MATH 3969 - MEASURE THEORY AND FOURIER ANALYSIS EXAM NOTES

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1. Measure Theory

Definition 1.1 (σ-algebra)**.** Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a σ-algebra if

- $\emptyset \in \mathcal{A}$
- If $A \subseteq X$ is in $\mathcal{A}$, then its complement $A^c = X \backslash A$ is in $\mathcal{A}$
- Whenever $A_0, A_1, \dots$ are subsets of X in $\mathcal{A}$, then their union
$$\bigcup_{k=0}^{\infty} A_k$$
also belongs to $\mathcal{A}$.

Definition 1.2 (Measure)**.** Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. Suppose that $\mu : \mathcal{A} \to [0, \infty]$ is a function. Then μ is a measure if

- $\mu(\emptyset) = 0$
- Whenever $A_0, A_1, \dots$ are *pairwise disjoint* subsets of X in $\mathcal{A}$, then
$$\mu(\bigcup_{k=0}^{\infty} A_k) = \sum_{k=0}^{\infty} \mu(A_k)$$

Proposition 1.3 (Properties of a σ-algebra)**.** *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. Then*

- $X, \emptyset \in \mathcal{A}$
- *If $A_k \in \mathcal{A}$, then $\bigcap_{k=0}^{\infty} A_k \in \mathcal{A}$*
- *If $A, B \in \mathcal{A}$, then $A \cup B, A \cap B \in \mathcal{A}$*

Definition 1.4 (Algebra)**.** A collection $\mathcal{A}$ of subsets A of X which satisfies the first two conditions of a σ-algebra and also

- If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$

is called an algebra. Every σ-algebra is an algebra, but not every algebra is a σ-algebra

Definition 1.5 (σ-algebra generated by $\mathcal{S}$)**.** Let $\mathcal{S}$ be a collection of subsets of X. Let
$$\mathcal{A}(\mathcal{S}) = \bigcap \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra , and } \mathcal{S} \subseteq \mathcal{A}\}$$
$\mathcal{A}(\mathcal{S})$ is called the *σ-algebra generated by $\mathcal{S}$*

Definition 1.6 (Borel σ-algebra)**.** Let X be a metric space and $\mathcal{S}$ the collection of all open sets in X. We call $\mathcal{B} = \mathcal{A}(\mathcal{S})$ the *Borel σ-algebra* . Sets in $\mathcal{B}$ are called *Borel sets.*

Corollary. *We have the following examples of Borel sets.*

- *Any open set is a Borel set.*
- *If B is a Borel set, then so is B^c. If $B_0, B_1, \ldots$ is a sequence of Borel sets, then so are $\bigcup_{k=0}^{\infty} B_k$ and $\bigcap_{k=0}^{\infty} B_k$.*

1.1. **Properties of Measures.**

Proposition 1.7 (The Monotonicity Property)**.** *If A and B are μ-measurable subsets of X with $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*

Proposition 1.8 (The Countable Subadditivity Property)**.** *If $A_0, A_1, \ldots$ are μ-measurable subsets of X, then*

$$\mu(\bigcup_{k=0}^{\infty} A_k) \leq \sum_{k=0}^{\infty} \mu(A_k)$$

Proposition 1.9 (Monotone Convergence Property of Measures)**.** *Let $A_0 \subseteq A_1 \subseteq A_2$ be an increasing sequence of measurable sets. Then*

$$\mu(\bigcup_{k=0}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k)$$

Proposition 1.10. *Let $A_0 \supseteq A_1 \supseteq A_2 \ldots$ be sets from some σ-algebra $\mathcal{A}$. If $\mu(A_0) < \infty$, then*

$$\mu(\bigcap_{k=0}^{\infty} A_k) = \lim_{k\to\infty} \mu_(A_k)$$

1.2. **Constructing σ-algebras and measures.**

Definition 1.11 (Lebesgue outer measure)**.** If $A \subseteq \mathbb{R}$, let

$$m^*(A) = \inf\left\{\sum_{k=0}^{\infty}(b_k - a_k) \,|\, a_k < b_k, A \subseteq \bigcup_{k=0}^{\infty}(a_k, b_k)\right\}$$

Proposition 1.12 (Properties of the Lebesgue outer measure)**.** *The Lebesgue measure obeys the following properties.*

- *$m^*(A)$ is defined, and $m^*(A) \in [0, \infty]$ for any subset of $\mathbb{R}$.*
- *$m^*(\emptyset) = 0$*
- *If $A \subseteq B$, $m^*(A) \leq m^*(B)$*
- *For every sequence $A_0, A_1, \ldots,$ we have*

$$m^*(\bigcup_{k=0}^{\infty} A_k) \leq \sum_{k=0}^{\infty} m^*(A_k)$$

Definition 1.13 (Outer Measure)**.** A function $\mu^* : \mathcal{P} \to [0, \infty]$ is such that

- $\mu^*(\emptyset) = 0$
- If $A \subseteq B$, $\mu^*(A) \leq \mu^*(B)$
- For every sequence $A_0, A_1, \dots$, we have

$$\mu^*(\bigcup_{k=0}^{\infty} A_k) \leq \sum_{k=0}^{\infty} \mu^*(A_k)$$

Then μ^* is called an *outer measure* on X.

Theorem 1.14 (Construction from outer measures)**.** *Let μ^* be an outer measure on a set X. Then*

$$\mathcal{A} = \{A \subseteq X \mid \mu * (S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \textit{ for all } S \subseteq X\}$$

Then $\mathcal{A}$ is a σ-algebra. Let $\mu(A) = \mu^*(A)$ when $A \in \mathcal{A}$. Then $\mu : \mathcal{A} \to [0, \infty]$ is a measure.

Proposition 1.15. *Let μ^* be an outer measure, and let $\mathcal{A}$ be the σ-algebra defined in the last theorem. Let $A \subseteq X$ satisfy $\mu^*(A) = 0$. Then $A \in \mathcal{A}$, and so $\mu(A)$ is defined, and equals 0.*

Definition 1.16 (Null set)**.** If $\mu : \mathcal{A} \to [0, \infty]$ is any measure, then a set $A \in \mathcal{A}$ satisfying $\mu(A) = 0$ is called a null set.

1.3. **Properties of the Lebesgue measure on $\mathbb{R}$.** Let

$$\mathcal{M} = \{A \subseteq \mathbb{R} \mid m * (S) = m^*(S \cap A) + m^*(S \cap A^c) \text{ for all } S \subseteq \mathbb{R}\}$$

The sets in $\mathcal{M}$ are called hte *Lebesgue measurable subsets* of $\mathbb{R}$. If $A \in \mathcal{M}$, the we write $m(A) = m^*(A)$. This real number is called the *Lebesgue measure of A*.

We now show that this σ-algebra $\mathcal{M}$ is very large.

Theorem 1.17. *Let m^* denote the Lebesgue outer measure on $\mathbb{R}$, and let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets. Then*

- *If $I \subseteq \mathbb{R}$ is an interval. Then $m^*(I) = l(I)$. That is, the outer measure is just its length.*
- *If $I \subseteq \mathbb{R}$ is an interval, then $I \in \mathcal{M}$.*

Proposition 1.18. *Any open subset of $\mathbb{R}$ is in $\mathcal{M}$. Any closed subset of $\mathbb{R}$ is in $\mathcal{M}$. That is, all open or closed sets in $\mathbb{R}$ are Lebesgue measurable.*

Corollary. *Every Borel subset of $\mathbb{R}$ is contained in $\mathcal{M}$.*

Proof. $\mathcal{M}$ is a σ-algebra which contains every open subset of $\mathbb{R}$. The σ-algebra $\mathcal{B}$ is by definition the smallest such σ-algebra. Thus $\mathcal{B} \subseteq \mathcal{M}$. □

2. Measurable Functions

Definition 2.1 (Measurable function)**.** Let $\mathcal{A}$ be σ-algebra of subsets of a set X. A function $f : X \to \overline{\mathbb{R}}$, is called *measurable* (or $\mathcal{A}$-measurable) if for every $\alpha \in \mathbb{R}$, the set

$$\{x \in X \mid f(x) > \alpha\}$$

is in $\mathcal{A}$.

Definition 2.2 (Indicator function)**.** . Let $S \subset X$. We define the *indicator function* of S to be the function $1_S : X \to \mathbb{R}$ given by

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Proposition 2.3. *Let $S \subset X$. Then 1_S is measurable if and only if $S \in \mathcal{A}$.*

Proof. Let $\alpha \in \mathbb{R}$. Then

$$\{x \in X \mid 1_S(x) > \alpha\} = \begin{cases} \emptyset & \text{if } \alpha \geq 1 \\ S & \text{if } 0 \leq \alpha < 1 \\ X & \text{if } \alpha < 0 \end{cases}$$

As $\emptyset, X$ are in $\mathcal{A}$, then 1_S is measurable if and only if $S \in \mathcal{A}$. □

Proposition 2.4 (Continuous functions are Lebesgue measurable)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Lebesgue measurable on $[a, b]$. More generally, if $X \subset \mathbb{R}$ is in $\mathcal{M}$ and $f : X \to \mathbb{R}$ is continuous, then f is Lebesgue measurable on X.*

2.1. **Basic properties of measurable functions.**

Lemma 2.5. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $f : X \to \overline{\mathbb{R}}$ be a function. Then f is measurable if and only if it satisfies one of the following conditions.*

- *For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) > \alpha\}$ in in $\mathcal{A}$.*
- *For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) < \alpha\}$ in in $\mathcal{A}$.*
- *For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) \leq \alpha\}$ in in $\mathcal{A}$.*
- *For each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid f(x) \geq \alpha\}$ in in $\mathcal{A}$.*

Proposition 2.6. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $f, g : X \to \overline{\mathbb{R}}$ be functions. Then*

- *$f + g$ is measurable (provided that $f(x) = \infty$ and $g(x) = -\infty$ or vice versa holds for no $x \in X$).*
- *cf is measurable for any constant $c \in \mathbb{R}$.*

- *fg is measurable.*
- *f/g is measurable (provided that $g(x)$ is nonzero and not infinity for all $x \in X$).*

Similarly, let $f_0, f_1, \dots : X \to \overline{\mathbb{R}}$. Then

- $\sup\{f_0, f_1, \dots\}$ *and* $\inf\{f_0, f_1, \dots\}$ *are measurable functions.*

Corollary. *Let f, g be measurable. Then* $\max\{f, g\}$ *and* $\min\{f, g\}$ *are measurable functions.*

Proposition 2.7. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $f_0, f_1, \dots : X \to \overline{\mathbb{R}}$ be measurable functions. Let $f(x) = \lim_{k\to\infty} f_k(x)$ for each $x \in X$. Then f is a measurable function.*

2.2. **Simple functions.**

Definition 2.8 (Simple function)**.** Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. A function $\varphi : X \to \mathbb{R}$ is called *simple* if it is measurable and only takes a finite number of values.

Proposition 2.9. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $f : X \to [0, \infty]$ be a nonnegative measurable function. Then there is a sequence (φ_n) of simple functions such that*

- $0 \le \varphi_1(x) \le \varphi_2(x) \le \dots \le f(x)$ *for all $x \in X$.*
- $f(x) = \lim_{n\to\infty} \varphi_n(x)$ *for all $x \in X$.*

Proof. Define the function φ_n as follows.

Let
$$A_{n,k} = \{x \in X \mid \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\}$$
and let $A_{n,2^n} = \{x \in X \mid f(x) \ge n\}$

Then the function
$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} 1_{A_{n,k}}$$
obeys the required properties. □

3. Integration

Definition 3.1 (Integration of simple functions)**.** Let $\varphi = \sum_{j=1}^m a_j 1_{A_j}$. Then the *integral* $\int_X \varphi \, d\mu$ *of φ over X with respect to μ* is given by
$$\int_X \varphi \, d\mu = \sum_{j=1}^m a_j \mu(A_j)$$

Proposition 3.2. *Let φ and ψ be nonnegative simple functions on X, and let $c \ge 0$ be constant. Then*

- $\int_X \varphi + \psi \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$
- *If* $0 \le \psi \le \varphi$*, then*
$$0 \le \int_0 \psi \, d\mu \le \int_X \psi \, d\mu$$

- $\int_X c\varphi \, d\mu = c \int_X \varphi \, d\mu$

Definition 3.3 (Integral over a subset of X)**.** Let φ be a nonnegative simple function, and let $S \subset X$ be measurable. Then *the integral of φ over S with respect to μ*, denoted $\int_S \varphi \, d\mu$, is given by

$$\int_S \varphi \, d\mu = \int_X \varphi \cdot 1_s \, d\mu$$

3.1. **Integration of nonnegative measurable functions.**

Definition 3.4. Let $f : X \to [0, \infty]$ be a nonnegative measurable function. We define the *integral $\int_X f \, d\mu$ of f over X with respect to μ* by

$$\int_X f \, d\mu = \sup \left\{ \int_X \varphi \, d\mu \,|\, \varphi \text{ is simple, and } 0 \leq \varphi \leq f \text{ on } X \right\}$$

Lemma 3.5. *Suppose that f, g are two nonnegative measurable functions, and $0 \leq g \leq f$ on X. Then*

$$0 \leq \int_X g \, d\mu \leq \int_X f \, d\mu$$

The following is an extremely important theorem in measure theory.

Theorem 3.6 (Monotone convergence theorem)**.** *Let (f_k) be a sequence of nonnegative measurable functions on X. Assume that*

- *$0 \leq f_0(x) \leq f_1(x) \leq \dots$ for each $x \in X$,*
- *$\lim_{k \to \infty} f_k(x) = f(x)$ for each $x \in X$.*

When these hold, we write $f_k \nearrow f$ pointwise.

Then f is measurable, and

$$\int_X f \, d\mu = \lim_{k \to \infty} \int_X f_k \, d\mu$$

Corollary. *Let f, g be measurable on X. THen*

$$\int_X f + g \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$

Theorem 3.7. *Suppose that f_k is a nonnegative measurable function for $k = 0, 1, \dots$. Then*

$$\int_X \left(\sum_{k=0}^{\infty} f_k \right) \, d\mu = \sum_{k=0}^{\infty} \left(\int_X f_k \, d\mu \right)$$

Theorem 3.8. *Suppose that X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, \infty]$ is a measure. Let f be nonnegative and measurable on X. Then define $\mu_f : \mathcal{A} \to [0, \infty]$ by*

$$\mu_f(A) = \int_A f \, d\mu = \int_X f \cdot 1_A \, d\mu$$

Then μ_f is a measure.

Proposition 3.9. *Suppose that f is nonnegative and measurable on X, and suppose that $\int_X f\,d\mu < \infty$. Then the set $\{x \in X \mid f(x) = \infty\}$ has measure 0.*

Proposition 3.10. *Suppose that X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, \infty]$ is a measure. Suppose that there is a set $N \in \mathcal{A}$ with $\mu(N) = 0$ and suppose that some property P holds for all $x \in X$ outside N. Then we say that the property P holds* almost everywhere *or* for almost all $x \in X$.

Proposition 3.11. *Suppose that f_k is a nonnegative measurable function on X for $k = 0, 1, \ldots$. Suppose that*

$$\sum_{k=0}^{\infty} \left(\int_X f_k\,d\mu \right) < \infty$$

Then

$$\sum_{k=0}^{\infty} < \infty \text{ for almost all } x \in X$$

Proposition 3.12. *Suppose that f is a nonnegative measurable function on X. Then*

$$\int_X f\,d\mu = 0 \iff f(x) = 0 \text{ almost everywhere.}$$

Theorem 3.13 (Fatou's Lemma)**.** *Suppose that f_k is a nonnegative measurable function on X, for $k = 0, 1, \ldots$, and that $\lim_{k\to\infty} f_k(x) = f(x)$ for each $x \in X$. Then*

$$\int_X f\,d\mu \le \liminf_{k\to\infty} \left(\int_X f_k\,d\mu \right)$$

3.2. **Integration of real and complex valued functions.**

Lemma 3.14. *Let $f : X \to \overline{\mathbb{R}}$ be a measurable function, and let f^+ and f^- be the positive and negative parts of f. Then $f = f^+ - f^-$, and $|f| = f^+ + f^-$. Moreover, $|f|$ is a measurable function, and*

$$f \text{ is integrable if and only if } \int_X |f|\,d\mu < \infty$$

Definition 3.15 (Integral of a complex valued function)**.** Let $f = u + iv$, where $u, v : X \to \overline{\mathbb{R}}$. Then

$$\int_X f\,d\mu = \int_X u\,d\mu + i \int_X v\,d\mu$$

Lemma 3.16. *Let $f : X \to \bar{\mathbb{C}}$. Then f is integrable if and only if $\int_X |f|\,d\mu < \infty$.*

The next theorem is probably the most important single theorem in these notes. It has many applications, both of a theoretical and practical nature.

Theorem 3.17 (Dominated convergence theorem)**.** *Let (f_k) be a sequence of real or complex valued measurable function on X. Assume that*

- $\lim_{k\to\infty} f_k(x) = f(x)$

and that there is a measurable function $g : X \to [0, \infty]$ such that

- $|f_k(x)| \leq g(x)$ *for each k and x, and*
- $\int_X g\, d\mu < \infty$

Then

$$\int_X f\, d\mu = \lim_{k\to\infty} \int_X f_k\, d\mu$$

Theorem 3.18 (Bounded convergence theorem)**.** *Let (f_k) be a sequence of real or complex valued measurable function on X. Assume that*

- $\lim_{k\to\infty} f_k(x) = f(x)$ *for each $x \in X$,*
- *There exists a constant $M < \infty$ such that $|f_k(x)| \leq M$ for each k and x,*
- $\mu(X) < \infty$.

Then

$$\int_X f\, d\mu = \lim_{k\to\infty} \int_X f_k\, d\mu$$

Theorem 3.19. *Suppose that f_k is a measurable real or complex valued function on X for $k = 0, 1, \dots$. Suppose that we have*

$$\sum_{k=0}^{\infty} \left(\int_X |f_k|\, d\mu \right) < \infty$$

or, equivalently,

$$\int_X \left(\sum_{k=0}^{\infty} |f_k| \right) d\mu < \infty$$

Then we have

$$\int_X \left(\sum_{k=0}^{\infty} f_k \right) d\mu = \sum_{k=0}^{\infty} \left(\int_X f_k\, d\mu \right)$$

Definition 3.20 (Integrable function)**.** We call $f : X \to \mathbb{K}$ μ -integrable if f is μ measurable and

$$\int_X |f|\, d\mu < \infty$$

We set

$$\mathcal{L}^1(X, \mathbb{K}) = \{ f : X \to \mathbb{K} \mid f \mu\text{-integrable} \}$$

Theorem 3.21. *$\mathcal{L}^1(X, \mathbb{K})$ is a vector space over $\mathbb{K}$*

Definition 3.22 (The Lebesgue-Stieltjes integral)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing right continuous function, that is $\lim_{s\to t^+} F(s) = F(t)$ for all $t \in \mathbb{R}$. Then for $A \subseteq \mathbb{R}$ let

$$\mu^\star{}_F(A) = \inf\{ \sum_{k=0}^{\infty} (F(b_k) - F(a_k)) \mid A \subseteq \bigcup_{k\in\mathbb{N}} (a_k, b_k)$$

The $\mu_F^\star$ is an outer measure inducing an inner measure on $\mathbb{R}$. Then we have

- μ_F is a Borel measure.
- $\mu_F((a,b]) = F(b) - F(a)$.

We then define $\int_A f\,dF = \int_A f\,d\mu_F$ as the Lebesgue-Stieltjes integral.

Lemma 3.23. *If μ is a finite measure on $\mathbb{R}$, then we define $F(t) = \mu((-\infty, t])$ as the* ***distribution*** *function of $\mathbb{R}$.*

Theorem 3.24. *There is a bijection from finite measures and some class of right-continuous increasing functions.*

Definition 3.25 (Measures from other measures)**.** Let $g : X \to [0,\infty]$ be a μ-measurable function. For $A \in \mathcal{A}$ define

$$\nu(A) = \int_A g\,d\mu$$

Then using the monotone convergence theorem one can show that ν is a measure defined on $\mathcal{A}$. Moreover, if $f : X \to \mathbb{K}$ is μ-measurable, then

$$\int_X f\,d\nu = \int_X fg\,d\mu$$

We call g the **density of ν with respect to μ.**

Proposition 3.26. *Let $f \in \mathcal{L}^1(X,\mathbb{K})$ with respect to the Lebesgue measure. Then*

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

3.3. **Parameter integrals.**

Definition 3.27 (Parameter integral)**.** Let $(X,\mathcal{A},\mu)$ be a measure space and Y a metric space. Suppose that $f : X \times Y \to \mathbb{K}$ is such that

- $x \mapsto f(x,y)$ is μ-integrable for all $y \in Y$,
- $y \mapsto f(x,y)$ is continuous at y_0 for almost all $x \in X$,
- there exists $g \in \mathcal{L}^1(X,\mathbb{R})$ such that

$$|f(x,y)| \le g(x)$$

  for almost all $x \in X$.

Define $F(y) = \int_X f(x,y)\,d\mu(x)$. Then F is continuous at $y_0 \in Y$.

Theorem 3.28 (Differentiation of parameter integrals.)**.** *Let $(X,\mathcal{A},\mu)$ be a measure space and $L \subset \mathbb{R}$ an interval. Suppose that $f : X \times L \to \mathbb{R}$ is such that*

- *$x \mapsto f(x,y)$ is μ-integrable for all $y \in Y$,*
- *$\frac{\partial}{\partial t} f(x,t)$ exists for all $t \in L$, for almost all $x \in X$, and is continuous ,*
- *there exits $g \in \mathcal{L}^1(X,\mathbb{R})$ with $|\frac{\partial}{\partial t} f(x,t)| < g(x)$ for almost all $x \in X$ and all $t \in L$.*

Define $F(t) = \int_X f(x,t)\,d\mu(x)$. *Then* $f : L \to \mathbb{K}$ *is differentiable and*

$$F'(t) = \int_X \frac{\partial}{\partial t} f(x,t)\,d\mu(x)$$

4. The L^p-spaces

Definition 4.1 (L^p-spaces)**.** Let $1 \leq p < \infty$ and $f : X \to \mathbb{K}$ measurable. We call

$$\|f\|_p = \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}}$$

the L^p*-norm* of f. We set

$$\mathcal{L}^p(X) = \{f : X \to \mathbb{K} \,|\, f \text{ measurable}, \|f\|_p < \infty$$

Theorem 4.2 (Hölder's inequality)**.** *Let* $p, q \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in \mathcal{L}^p(X)$ *and* $g \in \mathcal{L}^q(X)$*, then*

$$|\int_X fg\,d\mu| \leq \|f\|_p \|g\|_q$$

Proposition 4.3 (Minkowski's inequality)**.** *If* $f, g \in \ell p$*,* $1 \leq p \leq \infty$*, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Definition 4.4 (L^p-spaces)**.** Let $f \sim g$ if $f = g$ almost everywhere. Denote the equivalence class of f by $[f]$. Then

$$L^p(X) = \{[f] \,|\, f \in \mathcal{L}^p(X)\}$$

Definition 4.5 (Cauchy sequence)**.** A sequence $(f_n) \in L^p(X)$ is called **Cauchy** if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \epsilon$$

for all $n, m > n_0$.

Theorem 4.6 (Completeness of $L^p(X)$)**.** *Let* (f_n) *be a sequence in* $L^p(X)$*. Then* (f_n) *converges in* $L^p(X)$ *if and only if* (f_n) *is a Cauchy sequence.*

Remark. Introducing the metric $d(f,g) = \|f - g\|_p$, we have that $L^((X)X)$ is a **complete normed space** or a **Banach space**. If $p = 2$, then $\|f\|_2$ is induced by an inner product - hence $L^2(X)$ is a **complete inner product space**, or a **Hilbert space.**

Proposition 4.7. *Suppose that* $f_n, f \in \mathcal{L}^p(X)$ *with* $\|f_n - f\| \to 0$*. Then there exists a subsequence* (f_{n_k}) *with* f_{n_k} *converging pointwise to* f *for almost every* $x \in X$*.*

Theorem 4.8. *The simple functions are dense in* $L^p(X)$ *for* $1 \leq p < \infty$*.*

In $\mathbb{R}^N$ *and the Lebesgue measure, we can modify the statement to the simple function with bounded support are dense in* $\mathbb{R}^N$*.*

Theorem 4.9. *For $1 \leq p < \infty$*

$$\mathit{span}\{1_U \,|\, U \subseteq \mathbb{R}^N \text{ open and bounded}\}$$

is dense in $L^p(\mathbb{R}^N)$.

We can also use bounded rectangles in the place of open bounded sets here.

Definition 4.10 (Essential supremum)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ μ-measurable. We call $\text{ess-sup} f(x) = \inf\{t \in \mathbb{R} \,|\, \mu(\{x \in X \,|\, f(x) > t\}) = 0\}$ the *essential supremum* of f. The essential supremum of $|f|$ is denoted $\|f\|_\infty$

Theorem 4.11 (Completeness of $L^\infty(X)$)**.** $L^((X)\infty)$ *is a complete normed space.*

Lemma 4.12. *Hölder's inequality holds for $p = 1, q = \infty$. That is,*

$$|\inf_X fg \, d\mu| \leq \|f\|_p \|g\|_q$$

Lemma 4.13. *If $\mu(X) < \infty$, then $\lim_{p\to\infty} \|u\|_p = \|f\|_\infty$*

4.1. **Fubini's Theorem.**

Theorem 4.14 (Tonelli)**.** *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty]$ is measurable. Then there exist sets $N \subseteq \mathbb{R}^n$ and $M \subseteq \mathbb{R}^m$ of measure zero such that*

(i) $x \mapsto f(x, y)$ *is measurable for all* $y \in \mathbb{R}^m - M$,
(ii) $y \mapsto \int_{\mathbb{R}^n} f(x, y) \, dx$ *is measurable,*
(iii) $y \mapsto f(x, y)$ *is measurable for all* $x \in \mathbb{R}^m - N$,
(iv) $x \mapsto \int_{\mathbb{R}^m} f(x, y) \, dy$ *is measurable,*
(v)

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y) \, d(x, y) =$$
$$\int_{R^m - M} \left(\int_{\mathbb{R}}^n f(x, y) dx \right) dy =$$
$$\int_{R^n - N} \left(\int_{\mathbb{R}}^m f(x, y) dy \right) dx$$

Theorem 4.15 (Fubini)**.** *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty]$ is measurable. Let N, M be the sets from Theorem 4.14 applied to the function $|f|$ such that (v) holds with f replaced with $|f|$. Assume that one of these integrals is finite - and hence all of them. Then there exists sets N_1 of $\mathbb{R}^n$ and M_1 of $\mathbb{R}^m$ such that $(i) - (v)$ of Theorem 4.14 hold with N, M replaced with N_1, M_1.*

Definition 4.16 (Complete measure space)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. We call the measure μ *complete* if whenever $A \in \mathcal{A}$ has measure 0, then any subset of A is in $\mathcal{A}$, (and has measure 0).

Definition 4.17 (σ-finite measure space)**.**

5. Convolution

Definition 5.1 (Translation of a function)**.** Let $f : \mathbb{R}^N \to \mathbb{C}$ be a function and $t \in \mathbb{R}^N$ a fixed vector. We define the translation operator τ_t by

$$\tau_t f(x) = f(x - t)$$

Theorem 5.2 (Continuity of translation)**.** *Let* $1 \leq p < \infty$ *and* $f \in L^p(\mathbb{R}^N)$*. Then*

$$\lim_{t \to 0} \|\tau_t f - f\|_p = 0$$

Remark. This does not hold if $p = \infty$.

Lemma 5.3. *Let* $f : \mathbb{R}^N \to \mathbb{C}$ *be measurable and set*

$$F_1(x, y) = f(x)$$
$$F_2(x, y) = f(y - x)$$

Then $F_1, F_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ *are measurable.*

Definition 5.4 (Convolution)**.** Let $f, g : \mathbb{R}^N \to \mathbb{C}$ be measurable. We define the **convolution** $f \star g : \mathbb{R}^N \to \mathbb{C}$ by

$$(f \star g)(x) = \int_{\mathbb{R}^N} f(x - y) g(y)\, dy$$

wherever the integral exists

Definition 5.5 (Convex function)**.** A function $\varphi : (a, b) \to \mathbb{R}$ is called **convex** if

$$\varphi(\lambda s + (1 - \lambda)t) \leq \lambda\varphi(s) + (1 - \lambda)\varphi(t)$$

for all $s, t \in (a, b)$ and all $\lambda \in (0, 1)$

Lemma 5.6. *This is equivalent to the condition*

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

whenever $a < s < t < u < b$.

Theorem 5.7 (Jensen's inequality in $\mathcal{L}^p(X)$-spaces)**.** *Let* $f \in \mathcal{L}^p(X)$, $1 \leq p < \infty$, *and let* $g \in \mathcal{L}^1(X)$*. Then*

$$\left(\int_X |fg|\, d\mu\right)^p \leq \|g\|_1^{p-1} \int_X |f|^p |g|\, d\mu$$

Theorem 5.8 (Young's inequality)**.** *Let* $1 \leq p \leq \infty$*. If* $f \in L^p(\mathbb{R}^N, \mathbb{C})$ *and* $g \in L^1(\mathbb{R}^N, \mathbb{C})$*, then* $f \star g$ *exists almost everywhere and* $f \star g \in L^p(\mathbb{R}^N, \mathbb{C})$*. Moverover,*

$$\|f \star g\|_p \leq \|f\|_p \|g\|_1$$

Theorem 5.9. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, then*

$$f \star g \in BC(\mathbb{R}^N)$$

where BC is the vector space of bounded continuous functions.

5.1. **Approximate identities.**

Definition 5.10 (Approximate identity)**.** Let $\varphi : \mathbb{R}^N \to [0, \infty)$ be measurable with

$$\int_{\mathbb{R}^N} \varphi \, dx = 1$$

and set $\varphi_n(x) = n^N \varphi(nx)$ for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Then (φ_n) is called an **approximate identity**

Theorem 5.11. *Let (φ_n) be an approximate identity and $f \in L^p(\mathbb{R}^N), 1 \leq p < \infty$. Then*

$$f \star \varphi_n \to f$$

in $L^p(\mathbb{R}^N)$ as $n \to \infty$

Theorem 5.12. *Let $f \in L^\infty(\mathbb{R}^N)$ and (φ_n) an approximate identity. If f is continuous at x, then*

$$f(x) = \lim_{n \to \infty} (f \star \varphi_n)(x)$$

Definition 5.13 (Test function)**.** Let $U \subseteq \mathbb{R}^N$ be open. We let

$$C^\infty(U, \mathbb{K}) = \{f : U \to \mathbb{K} \,|\, f \text{ has partial derivatives of all orders}\}$$

and

$$C_c^\infty(U, \mathbb{K}) = \{f \in C^\infty(U, \mathbb{K}) \,|\, \operatorname{supp}(f) \subseteq U, \operatorname{supp}(f) \text{ compact}$$

The functions in $C_c^\infty(U, \mathbb{K})$ are called **test functions** on U.

Proposition 5.14. *Let $f : \mathbb{R}^N \to \mathbb{K}$ be measurable such that $f \in \mathcal{L}^1(B)$ for every bounded set $B \subseteq \mathbb{R}^N$. If $\varphi \in C_c^\infty(\mathbb{R}^N)$, then $f \star \varphi \in C^\infty(\mathbb{R}^N)$ and*

$$\frac{\partial}{\partial x_i}(f \star \varphi) = f \star \frac{\partial \varphi}{\partial x_i}$$

Theorem 5.15. *Let $U \subseteq \mathbb{R}^N$ open and $1 \leq p < \infty$. Then $C_c^\infty(U)$ is dense in $L^p(U)$.*

Remark. The above proposition does not hold for $p = \infty$.

6. The Fourier Transform

Definition 6.1 (Fourier transform)**.** Let $f \in L^1(\mathbb{R}^N, \mathbb{C})$. We call

$$\widehat{f}(t) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot t} \, dx$$

Theorem 6.2. *We have*

- $\widehat{f} : \mathbb{R}^N \to \mathbb{C}$ *is continuous,*
- $\|\widehat{f}\|_\infty \leq \|f\|_1$

Proposition 6.3. *Let* $\varphi(x) = e^{-\pi|x|^2}$. *Then* $\|\varphi\|_1 = 1$ *and* $\widehat{\varphi} = \varphi$.

Proposition 6.4. *Let* $f \in L^1(\mathbb{R}^N, \mathbb{C}), x_0 \in \mathbb{R}^N$ *and* $\alpha \in R, \alpha > 0$.

(i) If $g(x) = f(x - x_0)$*, then* $\widehat{g}(t) = e^{-2\pi i x_0 \cdot t}\widehat{f}(t)$,

(ii) If $g(x) = f(\alpha x)$*, then* $\widehat{g}(t) = \frac{1}{\alpha^N}\widehat{f}\left(\frac{x}{\alpha}\right)$,

(iii) If $g(x) = \overline{f(-x)}$*, then* $\widehat{g}(t) = \overline{\widehat{f}(t)}$

Definition 6.5. Let $C_0(\mathbb{R}^N, \mathbb{K}) = \{f \in C(\mathbb{R}^N, K) \mid \lim_{|x|\to\infty} f(x) = 0\}$, the set of continuous functions vanishing at infinity

Theorem 6.6 (Riemann-Lebesgue)**.** *If* $f \in L^1(\mathbb{R}^N, \mathbb{C})$*, then* $\widehat{f} \in C_0(\mathbb{R}^N, \mathbb{C})$ *and* $\|\widehat{f}\|_\infty \leq \|f\|_1$.

Theorem 6.7. *If* $f, g \in L^1(\mathbb{R}^N, \mathbb{C})$ *then* $f \star g \in L^1(\mathbb{R}^N, \mathbb{C})$ *and*

$$\widehat{f \star g} = \widehat{f}\,\widehat{g}$$

Proposition 6.8. *Let* $f, g \in L^1(\mathbb{R}^N, \mathbb{C})$. *Then*

$$\int_{\mathbb{R}^N} \widehat{f} g \, dx = \int_{\mathbb{R}^N} f \widehat{g} \, dx$$

Lemma 6.9. *Let* $\varphi(x) = e^{-\pi|x|^2}$ *and* $\varphi_n(x) = n^N \varphi(nx)$. *Then*

$$\int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} \varphi\left(\frac{t}{n}\right) dt = (f \star \varphi_n)(x)$$

for all $f \in L^1(\mathbb{R}^N, \mathbb{C}), x \in \mathbb{R}^N$*, and* $n \in \mathbb{N}$.

Theorem 6.10 (Fourier inversion formula)**.** *Let* $f \in L^1(\mathbb{R}^N, \mathbb{C})$. *Then*

(i)

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} \, dt = f$$

in $L^1(\mathbb{R}^N, \mathbb{C})$.

(ii) If f *is continuous at* x*, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} \, dt = f(x)$$

Corollary. *Let* $f, g \in L^1(\mathbb{R}^N)$ *with* $\widehat{f} = \widehat{g}$. *Then* $f = g$ *almost everywhere.*

6.1. **The Fourier transform on** $L^2(\mathbb{R}^N)$**.** We have defined the Fourier transform $\widehat{f}$ with $f \in L^1(\mathbb{R}^N)$. We have that $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$ as well as in $L^1(\mathbb{R}^N)$, so in particular $L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$. We can use this to extend the Fourier transform to $L^2(\mathbb{R}^N)$. The key for doing so is the following theorem.

Theorem 6.11 (Plancherel)**.** *Let* $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$*. Then* $\|\widehat{f}\|_2 = \|f\|_2$*.*

Proposition 6.12. *There is a unique coninuous linear operator*

$$\mathcal{F} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

such that $\mathcal{F}f = \widehat{f}$ *for all* $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$*. Moreover,* $\|f\|_2 = \|\mathcal{F}f\|_2$ *for all* $f \in L^2(\mathbb{R}^N)$*.*

Remark. We use the notation $\widehat{f} = \mathcal{F}f$ for $f \in L^2(\mathbb{R}^N)$.

Remark. Let $\varphi_n : \mathbb{R}^N \to [0,1]$ such that $\varphi_n \in L^2(\mathbb{R}^N)$ and $\varphi_n(x) \to 1$ for all $x \in \mathbb{R}^N$. If $f \in L^2(\mathbb{R}^N)$, then

$$\widehat{f} = \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)\varphi_n(x)e^{-2\pi i x\cdot t}\, dx$$

Common choices for φ_n are

- $\varphi_n(x) = 1_{B(0,n)}$,
- $\varphi_n(x) = e^{-\pi\frac{|x|^2}{n^2}}$

Theorem 6.13. *Let* $f \in L^2(\mathbb{R}^N, \mathbb{C})$*. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \widehat{f}(t)e^{2\pi i x\cdot t}e^{-\pi\frac{|t|^2}{n^2}}\, dt = f$$

in $L^2(\mathbb{R}^N, \mathbb{C})$*.*

Theorem 6.14. $\mathcal{F} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *is bijective with* $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$ *for all* $f \in L^2(\mathbb{R}^N)$*.*

Remark. Let

$$\langle f, g\rangle = \int_{\mathbb{R}^N} f(x)\overline{g(x)}\, dx$$

denote the inner product on $L^2(\mathbb{R}^N)$. Then the above theorem implies

$$\langle \widehat{f}, \widehat{g}\rangle = \langle f, g\rangle.$$

Moverover, by approximating f, g by functions in $L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ we also have

$$\langle \widehat{f}, g\rangle = \langle f, \overline{\widehat{g}}\rangle$$

7. The Radon-Nikodym Theorem

7.1. **The Reisz representation theorem.** Let H be an inner product space with inner product $(x|v)$. Then H is a normed space with norm

$$|u\| = \sqrt{(u|u)}$$

We call H a **Hilbert space** if H is complete with respect to $\|.\|$, that is, every Cauchy sequence in H converges.

Theorem 7.1 (Projections)**.** *Let H be a Hilbert space and $M \subseteq H$ a closed subspace of H. Let $u \in H$. Then there exists $m_0 \in M$ such that*

$$\|u - m_0\| = \min_{m \in M} \|u - m\|$$

Moreover,

$$(u - m_o|m) = 0$$

for all $m \in M$.

Remark. Fix $g \in H$ and consider the function $\varphi_g : H \to \mathbb{K}$ given by

$$\varphi_g(f) = (f|g)$$

Then φ_g is linear, and by the Cauchy-Swartz inequality,

$$|\varphi_g(f) = |(f|g)| \leq \|f\|\|g\|$$

for all $f \in H$. We say φ_g is a bounded linear functional on H.

Definition 7.2. Let H be a Hilbert space. We call a linear operator $\varphi : H \to \mathbb{K}$ a **bounded linear functional** on H if there exists $M > 0$ such that

$$|\varphi(f)| \leq M\|f\|$$

for all $f \in H$.

Theorem 7.3 (Riesz representation theorem)**.** *Let H be a Hilbert space over $\mathbb{K}$ and $\varphi(H \to K)$ a bounded linear function. Then there exists $g \in H$ such that*

$$\varphi(f) = (f|g)$$

for all $f \in H$.

7.2. **The Radon-Nikodym Theorem.** Suppose that μ is a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X. Given a measurable function $g : X \to [0, \infty]$ we define

$$\nu(A) = \int_A g \, d\mu$$

Then ν is a measure defined on the σ-algebra $\mathcal{A}$.

The converse does not necessarily hold - that is, given two measures μ and ν on a σ-algebra $\mathcal{A}$, there is not always a measurable function $g : X \to [0, \infty]$ such that the above equation holds.

Definition 7.4 (Absolute continuity)**.** Let ν, μ be the measures defined on a σ-algebra $\mathcal{A}$. We call ν **absolutely continuous with respect to** μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. In that case, we write $\nu << \mu$.

Proposition 7.5. *Let μ, ν be measures defined on a σ-albegra. Suppose that $\mu(X), \nu(X) < \infty$. Set $\lambda = \mu + \nu$. Then there exists a measurable function $h : X \to [0, \infty]$ such that*

$$\int_X f\, d\nu = \int_X fh\, d\lambda$$

for all $f \in L^2(X, \lambda)$

Theorem 7.6 (Radon-Nikodym)**.** *Let μ, ν be measures defined on a σ-algebra. Suppose that ν and μ are σ-finite and that $v << u$. Then there exits a measurable function $g : X \to [0, \infty)$ such that $\nu(A) = \int_A g\, d\mu$ for all $A \in \mathcal{A}$.*

Formally we can write

$$\int_X f\, d\nu = \int_X f \frac{d\nu}{d\mu}\, d\mu$$

if we define $g = \frac{d\nu}{d\mu}$, where g is the density function from the Radon-Nikodym theorem.

Remark. If g is the function in the Radon-Nikodym theorem, it is not hard to show that

$$\int_X f\, d\nu = \int_X fg\, d\mu$$

for all $f \in L^1(X, \nu)$.

8. Probability Theory

Definition 8.1 (Random variable)**.** Let $(\Omega, \mathcal{A}, P)$ be a probability space. A $\mathcal{A}$-measurable function

$$X : \Sigma \to \mathbb{R}$$

is called a **random variable**.

Definition 8.2. Let $X : \Sigma \to \mathbb{R}$ a random variable. We say that X has *finite expectation* if $X \in L^1(\Sigma)$ and call

$$E[X] = \int_\Sigma X\, dP$$

the **expectation** of X.

Definition 8.3 (Distrbution)**.** For every Borel set $A \subseteq \mathbb{R}$ we define

$$P_X[A] = P[\{\omega \in \Omega | X(\omega) \in A\}] = P[X \in A]$$

Since X is measurable, $X^{-1}[A]$ is measurable for all Borel sets $A \subseteq \mathbb{R}$.

Definition 8.4 (Distribution)**.** Let X be a random variable. The Borel measure defined above is called the **distribution** of X. The function

$$F(t) = P_X\left[(-\infty, t]\right] = P[X \leq t]$$

is called the **distribution function** of X.

Lemma 8.5. *Let X be a random variable and let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Then*

$$\int_\Sigma f \circ X \, dP = \int_\mathbb{R} f \, dP_X$$

8.1. **Conditional expectation.**

Definition 8.6 (Conditional expectation)**.** Let $X : \Omega \to \mathbb{R}$ a random variable with finite expectation. Let $\mathcal{A}_0$ be a σ-algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$. We call

$$X_0 : \Sigma \to \mathbb{R}$$

a **conditional expectation** given $\mathcal{A}_0$ if

- X_0 is $\mathcal{A}_0$-measurable
- $\int_A X_0 \, dP = \int_A X \, dP$ for all $A \subseteq \mathcal{A}_0$.

We write $X_0 = E[X|\mathcal{A}_0]$

Theorem 8.7. *Let X be a random variable with finite expectation. If $\mathcal{A}_0$ is a σ-algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$, then the conditional expectation $X_0 = E[X|\mathcal{A}_0]$ exists and is essentially unique.*

Remark.
- If X is $\mathcal{A}_0$-measurable, then $X = E[X|\mathcal{A}_0]$ almost everywhere.
- If we set $\mathcal{A}_0 = \{\varphi, \Omega\}$, then
$$E[X|\mathcal{A}_0] = E[X]$$