

MATH 3975 - FINANCIAL MATHEMATICS

EXAM NOTES

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1. INTRODUCTION TO MARKETS

1.1. Introduction.

Definition 1.1.1 (Time Value of Money). The **future value** $F(0, t)$ is defined to be the value at time $t > 0$ of \$1 invested at time 0.

The **present value** or **discount factor** $P(0, t)$ is the amount invested at time 0 such that its value at time t is equal to \$1.

To avoid arbitrage, we must have $P(0, t) = F^{-1}(0, t)$ for all t .

Proposition 1.1.2. *We must have the following relationship for $P(t, T)$ and $F(t, T)$.*

$$F(t, T) = \frac{F(0, T)}{F(0, t)}$$
$$P(t, T) = \frac{P(0, T)}{P(0, t)}$$

Definition 1.1.3 (Spot rate). The spot rate $r(t)$ is defined as

$$F(t, t + \Delta) = 1 + r(t)\Delta$$

Proposition 1.1.4. *We have*

$$P(0, T) = \prod_{i=1}^n (1 + r_i)^{-1}$$

in the discrete case, and in the continuous case, we have

$$P(t, T) = e^{-\int_t^T r(s)ds}$$

1.2. Riskless Securities and Bonds.

Theorem 1.2.1 (Fundamental Theorem of Riskless Security Pricing). *If interest rates are deterministic, the arbitrage free price of a riskless security is given by*

$$S_0 = \sum_{i=1}^n P(0, t_i) C_i$$

1

2. SINGLE PERIOD MARKET MODELS

A single period market model is the most elementary model. Only a single period is considered. At times $t = 0$ and 1 market prices are recorded.

Single period market models are the atoms of Financial Mathematics

We assume that we have a finite sample space

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$$

2.1. The most elementary market model. Assume the sample space consists of two states, H and T , with $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = 1 - p$. Define the price of the stock at time 0 to be S_0 , and let S_1 be a random variable depending on H and T . Let $u = \frac{S_1(H)}{S_0}$ and $d = \frac{S_1(T)}{S_0}$.

Definition 2.1.1 (Trading strategy). A trading strategy (x, ϕ) is a pair where x is the total initial investment at $t = 0$, and ϕ denotes the number of shares bought at $t = 0$. Given a strategy (x, ϕ) , the agent invests the remaining money $x - \phi S_0$ in a money market account. We note this amount may be negative (borrowing from the money market account).

Definition 2.1.2 (Value process). The value process of the trading strategy (x, ϕ) in our elementary market model is given by $(V_0(x, \phi), V_1(x, \phi))$ where $V_0(x, \phi) = x$ and

$$V_1 = (x - \phi S_0)(1 + r) + \phi S_1$$

Definition 2.1.3 (Arbitrage). An arbitrage is a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money.

More rigorously, we have a trading strategy (x, ϕ) is an arbitrage if

- $x = V_0(x, \phi) = 0$,
- $V_1(x, \phi) \geq 0$,
- $\mathbb{E}[V_1(x, \phi)] > 0$.

Proposition 2.1.4. *To rule out arbitrage in our model, we must have $d < 1 + r < u$.*

Proof. If this inequality is violated, consider the following strategies.

- If $d \geq 1 + r$, borrow S_0 from the money market.
- If $u \leq 1 + r$, short S_0 and invest in the money market.

These are both arbitrages and the proposition is proven. □

We have the following theorem, giving the converse of the above proposition

Theorem 2.1.5. *The condition $d < 1 + r < u$ is a necessary and sufficient no arbitrage condition. That is,*

$$\text{No arbitrage} \iff d < 1 + r < u$$

Definition 2.1.6 (Replicating strategy or hedge). A **replicating strategy** or **hedge** for the option $h(S_1)$ in our elementary single period market model is a trading strategy (x, ϕ) satisfying $V_1(x, \phi) = h(S_1)$. That is,

$$\begin{aligned}(x - \phi S_0)(1 + r) + \phi S_1(H) &= h(S_1(H)) \\ (x - \phi S_0)(1 + r) + \phi S_1(T) &= h(S_1(T))\end{aligned}$$

Theorem 2.1.7. Let $h(S_1)$ be an option in our market model, and let (x, ϕ) be a replicating strategy for $h(S_1)$. Then x is the only price for the option at time $t = 0$ which does not allow arbitrage.

To find a replicating strategy for an arbitrary option, define

$$\begin{aligned}\phi &= \frac{h(S_1(H)) - h(S_1(T))}{S_1(H) - S_1(T)} \\ \tilde{p} &= \frac{1 + r - d}{u - d}\end{aligned}$$

Then by solving the above two equations for x , we have

$$x = \mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1 + r} h(S_1) \right] = \frac{1}{1 + r} [\tilde{p} h(S_1(H)) + (1 - \tilde{p}) h(S_1(T))]$$

2.2. A general single period market model.

Definition 2.2.1 (Trading strategy). A **trading strategy** for an agent in our general single period market model is a pair (x, ϕ) , where $\phi = (\phi^1, \dots, \phi^n) \in \mathbb{R}^n$ specifying the initial investment in the i -th stock.

Definition 2.2.2 (Value and gains process). Let the **value process** of the trading strategy (x, ϕ) is given $(V_0(x, \phi), V_1(x, \phi))$ where $V_0(x, \phi) = x$ and

$$V_1(x, \phi) = (x - \sum_{i=1}^n \phi^i S_0^i)(1 + r) + \sum_{i=1}^n \phi^i S_1^i$$

The **gains process** is defined as

$$G(x, \phi) = (x - \sum_{i=1}^n \phi^i S_0^i)r + \sum_{i=1}^n \phi^i \Delta S^i$$

where ΔS^i is defined as

$$\Delta S^i = S_1^i - S_0^i$$

We have the simple result

$$V_1(x, \phi) = V_0(x, \phi) + G(x, \phi)$$

To study the prices of the stocks in relation to the money market account, we introduce the **discounted stock prices** \hat{S}_t^i defined as follows:

$$\begin{aligned}\hat{S}_0^i &= S_0^i \\ \hat{S}_1^i &= \frac{1}{1+r} S_1^i\end{aligned}$$

Definition 2.2.3 (Discounted value and gains process). We define the **discounted value process** $\hat{V}(x, \phi)$ by

$$\begin{aligned}\hat{V}_0(x, \phi) &= x \\ \hat{V}_1(x, \phi) &= (x - \sum_{i=1}^n \phi^i S_0^i) + \sum_{i=1}^n \phi^i \hat{S}_1^i\end{aligned}$$

and the discounted gains process $\hat{G}(x, \phi)$ as

$$\hat{G}(x, \phi) = \sum_{i=1}^n \phi^i \Delta \hat{S}^i$$

with $\Delta \hat{S}^i = \hat{S}_1^i - \hat{S}_0^i$.

We then have the relation

$$\hat{V}_1(x, \phi) = \hat{V}_0(x, \phi) + \hat{G}(x, \phi)$$

Definition 2.2.4 (Arbitrage). A trading strategy (x, ϕ) is an arbitrage in our general single period market model if

- $x = V_0(x, \phi) = 0$,
- $V_1(x, \phi) \geq 0$,
- $\mathbb{E}[V_1(x, \phi)] > 0$.

Alternatively, if a trading strategy satisfies the first two conditions above, it is an arbitrage if the following condition is satisfied:

There exists $\omega \in \Omega$ with $V_1(x, \phi) > 0$.

Alternatively, we can replace all references to V in the above definition with \hat{V} .

Definition 2.2.5 (Risk neutral measure). A measure $\tilde{\mathbb{P}}$ on Ω is a **risk neutral measure** if

- $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$
- $\mathbb{E}_{\tilde{\mathbb{P}}}[\Delta \hat{S}^i] = 0$ for all i

Theorem 2.2.6 (Fundamental Theorem of Asset Pricing). *In the general single period market model, there are no arbitrages if and only if there exists a risk neutral measure for the market model.*

Definition 2.2.7 (Alternative definition of arbitrage). Define the set \mathbb{W} by the following:

$$\mathbb{W} = \{X \in \mathbb{R}^k \mid X = \hat{G}(x, \phi) \text{ for some trading strategy } (x, \phi)\}$$

Then, letting \mathbb{A} be given as

$$\mathbb{A} = \{X \in \mathbb{R}^k \mid X \geq 0, X \neq 0\}$$

Then we have a definition of arbitrage:

$$\text{no arbitrage} \iff \mathbb{W} \cap \mathbb{A} = \emptyset$$

Definition 2.2.8 (Set of risk neutral measures). Now, consider the orthogonal component \mathbb{W}^\perp , defined as

$$\mathbb{W}^\perp = \{Y \in \mathbb{R}^k \mid \langle X, Y \rangle = 0 \text{ for all } X \in \mathbb{R}^k\},$$

the set of vectors in \mathbb{R}^k perpendicular to all elements of \mathbb{W} . Furthermore, defining \mathcal{P}^+ as

$$\mathcal{P}^+ = \{X \in \mathbb{R}^k \mid \sum_{i=1}^k X_i = 1, X_i > 0\}$$

Then we have the following theorem.

Theorem 2.2.9. A measure $\tilde{\mathbb{P}}$ is a risk neutral measure on Ω if and only if $\tilde{\mathbb{P}} \in \mathcal{P}^+ \cap \mathbb{W}^\perp$.

We denote the set of risk neutral measures $\mathbb{M} = \mathcal{P}^+ \cap \mathbb{W}^\perp$.

Definition 2.2.10 (Contingent claim). A **contingent claim** in our general single period market model is a random variable X on Ω representing a payoff at time $t = 1$.

Proposition 2.2.11. Let X be a contingent claim in our general single period market model, and let (x, ϕ) be a hedging strategy for X , so that $V_1(x, \phi) = X$ then the only price of X which complies with the no arbitrage principle is $x = V_0(x, \phi)$.

Definition 2.2.12 (Attainable contingent claim). A contingent claim is **attainable** if there exists a trading strategy (x, ϕ) which replicates X , so that $V_1(x, \phi) = X$.

Theorem 2.2.13. Let X be an attainable contingent claim and $\tilde{\mathbb{P}}$ be an arbitrary risk neutral measure. Then the price x of X at time $t = 0$ can be computed by the formula

$$x = \mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1+r} X \right]$$

Corollary. This theorem tells us that in particular, for any risk neutral measure in our model, we get the same value when taking the expectation above.

Definition 2.2.14. We say that a price x for the contingent claim X **complies with the no arbitrage principle** if the extended model, which consists of the original assets S^1, \dots, S^n and an additional asset S^{n+1} which satisfies $S_0^{n+1} = x$ and $S_1^{n+1} = X$ is arbitrage free.

The following proposition shows that when using the risk neutral measure to price a contingent claim, one obtains a price which complies with the no-arbitrage principle.

Proposition 2.2.15. *Let X be a possibly unattainable contingent claim and $\tilde{\mathbb{P}}$ any risk neutral measure for our general single period market model. Then*

$$x = \mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1+r} X \right]$$

defines a price for the contingent claim at time $t = 0$ which complies with the no-arbitrage principle.

Definition 2.2.16 (Complete market). A financial market is called **complete** if for any contingent claim X there exists a replicating strategy (x, ϕ) . A model which is not complete is called **incomplete**.

Proposition 2.2.17. *Assume a general single period market model consisting of stocks S^1, \dots, S^n and a money market account modelled on the state space $\Omega = \{\omega_1, \dots, \omega_k\}$ is arbitrage free. Then this model is complete if and only if the $k \times (n+1)$ matrix A given by*

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) & \dots & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \dots & S_1^n(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1^1(\omega_k) & \dots & S_1^n(\omega_k) \end{pmatrix}$$

has full rank, that is, $\text{rank}(A) = k$.

Proof. First, a matrix A has full rank if and only if for every $X \in \mathbb{R}^k$, the equation $AZ = X$ has a solution $Z \in \mathbb{R}^{n+1}$.

Secondly, we have

$$\begin{pmatrix} 1+r & S_1^1(\omega_1) & \dots & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \dots & S_1^n(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1^1(\omega_k) & \dots & S_1^n(\omega_k) \end{pmatrix} \begin{pmatrix} x - \sum_{i=1}^n \phi^i S_0^i \\ \phi^1 \\ \vdots \\ \phi^n \end{pmatrix} = \begin{pmatrix} V_1(x, \phi)(\omega_1) \\ V_1(x, \phi)(\omega_2) \\ \vdots \\ V_1(x, \phi)(\omega_k) \end{pmatrix}$$

This shows that computing a replicating strategy for a contingent claim X is the same as to solve the equation $AZ = X$, and the proposition follows \square

Proposition 2.2.18. *A contingent claim X is attainable, if and only if $\mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1+r} X \right]$ takes the same value for all $\tilde{\mathbb{P}} \in \mathbb{M}$.*

Theorem 2.2.19. *Under the assumption that the model is arbitrage free, it is complete, if and only if \mathbb{M} consists of only one element - i.e., there is a unique risk measure.*

2.3. Single Period Investment.

Definition 2.3.1. A continuously differentiable function $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ is called a risk averse utility function if it has the following two properties:

- u is strictly increasing - that is, $u'(x) > 0$ for all $x \in \mathbb{R}^+$
- u is strictly concave - that is, $u(\lambda x + (1 - \lambda)y) > \lambda u(x) + (1 - \lambda)u(y)$

In the case that $u''(x)$ exists, the second condition in the above definition is equivalent to $u''(x) < 0$ for all $x \in \mathbb{R}^+$. Sometimes one assumes in addition the condition:

- $\lim_{x \rightarrow 0} u'(x) = +\infty$ and $\lim_{x \rightarrow \infty} u'(x) = 0$

Example 2.3.2 (Utility functions). The following are all risk averse utility functions:

- (1) Logarithmic utility: $u(x) = \log(x)$
- (2) Exponential utility: $u(x) = 1 - e^{-\lambda x}$
- (3) Power utility: $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$ with $\gamma > 0, \gamma \neq 1$
- (4) Square root utility: $u(x) = \sqrt{x}$

Proposition 2.3.3 (Principle of expected utility). *We assume the following axiom of agents behaviour - that of maximising expected utility.*

$$X \text{ is preferred to } Y \iff \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

From Jensen's inequality, we have that for every risk averse utility function u and risky payoff X ,

$$\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$$

Definition 2.3.4 (Certainty equivalent price). The certain value $X_0 \in \mathbb{R}$ that makes an investor indifferent between X_0 and a risky payoff X is called the **certainly equivalent price** of X , that is,

$$u(X_0) = \mathbb{E}[u(X)]$$

or equivalently,

$$X_0 = u^{-1}(\mathbb{E}[u(X)])$$

Lemma 2.3.5. *The certainty equivalent price of a risky payoff is invariant under a positive linear transformation of the utility function $u(x)$.*

Definition 2.3.6 (Risk premium). We have that $X_0 < \mathbb{E}[X]$, and the difference between the two is called the **risk premium** ρ , that is,

$$\rho = \mathbb{E}[X] - u^{-1}(\mathbb{E}[u(X)])$$

We can write the equation above as

$$u(\mathbb{E}[X] - \rho) = \mathbb{E}[u(X)]$$

Definition 2.3.7 (Measures of risk aversion). We define the following risk aversion coefficients, as a measure of how risk averse the investor is. The **absolute risk aversion** ρ_{abs} , given by

$$\rho_{abs} = -\frac{u''(x)}{u'(x)}$$

and the **relative risk aversion** ρ_{rel} , given by

$$\rho_{rel} = -\frac{xu''(x)}{u'(x)}$$

We now seek to find the optimal investment in a market, which can be translated as finding a trading strategy (x, ϕ) such that $\mathbb{E}[u(V_1(x, \phi))]$ achieves an optimal value.

Definition 2.3.8. A trading strategy (x, ϕ^*) is a solution to the optimal portfolio problem with initial investment x and utility function u , if

$$\mathbb{E}[u(V_1(x, \phi^*))] = \max_{\phi} \mathbb{E}[u(V_1(x, \phi))]$$

Proposition 2.3.9. *If there exists a solution to the optimal portfolio problem, then there can not exist an arbitrage in the market.*

Proposition 2.3.10. *Let (x, ϕ) be a solution to the optimal portfolio problem with initial wealth x and utility function u , then the measure \mathbb{Q} defined by*

$$\mathbb{Q}(\omega) = \frac{\mathbb{P}(\omega)u'(V_1(x, \phi)(\omega_i))}{\mathbb{E}[u'(V_1(x, \phi))]}$$

Assume that our model is complete. In this case, there is a unique risk neutral measure which we denote by $\tilde{\mathbb{P}}$.

Definition 2.3.11. We define the set of attainable wealths from initial investment $x > 0$ by

$$\mathbb{W}_x = \left\{ W \in \mathbb{R}^k \mid \mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1+r} W \right] = x \right\}$$

Our optimisation problem is hence:

$$\begin{aligned} & \text{maximise} && \mathbb{E}[u(W)] \\ & \text{subject to} && W \in \mathbb{W}_x \end{aligned}$$

To solve this problem, we use the Lagrange multiplier method. To do this, consider the Lagrange function

$$\mathcal{L}(W, \lambda) = \mathbb{E}[u(W)] - \lambda \left(\mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1+r} W \right] - x \right)$$

By introducing the **state price density**

$$L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)},$$

we can write the Lagrange function as

$$\mathcal{L}(W, \lambda) = \sum_{i=1}^k \mathbb{P}(\omega_i) \left[u(W(\omega_i)) - \lambda \left(L(\omega_i) \frac{1}{1+r} W(\omega_i) - x \right) \right]$$

Computing partial derivatives with respect to $W_i = W(\omega_i)$ and setting them equal to zero, multiplying with $\mathbb{P}(\omega_i)$ and summing over i , we deduce that

$$\lambda = \mathbb{E}[(1+r)u'(W)]$$

and (denoting the inverse function of $u'(x)$ by $I(x)$)

$$W(\omega) = I\left(\lambda \frac{L(\omega)}{1+r}\right)$$

Since we have

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}W\right] = x$$

and substituting the expression from the above equation into the last equation, we obtain

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}I\left(\lambda \frac{L}{1+r}\right)\right] = x$$

3. MULTI PERIOD MARKET MODELS

3.1. The general model.

Definition 3.1.1 (Specification of the general model). The two most important new features of multi period market models are:

- Agents can buy and sell assets not only at the beginning of the trading period, but at any time t out of a discrete set of trading times $t \in \{0, 1, 2, \dots, T\}$.
- Agents can gather information over time, since they can observe prices. Hence, they can make their investment decisions at time $t = 1$ dependent on the prices of the asset at time $t = 1$.

Throughout, we assume we are working on a finite state space Ω on which there is defined a probability measure \mathbb{P} .

Definition 3.1.2 (σ -algebra). A collection \mathcal{F} of subsets of the state space Ω is called a σ -algebra if the following conditions hold:

- $\Omega \in \mathcal{F}$
- If $F \in \mathcal{F}$, then $F^c \in \mathcal{F}$
- If $F_i \in \mathcal{F}$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

Definition 3.1.3 (Partition of a σ -algebra). Let I denote some index set. A **partition** of a σ -algebra \mathcal{F} is a collection of sets $\emptyset \neq A_i \in \mathcal{F}$ for $i \in I$, such that

- Every set $F \in \mathcal{F}$ can be written as a union of some of the A_i .
- The sets A_i are pairwise disjoint.

Definition 3.1.4. A random variable $X : \Omega \rightarrow \mathbb{R}$ is called **\mathcal{F} -measurable**, if for every closed interval $[a, b] \subset \mathbb{R}$, the preimage under X belongs to \mathcal{F} , that is,

$$X^{-1}([a, b]) \in \mathcal{F}$$

Proposition 3.1.5. Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable and (A_i) a partition of the σ -algebra \mathcal{F} , then X is \mathcal{F} -measurable if and only if X is constant on each of the sets of the partition, that is, there exist $c_j \in \mathbb{R}$ for all $j \in I$ such that

$$X(\omega) = c_j \text{ for all } \omega \in A_j$$

Definition 3.1.6. A sequence $(\mathcal{F}_t)_{0 \leq t \leq T}$ of σ -algebras on Ω is called a **filtration** if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$.

Definition 3.1.7. A family (X_t) with $0 \leq t \leq T$ consisting of random variables, is called a **stochastic process**. If $(\mathcal{F}_t)_{0 \leq t \leq T}$ is a filtration, the stochastic process (X_t) is called **(\mathcal{F}_t) -adapted** if for all t we have that X_t is \mathcal{F}_t -measurable.

Definition 3.1.8. Let $(X_t)_{0 \leq t \leq T}$ be a stochastic process on $(\Sigma, \mathcal{F}, \mathbb{P})$. Define

$$\mathcal{F}_t^X = \sigma(X_u^{-1}([a, b]) \mid 0 \leq u \leq t, a \leq b)$$

This is the smallest σ -algebra which contains all the sets $X_u^{-1}([a, b])$ where $0 \leq u \leq t$ and $a \leq b$. Clearly (\mathcal{F}_s^X) is a filtration. It follows immediately from the definition that (X_t) is (\mathcal{F}_t^X) adapted. (\mathcal{F}_t^X) is called the filtration **generated** by the process X .

Definition 3.1.9 (Value process). The **value process** corresponding to the trading strategy $\phi = (\phi_t)_{0 \leq t \leq T}$ is the stochastic process $(V_t(\phi))_{0 \leq t \leq T}$ where

$$V_t(\phi) = \phi_t^0 B_t + \sum_{i=1}^n \phi_t^i S_t^i$$

Definition 3.1.10. A trading strategy $\phi = (\phi_t)$ is called **self financing** if for all $t = 0, \dots, T-1$,

$$\phi_t^0 B_{t+1} + \sum_{i=1}^n \phi_t^i S_{t+1}^i = \phi_{t+1}^0 B_{t+1} + \sum_{i=1}^n \phi_{t+1}^i S_{t+1}^i$$

Lemma 3.1.11. For a self-financing trading strategy $\phi = (\phi_t)$, the value process can be alternatively computed via

$$V_t(\phi) = \phi_{t-1}^0 B_t + \sum_{i=1}^n \phi_{t-1}^i S_t^i$$

Definition 3.1.12 (General multi period market model). A general multi period market model is given by the following data.

- A probability space $(\Sigma, \mathcal{F}, \mathbb{P})$ together with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ of \mathcal{F} .
- A money market account (B_t) which evolves according to $B_t = (1 + r)^t$.
- A number of financial assets $(S_t^1), \dots, (S_t^n)$ which are assumed to be (\mathcal{F}_t) -adapted stochastic processes.
- A set \mathcal{T} of self financing and (\mathcal{F}_t) -adapted trading strategies.

Definition 3.1.13. Assume we are given a general multi period market model as described above. The **increment process** (ΔS_t^i) is defined as

$$\Delta S_t^i = S_t^i - S_{t-1}^i$$

and

$$\Delta B_t = B_t - B_{t-1} = rB_{t-1}$$

Definition 3.1.14 (Gains process). Given a trading strategy ϕ , the corresponding **gains process** $(G_t(\phi))_{0 \leq t \leq T}$ is given by

$$G_t(\phi) = \sum_{s=0}^{t-1} \phi_s^0 \Delta B_{s+1} + \sum_{i=1}^n \sum_{s=0}^{t-1} \phi_s^i \Delta S_{s+1}^i$$

Proposition 3.1.15. An adapted trading strategy $\phi = (\phi)_{0 \leq t \leq T}$ is self financing, if and only if any of the two equivalent statements hold

- $V_t(\phi) = V_0(\phi) + G_t(\phi)$
- $\hat{V}_t(\phi) = \hat{V}_0(\phi) + \hat{G}_t(\phi)$

for all $0 \leq t \leq T$. Here \hat{V}_t and \hat{G}_t denote the discounted value and gains process, as defined in the following definition.

Definition 3.1.16 (Discount processes). The discounted prices are given by

$$\hat{S}_t^i = \frac{S_t^i}{B_t}$$

and discounted gains, discounted value process, and discounted gains process are all defined analogously.

3.2. Properties of the general multi period market model. Here, we redefine the general concepts of financial mathematics, such as arbitrage, hedging, in the context of a multi period market model.

Definition 3.2.1 (Arbitrage). A (self-financing) trading strategy $\phi = (\phi)_{0 \leq t \leq T}$ is called an arbitrage if

- $V_0(\phi) = 0$
- $V_T(\phi) \geq 0$
- $\mathbb{E}[V_T(\phi)] > 0$

Definition 3.2.2 (Contingent claim). A **contingent claim** in a multi period market model is an \mathcal{F}_T -measurable random variable X on Ω representing a payoff at terminal time T . A **hedging strategy** for X in our model is a trading strategy $\phi \in \mathcal{T}$ such that

$$V_T(\phi) = X,$$

that is, the terminal value of the trading strategy is equal to the payoff of the contingent claim.

Proposition 3.2.3. *Let X be a contingent claim in a multi period market model, and let $\phi \in \mathcal{T}$ be a hedging strategy for X , then the only price of X at time t which complies with the no arbitrage principle is $V_t(\phi)$. In particular, the price at the beginning of the trading period at time $t = 0$ is the total initial investment in the hedge.*

Definition 3.2.4 (Attainable contingent claim). A contingent claim X is called **attainable** in \mathcal{T} , if there exists a trading strategy $\phi \in \mathcal{T}$ which replicates X , that is, $V_T(\phi) = X$.

Definition 3.2.5 (Complete market). A general multi period market model is called **complete**, if and only if for any contingent claim X there exists a replicating strategy ϕ . A model which is not complete is called **incomplete**.

Definition 3.2.6 (Conditional expectation). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space and X an \mathcal{F} -measurable random variable. Assume that \mathcal{G} is a σ -algebra which is contained in \mathcal{F} . Denoting the unique partition of \mathcal{G} with $(A_i)_{i \in I}$, the **conditional expectation** $\mathbb{E}[X | \mathcal{G}]$ of X with respect to \mathcal{G} is defined as the random variable which satisfies

$$\mathbb{E}[X | \mathcal{G}](\omega) = \sum_x x \mathbb{P}(X = x | A_i)$$

whenever $\omega \in A_i$.

We then have the following identity:

$$\int_G X d\mathbb{P} = \int_G \mathbb{E}[X | \mathcal{G}] d\mathbb{P}$$

for any $G \in \mathcal{G}$.

Proposition 3.2.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space and X an \mathcal{F} -measurable random variable. Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ be sub- σ -algebras. Assume furthermore that $\mathcal{G}_2 \subset \mathcal{G}_1$. Then*

- **Tower property.**

$$\mathbb{E}[X | \mathcal{G}_2] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}_1] | \mathcal{G}_2]$$

- **Taking out what is known.** *If $Y : \Omega \rightarrow \mathbb{R}$ is \mathcal{G} -measurable, then*

$$\mathbb{E}[YX | \mathcal{G}] = Y\mathbb{E}[X | \mathcal{G}]$$

- *If $\mathcal{G} = \{\emptyset, \Omega\}$ is the trivial σ -algebra, then*

$$\mathbb{E}[X | \mathcal{G}] = \mathbb{E}[X]$$

Definition 3.2.8 (Risk neutral measure). A measure $\tilde{\mathbb{P}}$ on Ω is called a **risk neutral measure** for a general multi period market model if

- $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$
- $\mathbb{E}_{\tilde{\mathbb{P}}} [\Delta \hat{S}_t^i | \mathcal{F}_{t-1}] = 0$ for $i = 1, \dots, n$ and for all $1 \leq t \leq T$.

An alternative formulation of the second condition in the previous definition is

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{1}{1+r} S_{t+1}^i | \mathcal{F}_t \right] = S_t^i$$

Definition 3.2.9 (Martingale). A \mathcal{F}_t -adapted process (X_t) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **martingale** if for all $s < t$,

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s$$

Lemma 3.2.10. *Let $\tilde{\mathbb{P}}$ be a risk neutral measure. Then the discounted stock prices (\hat{S}_t^i) for $i = 1, \dots, n$ are martingales under $\tilde{\mathbb{P}}$.*

Proposition 3.2.11. *Let $\phi \in \mathcal{T}$ be a trading strategy. Then the discounted value process $(\hat{V}_t(\phi))$ and the discounted gains process $\hat{G}_t(\phi)$ are martingales under any risk neutral measure $\tilde{\mathbb{P}}$.*

Theorem 3.2.12 (Fundamental Theorem of Asset Pricing). *Given a general multi period market model, if there is a risk neutral measure, then there are no arbitrage strategies $\phi \in \mathcal{T}$. Conversely, if there are no arbitrages among self financing and adapted trading strategies, then there exists a risk neutral measure.*

Definition 3.2.13. We say that an adapted stochastic process (X_t) is a price process for the contingent claim X which **complies with the no arbitrage principle**, if there is no adapted and self financing arbitrage strategy in the extended model, which consists of the original stocks $(S_t^1), \dots, (S_t^n)$ and an additional asset given by $S_t^{n+1} = X_t$ for $0 \leq t \leq T-1$ and $S_T^{n+1} = X$.

Proposition 3.2.14. *Let X be a possibly unattainable contingent claim and $\tilde{\mathbb{P}}$ a risk neutral measure for a general multi period market model. Then*

$$X_t = B_t \mathbb{E}_{\tilde{\mathbb{P}}} \left[\frac{X}{B_T} | \mathcal{F}_t \right]$$

defines a price for the contingent claim consistent with the no arbitrage principle.

Theorem 3.2.15. *Under the assumption that a general multi period market model is arbitrage free, it is complete, if and only if there is a unique risk neutral measure.*