PMH8 - SPECTRAL THEORY AND PDES

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1. Preliminaries

In general, we cannot solve arbitrary PDEs. We generally seek to prove **existence** of solutions and various **properties** of these solutions.

Assessment Schedule:

(i) Assignments - 2 or 3 (40%)
(ii) Exam - (60%)

References

(i) M. Protter and Weinberger – *Maximum Principle ...*
(ii) M. Renardy – *Elliptic PDEs*
(iii) A. Friedman – *Elliptic PDEs*
(iv) F. John – *PDEs*

2. Introduction to Functional Analysis

Definition 2.1 (Quotient space)**.** If M is a closed subspace of a normed vector space E, then we define another normed space E/M, the **quotient space**. Elements of E/M are of the form $\{u + m \mid m \in M\}$ where $u \in E$.

We now define the vector space operations. Define $(u_1 + M) + (u_2 + M) = (u_1 + u_2) + M$. If $\lambda \in \mathbb{K}$, define $\lambda(u + M) = \lambda u + M$. These operations make E/M a vector space.

Exercise 2.2. Show these operations are well defined.

Definition 2.3 (Normed quotient space)**.** Define

$$\|u + M\| = \inf_{m \in M} \|u + m\|.$$

If $u \notin M$, $\|u + M\| > 0$. This is because if there exists $(m_n) \in M$ with $\|u + m_n\| \to 0$, then $m_n \to -u$, and so $-u \in M$, which implies $u \in M$.

We can also show that $\|\lambda u + M\| = |\lambda|\|u + m\|$, and

$$\|(u_1 + u_2) + M\| \leq \|u_1 + M\| + \|u_2 + M\|.$$

With this norm, E/M is a normed space.

Exercise 2.4. Check the triangle and scaling inequalities.

Lemma 2.5. *Define an operator P by*

$$\begin{aligned} P: \quad & E \to E/M \\ & x \mapsto x + M \end{aligned}$$

Then P is linear and bounded.

Proof.

$$\|Px\| = \|x + M\| = \inf_{m \in M} \|x + M\| \leq \|x\|$$

Hence $\|Px\| \leq \|x\|$ and so P is bounded with $\|P\| \leq 1$. □

Theorem 2.6. *If E is a Banach space, then so is E/M, where M is a closed subspace of E.*

Theorem 2.7. *If M is a closed subspace of a normed space E and $z \in E \backslash M$, there exists $f \in E'$ such that $f(m) = 0$ for all $m \in M$, and $f(z) \neq 0$.*

Proof. $z + M$ is not zero in E/M, and so by the Hahn-Banach theorem, there exists $h \in (E/M)'$ such that $h(z + M) \neq 0$. Then define $f : E \to \mathbb{K}$ by $f(x) = h(Px)$ where $P : E \to E/M$ is the projection operator defined previously.

As f is the composition of two continuous maps, we have that $f \in E'$. Now, not that $f(m) = 0$ if $m \in M$, as $m + M$ is the zero coset. If $z \in E \backslash M$, then $f(z) = h(z + M) \neq 0$ by definition. □

Theorem 2.8. *If $T \in \mathcal{L}(X, Y)$ and* $\textsc{Im}\ T$ *is closed, then*

$$\textsc{Im}\ T = \{y \in Y \mid f(y) = 0 \textit{ for all } f \in \textsc{Ker}\ T'\}$$

Remark.

(i) In fact, if $\textsc{Im}\ T$ is not closed, the above theorem holds with $\overline{\textsc{Im}\ T}$.

(ii) This gives a solution to the inverse problem, i.e. given $y \in Y$, does there exists $x \in X$ such that $Tx = y$.

Definition 2.9 (Dual mapping)**.** Let $T \in \mathcal{L}(X, Y)$. Define the dual mapping $T' \in \mathcal{L}(Y', X')$ with $(T'f)(x) = f(Tx)$ for all $f \in Y'$.

Proof of Theorem 2.8. Let $A = R(T)$, if $z \in A$ there exists$f \in Y'$ such that $f(y) = 0$ for all $y \in A$ and $f(z) \neq 0$. Let $B = \{y \in Y \mid f(y) = 0\, \forall f \in N(T')\}$.

Hence

$$\begin{aligned} f(y) &= 0 \forall y \in A \\ f(Tx) &= 0 \forall x \in X \\ (T'f)(x) &= 0 \forall x \in X \end{aligned}$$

so that $T'f = 0$, and so $f \in N(T')$. But $f(z) \neq 0$, so $z \notin B$, and so $B \subseteq A$.

If $v \in R(T)$, then $v = Tx$. If $f \in N(T')$, then $f(v) = f(Tx) = (T'f)(x) = 0$, and so $v \in B$. Hence $A \subseteq B$. □

Remark. If H is a Hilbert space, and $T \in \mathcal{L}(H)$, then

$$\langle Tx, y\rangle = \langle x, T^\star y\rangle \quad \forall x, y \in H$$

where $T^\star$ is the adjoint. NOte that $T^\star = J^{-1}T'J$ where $J : H \to H'$ and T' is the conjugate operator. In this case, if $R(T)$ is closed, then

$$R(T) = \{x \in H \mid \langle x, y\rangle = 0 \quad \forall y \in N(T^\star)\}.$$

Remark. When is $R(T)$ closed?

(i) If $\lambda \neq 0$ and $K \in \mathcal{K}(X)$, $\lambda I - K$ has closed range.
(ii) If $K \in \mathcal{K}(X)$, $R(K)$ is closed if and only if $R(K)$ is finite dimensional.
(iii) If $N(T) = \{0\}$, X, Y are Banach spaces, and $T \in \mathcal{L}(X, Y)$, then $R(T)$ is closed if and only if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$. Note that if $R(T)$ is closed, it is a Banach space.

Corollary (Corollary to Theorem 2.8)**.** *If X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$, then T is invertible if and only if* $\textsc{Ker}\ T = \{0\}$, $\textsc{Ker}\ T' = \{0\}$ *and* $\textsc{Im}\ T$ *is closed.*

Note that the open mapping theorems shows that T is invertible if and only if $\textsc{Im}\ T = Y$ *and* $\textsc{Ker}\ T = \{0\}$.

Proof. If $\textsc{Im}\ T$ is closed, then by Theorem 2.8,

$$\textsc{Im}\ T = Y \iff \textsc{Ker}\ T' = \{0\},$$

as $\textsc{Im}\ T = \{y \in Y \mid f(y) = 0 \text{ for all } f \in \textsc{Ker}\ T'\}$. □

In a Hilbert space $\mathcal{H}$, if $T \in \mathcal{L}(\mathcal{H})$, then $T^* = J^{-1}T'J$ where $J : \mathcal{H} \to \mathcal{H}'$ is an isomorphism of Hilbert spaces.

Definition 2.10 (Weak convergence)**.** Let (x_n) be a sequence in X. We say that $x_n \rightharpoonup x$ **weakly** if $f(x_n) \to f(x)$ for all $f \in X'$.

Lemma 2.11. *If $x_n \to x$ in the usual sense, then $x_n \rightharpoonup x$* ***weakly****.*

Lemma 2.12. *If $x_n \rightharpoonup x$ weakly, then $\{x_n\}$ is bounded. Furthermore, $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

Proof. By Hahn-Banach, there exists $f \in X'$ such that $\|f\| = 1$ and $f(x) = \|x\|$. So $\|x\| = f(x) = \lim_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f(x_n)$. But

$$\|f(x_n)\| \leq \|f\|\|x_n\| \leq \|x_n\|$$

as $\|f\| = 1$. So

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|$$

□

Exercise 2.13. If (x_n) is bounded, then $x_n \rightharpoonup x$ weakly if and only if $f(x_n) \to f(x)$ as $n \to \infty$ for all functions in a dense subset of X'.

In fact, if (x_n) is bounded, we only need prove that if $f(x_n) \to f(x)$ for a subset M of X', then $f(x_n) \to f(x)$ for all finite linear combinations of elements of M.

Example 2.14. Let $1 < p < \infty$, and consider the Banach space ℓ^p. Then let $e_n = (\overbrace{0, 0, 0, \dots}^{p-1}, 1, \dots)$. Then $\|e_n\|_p = 1$ and $e_n \rightharpoonup 0$ weakly in ℓ^p as $n \to \infty$. Let $(\ell^p)' = \ell^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. If fact, every $f \in (\ell^p)'$ can be uniquely written as

$$f(x) = \sum_{i=1}^{\infty} x_i y_i$$

where $(y_i) \in \ell^{p'}$. In $\ell^{p'}$, the set of finite linear combinations of the e_n are dense in $\ell^{p'}$, since we can approximate (x_n) by $(x_1, x_2, \dots, x_m, 0, 0 \dots)$, which is a finite linear combination of the (e_n).

Hence a **bounded** sequence in ℓ^p, say $x^1 = (x_1^1, x_2^1, \dots), x^2 = (x_1^2, x_1^2, \dots)$ converges weakly if and only if $e_i(x^n) = x_i^n$ converges as $n \to \infty$ for each i.

In particular, $e_i \rightharpoonup 0$ weakly in ℓ^p.

Theorem 2.15. *If $x_n \rightharpoonup x$ weakly in X and $T \in \mathcal{L}(X)$, then $Tx_n \rightharpoonup Tx$ weakly in X.*

*Note that this is **not true** for continuous non-linear maps.*

Proof. Let $f \in X'$. Then

$$f(Lx_n) = (L'f)(x_n) \rightharpoonup (L'f)(x) = f(Lx)$$

weakly as $x_n \to x$ weakly and $(L'f)$ is a bounded linear operator. □

Definition 2.16 (Bidual)**.** Let X be a normed vector space. Then X' is a Banach space. The dual of the dual space, $(X')' = X''$ is known as the **bidual** of X.

There is a natural map

$$\begin{aligned} c: \quad & X \to X'' \\ & x \mapsto \hat{x} \end{aligned}$$

of X into X'', defined as folllows. Let $\hat{x}(f) = f(x)$ for all $f \in X'$. Then we can see that $\hat{x}$ is a linear mapping, and we must show that it is a bounded map from X' to $\mathbb{R}$.

We have

$$\|\hat{x}\| = \sup_{\|f\|\leq 1} |\hat{x}(f)| = \sup_{\|f\|\leq 1} |f(x)| \leq \sup_{\|f\|\leq 1} \|f\|\|x\| \leq \|x\|.$$

Thus $\|\hat{x}\| \leq \|x\|$. (By Hahn-Banach, we can show $\|\hat{x}\| = \|x\|$.)

Exercise 2.17. Show that $\textsc{Ker}\ c = \{0\}$.

Thus c is a bounded linear map with a zero null-space.

Definition 2.18 (Reflexive)**.** A Banach space is refexive if this map of X onto X'' is bijective.

Example 2.19.

(i) Finite dimensional spaces are reflexive (as the bidual has the same dimension as the base space).
(ii) ℓ^p, L^p are reflexive if $1 < p < \infty$, and are not reflexive otherwise.
(iii) Hilbert spaces $\mathcal{H}$ are reflexive.
(iv) $\mathcal{C}(\Omega)$, the set of continuous operators on a compact set in $\mathbb{R}^n$.

Theorem 2.20 (Compactness property)**.** *A Banach space X is reflexive if and only if every bounded sequence in X has a subsequence that converges weakly in X.*

Remark (Closeness property). If C is a closed and convex subset of a Banach space X, and x_n is a sequence in C with $x_n \rightharpoonup y \in X$ weakly, then $y \in C$.

Proof. Uses the geometric version of the Hahn-Banach theorem.

Theorem 2.21 (Geometric Hahn-Banach)**.** *If C is a closed and convex subset in X and $z \notin C$, there exists $f \in X'$ and $m \in \mathbb{R}$, such that $f(x) \leq m$ for all $x \in C$ and $f(z) > m$.*

If $y \notin C$, there exists $f \in X'$ and $m \in \mathbb{R}$ such that $f(x) \leq m$ for all $x \in C$ and $f(y) > m$. But as $f(x_n) \leq m$ and $f(x_n) \to f(y)$ (by weak convergence), we must have $f(y) \leq m$. Thus we achieve our required result, $y \in C$. □

3. Linear Operators on Hilbert Spaces

Theorem 3.1 (Lax-Milgram theorem)**.** *If $T \in \mathcal{L}(\mathcal{H})$ and there exists $\mu > 0$ such that $Re\langle Tx, x\rangle \geq \mu\|x\|^2$ for all $x \in \mathcal{H}$, then T is invertible.*

Proof. If suffices to prove that $\textsc{Ker}\ T = \{0\}$, $\textsc{Im}\ T$ is closed, and $\textsc{Ker}\ T^\star = \{0\}$, by a corollary to Theorem 2.8.

By Cauchy-Swartz, we have

$$\mu\|x\|^2 \leq \operatorname{Re}\langle Tx, x\rangle \leq |\langle Tx, x\rangle| \leq \|Tx\|\|x\|.$$

If $x \neq 0$, then $\mu\|x\| \leq \|Tx\|$, so $\textsc{Ker}\ T = \{0\}$.

Secondly, $\|Tx\| \geq \mu\|x\|$ for $\mu > 0$ implies $\textsc{Im}\ T$ is closed.

Exercise 3.2. Prove this proposition.

Then finally, we have

$$\operatorname{Re}\langle Tx, x\rangle = \operatorname{Re}\langle x, T^\star x\rangle \leq |\langle x, T^\star x\rangle| \leq \|x\|\|T^\star x\|$$

by Cauchy-Swartz. So

$$\mu\|x\|^2 \leq \|x\|\|T^\star x\|$$

and so $\mu\|x\| \leq \|T^\star x\|$ and so $\textsc{Ker}\ T^\star = \{0\}$. □

Definition 3.3 (Coercive)**.** T is coercive if there exists $\mu > 0$ such that $\operatorname{Re}\langle Tx, x\rangle \geq \mu\|x\|^2$.

Definition 3.4 (Spectral radius)**.** Let X be a complex Banach space. If $T \in \mathcal{L}(X)$, then we can define the spectral radius $r(T)$ by the formula

$$r(T) = \limsup_{n\to\infty} \|T^n\|^{1/n}.$$

Theorem 3.5. *We have*

$$r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\}.$$

Note that $r(T) \leq \|T\|$.

Theorem 3.6. *If $T \in \mathcal{L}(\mathcal{H})$ and T is a self-adjoint operator then $r(T) = \|T\|$.*

Proof. We have

$$\|T\|^2 = \|T^\star T\| = \|T^2\|,$$

since for any linear operator T, we have

$$\|T\|^2 = \|T^\star T\|.$$

Then by induction, we have

$$r(T) = \limsup_{n\to\infty} \|T^{2^n}\|^{1/2^n} = \|T\|$$

□

Theorem 3.7 (Raleigh-Rety algorithm)**.** *If $T \in \mathcal{L}(\mathcal{H})$ is self-adjoint, then*

$$\sup\sigma(T) = \sup\{\langle Tx, x\rangle \mid \|x\| = 1\}$$
$$\inf\sigma(T) = \inf\{\langle Tx, x\rangle \mid \|x\| = 1\}$$

Proof. If suffices to prove the first statement (and then apply to $-T$). We first show $\sup\sigma(T) \leq \sup\{\langle Tx, x\rangle \mid \|x\| = 1\} \equiv \mu$.

If $\lambda > \mu$, then

$$\lambda\|x\|^2 - \langle Tx, x\rangle \geq \lambda - \mu > 0$$

if $\|x\| = 1$. Hence

$$\begin{aligned} \lambda - \mu &\leq \langle(\lambda I - T)x, x\rangle \quad \|x\| = 1 \\ &\leq \|(\lambda I - T)x\|\|x\| \end{aligned}$$

$$\Rightarrow \|(\lambda I - T)\|x\| \geq (\lambda - \mu)\|x\|$$

and hence Ker $\lambda I - T = \{0\}$, and as Im $\lambda I - T$ is closed by Exercise 3.8, we have that $\lambda I - T$ is invertible. Thus $\sup \sigma(T) \geq \sup\{\langle Tx, x\rangle \mid \|x\| = 1\}$.

Consequently, it suffices to assume $\sigma(T)$ is non-negative (replace T with $T + rI$). Then if $\mu \in \sigma(T_1)$ with T_1 self-adjoint, then there exists a sequence x_n with $\|x_n\| = 1$ such that

$$\|T_1x_n - \mu x_n\| \to 0$$

as $n \to \infty$. Existence of such a sequence is proven as if $\|T_1x - \mu x\| \geq \alpha\|x\|$, then $\mu \notin \sigma(T_1)$.

Thus

$$\begin{aligned} &\langle T_1x_n, x_n\rangle \to \mu \\ &\langle T_1x_n, x_n\rangle = \underbrace{\langle(T_1 - \mu I)x_n, x_n\rangle}_{\to 0} + \underbrace{\mu\,\langle x_n, x_n\rangle}_{\mu}. \end{aligned}$$

Thus

$$\sup\{\langle Tx, x\rangle \mid \|x\| = 1\} \geq \sup \sigma(T).$$

□

Exercise 3.8. If $\|Tx\| \geq m\|x\|$ for all x, then Im T is closed.

4. Generalised Derivatives

Definition 4.1 ($L^1_{loc}(\Omega)$)**.** Let $\Omega \subset \mathbb{R}^n$ be open. Then $u \in L^1_{loc}(\Omega)$ if u is measurable and $u|_K \in L^1(K)$ for every compact $K \subseteq \Omega$.

Definition 4.2 (Generalised derivative)**.** We say $u \in L^1_{loc}(\Omega)$ has a (weak) generalised j-th partial derivative if there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u\frac{\partial\phi}{\partial x_j} = -\int_\Omega g\phi \tag{4.1}$$

for all $\phi \in C_c^\infty(\Omega)$.

Note that g is defined only up to sets of measure zero.

Note. The motivation comes from the integration by parts formula, where if u is $C^1(\Omega)$, then

$$\int_\Omega u \frac{\partial \phi}{\partial x_j} = -\int_\Omega \frac{\partial u}{\partial x_j} \phi$$

for all $\phi \in C_c^1(\Omega)$ by integration by parts. Thus we can write $g = \frac{\partial u}{\partial x_j}$.

Lemma 4.3. *The function g, if it exists, is unique (up to sets of measure zero).*

Proof. If g_1, g_2 both satisfy (4.1), then

$$-\int_\Omega u \frac{\partial \phi}{\partial x_j} = \int_\Omega g_1 \phi = \int_\Omega g_2 \phi$$

for all $\phi \in C_c^\infty(\Omega)$. Thus

$$\int_\Omega (g_1 - g_2)\phi = 0 \tag{$\star$}$$

for all $\phi \in C_c^\infty(\Omega)$.

Suppose B is a ball with $\overline{B} \subseteq \Omega$. Then

$$(g_1 - g_2)|_B \in L^1(B).$$

Since ($\star$) holds for all $\phi \in C_c^\infty(B)$, consider the measurable function

$$\operatorname{sgn}(g_1 - g_2) = \begin{cases} 1 & (g_1 - g_2)(x) \geq 0 \\ -1 & (g_1 - g_2)(x) < 0. \end{cases}$$

We assume that there exists $(\phi_n) \in C_c^\infty(B)$ such that ϕ_n are uniformly bounded and $\phi_n(x) \to \operatorname{sgn}(g_1 - g_2)$ almost everywhere. This can be justified by Young's inequality, where if

$$f_n(x) = \int_B \psi_n(x-y) f(y)\, dy$$

then $\|f_n\|_\infty \leq \|\psi_n\| \; |f\|_\infty$, so our approximating function f_n are uniformly bounded.

Then

$$0 = \int_\Omega (g_1 - g_2)\phi_n \to \int_B (g_1 - g_2)\operatorname{sgn}(g_1 - g_2) = \int_B |g_1 - g_2|$$

as $n \to \infty$ by the dominate convergence theorem.

Thus $g_1 - g_2 = 0$ almost everywhere on B. By the Lindeloff property (Lemma 4.4), Ω is a countable union of balls, and so we can extend this result to the result,

$$g_1 - g_2 = 0$$

almost everywhere on ϕ. □

Lemma 4.4 (Lindeloff property)**.** *A separable metric space, such as R^n, any open set is a countable union of open balls.*

Remark.

(i) If g is the generalised j-th partial derivative of u on Ω and $\Omega_1 \subset \Omega$ is open, then $g|_{\Omega_1}$ is the j-th generalised partial derivative of $u|_{\Omega_1}$.

(ii) Assume $A \subseteq \Omega$, u has a generalised j-th partial derivative on Ω, A is open, and u is C^1 on A.

Then the generalised j-the partial derivative of u is equal to the classical partial derivative almost everywhere on A.

Example 4.5. Consider the function

$$u(x,y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}$$

If the generalised derivative $\frac{\partial u}{\partial x}$ exists, it must be zero when $y > 0$ and when $y < 0$.

It turns out that $\frac{\partial u}{\partial x}$ exists but $\frac{\partial u}{\partial y}$ does not.

Example 4.6. $f : \mathbb{R} \to \mathbb{R}$ define by $f(x) = |x|$ has a generalised derivative g defined by

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x > 0 \end{cases}$$

Note that f is C^1 if $x \neq 0$ so if the generalised derivative exists it must be equal to g.

Example 4.7. If B_1 is the open unit ball in $\mathbb{R}^2$ and

$$f(x) = \begin{cases} \ln(x^2 + y^2) & (x,y) \neq (0,0) \end{cases}$$

- thus $f(x) = 2 \ln r$ in polar coordinates.

Then

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$$
$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

are the generalised partial derivatives on $\mathbb{R}^2$.

Definition 4.8 (Generalised derivative)**.** We say that $u \in L^1_{loc}(\Omega)$ has a generalised derivative on Ω if all the generalised partial derivatives $\frac{\partial u}{\partial x_j}$ exist for $1 \leq j \leq n$ (where Ω is an open set in $\mathbb{R}^n$).

Remark.

(i) If u_1 and u_2 have generalised derivatives on Ω and C_1, C_2 are constant, then $C_1 u_1 + C_2 u_2$ has a generalised derivative on Ω, given by the appropriate linear combination.

(ii) If u has a generalised derivative on Ω and $\Psi \in C^\infty(\Omega)$, then $u\Psi$ has a generalised derivative on Ω and

$$\frac{\partial}{\partial x_j}(u\Psi) = \frac{\partial u}{\partial x_j}\Psi + u\frac{\partial \Psi}{\partial x_j}$$

Lemma 4.9. *If u_k has a generalised derivative on Ω and $u_k \to u$ in $L^1_{loc}(\Omega)$ as $k \to \infty$ and if $\frac{\partial u_k}{\partial x_l} \to g_l$ in $L^1_{loc}(\Omega)$ then u has a generalised derivative on Ω and*

$$\frac{\partial u}{\partial x_l} = g_l.$$

Proof.

$$\int_\Omega u_k \frac{\partial \phi}{\partial x_j} = -\int_\Omega \frac{\partial u}{\partial x_j}\phi \tag{$\star$}$$

if $\phi \in C_c^\infty(\Omega)$. Fix ϕ and choose K compact so the support of ϕ is contained in K. Then $u_k \frac{\partial \phi}{\partial x_j} \to u \frac{\partial \phi}{\partial x_j}$ in L^1 on K.

Then letting $k \to \infty$ in ($\star$) we obtain

$$\int_\Omega u \frac{\partial \phi}{\partial x_j} = -\int_\Omega g_j \phi.$$

□

Remark. If $g_j \in L^p(\Omega)$ and $g_j \to g$ in $L^p(\Omega), (1 \leq p \leq \infty)$, then $g_j \to g$ in $L^1_{loc}(\Omega)$.

Proof. If K is compact, then

$$\int_K (g_j - g) \leq \|g_j - g\|_{p,K}^{\frac{1}{p}} \|1\|_{p',K}^{1/p'}$$

by Hölder's inequality. □

5. Sobolev Spaces

Definition 5.1 (Sobolev spaces)**.** If $1 \leq p \leq \infty$ and Ω is open in $\mathbb{R}^n$, then the space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \underbrace{\frac{\partial u}{\partial x_i} \in L^p(\Omega)}_{\substack{\text{generalised}\\ \text{derivatives}}} \text{ for } 1 \leq i \leq n\}$$

equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_p$$

is a Banach space. We call $W^{1,p}$ a Sobolev space.

It is a linear space by linearity of the generalised derivatives. Similarly, the triangle inequality holds as all components of the norm $\|\cdot\|_{1,p}$ satisfy the triangle inequality. It can be shown that $W^{1,p} \subseteq L^p(\Omega)^{N+1}$ and $W^{1,p}$ is a closed subspace, which shows that $W^{1,p}$ is Banach, being the closed subspace of a Banach space.

Proposition 5.2. *$W^{1,p}$ is a Banach space. In fact, $W^{1,p}$ is a closed subspace of $L^p(\Omega)^{n+1}$.*

Proof. Consider the the map

$$(u_j, \frac{\partial u_j}{\partial x_1}, \ldots, \frac{\partial u_j}{\partial x_n}) \to (w_0, w_1, \ldots, w_n)$$

If $u_j \to w_0 \in L^p(\Omega)$, then $\frac{\partial u_j}{\partial x_1} \to w_1$ in $L^p(\Omega)$ which implies that $\frac{\partial u_j}{\partial x_1} \to w_1$ in $L^1_{loc}(\Omega)$.

By Lemma 4.9, $\frac{\partial w_0}{\partial x_1}$ exists on Ω and equals w^1. Similarly, $\frac{\partial w_0}{\partial x_l}$ exists and equals w_l. Then since $w_0 \in W^{1,p}(\Omega)$ and $\frac{\partial w_0}{\partial x_l} = w_l$ the closure property holds. Hence we have a Banach space. □

Note. Recall that all norms on a finite dimensional vector space are equivalent. For example,

$$\left(\|u\|_p^p + \sum_{j=1}^{n} \| \frac{\partial u}{\partial x_j} \|_p^p)^{1/p} \right)$$

and

$$\|u\|_p + \left(\sum_{j=1}^{n} \| \frac{\partial u}{\partial x_j} \|_p^p \right)^{1/p}$$

are equivalent.

Definition 5.3 (Higher Sobolev spaces)**.** We have

$$W^{2,p}(\Omega) = \{u \in L^p(\Omega) \,|\, \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i x_j} \in L^p(\Omega) \text{ for } 1 \leq i \leq n, 1 \leq j \leq n\}$$

Definition 5.4. $\dot{W}^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ in the norm $\| \cdot \|_{1,p}$. In general, $\dot{W}^{1,p}(\Omega) \subseteq W^{1,p}(\Omega)$.

Proposition 5.5. $\dot{W}^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

Proposition 5.6. $\dot{W}^{1,2}(\Omega), W^{1,2}(\Omega)$ *are Hilbert spaces under the inner product* $\langle \cdot, \cdot \rangle$ *defined by*

$$\langle u, v \rangle = (u, v) + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)$$

where $(u, v) = \int_\Omega u(x)\overline{v(x)}\, dx$.

6. Convolutions and Approximations

Recall that there exists $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi(x) > 0$ if $\|x\| < 1$ and $\phi(x) = 0$ if $\|x\| \geq 1$. We can assume that $\int_{\mathbb{R}^n} \phi = 1$.

If $f \in L^p_{loc}(\mathbb{R}^n)$ and $1 \leq p < \infty$, we define $T_\epsilon f$ by

$$(T_\epsilon f)(x) = \epsilon^{-N} \int \phi \left(\frac{x-y}{\epsilon} \right) f(y)\, dy = \phi_\epsilon \star f$$

where $\phi_\epsilon = \epsilon^{-N} \phi \left(\frac{x}{\epsilon} \right)$.

Proposition 6.1. *If $f \in L^p(\mathbb{R}^n)$ where $1 \le p < \infty$, then*

$$T_\epsilon f \to f$$

in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0$.

Lemma 6.2. *If f has support in a compact set K, then $T_\epsilon f$ has support in $\{x \in \mathbb{R}^n \mid d(x, K) \le \epsilon\}$.*

Lemma 6.3. *By Proposition 6.1, if $f \in L^p(\mathbb{R}^n)$, there exists $\epsilon_l \to 0$ such that $T_{\epsilon_l} f \to f$ almost everywhere as $l \to \infty$.*

Lemma 6.4.

$$\|T_\epsilon f\|_\infty \le \|f\|_\infty$$

if $f \in L^\infty(\mathbb{R}^n)$.

Proof. If $-1 \le f \le 1$ on $\mathbb{R}^n$, then as

$$T_\epsilon(-1) \le T_\epsilon f \le T_\epsilon 1$$

that is,

$$-1 \le T_\epsilon f(x) \le 1 \quad \forall x$$

then since $T_\epsilon f$ is linear we have

$$\|T_\epsilon f\|_\infty \le \|f\|_\infty$$

if $f \in L^\infty(\mathbb{R}^n)$. □

Proposition 6.5. *$C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,2}(\mathbb{R}^n)$ that is, if $f \in W^{1,2}(\mathbb{R}^n)$, there exists $(f_n) \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - f_n\|_{1,2} \to 0$ as $n \to \infty$.*

Note that this is non-trivial as if Ω is bounded the corresponding result is false.

Proof. Let $f \in W^{1,2}(\mathbb{R}^n)$ and $\delta > 0$. By a previous exercise, there exists $\tilde{f} \in W^{1,2}(\mathbb{R}^n)$ of compact support such that

$$\|f - \tilde{f}\| \le \frac{\delta}{2}.$$

Hence it suffices to find $f_n \in C_c^\infty(\mathbb{R}^n)$ such that $\|f_n - \tilde{f}\|_{1,2} \to 0$ as $n \to \infty$ (as this would imply $\|f_n - f\| \le \delta$ for large enough n).

We prove that $T_\epsilon \tilde{f} \in W^{1,2}(\mathbb{R}^n)$ and $T_\epsilon \tilde{f} \to \tilde{f}$ in $W^{1,2}(\mathbb{R}^n)$ as $\epsilon \to 0$. Recall that $T_\epsilon \tilde{f} \in C^\infty(\mathbb{R}^n)$. Suppose that $T_\epsilon \tilde{f} \subseteq B(\epsilon)\{\text{supp}(\tilde{f})\} = \{x \in \mathbb{R}^n \mid d(x, \text{supp}(\tilde{f})) \le \epsilon\}$. Recall that $T_\epsilon \tilde{f} \to \tilde{f}$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$ from MATH 3969.

We thus need to prove

$$\frac{\partial}{\partial x_l} T_\epsilon \tilde{f} = T_\epsilon \underbrace{\left(\frac{\partial \tilde{f}}{\partial x_l}\right)}_{\substack{\text{generalised} \\ \text{derivative}}} \tag{6.1}$$

If we prove that
$$\frac{\partial}{\partial x_l}\left(T_\epsilon \tilde{f}\right) = T_\epsilon\left(\frac{\partial \tilde{f}}{\partial x_l}\right) \to \frac{\partial \tilde{f}}{\partial x_l}$$
in $L^2(\mathbb{R}^n)$.

We have
$$\begin{aligned}\frac{\partial}{\partial x_l} T_\epsilon \tilde{f}(x) &= \frac{\partial}{\partial x_l}\left(\epsilon^{-n}\int \phi\left(\frac{x-y}{\epsilon}\right)\tilde{f}(y)\,dy\right)\\ &= \epsilon^{-n}\int \frac{\partial}{\partial x_l}\phi\left(\frac{x-y}{\epsilon}\right)\tilde{f}(y)\,dy\\ &= e^{-n}\int -\frac{\partial}{\partial y_l}\phi\left(\frac{x-y}{\epsilon}\right)\tilde{f}(y)\,dy\end{aligned}$$
where we use the fact that
$$\frac{\partial}{\partial x_l} g(x-y) = -\frac{\partial}{\partial y_l} g(x-y).$$
Continuing, we obtain
$$\begin{aligned}\frac{\partial}{\partial x_l} T_\epsilon \tilde{f}(x) &= -\epsilon^{-n}\int \frac{\partial}{\partial y_l}\left(\phi\left(\frac{x-y}{\epsilon}\right)\right)\tilde{f}(y)\,dy\\ &= \epsilon^{-n}\int \phi\left(\frac{x-y}{\epsilon}\right)\frac{\partial}{\partial y_l}\tilde{f}(y)\,dy \qquad (\star)\\ &= T_\epsilon\left(\frac{\partial \tilde{f}}{\partial y_l}\right)\end{aligned}$$
as $\phi\left(\frac{x-y}{\epsilon}\right)$ is a smooth function of compact support. Thus the generalised derivative exists as $\tilde{f} \in W^{1,2}(\mathbb{R}^n)$, and so the manipulation in $(\star)$ is justified.

□

7. Fourier Transforms and Weak Derivatives

Definition 7.1 (Fourier transform)**.** If $f \in L^1(\mathbb{R}^n), \lambda \in \mathbb{R}^n$, then the Fourier transform $\hat{f}$ of f is defined by
$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} f(t)e^{i\lambda t}\,dt. \tag{7.1}$$

Theorem 7.2. *The map $f \mapsto \hat{f}$ is a bijection on $L^2(\mathbb{R}^n)$.*

Theorem 7.3 (Parseval's theorem)**.** *If $f \in L^2(\mathbb{R}^n)$, then $(2\pi)^n\|f\|_2^2 = \|\hat{f}\|_2^2$. This can be generalised slightly to if $f, g \in L^2(\mathbb{R}^n)$, then*
$$(2\pi)^n\langle f, g\rangle = \left\langle \hat{f}, \hat{g}\right\rangle = \int_{\mathbb{R}^n} \hat{f}(x)\overline{\hat{g}(x)}\,dx$$

Theorem 7.4. *If $f \in L^2(\mathbb{R}^n)$, the following are equivalent:*

(i) $f \in W^{1,2}(\mathbb{R}^n)$,

(ii) $-i\lambda_j \hat{f}(\lambda) \in L^2(\mathbb{R}^n)$ *for* $1 \leq j \leq n$,
(iii) $1 + |\lambda|\hat{f}(\lambda) \in L^2(\mathbb{R}^n)$.

If any of these hold, the generalised derivative $\frac{\partial f}{\partial x_j}$ *exists and* $\frac{\hat{\partial f}}{\partial x_j}(\lambda) = -i\lambda_j \hat{f}(\lambda)$ *for* $1 \leq j \leq n$.

Proof. $(ii) \iff (iii)$ $|\lambda_j \hat{f}(\lambda)| \leq |\lambda||\hat{f}(\lambda)|$ and hence (ii) $\iff$ (iii).

$(i) \Rightarrow (ii)$ The only thing left to prove is that

$$\frac{\partial \hat{f}}{\partial x_j}(\lambda) = -i\lambda_j \hat{f}(\lambda)$$

for $f \in W^{1,2}(\mathbb{R}^n)$. We have

$$\begin{aligned} f \in W^{1,2}(\mathbb{R}^n) &\Rightarrow \frac{\partial f}{\partial x_j} \in L^2(\mathbb{R}^n) \\ &\Rightarrow \frac{\hat{\partial f}}{\partial x_j}(\lambda) \in L^2(\mathbb{R}^n) \end{aligned}$$

so to prove the previous result we choose $f_n \in C_c^\infty(\mathbb{R}^n)$ such that $\|f_n - f\|_{1,2} \to 0$ as $n \to \infty$. Since

$$\frac{\partial \hat{f}_n}{\partial x_j}(\lambda) = -i\lambda_j \hat{f}_n(\lambda),$$

and $f_n \to f$ in $W^{1,2}(\Omega)$, we have

$$\begin{aligned} &\Rightarrow f_n \to f \text{ in } L^2(\mathbb{R}^n) \\ &\Rightarrow \hat{f}_n \to \hat{f} \text{ in } L^2(\mathbb{R}^n) \\ &\Rightarrow \hat{f}_n(\lambda) \to \hat{f}(\lambda) \text{ a.e. (taking subsequences)} \\ &\Rightarrow \frac{\partial f_n}{\partial x_j} \to \frac{\partial f}{\partial x_j} \text{ in } L^2(\mathbb{R}^n) \\ &\Rightarrow \frac{\partial \hat{f}_n}{\partial x_j} \to \frac{\hat{\partial u}}{\partial x_j} \text{ in} L^2(\mathbb{R}^n) \\ &\Rightarrow -i\lambda_j \hat{f}_n(\lambda) \to \frac{\hat{\partial f}}{\partial x_j}(\lambda) \text{ in } L^2(\mathbb{R}^n) \\ &\Rightarrow \frac{\hat{\partial f}}{\partial x_j}(\lambda) = -i\lambda_j \hat{f}(\lambda) \text{ a.e.} \end{aligned}$$

$(ii) \Rightarrow (i)$ As $-i\lambda_j \hat{f}(\lambda) \in L^2(\mathbb{R}^n)$, and so there exists $g_j \in L^2(\mathbb{R}^n)$ such that

$$\hat{g}_j = -i\lambda_j \hat{f}(\lambda).$$

Thus we have

$$
\begin{aligned}
(2\pi)^n \left(f, \frac{\partial \phi}{\partial x_j}\right) &= \left(\hat{f}, \frac{\hat{\partial \phi}}{\partial x_j}\right) \qquad \phi \in C_c^\infty(\mathbb{R}^n) \\
&= \int_{\mathbb{R}^n} \hat{f}(\lambda)\overline{-i\lambda_j \hat{\phi}(\lambda)} \\
&= \int_{\mathbb{R}^n} \hat{f}(\lambda) i\lambda_j \overline{\hat{\phi}(\lambda)} \\
&= \int_{\mathbb{R}^n} i\lambda_j \hat{u}(\lambda)\overline{\hat{\phi}(\lambda)} \\
&= \int_{\mathbb{R}^n} \hat{g}_j \overline{\hat{\phi}(\lambda)} \\
&= -(2\pi)^n \left(g_j, \phi\right) \\
&\Rightarrow \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_j} = -\int g_j \phi
\end{aligned}
$$

and so g_j is the j-th generalised derivative of f and $g_j \in L^2(\mathbb{R}^n)$, thus $f \in W^{1,2}(\mathbb{R}^n)$ $\square$

Remark. As a consequence,

$$u \in W^{2,2}(\mathbb{R}^n) \iff \left(1 + |\lambda|^2\right) \hat{u}(\lambda) \in L^2(\mathbb{R}^n)$$

and a similar result can be obtained for $W^{k,2}(\mathbb{R}^n)$. This follows from the fact that

$$C_2 \leq \frac{1 + |\lambda|^2}{(1 + |\lambda|)^2} \leq C_1$$

on $\mathbb{R}^n$ where $C_1, C_2 > 0$.

Example 7.5. Consider the PDE

$$-\Delta u + u = f \tag{7.2}$$

on $\mathbb{R}^n$, where $f \in L^2(\mathbb{R})$ and we look for $u \in W^{2,2}(\mathbb{R}^n)$. Taking Fourier transformations, we have

$$
\begin{aligned}
\frac{\hat{\partial^2 u}}{\partial x_j \partial x_k} &= (-i\lambda_k)(-i\lambda_j)\hat{u}(\lambda) \\
&= -\lambda_k \lambda_j \hat{u}(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
-\left(-\sum_{k=1}^n \lambda_k^2\right) \hat{u}(\lambda) + \hat{u}(\lambda) &= \hat{f}(\lambda) \\
\left(1 + |\lambda|^2\right) \hat{u}(\lambda) &= \hat{f}(\lambda).
\end{aligned}
$$

So

$$\hat{u}(\lambda) = \frac{\hat{f}(\lambda)}{1 + |\lambda|^2}$$

and $u \in W^{2,2}(\mathbb{R}^n)$ (since $\left(1 + |\lambda|^2\right) \hat{u}(\lambda) = \hat{f}(\lambda) \in L^2(\mathbb{R}^n)$. This is the unique solution in $W^{2,2}$.

Example 7.6. Consider a slightly modified version of (7.2)

$$-\Delta u = f.$$

we obtain $\hat{u}(\lambda) = \frac{\hat{f}(\lambda)}{|\lambda|^2}$ and this is not well defined for λ near zero.

Example 7.7. Considering the equation (7.2), we take $u \in W^{1,2}(\mathbb{R}^n)$ such that

$$\int (\nabla u \nabla \phi + u\phi) = \int f\phi \quad \forall \phi \in C_c^\infty(\mathbb{R}^n). \tag{$\star$}$$

If this holds, it follows that $\phi \in W^{1,2}(\mathbb{R}^n)$. By Parseval's theorem, we have

$$\int \sum_j -i\lambda_j \hat{u}(\lambda)\overline{-i\lambda_j \hat{\phi}(\lambda)} + \int \hat{u}(\lambda)\overline{\hat{\phi}(\lambda)} = \int \hat{f}(\lambda)\overline{\hat{\phi}(\lambda)}$$

and this is solved by

$$\hat{u}(\lambda) = \frac{\hat{f}(\lambda)}{1 + |\lambda|^2}$$

Note that ($\star$) has at most one solution in $W^{1,2}(\mathbb{R}^n)$. If u_1, u_2 are solutions then we have

$$\int \nabla u_1 \nabla \phi + u_1 \phi = \int f\phi \quad \forall \phi \in W^{1,2}(\mathbb{R}^n).$$
$$\int \nabla u_2 \nabla \phi + u_2 \phi = \int f\phi.$$

Subtracting these obtains

$$\int \nabla u_1 - u_2 \nabla \phi + (u_1 - u_2 \phi) = 0.$$

Letting $\phi = u_1 - u_2 \in W^{1,2}(\mathbb{R}^n)$, we have

$$\int \underbrace{\nabla(u_1 - u_2)\nabla u_1 - u_2}_{\geq 0} + \underbrace{(u_1 - u_2)^2}_{\geq 0} = 0.$$

8. Poincaré Inequality and Applications

Lemma 8.1. *If $v \in C^1(\mathbb{R})$, $a \neq b$ and $v(a) = v(b) = 0$, then*

$$\int_a^b v^2(t)\,dt \leq (b-a)^2 \int_a^b \left(v'(t)\right)^2\,dt. \tag{8.1}$$

Proof. We have $v(x) = v(a) + \int_a^x v'(t)\,dt = \int_a^x v'(t)\,dt$ for $a < x < b$. So

$$\begin{aligned}|v(x)| &\le \left|\int_a^x v'(t)\,dt\right| \\ &\le \int_a^b |v'(t)|\,dt \\ &\le (b-a)^{1/2}\left(\int_a^b (v'(t))^2\,dt\right)^{1/2}.\end{aligned}$$

Squaring and integrating from a to b, we obtain our result,

$$\int_a^b v^2(t)\,dt \le (b-a)^2 \int_a^b (v'(t))^2\,dt.$$

□

Theorem 8.2 (Poincaré inequality)**.** *If Ω is a domain in $\mathbb{R}^n$ with $\Omega \subseteq C$ where C is a cube of side d, then*

$$\|w\|_{2,\Omega} \le d\|\nabla w\|_{2,\Omega} \tag{8.2}$$

for $w \in \dot{W}^{1,2}(\Omega)$.

Remark. Recall that $\cdot W^{1,2}(\Omega) \subset W^{1,2}(\Omega)$ if Ω is a bounded domain. Note that the identity function $1 \in W^{1,2}(\Omega)$ does not satisfy this inequality.

Proof. First assume $u \in C_c^\infty(\Omega)$. We can extend u to $\tilde{u}$ in $C_c^\infty(C)$ by defining $\tilde{u}(x) = 0$ if $x \in C \setminus \Omega$. Assume $C = [a,b]^n$. Then (identifying u with $\tilde{u}$),

$$\int_a^b u(x_1,\dots,x_n)^2\,dx_1 \le (b-a)^2 \int_a^b \left(\frac{\partial u}{\partial x_1}\right)^2\,dx_1$$

by Lemma 8.1. Integrating over the entire n-cube, we then have

$$\begin{aligned}\int_a^b \cdots \int_a^b u(x_1,\dots,x_n)^2\,dx_1\dots dx_n &\le (b-a)^2 \int_a^b \cdots \int_a^b \left(\frac{\partial u}{\partial x_1}\right)^2\,dx_1\dots dx_n \\ &\le (b-a)^2 \int_C |\nabla u|^2\end{aligned}$$

as $|\nabla u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_1}\right)^2$. As u is zero on $C \setminus \Omega$ we have the result

$$\int_\Omega u^2 \le d^2 \int_\Omega |\nabla u|^2 \tag{$\star$}$$

for $u \in C_c^\infty(\Omega)$.

Now, if $u \in \dot{W}^{1,2}(\Omega)$, there exists $u_n \in C_c^\infty(\Omega)$ such that $\|u_n - u\|_{1,2} \to 0$ as $n \to \infty$. For each n, we then have

$$\int_\Omega u_n^2 \le d^2 \int_\Omega |\nabla u_n|^2$$

by $(\star)$. Taking the limit, we obtain our required result,

$$\int_\Omega u^2 \leq d^2 \int_\Omega |\nabla u|^2. \qquad \square$$

Intuitively, $\dot{W}^{1,2}(\Omega)$ is the set of functions in $W^{1,2}(\Omega)$ vanishing on $\partial\Omega$. If Ω is a domain with a smooth boundary, then it can be proven there is a map T, known as the trace map,

$$T : W^{1,2}(\Omega) \to L^2(\partial\Omega)$$

such that $\dot{W}^{1,2}(\Omega) = \text{Ker } T$. The key difficulty in the proof is showing the inequality

$$\int_{\partial\Omega} (v|_{\partial\Omega})^2 \leq K\|v\|_{1,2}^2$$

if $v \in W^{1,2}(\Omega)$. By the Poincaré inequality, we can use $\|\nabla u\|_2$ as a norm on $\dot{W}^{1,2}(\Omega)$ if Ω is abounded domain. This is equivalent to $\|u\|_2 + \|\nabla u\|_2$.

Note that this norm is induced by the scalar product (assuming real u, v)

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v = \sum_{j=1}^n \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}$$

Proposition 8.3. *Consider the equation*

$$-\Delta u = f \tag{8.3}$$

in Ω, with boundary conditions $u = 0$ on $\partial\Omega$ and $f \in L^2(\Omega)$. If Ω is bounded, then this has a unique weak solution in $\dot{W}^{1,2}(\Omega)$.

That is, there exists a unique $u \in \dot{W}^{1,2}(\Omega)$ such that

$$\int_\Omega -\Delta u\phi = \int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\phi$$

for all $\phi \in C_c^\infty(\Omega)$. This equation follows from multiplying by a smooth function ϕ and integrating by parts.

Proof. Let $\langle u, v\rangle = \int_\Omega \nabla u \nabla v$ is a scalar product on $\dot{W}^{1,2}(\Omega)$ generalising the norm. The map $\phi \mapsto \int_\Omega f\phi$ is linear in ϕ. Our equation then reduces to

$$\langle u, \phi\rangle = (f, \phi)$$

where the right hand side is the L^2 inner product. Then we have

$$\begin{aligned} |(f,\phi)| &\leq \|f\|_2\|\phi\|_2 \\ &\leq C\|f\|_2\|\nabla\phi\|_2 \quad \text{by Poincaré inequality} \end{aligned}$$

and so (f, ϕ) is a bounded linear functional on $\dot{W}^{1,2}(\Omega)$.

So $(f, \phi) = \langle F, \phi\rangle$ where $F \in \dot{W}^{1,2}(\Omega)$ by the Reisz representation theorem. Thus setting $u = F$ we obtain our solution.

Uniqueness is clear from $\langle u - F, \phi \rangle = 0 \Rightarrow u - F = 0$. $\square$

Note. Consider looking for a solution $u \in C^2(\Omega) \wedge C(\overline{\Omega})$ for the equation

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

with $f \in L^2(\Omega)$.

We can prove existence of a weak solution quite generally. If $a(u,v) : \mathcal{H} \oplus \mathcal{H} \to k$ which is linear in v for fixed u and linear in u for fixed v (bilinear) and there exists K such that

$$|a(u,v)| \leq K\|u\|\|v\|,$$

then we can write $a(u,v) = \langle Lu, v\rangle$ where $L : \mathcal{H} \to \mathcal{H}$ is bounded and linear.

If a is of this class on on $\dot{W}^{1,2}(\Omega)$ and $f \in L^2(\Omega)$, then the equation

$$a(u,v) = (f,v) \tag{$\star$}$$

for all $v \in \dot{W}^{1,2}(\Omega)$ can be written as

$$\langle Lu, v\rangle = \langle F, v\rangle$$

where $L : \dot{W}^{1,2}(\Omega) \to \dot{W}^{1,2}(\Omega)$ is bounded linear. Thus $Lu = F$. Thus the equation is uniquely soluble if L is invertible. By the Lax-Milgram result (Theorem 3.1), L is invertible if

$$\operatorname{Re}\langle Lu, u\rangle \geq c\|u\|_{1,2}^2 \tag{$\star\star$}$$

on $\dot{W}^{1,2}(\Omega)$ where $c > 0$. Thus ($\star$) has a unique solution if ($\star\star$) holds and $\langle Lu, v\rangle$ is bilinear and bounded on $\dot{W}^{1,2}(\Omega)$. Notice that ($\star\star$) can be written as $\operatorname{Re} a(u,u) \geq c\|u\|_{1,2}^2$.

Recall that the equation

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

with $f \in L^2(\Omega)$. has a unique weak solution if Ω is bounded. We prove that $u \in W^{2,2}_{loc}(\Omega)$. To prove thius, suppose that $x_0 \in \Omega$ and choose $\phi \in C_c^\infty(\Omega)$ and that $\phi = 1$ in a neighbourhood of x_0. We prove that $u\phi$ is the weak solution of the problem

$$-\Delta(u\phi) + (u\phi) = w \tag{$\star\star\star$}$$

on $\mathbb{R}^n$ where

$$w = \underbrace{f\phi - 2\nabla u \nabla\phi - u\Delta\phi + u\phi}_{L^2(\mathbb{R}^n)}$$

But the solution of ($\star\star\star$) is $W^{2,2}(\mathbb{R}^n)$, which can be derived by Fourier transforms.

We now seek to prove $(\star\star\star)$. Choose $\psi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}\int \nabla(u\phi)\cdot\nabla\psi &= \int (\psi\nabla u + u\nabla\phi)\cdot\nabla\psi \\ &= \int u\nabla\phi\cdot\nabla\psi + \int \phi\nabla u\cdot\nabla\psi \qquad (\dagger)\end{aligned}$$

and similarly,

$$\begin{aligned}\int \nabla u\cdot\nabla(\phi\psi) &= \int \nabla u\cdot(\nabla\phi)\psi + \int \nabla u(\nabla\psi)\nabla\phi \\ &= \int_\Omega (f\phi\psi) \\ &= \int f\phi\psi\end{aligned}$$

and so

$$\int \nabla u\cdot(\nabla\phi)\psi = \int f\phi\psi - \int \nabla u\cdot(\nabla\psi)\phi \qquad (\ddagger)$$

Then we have

$$\begin{aligned}\int \nabla(u\phi)\cdot\nabla\psi &= \int u(\nabla\phi)\cdot\nabla\psi + \int \phi\nabla u\cdot\nabla\psi && \text{by } (\dagger)\\ &= \int u\nabla\phi\cdot\nabla\psi + \int f\phi\psi - \int \nabla u\cdot(\nabla\phi)\psi && \text{by } (\ddagger)\\ &= -\int \psi\nabla(u\nabla\phi) + \int f\phi\psi - \int \nabla u\cdot(\nabla\phi)\psi \\ &= -\int \psi\left(\nabla u\cdot\nabla\phi + u\Delta\phi\right) + \int f\phi\psi - \int \nabla u\cdot(\nabla\phi)\psi \\ &= \int (f\phi)\psi - 2(\nabla u\cdot\nabla\phi) - u(\Delta\phi)\psi.\end{aligned}$$

So

$$\int \left(\nabla(u\phi)\cdot\nabla\psi + u\phi\psi\right) = \int \left(f\phi\psi - (2\nabla u\cdot\nabla\phi)\psi - u(\Delta\phi)\psi + u\phi\psi\right).$$

Hence $u\phi$ is a weak solution on $\mathbb{R}^n$ of $-\Delta z + z = w$. We can use similar arguments to show that $f \in W^{k,2}(\Omega)$, which then implies that $u \in W^{k+2,2}_{loc}(\Omega)$.

It can be show that if $u \in L^p(\Omega)$ and $-\Delta u + u = f$ in Ω, then $u \in W^{2,p}_{loc}(\Omega)$.

Now, consider the weak solutions of

$$-\frac{\partial}{\partial x_l}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + b_l\frac{\partial u}{\partial x_l} + cu = f. \qquad (8.4)$$

on Ω, with $u \in \dot{W}^{1,2}(\Omega)$. We implicitly use the repeated index summation convention.

We seek to find u such that

$$\int_\Omega \left(a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \phi}{\partial x_l}\right) + \int_\Omega b_l\frac{\partial u}{\partial x_l}\phi + \int_C cu\phi = \int_\Omega f\phi$$

for all $\phi \in \dot{W}^{1,2}(\Omega)$. The left hand side is a bilinear operator $A(u,\phi)$ where

$$A : \dot{W}^{1,2}(\Omega) \times \dot{W}^{1,2}(\Omega) \to \mathbb{R}$$

is bounded if a_{ij}, b_j, $c \in L^\infty(\Omega)$ and hence are bounded on Ω. In this case, there is a generalised theorem that

$$A(u,\phi) = \langle Lu, \phi \rangle$$

where L is a bounded linear map $\dot{W}^{1,2}(\Omega) \to W^{1,2}(\Omega)$. Then our equation becomes

$$\langle Lu, \phi \rangle = \int_\Omega f\phi = \langle F, \phi \rangle$$

by the Reisz representation theorem. That is, $Lu = f$. Then our problem has a unique solution if L is invertible. By Lax-Milgram (Theorem 3.1), this is true if

$$A(u,u) \geq c\|u\|_{1,2}^2.$$

We now seek to find assumptions such that A satisfies these conditions. We assume that there exists $c_1 > 0$ such that

$$\langle a_{ij}(x)\eta_i, \eta_j \rangle \geq c_1|\eta|^2 \tag{††}$$

for all $\eta \in \mathbb{R}^n, x \in \Omega$.

Consider the operator

$$A(u,v) = \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \int_\Omega b_l \frac{\partial u}{\partial x_l} v + \int_\Omega cuv.$$

To bound the second term, we have have

$$\begin{aligned}
\left| \int_\Omega b_l \frac{\partial u}{\partial x_l} |v| \right| &\leq \int_\Omega |b_l| \left| \frac{\partial u}{\partial x_l} \right| |v| \\
&\leq K \int \left| \frac{\partial u}{\partial x_l} \right| |v| \\
&\leq K \left\| \frac{\partial u}{\partial x_l} \right\|_2 \|v\|_2 && \text{by Cauchy-Swartz} \\
&\leq K \left(\epsilon \left\| \frac{\partial u}{\partial x_l} \right\|_2^2 + \frac{1}{\epsilon}\|v\|_2^2 \right) && \text{by the inequality } |st| \leq \epsilon s^2 + \frac{t^2}{\epsilon}
\end{aligned}$$

Other terms are similar but easier to bound. Thus we have a bounded bilinear map.

With the above assumptions, we have

$$A(u,u) = \underbrace{\int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}}_{\geq \mu \int_\Omega |\nabla u|^2} + \int_\Omega b_l u \frac{\partial u}{\partial x_l} + \int cu^2$$

Estimating the final term, we have

$$\int cu^2 \geq \inf c \|u\|_2^2.$$

The first term is bounded by the assumption (††).

Coercivity is then given, $A(u,u) \geq \alpha \|u\|_{1,2}^2$ for $\alpha > 0$ if $b_l = 0$ and $\inf c \geq 0$. If $b_l = 0$ and Ω is bounded, then $\inf c \geq 0$ is sufficient.

If b_l does not vanish on Ω, then we have the estimate

$$A(u,u) \geq \mu \|\nabla u\|_2^2 - K\left(\epsilon \|\nabla u\|_2^2 + \frac{1}{\epsilon}\|u\|_2^2\right) + \inf c \|u\|_2^2.$$

Choose ϵ such that $K\epsilon < \mu$ and $\inf c > \frac{K}{\epsilon}$. Then we get

$$A(u,u) \geq \tilde{c}\left(\|\nabla u\|_2^2 + \|u\|_2^2\right),$$

and we obtain coercivity.

Lemma 8.4. *If $A(u,v) = \int_\Omega fv$ for all $v \in \dot{W}^{1,2}(\Omega)$, then there is a unique weak solution in $\dot{W}^{1,2}(\Omega)$ if A is bounded and bilinear, A is coercive, and $f \in L^2(\Omega)$.*

9. Compactness in Sobolev Spaces

Theorem 9.1. *If Ω is bounded and open in $\mathbb{R}^n$ the natural inclusion $i : \dot{W}^{1,2}(\Omega) \to L^2(\Omega)$ is compact. That is, bounded sets in $\dot{W}^{1,2}(\Omega)$ are contained in a compact set of $L^2(\Omega)$.*

Remark. The theorem does not hold for $\Omega = R^n$, but true for $W^{1,2}(\Omega)$ under minor assumptions on $\partial\Omega$. There is a similar result for $\dot{W}^{1,p}(\Omega)$ for $1 < p < \infty$.

Lemma 9.2. *For ϵ sufficiently small,*

$$\left|\left(\hat{\phi}(\epsilon s) - 1\right)\right| \leq r\sqrt{1 - |s|^2} \tag{9.1}$$

on $\mathbb{R}^n$.

Proof. We have $\hat{\phi}(0) - = 1$, $\hat{\phi}$ is continuous and bounded, and so $\left|\hat{\phi}(s)\right| \leq K$ on $\mathbb{R}^n$. So

$$\left|\hat{\phi}(\epsilon s) - 1\right| \leq K + 1 \leq r\sqrt{1 + |s|^2}$$

if

$$|s|^2 \geq \underbrace{\left(\frac{K+1}{r}\right)^2 - 1}_{\mu^2}.$$

And so this is true if $|s| \geq \mu$ (uniformly in ϵ). Thus (9.1) holds if $|s| \geq \mu$.

If $|s| \leq \mu$, ϵs is small, and so $\left|\hat{\phi}(\epsilon s) - 1\right|$ is close to $\hat{\phi}(0) - 1 = 0$. Note that $\left|\hat{\phi}(\epsilon s) - 1\right| \leq r$ if ϵ is small and $|s| \leq \mu$. Hence

$$\left|\hat{\phi}(\epsilon s) - 1\right| \leq r\sqrt{1 + |s|^2}$$

if $|s| \leq \mu$ and ϵ is small. Hence (9.1) holds and our lemma is proven. $\square$

Lemma 9.3. *Given $r > 0$, there exists $\epsilon_0 > 0$ such that $\|T_\epsilon u - u\| \leq r\|u\|_{1,2}$ if $0 < \epsilon \leq \epsilon_0$ and $u \in \dot{W}^{1,2}(\Omega)$.*

Proof. We choose a cube C such that $\overline{\Omega} \subset \textsc{Int}\ C$. Notice that $\dot{W}^{1,2}(\Omega)$ can be extended to $\dot{W}^{1,2}(C)$ by letting $u = 0$ on $C \backslash \Omega$.

Choose $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi \geq 0$, $\int \phi = 1$, and ϕ even. Let

$$T_\epsilon u = \epsilon^{-n} \int_\Omega \phi\left(\frac{x-y}{\epsilon}\right) u(y)\, dy = \phi_\epsilon \star u$$

where $\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right)$.

Taking Fourier transforms, we have

$$\begin{aligned}
\hat{\phi}_\epsilon(S) &= \int_{\mathbb{R}^n} e^{its}\phi_\epsilon(t)\, dt \\
&= \epsilon^{-n} \int e^{its}\phi\left(\frac{t}{\epsilon}\right)\, dt \\
&= \hat{\phi}(\epsilon s).
\end{aligned}$$

Then estimating $\|T_\epsilon u - u\|_2^2$ by Fourier transforms, we have

$$\begin{aligned}
A = \|T_\epsilon u - u\|_2^2 &= (2\pi)^{-n} \left\| T_\epsilon \widehat{u - u} \right\|_2^2 \\
&= \|\hat{T_\epsilon} u - \hat{u}\|_2^2.
\end{aligned}$$

But $\hat{T_\epsilon} u = \hat{\phi}_\epsilon \hat{u} = \hat{\phi}(\epsilon s)\hat{u}(s)$, and so

$$A = (2\pi)^{-n} \int \left| \left(\left(\hat{\phi}(\epsilon s) - 1 \right) \hat{u}(s) \right)^2 \right| ds$$

From Lemma 9.2, we have

$$\begin{aligned}
A &\leq (2\pi)^{-n} \int r^2 \left(1 - |s|^2\right)^2 |\hat{u}(s)|^2\, ds \\
&\leq r^2 (2\pi)^{-n} \int \left(1 + |s|^2\right) |\hat{u}(s)|^2\, ds \\
&= r^2 \|u\|_{1,2}^2
\end{aligned}$$

using the definition of the $\|u\|_{1,2}^2$ as $\|u\|_2^2 + \|\nabla u\|_2^2$.

Hence $\|T_\epsilon u - u\|_2^2 \leq r^2 \|u\|_{1,2}^2$. $\square$

Definition 9.4 (Finite ϵ-net)**.** A finite set $\{a_i\}_{i=1}^n$ in a metric space Y is a finite ϵ-net if $Y \subseteq \bigcup_{i=1}^n B_\epsilon(a_i)$.

Theorem 9.5. *A closed net Y in a compact metric space is compact if and only if it has a finite ϵ -net for every $\epsilon > 0$.*

Definition 9.6 (Precompact)**.** A subset T in a complete metric space is said to be precompact if $\overline{T}$ is compact.

T is precompact if and only if T has a finite ϵ net for every $\epsilon > 0$.

Proof of Theorem 9.1. It suffices to show that for any $\delta > 0$, the set

$$\{u \in \dot{W}^{1,2}(\Omega) \mid \|u\|_{1,2} \leq 1\}$$

lies in a compact set of $L^2(\Omega)$ and hence it suffices to prove if $\delta > 0$ it haas a finite δ-net in $L^2(\Omega)$. Recall that

$$\|T_\epsilon u - u\| \leq \frac{1}{2}\delta$$

if $u \in \dot{W}^{1,2}(\Omega)$, $\|u\|_{1,2} \leq 1$ by Lemma 9.3. Thus it suffices to get a finite $\frac{\delta}{2}$-net in $L^2(\Omega)$ for

$$\{T_\epsilon u \mid \|u\|_{1,2}\}$$

for ϵ small.

There are C^1 function on $\mathbb{R}^n$, and so

$$(T_\epsilon u)'(x) = \epsilon^{-n-1} \int \phi'\left(\frac{x-y}{\epsilon}\right) u(y)\, dy. \tag{$\star$}$$

It suffices to prove for a fixed ϵ,

$$\{T_\epsilon u \mid \|u\|_{1,2} \leq 1\}$$

is precompact in $C(C)$ (the set of continuous functions on the cube C.)

The map $i : C(C) \to L^2(C)$ is continuous and so maps compact sets to compact sets. For *fixed* ϵ, $|(T_\epsilon u)'(x)| \leq K$ if $\|u\|_{1,2} \leq 1$, as

$$\begin{aligned} |T'(u)(x)| \leq K \int |u(y)|\, dy &\leq K\|u\|_1 \\ &\leq K_1 \|u\|_2 \\ &\leq K_1 \|u\|_{1,2} \\ &\leq K_1. \end{aligned}$$

On C,

$$\begin{aligned} |T_\epsilon u(x_1) - T_\epsilon u(x_2)| &\leq \sup |(T_\epsilon u)'(x)||x_1 - x_2| \\ &\leq K_1 |x_1 - x_2| \end{aligned}$$

for any $x_1, x_2 \in C$. So $T_\epsilon u$ is uniformly bounded. This shows that $\{T_\epsilon u \mid \|u\|_{1,2} \leq 1\}$ is *equicontinuous*, in the sense that given $\mu > 0$, there exists $\tau > 0$ such that $\|T_\epsilon u(x_1) - T_\epsilon(x_2)\| \leq \mu$ if $|x_1 - x_2| \leq \tau$ for all u such that $\|u\|_{1,2} \leq 1$.

Lemma 9.7 (Anzela-Anscoli)**.** *A bounded set in $C(C)$ is precompact if and only if it is equicontinuous.*

Proof. See Simmond's book on Modern Analysis. □

Applying Anzela-Anscoli to our set $\{T_\epsilon u \,|\, \|u\|_{1,2} \leq 1\}$ then proves that it is precompact in $C(C)$. As a set that is precompact in $C(C)$ is precompact in $L^2(C)$, our theorem is proven. □

Remark. There are similar results for $i : \dot{W}^{1,p}(\Omega) \to L^p(\Omega)$ if $1 < p < \infty$ if Ω is bounded open.

Recall that $\dot{W}^{1,2}(\Omega) \to L^2(\Omega)$ is compact if Ω is bounded.

Consider the equation

$$\begin{aligned} -\Delta u &= \lambda u + f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{9.2}$$

with Ω a **bounded** domain in $\mathbb{R}^n$.

For a weak solution, we seek to find $u \in \dot{W}^{1,2}(\Omega)$ such that

$$\int \nabla u \cdot \nabla v = \lambda \int uv + \int fv$$

for all $v \in \dot{W}^{1,2}(\Omega)$. This is equivalent to asking that

$$\langle u, v \rangle = \lambda \langle Bu, v \rangle + \langle F, v \rangle \tag{$\star$}$$

where $\langle Bu, v \rangle = (u, v)$ is a bounded bilinear form on $\dot{W}^{1,2}(\Omega)$ and $\int fv = \langle F, v \rangle$. Note that ($\star$) is equivalent to

$$u = \lambda Bu + F \tag{$\star\star$}$$

whern $u \in \dot{W}^{1,2}(\Omega)$.

Recall that

$$\begin{aligned} |\langle Bu, v \rangle| &= \left| \int uv \right| \\ &\leq \|u\|_2 \|v\|_2 \\ &\leq C \|\nabla u\|_2 \|\nabla v\|_2 \end{aligned}$$

by Poincaré . Moreover, B is compact, as Ω is bounded. This is true as supposing that u_n is a bounded sequence in $\dot{W}^{1,2}(\Omega)$.. Then $\{u_n\}$ has a convergent subsequence in $\dot{W}^{1,2}(\Omega)$. But by Theorem 9.1, $\{u_n\}$ has a subsequence which converges in $L^2(\Omega)$. Restricting now to the subsequence, for any u_n, u_m, we have

$$\begin{aligned}\|Bu_n - Bu_m\|_{1,2} &= \sup_{\|v\|_{1,2}\le 1} |\langle Bu_n - Bu_m, v\rangle| \\ &= \sup_{\|v\|_{1,2}\le 1} |\langle B(u_n - u_m), v\rangle| \\ &\le \sup_{\|v\|_{1,2}\le 1} |(u_n - u_m, v)| \\ &\le \sup_{\|v\|_{1,2}\le 1} \underbrace{\|u_n - u_m\|_2}_{\to 0} \underbrace{\|v\|_2}_{\le C\|v\|_{1,2}}\end{aligned}$$

by convergence in $L^2(\Omega)$, Cauchy-Swartz and the Poincaré inequality.

So $\|Bu_n - Bu_m\|_{1,2} \to 0$ as $n, m \to \infty$, and so $\{Bu_n\}$ converges in $\dot{W}^{1,2}(\Omega)$ as required.

B is also self adjoint as $\langle Bu, v\rangle = \int uv$.

Theorem 9.8. *The problem $u = \lambda Bu$ on $\dot{W}^{1,2}(\Omega)$ has an infinite sequence of eigenvalues $\{\lambda_n\}$ which are all real and $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$. Moreover, $I - \lambda B$ is invertible if $\lambda \neq \lambda_n$ for all n.*

Proof.

(i) The eigenvalues are all real as B is self-adjoint.

(ii) Note that the null-space of B is $\{0\}$, since

$$\langle Bu, u\rangle = (u, u) = \int_\Omega u^2 > 0$$

if $u \neq 0$.

Hence

$$\begin{aligned}u = \lambda Bu \iff \underbrace{\langle u, u\rangle}_{>0} &= \langle \lambda Bu, u\rangle \\ &= \lambda \langle Bu, u\rangle \\ &= \lambda \underbrace{\int u^2}_{>0}\end{aligned}$$

and so all eigenvalues are greater than zero.

(iii) The smallest eigenvalue is $\inf_{u\neq 0} \frac{\int |\nabla u|^2}{\int u^2}$. By Theorem 3.7, for any operator T we have

$$\begin{aligned}\sup \sigma(T) &= \sup_{\|x\|=1} \langle Tx, x\rangle \\ &= \sup_{\|x\neq 0\|} \frac{\langle Tx, x\rangle}{\langle x, x\rangle} \\ \inf \sigma(B) &= \inf_{x\neq 0} \frac{\langle x, x\rangle}{\langle Bx, x\rangle} \\ &= \inf_{u\neq 0} \frac{\langle u, u\rangle}{\langle Bu, u\rangle} \\ &= \inf_{u\neq 0} \frac{\int |\nabla u|^2}{\int u^2}.\end{aligned}$$

(iv) If $\lambda \neq \lambda_n$, (9.2) has a unique weak solution for all $f \in L^2(\Omega)$. If $\lambda = \lambda_n$, (9.2) has a solution if and only if $(f, \phi_n) = 0$ for all eigenfunctions ϕ_n corresponding to $\lambda = \lambda_n$.

Recall from Theorem 2.8, $Tx = y$ has a solution if and only if $f(y) = 0$ for all $f \in \text{Ker } T'$. Note that the this is satisfied if and only if $\langle F, \phi_n\rangle = (f, \phi_l) = 0$ for all eigenfunctions ϕ_n.

(v) The set of eigenfunctions are an orthogonal basis for $L^2(\Omega)$ and $\dot{W}^{1,2}(\Omega)$.

This is true for any compact self-adjoint operator.

□

Consider the equation

$$\begin{aligned}-\Delta u &= \lambda u + f \quad \text{in } \Omega \qquad (9.3)\\ u &= 0 \quad \text{on } \partial\Omega \\ & \qquad (9.4)\end{aligned}$$

with $u \in \dot{W}^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Consider the equation

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + b_i\frac{\partial u}{\partial x_l} + cu &= \lambda u + f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

with Ω bounded. We can apply the previous theory to this case (modulo some complications.)

Let $f = 0$. Then if $\tilde{\lambda}$ is the least eigenvalue of (9.3) then there is a non-negative eigenfunction of (9.3) corresponding to $\lambda = \tilde{\lambda}$.

Theorem 9.9. *Recall that*

$$\tilde{\lambda} = \inf_{\substack{u\in \dot{W}^{1,2}(\Omega)\\ u\neq 0}} \frac{\int |\nabla u|^2}{\int u^2} \qquad (9.5)$$

If $\tilde{u} \in \dot{W}^{1,2}(\Omega)$ achieves this minimum, then

$$-\Delta \tilde{u} = \tilde{\lambda}\tilde{u}.$$

Proof. Consider test functions of the form $\tilde{u} + \epsilon\phi$ where $\phi \in \dot{W}^{1,2}(\Omega)$. Then

$$\frac{\int |\nabla(\tilde{u}+\epsilon\phi)|^2}{\int (\tilde{u}+\epsilon\phi)^2} \geq \tilde{\lambda}.$$

and

$$\frac{d}{d\epsilon} \left. \frac{\int |\nabla(\tilde{u}+\epsilon\phi)|^2}{\int (\tilde{u}+\epsilon\phi)^2} \right|_{\epsilon=1} = 0.$$

This implies that

$$\int \nabla u \tilde{\nabla} \phi - \tilde{\lambda}\tilde{u}\phi = 0$$

As this is true for all $\phi \in \dot{W}^{1,2}(\Omega)$, we have that $\tilde{u}$ is a weak solution of (9.3) for $\lambda = \tilde{\lambda}$ and $f = 0$. □

Lemma 9.10. *If $\tilde{u}$ is an eigenfunction corresponding to $\tilde{\lambda}$ then $|\tilde{u}|$ is in $\dot{W}^{1,2}(\Omega)$ and $|\tilde{u}$ is a minimiser of* (9.5)*, and hence, as in Lemma 9.9, $|\tilde{u}|$ is also an eigenfunction corresponding to $\tilde{\lambda}$.*

Proof. Recall that

$$|\tilde{u}|(x) = \begin{cases} \tilde{u}(x) & \tilde{u} \geq 0 \\ -\tilde{u}(x) & \tilde{u}(x) < 0 \end{cases}$$

By the next section,

$$\frac{\partial}{\partial x_i}|\tilde{u}|(x) = \begin{cases} \frac{\partial \tilde{u}}{\partial x_i} & \tilde{u}(x) \geq 0 \\ -\frac{\partial \tilde{u}}{\partial x_i} & \tilde{u}(x) < 0 \end{cases}$$

Then

$$\left|\frac{\partial}{\partial x_i}|\tilde{u}|(x)\right| = \left|\frac{\partial \tilde{u}}{\partial x_i}\right|$$

and so

$$\frac{\int (\nabla|\tilde{u}|)^2}{\int |\tilde{u}|^2} = \frac{\int |\nabla \tilde{u}|^2}{\int \tilde{u}^2} = \tilde{\lambda}.$$ □

Theorem 9.11. *If $f \in L^2(\Omega)$ is non-negative and*

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

for $u \in \dot{W}^{1,2}(\Omega)$, then $u \geq 0$ on $\partial\Omega$.

Proof. Consider u^- as a test function in the definition of the weak solution

$$u^-(x) = \begin{cases} 0 & u(x) \geq 0 \\ u(x) & u(x) < 0. \end{cases}$$

and

$$\frac{\partial}{\partial x_i}u^-(x) = \begin{cases} 0 & u(x) \geq 0 \\ \frac{\partial u}{\partial x_i} & u(x) < 0. \end{cases}$$

Since

$$\int \nabla u \cdot \nabla \phi = \int f\phi$$

letting $\phi = u^-$, we have

$$\underbrace{\int fu^-}_{\leq 0} = \int \nabla u \nabla u^- = \underbrace{\int \left|\nabla u^-\right|^2}_{\leq 0}$$

Thus $\nabla u^- = 0$ and so $u^- = 0$ by Poincaré inequality (Theorem 8.2). □

10. Further Properties of $\dot{W}^{1,2}(\Omega)$

Theorem 10.1. *If $u \in \dot{W}^{1,2}(\Omega)$ where Ω is bounded and open then $u^+ \in \dot{W}^{1,2}(\Omega)$ and*

$$\frac{\partial}{\partial x_i}u^+ = \begin{cases} \frac{\partial u}{\partial x_i} & u(x) > 0 \\ 0 & u(x) \leq 0 \end{cases}$$

Proof. If $f \in C^1(\Omega)$, $f(0) = 0$, and f' is bounded on $\mathbb{R}$, then $f(u) \in \dot{W}^{1,2}(\Omega)$ and $\frac{\partial}{\partial x_i}f(u) = f'(u)\frac{\partial u}{\partial x_i}$ if $u \in C_c^\infty(\Omega)$ by the chain rule.

If $u \in \dot{W}^{1,2}(\Omega)$, choose $u_n \in C_c^\infty(\Omega)$ so $u_n \to u$ in $\dot{W}^{1,2}(\Omega)$ as $n \to \infty$. Then

$$-\int f(u_n)\frac{\partial \phi}{\partial x_l} = \int f'(u_n)\frac{\partial u_n}{\partial x_i}\phi$$

Since $u_n \to u$ and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in L^2, taking subsequences gives

$$-\int f(u)\frac{\partial \phi}{\partial x_i} = \int f'(u)\frac{\partial u}{\partial x_i}\phi \qquad \square.$$

This can be shown as $|f(0) - f(t)| \leq K|s - t|$ by the mean value theorem, and so $|f(u_n(x)) - f(u(x))| \leq K|u_n(x) - u(x)|$. On the left hand side, it suffices to show that $f(u_n) \to f(u)$ in $L^1(\Omega)$, then we use the dominated convergence theorem. We have

$$\|f(u_n(x)) - f(u(x))\|_1 \leq K\|u_n - u\|$$

On the right hand side, we have L^2 convergence if we prove that each term ($\frac{\partial u_n}{\partial x_i}$, $f'(u_n)\phi$) converges in $L^2(\Omega)$, We have

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$$

by a lemma of generalised derivatives, and

$$f'(u_n)\phi \to f'(u)\phi$$

since they are both uniformly bounded and converge pointwise.

More explicitly, we have

$$\int f(u)\frac{\partial \phi}{\partial x_l} - \int f(u_n)\frac{\partial \phi}{\partial x_l} = \int \left(f(u_n) - f(u)\right)\frac{\partial \phi}{\partial x_i}$$
$$\leq \underbrace{\|f(u) - f(u_n)\|_2}_{\to 0}\left\|\frac{\partial \phi}{\partial x_i}\right\|_2$$

Now, consider the function $f_\epsilon(y)$ given by

$$f_\epsilon(y) = \begin{cases} \sqrt{y^2+\epsilon^2} & y \geq 0 \\ 0 & y < 0. \end{cases}$$

Then $f_\epsilon \in C^1$, $f_\epsilon'(y) = \frac{y}{\sqrt{y^2+\epsilon^2}}$, and $|f_\epsilon'(y)| \leq 1$.

If $u \in \dot{W}^{1,2}(\Omega)$, then

$$-\int f_\epsilon(u)\frac{\partial \phi}{\partial x_i} = \int f_\epsilon'(u)\frac{\partial u}{\partial x_i}\phi$$

by the previous step.

Note that $f_\epsilon(y) \to y^+$ uniformly in $\mathbb{R}$ as $\epsilon \to 0$, and hence converges in L^2. Thus

$$\int f_\epsilon(u)\phi = \int u^+\phi.$$

Next, note that $f_\epsilon'(y)$ is uniformly bounded and converges pointwise and in $L^2(\Omega)$ to $\mathbf{1}_{y>0}$. Thus by Cauchy-Swartz,

$$\int f_\epsilon'(u)\frac{\partial u}{\partial x_i}\phi \to \int \mathbf{1}_{u>0}\frac{\partial u}{\partial x_i}\phi.$$

Hence

$$\int u^+\frac{\partial \phi}{\partial x_l} = \int \mathbf{1}_{u>0}\frac{\partial u}{\partial x_i}\phi$$

and so $\frac{\partial u^+}{\partial x_i}$ exists and is $\mathbf{1}_{u>0}\frac{\partial u}{\partial x_i}$.

Remark. This tells us that $\frac{\partial u^+}{\partial x_i} = 0$ a.e. where $u = 0$, and also that $\frac{\partial u^-}{\partial x_i} = -\mathbf{1}_{u<0}\frac{\partial u}{\partial x_i}$.

Remark. If $u \in W^{1,2}(\Omega)$m then $u^+ \in W^{1,2}(\Omega)$ with the same formula for $\frac{\partial u}{\partial x_i}$.

If $N = 1$ and $\dot{W}^{1,2}([a,b])$ then there exists $\tilde{u}$ such that $\tilde{u} = u$ almost everywhere and $\tilde{u} \in C[a,b]$.

If $N = 2$ and $u \in \dot{W}^{1,2}(\Omega)$, $u \in L^p(\Omega)$ for all $p \geq 1$.

If $N \geq 3$ and $u \in \dot{W}^{1,2}(\Omega)$, then $u \in L^{2^*}(\Omega)$ where $2^\star = \frac{2N}{N-2}$.

Exercise 10.2. If $u \in \dot{W}^{1,2}(\Omega)$ and $a > 0$, then

$$(u-a)^+ \in \dot{W}^{1,2}(\Omega).$$

Remark. In general, if F is Lipschitz on $\mathbb{R}$ with $F(0) = 0$ and $u \in \dot{W}^{1,2}(\Omega)$, then

$$F(u) \in \dot{W}^{1,2}(\Omega).$$

Theorem 10.3. *If $n = 1$ and $u \in \dot{W}^{1,2}(\Omega)$, then $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$ (assuming Ω is bounded and open). More precisely, there exists $K > 0$ such that if $u \in \dot{W}^{1,2}(\Omega)$ there exists $v \in C(\Omega)$ such that $v = 0$ on $\partial\Omega$, $v = u$ almost everywhere, and $\|v\|_\infty \leq K\|u\|_{1,2}$.*

This is true for $\dot{W}^{1,p}(\Omega)$ if $n = 1$ and $p > 1$.

Proof. We prove for $\Omega = (a, b)$, as any open set in $\mathbb{R}$ is a countable union of disjoint intervals. We prove that there exists $K > 0$ such that if $u \in C_c^\infty((a, b))$, then

$$\|u\|_\infty \leq K\|u\|_{1,2}. \tag{$\star$}$$

This will be sufficient to prove the theorem. To show this, suppose that $w \in \dot{W}^{1,2}(\Omega)$ and $u_n \in C_c^\infty((a, b))$ such that $u_n \to w$ in $\dot{W}^{1,2}([a, b])$ as $n \to \infty$, then

$$\begin{aligned}
\|u_n - u_m\|_\infty &\leq K\|u_n - u_m\|_{1,2} \quad \text{by } (\star) \\
&\leq K\left(\|u_n - w\|_{1,2} + \|u_m - w\|_{1,2}\right) \\
&\to 0
\end{aligned}$$

as $m, n \to \infty$. Hence $\sup_{x\in[a,b]} |u_n(x) - u_m(x)| \to 0$ as $m, n \to \infty$, and so $\{u_n\}$ is Cauchy in $C([a, b])$. Hence there exists $v \in C([a, b])$ such that $u_n \to v$ uniformly as $n \to \infty$ and $v(a) = v(b) = 0$. Since $u_n \to w$ in $L^2([a, b])$ as $n \to \infty$, then $u_n(x) \to w(x)$ almost everywhere as $n \to \infty$, and so $v = w$ almost everywhere, with w continuous and $w(a) = w(b) = 0$.

So we have

$$\begin{aligned}
\|u_n\|_\infty &\leq K\|u_n\|_{1,2} \\
\|v\|_\infty &\leq K\|v\|_{1,2}
\end{aligned}$$

and this is what we need. It suffices to prove $(\star)$ for $u \in C_c^\infty((a, b))$. Let $x, y \in (a, b)$, with $x < y$. THen

$$u(x) - u(y) = \int_x^y u'(t)\, dt$$

So

$$\begin{aligned}
|u(x) - u(y)| &\leq \left|\int_x^y u'(t)\, dt\right| \\
&\leq \left(\int_x^y dt\right)^{1/2} \left(\int_x^y |u'(t)|^2\, dt\right)^{1/2} \\
&\leq (y - x)^{1/2} \left(\int_a^b |u'(t)|^2\, dt\right)^{1/2} \\
&\leq (b - a)^{1/2}\|\nabla u\|_2
\end{aligned}$$

□

Theorem 10.4. *If $n > 2$ and Ω is bounded and open, there exists C depending only on n such that if $u \in \dot{W}^{1,2}(\Omega)$ hen $u \in L^{2^\star}(\Omega)$ and $\|u\|_{2^\star} \le C\|\nabla u\|_{1,2}$. Here $2^\star = \frac{2n}{n-2}$.*

Remark. There is a similar theorem for $\dot{W}^{1,2}(\Omega)$ if $1 \le p < n$. Here $p^\star = \frac{np}{n-p}$.

Remark. $u \in L^2(\Omega)$ and $u \in L^{2^\star}(\Omega)$ implies that $u \in L^q(\Omega)$ for all $2 \le q \le 2^\star$ by Hölder's inequality.

Proof of Theorem 10.4. It suffices to assume that $u \in C_c^\infty(\Omega)$.

Step 1.

Lemma 10.5. *If $u \in \dot{W}^{1,1}(\Omega)$, then $\|u\|_{1^\star} \le \|\nabla u\|_1$, where $\mathbf{1}^\star = \frac{n}{n-1}$.*

Proof. By Hölder, we have

$$\int |g_1| \times |g_2| \dots |g_{n-1}| \le \left(\int |g_1|^{n-1}\right)^{\frac{1}{n-1}} \cdots \left(\int |g_{n-1}|^{n-1}\right)^{\frac{1}{n-1}}$$

$$\left(\int |g_1| \times |g_2| \dots |g_{n-1}\right)^{n-1} \le \prod_{i=1}^{n-1} \left(\int |g_i|^{n-1}\right) \quad \text{by induction.}$$

For $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$f(x_1, \dots, x_n) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)\, dt_i$$

Then we can estimate $|f|$ by

$$\begin{aligned} |f(x_1, \dots, x_n)| &\le \int_{-\infty}^{x_i} \left| \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n) \right| dt_i \\ &\le \int_{-\infty}^{\infty} |\nabla f(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)|\, dt_i \end{aligned}$$

and so

$$|f|^n \le \prod_{i=1}^n \int_{-\infty}^{\infty} |\nabla f(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)|\, dt_i$$

Then taking the $(n-1)$-th root, we obtain

$$|f|^{1^\star} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |\nabla f(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)|\, dt_i\right)^{\frac{1}{n-1}}.$$

Now, integrating in x_1, we have

$$\int |f|^{1^\star}\, dx_1 \le \left(\int_{-\infty}^{\infty} |\nabla f|\, dt_1\right)^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int |\nabla f(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)|\, dt_i\, dx_1\right)^{\frac{1}{n-1}}$$

Now integrating in x_2, we have

$$\int |\nabla f|^{1^\star} dx_1\, dt_2 \le \left(\iint |\nabla f|\ dt_1\, dx_2\right)^{\frac{1}{n-1}} \left(\iint |\nabla f|\ dt_1\, dx_2\right)^{\frac{1}{n-1}} \left(\prod_{i=3}^{n} \iint |\nabla f|\, dx_1\, dx_2\, dt_i\right)^{\frac{1}{n-1}}$$

By induction, we obtain

$$\begin{aligned}\int |f|^{1^\star} dx_1 \dots dx_n &\le \left(\prod_{i=1}^{n} \int |\nabla f|\, dx_1 \dots dx_n\right)^{\frac{1}{n-1}} \\ &\le \left(\int |\nabla f|\, dx_1 \dots dx_n\right)^{\frac{n}{n-1}}\end{aligned}$$

and taking $\frac{n-1}{n}$-th roots obtains the required result. $\square$

We can also show

(i) If $p > n$, then $\dot{W}^{1,p}(\Omega) \subseteq C(\overline{\Omega})$ and $\|u\|_\infty \le C\|\nabla u\|_p$.
(ii) If $n \le 3$, function in $W^{2,2}(\Omega)$ are continuous on the interior.
(iii) If $u \in W^{1,2}(\Omega)$ and $u(x) \to 0$ as $x \to \partial\Omega$, then $u \in \dot{W}^{1,2}(\Omega)$.

Step 2.

Complete the proof. As before, we construct $u \in C_c^\infty(\mathbb{R}^n)$. Applying **Step 1.** to $|u|^\gamma$ where $\gamma > 1$ and γ is to be chosen, then

$$\begin{aligned}\left(\int (|u|^\gamma)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} &\le \left\|\gamma |u|^{\gamma-1}(\nabla u)\right\|_1 \\ &\le \gamma \left\||u|^{\gamma-1}\nabla u\right\|_1 \\ &\le \gamma \left(\int u^{(\gamma-1)\cdot\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int |\nabla u|^p\right)^{1/p} \quad \text{by Hölder}\end{aligned}$$

Then choosing γ such that $\gamma\frac{n}{n-1} = (\gamma-1)\frac{p}{p-1} = p^\star$, we have

$$\left(\int |u|^{p^\star}\right)^{\frac{n-1}{n}} \le \gamma \left(|u|^{p^\star}\right)^{\frac{p-1}{p}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}}$$

Then dividing both sides by $\left(\int |u|^{p^\star}\right)^{\frac{p-1}{p}}$, we have

$$\left(\int |u|^{p^\star}\right)^{\frac{1}{p^\star}} \le C(\gamma) \left(\int |\nabla u|^p\right)^{\frac{1}{p}}$$

$\square$

Lemma 10.6. *Consider the differential equation*

$$\nabla u = f \quad \textit{on } \Omega$$
$$u = 0 \quad \textit{on } \partial\Omega$$

with Ω bounded. This has a weak solution in $\dot{W}^{1,2}(\Omega)$. We claim that any classical solution is also a weak solution.

Consider a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then a classical solution (if it exists) is a weak solution $u \in \dot{W}^{1,2}(\Omega)$ and

$$\int \nabla u \cdot \nabla \phi = \int f\phi$$

for all $\phi \in C_c^\infty(\Omega)$. If u is smooth and ϕ has compact support, then

$$\int \nabla u \cdot \nabla \phi = -\int \Delta u \phi = \int f\phi$$

if $\nabla u = f$.

We need to check $\nabla u \in L^2(\Omega)$ and $u \in \dot{W}^{1,2}(\Omega)$. Consider the function $(u-a)^+ \in W^{1,2}(\Omega)$ if $a > 0$ that vanishes near $\partial\Omega$. Then $u - a \in W^{1,2}(\Omega)$ and so $(u-a)^+ \in W^{1,2}(\Omega)$ on compact sets, as

$$\frac{\partial}{\partial x_i}(u-a)^+ = \frac{\partial u}{\partial x_i}\mathbf{1}_{\{u>a\}}.$$

Then $(u-a)^+ \in \dot{W}^{1,2}(\Omega)$ (by an exercise.)

Using $(u-a)^+$ as a text function, we have

$$\begin{aligned}
\int \nabla u \cdot \nabla (u-a)^+ &= \int f(u-a)^+ \\
&\leq\leq \|f\|_2 \|(u-a)^+\|_2 \quad \text{by Cauchy-Swartz} \\
&= K
\end{aligned}$$

but

$$\begin{aligned}
\int \nabla u \cdot \nabla (u-a)^+ &= \int |\nabla u|^2 \mathbf{1}_{\{u>a\}} \leq K \\
&\to \int_{u \geq 0} |\nabla u|^2
\end{aligned}$$

by monotone convergence theorem as $a \to 0$, and so $u^+ \in L^2$. Similarly, $\nabla u^- \in L^2(\Omega)$, and so $\nabla u \in L^2(\Omega)$.

11. Applications to Nonlinear Equations

Consider the differential equation

$$-\Delta u = g(u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous - so $-\Delta u(x) = g(u(x))$.

We look for weak solutions, that is $u \in \dot{W}^{1,2}(\Omega)$ satisfying

$$\int \nabla u \cdot \nabla \phi = \int g(u)\phi \quad \text{for all } \phi \in C_c^\infty(\Omega). \tag{11.1}$$

12. Variational Methods

Assume that Ω is bounded and g is continuous and satisfies

$$|g(y)| \leq K_1|y| + K_2$$

on $\mathbb{R}$, and if $G' = g$ we assume that there exists $\mu < \lambda_1$ such that[1]

$$G(y) \leq \frac{1}{2}\mu y^2$$

for $|y|$ large. Equivalently, $G(y) \leq \frac{1}{2}\mu y^2 + K_3$.

Consider the *energy* function $E : \dot{W}^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 - G(u)\right). \tag{12.1}$$

We prove that there exists $w \in \dot{W}^{1,2}(\Omega)$ such that

$$E(u) \geq E(w)$$

for all $u \in \dot{W}^{1,2}(\Omega)$ and that such a w is a weak solution of our equation.

[1]Here, λ_1 is the minimal eigenvalue of the eigenvalue equation

$$-\Delta u = \lambda u \quad \text{on } \Omega.$$
$$u = 0 \quad \text{on } \Omega.$$

Indeed,

$$\lambda_1 = \inf \frac{\int |\nabla u|^2}{\int u^2}.$$

Step 1. We prove that there exists $C_1 > 0$ such that $E(u) \geq -C_1$ for all $u \in \dot{W}^{1,2}(\Omega)$. From (11.1), we have

$$\begin{aligned} E(u) &\geq \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{2}\mu u^2 - K_3 \right) \qquad (\star\star) \\ &\geq \frac{1}{2} \underbrace{\int \left(|\nabla u|^2 - \mu u^2 \right)}_{\geq 0} - \tilde{K}_3 \\ &\geq -\tilde{K}_3 \\ &\equiv \gamma. \end{aligned}$$

since $\lambda_1 = \inf \frac{\int |\nabla u|^2}{\int u^2}$ and so $\int |\nabla u|^2 \geq \lambda_1 \int u^2$. Hence

$$\int \left(|\nabla u|^2 - \mu u^2 \right) \geq (\lambda_1 - \mu) \int u^2 \geq 0.$$

We get a little more,

$$E(u) \geq (\lambda_1 - \mu) \left(\int u^2 \right) - K_3$$

so if $E(u) \leq K_4$, we have that $\int u^2$ is bounded. Thus by $(\star\star)$,

$$\int |\nabla u|^2$$

is bounded. Thus if

$$E(u_n) \to \inf \left\{ E(u) \,|\, u \in \dot{W}^{1,2}(\Omega) \right\},$$

then $\{u_n\}$ is bounded in $\dot{W}^{1,2}(\Omega)$.

Lemma 12.1. *The sequence $\{u_n\}$ has a subsequence which converges weakly to $w \in \dot{W}^{1,2}(\Omega)$ and w is a minimiser of E.*

Proof. Recall from §3 that every bounded sequence in a Hilbert space $\mathcal{H}$ has a subsequence which converges weakly. Thus our sequence $\{u_n\}$ has a subsequence that converges weakly to w.

We now need only show that w is a minimiser of E. Let $u_n \rightharpoonup w$ in $\dot{W}^{1,2}(\Omega)$. Let $i : \dot{W}^{1,2}(\Omega) \to L^2(\Omega)$ be the inclusion mapping. Then i is a bounded linear operator, and

$$i(u_n) \rightharpoonup i(w)$$

in $L^2(\Omega)$. That is, $u_n \to w$ in $L^2(\Omega)$. Since bounded sets in $\dot{W}^{1,2}(\Omega)$ are precompact sets in $L^2(\Omega)$, we can choose a subsequence such that $u_n \to w$ (strongly) in $L^2(\Omega)$. Hence the weak convergence in $\dot{W}^{1,2}(\Omega)$ can be "converted" into strong convergence in $L^2(\Omega)$.

We now need to show that w minimises E and w is a solution to our equation. We need to show that $E(u_n) \to E(w) = \gamma$. Recall that in a Banach space, if $u_n \rightharpoonup u$ weakly, then

$$\|u\| \leq \| \liminf_{n\to\infty} \|u_n\|.$$

Now, we then have

$$\|w\|_{1,2} \leq \liminf_{n\to\infty} \|u_n\|_{1,2}$$

Taking squares, we obtain

$$\|\nabla w\|_2^2 \leq \liminf_{n\to\infty} \|\nabla u_n\|_2^2.$$

We also need to prove that

$$\int G(u_n) \to \int G(w) \quad \text{as } n \to \infty \tag{$\star\star\star$}$$

Then we can show that

$$E(u_n) = \frac{1}{2}\int |\nabla u_n|^2 - \int G(u_n) \to \gamma$$

and hence

$$E(w) = \frac{1}{2}\int |\nabla w|^2 - \int G(w) \leq \gamma.$$

But $E(w) \geq \gamma$, and so $E(w) = \gamma$, that is, w is a minimiser.

It thus remains to prove $(\star\star\star)$. Since $u_n \to w$ in $L^2(\Omega)$, we can show that $u_n \to w$ a.e by taking subsequences. By a result in analysis (Ergerov's theorem), there exists sets V_k of arbitrarily small measure such that

$$u_n(x) \to w(x)$$

uniformly on $\Omega \setminus V_k$ as $n \to \infty$, again taking subsequences. We know that w is bounded off a set of small measure and hence we can find a set Z of small measure so $u_n \to w$ uniformly on $\Omega \setminus Z$ and w is bounded on $\Omega \setminus Z$. This implies that

$$G(u_n) \to G(w)$$

uniformly on $\Omega \setminus Z$ the fact that a continuous function on $\mathbb{R}$ is uniformly continuous on bounded sets. Hence,

$$\int_{\Omega\setminus Z} G(u_n) \to \int_{\Omega\setminus Z} G(w).$$

We now prove $\int_Z G(u_n), \int_Z G(w)$ are uniformly small in Z if Z has small measure. We have

$$\begin{aligned}\int_Z G(u_n) &\leq \int_Z \left(\frac{1}{2}\mu u_n^2 + K_3\right) \\ &\leq \frac{1}{2}\mu \int_Z u_n^2 + K_3 m(Z)\end{aligned}$$

where $m(Z)$ is the measure of Z.

Since u_n is bounded in $\dot{W}^{1,2}(\Omega)$, by Sobolev's embedding theorem, we can show that $\|u_n\|_{p^\star}$ is bounded for $p^\star > 2$. So the first term is less than or equal to $\frac{1}{2}\mu\|u_n\|_{2,Z}^2$. Since

$$\int_Z u_n^2 \leq \left(\int_Z \left(|u_n|^2\right)^q\right)^{\frac{1}{q}} \left(\int_Z 1^q\right)^{\frac{1}{q'}}$$

for q, q' Hölder pairs, so letting $p^\star = 2q$ for $q > 1$, we have

$$\begin{aligned}\int_Z u_n^2 &\leq \left(\int_Z |u_n|^{p^\star}\right)^{\frac{1}{q}} (m(Z))^{\frac{1}{q'}} \\ &\leq \left(\|u_n\|_{p^\star}\right)^{\frac{p^\star}{2}} (m(Z))^{\frac{1}{q'}}\end{aligned}$$

as required.

Recall that since w is a minimizer, we have

$$\begin{aligned}E(w + t\phi) \geq E(w) \quad \forall \phi \in C_c^\infty(\Omega) \quad \forall t \\ \frac{d}{dt}E(w + t\phi)\bigg|_{t=0} = 0\end{aligned}$$

if it exists. We will now prove that the derivative exists and equals

$$\int_\Omega \nabla w \cdot \nabla \phi - g(w)\phi.$$

In this case,

$$\int \nabla w \cdot \nabla \phi = g(w)\phi \quad \forall \phi \in C_c^\infty(\Omega),$$

and so $-\Delta w = g(w)$.

We have

$$\begin{aligned}E(w + t\phi) &= \frac{1}{2}\int_\Omega \nabla(w + t\phi) \cdot \nabla(w + t\phi) - \int_\Omega G(w + t\phi) \\ &= \frac{1}{2}\int_\Omega |\nabla w|^2 + 2t\nabla w \cdot \nabla \phi + t^2|\nabla \phi|^2 - \int G(w + t\phi).\end{aligned}$$

Therefore

$$\frac{d}{dt}E(w+t\phi) = \int_\Omega \nabla w\cdot\nabla\phi + t\int_\Omega |\nabla\phi|^2 - \frac{d}{dt}\int G(w+t\phi)$$

$$\frac{d}{dt}E(w+t\phi)|_{t=0} = \int_\Omega \nabla w\cdot\nabla\phi - \frac{d}{dt}\int G(w+t\phi).$$

We thus need only prove that

$$\frac{d}{dt}(\int G(w+t\phi))|_{t=0} = \int g(w)\phi.$$

Now

$$\frac{\int G(w+t\phi)-G(w)}{t} = \int G'(w+\theta(x)t\phi(x))\phi(x)$$

where $0\le\theta(x)\le 1$. We need to prove (remembering $G'=g$), that

$$\int G'(w+\theta(x)t\phi(x))\phi(x) \to \int g(w)\phi(x)$$

Choose a set T so that $\mu(\Omega - T)$ is small and w,ϕ are bounded on T. Then

$$g(w+t\theta(x)\phi(x)) \to g(w(x))\phi(x)$$

uniformly on T as $t\to 0$ as g is uniformly continuous on bounded sets. We need only prove that

$$\int_{\Omega\setminus T} g(w+t\theta(x)\phi(x))\phi(x)$$

is small for all t small.

.... *CBF finishing this.*

Remark.

(i) If $g(0)=0$, our minimum may be $u(x)=0$.

(ii) If $g(0)=0$ and $g'(0)>\lambda$, 0 may not be the minimum and we must have a non-trivial solution. We only need to find $z\in\dot{W}^{1,2}(\Omega)$ with $E(Z)<0$. We choose $z=t\phi$, where t is small and positive and ϕ_1 is the eigenfunctino corresponding to λ_1. Then

$$G(s) = \frac{1}{2}g'(0)s^2 + m(s),$$

where $\frac{m(s)}{s^2}\to 0$ as $s\to 0$. Then

$$E(t\phi_1) = \frac{1}{2}t^2\left(\lambda_1 - g'(0)\right)\int_\Omega \phi_1^2 + o(t^2) < 0$$

if t is small.

□

13. Fixed Point Methods

Theorem 13.1 (Brower)**.** *B^n is the closed ball in $\mathbb{R}^n$ and $f : B^n \to B^n$ is continouus then there exists $x \in B^n$ such that $f(x) = x$.*

Definition 13.2 (Completely continuous)**.** $A : E \to E$ is completely continuous (cc) if A is continuous and if D is bounded in E, then $A(D)$ is compact in E.

Lemma 13.3. *If E is an infinite dimensional Banach space hen $I : E \to E$ is not cc.*

If A is linear, $A : E \to E$, then A is cc if and only if A is compact.

(Shauder). If D is closed, bounded and convex in a Banach space E and $A : D \to E$ is cc and $A(D) \subseteq D$, then there exists $x \in D$ such that $A(x) = x$ (fixed point).

Example 13.4 (Example of fixed point methods)**.** Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and $\frac{g(y)}{y} \to \tau$ as $|y| \to \infty$ where τ is not an eigenvalue of

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

We prove the problem

$$\begin{aligned} -\Delta u &= g(u) \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a *weak* solution

$$g(y) = \tau y + h(y)$$

where $\frac{h(y)}{y} \to 0$ as $|y| \to \infty$.

Note that if such a solution exists, then we have

$$\begin{aligned} & & -\Delta u &= \tau u + h(u) \\ (\Rightarrow) & & (-\Delta - \tau I)u &= h(u) \\ (\Rightarrow) & & u &= (-\Delta - \tau I)^{-1} h(u) \equiv H(u). \end{aligned}$$

Proof. For simplicity, assume $\tau = 0$. We prove that for large M, H maps the set $Z = \{u \in L^2(\Omega) \mid \|u\|_2 \leq M\}$ into itself and is cc.

If we do this then by the Schauder theorem, we can show that H has a fixed point which is our solution.

Aside. Consider

$$\begin{aligned} -\Delta u &= f(x) \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Then a weak solution satisfies $u \in \dot{W}^{1,2}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \phi = \underbrace{\int_\Omega f\phi}_{\substack{\text{bounded linear functional} \\ \text{on } \dot{W}^{1,2}(\Omega) \text{ if } f \in L^2(\Omega)}} \qquad \forall \phi \in \dot{W}^{1,2}(\Omega).$$

Thus

$$\langle u, \phi \rangle = \langle F, \phi \rangle$$

and so our solution is $u = F$.

If $n \geq 3$, and if $f \in L^{\frac{2n}{n+2}}(\Omega)$ with Ω bounded, then it suffices to prove $\int_\Omega f\phi$ is a bounded linear functional on $\dot{W}^{1,2}(\Omega)$. We have

$$\begin{aligned} \left| \int f\phi \right| &\leq \|f\|_{\frac{2n}{n+2}} \|\phi\|_{\frac{2n}{n-2}} \quad \text{by Hölder} \\ &\leq K\|\|_{\frac{2n}{n+2}} \|\nabla \phi\|_2 \quad \text{by Sobolev embedding} \end{aligned}$$

and so

$$\|\nabla u\|_2^2 = \int |\nabla u|^2 \leq C\|f\|_{\frac{2n}{n+2}} \|\nabla u\|_2$$

and hence

$$\|\nabla u\|_2 \leq C\|f\|_{\frac{2n}{n+2}}$$

Proof of example. We now show that H has the desired properties. Let $\epsilon > 0$. Then there exists $K > 0$ such that

$$|h(y)| \leq \epsilon|y| + K$$

So we have

$$\begin{aligned} \|h(u)\|_2 &\leq \|\epsilon|u| + K\|_2 \\ &\leq \|\epsilon u\|_2 + \|K\|_2 \\ &\leq \epsilon\|u\|_2 + Km(\Omega)^{1/2}. \end{aligned} \tag{$\star$}$$

Then we have

$$\begin{aligned} \|H(u)\|_2 &= \|(-\Delta^{-1})h(u)\| \\ &\leq K_1\|h(u)\|_2 \\ &\leq K_1\left(\epsilon\|u\|_2 + Km(\Omega)^{1/2}\right) \\ &\leq \frac{1}{2}\|u\|_2 + \underbrace{K_2}_{=K_1Km(\Omega)^{1/2}} \qquad \text{letting } \epsilon = \frac{1}{2K_1} \end{aligned}$$

Then H maps the set $Z = \{u \in L^2(\Omega) \,|\, \|u\|_2 \leq 2K_2\}$ into itself (that is, $H(Z) \subseteq Z$.)

Secondly, the image under H of this ball lies in a compact set in $L^2(\Omega)$. It suffices to prove H of this set lies in a bounded set in $\dot{W}^{1,2}(\Omega)$ and then use the result that the inclusion mapping $i : \dot{W}^{1,2}(\Omega) \to L^2(\Omega)$ is compact.

This is easy since $\{h(u) \,|\, u \in Z\}$ lies in a bounded set in $L^2(\Omega)$ by $(\star)$ and $(-\Delta)^{-1}$ maps bounded sets in $L^2(\Omega)$ to bounded sets in $\dot{W}^{1,2}(\Omega)$.

Finally, H is continuos. We prove that the map $u \to h(u)$ is continuous and $L^2(\Omega) \to L^{\frac{2n}{n+2}}(\Omega)$. This suffices since $H = (-\Delta)^{-1} \circ h$.

Suppose that $u_n \to u$ in $L^2(\Omega)$. As before, there exists T a set such that $\Omega - T$ has small measure such that u is bounded on T and $u_n \to u$ uniformly on T. Hence $h(u_n) \to h(u)$ uniformly on T and so

$$\int_T |h(u_n) - h(u)|^{\frac{2n}{n+2}} \to 0.$$

We now need only prove

$$\begin{aligned}
&\int_{\Omega \setminus T} |h(u_m) - h(u)|^{\frac{2n}{n+2}} \quad \text{is small for large } m \\
&= \|h(u_m) - h(u)\|_{\frac{2n}{n+2}, \Omega \setminus T} \\
&\leq \|h(u_m) - h(u)\|^{\alpha}_{2, \Omega \setminus T} \left(\int_{\Omega \setminus T} 1 \right)^{\beta} \quad \text{by Hölder}
\end{aligned}$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. We need only then bound

$$\begin{aligned}
\|h(u_m) - h(u)\|_{2, \Omega \setminus T} &\leq \|h(u_m)\|_2 + \|h(u)\|_2 \\
&\leq K_1 \quad \text{by } (\star).
\end{aligned}$$

This result can also be shown using the result that if $u \in L^1(\Omega)$, Ω bounded, then given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |u| \leq \epsilon$$

if $m(A) \leq \delta$. $\square$

Consider the equation

$$\begin{aligned}
-\Delta u &= g(u, \nabla u) \quad \text{on } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

This has a weak solution if g is continuous and bounded on $\mathbb{R} \times \mathbb{R}^n$ and Ω is bounded (by Schauder). It is possible to show that this equation is a mapping of

$$\{u \in \dot{W}^{1,2}(\Omega) \,|\, \|u\|_{1,2} \leq K\}$$

into itself. We need to show that this mapping is compact, as above.

Lemma 13.5 (Schauder). *If A is a Banach*

(i) $A : E \times [0,1] \to E$ *is completely continuous, and*
(ii) $A(x,1) = L$ *where L is linear and $I - L$ is invertible, and*
(iii) *if* $x = A(x,t)$ *where* $0 \le t \le 1$, *then* $\|x\| \le M$,

then the equation $x = A(x,0)$ *has a solution.*

14. Other Types of Problems

If Ω is a bounded domain with smooth boundary, and consider the equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \quad \text{on } \Omega \\ u(x,t) &= 0 \quad \text{if } x \in \partial\Omega \end{aligned}$$

with $u(x,0) = u_0(x) \in L^2(\Omega)$ given.

Suppose ϕ_i are the weak eigenfunctions of $-\Delta$ for the Dirichlet Boundary condition $u(x,t) = 0$ for $x \in \partial\Omega$. Then $\|\phi_i\|_2 = 1$ and they form a complete orthonormal basis for $L^2(\Omega)$. Then we can write

$$u(x,0) = \sum_{i=1}^{\infty} c_i \phi_i(x)$$

where $\sum c_i^2 < \infty$.

The solution can be then be uniquely written as

$$u(x,t) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i t} \phi_i(x)$$

We can trivially see that

$$\|u(x,t) - u(x,0)\|_2 \to 0$$

as

$$\begin{aligned} \|u(x,t) - u(x,0)\|_2^2 &= \left\| \sum c_i \left(e^{-\lambda_i t} - 1 \right) \phi_i(x) \right\|_2^2 \\ &= \sum c_i^2 \left(e^{-\lambda_i t} - 1 \right)^2 \to 0. \end{aligned}$$

Note that $u_0 \in L^2$, but $u(x,t) \in C^\infty$ for all $t > 0$.

Consider now the differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\Delta u \quad \text{on } \Omega \\ u(x,t) &= 0 \quad \text{if } x \in \partial\Omega \end{aligned}$$

for $t \geq 0$. This is equivalent to running the heat equation backwards in time. Formally, the solution is

$$\sum c_i e^{\lambda_i t} \phi_i(x)$$

for $t \geq 0$, which does not converge in L^2.

It can be shown that there is at most one solution. This is an *ill-posed* problem.

15. Various Other Results

Theorem 15.1. *Eigenfunctions of a compact self-adjoint operator form a complete set*

Theorem 15.2. *The inverse of the Laplacian is a compact, self-adjoint operator.*

Comments on the exam.

(i) Asked some definitions.
(ii) Asked some simple proofs.
(iii) Asked some problem questions, possibly similar to assignments.
(iv) Look at the assignments for questions.