

**MATH 3969 - MEASURE THEORY AND FOURIER ANALYSIS
EXAM NOTES**

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1. MEASURE THEORY

Definition 1.1 (σ -algebra). Let X be a set. A collection \mathcal{A} of subsets of X is called a σ -algebra if

- $\emptyset \in \mathcal{A}$
- If $A \subseteq X$ is in \mathcal{A} , then its complement $A^c = X \setminus A$ is in \mathcal{A}
- Whenever A_0, A_1, \dots are subsets of X in \mathcal{A} , then their union

$$\bigcup_{k=0}^{\infty} A_k$$

also belongs to \mathcal{A} .

Definition 1.2 (Measure). Let X be a set, and let \mathcal{A} be a σ -algebra of subsets of X . Suppose that $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a function. Then μ is a measure if

- $\mu(\emptyset) = 0$
- Whenever A_0, A_1, \dots are *pairwise disjoint* subsets of X in \mathcal{A} , then

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} \mu(A_k)$$

Proposition 1.3 (Properties of a σ -algebra). *Let \mathcal{A} be a σ -algebra of subsets of a set X . Then*

- $X, \emptyset \in \mathcal{A}$
- If $A_k \in \mathcal{A}$, then $\bigcap_{k=0}^{\infty} A_k \in \mathcal{A}$
- If $A, B \in \mathcal{A}$, then $A \cup B, A \cap B \in \mathcal{A}$

Definition 1.4 (Algebra). A collection \mathcal{A} of subsets A of X which satisfies the first two conditions of a σ -algebra and also

- If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$

is called an algebra. Every σ -algebra is an algebra, but not every algebra is a σ -algebra

Definition 1.5 (σ -algebra generated by \mathcal{S}). Let \mathcal{S} be a collection of subsets of X . Let

$$\mathcal{A}(\mathcal{S}) = \bigcap \{ \mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra, and } \mathcal{S} \subseteq \mathcal{A} \}$$

$\mathcal{A}(\mathcal{S})$ is called the σ -algebra generated by \mathcal{S}

Definition 1.6 (Borel σ -algebra). Let X be a metric space and \mathcal{S} the collection of all open sets in X . We call $\mathcal{B} = \mathcal{A}(\mathcal{S})$ the *Borel σ -algebra*. Sets in \mathcal{B} are called *Borel sets*.

Corollary. We have the following examples of Borel sets.

- Any open set is a Borel set.
- If B is a Borel set, then so is B^c . If B_0, B_1, \dots is a sequence of Borel sets, then so are $\bigcup_{k=0}^{\infty} B_k$ and $\bigcap_{k=0}^{\infty} B_k$.

1.1. Properties of Measures.

Proposition 1.7 (The Monotonicity Property). If A and B are μ -measurable subsets of X with $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

Proposition 1.8 (The Countable Subadditivity Property). If A_0, A_1, \dots are μ -measurable subsets of X , then

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \mu(A_k)$$

Proposition 1.9 (Monotone Convergence Property of Measures). Let $A_0 \subseteq A_1 \subseteq A_2$ be an increasing sequence of measurable sets. Then

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

Proposition 1.10. Let $A_0 \supseteq A_1 \supseteq A_2 \dots$ be sets from some σ -algebra \mathcal{A} . If $\mu(A_0) < \infty$, then

$$\mu\left(\bigcap_{k=0}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

1.2. Constructing σ -algebras and measures.

Definition 1.11 (Lebesgue outer measure). If $A \subseteq \mathbb{R}$, let

$$m^*(A) = \inf \left\{ \sum_{k=0}^{\infty} (b_k - a_k) \mid a_k < b_k, A \subseteq \bigcup_{k=0}^{\infty} (a_k, b_k) \right\}$$

Proposition 1.12 (Properties of the Lebesgue outer measure). The Lebesgue measure obeys the following properties.

- $m^*(A)$ is defined, and $m^*(A) \in [0, \infty]$ for any subset of \mathbb{R} .
- $m^*(\emptyset) = 0$
- If $A \subseteq B$, $m^*(A) \leq m^*(B)$
- For every sequence A_0, A_1, \dots , we have

$$m^*\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} m^*(A_k)$$

Definition 1.13 (Outer Measure). A function $\mu^* : \mathcal{P} \rightarrow [0, \infty]$ is such that

- $\mu^*(\emptyset) = 0$
- If $A \subseteq B$, $\mu^*(A) \leq \mu^*(B)$
- For every sequence A_0, A_1, \dots , we have

$$\mu^*\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \mu^*(A_k)$$

Then μ^* is called an *outer measure* on X .

Theorem 1.14 (Construction from outer measures). Let μ^* be an outer measure on a set X . Then

$$\mathcal{A} = \{A \subseteq X \mid \mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \text{ for all } S \subseteq X\}$$

Then \mathcal{A} is a σ -algebra. Let $\mu(A) = \mu^*(A)$ when $A \in \mathcal{A}$. Then $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a measure.

Proposition 1.15. Let μ^* be an outer measure, and let \mathcal{A} be the σ -algebra defined in the last theorem. Let $A \subseteq X$ satisfy $\mu^*(A) = 0$. Then $A \in \mathcal{A}$, and so $\mu(A)$ is defined, and equals 0.

Definition 1.16 (Null set). If $\mu : \mathcal{A} \rightarrow [0, \infty]$ is any measure, then a set $A \in \mathcal{A}$ satisfying $\mu(A) = 0$ is called a null set.

1.3. Properties of the Lebesgue measure on \mathbb{R} . Let

$$\mathcal{M} = \{A \subseteq \mathbb{R} \mid m^*(S) = m^*(S \cap A) + m^*(S \cap A^c) \text{ for all } S \subseteq \mathbb{R}\}$$

The sets in \mathcal{M} are called the *Lebesgue measurable subsets* of \mathbb{R} . If $A \in \mathcal{M}$, then we write $m(A) = m^*(A)$. This real number is called the *Lebesgue measure* of A .

We now show that this σ -algebra \mathcal{M} is very large.

Theorem 1.17. Let m^* denote the Lebesgue outer measure on \mathbb{R} , and let \mathcal{M} be the σ -algebra of Lebesgue measurable sets. Then

- If $I \subseteq \mathbb{R}$ is an interval. Then $m^*(I) = l(I)$. That is, the outer measure is just its length.
- If $I \subseteq \mathbb{R}$ is an interval, then $I \in \mathcal{M}$.

Proposition 1.18. Any open subset of \mathbb{R} is in \mathcal{M} . Any closed subset of \mathbb{R} is in \mathcal{M} . That is, all open or closed sets in \mathbb{R} are Lebesgue measurable.

Corollary. Every Borel subset of \mathbb{R} is contained in \mathcal{M} .

Proof. \mathcal{M} is a σ -algebra which contains every open subset of \mathbb{R} . The σ -algebra \mathcal{B} is by definition the smallest such σ -algebra. Thus $\mathcal{B} \subseteq \mathcal{M}$. \square

2. MEASURABLE FUNCTIONS

Definition 2.1 (Measurable function). Let \mathcal{A} be σ -algebra of subsets of a set X . A function $f : X \rightarrow \overline{\mathbb{R}}$, is called *measurable* (or \mathcal{A} -measurable) if for every $\alpha \in \mathbb{R}$, the set

$$\{x \in X \mid f(x) > \alpha\}$$

is in \mathcal{A} .

Definition 2.2 (Indicator function). Let $S \subset X$. We define the *indicator function* of S to be the function $1_S : X \rightarrow \mathbb{R}$ given by

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Proposition 2.3. Let $S \subset X$. Then 1_S is measurable if and only if $S \in \mathcal{A}$.

Proof. Let $\alpha \in \mathbb{R}$. Then

$$\{x \in X \mid 1_S(x) > \alpha\} = \begin{cases} \emptyset & \text{if } \alpha \geq 1 \\ S & \text{if } 0 \leq \alpha < 1 \\ X & \text{if } \alpha < 0 \end{cases}$$

As \emptyset, X are in \mathcal{A} , then 1_S is measurable if and only if $S \in \mathcal{A}$. □

Proposition 2.4 (Continuous functions are Lebesgue measurable). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then f is Lebesgue measurable on $[a, b]$. More generally, if $X \subset \mathbb{R}$ is in \mathcal{M} and $f : X \rightarrow \mathbb{R}$ is continuous, then f is Lebesgue measurable on X .

2.1. Basic properties of measurable functions.

Lemma 2.5. Let \mathcal{A} be a σ -algebra of subsets of a set X , and let $f : X \rightarrow \overline{\mathbb{R}}$ be a function. Then f is measurable if and only if it satisfies one of the following conditions.

- For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) > \alpha\}$ is in \mathcal{A} .
- For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) < \alpha\}$ is in \mathcal{A} .
- For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) \leq \alpha\}$ is in \mathcal{A} .
- For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) \geq \alpha\}$ is in \mathcal{A} .

Proposition 2.6. Let \mathcal{A} be a σ -algebra of subsets of a set X , and let $f, g : X \rightarrow \overline{\mathbb{R}}$ be functions. Then

- $f + g$ is measurable (provided that $f(x) = \infty$ and $g(x) = -\infty$ or vice versa holds for no $x \in X$).
- cf is measurable for any constant $c \in \mathbb{R}$.

- fg is measurable.
- f/g is measurable (provided that $g(x)$ is nonzero and not infinity for all $x \in X$).

Similarly, let $f_0, f_1, \dots : X \rightarrow \overline{\mathbb{R}}$. Then

- $\sup\{f_0, f_1, \dots\}$ and $\inf\{f_0, f_1, \dots\}$ are measurable functions.

Corollary. Let f, g be measurable. Then $\max\{f, g\}$ and $\min\{f, g\}$ are measurable functions.

Proposition 2.7. Let \mathcal{A} be a σ -algebra of subsets of a set X , and let $f_0, f_1, \dots : X \rightarrow \overline{\mathbb{R}}$ be measurable functions. Let $f(x) = \lim_{k \rightarrow \infty} f_k(x)$ for each $x \in X$. Then f is a measurable function.

2.2. Simple functions.

Definition 2.8 (Simple function). Let \mathcal{A} be a σ -algebra of subsets of a set X . A function $\varphi : X \rightarrow \mathbb{R}$ is called *simple* if it is measurable and only takes a finite number of values.

Proposition 2.9. Let \mathcal{A} be a σ -algebra of subsets of a set X , and let $f : X \rightarrow [0, \infty]$ be a nonnegative measurable function. Then there is a sequence (φ_n) of simple functions such that

- $0 \leq \varphi_1(x) \leq \varphi_2(x) \leq \dots \leq f(x)$ for all $x \in X$.
- $f(x) = \lim_{n \rightarrow \infty} \varphi_n(x)$ for all $x \in X$.

Proof. Define the function φ_n as follows.

Let

$$A_{n,k} = \{x \in X \mid \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}$$

and let $A_{n,2^n} = \{x \in X \mid f(x) \geq n\}$

Then the function

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} 1_{A_{n,k}}$$

obeys the required properties. □

3. INTEGRATION

Definition 3.1 (Integration of simple functions). Let $\varphi = \sum_{j=1}^m a_j 1_{A_j}$. Then the *integral* $\int_X \varphi d\mu$ of φ over X with respect to μ is given by

$$\int_X \varphi d\mu = \sum_{j=1}^m a_j \mu(A_j)$$

Proposition 3.2. Let φ and ψ be nonnegative simple functions on X , and let $c \geq 0$ be constant. Then

- $\int_X \varphi + \psi d\mu = \int_X \varphi d\mu + \int_X \psi d\mu$
- If $0 \leq \psi \leq \varphi$, then

$$0 \leq \int_X \psi d\mu \leq \int_X \varphi d\mu$$

- $\int_X c\varphi d\mu = c \int_X \varphi d\mu$

Definition 3.3 (Integral over a subset of X). Let φ be a nonnegative simple function, and let $S \subset X$ be measurable. Then the integral of φ over S with respect to μ , denoted $\int_S \varphi d\mu$, is given by

$$\int_S \varphi d\mu = \int_X \varphi \cdot 1_S d\mu$$

3.1. Integration of nonnegative measurable functions.

Definition 3.4. Let $f : X \rightarrow [0, \infty]$ be a nonnegative measurable function. We define the integral $\int_X f d\mu$ of f over X with respect to μ by

$$\int_X f d\mu = \sup \left\{ \int_X \varphi d\mu \mid \varphi \text{ is simple, and } 0 \leq \varphi \leq f \text{ on } X \right\}$$

Lemma 3.5. Suppose that f, g are two nonnegative measurable functions, and $0 \leq g \leq f$ on X . Then

$$0 \leq \int_X g d\mu \leq \int_X f d\mu$$

The following is an extremely important theorem in measure theory.

Theorem 3.6 (Monotone convergence theorem). Let (f_k) be a sequence of nonnegative measurable functions on X . Assume that

- $0 \leq f_0(x) \leq f_1(x) \leq \dots$ for each $x \in X$,
- $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ for each $x \in X$.

When these hold, we write $f_k \nearrow f$ pointwise.

Then f is measurable, and

$$\int_X f d\mu = \lim_{k \rightarrow \infty} \int_X f_k d\mu$$

Corollary. Let f, g be measurable on X . Then

$$\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu$$

Theorem 3.7. Suppose that f_k is a nonnegative measurable function for $k = 0, 1, \dots$. Then

$$\int_X \left(\sum_{k=0}^{\infty} f_k \right) d\mu = \sum_{k=0}^{\infty} \left(\int_X f_k d\mu \right)$$

Theorem 3.8. Suppose that X is a set, \mathcal{A} is a σ -algebra of subsets of X and $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a measure. Let f be nonnegative and measurable on X . Then define $\mu_f : \mathcal{A} \rightarrow [0, \infty]$ by

$$\mu_f(A) = \int_A f d\mu = \int_X f \cdot 1_A d\mu$$

Then μ_f is a measure.

Proposition 3.9. Suppose that f is nonnegative and measurable on X , and suppose that $\int_X f d\mu < \infty$. Then the set $\{x \in X \mid f(x) = \infty\}$ has measure 0.

Proposition 3.10. Suppose that X is a set, \mathcal{A} is a σ -algebra of subsets of X and $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a measure. Suppose that there is a set $N \in \mathcal{A}$ with $\mu(N) = 0$ and suppose that some property P holds for all $x \in X$ outside N . Then we say that the property P holds almost everywhere or for almost all $x \in X$.

Proposition 3.11. Suppose that f_k is a nonnegative measurable function on X for $k = 0, 1, \dots$. Suppose that

$$\sum_{k=0}^{\infty} \left(\int_X f_k d\mu \right) < \infty$$

Then

$$\sum_{k=0}^{\infty} f_k(x) < \infty \text{ for almost all } x \in X$$

Proposition 3.12. Suppose that f is a nonnegative measurable function on X . Then

$$\int_X f d\mu = 0 \iff f(x) = 0 \text{ almost everywhere.}$$

Theorem 3.13 (Fatou's Lemma). Suppose that f_k is a nonnegative measurable function on X , for $k = 0, 1, \dots$, and that $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ for each $x \in X$. Then

$$\int_X f d\mu \leq \liminf_{k \rightarrow \infty} \left(\int_X f_k d\mu \right)$$

3.2. Integration of real and complex valued functions.

Lemma 3.14. Let $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function, and let f^+ and f^- be the positive and negative parts of f . Then $f = f^+ - f^-$, and $|f| = f^+ + f^-$. Moreover, $|f|$ is a measurable function, and

$$f \text{ is integrable if and only if } \int_X |f| d\mu < \infty$$

Definition 3.15 (Integral of a complex valued function). Let $f = u + iv$, where $u, v : X \rightarrow \overline{\mathbb{R}}$. Then

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu$$

Lemma 3.16. Let $f : X \rightarrow \overline{\mathbb{C}}$. Then f is integrable if and only if $\int_X |f| d\mu < \infty$.

The next theorem is probably the most important single theorem in these notes. It has many applications, both of a theoretical and practical nature.

Theorem 3.17 (Dominated convergence theorem). Let (f_k) be a sequence of real or complex valued measurable function on X . Assume that

- $\lim_{k \rightarrow \infty} f_k(x) = f(x)$

and that there is a measurable function $g : X \rightarrow [0, \infty]$ such that

- $|f_k(x)| \leq g(x)$ for each k and x , and
- $\int_X g \, d\mu < \infty$

Then

$$\int_X f \, d\mu = \lim_{k \rightarrow \infty} \int_X f_k \, d\mu$$

Theorem 3.18 (Bounded convergence theorem). *Let (f_k) be a sequence of real or complex valued measurable function on X . Assume that*

- $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ for each $x \in X$,
- *There exists a constant $M < \infty$ such that $|f_k(x)| \leq M$ for each k and x ,*
- $\mu(X) < \infty$.

Then

$$\int_X f \, d\mu = \lim_{k \rightarrow \infty} \int_X f_k \, d\mu$$

Theorem 3.19. *Suppose that f_k is a measurable real or complex valued function on X for $k = 0, 1, \dots$. Suppose that we have*

$$\sum_{k=0}^{\infty} \left(\int_X |f_k| \, d\mu \right) < \infty$$

or, equivalently,

$$\int_X \left(\sum_{k=0}^{\infty} |f_k| \right) \, d\mu < \infty$$

Then we have

$$\int_X \left(\sum_{k=0}^{\infty} f_k \right) \, d\mu = \sum_{k=0}^{\infty} \left(\int_X f_k \, d\mu \right)$$

Definition 3.20 (Integrable function). We call $f : X \rightarrow \mathbb{K}$ μ -integrable if f is μ measurable and

$$\int_X |f| \, d\mu < \infty$$

We set

$$\mathcal{L}^1(X, \mathbb{K}) = \{f : X \rightarrow \mathbb{K} \mid f \mu\text{-integrable}\}$$

Theorem 3.21. $\mathcal{L}^1(X, \mathbb{K})$ is a vector space over \mathbb{K}

Definition 3.22 (The Lebesgue-Stieltjes integral). Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing right continuous function, that is $\lim_{s \rightarrow t+} F(s) = F(t)$ for all $t \in \mathbb{R}$. Then for $A \subseteq \mathbb{R}$ let

$$\mu_F^*(A) = \inf \left\{ \sum_{k=0}^{\infty} (F(b_k) - F(a_k)) \mid A \subseteq \bigcup_{k \in \mathbb{N}} (a_k, b_k) \right\}$$

The μ_F^* is an outer measure inducing an inner measure on \mathbb{R} . Then we have

- μ_F is a Borel measure.
- $\mu_F((a, b]) = F(b) - F(a)$.

We then define $\int_A f dF = \int_A f d\mu_F$ as the Lebesgue-Stieltjes integral.

Lemma 3.23. *If μ is a finite measure on \mathbb{R} , then we define $F(t) = \mu((-\infty, t])$ as the **distribution function** of \mathbb{R} .*

Theorem 3.24. *There is a bijection from finite measures and some class of right-continuous increasing functions.*

Definition 3.25 (Measures from other measures). Let $g : X \rightarrow [0, \infty]$ be a μ -measurable function. For $A \in \mathcal{A}$ define

$$\nu(A) = \int_A g d\mu$$

Then using the monotone convergence theorem one can show that ν is a measure defined on \mathcal{A} . Moreover, if $f : X \rightarrow \mathbb{K}$ is μ -measurable, then

$$\int_X f d\nu = \int_X fg d\mu$$

We call g the **density of ν with respect to μ** .

Proposition 3.26. *Let $f \in \mathcal{L}^1(X, \mathbb{K})$ with respect to the Lebesgue measure. Then*

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

3.3. Parameter integrals.

Definition 3.27 (Parameter integral). Let (X, \mathcal{A}, μ) be a measure space and Y a metric space. Suppose that $f : X \times Y \rightarrow \mathbb{K}$ is such that

- $x \mapsto f(x, y)$ is μ -integrable for all $y \in Y$,
- $y \mapsto f(x, y)$ is continuous at y_0 for almost all $x \in X$,
- there exists $g \in \mathcal{L}^1(X, \mathbb{R})$ such that

$$|f(x, y)| \leq g(x)$$

for almost all $x \in X$.

Define $F(y) = \int_X f(x, y) d\mu(x)$. Then F is continuous at $y_0 \in Y$.

Theorem 3.28 (Differentiation of parameter integrals.). *Let (X, \mathcal{A}, μ) be a measure space and $L \subset \mathbb{R}$ an interval. Suppose that $f : X \times L \rightarrow \mathbb{R}$ is such that*

- $x \mapsto f(x, y)$ is μ -integrable for all $y \in Y$,
- $\frac{\partial}{\partial t} f(x, t)$ exists for all $t \in L$, for almost all $x \in X$, and is continuous ,
- there exists $g \in \mathcal{L}^1(X, \mathbb{R})$ with $|\frac{\partial}{\partial t} f(x, t)| < g(x)$ for almost all $x \in X$ and all $t \in L$.

Define $F(t) = \int_X f(x, t) d\mu(x)$. Then $f : L \rightarrow \mathbb{K}$ is differentiable and

$$F'(t) = \int_X \frac{\partial}{\partial t} f(x, t) d\mu(x)$$

4. THE L^p -SPACES

Definition 4.1 (L^p -spaces). Let $1 \leq p < \infty$ and $f : X \rightarrow \mathbb{K}$ measurable. We call

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

the L^p -norm of f . We set

$$\mathcal{L}^p(X) = \{f : X \rightarrow \mathbb{K} \mid f \text{ measurable, } \|f\|_p < \infty\}$$

Theorem 4.2 (Hölder's inequality). Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathcal{L}^p(X)$ and $g \in \mathcal{L}^q(X)$, then

$$\left| \int_X fg d\mu \right| \leq \|f\|_p \|g\|_q$$

Proposition 4.3 (Minkowski's inequality). If $f, g \in \mathcal{L}^p$, $1 \leq p \leq \infty$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Definition 4.4 (L^p -spaces). Let $f \sim g$ if $f = g$ almost everywhere. Denote the equivalence class of f by $[f]$. Then

$$L^p(X) = \{[f] \mid f \in \mathcal{L}^p(X)\}$$

Definition 4.5 (Cauchy sequence). A sequence $(f_n) \in L^p(X)$ is called **Cauchy** if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \epsilon$$

for all $n, m > n_0$.

Theorem 4.6 (Completeness of $L^p(X)$). Let (f_n) be a sequence in $L^p(X)$. Then (f_n) converges in $L^p(X)$ if and only if (f_n) is a Cauchy sequence.

Remark. Introducing the metric $d(f, g) = \|f - g\|_p$, we have that $L^p(X)$ is a **complete normed space** or a **Banach space**. If $p = 2$, then $\|f\|_2$ is induced by an inner product - hence $L^2(X)$ is a **complete inner product space**, or a **Hilbert space**.

Proposition 4.7. Suppose that $f_n, f \in \mathcal{L}^p(X)$ with $\|f_n - f\| \rightarrow 0$. Then there exists a subsequence (f_{n_k}) with f_{n_k} converging pointwise to f for almost every $x \in X$.

Theorem 4.8. The simple functions are dense in $L^p(X)$ for $1 \leq p < \infty$.

In \mathbb{R}^N and the Lebesgue measure, we can modify the statement to the simple function with bounded support are dense in \mathbb{R}^N .

Theorem 4.9. For $1 \leq p < \infty$

$$\text{span}\{1_U \mid U \subseteq \mathbb{R}^N \text{ open and bounded}\}$$

is dense in $L^p(\mathbb{R}^N)$.

We can also use bounded rectangles in the place of open bounded sets here.

Definition 4.10 (Essential supremum). Let (X, \mathcal{A}, μ) be a measure space and $f : X \rightarrow \mathbb{R}$ μ -measurable. We call $\text{ess-sup} f(x) = \inf\{t \in \mathbb{R} \mid \mu(\{x \in X \mid f(x) > t\}) = 0\}$ the *essential supremum* of f . The essential supremum of $|f|$ is denoted $\|f\|_\infty$

Theorem 4.11 (Completeness of $L^\infty(X)$). $L^\infty(X)$ is a complete normed space.

Lemma 4.12. Hölder's inequality holds for $p = 1, q = \infty$. That is,

$$\left| \int_X fg \, d\mu \right| \leq \|f\|_p \|g\|_q$$

Lemma 4.13. If $\mu(X) < \infty$, then $\lim_{p \rightarrow \infty} \|u\|_p = \|f\|_\infty$

4.1. Fubini's Theorem.

Theorem 4.14 (Tonelli). Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [0, \infty]$ is measurable. Then there exist sets $N \subseteq \mathbb{R}^n$ and $M \subseteq \mathbb{R}^m$ of measure zero such that

- (i) $x \mapsto f(x, y)$ is measurable for all $y \in \mathbb{R}^m - M$,
- (ii) $y \mapsto \int_{\mathbb{R}^n} f(x, y) \, dx$ is measurable,
- (iii) $y \mapsto f(x, y)$ is measurable for all $x \in \mathbb{R}^n - N$,
- (iv) $x \mapsto \int_{\mathbb{R}^m} f(x, y) \, dy$ is measurable,
- (v)

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y) \, d(x, y) &= \\ \int_{\mathbb{R}^m - M} \left(\int_{\mathbb{R}^n} f(x, y) \, dx \right) \, dy &= \\ \int_{\mathbb{R}^n - N} \left(\int_{\mathbb{R}^m} f(x, y) \, dy \right) \, dx \end{aligned}$$

Theorem 4.15 (Fubini). Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [0, \infty]$ is measurable. Let N, M be the sets from Theorem 4.14 applied to the function $|f|$ such that (v) holds with f replaced with $|f|$. Assume that one of these integrals is finite - and hence all of them. Then there exists sets N_1 of \mathbb{R}^n and M_1 of \mathbb{R}^m such that (i) - (v) of Theorem 4.14 hold with N, M replaced with N_1, M_1 .

Definition 4.16 (Complete measure space). Let (X, \mathcal{A}, μ) be a measure space. We call the measure μ *complete* if whenever $A \in \mathcal{A}$ has measure 0, then any subset of A is in \mathcal{A} , (and has measure 0).

Definition 4.17 (σ -finite measure space).

5. CONVOLUTION

Definition 5.1 (Translation of a function). Let $f : \mathbb{R}^N \rightarrow \mathbb{C}$ be a function and $t \in \mathbb{R}^N$ a fixed vector. We define the translation operator τ_t by

$$\tau_t f(x) = f(x - t)$$

Theorem 5.2 (Continuity of translation). Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^N)$. Then

$$\lim_{t \rightarrow 0} \|\tau_t f - f\|_p = 0$$

Remark. This does not hold if $p = \infty$.

Lemma 5.3. Let $f : \mathbb{R}^N \rightarrow \mathbb{C}$ be measurable and set

$$\begin{aligned} F_1(x, y) &= f(x) \\ F_2(x, y) &= f(y - x) \end{aligned}$$

Then $F_1, F_2 : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{C}$ are measurable.

Definition 5.4 (Convolution). Let $f, g : \mathbb{R}^N \rightarrow \mathbb{C}$ be measurable. We define the **convolution** $f \star g : \mathbb{R}^N \rightarrow \mathbb{C}$ by

$$(f \star g)(x) = \int_{\mathbb{R}^N} f(x - y)g(y) dy$$

wherever the integral exists

Definition 5.5 (Convex function). A function $\varphi : (a, b) \rightarrow \mathbb{R}$ is called **convex** if

$$\varphi(\lambda s + (1 - \lambda)t) \leq \lambda \varphi(s) + (1 - \lambda)\varphi(t)$$

for all $s, t \in (a, b)$ and all $\lambda \in (0, 1)$

Lemma 5.6. This is equivalent to the condition

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

whenever $a < s < t < u < b$.

Theorem 5.7 (Jensen's inequality in $\mathcal{L}^p(X)$ -spaces). Let $f \in \mathcal{L}^p(X)$, $1 \leq p < \infty$, and let $g \in \mathcal{L}^1(X)$. Then

$$\left(\int_X |fg| d\mu \right)^p \leq \|g\|_1^{p-1} \int_X |f|^p |g| d\mu$$

Theorem 5.8 (Young's inequality). Let $1 \leq p \leq \infty$. If $f \in L^p(\mathbb{R}^N, \mathbb{C})$ and $g \in L^1(\mathbb{R}^N, \mathbb{C})$, then $f \star g$ exists almost everywhere and $f \star g \in L^p(\mathbb{R}^N, \mathbb{C})$. Moreover,

$$\|f \star g\|_p \leq \|f\|_p \|g\|_1$$

Theorem 5.9. Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, then

$$f \star g \in BC(\mathbb{R}^N)$$

where BC is the vector space of bounded continuous functions.

5.1. Approximate identities.

Definition 5.10 (Approximate identity). Let $\varphi : \mathbb{R}^N \rightarrow [0, \infty)$ be measurable with

$$\int_{\mathbb{R}^N} \varphi dx = 1$$

and set $\varphi_n(x) = n^N \varphi(nx)$ for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Then (φ_n) is called an **approximate identity**

Theorem 5.11. Let (φ_n) be an approximate identity and $f \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Then

$$f \star \varphi_n \rightarrow f$$

in $L^p(\mathbb{R}^N)$ as $n \rightarrow \infty$

Theorem 5.12. Let $f \in L^\infty(\mathbb{R}^N)$ and (φ_n) an approximate identity. If f is continuous at x , then

$$f(x) = \lim_{n \rightarrow \infty} (f \star \varphi_n)(x)$$

Definition 5.13 (Test function). Let $U \subseteq \mathbb{R}^N$ be open. We let

$$C^\infty(U, \mathbb{K}) = \{f : U \rightarrow \mathbb{K} \mid f \text{ has partial derivatives of all orders}\}$$

and

$$C_c^\infty(U, \mathbb{K}) = \{f \in C^\infty(U, \mathbb{K}) \mid \text{supp}(f) \subseteq U, \text{supp}(f) \text{ compact}\}$$

The functions in $C_c^\infty(U, \mathbb{K})$ are called **test functions** on U .

Proposition 5.14. Let $f : \mathbb{R}^N \rightarrow \mathbb{K}$ be measurable such that $f \in \mathcal{L}^1(B)$ for every bounded set $B \subseteq \mathbb{R}^N$. If $\varphi \in C_c^\infty(\mathbb{R}^N)$, then $f \star \varphi \in C^\infty(\mathbb{R}^N)$ and

$$\frac{\partial}{\partial x_i} (f \star \varphi) = f \star \frac{\partial \varphi}{\partial x_i}$$

Theorem 5.15. Let $U \subseteq \mathbb{R}^N$ open and $1 \leq p < \infty$. Then $C_c^\infty(U)$ is dense in $L^p(U)$.

Remark. The above proposition does not hold for $p = \infty$.

6. THE FOURIER TRANSFORM

Definition 6.1 (Fourier transform). Let $f \in L^1(\mathbb{R}^N, \mathbb{C})$. We call

$$\widehat{f}(t) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot t} dx$$

Theorem 6.2. We have

- $\widehat{f}: \mathbb{R}^N \rightarrow \mathbb{C}$ is continuous,
- $\|\widehat{f}\|_\infty \leq \|f\|_1$

Proposition 6.3. Let $\varphi(x) = e^{-\pi|x|^2}$. Then $\|\varphi\|_1 = 1$ and $\widehat{\varphi} = \varphi$.

Proposition 6.4. Let $f \in L^1(\mathbb{R}^N, \mathbb{C})$, $x_0 \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}, \alpha > 0$.

- (i) If $g(x) = f(x - x_0)$, then $\widehat{g}(t) = e^{-2\pi i x_0 \cdot t} \widehat{f}(t)$,
- (ii) If $g(x) = f(\alpha x)$, then $\widehat{g}(t) = \frac{1}{\alpha^N} \widehat{f}\left(\frac{t}{\alpha}\right)$,
- (iii) If $g(x) = \overline{f(-x)}$, then $\widehat{g}(t) = \overline{\widehat{f}(t)}$

Definition 6.5. Let $C_0(\mathbb{R}^N, \mathbb{K}) = \{f \in C(\mathbb{R}^N, \mathbb{K}) \mid \lim_{|x| \rightarrow \infty} f(x) = 0\}$, the set of continuous functions vanishing at infinity

Theorem 6.6 (Riemann-Lebesgue). If $f \in L^1(\mathbb{R}^N, \mathbb{C})$, then $\widehat{f} \in C_0(\mathbb{R}^N, \mathbb{C})$ and $\|\widehat{f}\|_\infty \leq \|f\|_1$.

Theorem 6.7. If $f, g \in L^1(\mathbb{R}^N, \mathbb{C})$ then $f \star g \in L^1(\mathbb{R}^N, \mathbb{C})$ and

$$\widehat{f \star g} = \widehat{f} \widehat{g}$$

Proposition 6.8. Let $f, g \in L^1(\mathbb{R}^N, \mathbb{C})$. Then

$$\int_{\mathbb{R}^N} \widehat{f} g \, dx = \int_{\mathbb{R}^N} f \widehat{g} \, dx$$

Lemma 6.9. Let $\varphi(x) = e^{-\pi|x|^2}$ and $\varphi_n(x) = n^N \varphi(nx)$. Then

$$\int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} \varphi\left(\frac{t}{n}\right) dt = (f \star \varphi_n)(x)$$

for all $f \in L^1(\mathbb{R}^N, \mathbb{C})$, $x \in \mathbb{R}^N$, and $n \in \mathbb{N}$.

Theorem 6.10 (Fourier inversion formula). Let $f \in L^1(\mathbb{R}^N, \mathbb{C})$. Then

(i)

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} dt = f$$

in $L^1(\mathbb{R}^N, \mathbb{C})$.

(ii) If f is continuous at x , then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} dt = f(x)$$

Corollary. Let $f, g \in L^1(\mathbb{R}^N)$ with $\widehat{f} = \widehat{g}$. Then $f = g$ almost everywhere.

6.1. The Fourier transform on $L^2(\mathbb{R}^N)$. We have defined the Fourier transform \widehat{f} with $f \in L^1(\mathbb{R}^N)$. We have that $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$ as well as in $L^1(\mathbb{R}^N)$, so in particular $L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$. We can use this to extend the Fourier transform to $L^2(\mathbb{R}^N)$. The key for doing so is the following theorem.

Theorem 6.11 (Plancherel). *Let $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Then $\|\widehat{f}\|_2 = \|f\|_2$.*

Proposition 6.12. *There is a unique continuous linear operator*

$$\mathcal{F} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$$

such that $\mathcal{F}f = \widehat{f}$ for all $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Moreover, $\|f\|_2 = \|\mathcal{F}f\|_2$ for all $f \in L^2(\mathbb{R}^N)$.

Remark. We use the notation $\widehat{f} = \mathcal{F}f$ for $f \in L^2(\mathbb{R}^N)$.

Remark. Let $\varphi_n : \mathbb{R}^N \rightarrow [0, 1]$ such that $\varphi_n \in L^2(\mathbb{R}^N)$ and $\varphi_n(x) \rightarrow 1$ for all $x \in \mathbb{R}^N$. If $f \in L^2(\mathbb{R}^N)$, then

$$\widehat{f} = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x) \varphi_n(x) e^{-2\pi i x \cdot t} dx$$

Common choices for φ_n are

- $\varphi_n(x) = 1_{B(0, n)}$,
- $\varphi_n(x) = e^{-\pi \frac{|x|^2}{n^2}}$

Theorem 6.13. *Let $f \in L^2(\mathbb{R}^N, \mathbb{C})$. Then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} dt = f$$

in $L^2(\mathbb{R}^N, \mathbb{C})$.

Theorem 6.14. $\mathcal{F} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ *is bijective with $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$ for all $f \in L^2(\mathbb{R}^N)$.*

Remark. Let

$$\langle f, g \rangle = \int_{\mathbb{R}^N} f(x) \overline{g(x)} dx$$

denote the inner product on $L^2(\mathbb{R}^N)$. Then the above theorem implies

$$\langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle.$$

Moreover, by approximating f, g by functions in $L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ we also have

$$\langle \widehat{f}, g \rangle = \langle f, \widehat{g} \rangle$$

7. THE RADON-NIKODYM THEOREM

7.1. The Riesz representation theorem. Let H be an inner product space with inner product $(x|v)$. Then H is a normed space with norm

$$\|u\| = \sqrt{(u|u)}$$

We call H a **Hilbert space** if H is complete with respect to $\|\cdot\|$, that is, every Cauchy sequence in H converges.

Theorem 7.1 (Projections). *Let H be a Hilbert space and $M \subseteq H$ a closed subspace of H . Let $u \in H$. Then there exists $m_0 \in M$ such that*

$$\|u - m_0\| = \min_{m \in M} \|u - m\|$$

Moreover,

$$(u - m_0 | m) = 0$$

for all $m \in M$.

Remark. Fix $g \in H$ and consider the function $\varphi_g : H \rightarrow \mathbb{K}$ given by

$$\varphi_g(f) = (f | g)$$

Then φ_g is linear, and by the Cauchy-Swartz inequality,

$$|\varphi_g(f)| = |(f | g)| \leq \|f\| \|g\|$$

for all $f \in H$. We say φ_g is a bounded linear functional on H .

Definition 7.2. Let H be a Hilbert space. We call a linear operator $\varphi : H \rightarrow \mathbb{K}$ a **bounded linear functional** on H if there exists $M > 0$ such that

$$|\varphi(f)| \leq M \|f\|$$

for all $f \in H$.

Theorem 7.3 (Riesz representation theorem). *Let H be a Hilbert space over \mathbb{K} and $\varphi(H \rightarrow \mathbb{K})$ a bounded linear function. Then there exists $g \in H$ such that*

$$\varphi(f) = (f | g)$$

for all $f \in H$.

7.2. The Radon-Nikodym Theorem. Suppose that μ is a measure defined on the σ -algebra \mathcal{A} of subsets of X . Given a measurable function $g : X \rightarrow [0, \infty]$ we define

$$\nu(A) = \int_A g d\mu$$

Then ν is a measure defined on the σ -algebra \mathcal{A} .

The converse does not necessarily hold - that is, given two measures μ and ν on a σ -algebra \mathcal{A} , there is not always a measurable function $g : X \rightarrow [0, \infty]$ such that the above equation holds.

Definition 7.4 (Absolute continuity). Let ν, μ be the measures defined on a σ -algebra \mathcal{A} . We call ν **absolutely continuous with respect to μ** if $\nu(A) = 0$ whenever $\mu(A) = 0$. In that case, we write $\nu \ll \mu$.

Proposition 7.5. Let μ, ν be measures defined on a σ -algebra. Suppose that $\mu(X), \nu(X) < \infty$. Set $\lambda = \mu + \nu$. Then there exists a measurable function $h : X \rightarrow [0, \infty]$ such that

$$\int_X f d\nu = \int_X fh d\lambda$$

for all $f \in L^2(X, \lambda)$

Theorem 7.6 (Radon-Nikodym). Let μ, ν be measures defined on a σ -algebra. Suppose that ν and μ are σ -finite and that $\nu \ll \mu$. Then there exists a measurable function $g : X \rightarrow [0, \infty)$ such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{A}$.

Formally we can write

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu$$

if we define $g = \frac{d\nu}{d\mu}$, where g is the density function from the Radon-Nikodym theorem.

Remark. If g is the function in the Radon-Nikodym theorem, it is not hard to show that

$$\int_X f d\nu = \int_X fg d\mu$$

for all $f \in L^1(X, \nu)$.

8. PROBABILITY THEORY

Definition 8.1 (Random variable). Let (Ω, \mathcal{A}, P) be a probability space. A \mathcal{A} -measurable function

$$X : \Sigma \rightarrow \mathbb{R}$$

is called a **random variable**.

Definition 8.2. Let $X : \Sigma \rightarrow \mathbb{R}$ a random variable. We say that X has *finite expectation* if $X \in L^1(\Sigma)$ and call

$$E[X] = \int_{\Sigma} X dP$$

the **expectation** of X .

Definition 8.3 (Distribution). For every Borel set $A \subseteq \mathbb{R}$ we define

$$P_X[A] = P[\{\omega \in \Omega | X(\omega) \in A\}] = P[X \in A]$$

Since X is measurable, $X^{-1}[A]$ is measurable for all Borel sets $A \subseteq \mathbb{R}$.

Definition 8.4 (Distribution). Let X be a random variable. The Borel measure defined above is called the **distribution** of X . The function

$$F(t) = P_X [(-\infty, t]] = P[X \leq t]$$

is called the **distribution function** of X .

Lemma 8.5. *Let X be a random variable and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable. Then*

$$\int_{\Sigma} f \circ X \, dP = \int_{\mathbb{R}} f \, dP_X$$

8.1. Conditional expectation.

Definition 8.6 (Conditional expectation). Let $X : \Omega \rightarrow \mathbb{R}$ a random variable with finite expectation. Let \mathcal{A}_0 be a σ -algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$. We call

$$X_0 : \Sigma \rightarrow \mathbb{R}$$

a **conditional expectation** given \mathcal{A}_0 if

- X_0 is \mathcal{A}_0 -measurable
- $\int_A X_0 \, dP = \int_A X \, dP$ for all $A \in \mathcal{A}_0$.

We write $X_0 = E[X|\mathcal{A}_0]$

Theorem 8.7. *Let X be a random variable with finite expectation. If \mathcal{A}_0 is a σ -algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$, then the conditional expectation $X_0 = E[X|\mathcal{A}_0]$ exists and is essentially unique.*

Remark. • If X is \mathcal{A}_0 -measurable, then $X = E[X|\mathcal{A}_0]$ almost everywhere.

- If we set $\mathcal{A}_0 = \{\varnothing, \Omega\}$, then

$$E[X|\mathcal{A}_0] = E[X]$$