

# PMH3 - FUNCTIONAL ANALYSIS

## LECTURE NOTES

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### 1. LECTURE 1 - MONDAY 28 FEBRUARY

**Definition 1.1** (Norm). Let  $X$  be a vector space. A norm on  $X$  is a function  $\|\cdot\| : X \mapsto \mathbb{R}$  satisfying

- $\|x\| \geq 0$  with equality if and only if  $x = 0$ .
- $\|\alpha x\| = |\alpha|\|x\|$ .
- $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .

We call the pair  $(X, \|\cdot\|)$  a **normed vector space**.

**Theorem 1.2** (Reverse triangle inequality). *Let  $X$  be a normed vector space. For any  $x, y \in X$ , we have*

$$|||x| - |y|| \leq \|x - y\|$$

**Definition 1.3** (Complete space). Let  $X$  be a normed vector space. Then  $X$  is **complete** if every Cauchy sequence in  $X$  converges to some  $x \in X$ .

**Definition 1.4** (Banach space). A **Banach space** is a complete normed vector space.

### 2. LECTURE 2 - WEDNESDAY 2 MARCH

**Proposition 2.1** (Convergence). *Let  $(V, \|\cdot\|)$  be a normed vector space. A sequence  $(x_n)$  in  $V$  converges to  $x \in V$  if given  $\epsilon > 0$ , there exists  $N$  such that  $\|x - x_n\| < \epsilon$  whenever  $n < N$ .*

**Lemma 2.2.** *If  $x_n \rightarrow x$ , then  $\|x_n\| \rightarrow \|x\| \in \mathbb{R}$ .*

*Proof.*  $|||x_n| - |x|| \leq \|x - x_n\| \rightarrow 0$ . □

**Proposition 2.3.** *Every convergent sequence is Cauchy.*

**Definition 2.4** (Banach space). A complete, normed, vector space is called a **Banach space**.

**Proposition 2.5.**  $(\mathbb{K}, |\cdot|)$  is complete.

**Proposition 2.6.**  $(\ell^p, \|\cdot\|_p)$  is a Banach space for all  $1 \leq p \leq \infty$ .

*Proof.* A general proof outline follows.

- Use completeness of  $\mathbb{R}$  to find a candidate for the limit.
- Show this limit function is in  $V$ .
- Show that  $x_n \rightarrow x$  in  $V$ .

Let  $x^{(n)}$  be a Cauchy sequence in  $\ell^p$ . Since  $|x_j^{(n)} - x_j^{(m)}| \leq \|x^{(n)} - x^{(m)}\|$ , we know that  $x_j^{(n)}$  is a Cauchy sequence in  $\mathbb{K}$ . Hence,  $\lim_{n \rightarrow \infty} x_j^{(n)} := x_j$  exists, and is our limit candidate.

We now show that  $\sum_{j=1}^{\infty} |x_j|^p < \infty$ . We have  $\square$

**Proposition 2.7.**  $(\ell([a, b]), \|\cdot\|_{\infty})$  is a Banach space

**Proposition 2.8.** If  $1 \leq p < \infty$ , then  $(\ell([a, b]), \|\cdot\|_p)$  is **not** a Banach space.

*Proof.* Consider a sequence of functions that is equal to one on  $[0, \frac{1}{2}]$ , zero on  $[\frac{1}{2} + \frac{1}{n}, 1]$ , and linear between. This is a Cauchy sequence that does not converge to a continuous function.  $\square$

### 3. LECTURE 3 - MONDAY 7 MARCH

We've seen that  $(\ell([a, b]), \|\cdot\|_p)$  is not complete for  $1 \leq p < \infty$ .

**Theorem 3.1** (Completion). Let  $(V, \|\cdot\|)$  be a normed vector space over  $\mathbb{K}$ . There exists a Banach space  $(V_1, \|\cdot\|_1)$  such that  $(V, \|\cdot\|)$  is isometrically isomorphic to a dense subspace of  $(V_1, \|\cdot\|_1)$ .

Furthermore, the space  $(V_1, \|\cdot\|_1)$  is unique up to isometric isomorphisms.

*Proof.* Rather straightforward - construct Cauchy sequences, append limits, quotient out (as different sequences may converge to the same limit).  $\square$

**Definition 3.2.**  $(V_1, \|\cdot\|_1)$  is called **the completion** of  $(V, \|\cdot\|)$ .

**Definition 3.3** (Dense). If  $X$  is a topological space and  $Y \subseteq X$ , then  $Y$  is **dense** in  $X$  if the closure of  $Y$  in  $X$  equals  $X$ , that is,  $\bar{Y} = X$ .

Alternatively, for each  $x \in X$ , there exists  $(y_n)$  in  $Y$  such that  $y_n \rightarrow x$ .

**Definition 3.4** (Isomorphism of vector spaces). Two normed vector spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are **isometrically isomorphic** if there is a vector space isomorphism  $\Psi : X \rightarrow Y$  such that

$$\|\Psi(x)\|_Y = \|x\|_X \quad \forall x \in X$$

**Example 3.5.** Let  $\ell_0 = \{(x_i) \mid \#\{i, x_i \neq 0\} < \infty\}$ . The completion of  $\ell_0, \|\cdot\|_p$  is  $(\ell^p, \|\cdot\|_p)$ , because,

- $\ell_0$  is a subspace of  $\ell^p$ ,
- It is dense, since we can easily construct a sequence in  $\ell_0$  converging to arbitrary  $x \in \ell^p$ .

**Example 3.6** ( $L^p$  spaces). Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$ . Let

$$\mathcal{L}^p([a, b]) = \{\text{measurable } f : [a, b] \rightarrow \mathbb{K} \mid \int_a^b |f|^p d\mu < \infty\}$$

Let  $\|f\|_p = \left( \int_a^b |f|^p d\mu \right)^{1/p}$ . Since  $\|f\|_p = 0 \iff f = 0 \text{ a.e.}$ , we quotient out by the rule  $f \equiv g \iff f - g = 0 \text{ a.e.}$ , and then our space of equivalence classes forms a normed vector space, denoted  $L^p([a, b])$ .

**Theorem 3.7** (Riesz-Fischer).  *$(L^p([a, b]), \|\cdot\|_p)$  is the completion of  $(\mathcal{C}[a, b], \|\cdot\|_p)$ , and is a Banach space.*

*Proof.* Properties of the Lebesgue integral. □

*Remark.*

- Let  $X$  be any compact topological space, let  $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{K} \mid f \text{ is continuous}\}$ , and let  $\|f\|_\infty = \sup_{x \in X} \|f(x)\|$ . Then  $\mathcal{C}(X, \|\cdot\|_\infty)$  is Banach.
- Let  $X$  be any topological space. Then the set of all continuous and bounded functions with the supremum norm forms a Banach space.
- Let  $(S, \mathcal{A}, \mu)$  be a measure space. Then we can define the  $\mathcal{L}^p$  and  $L^p$  analogously, and they are also Banach.

**Definition 3.8** (Linear operators on normed vector spaces). Let  $X, Y$  be vector spaces over  $\mathbb{K}$ . A linear operator is a function  $T : X \rightarrow Y$  such that

$$\begin{aligned} T(x + y) &= T(x) + T(y) \\ T(\alpha x) &= \alpha T(x) \end{aligned}$$

for all  $x, y, \alpha$ .

We write  $\text{Hom}(X, Y) = \{T : X \rightarrow Y \mid T \text{ is linear}\}$

**Definition 3.9.**  $T : X \rightarrow Y$  is continuous at  $x \in X$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\|x - y\|_X < \delta \Rightarrow \|Tx - Ty\|_Y < \epsilon$$

**Definition 3.10.**

$$\mathcal{L}(X, Y) = \{T : X \rightarrow Y \mid T \text{ is linear and continuous}\}$$

*Remark.* If  $\dim(X) < \infty$  then  $\text{Hom}(X, Y) = \mathcal{L}(X, Y)$ . This is **not** true if  $X$  has infinite dimension.

**Definition 3.11** (Bounded linear operator). Let  $T : X \rightarrow Y$  be linear, then  $T$  is **bounded** if  $T$  maps bounded sets in  $X$  to bounded sets in  $Y$ . That is: for each  $M > 0$  there exists  $M' > 0$  such that

$$\|x\|_X \leq M \Rightarrow \|Tx\|_Y \leq M'$$

#### 4. LECTURE 4 - WEDNESDAY 9 MARCH

Consider the space  $\mathcal{L}(X, Y)$ , the set of all linear and continuous maps between two normed vector spaces  $X$  and  $Y$ .

**Theorem 4.1** (Fundamental theorem of linear operators). *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed vector spaces. Let  $T \in \text{Hom}(X, Y)$ , the set of all linear maps from  $X$  to  $Y$ . Then the following are all equivalent.*

- 1)  $T$  is uniformly continuous
- 2)  $T$  is continuous
- 3)  $T$  is continuous at 0
- 4)  $T$  is bounded
- 5) There exists a constant  $c > 0$  such that

$$\|Tx\|_Y \leq c\|x\|_X \quad \forall x \in X$$

*Proof.* 1)  $\Rightarrow$  2)  $\Rightarrow$  3) is clear.

3)  $\Rightarrow$  4). Since  $T$  is continuous at 0, given  $\epsilon = 1 > 0$ , there exists  $\delta$  such that

$$\|Tx - T0\| \leq 1 \quad \text{whenever} \quad \|x - 0\| \leq \delta,$$

i.e. that  $\|x\| \leq \delta \Rightarrow \|Tx\| \leq 1$ . Let  $y \in X$ . The  $\|\frac{\delta y}{\|y\|}\| \leq \delta$ , and so  $\|T\left(\frac{\delta y}{\|y\|}\right)\| \leq 1$ . Hence,

$$\frac{\delta}{\|y\|} \|Ty\| \leq 1$$

and so

$$\|Ty\| \leq \frac{\|y\|}{\delta}$$

for all  $y \in X$ . Thus, for all  $\|y\| \leq M$ , we have  $\|Ty\| \leq M'$ , where  $M' = \frac{M}{\delta}$ , and so  $T$  is **bounded**.

4)  $\Rightarrow$  5). If  $T$  is bounded, given  $M = 1 > 0$ , there exists  $c \geq 0$  such that  $\|x\| \leq 1 \Rightarrow \|Tx\| \leq c$ . Then

$$\left\| T \left( \frac{x}{\|x\|} \right) \right\| \leq c$$

Hence,  $\|Tx\| \leq c\|x\|$ .

5)  $\Rightarrow$  1). If 5) holds, then

$$\|Tx - Ty\| = \|T(x - y)\| \leq c\|x - y\|.$$

So if  $\epsilon$  is given, taking  $\delta = \frac{\epsilon}{c}$ , we have

$$\|Tx - Ty\| \leq c\|x - y\| < c \frac{\epsilon}{c} = \epsilon. \quad \square$$

**Corollary.** *If  $T \in \text{Hom}(X, Y)$ , then  $T$  continuous  $\iff T$  bounded  $\iff \|Tx\| \leq c\|x\|$  for all  $x \in X$ .*

**Definition 4.2** (Operator norm). The **operator norm** of  $T \in \mathcal{L}(x, y)$ ,  $\|T\|$  is defined by any one of the following equivalent expressions.

- (a)  $\|T\| = \inf\{c > 0 \mid \|Tx\| < c\|x\|\}$ .
- (b)  $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$ .

- (c)  $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$ .  
 (d)  $\|T\| = \sup_{\|x\|=1} \|Tx\|$ .

**Proposition 4.3.** *The operator norm is a norm on  $\mathcal{L}(x, y)$ .*

*Proof.* The following are simple to verify.

- (a)  $\|T\| \geq 0$ , with equality if and only if  $T = 0$ .  
 (b)  $\|\alpha T\| = |\alpha| \|T\|$ .  
 (c)  $\|S + T\| \leq \|S\| + \|T\|$ .

□

**Example 4.4** (Calculating  $\|T\|$ ). To calculate  $\|T\|$ , try the following.

- 1) Make sensible calculations to find  $c$  such that

$$\|Tx\| \leq c\|x\|$$

for all  $x \in X$ .

- 2) Find  $x \in X$  such that  $\|Tx\| = c\|x\|$ .

## 5. LECTURE 5 - TUESDAY 15 MARCH

*Remark.* Ignore !2, Q3(b), Q8 on the practice sheet, as we will be ignoring Hilbert space theory for the time being.

**Definition 5.1** (Algebraic dual). Let  $(X, \|\cdot\|)$  be a normed vector space over  $\mathbb{K}$ . The **algebraic dual** of  $X$  is

$$X^* = \text{Hom}(X, \mathbb{K}) = \{\varphi : X \rightarrow \mathbb{K} \mid \varphi \text{ is linear}\}.$$

Elements of  $X^*$  are called linear functionals.

**Definition 5.2** (Continuous dual (just **dual**)). The **continuous dual** (just dual) of  $X$  is

$$X' = \mathcal{L}(X, \mathbb{K}) = \{\varphi : X \rightarrow \mathbb{K} \mid \varphi \text{ is linear and continuous}\}.$$

*Remark.*  $X^* \supsetneq X'$  if  $\dim(X) = \infty$ .

**Example 5.3.** Let  $(\wp([a, b]), \|\cdot\|_\infty)$  be the normed vector space of polynomials  $p : [a, b] \rightarrow \mathbb{K}$ .

- (a) The functional  $D : \wp([0, 1]) \rightarrow \mathbb{K}$  given by  $D(p) = p'(1)$  is linear, but **not** continuous.  
 (b) The functional  $I : \wp([0, 1]) \rightarrow \mathbb{K}$  given by  $I(p) = \int_0^1 p(t) dt$  is linear **and** continuous.

*Proof.* (a) Linearity is clear. The  $p_n(t) = t^n$  for all  $t \in [0, 1]$ . Then  $|D(p_n)| = n\|p_n\|_\infty$ . So  $D$  is not continuous, as continuity implies that there exists  $c$  such that

$$\|Tx\| \leq c\|x\|.$$

(b) Exercise: Show  $\|I\| = 1$ .

□

Describing the continuous dual space  $X'$  is one of the first things to do when trying to understand a normed vector space. It is generally pretty difficult to describe  $X'$ .

**Proposition 5.4** (Dual of the  $\ell^p$  space for  $(1 < p < \infty)$ ). *Let  $1 < p < \infty$ . Let  $q$  be the “dual” of  $p$ , defined by  $\frac{1}{q} + \frac{1}{p} = 1$ . Then  $(\ell^p)'$  is isometrically isomorphic to  $\ell^q$ .*

*Remark* (Observation before proof). Let  $1 \leq p < \infty$ . Let  $e_i = (0, 0, \dots, 1, 0, \dots)$  where 1 is in the  $i$ -th place.

1) If  $x = (x_i) \in \ell^p$ , then

$$x = \sum_{i=1}^{\infty} x_i e_i$$

in the sense that the partial sums converge to  $x$ .

2) If  $\varphi : \ell^p \rightarrow \mathbb{K}$  is linear and continuous, then

$$\varphi(x) = \sum_{i=1}^{\infty} x_i \varphi(e_i)$$

*Proof of observations.* Let  $S_n = \sum_{i=1}^n x_i e_i$ . Then

$$\begin{aligned} \|x - S_n\|_p^p &= \|(0, 0, \dots, x_{n+1}, x_{n+2}, \dots)\|_p^p \\ &= \sum_{i=n+1}^{\infty} |x_i|^p \\ &\rightarrow 0 \quad \text{as it is the tail of a convergent sum.} \end{aligned}$$

Write  $\varphi(x)$  as

$$\begin{aligned} \varphi(x) &= \varphi\left(\lim_{n \rightarrow \infty} S_n\right) \quad (\text{continuity}) \\ &= \lim_{n \rightarrow \infty} (\varphi(S_n)) \\ &= \lim_{n \rightarrow \infty} \varphi\left(\sum_{i=1}^n x_i e_i\right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i \varphi(e_i) \quad (\text{linearity}) \\ &= \sum_{i=1}^{\infty} x_i \varphi(e_i) \end{aligned}$$

□

*Proof.* Define a map  $\theta$  by

$$\begin{aligned}\theta : \ell^q &\rightarrow (\ell^p)' \\ y &\mapsto \varphi_y\end{aligned}$$

where  $\varphi_y(x) = \sum x_i y_i$  for all  $x \in \ell^p$ .

- (1)  $\varphi_y$  is linear, as  $\varphi_y(x + x') = \varphi_y(x) + \varphi_y(x')$  (valid as sums converge absolutely.)
- (2)  $\varphi_y$  is continuous, as

$$|\varphi_y(x)| = \left| \sum x_i y_i \right| \leq \sum |x_i y_i| \leq \|x\|_p \|y\|_q$$

by Hölder's inequality. From the fundamental theorem of linear operators, as  $|\varphi_y(x)| \leq \|x\|_p \|y\|_q$ , we have that  $\varphi_y$  is continuous, and that

$$\|\varphi_y\| \leq \|y\|_q \quad (\star)$$

- (3)  $\theta$  is linear.
- (4)  $\theta$  is injective, as

$$\begin{aligned}\theta(y) = \theta(y') &\Rightarrow \varphi_y = \varphi_{y'} \Rightarrow \varphi_y(x) = \varphi_{y'}(x) \quad \forall x \in \ell^p \\ &\Rightarrow \varphi_y(e_i) = \varphi_{y'}(e_i) \quad \forall i \in \mathbb{N} \Rightarrow y_i = y'_i \quad \forall i \in \mathbb{N} \Rightarrow y = y'\end{aligned}$$

- (5)  $\theta$  is surjective. Let  $\varphi \in (\ell^p)'$ . Let  $y = (\varphi(e_1), \dots, \varphi(e_n), \dots) = (y_1, \dots, y_n, \dots)$ . We now show  $y \in \ell^q$ .

Let  $x^{(n)} \in \ell^p$  be defined by

$$x_i^{(n)} = \begin{cases} \frac{|y_i|^q}{y_i} & \text{if } i \leq n \text{ and } y_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\varphi(x^{(n)}) = \sum_{i=1}^{\infty} x_i^{(n)} \varphi(e_i) = \sum_{i=1}^n |y_i|^q \quad (\dagger)$$

by Observation 2) above.

On the other hand, we know

$$\begin{aligned}
 \|\varphi(x^{(n)})\| &\leq \|\varphi\| \|x^{(n)}\|_p \\
 &= \|\varphi\| \left( \sum_{i=1}^{\infty} |x_i^{(n)}|^p \right)^{1/p} \\
 &= \|\varphi\| \left( \sum_{i=1}^n |y_i|^{(q-1)p} \right)^{1/p} \\
 &= \|\varphi\| \left( \sum_{i=1}^n |y_i|^q \right)^{1/p} \quad \text{as } 1/p + 1/q = 1.
 \end{aligned} \tag{**}$$

Now, using (†) and (\*\*), we have

$$\sum_{i=1}^n |y_i|^q \leq \|\varphi\| \left( \sum_{i=1}^n |y_i|^q \right)^{1/p}$$

and so we must have

$$\|y\|_q \leq \|\varphi\| \tag{***}$$

and so  $y \in \ell^q$ .

We also have, by (\*\*),

$$\|y\|_q \leq \|\varphi_y\|$$

(6) Finally, we show that  $\theta$  is an isometry. By (\*) and (\*\*\*), we have

$$\|\theta(y)\| = \|\varphi_y\| = \|y\|_q$$

as required.  $\square$

## 6. LECTURE 6 - WEDNESDAY 16 MARCH

How big is  $X'$ ? When is  $X' \neq \{0\}$ ? Examples suggest that  $X'$  is big with a rich structure.

**6.1. The Hahn-Banach theorem.** The Hahn-Banach theorem is a cornerstone of functional analysis. It is all about extending linear functionals defined on a subspace to linear functionals on the whole space, while preserving certain properties of the original functional.

**Definition 6.1** (Seminorm). A let  $X$  be a vector space over  $\mathbb{K}$ . A seminorm on  $X$  is a function  $p : X \rightarrow \mathbb{R}$  such that

- (1)  $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X$
- (2)  $p(\lambda x) = |\lambda|p(x) \quad \forall x \in X, \lambda \in \mathbb{K}$



**Theorem 6.2** (General Hahn-Banach). *Let  $X$  be a vector space over  $\mathbb{K}$ . Let  $p : X \rightarrow \mathbb{R}$  be a seminorm on  $X$ . Let  $Y \subseteq X$  be a subspace of  $X$ . If  $f : Y \rightarrow \mathbb{K}$  is a linear functional such that*

$$|f(y)| \leq p(y) \quad \forall y \in Y$$

*then there is an extension  $\tilde{f} : X \rightarrow \mathbb{K}$  such that*

- $\tilde{f}$  is linear
- $\tilde{f}(y) = f(y) \quad \forall y \in Y$
- $|\tilde{f}(x)| \leq p(x) \quad \forall x \in X$

*Remark.* This is great.

- $Y$  can be finite dimensional (and we know about linear functionals on finite dimensional spaces)
- If  $p(x) = \|x\|$ , then

$$|\tilde{f}(x)| \leq \|x\| \quad \forall x \in X$$

and so  $\tilde{f} \in X'$

**Corollary.** *Let  $(X, \|\cdot\|)$  be a normed vector space over  $\mathbb{K}$ . For each  $y \in X$ , with  $y \neq 0$ , there is  $\varphi \in X'$  such that*

$$\varphi(y) = \|y\| \quad \text{and} \quad \|\varphi\| = 1$$

*Proof.* Fix  $y \neq 0$  in  $X$ . Let  $Y = \{\mathbb{K}y\} = \{\lambda y | \lambda \in \mathbb{K}\}$ , a one-dimensional subspace.

Define  $f : Y \rightarrow \mathbb{K}$ ,  $f(\lambda y) = \lambda \|y\|$ . This is linear. Set  $p(x) = \|x\|$ . Then

$$|f(\lambda y)| = p(\lambda y)$$

and so by Hahn-Banach, there exists  $\tilde{f} : X \rightarrow \mathbb{K}$  such that

- $\tilde{f}$  is linear
- $\tilde{f}(\lambda y) = f(\lambda y) \quad \forall \lambda \in \mathbb{K}$
- $|\tilde{f}(x)| \leq \|x\| \quad \forall x \in X$

Then we have  $\tilde{f} \in X'$  and  $\|\tilde{f}\| = 1$  as required. □

## 6.2. Zorn's Lemma.

**Theorem 6.3** (Axiom of Choice is equivalent to Zorn's Lemma). *See handout for proof that*

$$A.C. \Rightarrow Z.L.$$

**Definition 6.4** (Partially ordered set). A **partially ordered set** (poset) is a set  $A$  with a relation  $\leq$  such that

- (1)  $a \leq a$  for all  $a \in A$ ,
- (2) If  $a \leq b$  and  $b \leq a$  then  $a = b$ ,

(3) If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$

**Definition 6.5** (Totally ordered set). A **totally ordered set** is a poset  $(A, \leq)$  such that if  $a, b \in A$  then either  $a \leq b$  or  $b \leq a$ .

**Definition 6.6** (Chain). A **chain** in a poset  $(A, \leq)$  is a totally ordered subset of  $A$ .

**Definition 6.7** (Upper bound). Let  $(A, \leq)$  be a poset. An **upper bound** for  $B \subseteq A$  is an element  $u \in A$  such that  $b \leq u$  for all  $b \in B$ .

**Definition 6.8** (Maximal element). A **maximal element** of a poset  $(A, \leq)$  is an element  $m \in A$  such that  $m \leq x$  implies  $x = m$ , that is,

$$m \leq x \Rightarrow x = m$$

**Example 6.9.** Let  $S$  be any set. Let  $\mathcal{P}(S)$  be the power set of  $S$  (the set of all subsets of  $S$ ). Define  $a \leq b \iff a \subseteq b$ . Maximal element is  $S$

**Theorem 6.10** (Zorn's Lemma). *Let  $(A, \leq)$  be a poset. Suppose that every chain in  $A$  has an upper bound. Then  $A$  has (at least one) maximal element.*

**Example 6.11** (Application - all vector spaces have a basis).

**Definition 6.12** (Linearly independent). Let  $X$  be a vector space over  $\mathbb{F}$ . We call  $B \subseteq X$  **linearly independent** if

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0$$

for all finite  $\{x_1, \dots, x_n\} \subseteq B$ .

**Definition 6.13** (Span). We say  $B \subseteq X$  **spans**  $X$  if each  $x \in X$  can be written as

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

for some  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  and  $\{x_1, \dots, x_n\} \subseteq B$ .

**Definition 6.14** (Hamel basis). A Hamel basis is a linearly independent spanning set. Equivalently,  $B \subseteq X$  is a Hamel basis if and only if each  $x \in X$  can be written in exactly one way as a finite linear combination of elements of  $B$ .

**Theorem 6.15.** *Every vector space has a Hamel basis*

*Proof.* Let  $L = \{\text{linearly independent subsets}\}$ , with subset ordering. Let  $C$  be a chain in  $L$ . Let  $u = \bigcup_{a \in C} a$ . Then

- (1)  $u \in L$ ,
- (2)  $u$  is an upper bound for  $C$ .

So Zorn's Lemma says that  $L$  has a maximal element  $\mathbf{b}$ .

Then  $\mathbf{b}$  is a Hamel basis.

- $\mathbf{b}$  is linearly independent.
- If  $\text{Span}(\mathbf{b}) \neq X$ , there exists  $X \in X \setminus \text{Span}(\mathbf{b})$ , and  $\mathbf{b}' = \mathbf{b} \cup \{x\} \in L$  is linearly independent, contradicting maximality of  $\mathbf{b}$ .

□

*Remark.* If  $X, \|\cdot\|$  is Banach, every Hamel basis is uncountable.

## 7. LECTURE 7 - MONDAY 21 MARCH

Proof of Hahn-Banach Theorem Discussion of Dual operators

**Theorem 7.1** (Hahn-Banach theorem over  $\mathbb{R}$ ). *Let  $X$  be a real linear space and let  $p(x)$  be a seminorm on  $X$ . Let  $M$  be a real linear subspace of  $X$  and  $f_0$  a real-valued linear functional defined on  $M$ . Let  $f_0$  satisfy  $f_0(x) \leq p(x)$  on  $M$ . Then there exists a real valued linear functional  $F$  defined on  $X$  such that*

- (i)  $F$  is an extension of  $f_0$ , that is,  $F(x) = f_0(x)$  for all  $x \in M$ , and
- (ii)  $F(x) \leq p(x)$  on  $X$ .

*Proof.* We first show that  $f_0$  can be extended if  $M$  has codimension one. Let  $x_0 \in X \setminus M$  and assume that  $\text{span}(M \cup \{x_0\}) = X$ . As  $x_0 \notin M$  we can write  $x \in X$  uniquely in the form

$$x = m + \alpha x_0$$

for  $\alpha \in \mathbb{R}$ . Then for every  $c \in \mathbb{R}$ , the map  $f_c \in \text{Hom}(X, \mathbb{R})$  given by  $f_c(m + \alpha x) = f_0(m) + c\alpha$  is well defined, and  $f_c(m) = f_0(m)$  for all  $m \in M$ . We now show that we can choose  $c \in \mathbb{R}$  such that  $f_c(x) \leq p(x)$  for all  $x \in X$ . Equivalently, we must show

$$f_0(m) + c\alpha \leq p(m + \alpha x_0)$$

for all  $m \in M$  and  $\alpha \in \mathbb{R}$ . By positive homogeneity of  $p$  and linearity of  $f$  we have

$$\begin{aligned} f_0(m/\alpha) + c &\leq p(x_0 + m/\alpha) \quad \alpha > 0 \\ f_0(-m/\alpha) - c &\leq p(-x_0 - m/\alpha) \quad \alpha < 0 \end{aligned}$$

Hence we need to choose  $c$  such that

$$\begin{aligned} c &\leq p(x_0 + m) - f_0(m) \\ c &\geq -p(-x_0 + m) + f_0(m). \end{aligned}$$

This is possible if

$$-p(-x_0 + m_1) + f_0(m_1) \leq p(x_0 + m_2) - f_0(m_2)$$

for all  $m_1, m_2 \in M$ . By subadditivity of  $p$  we can verify this condition since

$$f_0(m_1 + m_2) \leq p(m_1 m_2) = p(m_1 - x_0 + m_2 - x_0) \leq p(m_1 - x_0) + p(m_2 + x_0)$$

for all  $m_1, m_2 \in M$ . Hence  $c$  can be chosen as required.

Hence  $D(F) = X$ , and the theorem is proven.  $\square$

**Theorem 7.2** (Hahn-Banach over  $\mathbb{C}$ ). *Suppose that  $c$  is a seminorm on a complex vector space  $X$  and let  $M$  sub a subspace of  $X$ . If  $f_0 \in \text{Hom}(M, \mathbb{C})$  is such that  $|f_0(x)| \leq p(x)$  for all  $x \in M$ , then there exists an extension  $f \in \text{Hom}(X, \mathbb{C})$  such that  $f|_M = f_0$  and  $|f(x)| \leq p(x)$  for all  $x \in X$ .*

*Proof.* Split  $f_0$  into real and imaginary parts

$$f_0(x) = g_0(x) + ih_0(x).$$

By linearity of  $f_0$  we have

$$\begin{aligned} 0 &= if_0(x) - f_0(ix) = ig_0(x) - h_0(x) - g_0(ix) - ih_0(ix) \\ &= -(g_0(ix) + h_0(x)) + i(g_0(x) - h_0(ix)) \end{aligned}$$

and so  $h_0(x) = -g_0(ix)$ . Therefore,

$$f_0(x) = g_0(x) - ig_0(ix)$$

for all  $x \in M$ . We now consider  $X$  as a vector space over  $\mathbb{R}$ ,  $X_{\mathbb{R}}$ . Now considering  $M_{\mathbb{R}}$  as a subspace of  $X_{\mathbb{R}}$ . Since  $g_0 \in \text{Hom}(M_{\mathbb{R}}, \mathbb{R})$  and  $g_0(x) \leq |f_0(x)| \leq p(x)$  and so by the real Hahn-Banach, there exists  $g \in \text{Hom}(X_{\mathbb{R}}, \mathbb{R})$  such that  $g|_{M_{\mathbb{R}}} = g_0$  and  $g(x) \leq p(x)$  for all  $x \in X_{\mathbb{R}}$ . Now set  $F(x) = g(x) - ig(ix)$  for all  $x \in X_{\mathbb{R}}$ . Then by showing  $f(ix) = if(x)$ , we have that  $f$  is linear.

We now show  $|f(x)| \leq p(x)$ . For a fixed  $x \in X$  choose  $\lambda \in \mathbb{C}$  such that  $\lambda f(x) = |f(x)|$ . Then since  $|f(x)| \in \mathbb{R}$  and by definition of  $f$ , we have

$$|f(x)| = \lambda f(x) = f(\lambda x) = g(\lambda x) \leq p(\lambda x) = |\lambda| p(x) = p(x)$$

as required.  $\square$

## 8. LECTURE 8 - WEDNESDAY 23 MARCH

**Definition 8.1** (Inner product). Let  $X$  be a vector space over  $\mathbb{K}$ . An **inner product** is a function

$$\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$$

such that

- (1)  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- (2)  $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$
- (3)  $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- (4)  $\langle x, x \rangle \geq 0$  with equality if and only if  $x = 0$

We then have

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

and

$$\langle x, \alpha z \rangle = \bar{\alpha} \langle x, z \rangle$$

**Definition 8.2** (Inner product space). Let  $(X, \langle \cdot, \cdot \rangle)$  be an **inner product space**. Defining  $\|x\| = \sqrt{\langle x, x \rangle}$  turns  $X$  into a normed vector space. To prove the triangle inequality, we use the Cauchy-Swartz theorem.

**Theorem 8.3** (Cauchy-Schwarz). *In an inner product space  $(X, \langle \cdot, \cdot \rangle)$ , we have*

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \forall x, y \in X$$

*Proof.*

$$\begin{aligned} 0 &\leq \langle x - \lambda y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \langle x, \lambda y \rangle - \langle \lambda y, x \rangle + \langle \lambda y, \lambda y \rangle \\ &= \|x\|^2 - \bar{\lambda} \langle x, y \rangle - \lambda \langle y, x \rangle + |\lambda|^2 \|y\|^2 \\ &= \|x\|^2 - 2\operatorname{Re}(\lambda \langle y, x \rangle) + |\lambda|^2 \|y\|^2 \end{aligned}$$

Set  $\lambda = \frac{\langle x, y \rangle}{\|y\|^2}$ . Then

$$\begin{aligned} 0 &\leq \|x\|^2 - 2\operatorname{Re}\left(\frac{|\langle x, y \rangle|^2}{\|y\|^2}\right) + \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \end{aligned}$$

as required. □

**Corollary.**

$$\|x + y\| \leq \|x\| + \|y\|$$

**Definition 8.4** (Hilbert space). If  $(X, \langle \cdot, \cdot \rangle)$  is complete with respect to  $\|\cdot\|$  then it is called a **Hilbert space**.

**Example 8.5.** (a)  $\ell^2$ , where  $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$ .

Cauchy-Schwarz then says

$$\left| \sum_{i=1}^{\infty} x_i \bar{y}_i \right| \leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \sqrt{\sum_{i=1}^{\infty} |y_i|^2}$$

(b)  $L^2([a, b])$ , where  $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$ .

Cauchy-Swartz then says

$$\left| \int_a^b f(x) \overline{g(x)} dx \right| \leq \dots$$

**Definition 8.6** (Orthogonality). Let  $(X, \langle \cdot, \cdot \rangle)$  be inner product spaces. Then  $x, y \in X$  are orthogonal if  $\langle x, y \rangle = 0$  where  $x, y \neq 0$ .

**Theorem 8.7.** Let  $x_1, \dots, x_n$  be pairwise orthogonal elements in  $(X, \langle \cdot, \cdot \rangle)$ . Then

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

**Theorem 8.8** (Parallelogram identity). In  $(X, \langle \cdot, \cdot \rangle)$  we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (\star)$$

for all  $x, y \in X$ .

*Remark.* If  $(X, \|\cdot\|)$  is a normed vector space which satisfies parallelogram identity then  $X$  is an inner product space with inner products defined by the polarisation equation

$$\langle x, y \rangle = \begin{cases} \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) & \mathbb{K} = \mathbb{C} \end{cases}$$

**Definition 8.9** (Projection). Let  $X$  be a vector space over  $\mathbb{K}$ . A subset  $M$  of  $X$  is convex if for any  $x, y \in M$ , then

$$tx + (1 - t)y \in M \quad \forall t \in [0, 1]$$

**Theorem 8.10** (Projection). Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Let  $M \subseteq \mathcal{H}$  be closed and convex. Let  $x \in \mathcal{H}$ . Then there exists a unique point  $m_x \in M$  which is closest to  $x$ , i.e.

$$\|x - m_x\| = \inf_{m \in M} \|x - m\| = d$$

*Proof.* For each  $k \geq 1$  choose  $m_k \in M$  such that

$$d^2 \leq \|x - m_k\|^2 \leq d^2 + \frac{1}{k}$$

Each  $m_k$  exists as  $d$  is defined as the infimum over all  $m$ .

Then

$$\begin{aligned} \|m_k - m_l\|^2 &= \|(m_k - x) - (m_l - x)\|^2 \\ &= 2\|m_k - x\|^2 + 2\|m_l - x\|^2 - \|m_k + m_l - 2x\|^2 \\ &\leq 2d^2 + \frac{2}{l} + 2d^2 + \frac{2}{k} - 4\left\|\frac{m_k + m_l}{2} - x\right\|^2 \end{aligned}$$

and as  $m_k/2 + m_l/2 \in M$ , we have  $\|\frac{m_k+m_l}{2} - x\|^2 \geq d^2$ . Then

$$\|m_k - m_l\|^2 \leq 2\left(\frac{1}{k} + \frac{1}{l}\right)$$

Thus  $(m_k)$  is Cauchy. So  $m_k \rightarrow m_x \in M$  as  $\mathcal{H}$  is complete and  $M$  is closed. We then have

$$\|x - m_x\| = d$$

and so now we show that  $m_x$  is unique.

Suppose that there exists  $m'_x \in M$  with  $\|x - m'_x\| = d$ . Then by the above inequality, we have

$$\|m_x - m'_x\|^2 = 2\|m_x - x\|^2 + 2\|m'_x - x\|^2 - 4\left\|\frac{m_x - m'_x}{2} - x\right\|^2 \leq 0$$

from above. □

**Definition 8.11** (Projection operator). Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Let  $M \subseteq \mathcal{H}$  be closed and convex. Define

$$P_M : \mathcal{H} \rightarrow \mathcal{H}$$

by  $P_M(x) = m_x$  from above. This is the projection of  $\mathcal{H}$  onto  $M$ .

**Definition 8.12** (Orthogonal decomposition). If  $S \subseteq \mathcal{H}$ , let

$$S^\perp = \{x \in \mathcal{H} \mid \langle x, y \rangle = 0 \quad \forall y \in S\}.$$

We call  $S^\perp$  the orthogonal component.

## 9. LECTURE 9 - MONDAY 28 MARCH

**Theorem 9.1** (From previous lecture). *If  $M \subseteq \mathcal{H}$ , then the projection of  $\mathcal{H}$  onto  $M$  is*

$$P_m : \mathcal{H} \rightarrow \mathcal{H}$$

$$x \mapsto m_x$$

where  $m_x \in M$  is the unique element with  $\|x - m_x\| = \inf_{m \in M} \|x - m\|$ .

**Lemma 9.2.** *Let  $M \subseteq \mathcal{H}$  be closed subspace. Then  $x - P_M x \in M^\perp$  for all  $x \in \mathcal{H}$ .*

*Proof.* Let  $m \in M$ . We need to show  $\langle x - P_M x, m \rangle = 0$ . This is clear if  $m = 0$ . Without loss of generality, assuming  $m \neq 0$ , we can assume  $\|m\| = 1$ . Then write

$$x - P_M x = x - (P_M x + \langle x - P_M x, m \rangle m) + \langle x - P_M x, m \rangle m.$$

Let the bracketed term be  $m'$ . Then  $x - m' \perp \langle x - P_M x, m \rangle m$  because

$$\begin{aligned} \langle x - m', \langle x - P_M x, m \rangle m \rangle &= \overline{\langle x - P_M x, m \rangle} \langle x - m', m \rangle \\ &= C \langle x - P_M x - \langle x - P_M x, m \rangle m, m \rangle \\ &= C (\langle x - P_M x, m \rangle - \langle x - P_M x, m \rangle \|m\|) \\ &= 0. \end{aligned}$$

So  $\|x - P_M x\|^2 = \|x - m'\|^2 + |\langle x - P_M x, m \rangle|^2$ . So  $\|x - P_M x\|^2 \geq \|x - P_M x\|^2 + |\langle x - P_M x, m \rangle|^2$  by definition of  $P_M x$ . Thus,

$$\langle x - P_M x, m \rangle = 0$$

and thus  $x - P_M x \in M^\perp$ . □

**Theorem 9.3.** *The following theorem is the key fundamental result. Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Let  $M$  be a closed subspace of  $\mathcal{H}$ . Then*

$$\mathcal{H} = M \oplus M^\perp.$$

*That is, each  $x \in \mathcal{H}$  can be written in exactly one way as  $x = m + m^\perp$  with  $m \in M$ ,  $m^\perp \in M^\perp$ .*

*Proof. Existence* - Let  $x = P_M x + (x - P_M x)$ .

**Uniqueness** - Let  $x = x_1 + x_1^\perp$ ,  $x = x_2 + x_2^\perp$  with  $x_1, x_2 \in M$ ,  $x_1^\perp, x_2^\perp \in M^\perp$ . Then

$$x_1 - x_2 = x_2^\perp - x_1^\perp \in M^\perp$$

Then

$$\langle x_1 - x_2, x_1 - x_2 \rangle = 0 \Rightarrow x_1 = x_2.$$

Thus  $x_1^\perp = x_2^\perp$ . □

**Corollary.** *Let  $M \subseteq \mathcal{H}$  be a closed subspace. Then we have*

- (a)  $P_M \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ .
- (b)  $\|P_M\| \leq 1$ .
- (c)  $\text{Im } P_M = M$ ,  $\text{KER } P_M = M^\perp$ .
- (d)  $P_M^2 = P_M$ .
- (e)  $P_{M^\perp} = I - P_M$ .

*Proof.* (c), (d), (e) exercises.

(a). Let  $x, y \in H$ . Write  $x = x_1 + x_1^\perp$  and  $y = y_1 + y_1^\perp$  with  $x_1, y_1 \in M$  and  $x_1^\perp, y_1^\perp \in M^\perp$ . Then

$$x = y = (x_1 + y_1) + (x_1^\perp + y_1^\perp)$$

and so

$$P_M(x + y) = x_1 + y_1$$



and similarly  $P_M(\alpha x) = \alpha P_M x$ . We also have

$$\begin{aligned}\|x\|^2 &= \|P_M x + (x - P_M x)\|^2 \\ &= \|P_M x\|^2 + \|x - P_M x\|^2 \\ &\geq \|P_M x\|^2\end{aligned}$$

and so  $\|P_M\| \leq 1$ . □

**9.1. The dual of a Hilbert space.** If  $y \in \mathcal{H}$  is fixed, then the map

$$\begin{aligned}\varphi_y : \mathcal{H} &\rightarrow \mathbb{K} \\ x &\mapsto \langle x, y \rangle\end{aligned}$$

is in  $\mathcal{H}'$ . Linearity is clear, and continuity is proven by Cauchy-Swartz,

$$|\varphi_y(x)| = |\langle x, y \rangle| \leq \|y\| \|x\|.$$

So  $\|\varphi_y\| \leq \|y\|$ . Since  $|\varphi_y(y)| = \|y\|^2$ , we then have

$$\|\varphi_y\| = \|y\|.$$

**Theorem 9.4** (Riesz Representation Theorem). *Let  $\mathcal{H}$  be a Hilbert space. The map*

$$\begin{aligned}\theta : \mathcal{H} &\rightarrow \mathcal{H}' \\ y &\mapsto \varphi_y\end{aligned}$$

*is a conjugate linear bijection, and  $\|\varphi_y\| = \|y\|$ .*

*Proof.* Conjugate linearity is clear.

**Injectivity**

$$\varphi_y = \varphi_{y'} \Rightarrow \varphi_y(x) = \varphi_{y'}(x) \quad \forall x$$

so

$$\langle x, y - y' \rangle = 0 \quad \Rightarrow \langle y - y', y - y' \rangle = 0$$

and so  $y = y'$ .

**Surjectivity** Let  $\varphi \in \mathcal{H}'$ . We now find  $y \in \mathcal{H}$  with  $\varphi = \varphi_y$ . If  $\varphi = 0$ , take  $y = 0$ . Suppose  $\varphi \neq 0$ . Then  $\text{KER } \varphi \neq \mathcal{H}$ . But  $\text{KER } \varphi$  is a closed subspace of  $\mathcal{H}$ . So

$$H = (\text{KER } \varphi) \oplus (\text{KER } \varphi)^\perp.$$

Hence  $(\text{KER } \varphi)^\perp \neq \{0\}$ . Pick  $z \in (\text{KER } \varphi)^\perp, z \neq 0$ . For each  $x \in \mathcal{H}$ , the element

$$x - \frac{\varphi(x)}{\varphi(z)} z \in \text{KER } \varphi$$

Note that  $\varphi(z) \neq 0$  since  $z \notin \text{KER } \varphi$ . Then

$$\begin{aligned} 0 &= \langle x - \frac{\varphi(x)}{\varphi(z)}z, z \rangle \\ &= \langle x, z - \frac{\varphi(x)}{\varphi(z)}\|z\|^2 \rangle \end{aligned}$$

and so

$$\varphi(x) = \langle x, \frac{\overline{\varphi(z)}}{\|z\|^2}z \rangle \quad \forall x \in \mathcal{H},$$

and so letting  $y = \frac{\overline{\varphi(z)}}{\|z\|^2}z$ , we have  $\varphi = \varphi_y$ . □

**Example 9.5.** From Hahn-Banach given  $y \in \mathcal{H}$  there exists  $\varphi \in \mathcal{H}'$  such that

$$\|\varphi\| = 1$$

and  $\varphi(y) = \|y\|$ . We can be very constructive in the Hilbert case, and let

$$\varphi(x) = \langle x, \frac{y}{\|y\|} \rangle$$

**Example 9.6.** All continuous linear functionals on  $L^2([a, b])$  are of the form

$$\varphi(f) = \int_a^b f(x) \overline{g(x)} dx$$

for some  $g \in L^2([a, b])$ .

**Example 9.7** (Adjoint operators). Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces. Let  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ . The **adjoint** of  $T$  is  $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$  given by

$$\langle Tx, y \rangle_2 = \langle x, T^*y \rangle_1$$

for all  $x \in \mathcal{H}_1, y \in \mathcal{H}_2$

**Exercise 9.8.** Check all of the above.

**Exercise 9.9.** Prove  $T^* = \overline{T^t}$  where  $T^t$  is the transpose.

## 10. LECTURE 10 - WEDNESDAY 30 MARCH

**Definition 10.1** (Orthonormal system). A subset  $S \subseteq \mathcal{H}$  is an **orthonormal system** (orthonormal) if

$$\langle e, e' \rangle = \delta_{e, e'} \quad \forall e, e' \in S$$

**Definition 10.2** (Complete orthonormal system or Hilbert basis). An orthonormal system  $S$  is **complete** or a **Hilbert basis** if

$$\overline{\text{span } S} = \mathcal{H}$$

*Remark.* By Gram-Schmidt and Zorn's Lemma, every Hilbert space has a complete orthonormal system.

**Example 10.3.** (1)  $\ell^2$ . Then

$$S = \{e_i \mid i \geq 1\}$$

is orthonormal and is complete.

(2)  $L^2_{\mathbb{C}}([0, 2\pi])$ . Then

$$S = \left\{ \frac{1}{2\pi} e^{int} \mid n \in \mathbb{Z} \right\}$$

is orthonormal and is complete. Completeness follows from Stone-Weierstrass theorem.

(3)  $L^2_{\mathbb{R}}([0, 2\pi])$ . Then

$$S = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt \mid n \geq 1 \right\}$$

is orthonormal and is complete, again by Stone-Weierstrass.

We want to look at series  $\sum_{e \in S} \dots$ , which is tricky if  $S$  is not countable.

**Lemma 10.4.** *If  $\{e_k \mid k \geq 0\}$  is orthonormal, then*

$$\sum_{k=0}^{\infty} a_k e_k$$

*converges in  $\mathcal{H}$  if and only if*

$$\sum_{k=0}^{\infty} |a_k|^2$$

*converges in  $\mathbb{K}$ .*

*If either series converges, then*

$$\left\| \sum_{k=0}^{\infty} a_k e_k \right\|^2 = \sum_{k=0}^{\infty} |a_k|^2$$

*Note.* If  $x_n \rightarrow x, y_n \rightarrow y$ , then

$$\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$$

*Proof.* If  $\sum_{k=0}^{\infty} a_k e_k$  converges to  $x$ , then

$$\begin{aligned} \langle x, x \rangle &= \lim_{n \rightarrow \infty} \left\langle \sum_{k=0}^n a_k e_k, \sum_{k=0}^n a_k e_k \right\rangle \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n |a_k|^2 \end{aligned}$$

Conversely, if  $\sum_{k=0}^{\infty} |a_k|^2$  converges, then writing  $x_n = \sum_{k=0}^n a_k e_k$ , we have

$$\begin{aligned} \|x_m - x_n\|^2 &= \left\| \sum_{k=n+1}^m a_k e_k \right\|^2 \\ &= \sum_{k=n+1}^m \|a_k e_k\|^2 \text{ by Pythagoras} \\ &= \sum_{k=n+1}^m |a_k|^2 \rightarrow 0 \end{aligned}$$

and so  $(x_n)$  is Cauchy, and hence converges by completeness of  $\mathcal{H}$ . □

**Lemma 10.5.** *Let  $\{e_1, \dots, e_n\}$  be orthonormal. Then*

$$\sum_{k=1}^n |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

for each  $x \in \mathcal{H}$ .

*Proof.* Let  $y = \sum_{k=1}^n \langle x, e_k \rangle e_k$ . Let  $z = x - y$ . We claim that  $z \perp y$ . We have

$$\begin{aligned} \langle x, y \rangle &= \langle x - y, y \rangle \\ &= \langle x, y \rangle - \|y\|^2 \\ &= \sum_{k=1}^n \overline{\langle x, e_k \rangle} \langle x, e_k \rangle - \sum_{k=1}^n |\langle x, e_k \rangle|^2 \\ &= 0. \end{aligned}$$

So

$$\begin{aligned} \|x\|^2 &= \|y + z\|^2 \\ &= \|y\|^2 + \|z\|^2 \text{ Pythagoras} \\ &\geq \|y\|^2 = \sum_{k=1}^n |\langle x, e_k \rangle|^2 \end{aligned}$$

□

We want to write expressions like  $\sum_{e \in S} \langle x, e \rangle e$ .

**Corollary.** *Let  $x \in \mathcal{H}$  and  $S$  orthonormal. Then*

$$\{e \in S \mid \langle x, e \rangle \neq 0\}$$

*is countable.*

*Proof.*

$$\{e \in S \mid \langle x, e \rangle \neq 0\} = \bigcup_{k \geq 1} \{e \in S \mid |\langle x, e \rangle| > \frac{1}{k}\}$$

From the lemma,

$$\#\{e \in S \mid |\langle x, e \rangle| > \frac{1}{k}\} \leq k^2 \|x\|^2$$

For if this number were greater than  $k^2 \|x\|^2$ , then the LHS in Lemma is greater than  $\frac{1}{k^2} k^2 \|x\|^2$ .  $\square$

Therefore:

**Corollary** (Bessel's Inequality). *If  $S$  is orthonormal, then*

$$\sum_{e \in S} |\langle x, e \rangle|^2 \leq \|x\|^2$$

*for all  $x \in \mathcal{H}$*

*Proof.*  $\sum_{e \in S} |\langle x, e \rangle|^2$  is a sum of countably many positive terms, and so order is not important.  $\square$

We want to write  $\sum_{e \in S} \langle x, e \rangle e$ . This sum is over a countable set, but is the order important?

**Theorem 10.6.** *Let  $S$  be orthonormal. Let  $M = \overline{\text{span } S}$ . Then*

$$P_M x = \sum_{e \in S} \langle x, e \rangle e$$

*where the sum can be taken in any order.*

*Proof.* Fix  $x \in H$ . Choose an enumeration

$$\{e_k \mid k \geq 0\} = \{e \in S \mid \langle x, e \rangle \neq 0\}.$$

By Bessel's inequality, we have

$$\sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

and so the LHS converges. By Lemma 10.4, we know

$$y = \sum_{k=0}^{\infty} \langle x, e_k \rangle e_k \in M$$

converges in  $\mathcal{H}$ .

Write  $x = y + (x - y) = M + M^\perp$ . We claim  $(x - y) \in M^\perp$ . Then  $P_M x = y$  from characterisation of projection operator. Let  $e \in S$ . Then

$$\begin{aligned}\langle x - y, e \rangle &= \lim_{n \rightarrow \infty} \left\langle x - \sum_{k=0}^n \langle x, e_k \rangle e_k, e \right\rangle \\ &= \lim_{n \rightarrow \infty} (\langle x, e \rangle - \sum_{k=0}^n \langle x, e_k \rangle \langle e_k, e \rangle) \\ &= \langle x, e \rangle - \sum_{k=0}^{\infty} \langle x, e_k \rangle \langle e_k, e \rangle.\end{aligned}$$

If  $e \in \{e' \in S \mid \langle x, e' \rangle \neq 0\}$ , then  $e = e_j$  for some  $j$ , and so

$$\langle x - y, e \rangle = \langle x, e_j \rangle - \langle x, e_j \rangle = 0$$

If  $\langle x, e \rangle = 0$ , then  $e \neq e_j$  for all  $j$ , and so  $\langle e_j, e \rangle = 0$ , and so

$$\langle x - y, e \rangle = 0 - 0 = 0.$$

Thus  $x - y \in (\text{span } S)^\perp$ .

**Exercise 10.7.** Show that

$$x - y \in \overline{(\text{span } S)}^\perp = M^\perp.$$

□

## 11. LECTURE 11 - MONDAY 4 APRIL

Recall that if  $\{x_1, \dots\}$  is a countable orthonormal system in a Hilbert space  $\mathcal{H}$ . Then

$$\sum_{k=1}^{\infty} a_k e_k < \infty \iff \sum_{k=1}^{\infty} |a_k|^2 < \infty$$

and

$$\left\| \sum_{k=1}^{\infty} a_k e_k \right\|^2 = \sum_{k=1}^{\infty} |a_k|^2 \quad (\star)$$

We also had the following.

**Theorem 11.1.** *Let  $S$  be orthonormal in  $\mathcal{H}$ . Let  $M = \overline{\text{span } S}$ . Then*

$$P_M x = \sum_{e \in S} \langle x, e \rangle e \quad \forall x \in \mathcal{H}$$

*where the sum has only countable many terms and convergence is unconditional.*

**Theorem 11.2.** *Let  $S$  be orthonormal in  $\mathcal{H}$ . Then following are equivalent.*

(a)  *$S$  is a complete orthonormal system ( $\overline{\text{span } S} = \mathcal{H}$ ).*

(b)  $x = \sum_{e \in S} \langle x, e \rangle e$  for all  $x$  (Fourier series).

(c)  $\|x\|^2 = \sum_{e \in S} |\langle x, e \rangle|^2$  for all  $x$  (Parseval's formula).

*Proof.* (a)  $\Rightarrow$  (b). If  $M = \overline{\text{span } S} = \mathcal{H}$ , then

$$P_M x = x = \sum_{e \in S} \langle x, e \rangle e$$

by Theorem 11.1.

(b)  $\Rightarrow$  (c). By the infinite Pythagoras theorem ( $\star$ ).

(c)  $\Rightarrow$  (a). Let  $M = \overline{\text{span } S}$ . Suppose that  $z \in M^\perp$ . Then  $z = 0 + z \in M + M^\perp$ . Hence

$$0 = \|P_M z\|^2 = \left\| \sum_{e \in S} \langle z, e \rangle e \right\|^2 = \sum_{e \in S} |\langle z, e \rangle|^2 = \|z\|^2$$

which implies  $z = 0$ , so  $M = \mathcal{H}$ , and so  $S$  is complete.  $\square$

*Remark.* Consider  $L^2([0, 2\pi])$ , and let  $S = \{e_n \mid n \in \mathbb{Z}\}$ . Then we can write

$$f = \sum_{n \in \mathbb{Z}} c_n e_n$$

where  $c_n = \langle f, e_n \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) e^{-int} dt$ .

We do not claim that convergence is pointwise, what we have proven is convergence in  $L^2$ ,

$$\left\| f - \sum_{|n| \leq N} c_n e_n \right\|_2 \rightarrow 0$$

as  $N \rightarrow \infty$ . This is not the same as pointwise or uniform convergence ( $\|\cdot\|_\infty$ ).

**11.1. Stone-Weierstrass theorem.** This is a useful tool to show an orthonormal system is complete. In fact, this theorem is about uniformly approximating elements of  $\mathcal{C}(X)$ , where  $X$  is a compact Hausdorff space. It is a generalisation of the Weierstrass approximation theorem.

**Theorem 11.3** (Weierstrass approximation theorem). *Let  $f \in \mathcal{C}([a, b])$  and let  $\epsilon > 0$  be given. Then there exists a polynomial  $p(x)$  such that*

$$|f(x) - p(x)| < \epsilon \quad \forall x \in [a, b],$$

*that is,  $\|f - p\|_\infty < \epsilon$ .*

**Corollary.** *This implies the following important results:*

- Continuous functions can be uniformly approximated by polynomials.
- $\mathcal{P}([a, b])$ , the space of polynomials on  $[a, b]$ , is dense in  $\mathcal{C}([a, b])$ .
- $\overline{\mathcal{P}([a, b])} = \mathcal{C}([a, b])$ .

We now prove Stone's 1930's generalisation.

**First some setup:** Let  $X$  be a compact Hausdorff space throughout. We then know that  $\mathcal{C}(X)$  is a vector space. It also has sensible vector multiplication,

$$(fg)(x) = f(x)g(x).$$

Thus  $\mathcal{C}(X)$  is a unital, commutative, associative ring. As we have

$$f(\lambda g) = \lambda(fg)$$

then  $\mathcal{C}(X)$  is a unital, commutative, associative algebra over  $\mathbb{K}$ .

**Definition 11.4** (Subalgebra). A subalgebra of  $\mathcal{C}(X)$  is a subset  $\mathcal{A}$  which is closed under scalar multiplication, vector addition, and vector multiplication.  $\mathcal{A}$  is unital if it contains the constant function  $f(x) = 1$ .

**Example 11.5.**  $\mathcal{P}([a, b])$  is a subalgebra of  $\mathcal{C}([a, b])$ .

When is  $\mathcal{A}$  dense in  $\mathcal{C}(X)$ ?

**Theorem 11.6** (Stone-Weierstrass theorem). *Let  $X$  be a compact Hausdorff space, and let  $\mathcal{A}$  be a subalgebra of  $\mathcal{C}(X)$ . If*

- (1)  $\mathcal{A}$  is unital,
- (2)  $f \in \mathcal{A} \Rightarrow f^* \in \mathcal{A}$ , where  $f^*(x) = \overline{f(x)}$ ,
- (3)  $\mathcal{A}$  separates points of  $X$ .

*Then  $\overline{\mathcal{A}} = \mathcal{C}(X)$ .*

**Definition 11.7.**  $\mathcal{A}$  separates points of  $X$  if, given  $x \neq y$ , there is a function  $f \in \mathcal{A}$  with  $f(x) \neq f(y)$ .

**Corollary.** (a)  $\mathcal{P}([a, b])$  is dense in  $\mathcal{C}([a, b])$ , as  $f(x) = x$  separates points.  
 (b) Trigonometric polynomials are dense in

$$\{f \in \mathcal{C}([0, 2\pi]) \mid f(0) = f(2\pi)\}.$$

(c) Trigonometric polynomials are dense in  $L^2([0, 2\pi])$ , and

$$S = \{e_n \mid n \in \mathbb{Z}\}$$

*is complete.*

### Setup

**Lemma 11.8.** *The function  $f(t) = |t|$  can be uniformly approximated by polynomials on  $[-1, 1]$*

*Proof.* The binomial theorem says

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \quad \forall x \in [-1, 1]$$



We then have

$$|t| = \sqrt{t^2} = \sqrt{1 + (t^2 - 1)} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (t^2 - 1)^n \quad t \in [-\sqrt{2}, \sqrt{2}]$$

Now let  $p_N(t) = \sum_{n=0}^N \binom{\frac{1}{2}}{n} (t^2 - 1)^n$ , and

$$||t| - p_N(t)| = \left| \sum_{n=N+1}^{\infty} \binom{\frac{1}{2}}{n} (t^2 - 1)^n \right| \leq \sum_{n=N+1}^{\infty} \left| \binom{\frac{1}{2}}{n} \right|$$

and so  $||t| - p_N|_{\infty} \rightarrow 0$  as  $N \rightarrow \infty$  on  $[-1, 1]$ .  $\square$

## 12. LECTURE 12 - WEDNESDAY 6 APRIL

**Theorem 12.1** (Stone-Weierstrass theorem). *Let  $X$  be a compact Hausdorff space, and let  $\mathcal{A}$  be a subalgebra of  $\mathcal{C}(X)$ . If*

- (1)  $\mathcal{A}$  is unital,
- (2)  $f \in \mathcal{A} \Rightarrow f^* \in \mathcal{A}$ , where  $f^*(x) = \overline{f(x)}$ ,
- (3)  $\mathcal{A}$  separates points of  $X$ .

*Then  $\overline{\mathcal{A}} = \mathcal{C}(X)$ .*

*Proof.* We first prove for  $\mathcal{C}_{\mathbb{R}}(X)$ .

**Lemma 12.2.** *Let  $\mathcal{A}$  be a unital subalgebra of  $\mathcal{C}_{\mathbb{R}}(X)$ . Then*

- (a)  $|f| \in \overline{\mathcal{A}}$ ,
- (b)  $\min(f_1, \dots, f_n), \max(f_1, \dots, f_n) \in \overline{\mathcal{A}}$

*for all  $f, f_1, \dots, f_n \in \mathcal{A} \subseteq \mathcal{C}_{\mathbb{R}}(X)$ .*

*Proof.* (a) Replace  $f$  by  $\frac{f}{\|f\|_{\infty}}$  so we can assume that  $\|f\|_{\infty} = 1$ . From the previous lemma, we know for each  $n \geq 1$  there is a polynomial  $p_n : [-1, 1] \rightarrow \mathbb{R}$  such that  $||t| - p_n(t)| < \frac{1}{n}$  for all  $t \in [-1, 1]$ .

Since  $|f(x)| \leq \|f\|_{\infty} = 1$  for all  $x \in X$ , we have

$$||f| - p_n(f)| \leq \frac{1}{n}$$

But  $p_n(f)$  is a finite linear combination of  $1, f, f^2, f^3, \dots$  and so in  $\mathcal{A}$ , as  $\mathcal{A}$  is unital. Thus  $|f| \in \overline{\mathcal{A}}$ .

(b) Use the formulas

$$\max(f, g) = \frac{f + g + |f - g|}{2}, \quad \min(f, g) = \frac{f + g - |f - g|}{2} \in \overline{\mathcal{A}}$$

and induction.  $\square$

*Proof of Stone-Weierstrass for  $C_{\mathbb{R}}(X)$ .* Let  $f \in C_{\mathbb{R}}(X)$  and let  $\epsilon > 0$  be given. We need to find  $g \in \mathcal{A}$  such that

$$|f(z) - g(z)| < \epsilon \quad \forall z \in X$$

**Step 0.** We can assume that  $\mathcal{A}$  is closed.

**Exercise 12.3.** Why?

**Step 1.** Let  $x, y \in X$  be fixed.

**Proposition 12.4.** *There exists  $f_{xy} \in \mathcal{A}$  with*

$$f_{xy}(x) = f(x), \quad f_{xy}(y) = f(y)$$

*Proof.* If  $x = y$  then trivial (take  $f_{xy}(z) = f(x)\mathbf{1}(z)$ ).

If  $x \neq y$ , since  $\mathcal{A}$  separates points, there is  $h \in \mathcal{A}$  with  $h(x) \neq h(y)$ . Then take

$$f_{xy} = ah + b\mathbf{1} \in \mathcal{A}$$

we can invert the coefficient matrix to find our coefficients  $a$  and  $b$ . □

**Step 2.** Let  $x \in X$  be fixed.

**Proposition 12.5.** *There exists  $f_x \in \mathcal{A}$  such that*

- $f_x(x) = f(x)$ .
- $f_x(z) < f(z) + \epsilon$

*Proof.* For each  $y \in X$ , let

$$O_y = \{z \in X \mid f_{xy}(z) < f(z) + \epsilon\}$$

where  $f_{xy}$  is the function from Step 1. These are all open sets (why?) and thus

$$X = \bigcup_{y \in X} O_y$$

since  $y \in O_y$ .

By compactness of  $X$ , we have

$$X = \bigcup_{i=1}^m O_{y_i}$$

Letting  $f_x = \min(f_{xy_1}, \dots, f_{xy_m})$ . Then

- Since  $f_{xy_i}(x) = f(x)$  for all  $i$ ,

$$f_x(x) = f(x)$$

- If  $z \in X$ , then  $z \in O_{y_i}$  for some  $i$ , and so

$$f_x(z) \leq f_{xy_i}(z) < f(z) + \epsilon$$

as required.

□

**Step 3.****Proposition 12.6.** *There exists a function  $g \in \mathcal{A}$  such that*

$$|f(z) - g(z)| < \epsilon$$

*for all  $z \in X$ .**Proof.* For each  $x \in X$ , let

$$U_x = \{z \in X \mid f_x(z) > f(x) - \epsilon\}$$

where  $f_x$  is from Step 2. These sets  $U_x$  are open and since  $x \in U_x$ , for an open cover, we can write

$$X = \bigcup_{x \in X} U_x = \bigcup_{j=1}^n U_{x_j}.$$

Define  $g = \max(f_{x_1}, \dots, f_{x_n})$ . If  $z \in X$ ,

- $g(z) = f_{x_i}(z)$  for some  $i$ , which is less than  $f(z) + \epsilon$  from Step 2.
- If  $z \in U_{x_j}$  for some  $j = 1, \dots, n$ , then

$$g(z) \geq f_{x_j}(z) > f(x) - \epsilon.$$

□

□

**Exercise 12.7.** Where did we use the Hausdorff property?We now prove for  $\mathcal{C}_{\mathbb{C}}(X)$ .

Let

$$\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} \mid f \text{ is real valued}\}.$$

Then  $\mathcal{A}_{\mathbb{R}}$  is an  $\mathbb{R}$ -subalgebra of  $\mathcal{C}_{\mathbb{R}}(X)$ . It is unital, as  $1 \in \mathcal{A}$  and it is real valued.We now show  $\mathcal{A}_{\mathbb{R}}$  separates points. If  $x \neq y$ , there is  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ . Write  $f = u + iv$  with  $u, v$  real valued. Either  $u(x) \neq u(y)$  or  $v(x) \neq v(y)$ , and so  $\mathcal{A}_{\mathbb{R}}$  separates points.Hence  $\mathcal{A}_{\mathbb{R}}$  is dense in  $\mathcal{C}_{\mathbb{R}}$ .Now, let  $f \in \mathcal{C}_{\mathbb{C}}(X)$ . Then write  $f = u + iv$ . Then  $u, v \in \mathcal{C}_{\mathbb{R}}(X)$ . Then given  $\epsilon > 0$ , there exists  $u_1, v_1 \in \mathcal{A}_{\mathbb{R}}$  such that

$$\|u - u_1\|_{\infty} \leq \frac{\epsilon}{2}, \quad \|v - v_1\|_{\infty} \leq \frac{\epsilon}{2}$$

Writing  $f_1 = u_1 + iv_1 \in \mathcal{A}$ , we have

$$\|f - f_1\|_{\infty} \leq \|(u - u_1) + i(v - v_1)\|_{\infty} \leq \|u - u_1\|_{\infty} + \|v - v_1\|_{\infty} < \epsilon$$

and thus  $\mathcal{A}$  is dense in  $\mathcal{C}_{\mathbb{C}}(X)$ .

□

## 13. LECTURE 13 - MONDAY 11 APRIL

## 13.1. Applications of Stone-Weierstrass theorem.

**Corollary.** *Polynomials are dense in  $\mathcal{C}([a, b])$ .*

*Proof.*  $\mathcal{A} = \mathcal{P}([a, b])$  is an algebra, is unital, is closed under complex conjugation, and separates points. Thus,  $\mathcal{A}$  is dense in  $\mathcal{C}([a, b])$ .  $\square$

**Definition 13.1** (Trigonometric polynomials). A **trigonometric polynomial** is an expression

$$\sum_{n \in \mathbb{Z}} c_n e^{int}$$

with finitely many  $c_n \neq 0$ . So these are polynomials in  $s = e^{it}$  and  $s^{-1} = \bar{s} = e^{-it}$ .

**Corollary.** *The space  $\mathcal{A}$  of all trigonometric polynomials is dense in  $\mathcal{C}(\Pi)$ , where  $\Pi = \{z \in \mathbb{C} \mid |z| = 1\}$*

*Proof.*  $\mathcal{A}$  is a sub-algebra of  $\mathcal{C}(\Pi)$ , it is unital, closed under complex conjugation,

$$\overline{\sum_{n \in \mathbb{Z}} c_n e^{int}} = \sum_{n \in \mathbb{Z}} \overline{c_{-n}} e^{int}$$

and separates points.  $T$  is a compact Hausdorff space, and thus Stone-Weierstrass states that  $\mathcal{A}$  is dense in  $\mathcal{C}(\Pi)$ .  $\square$

**Corollary.** *The orthonormal system*

$$S = \left\{ \frac{1}{\sqrt{2\pi}} e^{int} \mid n \in \mathbb{Z} \right\}$$

*is complete in  $L^2([0, 2\pi])$ .*

*Proof.*  $\text{span } S = \mathcal{A}$  is a space of trigonometric polynomials, which is dense in  $\mathcal{C}(\Pi)$ . Define

$$\begin{aligned} \Phi : \mathcal{C}_p([0, 2\pi]) &\rightarrow \mathcal{C}(\Pi) \\ f &\mapsto \tilde{f} \end{aligned}$$

where  $\mathcal{C}_p([0, 2\pi]) = \{f \in \mathcal{C}([0, 2\pi]) \mid f(0) = f(2\pi)\}$ . Then  $\Phi$  is an isometric isomorphism, and therefore functions of the form  $f(t) = \sum c_n e^{int}$  is dense in  $\mathcal{C}_p([0, 2\pi])$ .

By the construction of the Lebesgue integral, **simple** functions

$$\sum_{i=1}^n a_i \mathbf{1}_{A_i}$$

are dense in  $L^2([0, 2\pi])$ .

**Exercise 13.2.** Given  $f \in L^2([0, 2\pi])$  and  $\epsilon > 0$ , there exists  $g \in \mathcal{C}_p([0, 2\pi])$  such that  $\|f - g\|_2 < \epsilon$ .

Thus  $\mathcal{A}$  is dense in  $L^2([0, 2\pi])$ . □

**Corollary.** *The following are separable (have a countable dense subset):*

- (a)  $\mathcal{C}([a, b])$ ,
- (b)  $L^p([a, b])$ ,  $1 \leq p < \infty$

*Proof.* (a) We have  $\mathcal{P}([a, b])$  is dense in  $\mathcal{C}([a, b])$  and set  $\mathcal{P}_{\mathbb{Q}}([a, b])$  with rational coefficients is dense in  $\mathcal{P}([a, b])$ . Clearly,  $\mathcal{P}_{\mathbb{Q}}([a, b])$  is countable, and thus is dense in  $\mathcal{C}([a, b])$ .

(b) Use the fact that  $\mathcal{C}([a, b])$  is dense in  $L^p([a, b])$ . □

**Corollary.** *Let  $X$  be a compact metric space. Then  $\mathcal{C}(X)$  is separable.*

*Proof.* As  $X$  is a compact metric space, then  $X$  is separable.

**Exercise 13.3.** Why?

Let  $\{x_n \mid n \geq 1\}$  be a countable dense subset of  $X$ . For each  $n \geq 1$  and  $m \geq 1$  define

$$f_{n,m} : X \rightarrow \mathbb{K}$$

by

$$f_{n,m}(x) = \inf_{z \notin B(x_n, \frac{1}{m})} d(x, z)$$

We then claim  $f_{n,m}$  is continuous. Now, let  $\mathcal{A}$  be the space of all  $\mathbb{K}$ -linear combinations of

$$f_{n_1, m_1}^{k_1}, \dots, f_{n_l, m_l}^{k_l}, k_1, \dots, k_l \in \mathcal{N}. \quad (\star)$$

This is a sub-algebra of  $\mathcal{C}(X)$ , as  $\mathcal{A}$  is unital, closed under conjugation, and separates points - if  $z_1, z_2 \in X$  with  $z_1 \neq z_2$ , Choose  $n, m$  such that  $z_1 \in B(x_n, \frac{1}{m})$ ,  $z_2 \notin B(x_n, \frac{1}{m})$ . Thus the sub-algebra  $\mathcal{A}$  is dense by Stone-Weierstrass.

The subset of  $\mathbb{Q}$ -linear combinations of  $(\star)$  is countable and dense. □

**Lemma 13.4.** *If  $X$  is compact metric space then  $X$  is separable.*

*Proof.* For each  $m \geq 1$ ,

$$X = \bigcup_{x \in X} B(x; \frac{1}{m})$$

has a finite subcover

$$X = \bigcup_{n=1}^{N_m} B(x_{m,n}, \frac{1}{m})$$

and thus the subset of all  $\{x_{m,n}\}$  is a countably dense subset. □

**Corollary.**

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

*Proof.*  $S = \{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\}$  is complete, and so Parseval's formula holds,

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2.$$

Apply to  $f(x) = x$ . □

A common strategy is to prove for polynomials, and then Stone-Weierstrass proves it for continuous functions.

**Corollary.** *If  $f \in \mathcal{C}([a, b] \times [c, d])$  then*

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

*Proof.* By direct calculation, the result is true for two-variable polynomials. Let  $f \in \mathcal{C}([a, b] \times [c, d])$  and  $\epsilon > 0$  be given. By Stone-Weierstrass, the space of polynomials in 2 variables is dense in  $\mathcal{C}([a, b] \times [c, d])$  and so there exists a polynomial  $p(x, y)$  with

$$|f(x, y) - p(x, y)| < \frac{\epsilon}{(b-a)(d-c)}.$$

The result then follows by direct calculation. □

#### 14. LECTURE 14 - WEDNESDAY 13 APRIL

The following is at the core of two of the cornerstone theorems of functional analysis - the uniform boundedness principle and the open mapping theorem.

**Theorem 14.1** (Baire's theorem). *Let  $X$  be a complete metric space. If  $U_1, U_2, \dots$  are open dense subsets of  $X$ , then*

$$U = \bigcap_{n=1}^{\infty} U_n$$

*is dense in  $X$ .*

*Proof.* Let  $x \in X$  and  $\epsilon > 0$  be given. We need to show that

$$B(x, \epsilon) \cap U \neq \emptyset.$$

**Lemma 14.2.** *There exists sequences  $(x_n)$  in  $X$  and  $(\epsilon_n)$  in  $\mathbb{R}^+$  with the property that*

- (a)  $x_1 = x, \epsilon_1 = \epsilon$ .
- (b)  $\epsilon_n \downarrow 0$
- (c)  $\overline{B(x_{n+1}, \epsilon_{n+1})} \subseteq B(x_n, \epsilon_n) \cap U_n$  for all  $n \geq 1$ .

*Proof.* Let  $x_1, \dots, x_n$  and  $\epsilon_1, \dots, \epsilon_n$  be chosen. By density of  $U_n$ ,

$$B(x_n, \epsilon_n) \cap U_n \neq \emptyset.$$

Choose  $x_{n+1} \in B(x_n, \epsilon_n) \cap U_n$ . Choose  $\epsilon'_{n+1} > 0$  such that  $B(x_{n+1}, \epsilon'_{n+1}) \subseteq B(x_n, \epsilon_n) \cap U_n$  (openness). We have  $\epsilon'_{n+1} \leq \epsilon_n$ . Choose  $0 \leq \epsilon_{n+1} \leq \min(\frac{\epsilon'_{n+1}}{2}, \frac{1}{n+1})$ , then we have

$$\begin{aligned}\overline{B}(x_{n+1}, \epsilon_{n+1}) &\subseteq B(x_{n+1}, \epsilon'_{n+1}) \\ &\subseteq B(x_n, \epsilon_n) \cap U_n\end{aligned}$$

and  $\epsilon_{n+1} < \epsilon_n$  with  $\epsilon_{n+1} < \frac{1}{n+1}$ . □

Given the lemma, the theorem follows. If  $m \geq n$ , then by (c),

$$B(x_m, \epsilon_m) \subseteq B(x_n, \epsilon_n) \cap U_n \quad (\star)$$

In particular,  $x_m \in B(x_n, \epsilon_n)$ . Thus,  $d(x_n, x_m) < \epsilon_n$  for all  $m \geq n$ . Thus  $(x_n)$  is Cauchy, and so  $x_n \rightarrow \zeta$  in  $X$  by completeness. By  $(\star)$ , we then have  $d(x_n, \zeta) \leq \epsilon_n$  for all  $n \geq 1$ . So  $\zeta \in \overline{B}(x_n, \epsilon_n)$ . So by (c),  $\zeta \in \overline{B}(x_{n+1}, \epsilon_{n+1}) \subseteq B(x_n, \epsilon_n) \cap U_n$ .

Thus  $\zeta \in B(x, \epsilon)$  and thus  $\zeta \in U = \bigcap_{n=1}^{\infty} U_n$ . □

The following corollary is often used

**Corollary.** *Let  $X$  be a complete metric space. If  $C_1, C_2, \dots$  are closed with  $X = \bigcup_{n=1}^{\infty} C_n$  then  $\text{Int}(C_n) \neq \emptyset$  for some  $n$ .*

*Proof.* If  $\text{Int}(C_n) = \emptyset$  for all  $n$  then  $U_n = X \setminus C_n$  are open and dense. So by Baire's theorem,  $\bigcap_{n=1}^{\infty} U_n$  is dense, and in particular,  $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$ . We have

$$\begin{aligned}X &= \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (X \setminus U_n) \\ &= X \setminus \left( \bigcap_{n=1}^{\infty} U_n \right) \\ &\subsetneq X,\end{aligned}$$

a contradiction. □

There are three cornerstone theorems.

- Hahn-Banach,
- Uniform Boundedness,
- Open Mapping.

**Theorem 14.3** (Uniform boundedness). *Let  $X, Y$  be Banach spaces. Let  $T_\alpha$ ,  $\alpha \in A$ , a family of continuous linear operators  $T_\alpha : X \rightarrow Y$ . Then if*

$$\sup_{\alpha \in A} \|T_\alpha x\| < \infty$$

for each fixed  $x \in X$ , then

$$\sup_{\alpha \in A} \|T_\alpha\| < \infty$$

*Remark.* Rather amazing - you get a global bound from pointwise bounds.

*Proof.* For each  $n \geq 1$ , let

$$X_n = \{x \in X \mid \|T_\alpha x\| \leq n \forall \alpha \in A\}$$

These are **closed** ( $T_\alpha$  is continuous) and

$$X = \bigcup_{n=1}^{\infty} X_n$$

by the hypothesis.

By the corollary to Baire's theorem, we know there exists  $n_0 \geq 1$  with  $\text{Int}(X_{n_0}) \neq \emptyset$ . Choose  $x_0 \in \text{Int}(X_{n_0})$ , and let  $r > 0$  such that

$$B(x_0, r) \subseteq \text{Int}(X_{n_0}).$$

If  $\|z\| \leq 1$  then  $x_0 + rz \in \overline{B}(x_0, r)$ . So  $x_0 + rz \in X_{n_0}$ , and

$$\|T_\alpha(x_0 + rz)\| \leq n_0 \forall \alpha \in A,$$

but  $\|a\| - \|b\| \leq \|a + b\|$ , so

$$\|T_\alpha(rz)\| - \|T_\alpha(x_0)\| \leq \|T_\alpha(x_0 + rz)\| \leq n_0.$$

So  $r\|T_\alpha z\| \leq n_0 + n_0$ , and

$$\|T_\alpha z\| \leq \frac{2n_0}{r} \forall \|z\| \leq 1, \forall \alpha \in A$$

For a general  $x \in X$ ,

$$\|T_\alpha x\| = \|T_\alpha(\frac{x}{\|x\|})\| \|x\| \leq \frac{2n_0}{r} \|x\|$$

and thus  $\|T_\alpha\| \leq \frac{2n_0}{r}$ , which implies

$$\sup_{\alpha \in A} \|T_\alpha\| < \infty$$

□

## 15. LECTURE 15 - MONDAY 18 APRIL

Recall, the Fourier series of  $f \in L^2([-\pi, \pi])$  is

$$\sum_{k \in \mathbb{Z}} \langle f, e_k \rangle e_k$$

where  $e_k(t) = \frac{e^{ikt}}{\sqrt{2\pi}}$ . This converges to  $f$  in the  $L^2$  norm.

**Exercise 15.1.** If  $f$  is  $2\pi$ -periodic and continuous, does the Fourier series converge pointwise?



There are explicit (complicated) examples, but the easiest existence is using the uniform boundedness principle.

**Proposition 15.2.** *There is a  $2\pi$  periodic continuous function whose Fourier series does not converge at 0.*

*Proof.* Let  $\mathcal{C}_p([-\pi, \pi]) = \{f \in \mathcal{C}([-\pi, \pi]) \mid f(-\pi) = f(\pi)\}$ . This is a Banach space with  $\|\cdot\|_\infty$ . If  $f \in \mathcal{C}_p$ , let

$$f_n = \sum_{|k| \leq n} \langle f, e_k \rangle e_k.$$

*Remark.* We can now define, for each  $n \geq 1$ , a linear operator  $T_n : \mathcal{C}_p \rightarrow \mathbb{K}$  by

$$T_n(f) = f_n(0).$$

If  $f_n(0)$  converges (as  $n \rightarrow \infty$ ) for each  $f \in \mathcal{C}_p$ , then

$$\sup_{n \geq 1} |T_n f| = \sup_{n \geq 1} |f_n(0)| < \infty$$

for all  $f \in \mathcal{C}_p$ , which by uniform boundedness implies

$$\sup_{n \geq 1} \|T_n\| \leq \infty. \quad (\star)$$

We now show that  $(\star)$  is false.

We have

$$\begin{aligned} f_n(x) &= \sum_{|k| \leq n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left( \sum_{|k| \leq n} e^{-ik(x-t)} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) dt \end{aligned}$$

where  $D_n(t) = \sum_{|k| \leq n} e^{ikt}$  is the **Dirichlet Kernel**. The Dirichlet kernel is real, and even, with

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}}.$$

*Note.*  $T_n$  is continuous, with norm  $\|T_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt$ .

*Proof.*

$$\begin{aligned} |T_n(f)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| |D_n(t)| dt \\ &\leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \right) \|f\|_\infty \end{aligned}$$

and so  $\|T_n\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt$ .

Going the other way, let

$$s_t = \begin{cases} 1 & D_n(t) \geq 0 \\ -1 & D_n(t) < 0 \end{cases}.$$

We have seen that set functions can be approximated in  $L^1$ -norm by continuous (periodic) functions.

So if  $\epsilon > 0$  is given, there is a  $g \in \mathcal{C}_p$  such that

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (g(t) - s(t)) D_n(t) dt \right| < \epsilon.$$

$g$  can be chosen with  $\|g\|_{\infty} = 1$ .

So

$$\left| T_n(g) - \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \right| < \epsilon.$$

Thus

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt - |T_n(g)| < \epsilon.$$

So

$$|T_n(g)| \geq \frac{\|g\|_{\infty}}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt - \epsilon.$$

Since  $\epsilon > 0$  was arbitrary,

$$\|T_n\| \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt.$$

□

All that remains is to show that

$$\|T_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \rightarrow \infty$$

We have

$$\begin{aligned}
\|T_n\| &= \frac{1}{\pi} \int_0^\pi |D_n(t)| dt \\
&= \frac{1}{\pi} \int_0^\pi \frac{|\sin(n + \frac{1}{2})t|}{|\sin \frac{t}{2}|} dt \\
&\geq \frac{2}{\pi} \int_0^\pi \frac{|\sin(n + \frac{1}{2})t|}{t} dt \\
&= \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{\sin v}{v} dv \\
&\geq \frac{2}{\pi} \int_0^{n\pi} \frac{\sin v}{v} dv \\
&= \frac{2}{\pi} \sum_{k=1}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin v|}{v} dv \\
&\geq \sum_{k=1}^n \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin v| dv \\
&= \frac{4}{\pi^2} \sum_{k=1}^n \frac{1}{k} \rightarrow \infty
\end{aligned}$$

as  $n \rightarrow \infty$ .

Thus there exists  $f \in \mathcal{C}_p$  such that the Fourier series of  $f$  diverges at  $x = 0$ .  $\square$

**15.1. The open mapping theorem.** This theorem is *tailor-made* to deal with inverse operators.

**Definition 15.3** (Open mapping). Let  $X, Y$  be metric spaces. A function  $f : X \rightarrow Y$  is **open** if open sets in  $X$  are mapped to open sets in  $Y$ .

**Theorem 15.4** (Open mapping theorem). Let  $X, Y$  be Banach spaces. If  $T \in \mathcal{L}(X, Y)$  is surjective then  $T$  is open.

**Corollary** (Bounded inverse theorem). Let  $X, Y$  be Banach spaces. If  $T \in \mathcal{L}(X, Y)$  is bijective, then

$$T^{-1} \in \mathcal{L}(Y, X).$$

*Proof.* Let  $O \subseteq X$  be open. Then  $(T^{-1})^{-1}(O) = T(O)$  is open (by the open mapping theorem). Thus  $T^{-1}$  is continuous.  $\square$

**Corollary.** Let  $(X, \|\cdot\|_1)$  and  $(X, \|\cdot\|_2)$  be Banach spaces. If

$$\|x\|_1 \leq C\|x\|_2 \quad \forall x \in X$$

then  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.

*Proof.*

$$\begin{aligned} i : (X, \|\cdot\|_2) &\rightarrow (X, \|\cdot\|_1) \\ x &\mapsto x \end{aligned}$$

is linear, surjective and injective, and also continuous, as

$$\|i(x)\| = \|x\|_1 \leq C\|x\|_2.$$

So the bounded inverse theorem gives

$$i^{-1} : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$$

is continuous. Thus there exists  $A > 0$  such that  $\|i^{-1}(x)\|_2 \leq A\|x\|_1$ , which implies  $\|x\|_2 \leq A\|x\|_1$ . So

$$\frac{1}{A}\|x\|_2 \leq \|x\|_1 \quad \forall x \in X \quad \square$$

More generally, if  $T \in \mathcal{L}(X, Y)$  is bijective, then by the bounded inverse theorem,

$$c\|x\| \leq \|Tx\| \leq C\|x\|$$

where  $c = \frac{1}{\|T^{-1}\|}$ ,  $C = \|T\|$ .

## 16. LECTURE 16 - WEDNESDAY 20 APRIL

**Lemma 16.1.** *Let  $X$  be a Banach space and  $Y$  a normed space. Then for  $T \in \mathcal{L}(X, Y)$ , the following are equivalent.*

- (a)  $T$  is open
- (b) There exists  $r > 0$  such that  $B(0, r) \subseteq T(\overline{B(0, 1)})$
- (c) There exists  $r > 0$  such that  $B(0, r) \subseteq T(\overline{\overline{B(0, 1)}})$ .

*Proof.* (a)  $\Rightarrow$  (b), (c). As  $B(0, 1)$  is open, the set  $T(B(0, 1))$  is open in  $Y$ . Since  $0 \in T(B(0, 1))$  there exists  $> 0$  such that the set

$$B(0, r) \subseteq T(B(0, 1)) \subseteq T(\overline{B(0, 1)}) \subseteq \overline{\overline{T(\overline{B(0, 1)})}}.$$

(c)  $\Rightarrow$  (b). Assume that there exists  $r > 0$  such that

$$B(0, r) \subseteq \overline{\overline{T(\overline{B(0, 1)})}}.$$

We now show that  $B(0, \frac{r}{2}) \subseteq T(\overline{B(0, 1)})$  which proves (b). Let  $y \in B(0, \frac{r}{2})$ . Then  $2y \in B(0, r)$  and since  $B(0, r) \subseteq \overline{\overline{T(\overline{B(0, 1)})}}$  there exists  $x_1 \in \overline{B(0, 1)}$  such that

$$\|2y - Tx_1\| \leq \frac{r}{2}$$

Hence  $4y - 2Tx_1 \in B(0, r)$  and by the same argument as before there exists  $x_2 \in \overline{B(0, 1)}$  such that

$$\|4y - 2Tx_1 - Tx_2\| \leq \frac{r}{2}$$

Continuing this way we construct a sequence  $(x_n) \in \overline{B(0, 1)}$  such that

$$\|2^n y - 2^{n-1}Tx_1 - \cdots - 2Tx_{n-1} - Tx_n\| \leq \frac{r}{2}$$

for all  $n$ . Dividing by  $2^n$  we obtain

$$\|y - \sum_{k=1}^n 2^{-k}Tx_k\| \leq \frac{r}{2^{n+1}}$$

Hence  $y = \sum_{k=1}^{\infty} 2^{-k}Tx_k$ . Since  $\|x_k\| \leq 1$  for all  $k \in \mathbb{N}$  we have that

$$\sum_{k=1}^{\infty} 2^{-k}\|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k} = 1$$

and so the series

$$x = \sum_{k=1}^{\infty} 2^{-k}x_k$$

converges absolutely in  $X$  as  $X$  is Banach and hence complete. We have also that  $\|x\| \leq 1$  and so  $x \in \overline{B(0, 1)}$ . Because  $T$  is continuous we have

$$Tx = \lim_{n \rightarrow \infty} \sum_{k=1}^n 2^{-k}Tx_k = y$$

by construction of  $x$ . Hence  $y \in T(\overline{B(0, 1)})$  and (b) follows.

(b)  $\Rightarrow$  (a). By (b) and the linearity of  $T$  we have

$$T(\overline{B(0, \epsilon)}) = \epsilon T(\overline{B(0, 1)})$$

for all  $\epsilon > 0$ . Since the map  $x \mapsto \epsilon x$  is a homeomorphism on  $Y$  the set  $T(\overline{B(0, \epsilon)})$  is a neighbourhood of zero for all  $\epsilon > 0$ . Now let  $U \subseteq X$  be open and  $y \in T(U)$ . As  $U$  is open there exists  $\epsilon > 0$  such that

$$\overline{B(x, \epsilon)} = x + \overline{B(0, \epsilon)} \subseteq U$$

where  $y = Tx$ . Since  $z \mapsto x + z$  is a homeomorphism and  $T$  is linear we have

$$T(\overline{B(x, \epsilon)}) = Tx + T(\overline{B(0, \epsilon)}) = y + T(\overline{B(0, \epsilon)}) \subseteq T(U).$$

Hence  $T(\overline{B(x, \epsilon)})$  is a neighbourhood of  $y$  in  $T(U)$ . As  $y$  was arbitrary in  $T(U)$  it follows that  $T(U)$  is open.  $\square$

**Lemma 16.2.** *Let  $X$  be a normed vector space and  $S \subseteq X$  convex with  $S = -S$ . If  $\overline{S}$  has a non-empty interior, then  $\overline{S}$  is a neighbourhood of zero.*

*Proof.* First note that  $\bar{S}$  is convex. If  $x, y \in S$  and  $x_n, y_n \in S$  with  $x_n, y_n \rightarrow x, y$  then  $tx_n + (1 - ty_n) \in S$  for all  $n$  and  $t \in [0, 1]$ . Letting  $n \rightarrow \infty$  we get  $tx + (1 - t)y \in \bar{S}$  for all  $t \in [0, 1]$  and so  $\bar{S}$  is convex. We also easily have  $\bar{S} = -\bar{S}$ . If  $\bar{S}$  has a non-empty interior, there exists  $z \in \bar{S}$  and  $\epsilon > 0$  such that  $B(z, \epsilon) \subseteq \bar{S}$ . Therefore  $z \pm h \in \bar{S}$  whenever  $\|h\| < \epsilon$  and since  $\bar{S} = -\bar{S}$  we also have  $-(z \pm h) \in \bar{S}$ . By the convexity of  $\bar{S}$  we have

$$y = \frac{1}{2}((x + h) + (-x + h)) \in \bar{S}$$

whenever  $\|h\| < \epsilon$ . Hence  $B(0, \epsilon) \subseteq \bar{S}$ , and so  $\bar{S}$  is a neighbourhood of zero.  $\square$

**Theorem 16.3** (Open mapping theorem). *Suppose that  $X$  and  $Y$  are Banach spaces. If  $T \in \mathcal{L}(X, Y)$  is surjective, then  $T$  is open.*

*Proof.* As  $T$  is surjective we have

$$Y = \bigcup_{n \in \mathbb{N}} \overline{T(B(0, n))}$$

with  $\overline{T(B(0, n))}$  closed for all  $n \in \mathbb{N}$ . Since  $Y$  is complete, by a corollary to Baire's theorem, there exists  $n \in \mathbb{N}$  such that  $\overline{T(B(0, n))}$  has non-empty interior. Since the map  $x \mapsto nx$  is a homeomorphism and  $T$  is linear, the set  $\overline{T(B(0, 1))}$  has non-empty interior as well. Now  $\overline{B(0, 1)}$  is convex and  $\overline{B(0, 1)} = -\overline{B(0, 1)}$ . By linearity of  $T$  we have that

$$T(\overline{B(0, 1)}) = -T(\overline{B(0, 1)})$$

is convex as well. Since we know that  $\overline{T(B(0, 1))}$  has non-empty interior, the previous lemma implies that  $\overline{T(B(0, 1))}$  is a neighbourhood of zero, and thus there exists  $r > 0$  such that

$$B(0, r) \subseteq \overline{T(B(0, 1))}$$

and since  $X$  is Banach the previous lemma shows that  $T$  is open.  $\square$

## 17. LECTURE 17 - MONDAY 1 MAY

**Exercise 17.1.** If  $X, Y$  are vector spaces, and if  $T : X \rightarrow Y$  is linear, then  $\Gamma(T)$  is a subspace of  $X \times Y$ . Moreover, if  $X, Y$  are normed vector spaces, with

$$\|(x, Tx)\|_{\Gamma} = \|x\| + \|Tx\|.$$

**Theorem 17.2** (Closed Graph theorem). *Let  $X, Y$  be Banach spaces, and  $T \in \text{Hom}(X, Y)$ . Then  $T \in \mathcal{L}(X, Y)$  if and only if  $\Gamma(T)$  is closed in  $X \times Y$ .*

*Proof.* Suppose  $T \in \mathcal{L}(X, Y)$ . If  $x_n \rightarrow x$  in  $X$ , then

$$(x_n, Tx_n) \rightarrow (x, Tx)$$

by continuity of  $T$ , and so  $\Gamma(T)$  is closed.

Conversely, suppose that  $\Gamma(T)$  is closed in  $X \times Y$ . Define a norm  $\|\cdot\|_\Gamma$  on  $X$  by  $\|x\|_\Gamma = \|x\| + \|Tx\|$ . Since  $\Gamma(T)$  is closed, and since  $(X, \|\cdot\|)$  is Banach, then  $(X, \|\cdot\|_\Gamma)$  is also a Banach space (exercise). Note that  $\|x\| \leq \|x\|_\Gamma$ . So by a corollary to the Open Mapping theorem,  $\|\cdot\|$  and  $\|\cdot\|_\Gamma$  are equivalent. So there is  $c > 0$  with

$$\|x\|_\Gamma \leq c\|x\| \quad \forall x \in X.$$

So  $\|x\| + \|Tx\| \leq c\|x\|$ , and so  $\|Tx\| \leq (c-1)\|x\|$ , and so  $T$  is continuous.  $\square$

**17.1. Spectral Theory.** The eigenvalues of an  $n \times n$  matrix  $T$  over  $\mathbb{C}$  are the  $\lambda \in \mathbb{C}$  with

$$\det(\lambda I - T) = 0$$

that is,  $\lambda I - T$  is not invertible.

*Remark.* Showing existence of eigenvalues is equivalent to the fundamental theorem of algebra.

*Remark.* We need our base field to be  $\mathbb{C}$  to get reasonable spectral theory.

**Definition 17.3.** Write  $\mathcal{L}(X) = \mathcal{L}(X, X)$ .

**Definition 17.4.** Let  $X$  be a Banach space over  $\mathbb{K}$ , and let  $T \in \mathcal{L}(X)$ . Then the spectrum of  $T$  is

$$\sigma(T) = \{\lambda \in \mathbb{K} \mid \lambda I - T \text{ is not invertible}\}.$$

*Remark.*  $\lambda I - T$  is non invertible if either  $\lambda I - T$  is not injective, or  $\lambda I - T$  is not surjective.

*Remark.* If  $\dim(X) < \infty$ , then  $X \setminus \text{KER}(T) \simeq \text{im}(T)$ , and so  $T$  is injective if and only if  $T$  is surjective. This fails in the infinite dimensional case - consider the left and right shift operators on  $\ell^2$ .

**Definition 17.5** (Eigenvalue).  $\lambda \in \mathbb{K}$  is an eigenvalue of  $T \in \mathcal{L}(X)$  if there is  $x \neq 0$  with  $Tx = \lambda x$ , i.e.  $\lambda$  is an eigenvalue if and only if  $\lambda I - T$  is not injective.

**Theorem 17.6.** Let  $X \neq \{0\}$  be a Banach space over  $\mathbb{C}$ , and let  $T \in \mathcal{L}(X)$ . Then  $\sigma(T)$  is a non-empty, compact (closed and bounded) subset of

$$\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$$

**Example 17.7.** Let  $L, R : \ell^2 \rightarrow \ell^2$  be the left and right shift operators.

Then  $\|L\| = 1$ , and so  $\sigma(L) \subseteq \overline{D}(0, 1)$ . If  $|\lambda| < 1$ , then

$$L(\lambda, \lambda^2, \lambda^3, \dots) = (\lambda^2, \lambda^3, \lambda^4, \dots) = \lambda(\lambda, \lambda^2, \lambda^3, \dots)$$

and so  $\lambda$  is an eigenvalue. Thus  $D(0, 1) \subseteq \sigma(L) \subseteq \overline{D}(0, 1)$ . But  $\sigma(L)$  is closed, and so  $\sigma(L) = \overline{D}(0, 1)$ . Are the  $\lambda$  with  $|\lambda| = 1$  eigenvalues? No - suppose  $|\lambda| = 1$  and  $x \neq 0$  with  $Lx = \lambda x$ .

Then

$$L^n(x) = \lambda^n x.$$

Thus,  $x_{n+1} = \lambda^n x_1$ . Then  $x = (x_1, \lambda, x_1, \lambda^2 x_1, \dots)$  which is not in  $\ell^2$ .

Then  $\|R\| = 1$ , and so  $\sigma(R) \subseteq \overline{D}(0, 1)$ .

*Note.*  $LRx = L(0, x_1, \dots) = (x_1, x_2, \dots)$ , so

$$LR = I \quad (\star)$$

*Remark.* Unlike  $\dim(X) < \infty$ ,  $(\star)$  does NOT say that  $R$  is invertible ( $RL = I$ ).

Consider the operator  $L(\lambda I - R) = \lambda L - I = -\lambda(\lambda^{-1}I - L)$ . If  $0 < |\lambda| < 1$ , then we know that  $\lambda^{-1}I - L$  is invertible (as  $\lambda^{-1} \notin \sigma(L)$ ). So if  $\lambda I - R$  were invertible, then  $L$  is invertible, which is false. Thus  $\lambda \in \sigma(R)$ . Hence

$$D(0, 1) \setminus \{0\} \subseteq \sigma(R) \subseteq \overline{D}(0, 1).$$

Since  $\sigma(R)$  is closed,  $\sigma(R) = \overline{D}(0, 1)$ .

## 18. LECTURE 18 - WEDNESDAY 4 MAY

**Theorem 18.1.** *Let  $X \neq \{0\}$  be a Banach space over  $\mathbb{C}$ . Let  $T \in \mathcal{L}(X)$ . Then  $\sigma(T)$  is a nonempty, compact subset of*

$$\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}.$$

**Lemma 18.2.** *With above assumptions  $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$ .*

*Proof.* We need to show that if  $|\lambda| > \|T\|$  then  $\lambda I - T$  is invertible.

**Technique: Geometric series.** We guess

$$(\lambda I - T)^{-1} = \frac{1}{\lambda I - T} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

We now verify this guess. Since

$$\sum_{k=0}^{\infty} \frac{\|T^k\|}{|\lambda|^{k+1}} \leq \sum_{k=0}^{\infty} \frac{\|T\|^k}{|\lambda|^{k+1}} < \infty,$$

the series  $S = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}$  converges in  $X$ .



We now show that  $S$  is the inverse of  $\lambda I - T$ . As we are working in infinite dimensions, we need to check left and right inverses. Let  $S_n = \sum_{k=0}^{n-1} \frac{T^k}{\lambda^{k+1}}$ . Then

$$\begin{aligned} S_n(\lambda I - T) &= \left( \sum_{k=0}^{n-1} \frac{T^k}{\lambda^{k+1}} \right) (\lambda I - T) \\ &= I - \frac{T^n}{\lambda^n} \rightarrow I \\ (\lambda I - T)S_n &= I - \frac{T^n}{\lambda^n} \rightarrow I \end{aligned}$$

and so  $S(\lambda I - T) = (\lambda I - T)S$  and so  $\lambda I - T$  is invertible.  $\square$

**Exercise 18.3.** Show that if  $\|I - T\| < 1$  then  $T$  is invertible with inverse  $\sum_{k=0}^{\infty} (I - T)^k$  *Hint: Consider*

$$\frac{1}{\overline{T}} = \frac{1}{I - (I - T)}.$$

In particular, the ball  $B(I, 1)$  in  $\mathcal{L}(X)$  consists of invertible elements.

The following is used to show  $\sigma(T)$  is closed and nonempty, it is also interesting in its own right.

**Proposition 18.4.** *Let  $X$  be Banach over  $\mathbb{K}$ . Let  $GL(X) = \{T \in \mathcal{L}(X) \mid T \text{ invertible}\}$ . Then*

- (a)  $GL(X)$  is a group under composition of operators.
- (b)  $GL(X)$  is open in  $\mathcal{L}(X)$ .
- (c) The map

$$\begin{aligned} \varphi : GL(X) &\rightarrow GL(X) \\ T &\mapsto T^{-1} \end{aligned}$$

is continuous.

*Proof.* (a) The open mapping theorem tells us that if  $T \in GL(X)$  then  $T^{-1} \in \mathcal{L}(X)$ , and so  $T^{-1} \in GL(X)$ . The rest is clear.

(b) Let  $T_0 \in GL(X)$ . We claim

$$B\left(T_0, \frac{1}{\|T_0^{-1}\|}\right) \subseteq GL(X).$$

We have

$$\begin{aligned} \|I - T_0^{-1}T\| &= \|T_0^{-1}(T_0 - T)\| \\ &\leq \|T_0^{-1}\| \|T_0 - T\| \\ &< 1 \quad \text{as } T \in B\left(T_0, \frac{1}{\|T_0^{-1}\|}\right) \end{aligned}$$

(c) We have

$$\begin{aligned}\|T_0^{-1} - T^{-1}\| &= \|T^{-1}(T - T_0)T_0^{-1}\| \\ &\leq \|T^{-1}\| \|T - T_0\| \|T_0^{-1}\|\end{aligned}\quad (\star)$$

If  $\|T - T_0\| \leq \frac{1}{2\|T_0^{-1}\|}$ , then

$$\begin{aligned}\|I - TT_0^{-1}\| &= \|(T_0 - T)T_0^{-1}\| \\ &\leq \|T_0 - T\| \|T_0^{-1}\| \\ &\leq \frac{1}{2}.\end{aligned}$$

We then have

$$\begin{aligned}\|T_0 T^{-1}\| &= \|(TT_0^{-1})^{-1}\| \\ &= \left\| \sum_{k=0}^{\infty} (I - TT_0^{-1})^k \right\| \\ &\leq \sum_{k=0}^{\infty} \|I - TT_0^{-1}\|^k \\ &\leq 2\end{aligned}$$

Hence  $\|T^{-1}\| = \|T_0^{-1}(T_0 T^{-1})\| \leq \|T_0^{-1}\| \|T_0 T^{-1}\| \leq 2\|T_0^{-1}\|$ , and from  $(\star)$ , we have

$$\|T_0^{-1} - T^{-1}\| \leq 2\|T_0^{-1}\|^2 \|T - T_0\|$$

and so  $T \mapsto T^{-1}$  is continuous.

□

**Corollary.**  $\sigma(T)$  is closed.

*Proof.* Let

$$\begin{aligned}f : \mathbb{C} &\rightarrow \mathcal{L}(X) \\ \lambda &\mapsto \lambda I - T\end{aligned}$$

This is continuous, as

$$\begin{aligned}\|f(\lambda) - f(\lambda_0)\| &= \|(\lambda - \lambda_0)I\| \\ &= |\lambda - \lambda_0|\end{aligned}$$

and

$$\sigma(T) = f^{-1}(\mathcal{L}(X) \setminus \text{GL}(X))$$

which is the inverse image of a closed set, and hence is closed.

□

So  $\sigma(T)$  is a compact subset of  $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$ . Write  $\rho(T) = \mathbb{C} \setminus \sigma(T)$  (the **resolvent set**), and let  $R_T = R : \rho(T) \rightarrow \mathcal{L}(X)$  with  $R_T(\lambda) = (\lambda I - T)^{-1}$ .

**Theorem 18.5.** *Let  $\mathbb{K} = \mathbb{C}$  and  $X \neq \{0\}$  and  $T \in \mathcal{L}(X)$ . Then  $\sigma(T) \neq \emptyset$ .*

*Proof.* We use Liouville's theorem - a bounded entire function must be constant.

Let  $\varphi = \mathcal{L}(X)'$  (hence  $\varphi : \mathcal{L}(X) \rightarrow C$ .) Let

$$\begin{aligned} f_\varphi : \rho(T) &\rightarrow \mathbb{C} \\ \lambda &\mapsto \varphi(R(\lambda)) \end{aligned}$$

**Lemma 18.6.**  *$f_\varphi$  is analytic on  $\rho(T)$ .*

*Proof.* We show  $f_\varphi$  is differentiable. Consider

$$\begin{aligned} \frac{f_\varphi(\lambda) - f_\varphi(\lambda_0)}{\lambda - \lambda_0} &= \varphi \left( \frac{R(\lambda) - R(\lambda_0)}{\lambda - \lambda_0} \right) \\ &= \varphi \left( \frac{(\lambda I - T)^{-1} - (\lambda_0 I - T)^{-1}}{\lambda - \lambda_0} \right) \\ &= \varphi \left( \frac{(\lambda_0 I - T)^{-1}((\lambda_0 - \lambda)I)(\lambda I - T)^{-1}}{\lambda - \lambda_0} \right) \\ &= -\varphi((\lambda_0 I - T)^{-1}(\lambda I - T)^{-1}) \\ &\rightarrow -\varphi((\lambda_0 I - T)^{-2}) \end{aligned}$$

as  $\lambda \rightarrow \lambda_0$ , where we use the fact that  $\varphi$  is continuous and  $T \rightarrow T^{-1}$  is continuous. So  $f_\varphi$  is analytic on  $\rho(T)$  for all  $\varphi \in \mathcal{L}(X)'$ .  $\square$

Now suppose that  $\sigma(T) = \emptyset$ . Then  $f_\varphi : \mathbb{C} \rightarrow \mathbb{C}$  is analytic.

**Lemma 18.7.**  *$f_\varphi$  is bounded.*

*Proof.* If  $|\lambda| > \|T\|$ , then

$$\begin{aligned} f_\varphi(\lambda) &= \varphi((\lambda I - T)^{-1}) \\ &= \varphi \left( \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \right) \\ &\leq \|\varphi\| \left\| \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \right\| \\ &\leq \|\varphi\| \sum_{k=0}^{\infty} \frac{\|T\|^k}{|\lambda|^{k+1}} \\ &= \frac{\|\varphi\|}{|\lambda| - \|T\|} \rightarrow 0 \end{aligned}$$

as  $|\lambda| \rightarrow \infty$ . So  $f_\varphi$  is bounded, entire, and thus  $f_\varphi = c$  by Liouville's theorem. By the above,  $f_\varphi(\lambda) = 0$  for all  $\lambda$ . Hence  $\varphi(R(\lambda)) = 0$  for all  $\lambda, \varphi$ .

Thus from Hahn-Banach,  $R(\lambda) = 0$  for all  $\lambda$  which is a contradiction, as the zero operator is not invertible if  $X \neq \{0\}$ . □

□

## 19. LECTURE 19 - MONDAY 9 MAY

**Theorem 19.1** (Spectral mapping theorem (polynomials)). *Let  $T$  be an  $n \times n$  matrix over  $\mathbb{C}$ . If we know all the eigenvalues of  $T$ , then we know the eigenvalues of every polynomial  $p(T) = a_0 + a_1T + \dots + a_nT^n$ . Specifically,*

$$\{\text{eigenvalues of } p(T)\} = \{p(\lambda) \mid \lambda \text{ is an eigenvalue of } T\}$$

Therefore

$$\sigma(p(T)) = p(\sigma(T)).$$

This is called the **spectral mapping theorem** (for matrices/polynomials).

This also holds for  $X$  Banach over  $\mathbb{C}$ , and  $T \in \mathcal{L}(X)$ .

**Lemma 19.2.** *Let  $\mathbb{C}[t]$  be the algebra of polynomials in  $t$  with complex coefficients. Multiplication is defined as usual.*

**Lemma 19.3.** *Let  $X$  be Banach over  $\mathbb{C}$ . Let  $T \in \mathcal{L}(X)$ . Then*

$$\begin{aligned} \varphi : \mathbb{C}[t] &\rightarrow \mathcal{L}(X) \\ p &\mapsto p(T) \end{aligned}$$

*is an algebra homomorphism (multiplication corresponds to composition in  $\mathcal{L}(X)$ ).*

*Proof.* Simply check

$$\begin{aligned} \varphi(p_1 + p_2) &= \varphi(p_1) + \varphi(p_2) \\ \varphi(p_1 p_2) &= \varphi(p_1) \varphi(p_2) \\ \varphi(\alpha p) &= \alpha \varphi(p) \end{aligned}$$

for all  $p_1, p_2, p \in \mathbb{C}[t], \alpha \in \mathbb{C}$ . □

**Theorem 19.4.** *Let  $X$  be Banach over  $\mathbb{C}$ , and let  $T \in \mathcal{L}(X)$ . Then*

$$\sigma(p(T)) = p(\sigma(T)).$$

*Proof.* If  $p = c$  is constant, then  $p(T) = cI$  has spectrum

$$\sigma(p(T)) = \sigma(cI) = \{c\}$$

On the other hand,

$$p(\sigma(T)) = \{c\}$$

Now, suppose that  $p$  is non constant. Let  $\mu \in \mathbb{C}$  fixed. By the fundamental theorem of algebra, we can factorise  $\mu - p(t)$  as

$$\alpha(t - \lambda_1)^{m_1} \dots (t - \lambda_n)^{m_n}$$

where  $\lambda_1, \dots, \lambda_n$  are the distinct roots of  $\mu - p(t)$ . Note that  $\mu = p(\lambda_i)$  for each  $i$ . Applying  $\psi : \mathbb{C}[t] \rightarrow \mathcal{L}(X)$  from above, we have

$$\mu I - p(T) = \alpha(T - \lambda_1 I)^{m_1} \dots (T - \lambda_n I)^{m_n}$$

**Exercise 19.5.** If  $T_1, \dots, T_n \in \mathcal{L}(X)$  which commute with each other, then  $T_1 \dots T_n$  is invertible if and only if the individual elements are invertible.

We know

$$\begin{aligned} \mu \in \sigma(p(T)) &\iff \mu - p(T) \text{ is not invertible} \\ &\iff T - \lambda I \text{ non invertible for some } i \\ &\iff \lambda \in \sigma(T) \text{ for some } i \\ &\iff \mu = p(\lambda_i) \in p(\sigma(T)) \end{aligned}$$

and so

$$\sigma(p(T)) = p(\sigma(T))$$

□

**Definition 19.6** (Spectral radius). Let  $X \neq \{0\}$  be a Banach space over  $\mathbb{C}$ . The **spectral radius** of  $T \in \mathcal{L}(X)$  is

$$\begin{aligned} r(T) &= \sup\{|\lambda| \mid \lambda \in \sigma(T)\} \\ &= \max\{|\lambda| : \lambda \in \sigma(T)\} \end{aligned}$$

*Note.*

$$r(T) \leq \|T\|$$

since  $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$ . Strict inequality can (and often does) occur.

**Example 19.7.** Let

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then consider  $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  where  $\|(x, y)\|_2 = \sqrt{|x|^2 + |y|^2}$ . Then

$$\begin{aligned} \|T\| &= \sup\{\|Tx\|_2 \mid x \in \mathbb{C}^2\} \\ &= \sqrt{\lambda_{\max}(T^*T)} \end{aligned}$$

where

$$T^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is conjugate transpose. Then  $\|T\| = 1$ . But  $\sigma(T) = \{0\}$ , and so  $r(T) = 0 < 1 = \|T\|$ .

**Theorem 19.8** (Gelfand, 1941). *Let  $X \neq \{0\}$  be Banach over  $\mathbb{C}$ , and let  $T \in \mathcal{L}(X)$ . Then*

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}.$$

*In particular, the limit exists.*

*Proof.* By the spectral mapping theorem,

$$\sigma(T^n) = \{\sigma(T)\}^n = \{\lambda^n \mid \lambda \in \sigma(T)\}.$$

So

$$\begin{aligned} r(T) &= r(T^n)^{1/n} \\ &\leq \|T^n\|^{1/n}. \end{aligned}$$

So

$$r(T) \leq \liminf_{n \rightarrow \infty} \|T^n\|^{1/n}$$

Now, we must show that

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq r(T).$$

Let  $\varphi \in \mathcal{L}(X)$  and let

$$\begin{aligned} f_\varphi : \rho(T) &\rightarrow \mathbb{C} \\ \lambda &\mapsto \varphi((\lambda I - T)^{-1}) \end{aligned}$$

We saw that  $f_\varphi$  is analytic on  $\rho(T)$ . We also have

$$f_\varphi(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \varphi(T^n) \tag{*}$$

if  $|\lambda| > \|T\|$ . By general theory of Laurent series, (\*) actually holds for all  $\lambda \in \rho(T)$ . In particular, it holds if  $|\lambda| > r(T)$ .

Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda^{n+1}} \varphi(T^n) = 0 \quad \boxed{|\lambda| > r(T)}$$

Sp for each  $\varphi \in \mathcal{L}(X)'$ , and each  $|\lambda| > r(T)$ , there is  $C_{\lambda, \varphi}$  such that

$$\left| \varphi \left( \frac{1}{\lambda^{n+1}} T^n \right) \right| \leq C_{\lambda, \varphi} \quad \forall n \geq 0$$

Then by the principle of uniform boundedness, there exists a constant  $C_\lambda$  such that

$$\left\| \frac{1}{\lambda^{n+1}} T^n \right\| \leq C_\lambda \quad \forall n \geq 0$$

So  $\|T^n\|^{1/n} \leq |\lambda|(C_\lambda|\lambda|)^{1/n}$ , which gives

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq \lambda$$

for all  $|\lambda| > r(T)$ . So

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq r(T)$$

We used the following lemma.

**Lemma 19.9.** *Let  $X$  be a normed vector space,  $A \subseteq X$  a subset. We say that*

- (1)  *$A$  is **bounded** if there exists  $C > 0$  with  $\|x\| \leq C$ , for all  $x \in A$ .*
- (2)  *$A$  is **weakly bounded** if for each  $\varphi \in X'$ , there exists  $C_\varphi > 0$  such that*

$$|\varphi(x)| \leq C_\varphi$$

*for all  $x \in A$ .*

*Then we have*

$$A \subseteq X \text{ is bounded} \iff \text{weakly bounded}$$

*Proof.*  $A$  bounded  $\Rightarrow \|x\| \leq C$  for all  $x \in A \Rightarrow |\varphi(x)| \leq \|\varphi\| \|x\| \leq \|\varphi\| C$ . So  $A$  is weakly bounded.

Now, suppose  $A$  is weakly bounded. For each  $x \in X$ , let  $\hat{x} \in X''$  with

$$\hat{x}(\varphi) = \varphi(x).$$

So  $|\hat{x}(\varphi)| \leq C_\varphi$  for all  $x \in A$ . By the principle of uniform boundedness,

$$\|\hat{x}\| \leq C$$

for all  $x \in A$ , and since  $\|\hat{x}\| = \|x\|$ . Thus  $A$  is bounded. □

□

## 20. LECTURE 20 - WEDNESDAY 11 MAY

We now turn to compact operators. In general, calculating  $\sigma(T)$  is difficult, but for compact operators on a complex Banach space, we have a fairly explicit theory.

**Theorem 20.1.** *Let  $X$  be a complex Banach space, with  $\dim(X) = \infty$ . Let  $T : X \rightarrow X$  be a compact operator. Then*

- (1)  $0 \in \sigma(T)$ .
- (2)  $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$ , that is, each  $\lambda \in \sigma(T) \setminus \{0\}$  is an eigenvalue of  $T$  (0 may or may not be an eigenvalue.)

(3) We are in exactly one of the cases:

- $\sigma(T) = \{0\}$ .
- $\sigma(T) \setminus \{0\}$  is finite (nonempty).
- $\sigma(T) \setminus \{0\}$  is a sequence of points converging to 0.

(4) Each  $\lambda \in \sigma(T) \setminus \{0\}$  is isolated, and the eigenspace  $\text{KER}(\lambda I - T)$  is finite dimensional.

where  $\sigma_p(T)$  is the **point spectrum** of  $T$ , where

$$\begin{aligned}\sigma_p(T) &= \{\lambda \in \mathbb{K} \mid \lambda I - T \text{ is not injective}\} \\ &= \{\lambda \in \mathbb{K} \mid \text{there exists nonzero vector } x \text{ with } (\lambda I - T)x = 0\} \\ &= \{\text{eigenvalues of } T\}\end{aligned}$$

*Proof.* We shall prove these results next week. □

**Definition 20.2.** Let  $X, Y$  be normed vector spaces. An operator  $T : X \rightarrow Y$  is **compact** if  $T$  is linear, and if  $B \subseteq X$  is bounded then  $T(B)$  is relatively compact (a set is relatively compact if its closure is compact.) Symbolically,

$$B \subseteq X \text{ bounded} \Rightarrow \overline{T(B)} \text{ compact}$$

**Lemma 20.3.** If  $T$  is compact, then  $T$  is continuous.

*Proof.* The closed ball  $B = \{x \in X \mid \|x\| \leq 1\}$  is bounded, and so if  $T$  is a compact operator, then  $\overline{T(B)}$  is compact, and hence bounded. Hence  $\|Tx\| \leq M$  for all  $\|x\| \leq 1$ , so  $T$  is continuous, with  $\|T\| \leq M$ . □

We now recall definitions of compactness

**Theorem 20.4** (Characterisations of compactness). Let  $X$  be a metric space. The following are equivalent.

- (1)  $X$  is **compact** (every open cover has a finite subcover).
- (2)  $X$  is **sequentially compact** (every sequence in  $X$  has a convergent subsequence)

**Lemma 20.5.** Let  $X$  be a compact set. Let  $Y \subseteq X$ . If  $Y \subseteq X$  is closed, then  $Y$  is compact.

**Lemma 20.6.** Let  $V$  be a finite dimensional vector space. If  $X \subseteq V$  is closed and bounded, then  $X$  is compact.

**Theorem 20.7** (Characterisations of compact operators). Let  $X, Y$  be normed vector spaces over  $\mathbb{K}$ . Let  $T \in \mathcal{L}(X, Y)$ . Then the following are equivalent.

- (a)  $T$  is compact.
- (b)  $\overline{T(B)}$  is compact, where  $B = \{x \in X \mid \|x\| \leq 1\}$ .
- (c) If  $(x_n)_{n \geq 1}$  is bounded in  $X$ , then  $(Tx_n)_{n \geq 1}$  has a convergent subsequence (**sequentially compact**).



*Proof.* (a)  $\Rightarrow$  (b) by definition.

(b)  $\Rightarrow$  (a). Suppose (b) holds. Let  $B_1 \subseteq X$  be bounded. Then  $B_1 \subseteq \alpha B$  for some  $\alpha > 0$ . So

$$\overline{T(B)} \subseteq \overline{T(\alpha B)} = \alpha \overline{T(B)}$$

which is a closed subset of a compact set, and hence compact.

(a)  $\Rightarrow$  (c). Suppose  $T$  is compact. Let  $(x_n)_{n \geq 1}$  be bounded sequence in  $X$ . Then  $T(B) = \{Tx_n \mid n \geq 1\}$  is relatively compact. So  $\overline{T(B)}$  is compact, and hence is sequentially compact, and so has a convergence subsequence.

(c)  $\Rightarrow$  (a). Let  $B \subseteq X$  be bounded. Let  $(y_n)_{n \geq 1}$  be a sequence in  $T(B)$ . Then there is  $x_n \in B$  with  $Tx_n = y_n$ . So  $(x_n)_{n \geq 1}$  is a bounded sequence. By assumption  $(Tx_n)_{n \geq 1}$  has a convergent subsequence. So  $\overline{T(B)}$  is sequentially compact, and hence compact.  $\square$

**Corollary.** *The set  $\{\text{compact operators } T : X \rightarrow Y\}$  is a vector space. That is, if  $T_1, T_2$  are compact, then  $T_1 + T_2$  and  $\alpha T_1$  are compact.*

*Proof.* Exercise. Use (c) from the characterisation of compact operators.  $\square$

**Corollary.**

$$\mathcal{K}(X, Y) \subseteq \mathcal{L}(X, Y) \subseteq \text{Hom}(X, Y)$$

where  $\mathcal{K}(X, Y)$  is the set of compact operators  $T : X \rightarrow Y$ .

**Example 20.8** (Finite rank operators). Let  $X, Y$  be normed vector spaces, and let  $T \in \mathcal{L}(X, Y)$ . If  $\dim(\text{Im } T) < \infty$ , then  $T$  is said to have **finite rank**. Then if  $T$  has finite rank, then  $T$  is compact.

*Proof.* Let  $(x_n)$  be a bounded sequence in  $X$ . Then  $\|Tx_n\| \leq \|T\|\|x_n\|$  so  $(Tx_n)$  is a bounded sequence in  $\text{Im } T$ . But  $\text{Im } T$  is finite dimensional, and so  $\{Tx_n \mid n \geq 1\}$  is compact (closed and bounded), and so  $(Tx_n)_{n \geq 1}$  has a convergent subsequence. By (c) in Theorem 20.7,  $T$  is compact.  $\square$

**Lemma 20.9.** *Let  $X, Y$  be normed vector spaces. If  $T \in \mathcal{L}(X, Y)$  has finite rank, then there exists  $y_1, \dots, y_n \in \text{Im } T$  and  $\varphi_1, \dots, \varphi_n \in X'$  with  $Tx = \sum_{j=1}^n \varphi_j(x)y_j$  for all  $x \in X$ , with  $n = \dim(\text{Im } T)$ .*

*Proof.* Choose a basis  $y_1, \dots, y_n$  of  $\text{Im } T$ . For each  $j = 1, \dots, n$ , define  $\alpha_j \in (\text{Im } T)'$  by

$$\alpha_j(a_1 y_1 + \dots + a_n y_n) = a_j$$

i.e. coordinate projection. By Hahn-Banach, we can extend  $\alpha_j$  to a continuous linear functional  $\tilde{a}_j \in Y'$ . Let  $\varphi_j = \tilde{a}_j \circ T : X \rightarrow \mathbb{K}$ . So  $\varphi_j \in X'$ . Since

$$y = \sum_{j=1}^n \tilde{a}_j(y)y_j \quad \forall y \in \text{Im } T$$

we have

$$\begin{aligned}
 Tx &= \sum_{j=1}^n \tilde{a}_j(Tx)y_j \\
 &= \sum_{j=1}^n (\alpha_j \circ T)(x)y_j \\
 &= \sum_{j=1}^n \varphi_j(x)y_j \quad \forall x \in X.
 \end{aligned}$$

□

## 21. LECTURE 21 - MONDAY 16 MAY

Recall that the closed unit ball in  $X$  is compact if and only if  $\dim(X) < \infty$ . Then it follows that the identity map  $I : X \rightarrow X$  is compact if and only if  $\dim(X) < \infty$ . Hence,

$$\mathcal{K}(X) \subsetneq \mathcal{L}(X) \subsetneq \text{Hom}(X, X)$$

when  $\dim(X) = \infty$ .

Consider a sequence of compact operators  $T_n$ . If  $T_n$  is compact and  $T_n \rightarrow T$ , then  $T$  is compact.

**Lemma 21.1** (Riesz's Lemma). *Let  $X$  be a normed vector space. Let  $Y \subsetneq X$  be a proper **closed** subspace. Let  $\theta \in (0, 1)$  be given. Then there exists  $x$  with  $\|x\| = 1$  such that  $\|x - y\| \geq \theta$  for all  $y \in Y$ .*

*Proof.* Pick any  $z \in X \setminus Y$ . Let  $\alpha = \inf_{y \in Y} \|z - y\| > 0$  since  $Y$  is closed. Then by the definition of the infimum, there is  $y_0 \in Y$  with  $\alpha \leq \|z - y_0\| \leq \frac{\alpha}{\theta}$ . Now let  $x = \frac{z - y_0}{\|z - y_0\|}$ . Then  $\|x\| = 1$ .

Now,

$$\begin{aligned}
 \|x - y\| &= \left\| \frac{z - y_0}{\|z - y_0\|} - y \right\| \\
 &= \frac{1}{\|z - y_0\|} \|z - y_0 - \|z - y_0\|y\| \\
 &\geq \frac{\theta}{\alpha} \alpha = \theta
 \end{aligned}$$

□

**Corollary.** *Let  $X$  be a normed vector space. The closed unit ball  $\overline{B}(0, 1)$  is compact if and only if  $\dim(X) < \infty$ .*

*Proof.* If  $\dim(X) < \infty$  then  $\overline{B}(0, 1)$  is compact (since closed and bounded if and only if compact in finite dimensions). Now suppose  $\dim(X) = \infty$ . Build a sequence  $(x_n)$  with  $\|x_n\| = 1$  with no convergent subsequence. Choose finite dimensional subspaces

$$\{0\} = X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \dots$$

These are all closed (finite dimensional spaces are complete, and hence closed). Use the lemma to choose  $x_k \in X_k$  with  $\|x_k\| = 1$ ,  $\|x_k - x\| \geq \frac{1}{2}$  for all  $x \in X_{k-1}$ . So  $\|x_k - x\| \geq \frac{1}{2}$  for all  $x \in X_j$  ( $j \leq k-1$ ). So  $\|x_n - x_m\| \geq \frac{1}{2}$  for all  $m, n \geq 1$ . So  $(x_n)$  has no convergent subsequence, and so  $\overline{B}(0, 1)$  is not compact.  $\square$

**Corollary.**  $I : X \rightarrow X$  is compact if and only if  $\dim(X) < \infty$ .

*Proof.* Recall  $T$  is compact if and only if  $T(\overline{B}(0, 1))$  is relatively compact.  $\square$

**21.1. Limits of compact operators.** One way to show that an operator is compact is to apply the following.

**Proposition 21.2.** Let  $X$  be a normed vector space, and let  $Y$  be Banach. Suppose that  $T_n \in \mathcal{K}(X, Y)$  for each  $n \geq 1$ . If  $T_n \rightarrow T$  (in operator norm,  $\|T_n - T\| \rightarrow 0$ ) then  $T$  is compact.

*Proof.* Let  $(x_n)$  be a bounded sequence in  $X$ . We now construct a subsequence  $(x'_n)$  for which  $(Tx'_n)$  converges.

- Since  $T_1$  is compact,  $(x_n)$  has a subsequence  $x_n^{(1)}$  such that  $(T_1 x_n^{(1)})$  converges.
- Since  $T_2$  is compact and  $x_n^{(1)}$  is bounded, there is a subsequence  $x_n^{(2)}$  such that  $T_2 x_n^{(2)}$  converges.
- Continuing, we can form a subsequence  $x_n^{(k)}$  such that  $T_k x_n^{(k)}$  converges.

Let  $x'_n = x_n^{(n)}$ . Then  $(x'_n)$  is a subsequence of  $(x_n^{(1)})$ , and  $(x'_n)_{n \geq 2}$  is a subsequence of  $(x_n^{(2)})$ , etc. So for each fixed  $k \geq 1$ ,  $(T_k x'_n)$  converges.

We now show  $Tx'_n$  is Cauchy, and hence converges. We have

$$\|Tx'_m - Tx'_n\| \leq \|Tx'_m - T_k x'_m\| + \|T_k x'_m - T_k x'_n\| + \|T_k x'_n - Tx'_n\|$$

where  $k$  is to be chosen. Suppose  $\|x_n\| \leq M$  for all  $n \geq 1$ . Then

$$\|Tx'_m - Tx'_n\| \leq 2M\|T - T_k\| + \|T_k x'_m - T_k x'_n\|$$

Let  $\epsilon > 0$  be given. Since  $\|T - T_k\| \rightarrow 0$  as  $k \rightarrow \infty$ , fix a  $k$  for which  $\|T - T_k\| \leq \frac{\epsilon}{3M}$ . For this fixed  $k$ , we know  $(T_k x'_n)$  converges, and so is Cauchy. So there exists  $N < \infty$  such that  $\|T_k x'_m - T_k x'_n\| < \epsilon$  for all  $m, n > N$ . Hence  $\|Tx'_m - Tx'_n\| \leq \frac{2M}{\epsilon} \epsilon + \frac{\epsilon}{3} = \epsilon$  for all  $m, n > N$ , so is Cauchy, and so converges.  $\square$

**Example 21.3.** Let  $K(x, y) \in L^2(\mathbb{R}^2)$ . Define  $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  by

$$Tf(x) = \int_{\mathbb{R}} K(x, y)f(y) dy$$

(Hilbert-Schmidt Integral operator)

**Proposition 21.4.**  $T$  is compact.

*Proof.* Note that  $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$  for all  $f \in L^2(\mathbb{R})$ , where  $\|K\|_2 = (\iint_{\mathbb{R}^2} |K(x, y)|^2 dx dy)^{1/2}$ . So  $T$  is continuous, with  $\|T\| \leq \|K\|_2$ . We now exhibit  $T$  as a limit of finite rank (hence compact) operators, with  $T_n : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . One can see that there is a sequence  $K_n \in L^2(\mathbb{R}^2)$  of the form

$$K_n(x, y) = \sum_{k=1}^{N_n} \alpha_k^{(n)}(x) \beta_k^{(n)}(y)$$

with  $K_n \rightarrow K$  in  $L^2(\mathbb{R}^2)$ . Then  $\|T_n - T\| \leq \|K_n - K\|_2 \rightarrow 0$ , and so  $T_n \rightarrow T$ . Hence

$$\begin{aligned} T_n f(x) &= \sum_{k=1}^{N_n} \int_{\mathbb{R}} \alpha_k^{(n)}(x) \beta_k^{(n)}(y) f(y) dy \\ &= \sum_{k=1}^{N_n} \left\langle f, \overline{\beta_k^{(n)}} \right\rangle \alpha_k^{(n)}(x) \end{aligned}$$

and so  $T_n f = \sum_{k=1}^{N_n} \langle f, \overline{\beta_k^{(n)}} \rangle \alpha_k^{(n)}$  from which we use that  $T_n$  has finite rank.  $\square$

## 22. LECTURE 22 - WEDNESDAY 18 MAY

**Theorem 22.1.** *Let  $X$  be a complex Banach space, with  $\dim(X) = \infty$ . Let  $T : X \rightarrow X$  be a compact operator. Then*

- (1)  $0 \in \sigma(T)$ .
- (2)  $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$ , that is, each  $\lambda \in \sigma(T) \setminus \{0\}$  is an eigenvalue of  $T$  (0 may or may not be an eigenvalue.)
- (3) We are in exactly one of the cases:
  - $\sigma(T) = \{0\}$ .
  - $\sigma(T) \setminus \{0\}$  is finite (nonempty).
  - $\sigma(T) \setminus \{0\}$  is a sequence of points converging to 0.
- (4) Each  $\lambda \in \sigma(T) \setminus \{0\}$  is isolated, and the eigenspace  $\text{KER}(\lambda I - T)$  is finite dimensional.

where  $\sigma_p(T)$  is the **point spectrum** of  $T$ , where

$$\begin{aligned} \sigma_p(T) &= \{\lambda \in \mathbb{K} \mid \lambda I - T \text{ is not injective}\} \\ &= \{\lambda \in \mathbb{K} \mid \text{there exists nonzero vector } x \text{ with } (\lambda I - T)x = 0\} \\ &= \{\text{eigenvalues of } T\} \end{aligned}$$

Compact operators are very well behaved with respect to composition.

**Proposition 22.2.** *Let  $X, Y, Z$  be normed vector spaces.*

- (a) *If  $T \in \mathcal{K}(X, Y)$  and  $S \in \mathcal{L}(Y, Z)$ , then  $ST \in \mathcal{K}(X, Z)$ .*
- (b) *If  $S \in \mathcal{L}(X, Y)$  and  $T \in \mathcal{K}(Y, Z)$ , then  $TS \in \mathcal{K}(X, Z)$ .*

- Proof.* (a) Let  $(x_n)$  be a bounded sequence in  $X$ . Since  $T$  is compact,  $Tx_n$  has a convergent subsequence, say  $Tx_{n_k} \rightarrow y \in Y$ . Then  $(STx_n)$  has a convergent subsequence, namely  $STx_{n_k} = S(Tx_{n_k}) \rightarrow Sy$  by continuity of  $S$ . So  $ST$  is compact.
- (b) Let  $B \subseteq X$  be bounded. Then  $S(B)$  is bounded in  $Y$ , as  $S$  is continuous. So  $TS(B) = T(S(B))$  is relatively compact since  $T$  is compact. Hence  $TS$  is compact.  $\square$

**Corollary** (Part (1) of theorem). *If  $X$  is infinite dimensional Banach space, then  $0 \in \sigma(T)$ .*

*Proof.* If  $0 \notin \sigma(T)$  then  $T$  is invertible. By bounded inverse theorem  $T^{-1}$  is continuous, and then  $I = TT^{-1}$  is compact, which is a contradiction.  $\square$

**Theorem 22.3** (Part (3) of theorem). *Let  $X$  be a normed vector space. Let  $T \in \mathcal{K}(X)$ . Then  $T$  has at most countably many eigenvalues. If  $T$  has infinitely many eigenvalues, then they can be arranged in a sequence converging to zero.*

*Proof.* We show that for each  $N > 0$ , we have

$$\#\{\lambda \in \sigma_p(T) \mid |\lambda| \geq N\} < \infty \quad (\star)$$

Suppose that there is  $N > 0$  such that  $(\star)$  fails. So  $\lambda_1, \lambda_2, \dots$  are distinct eigenvalues with  $|\lambda_n| \geq N$  for  $n = 1, 2, \dots$ . Let  $x_n \neq 0$  be an eigenvector.  $Tx_n = \lambda_n x_n$ ,  $n = 1, 2, \dots$ . Let  $X_n = \text{span}\{x_1, \dots, x_n\}$ . Since  $\{x_n \mid n \geq 1\}$  are linearly independent, we have

$$X_1 \subsetneq X_2 \subsetneq \dots$$

and each  $X_n$  is closed (finite dimensional).

By Reisz's Lemma from previous lecture, choose  $y_n \in X_n$  such that  $\|y_n\| = 1$  and  $\|y_n - x\| \geq \frac{1}{2}$  for all  $x \in X_{n-1}$ . So  $(y_n)$  is bounded in  $X$ . We show that  $Ty_n$  has no convergence subsequence, contradicting compactness of  $T$ .

Let  $m > n$ . Then

$$\begin{aligned} \|Ty_m - Ty_n\| &= \|\lambda_m y_m - (\lambda_m y_m - Ty_m + Ty_n)\| \\ &= |\lambda_m| \|y_m - (\text{something in } X_{m-1})\| \\ &\geq \frac{1}{2} |\lambda_m| \geq \frac{1}{2} N \end{aligned}$$

as required.

Note that  $y_m = a_1 x_1 + \dots + a_m x_m$ . Then

$$\begin{aligned} \lambda_m y_m - Ty_m &= \lambda_m a_1 x_1 + \dots + \lambda_m a_m x_m - (a_1 \lambda_1 x_1 + \dots + a_m \lambda_m x_m) \\ &= a_1 (\lambda_m - \lambda_1) x_1 + \dots + a_{m-1} (\lambda_m - \lambda_{m-1}) x_{m-1} \in X_{m-1} \end{aligned}$$

and  $Ty_n \in X_{m-1}$  since  $n < m$ .

□

### 22.1. Projections.

**Definition 22.4** (Projection operator). Let  $X$  be a vector space. A linear operator  $P : X \rightarrow X$  is called a projection if  $P^2 = P$ .

**Proposition 22.5.** *If  $P : X \rightarrow X$  is a projection then  $I - P$  is a projection, and*

$$\text{IM } I - P = \text{KER } P, \quad \text{KER } I - P = \text{IM } P$$

*Proof.* If  $P^2 = P$  then  $(I - P)^2 = I - 2P + P^2 = I - P$  and so  $I - P$  is a projection. Let  $x \in \text{IM } I - P$ . Then  $x = (I - P)y$  for some  $y \in X$ . So  $Px = P(I - P)y = (P - P^2)y = 0$ . So  $x \in \text{KER } P$  and  $\text{IM } I - P \subseteq \text{KER } P$ . If  $x \in \text{KER } P$  then  $Px = 0$ . So  $(I - P)x = x$ , and  $x \in \text{IM } (I - P)$ . □

**Definition 22.6** (Direct sum). Let  $X$  be a vector space, and let  $X_1, X_2$  be subspaces. Then  $X = X_1 \oplus X_2$  (direct sum) if

$$X = X_1 + X_2$$

and  $X_1 \cap X_2 = \{0\}$ . Equivalently,  $X = X_1 \oplus X_2$  if and only if each  $x \in X$  can be written in exactly one way as  $x = x_1 + x_2$  with  $x_1 \in X_1, x_2 \in X_2$ .

**Theorem 22.7** (Equivalence of direct sums and projections). *Let  $X$  be a vector space.*

(a) *If  $P : X \rightarrow X$  is a projection, then*

$$X = (\text{IM } P) \oplus (\text{KER } P)$$

(b) *If  $X = X_1 \oplus X_2$ , there exists a unique projection with*

$$\text{IM } P = X_1, \quad \text{KER } P = X_2.$$

*Specifically,  $Px = x_1$  if  $x = x_1 + x_2$ .*

*Proof.* (a) Let  $P : X \rightarrow X$  be a projection. Then we show  $X = (\text{IM } P) \oplus (\text{IM } I - P)$ ,  $x = Px + (I - P)x$ . This shows that  $X = \text{IM } P + \text{IM } I - P$ . If  $x \in \text{IM } P \cap \text{KER } P$  then  $x = Py$  and  $Px = 0$ . Hence,  $Px = P^2y = Py = 0$  and so  $x = 0$ .

(b) Exercise. □

**Proposition 22.8.** *Let  $X$  be Banach. Let  $X = X_1 \oplus X_2$ . Let  $P : X \rightarrow X$  be the corresponding projection operator. Then*

$$P \in \mathcal{L}(X) \iff X_1, X_2 \text{ closed}$$

*Proof.* ( $\Rightarrow$ ). Suppose  $P$  is continuous. Then  $X_1 = \text{IM } P = \text{KER } I - P$  and  $X_2 = \text{KER } P$  are both closed. For example, if  $x_n \in \text{KER } P$  and  $x_n \rightarrow x$ , then  $0 = Px_n \rightarrow Px$  and so  $x \in \text{KER } P$ .

( $\Leftarrow$ ). Suppose that  $X_1, X_2$  are closed. Since  $X = X_1 \oplus X_2$ , we can define a new norm  $\|\cdot\|'$  by  $\|x\|' = \|x_1\| + \|x_2\|$  where  $x = x_1 + x_2$ .

**Exercise 22.9.**

- (a) Show that  $\|\cdot\|'$  is a norm.
- (b) Show that  $(X, \|\cdot\|')$  is Banach. This relies on the fact that  $(X, \|\cdot\|)$  is Banach and  $X_1, X_2$  are closed.

Note that  $\|x\| = \|x_1 + x_2\| \leq \|x_1\| + \|x_2\| = \|x\|'$ , and so by a corollary to the open mapping theorem, there is a  $c > 0$  with  $\|x\|' \leq c\|x\|$  for all  $x \in X$ , and so

$$\|Px\| = \|x_1\| \leq \|x_1\| + \|x_2\| = \|x\|' \leq c\|x\|$$

and hence  $P$  is continuous. □

23. LECTURE 23 - MONDAY 23 MAY

**Corollary.** *Let  $X$  be Banach, and let  $M$  be a finite dimensional subspace. Then there exists a **closed**  $N$  with*

$$X = M \oplus N.$$

*Proof.* Let  $v_1, \dots, v_n$  be a basis of  $M$ . Define, for each  $j = 1, \dots, n$ ,  $\varphi_j \in M'$  by  $\varphi_j(a_1v_1 + \dots + a_nv_n) = a_j$ . Then using Hahn-Banach to extend  $\tilde{\varphi}_j \in X'$ . Let  $P : X \rightarrow X$  be defined by

$$Px = \sum_{j=1}^n \tilde{\varphi}_j(x)v_j.$$

Then we need only check that  $P$  is linear and continuous,  $\text{Im } P = M$ , and  $P^2 = P$ . Now take  $N = \text{Ker } P$  and then  $X = M \oplus N$ . □

We are now ready to prove the following theorem.

**Theorem 23.1.** *Let  $X$  be Banach, and let  $T \in \mathcal{K}(X)$ , and let  $\lambda \in \mathbb{K} \setminus \{0\}$ . For all  $k \in \mathbb{N}$ , we have*

- (a)  $\underbrace{\text{Ker } (\lambda I - T)^k}_{\text{generalised eigenspace}}$  is finite dimensional.
- (b)  $\text{Im } (\lambda I - T)^k$  is closed.

*Proof. Reductions.* Since  $\text{Ker } (\lambda I - T)^k = \text{Ker } (I - \lambda^{-1}T)^k$ , and similarly for the image, by replacing  $T \in \mathcal{K}(X)$  by  $\lambda T \in \mathcal{K}(X)$ , we can assume that  $\lambda = 1$ .

Also, we have

$$\begin{aligned}
 (I - T)^k &= \sum_{n=0}^k \binom{k}{n} (-1)^n T^n \\
 &= I - T \underbrace{\sum_{n=1}^k \binom{k}{n} (-1)^{n-1} T^{n-1}}_{\text{continuous}} \\
 &= I - \tilde{T}.
 \end{aligned}$$

where  $\tilde{T}$  is the composition of compact and continuous operators, and so is compact. So we can take  $\lambda = 1, k = 1$ .

(a) The closed unit ball in  $\text{KER } I - T$  is

$$\begin{aligned}
 \{x \in \text{KER } I - T \mid \|x\| \leq 1\} &= \{Tx \mid x \in \text{KER } I - T, \|x\| \leq 1\} \\
 &\subseteq \overline{T(\overline{B(0, 1)})}
 \end{aligned}$$

which is compact as  $T$  is compact. Hence, the closed unit ball in  $\text{KER } I - T$  is compact, and thus  $\text{KER } I - T$  is finite dimensional.

(b) Let  $S = I - T$ . We then need to show that  $\text{IM } S$  is closed. Since  $\text{KER } S$  is finite dimensional from above, there is a **closed** subspace  $N$  with

$$X = (\text{KER } S) \oplus N$$

Note that  $\text{IM } S = S(X) = S(N)$ , and that  $S|_N : N \rightarrow X$  is injective.

Suppose that  $S(N)$  is not closed. So there is a sequence  $(x_n)$  in  $N$  such that  $Sx_n \rightarrow y \in X \setminus S(N)$ . Then there are two cases

*Case 1* ( $\|x_n\| \rightarrow \infty$ ). Let  $y_n = \frac{1}{\|x_n\|} x_n$ . Then  $Sy_n = \frac{1}{\|x_n\|} Sx_n \rightarrow 0$ . But  $(y_n)_{n \geq 1}$  is bounded in  $X$ , and so there exists a subsequence  $y_{n_k}$  such that  $Ty_{n_k} \rightarrow z$  (as  $T$  is compact). Hence  $y_{n_k} = Sy_{n_k} + Ty_{n_k} \rightarrow 0 + z$ . Thus  $z \in N$  (as  $y_{n_k} \in N$ , and  $N$  is closed), and  $\|z\| = 1$ .

So  $Sy_{n_k} \rightarrow 0$ , but  $Sy_{n_k} \rightarrow Sz$  with  $z \in N \setminus \{0\}$ , by the continuity of  $S$ . This contradicts the injectivity of  $S|_N$ .

*Case 2* ( $\|x_n\|$  does not tend to infinity). So  $(x_n)$  has a bounded subsequence  $(x_{n_k})$ . Since  $T$  is compact,  $(x_{n_k})$  has a subsequence such that  $(Tx_{n_{k_l}})$  converges, to  $z_1$  say. By replacing  $x_n$  by this subsequence we can assume that  $Sx_n \rightarrow y$ , and that  $Tx_n \rightarrow z$ . A before, we can write

$$x_n = Sx_n + Tx_n \rightarrow y + z.$$

So  $x_n$  converges to  $x \in N$ . So  $Sx_n \rightarrow Sx \in S(N)$  by continuity, but we assume that  $Sx_n \rightarrow y \in X \setminus S(N)$ , which achieves our contradiction.



□

## 24. LECTURE 24 - WEDNESDAY 25 MAY

Let  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a linear operator. Then in the simplest case,  $T$  has  $n$  distinct eigenvalues, and the corresponding eigenvectors are linearly independent, forming a basis for  $\mathbb{C}^n$ .

Hence,  $\mathbb{C}^n = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$  and the matrix of  $T$  relative to this basis is simply diagonal with  $\lambda_1, \dots, \lambda_n$ .

This is not always possible, because there is not always a basis of eigenvectors. Instead look at the generalised eigenspace,

$$\{x \in \mathbb{C}^n \mid (\lambda I - T)^k x = 0 \text{ for some } k \geq 1\}.$$

But  $\{0\} \subseteq \text{KER } (\lambda I - T)^1 \subseteq \text{KER } (\lambda I - T)^2 \subseteq \dots$  and since  $\dim(\mathbb{C}^n) < \infty$  this must stabilise. Let  $r \geq 1$  be the first time that  $\text{KER } (\lambda I - T)^r = \text{KER } (\lambda I - T)^{r+1}$ . Then the generalised  $\lambda$ -eigenspace is just  $\text{KER } (\lambda I - T)^r$ . There is a basis of  $\mathbb{C}^n$  consisting of generalised eigenvectors, and the matrix of  $T$  relative to this basis is in block form.

**Definition 24.1** (Complete reduction). Let  $T : X \rightarrow X$  be linear. If  $X = X_1 \oplus X_2$  we can write

$$Tx = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where we identify  $x_1 + x_2 \iff (x_1, x_2)$ . Here,

$$T_{11} : X_1 \rightarrow X_1$$

$$T_{12} : X_2 \rightarrow X_1$$

$$T_{21} : X_1 \rightarrow X_2$$

$$T_{22} : X_2 \rightarrow X_2$$

we say that  $X = X_1 \oplus X_2$  **completely reduces**  $T$  (well adapted to  $T$ ) if

$$Tx = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We write  $T = T_1 \oplus T_2$ .

**Exercise 24.2.** If  $X = X_1 \oplus X_2$  completely reduces  $T = T_1 \oplus T_2$ , then

- (a)  $\text{KER } T = \text{KER } T_1 \oplus \text{KER } T_2$
- (b)  $\text{IM } T = \text{IM } T_1 \oplus \text{IM } T_2$
- (c)  $T$  is injective if and only if  $T_1, T_2$  are injective
- (d)  $T$  is surjective if and only if  $T_1, T_2$  are surjective
- (e) If  $T$  is bijective, then  $X = X_1 \oplus X_2$  completely reduces  $T^{-1} = T_1^{-1} \oplus T_2^{-1}$ .

**Corollary.** Let  $X = X_1 \oplus X_2$  be Banach, with  $X_1, X_2$  closed subspaces. If  $X = X_1 \oplus X_2$  completely reduces  $T = T_1 \oplus T_2 \in \mathcal{L}(X)$ , then

- (a)  $T_1 \in \mathcal{L}(X_1), T_2 \in \mathcal{L}(X_2)$
- (b)  $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$
- (c)  $\sigma_p(T) = \sigma_p(T_1) \cup \sigma_p(T_2)$

*Proof.* Exercise. □

Consider the following chains

$$\begin{aligned} \{0\} &\subseteq \text{KER } S^1 \subseteq \text{KER } S^2 \subseteq \dots \\ X &\supseteq \text{IM } S^1 \supseteq \text{IM } S^2 \supseteq \dots \end{aligned}$$

where  $X$  is a vector space and  $S \in \text{Hom}(X, X)$ . It is easy to see that if  $\text{KER } S^r = \text{KER } S^{r+1}$  then  $\text{KER } S^r = \text{KER } S^{r+k}$ . Similarly for images (p. 109 in Daners.)

There is no reason that these should stabilise in general.

**Theorem 24.3.** Let  $X$  be Banach,  $T \in \mathcal{K}(X), \lambda \neq 0$ . Then both chains (with  $S = \lambda I - T$ ) stabilise.

*Proof.* Without loss of generality, assume  $\lambda = 1$ , so we can write  $S = I - T$ . Suppose that the kernel chain does not stabilise. Since we assume

$$\text{KER } S^1 \subsetneq \text{KER } S^2 \subsetneq \text{KER } S^3 \subsetneq \dots$$

We know that these are closed (being finite dimensional) subspaces. So Reisz's Lemma gives  $x_n \in \text{KER } S^n$  with  $\|x_n\| = 1, \|x_n - x\| \geq \frac{1}{2}$  for all  $x \in \text{KER } S^{n+1}$ . This is a bounded sequence. We claim that  $Tx_n$  has no convergent subsequence.

Let  $m > n$ . Then

$$\begin{aligned} \|Tx_m - Tx_n\| &= \|(I - T)x_n - (I - T)x_m + x_m - x_n\| \\ &= \|Sx_n - Sx_m - x_m - x_n\| \\ &= \|x_m - \underbrace{(Sx_m - Sx_n + x_n)}_{\text{in KER } S^{m-1}}\| \\ &\geq \frac{1}{2} \end{aligned}$$

The image argument is similar - using the fact that the images are closed - proved in the previous lecture. □

**Theorem 24.4.** Let  $X$  be a vector space,  $S \in \text{Hom}(X, X)$ . Suppose that

$$\begin{aligned} \alpha(S) &= \inf\{r \geq 1 \mid \text{KER } S^r = \text{KER } S^{r+1}\} \\ \delta(S) &= \inf\{r \geq 1 \mid \text{IM } S^r = \text{IM } S^{r+1}\}, \end{aligned}$$

the **ascent** and **descent** of  $S$  respectively, are both finite.

Then

(a)  $\alpha(S) = \delta(S) = r$ , say

(b)  $X = \text{KER } S^r \oplus \text{IM } S^r$

(c) The direct sum in (b) completely reduces  $S$ .

*Proof.* Daner's notes, p. 109. □

**Corollary.** Let  $X$  be Banach,  $T \in \mathcal{K}(X)$ ,  $\lambda \neq 0$ . Let  $r = \alpha(\lambda I - T) = \delta(\lambda I - T)$ . Then  $X = \text{KER } (\lambda I - T)^r \oplus \text{IM } (\lambda I - T)^r$  and this completely reduces  $\mu I - T, \mu \in \mathbb{K}$ .

**Corollary.** If  $X$  is Banach,  $T \in \mathcal{K}(X)$ ,  $\lambda \neq 0$  then  $\lambda I - T$  is injective if and only if  $\lambda I - T$  is surjective.

*Proof.*

$$\begin{aligned}
 & \lambda I - T \text{ injective} \\
 \Rightarrow & 0 \in \text{KER } (\lambda I - T)^1 = \text{KER } (\lambda I - T)^2 \\
 \Rightarrow & \alpha(\lambda I - T) = 1 \\
 \Rightarrow & \delta(\lambda I - T) = 1 \\
 \Rightarrow & X = \underbrace{\text{KER } (\lambda I - T)}_{=\{0\}} \oplus \text{IM } (\lambda I - T) \\
 \Rightarrow & X = \text{IM } (\lambda I - T) \\
 \Rightarrow & X \text{ is surjective}
 \end{aligned}$$

The other direction is similar. □

**Corollary.** Let  $X$  be Banach,  $T \in \mathcal{K}(X)$ . Thus each  $\lambda \in \sigma(T) \setminus \{0\}$  is an eigenvalue.

*Proof.* Immediate from the previous corollary. □

## 25. LECTURE 25 - MONDAY 30 MAY

Recall the following.

**Corollary.** Let  $X$  be Banach,  $T \in \mathcal{K}(X)$ ,  $\lambda \neq 0$ . Let  $r = \alpha(\lambda I - T) = \delta(\lambda I - T)$ . Then  $X = \text{KER } (\lambda I - T)^r \oplus \text{IM } (\lambda I - T)^r$  and this completely reduces  $\mu I - T, \mu \in \mathbb{K}$ .

Also note that  $\text{IM } \text{KER } (\lambda I - T)^r$  is closed, and  $\text{KER } (\lambda I - T)^r$  is finite dimensional.

**Exercise 25.1.** Let  $\lambda_1, \dots, \lambda_n \in \sigma(T) \setminus \{0\}$ . Let  $N_j = \text{KER } (\lambda_j I - T)_j^r$  be the generalised  $\lambda_j$ -eigenspace. Show that there exists closed subspaces  $M$  with

$$X = N_1 \oplus N_2 \oplus \dots \oplus M$$

with  $T = T_1 \oplus T_2 \oplus \cdots \oplus T_M$ , and so spectral theory tells us how to *diagonalise*  $T$ .

In Hilbert spaces we can say even more. Recall that the adjoint of  $T \in \mathcal{L}(\mathcal{H})$  is defined by

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x, y \in \mathcal{H}$$

Then  $T^* \in \mathcal{L}(\mathcal{H})$ .

**Definition 25.2** (Self-adjoint).  $T \in \mathcal{L}(\mathcal{H})$  is

- (a) **Hermitian (self-adjoint)** if  $T^* = T$ .
- (b) **Unitary** if  $T^*T = TT^* = I$ .
- (c) **Normal** if  $T^*T = TT^*$ .

*Remark.* For matrices, we have

- (a) Hermitian if and only if  $\overline{A^T} = A$ .
- (b) Unitary if and only if the columns of  $A$  are orthonormal.
- (c) Hermitian and unitary operators are normal.

**Proposition 25.3.** Let  $\mathcal{H}$  be Hilbert over  $\mathbb{C}$ . If  $T \in \mathcal{L}(\mathcal{H})$  is normal, then  $r(T) = \|T\|$ .

*Proof.* For Hermitian operators it is easy. We have

$$\|T\|^2 = \|T^*T\| = \|T^2\|.$$

By induction, we then have  $\|T\|^{2^n} = \|T^{2^n}\|$ . So

$$\begin{aligned} r(T) &= \lim_{n \rightarrow \infty} \|T^n\|^{1/n} \\ &= \lim_{n \rightarrow \infty} \|T^{2^n}\|^{1/2^n} \\ &= \|T\|. \end{aligned}$$

For normal operators, we have

$$\begin{aligned} \|T^2\|^2 &= \|(T^2)^*T^2\| \\ &= \|T^*(T^*T)T\| \\ &= \|T^*TT^*T\| \quad \text{normal} \\ &= \|(T^*T)^*(T^*T)\| \\ &= \|T^*T\|^2 \\ &= \|T^4\| \end{aligned}$$

and then we have  $\|T^2\| = \|T\|^2$  and the proof follows by induction. □

**Corollary.** Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{C}$ .

(a) If  $T \in \mathcal{L}(\mathcal{H})$  is unitary, then

$$\sigma(T) \subseteq \mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$

(b) If  $T \in \mathcal{L}(\mathcal{H})$  is Hermitian, then

$$\sigma(T) \subseteq \mathbb{R}.$$

*Proof.*

(a) On practice sheet. Use the fact that  $\sigma(T^*) = \overline{\sigma(T)}$ .

(b) Let  $\lambda = a + ib \in \sigma(T)$ . So  $\lambda I - T$  is not invertible. Hence,  $(\lambda + it)I - (T + itI)$  is not invertible for all  $t \in \mathbb{R}$ . Then

$$\begin{aligned} \|\lambda + it\|^2 &\leq \|(\lambda + it)I - (T + itI)\|^2 \\ &\leq \|T + itI\|^2 \\ &= \|(T + itI)^*(T + itI)\| \\ &= \|(T - itI)(T + itI)\| \\ &= \|T^2 + t^2 I\| \\ &\leq \|T^2 + t^2\| \end{aligned}$$

However, the left hand side is equal to

$$a^2 + b^2 + 2bt + t^2,$$

and so we obtain

$$a^2 + b^2 + 2bt \leq \|T\|^2 \quad \forall t \in \mathbb{R}$$

and so  $b = 0$ .

□

**Lemma 25.4.** Let  $\mathcal{H}$  be Hilbert over  $\mathbb{C}$ . Let  $T \in \mathcal{L}(\mathcal{H})$ , and let

$$M_\lambda = \{x \in \mathcal{H} \mid Tx = \lambda x\} = \text{KER } \lambda I - T$$

be the  $\lambda$ -eigenspace of  $T$ . Then

(a)  $M_\lambda \perp M_\mu$  if  $\lambda \neq \mu$ .

(b) If  $T$  is normal, each  $M_\lambda$  is  $T$  and  $T^*$  invariant. That is,

$$T(M_\lambda) \subseteq M_\lambda, \quad T^*(M_\lambda) \subseteq M_\lambda.$$

*Proof.*

(a) Let  $u \in M_\lambda, v \in M_\mu$ . Then

$$\begin{aligned} (\lambda - \mu) \langle u, v \rangle &= \langle \lambda u, v \rangle - \langle u, \mu v \rangle \\ &= \langle Tu, v \rangle - \langle u, T^*v \rangle \\ &= \langle Tu, v \rangle - \langle Tu, v \rangle \\ &= 0 \end{aligned}$$

and so  $\langle u, v \rangle = 0$ .

(b) If  $T$  is normal, then  $\text{KER } T = \text{KER } T^*$  as

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle \\ &= \langle x, TT^*x \rangle = \langle T^*x, T^*x \rangle \\ &= \|T^*x\|^2. \end{aligned}$$

Similarly, if  $T$  is normal then  $\lambda I - T$  is normal. Then

$$\begin{aligned} M_\lambda &= \text{KER } \lambda I - T \quad (T \text{ invariant}) \\ &= \text{KER } \bar{\lambda} I - T^* \quad (T^* \text{ invariant}). \end{aligned}$$

□

The spectral theory for compact normal operators in a Hilbert space is particularly nice, as the following theorem demonstrates.

**Theorem 25.5.** *Let  $T \in \mathcal{L}(\mathcal{H})$  be compact and normal. Then*

$$\mathcal{H} = \overline{\bigoplus_{\lambda \in \sigma(T)} M_\lambda},$$

*the closure of the span of the eigenspaces, and  $\mathcal{H}$  has an orthonormal basis consisting of eigenvectors. Moreover,  $T$  acts diagonally with respect to this basis.*

*Proof.* Let

$$M = \overline{\bigoplus_{\lambda \in \sigma(T)} M_\lambda},$$

a closed subspace. Hence  $H = M \oplus M^\perp$ , where

$$M^\perp = \{x \in \mathcal{H} \mid \langle x, m \rangle = 0 \forall m \in M\}.$$

We must show that  $M^\perp = \{0\}$ . Assume the contrary. Then consider  $\tilde{T} = M^\perp \rightarrow \mathcal{H}$  be the restriction of  $T$  to  $M^\perp$ . Then we have

$$\tilde{T} : M^\perp \rightarrow M^\perp$$

is compact and normal (Exercise). Then

- (a)  $\sigma(\tilde{T}) = \{0\}$ . Then  $r(\tilde{T}) = 0$ , and so  $\|\tilde{T}\| = 0$ , and so  $\tilde{T} = 0$ . Then each  $x \in M^\perp \setminus \{0\}$  satisfies  $\tilde{T}x = 0 = 0x$ , and so  $x \in M_0$  with  $M^\perp \subseteq M_0 \subseteq M$ , a contradiction (from direct sum decomposition). Hence  $M = \{0\}$ .
- (b)  $\sigma(\tilde{T}) \neq \{0\}$ . So there is an eigenvalue  $\lambda \in \sigma(T) \setminus \{0\}$ . So there is  $x \in M^\perp \setminus \{0\}$  with  $\tilde{T}x = \lambda x$ .  
 $\circ Tx = \lambda x$ , and so  $x \in (M_\lambda \cap M^\perp) \setminus \{0\}$ , a contradiction. Hence  $M^\perp = \{0\}$ .

Choose an orthonormal basis for each  $M_\lambda$ , and combine to get an orthonormal basis of  $\mathcal{H}$ , using  $M_\lambda \perp M_\mu$ .  $\square$

## 26. LECTURE 26 - WEDNESDAY 1 JUNE

**26.1. Fredholm alternative for compact operators on a Hilbert space.** Recall that for matrices, we have the following result, known as the Fredholm alternative.

**Theorem 26.1** (Fredholm alternative (Finite dimensional spaces)). *Let  $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be linear. Then exactly one of the following two things occur:*

- (1)  $Ax = 0$  has only the trivial solution  $x = 0$ , in which case  $Ax = b$  has a unique solution for each  $b \in \mathbb{C}^n$ .
- (2)  $Ax = 0$  has a non-trivial solution, in which case  $Ax = b$  has either no solutions, or infinitely many solutions.

**Definition 26.2** (Hilbert-Schmidt integral operators).

$$T : L^2([a, b]) \rightarrow L^2([a, b])$$

$$(Tf)(x) \mapsto \int_a^b K(x, y)f(y) dy$$

where  $\|K\|_2$  is finite. These are compact operators.

Consider equations of the following form

$$\lambda f(x) - \int_a^b K(x, y)f(y) dy = g(x),$$

where  $\lambda \neq 0$  and  $g \in L^2$  are given. This can be rewritten in the form

$$(\lambda I - T)f = g.$$

Then we have the following theorem, due to Fredholm.

**Theorem 26.3** (Fredholm alternative (Hilbert spaces)). *Let  $\mathcal{H}$  be Hilbert over  $\mathbb{C}$ , and let  $T \in \mathcal{K}(\mathcal{H})$ . Then exactly one of the following occurs.*

- (a)  $(\lambda I - T) = 0$  has only the trivial solution, in which case  $(\lambda I - T)x = b$  has a unique solution for each  $b \in \mathcal{H}$ .

- (b)  $(\lambda I - T)x = 0$  has a non trivial solution, in which case  $(\lambda I - T)x = b$  has a solution if and only if  $b \perp y$  for every solution  $y$  of the equation

$$(\bar{\lambda}I - T^*)y = 0$$

This is finite dimensional, as it is the kernel of  $(\lambda I - T)^*$ .

*Proof.*

- (a) If  $(\lambda I - T)x = 0$  has only the trivial solution, then  $\text{KER } \lambda I - T = \{0\}$  and so it is injective. Hence  $\lambda$  is not an eigenvalue, and so  $\lambda$  is not a spectral value. So  $\lambda I - T$  is invertible, and so  $(\lambda I - T)x = b$  has a unique solution  $x = (\lambda I - T)^{-1}b$ , which can be expanded into a series expression if  $|\lambda| > r(T)$ .
- (b) Suppose  $(\lambda I - T)x = 0$  has a non-trivial solution. Then

$$\begin{aligned} & (\lambda I - T)x = b \text{ has a solution} \\ \iff & b \in \text{IM } \lambda I - T \text{ which is closed} \\ \iff & b \in ((\text{IM } \lambda I - T)^\perp)^\perp \\ \iff & b \in (\text{KER } \bar{\lambda}I - T^*)^\perp \\ \iff & b \perp y \quad \forall y \in \text{KER } \bar{\lambda}I - T^*. \end{aligned}$$

□

**Proposition 26.4** (Miscellaneous).

- (a) If  $M$  is a closed subspace of  $\mathcal{H}$ , then  $M = M^{\perp\perp}$ .
- (b) IF  $S : \mathcal{H} \rightarrow \mathcal{H}$  and  $S \in \mathcal{L}(\mathcal{H})$ , then  $(\text{IM } S)^\perp = \text{KER } S^*$ .

*Proof.*

- (a) Let  $m \in M$ , then  $\langle m, x \rangle = 0$  for all  $x \in M^\perp$ , and so  $m \in (M^\perp)^\perp = M^{\perp\perp}$ , and so  $M \subseteq M^{\perp\perp}$ .  
Let  $x \in M^{\perp\perp}$ . Since  $M$  is closed,  $\mathcal{H} = M \oplus M^\perp$ , and so  $x = m + m^\perp$ . So  $x - m \in M^{\perp\perp} + M \subseteq M^{\perp\perp}$ , and so  $x - m = m^\perp \in M^{\perp\perp}$ . But  $M^\perp$  is closed, and so  $\mathcal{H} = M^\perp \oplus M^{\perp\perp}$ . So  $x - m = 0$ , and  $x = m \in M$ .
- (b)

$$\begin{aligned} (\text{IM } S)^\perp &= \{x \in \mathcal{H} \mid \langle x, sy \rangle = 0 \quad \forall y \in \mathcal{H}\} \\ &= \{x \in \mathcal{H} \mid \langle S^*x, y \rangle = 0 \quad \forall y \in \mathcal{H}\} \\ &= \{x \in \mathcal{H} \mid S^*x = 0\} \\ &= \text{KER } S^* \end{aligned}$$

□