,Y](t) = dX_t \cdot dY_t = 0$.

Example 15.3. Let

$$dX_t = \mu_1(t)dt + \sigma_1(t)\, dB_1(t)$$
$$dY_t = \mu_2(t)dt + \sigma_2(t)\, dB_1(t).$$

Then

$$d(X_t \cdot Y_t) = Y_t\, dX_t + X_t\, dY_t + \sigma_1(t)\sigma_2(t)\, dt.$$

In particular, if $\mu_1 = \mu_2 = 0$, then

$$X_t\, Y_t = \int_0^t [\sigma_1(s)Y_s + \sigma_2(s)X_s]\, dB_s + \int_0^t \sigma_1(s)\sigma_2(s)\, dt$$

Thus, $Z_t = X_t Y_t - \int_0^t \sigma_1(s)\sigma_2(s)\, dt$ is a martingale.

Theorem 15.4 (Tanaka's formula)**.** *We have*

$$|B_t - a| = |a| + \int_0^t sgn(B_s - a)\, dB_s + \mathcal{L}(t,a)$$

where

$$\mathcal{L}(t,a) = \lim_{\epsilon > 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{|B_s - a| \leq \epsilon}\, ds$$

Proof. Let

$$f_\epsilon(x) = \begin{cases} |x-a| - \frac{\epsilon}{2} & |x-a| > \epsilon \\ \frac{1}{2\epsilon}(x-a)^2 & |x-a| \leq \epsilon \end{cases}$$

Then we have

$$f'_\epsilon(x) = \begin{cases} 1 & x > a + \epsilon \\ \frac{1}{\epsilon}(x-a) & |x-a| \leq \epsilon \\ -1 & x < a - \epsilon \end{cases}$$

and

$$f''_\epsilon(x) = \begin{cases} 0 & |x-a| > \epsilon \\ \frac{1}{\epsilon} & |x-a| \leq \epsilon \end{cases}$$

Then by Itô's formula, we have

$$f_\epsilon(B_t) = f_\epsilon(0) + \int_0^t f'_\epsilon(B-s)\, dB_s + \frac{1}{2}\int_0^t f''_\epsilon(B_s)\, ds$$

Obviously

$$\lim_{\epsilon \to 0} f_\epsilon(0) = |a|$$

and

$$\lim_{\epsilon \to 0} \frac{1}{2} \int_0^t f_\epsilon''(B_s)\, ds = \frac{1}{\epsilon} \int_0^t \mathbf{1}_{|B_s - a| \le \epsilon}\, ds = \mathcal{L}(t,a)$$

Note that

$$\int_0^t |f_\epsilon'(B_s) - \operatorname{sgn}(B_s - a)|^2 \, ds = \int_{|B_s - a \le \epsilon} \left| \frac{1}{\epsilon}(B_s - a) - \operatorname{sgn}(B_s - a) \right|^2 \, ds \to 0 a.s.$$

□

Theorem 15.5.

$$\mathcal{L}(t,a) = \int_0^t \delta_a(B_s)\, ds$$

Theorem 15.6. *If f is integrable on $\mathbb{R}$, then*

$$\int_{-\infty}^{\infty} \mathcal{L}(t,s) f(s)\, ds = \int_0^t f(B_s) \, ds$$

16. Lecture 16 - Thursday 21 March

Definition 16.1 (Linear cointegration)**.** Consider two non stationary time series X_t and Y_t. If there exist coefficients α and β such that

$$\alpha X_t + \beta Y_t = u_t$$

with u_t stationary, then we say that X_t and Y_t are **cointegrated**.

Definition 16.2 (Nonlinear cointegration)**.** If $Y_t - f(X_t) = u_t$ is stationary, with $f(\cdot)$ a nonlinear function.

17. Lecture 17 - Tuesday 3 May

17.1. **Stochastic integrals for martingales.** We now seek to define stochastic integrals with respect to processes other than Brownian motion.

Example 17.1. Let $X_t = \int_0^t Y_s \, dB_s$, and thus $dX_s = Y_s \, dB_s$. Then

$$Z_t = \int_0^t Y_s' \, dX_s = \int_0^t Y_s' \cdot Y_s \, dB_s$$

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Definition 17.2 (Martingale)**.** A martingale with respect

(1) M_t adapted to $\mathcal{F}_t$.
(2) $\mathbb{E}(|M_t|) < \infty$.

(3) $\mathbb{E}(M_t \,|\, \mathcal{F}_s) = M_s$ a.s.

A process is a submartingale if (3) is replace with $\mathbb{E}(M_t \,|\, \mathcal{F}_s) \geq M_s$. A process is a supermartingale if (3) is replaced with $\mathbb{E}(M_t \,|\, \mathcal{F}_s) \leq M_s$.

Example 17.3. Let N_t be a poisson process with intensity λ. Then

(1) N_t is a submartingale with respect to the natural filtration.
(2) $N_t - \lambda t$ is a martingale with respect to the natural filtration.

Theorem 17.4. *If $\int_0^t \mathbb{E}(H^2(s))\, ds < \infty$ and $H(s)$ is adapted to $\mathcal{F}_s = \sigma(B_t, t \leq s)$, then*

$$Y_t = \int_0^t H(s)\, dB_s, t \geq 0$$

is a continuous, square integrable martingale - that is, $\mathbb{E}(Y_t^2 < \infty)$.

Theorem 17.5. *Let M_t be a continuous, square integrable martingale with respect to $\mathcal{F}_t$. Then there exists an adapted process $H(s)$ such that $\int_0^t \mathbb{E}(H^2(s))\, ds < \infty$ and*

$$M_t = M_0 + \int_0^t H(s)\, dB_s$$

where B_t is a Brownian motion with respect to $\mathcal{F}_t$.

Theorem 17.6. *$M_t, t \geq 0$ is a Brownian motion if and only if it is a local continuous martingale with $[M, M](t) = t, t \geq 0$ under some probability measure Q.*

Proof. A local continuous martingale is of the form $M_t = M_0 + \int_0^t H(s)\, dB_s$. Then we have

$$[M, M](t) = \int_0^t H^2(s)\, ds = t \Rightarrow H(s) = 1 a.s. \Rightarrow M_t = B_t.$$

□

Theorem 17.7. *Let $M_t, t \geq 0$ e a continuous local martingale such that $[M, M](t) \uparrow \infty$. Let*

$$\tau_t = \inf\{s : [M, M](s) \geq t\}.$$

Then $M(\tau_t)$ is a Brownian motion. Moreover, $M(t) = B([M, M](t)), t \geq 0$.

This is an application of the ***change of time*** *method.*

Example 17.8. B_t is a Brownian motion - and then $Y_t = B_t^2 - t$ is a martingale. We have

$$dY_t = H(s)\, dB_s = 2B_s\, dB_s.$$

Thus,

$$B_t^2 - t = 2 \int_0^t B_s\, dB_s.$$

Definition 17.9 (Predictability of a stochastic process)**.** A stochastic process $X_t, t \geq 0$ is said to be predictable with respect to $\mathcal{F}_t$ if $X_t \in \mathcal{F}_{t-}$ for all $t \geq 0$, where

$$\mathcal{F}_{t-} = \bigcap_{h \geq 0}^{\infty} \mathcal{F}_{t+h}, \qquad \mathcal{F}_{t-} = \sigma\left(\bigcap_{h>0}^{\infty} \mathcal{F}_{t-h}\right).$$

Example 17.10. Let g_t be a step process, with

$$g_t = \sum_{i=1}^{n} \zeta_j \mathbf{1}_{[t_j, t_{j+1})}(t)$$

Then g_t is not predictable.

Let g_t be a step process, with

$$g_t = \sum_{i=1}^{n} \zeta_j \mathbf{1}_{(t_j, t_{j+1}]}(t)$$

Then g_t is predictable.

Example 17.11. Let N_t be a Poisson process. Then N_{t-} is predictable, but N_t is not predictable.

From now on, assume $M_t, t \geq 0$ is right continuous, square integrable martingale with left hand limits.

Lemma 17.12. *M_t^2 is a submartingale.*

Proof.

$$\begin{aligned}\mathbb{E}(M_t^2 \mid \mathcal{F}_s) = \mathbb{E}(M_s^2 + 2(M_t - M_s) + (M_t - M_s)^2 \mid \mathcal{F}_s) \\ &= M_s^2 + \mathbb{E}((M_t - M_s)^2 \mid \mathcal{F}_s) \\ &\geq M_s^2\end{aligned}$$

□

Theorem 17.13 (Doob-Myer decomposition)**.** *By Doob-Myer we can write*

$$M_t^2 = L_t + A_t$$

where L_t is a martingale, and A_t is a predictable process, right continuous, and increasing, such that $A_0 = 0, \mathbb{E}(A_t) < \infty, t \geq 0$.

*A_t is called the **compensator** of M_t^2, and is denoted by $\langle M, M\rangle(t)$.*

Example 17.14. Consider $B_t^2 = B_t^2 - t + t$. Then

$$\langle B, B\rangle(t) = t = [B, B](t)$$

Theorem 17.15. *If M_t is continuous then*

$$[M, M](t) = \langle M, M\rangle(t).$$

Example 17.16. Let N_t be a Poisson process. Then we know $\tilde{N}_t = N_t - \lambda t$ is a martingale. We may prove

$$\tilde{N}_t^2 - \lambda t$$

is a martingale, that is, $\langle \tilde{N}, \tilde{N} \rangle(t) = \lambda t$. However,

$$[\tilde{N}, \tilde{N}](t) = \lambda t + \tilde{N}_t \neq \langle \tilde{N}, \tilde{N} \rangle(t)$$

Example 17.17. $X_t = \int_0^t f(s)\, dB_s$ is a continuous martingale. Thus,

$$[X, X](t) = \int_0^t f^2(s)\, ds = \langle X, X \rangle(t)$$

Theorem 17.18. *If M_t is a continuous, square integrable martingale, then*

$$M_t^2 - [M, M](t)$$

is a martingale, and so

$$[M, M](t) = \langle M, M \rangle(t) + \mathit{martingale}$$

which implies

$$\mathbb{E}[M, M](t) = \mathbb{E}\langle M, M \rangle(t) = \mathbb{E}M_t^2$$

We now turn to defining integrals such as

$$\int_0^t X_s\, dM_s$$

where M_s is a martingale.

Definition 17.19. Let L^2_{pred} be the space of all predictable stochastic process X_s satisfying the condition

$$\int_0^t X_s^2\, d\langle M, M \rangle(s) < \infty.$$

Then the integral $\int_0^t X_s\, dMs$ is defined as before in two steps.

(1) If $X_s \in L^2_{pred}$ and $X_s = \sum_{j=1}^n \zeta_j \mathbf{1}_{[t_j, t_{j+1})}$. Define

$$I(X) = \sum_{j=1}^n \zeta_j (M_{t_{j+1}} - M_{t_j}).$$

(2) For all $X_s \in L^2_{pred}$, there exists a sequence of step process X_s^n such that $X_s^n \to X_s$ in L^2. Define $I(x)$ to be the limit in such situations such that

$$\mathbb{E}(I(X) - I(X^n))^2 \to 0.$$

Proposition 17.20. *Properties of the integral.*

(1) *If M_s is a (local) martingale, then*

$$\int_0^t f(s)\,dM_s$$

is a (local) martingale.

Proof.

$$\mathbb{E}(\int_s^t f(u)\,dM_u \,|\, \mathcal{F}_s) = 0$$

□

(2) *If M_t is a square integrable martingale and satisfies*

$$\mathbb{E}(\int_0^t f^2(s)\,d\langle M, M\rangle(t)) < \infty$$

then

$$I(f) = \int_0^t f(s)\,dM_s$$

is square integrable with $E(I(f)) = 0$, and $E(I^2(f)) = \int_0^t f^2(s)\,d\langle M, M\rangle(s)$.

In particular, if $M(s) = \int_0^s \sigma(u)\,dB_u$, then

$$\int_0^t X_s\,dM_s = \int_0^t X_s \sigma(s)\,dB_s$$

provided $\int_0^t X_s^2 \sigma^2(s)\,ds < \infty$ and $\int_0^t \sigma^2(s)\,ds$

(3) *If $X_t = \int_0^t f(x)\,dM_s$ and M_s is a continuous, square integrable martingale, then*

$$[X, X](t) = \int_0^t f^2(s)\,d[M, M](s) = \langle X, X\rangle(t)$$

18. Lecture 18 - Tudsay 10 May

Let M_t be a martingale. Then $M_t^2 - [M, M](t)$ is a martingale, and

$$M_t^2 = \text{martingale} + \langle M, M\rangle(t)$$

Recall that if M_t is a continuous square integrable martingale, then

$$\langle M, M\rangle(t) = [M, M](t)$$

Generally speaking,

$$[M, M](t) = \langle M, M\rangle(t) + \text{martingale}$$

Theorem 18.1. *If*

$$\in_0^t f^2(s)\,d\langle M, M\rangle(s) < \infty$$

a.s. then

$$Y_t = \int_0^t f(s)\, dM_s$$

is well defined, and

$$\begin{aligned} \mathbb{E}Y_t^2 &= \int_0^t f^2(s)\, d\langle M, M\rangle(t) \\ &= \int_0^t f^2(s)\, d[M, M](t) \quad \textit{if } M_t\textit{-continuous} \end{aligned}$$

and

$$\langle Y, Y\rangle(t) = [Y, Y](t) = \int_0^t f^2(s)\, d[M, M](s).$$

Proof.

$$\begin{aligned} dY_t &= f(t)\, dM_t \\ d[Y, Y](t) &= dY_t dY_t \\ &= f^2(t)\, dM_t dM_t \\ &= f^2(t)\, d[M, M](t) \\ [Y, Y](t) &= \int_0^t f^2(s)\, d[M, M](s) = \langle Y, Y\rangle(t) \end{aligned}$$

By Itô's formula, we also have

$$\begin{aligned} Y_t^2 &= Y_0^2 + 2\int_0^t Y_s\, dY_s + \int_0^t 1 \cdot [Y, Y](t) \\ &= 2\int_0^t Y_s \cdot f(s)\, dM_s + \langle Y, Y\rangle(t) \\ dY_t^2 &= 2Y_t f(t)\, dM_t + d\langle Y, Y\rangle(t) \end{aligned}$$

Hence

$$\langle Y, Y\rangle(t) = Y_t^2 - 2\int_0^t Y_s f(s)\, dM_s = [Y, Y](t)$$

since $\int_0^t Y_s f(s)\, dM_s$ is a martingale. □

18.1. **Itô's integration for continuous semimartingales.**

Definition 18.2. Let $(X_t, \mathcal{F}_t)$ be a continuous semimartingale. Then

$$X_t = M_t + A_t$$

where M_t is a martingale and A_t is a continuous adapted process of bounded variation ($\lim_{\delta\to 0} \sum_{j=1}^n |A_{t_{j+1}} - A_{t_j}| < \infty$).

Definition 18.3 (Integrals for semimartingales)**.**

$$\int_0^t c_s\,dX_s = \int_0^t c_s\,dM_s + \int_0^t c_s\,dA_s$$

and since A_t has bounded variation, the second integral is defined in the Riemann-Stiljes (R.S.) sense.

Theorem 18.4 (Itô's formula)**.** *Let X_t be a continuous semimartingale. Let $f(x)$ have twice continuous derivatives. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d[X,X](s)$$

Proof. Partition the interval $[0,t]$, and use a Taylor expansion to express $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2$. □

18.2. **Stochastic differential equations.** Consider the equation

$$dX_t = \sigma(t, X_t)\,dB_t = \mu(t, X_t)\,dt$$

We seek to solve for a function $f(t,x)$ such that

$$X_t = f(t, B_t).$$

Such an $f(t, B_t)$ is a **solution** to the stochastic differential equation.

Definition 18.5 (Strong solution)**.** $X_t = X_0 + \int_0^t \sigma(s, X_s)\,dB_s + \int_0^t \mu(s, B_s)\,ds$

19. Lecture 19 - Thursday 12 May

Theorem 19.1. *Let*

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dB_t$$

Assume $\mathbb{E}X_0 < \infty$. X_0 is independent of B_s and there exists a constant $c > 0$ such that

(1) $|a(t,x)| + |b(t,x)| \leq C(1 + |x|)$.
(2) $a(t,x), b(t,x)$ *satisfy the Lipschitz condition in x, i.e.*

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq C|x - y|$$

for all $t \in (0,T)$.

*Then there exists a **unique (strong) solution**.*

Example 19.2. Let

$$dX_t = c_1 X_t\,dt + c_2 X_t\,dB_t,$$

with c_1, c_2 constants.

Example 19.3. Let

$$dX_t = [c_1(t)X_t + c_2(t)]\,dt + [\sigma_1(t)X_t + \sigma_2(t)]\,dB_t$$

Let $a(t,x) = c_1(t)x + c_2(t)x, b(t,x) = \sigma_1(t)x + \sigma_2(t)$. Just follow Kuo p. 233.

Let

$$H_t = e^{-Y_t}, \quad Y_t = \int_0^t \sigma_1(s)\,ds + \int_0^t c_1(s)\,dB_s - \frac{1}{2}\int_0^t c_1^2(s)\,ds$$

Then by the Itô product formula, we have

$$d(H_tX_t) = H_t\left(dX_t - \sigma_1(t)X_t\,dt - c_1(t)X_t\,dB_t - c_2(t)c_1(t)\,dt\right)$$

Then by definition of the X_t, we obtain

$$d(H_tX_t) = H_t\left(c_2(t)\,dB_t + \sigma_2(t)\,dt - c_1(t)c_2(t)\,dt\right)$$

which can be integrated to yield

$$H_tX_t = C + \int_0^t H_s c_2(s)\,dB_s + \int_0^t H_s(\sigma_2(s) - c_1(t)c_2(t))\,ds$$

Dividing both sides by H_t we obtain our solution X_t.

Theorem 19.4. *The solution to the linear stochastic differential equation*

$$dX_t = [c_1(t)X_t + c_2(t)]\,dt + [\sigma_1(t)X_t + \sigma_2(t)]\,dB_t$$

is given by

$$X_t = Ce^{-Y_t} + \int_0^t e^{Y_t - Y_s} c_2(t)\,dB_s + \int_0^t e^{Y_t - Y_s}(\sigma_2(s) - c_1(t)c_2(t))\,dt$$

where $Y_t = \int_0^t c_1(s)\,dB_s + \int_0^t \left(\sigma_1(s) - \frac{1}{2}c_1^2(s)\right)\,ds$

20. Lecture 20 - Tuesday 17 May

20.1. **Numerical methods for stochastic differential equations.**

Theorem 20.1 (Euler's method)**.** *For the stochastic differential equation*

$$dX_t = a(X_t)\,dt + b(X_t)\,dB_t,$$

we simulate X_t according to

$$X_{t_j} = X_{t_{j-1}} + a(X_{t_{j-1}})\Delta t_j + b(X_{t_{j-1}})\,\Delta B_{t_j}$$

Theorem 20.2 (Milstein scheme)**.** *For the stochastic differential equation*

$$dX_t = a(X_t)\,dt + b(X_t)\,dB_t,$$

we simulate X_t according to

$$X_{t_j} = X_{t_{j-1}} + a(X_{t_{j-1}})\Delta t_j + b(X_{t_{j-1}})\,\Delta B_{t_j} + \frac{1}{2}b'(X_{t_{j-1}})(\Delta B_{t_j}^2 - \Delta t_j)$$

20.2. **Applications to mathematical finance.**

20.3. **Martingale method.** Consider a market with risky security S_t and riskless security β_t.

Definition 20.3 (Contingent claim)**.** A random variable $C_T : \Omega \to \mathbb{R}$, $\mathcal{F}_T$-measurable is called a contingent claim. If C_T is $\sigma(S_T)$-measurable it is **path-independent.**

Definition 20.4 (Strategy)**.** Let a_t represent number of units of S_t, and b_t represent number of units of β_t. If a_t, b_t are $\mathcal{F}_t$-adapted, then they are strategies in our market model. Our strategy value V_t at time t is

$$V_t = a_t X_t + b_t \beta_t$$

Definition 20.5 (Self-financing strategy)**.** A strategy (a_t, b_t) is self financing if

$$dV_t = a_t\, dS_t + b_t\, d\beta_t$$

The intuition is that we make one investment at $t = 0$, and after that only rebalance between S_t and β_t.

Definition 20.6 (Admissible strategy)**.** (a_t, b_t) is an **admissible strategy** if it is self financing and $V_t \geq 0$ for all $0 \leq t \leq T$.

Definition 20.7 (Arbitrage)**.** An arbitrage is an admissible strategy such that $V_0 = 0, V_T \geq 0$ and $\mathbb{P}(V_T > 0) > 0$. Alternatively, an arbitrage is a trading strategy with $V_0 = 0$, and $\mathbb{E}(V_T) > 0$.

Definition 20.8 (Attainable claim)**.** A contingent claim C_T is said to be attainable if there exists an admissible strategy (a_t, b_t) such that $V_T = C_T$. In this case, the portfolio is said to replicate the claim. By the law of one price, $C_t = V_t$ at all t.

Definition 20.9 (Complete)**.** The market is said to be **complete** if every contingent claim is attainable

Theorem 20.10 (Harrison and Pliska)**.** *Let $\mathbb{P}$ denote the real world measure of the underlying asset price X_t. If the market is arbitrage free, there exists an equivalent measure $\mathbb{P}^\star$, such that the discounted asset price $\hat{X}_t$ and every discounted attainable claim $\hat{C}_t$ are $\mathbb{P}^\star$-martingales. Further, if the market is complete, then $\mathbb{P}^\star$ is unique. In mathematical terms,*

$$C_t = \beta_t \mathbb{E}^\star(\beta_T^{-1} C_T \mid \mathcal{F}_t).$$

$\mathbb{P}^\star$ is called the equivalent martingale measure (EMM) or the risk-neutral measure.

21. Lecture 21 - Thursday 19 May

For a trading strategy (a_t, b_t), then the value V_t satisfies

$$V_t = V_0 + \int_0^t \alpha_s \, dS_s + \int_0^t b_s \, d\beta_s$$

where B_s is the riskless asset.

To price an attainable option X, let (a_t, b_t) be a trading strategy with value V_t that replicates X. Then to avoid arbitrage, the value of X at time $t = 0$ is given by V_0.

21.1. **Change of Measure.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 21.1 (Equivalent measure)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be measures on $(\Omega, \mathcal{F})$. Then for any $A \in \mathcal{F}$, if

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$$

then we say the measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. If $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$, we write $Q << P$.

Theorem 21.2 (Radon-Nikodyn)**.** *Let $Q << P$. Then there exists a random variable λ such that $\lambda \geq 0$, $\mathbb{E}_{\mathbb{P}}(\lambda) = 1$ and*

$$Q(A) = \int_A d\mathbb{P} = \mathbb{E}_p(\lambda \mathbf{1}_A)$$

for any $A \in \mathcal{F}$. λ is $\mathbb{P}$=almost surely unique.

Conversely, if there exists λ such that $\lambda \geq 1$, $\mathbb{E}_{\mathbb{P}}(\lambda) = 1$, then defining

$$\mathbb{Q}(A) = \int_A \lambda \, d\mathbb{P}$$

and then $\mathbb{Q}$ is a probability measure and $\mathbb{Q} << \mathbb{P}$. Consequently, if $Q << P$, then

$$\mathbb{E}_{\mathbb{Q}}(Z) = \mathbb{E}_{\mathbb{P}}(\lambda Z)$$

whenever $\mathbb{E}_{\mathbb{Q}}(|Z|) < \infty$.

The random variable λ is called the density of $\mathbb{Q}$ with respect to $\mathbb{P}$, and denoted by

$$\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

Example 21.3. Let $X \sim N(0,1)$ and $Y \sim N(\mu, 1)$ under probability $\mathbb{P}$. Then there exists a $\mathbb{Q}$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $Y \sim N(0,1)$ under $\mathbb{Q}$.

Proof.

$$
\begin{aligned}
\mathbb{P}(X \in A) &= \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2}}\, dt \\
\text{Define } \mathbb{Q}(A) &= \int_A e^{-\mu X - \frac{\mu^2}{2}}\, d\mathbb{P} \\
&= \frac{1}{\sqrt{2\pi}} \int_A e^{-\mu x - \frac{\mu^2}{2}} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{(\mu+x)^2}{2}}\, dx
\end{aligned}
$$

□

Then $\mathbb{Q} << \mathbb{P}, \mathbb{P} << \mathbb{Q}$ and let

$$
\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\mu X - \frac{\mu^2}{2}}
$$

Then λ satisfies the conditions of Radon-Nikodyn theorem.

Then we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(Y) &= \mathbb{E}_{\mathbb{P}}((X+\mu)\lambda) \\
&= \int (X+\mu) e^{-\mu X - \frac{\mu^2}{2}}\, d\mathbb{P} \\
&= \frac{1}{\sqrt{2\pi}} (x+\mu) e^{-\frac{(\mu+x)^2}{2}}\, dx = 0
\end{aligned}
$$

22. Lecture 22 - Tuesday 24 May

Theorem 22.1. *Let $\lambda(t), 0 \leq t \leq T$ be a positive martingale with respect to $\mathcal{F}_t$ such that*

$$
\mathbb{E}_{\mathbb{P}}(\lambda(T)) = 1.
$$

Define a new probability measure $\mathbb{Q}$ by

$$
\mathbb{Q}(A) = \int_A \lambda(T)\, d\mathbb{P}
$$

Then $\mathbb{Q} << \mathbb{P}$ and for any random variable X, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(X) &= \mathbb{E}_{\mathbb{P}}(\lambda(T) X) \\
\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}\left(\frac{\lambda(T) X}{\lambda(t)} \,|\, \mathcal{F}_t \right) \qquad (\star) \\
&= \frac{\mathbb{E}_{\mathbb{P}}(\lambda(T) X \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\lambda(T) \,|\, \mathcal{F}_t)} a.s.
\end{aligned}
$$

and if $X \in \mathcal{F}_t$, then for any $s \leq t$, we have

$$
\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}\left(\frac{\lambda(t) X}{\lambda(s)} \,|\, \mathcal{F}_s \right) \qquad (\star\star)
$$

Consequently a process $S(t)$ is a $\mathbb{Q}$-martingale if and only if

$$S(t)\lambda(t) \tag{$\dagger$}$$

is a $\mathbb{P}$-martingale

Proof. $(\star)$. We have

$$\begin{aligned}\mathbb{Q}(\mathbb{E}_{\mathbb{P}}(\lambda(T))\,|\,\mathcal{F}_t = 0) &= \mathbb{E}_p(\lambda(T)\mathbf{1}_{\mathbb{E}_{\mathbb{P}}(\lambda(T)\,|\,\mathcal{F}_t=0)}) \\ &= 0\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\frac{\mathbb{E}_{\mathbb{P}}(\lambda(T)X\,|\,\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\lambda(T)\,|\,\mathcal{F}_t)}\mathbf{1}_A\right) &= \mathbb{E}_{\mathbb{Q}}\left(\lambda(t)\frac{\mathbb{E}_{\mathbb{P}}(\lambda(T)X\,|\,\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\lambda(T)\,|\,\mathcal{F}_t)}\mathbf{1}_A\right) \\ &= \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(\lambda(T)X\,|\,\mathcal{F}_t)\mathbf{1}_A\right) \\ &= \mathbb{E}_{\mathbb{P}}(\lambda(T)X\mathbf{1}_A) \\ &= \mathbb{E}_{\mathbb{Q}}(X\mathbf{1}_A)\end{aligned}$$

$(\star\star)$. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(\frac{\lambda(t)X}{\lambda(s)}\,|\,\mathcal{F}_s\right)&\frac{1}{\lambda(s)}\mathbb{E}_{\mathbb{P}}(\lambda(t)X\,|\,\mathcal{F}_s) \\ &= \frac{1}{\lambda(s)}\mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(\lambda(T)X\,|\,\mathcal{F}_t)\,|\,\mathcal{F}_s\right) \\ &= \frac{1}{\lambda(s)}\mathbb{E}_{\mathbb{P}}(\lambda(T)X\,|\,\mathcal{F}_s) \\ &= \mathbb{E}_{\mathbb{Q}}(X\,|\,\mathcal{F}_s)\end{aligned}$$

because of $(\star)$.

$(\dagger)$. We have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}(S(t)\,|\,\mathcal{F}_u) = S(u) \\ \iff &\mathbb{E}_{\mathbb{P}}\left(\frac{\lambda(t)S(t)}{\lambda(u)}\,|\,\mathcal{F}_u\right) = S(u) \\ \iff &\mathbb{E}_{\mathbb{P}}(\lambda(t)S(t)\,|\,\mathcal{F}_u) = \lambda(u)S(u)\end{aligned}$$

as required. ∎

Theorem 22.2. *Let $B_s, 0 \le s \le T$ be a Brownian motion under $\mathbb{P}$. Let $S(t) = B_t + \mu_t, u \neq 0$. Then there exists a $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that$S(t)$ is a $\mathbb{Q}$-Brownian motion and*

$$\lambda(T) = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\mu B_T - \frac{1}{2}\mu^2 T}.$$

Note that $S(t)$ is not a martingale under $\mathbb{P}$, but it is a martingale under $\mathbb{Q}$.

Proof. Under $\mathbb{Q}$,

$$\mathbb{Q}(B_0 = 0) = \int_{B_0=0} \lambda(T)\, d\mathbb{P} = 1$$

$S(t)$ is a $\mathbb{Q}$-martingale if and only if $S(t)\lambda(t)$ is a $\mathbb{P}$-martingale. But we have

$$X_t = S(t)\lambda(t) = (B_t + \mu t)\, e^{-\mu B_t - \frac{1}{2}\mu^2 t}$$

is a martingale.

Finally, note that

$$[S, S](t) = [B, B](t) = t$$

□

22.1. **Black-Scholes model.**

Definition 22.3 (Black-Scholes model)**.** The Black-Scholes model assumes the risky asset S_t follows the diffusion process given by

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t$$

and the riskless asset follows the diffusion

$$\frac{d\beta_t}{\beta_t} = r\, dt$$

Define the discounted process as follows:

$$\hat{S}_t = \frac{S_t}{\beta_t}, \quad \hat{V}_t = \frac{V_t}{\beta_t}, \quad \hat{C}_t = \frac{C_t}{\beta_t}.$$

23. Lecture 23 - Thursday 26 May

Lemma 23.1.

(a) By a simple application of Itô's lemma,

$$\frac{d\hat{S}_t}{\hat{S}_t} = (\mu - r)\, dt + \sigma\, dB_t.$$

(b) $\hat{S}_t$ is a $\mathbb{Q}$-martingale with

$$\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-qB_T - \frac{1}{2}q^2 T}$$

with $q = \frac{\mu - r}{\sigma}$.

(c) Note that

$$\frac{d\hat{S}_t}{\hat{S}_t} = \sigma d(B_t + \frac{\mu - r}{\sigma} t) = \sigma d\hat{B}_t$$

where $\hat{B}_t = B_t + qt$ is a Brownian motion under $\mathbb{Q}$.

(d) $d\hat{S}_t = \sigma \hat{S}_t\, d\hat{B}_t$.

(e) *In a finite market, where S_t takes only finitely many values, $\hat{S}_t$ is a $\mathbb{Q}$-martingale is a necessary condition for no-arbitrage.*

(f) *Note that*

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t = r\, dt + \sigma\, d\hat{B}_t$$

Theorem 23.2. *A value process V_t is self financing if and only if the discounted value process $\hat{V}_t$ is a $\mathbb{Q}$-martingale.*

$$\begin{aligned} d\hat{V}_t = a_t\, d\hat{S}_t & \qquad\qquad \iff dV_t = \alpha dS_t + b_t\, d\beta_t \\ &\iff V_t = V_0 + \int_0^t a_s\, dS_s + \int_0^t b_s\, d\beta_s \\ &\iff V_t \text{ is self financing.} \end{aligned}$$

Proof. By Itô's formula, we have

$$\begin{aligned} d\hat{V}_t &= e^{-rt}\, dV_t - re^{-rt} V_t\, dt \\ &= e^{-rt}\left(a_t\, dS_t + b_t\, d\beta_t\right) - re^{-rt}\left(a_t S_t + b_t \beta_t\right)\, dt \\ & a_t(e^{-rt}\, dS_t - re^{-rt}\, dt) \\ &= a_t\, d\hat{S}_t \end{aligned}$$

□

Theorem 23.3. *In the Black-Scholes model, there are no arbitrage opportunities.*

Proof. For any admissible trading strategy (a_t, b_t), we have that the discounted value process $\hat{V}_t$ is a Q-martingale. So if $V_0 = 0$, then $\mathbb{E}(\hat{V}_0) = 0$, and we have

$$\mathbb{E}_{\mathbb{Q}}(\hat{V}_T) = \mathbb{E}_{\mathbb{Q}}(\hat{V}_T \,|\, \mathcal{F}_0) = \hat{V}_0 = 0$$

which implies $\mathbb{Q}(\hat{V}_T > 0) = 0$, which implies $\mathbb{P}(V_T > 0) = 0$, which implies that $\mathbb{E}_{\mathbb{P}}(V_T) = 0$, which then implies no arbitrage. □

Theorem 23.4. *For any self financing strategy,*

$$V_t = a_t S_t + b_t \beta_t = V_0 + \int_0^t a_u\, dS_u + \int_0^t b_u\, d\beta_u$$

And so a strategy is self financing if

$$S_t\, da_t + \beta_t\, db_t + d[a, S](t) = 0$$

We now consider several cases.

(1) *If a_t is of bounded variation, then $[a, S](t) = 0$. Hence*

$$S_t\, da_t + \beta_t\, db_t = 0,$$

which implies

$$db_t = -\frac{S_t}{\beta_t}\, da_t$$

Hence $da_t \cdot db_t < 0$.

(2) *If* a_t *is a semi-martingale* $a_t = a_t^2 + A_t$*, then* b_t *must be a semi-martingale, where* $b_t = b_t^2 + B_t$ *where* a_t^2, b_t^2 *are the martingale parts and* A_t, B_t *are of bounded variation.*

24. Lecture 24 - Tuesday 31 May

Theorem 24.1. *Given a claim* C_T *under the self-financing assumption, there exists a* $\mathbb{Q}$*-martingale such that*

$$V_t = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)C_T \,|\, \mathcal{F}_t}\right), \mathcal{F}_t = \sigma(B_s, 0 \le s \le t).$$

In particular, we have

$$V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT}C_T)$$

Proof. For any $\mathbb{Q}$-martingale $\hat{V}_t$, we have $\hat{V}_t = \mathbb{E}_{\mathbb{Q}}(\hat{V}_T \,|\, \mathcal{F}_t)$. Then

$$V_t = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}C_T \,|\, \mathcal{F}_t),$$

as $V_T = C_T$. □

Theorem 24.2. *A claim is attainable (there exists a trading strategy replicating the claim), that is,*

$$V_t = V_0 + G(t)$$

$$G(t) = \int_0^t a_u\, dS_u + \int_0^t b_u\, d\beta_u$$

$$V_T \ge 0, V_T = C_T$$

Theorem 24.3. *Suppose that* C_T *is a non-negative random variable,* $C_t \in \mathcal{F}_T$ *and* $\mathbb{E}_{\mathbb{Q}}(C_T^2 f < \infty)$*, where* $\mathbb{Q}$ *is defined as before. Then*

(a) *The claim is replicable.*

(b)

$$V_t = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)}C_T \,|\, \mathcal{F}_t\right) \iff \hat{V}_t = \mathbb{E}_{\mathbb{Q}}(\hat{C}_T \,|\, \mathcal{F}_t)$$

where $\hat{C}_T = e{-}rTC_T$.

In particular, $V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT}C_T) = \mathbb{E}_{\mathbb{Q}}(\hat{C}_T)$.

Theorem 24.4. *Assume* $\hat{V}_t = \mathbb{E}_{\mathbb{Q}}(\hat{C}_T \,|\, \mathcal{F}_t)$*. Using the martingale representation theorem, there exists an adapted process* $H(s)$ *such that*

$$\hat{V}_t = \hat{V}_0 \int_0^t H(s)\, d\hat{B}_s \iff d\hat{V}_t = H(t)\, d\hat{B}_t.$$

On the other hand, $d\hat{V}_t = a_t\, d\hat{S}_t = a_t \cdot \sigma \hat{S}_t\, d\hat{B}_t$. *Hence we obtain our required result,*

$$a_t = \frac{H(t)}{\sigma \hat{S}_t},$$

and then solve for b_t.

Example 24.5. Let $C_T = f(S_T)$. Then $V_t = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)f(S_T)\,|\,\mathcal{F}_t}\right)$. Since $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t) = \sigma(\hat{B}_s, 0 \le s \le t)$, and $\hat{S}_t$ is a $\mathbb{Q}$-martingale, we have

$$\hat{S}_t = \hat{S}_0 e^{-\frac{\sigma^2}{2}t + \sigma \hat{B}_t}$$

and so

$$S_T = e^{rT}\hat{S}_T = \hat{S}_t e^{rT} e^{-\frac{\sigma^2}{2}(T-t) + \sigma(\hat{B}_T - \hat{B}_t)}.$$

Then

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} f\left(e^{rT}\hat{S}_t e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\hat{B}_T-\hat{B}_t)}\right)\right]$$

and so

$$V_t = F(t, S_t)$$

where

$$\begin{aligned} F(t,x) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} f\left(e^{(-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}Z}\right)\right] \\ &= e^{-r(T-t)} \int_{\mathbb{R}} f\left(xe^{-\frac{\sigma^2}{2}(T-t)+\sigma z\sqrt{T-t}}\right) \phi(z)\, dz \end{aligned}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

Example 24.6. In particular, if $f(y) = (y-K)^+$, then we obtain

$$\begin{aligned} F(t,x) &= e^{-r\theta} \int_{-d_1'}^{\infty} xe^{-\frac{\sigma^2}{2}\theta + xz\sqrt{\theta} - \frac{z^2}{2}}\, dz - K\int_{-d_1'}^{\infty} e^{-r\theta - \frac{z^2}{2}}\, dz \\ &= x\Phi(d_1' + \sigma\sqrt{\theta}) - Ke^{-r\theta}\Phi(d_1') \\ &= x\Phi(d_1) - Ke^{-r\theta}\Phi(d_2), \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + (r + \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}}, \quad d_2 = d_1 - \sigma\sqrt{\theta}$$

Theorem 24.7 (Black-Scholes model summary)**.** $V_t = a_t S_t + b_t \beta_t$, *where*

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t$$

$$\frac{d\beta_t}{\beta_t} = r\, dt$$

(a) $\hat{S}_t = e^{-rt}S_t$ *is a* $\mathbb{Q}$*-martingale, where*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-qB_T - \frac{1}{2}q^2 T}$$

and $q = \frac{\mu - r}{\sigma}$.

Then $\hat{B}_t = B_t + \frac{\mu-r}{\sigma}t$ *is a* $\mathbb{Q}$*-Brownian motion, and* $d\hat{S}_t = \sigma \hat{S}_t \, d\hat{B}_t$.

(b) V_t *is self-financing if* (a_t, b_t) *satisfies*

$$S_t \, da_t + \beta_t \, db_t + d[a, S](t) = 0$$

which then implies $\hat{V}_t$ *is a* $\mathbb{Q}$*-martingale,* $\hat{V}_t = e^{-rt}V_t$.

(c) There are no arbitrage opportunities in the Black-Scholes model.

25. Lecture 25 - Thursday 2 June

Theorem 25.1 (Feyman-Kac formula)**.**

(1) Suppose the function $F(x,t)$ *solves the boundary value problem*

$$\frac{\partial F(t,x)}{\partial t} + \mu(t,x)\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F(t,x)}{\partial x^2} = 0$$

such that $F(T,x) = \Psi(x)$.

(2) Let S_t *be a solution of the SDE*

$$dS_t = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dB_t \qquad (\star)$$

where B_t *is a* $\mathbb{Q}$*-Brownian motion*

(3) Assume

$$\int_0^T \mathbb{E}(\sigma(t,S_t)\frac{\partial^2 F(t,S_t)}{\partial x^2})\, dt < \infty$$

Then

$$F(t, S_t) = \mathbb{E}_{\mathbb{Q}}\left(\Psi(S_T) \,|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(F(T, S_T) \,|\, \mathcal{F}_t\right).$$

where $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$.

Proof. It is enough to show that $F(t, S_t)$ is a martingale with respect to $\mathcal{F}_t$ under $\mathbb{Q}$. By Itô's lemma, we have

$$\begin{aligned}
dF(t,S_t) &= \frac{\partial f(t,S_t)}{\partial t}\, dt + \frac{\partial F(t,S_t)}{\partial x}\, dS_t + \frac{1}{2}\frac{\partial^2 F(t,S_t)}{\partial x^2} \cdot (dS_t)^2 \\
&= \left[\frac{\partial F}{\partial t} + \mu(t,S_t)\frac{\partial F}{\partial x} + \frac{\sigma^2(t,S_t)}{2}\frac{\partial^2 F(t,S_t)}{\partial x^2}\right] dt + \frac{\partial F(t,S_t)}{\partial x}\sigma(t,S_t)\, dB_t \\
&= \frac{\partial F(t,S_t)}{\partial x}\sigma(t,S_t)\, dB_t
\end{aligned}$$

which is a $\mathbb{Q}$-martingale. ∎

Theorem 25.2 (General Feyman-Kac formula)**.** *Let S_t be a solution of the SDE* $(\star)$. *Assume that there is a solution to the PDE*

$$\frac{\partial F(t,x)}{\partial t} + \mu(t,x)\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F(t,x)}{\partial x^2} = r(t,x)F(t,x).$$

Then

$$F(t,S_t) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r(u,S_u)\,du} F(T,S_T) \,|\, \mathcal{F}_t\right)$$

Proof. Again by Itô's lemma,

$$\begin{aligned} dF(t,S_t) &= \left(\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2}\right)dt + \frac{\partial F}{\partial x}\cdot\sigma(t,S_t)\,dB_t \\ &= r(t,x)F(t,S_t)\,dt + dM_t \end{aligned}$$

where $M_t = \int_0^t \frac{\partial F}{\partial x}\sigma(u,S_u)\,dB_u$. Hence we have

$$\begin{aligned} dF(t,S_t) &= r(t,S_t)X_t\,dt + dM_t \\ d\left[e^{-\int_t^\nu r(u,s_u)\,du}X_\nu\right] &= e^{-\int_t^\nu r(u,S_u)\,du}dM_\nu && (\Rightarrow) \\ e^{-\int_t^T r(u,S_u)\,du}F(T,S_T) &= F(t,S_t) + \int_t^T e^{-\int_t^\nu r(u,S_u)\,du}\,dM_\nu && (\Rightarrow) \\ \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r(u,S_u)\,du}F(T,S_T)\,|\,\mathcal{F}_t\right) &= F(t,S_t) && (\Rightarrow) \\ &\quad + \underbrace{\mathbb{E}_{\mathbb{Q}}\left(\int_t^T e^{-\int_t^\nu r(u,S_u)\,du}\,\frac{\partial F}{\partial x}\sigma(\nu,S_\nu)\,dB_\nu\,|\,\mathcal{F}_t\right)}_{=0} \end{aligned}$$

and so we obtain our result,

$$F(t,S_t) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r(u,S_u)\,du} F(T,S_T) \,|\, \mathcal{F}_t\right)$$

□