MATH 3975 - FINANCIAL MATHEMATICS EXAM NOTES

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1. Introduction to Markets

1.1. Introduction.

Definition 1.1.1 (Time Value of Money)**.** The **future value** $F(0,t)$ is defined to be the value at time $t > 0$ of \$1 invested at time 0.

The **present value** or **discount factor** $P(0,t)$ is the amount invested at time 0 such that its value at time t is equal to \$1.

To avoid arbitrage, we must have $P(0,t) = F^{-1}(0,t)$ for all t.

Proposition 1.1.2. *We must have the following relationship for $P(t,T)$ and $F(t,T)$.*

$$F(t,T) = \frac{F(0,T)}{F(0,t)}$$
$$P(t,T) = \frac{P(0,T)}{P(0,t)}$$

Definition 1.1.3 (Spot rate)**.** The spot rate $r(t)$ is defined as

$$F(t,t+\Delta) = 1 + r(t)\Delta$$

Proposition 1.1.4. *We have*

$$P(0,T) = \prod_{i=1}^{n}(1+r_i)^{-1}$$

in the discrete case, and in the continuous case, we have

$$P(t,T) = e^{-\int_t^T r(s)ds}$$

1.2. Riskless Securities and Bonds.

Theorem 1.2.1 (Fundamental Theorem of Riskless Security Pricing)**.** *If interest rates are deterministic, the arbitrage free price of a riskless security is given by*

$$S_0 = \sum_{i=1}^{n} P(0,t_i)C_i$$

2. Single Period Market Models

A single period market model is the most elementary model. Only a single period is considered. At times $t = 0$ and 1 market prices are recorded.

Single period market models are the atoms of Financial Mathematics

We assume that we have a finite sample space

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$$

2.1. **The most elementary market model.** Assume the sample space consists of two states, H and T, with $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = 1 - p$. Define the price of the stock at time 0 to be S_0, and let S_1 be a random variable depending on H and T. Let $u = \frac{S_1(H)}{S_0}$ and $d = \frac{S_1(T)}{S_0}$.

Definition 2.1.1 (Trading strategy)**.** A trading strategy (x, ϕ) is a pair wheere x is the total initial investment at $t = 0$, and ϕ denotes the number of shares bought at $t = 0$. Given a strategy (x, ϕ), the agent invests the remaining money $x - \phi S_0$ in a money market account. We note this amount may be negative (borrowing from the money market account).

Definition 2.1.2 (Value process)**.** The value process of the trading strategy (x, ϕ) in ou elementary market model is given by $(V_0(x, \phi), V_1(x, \phi))$ where $V_0(x, \phi) = x$ and

$$V_1 = (x - \phi S_0)(1 + r) + \phi S_1$$

Definition 2.1.3 (Arbitrage)**.** An arbitrage is a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money.

More rigorously, we have a trading strategy (x, ϕ) is an arbitrage if

- $x = V_0(x, \phi) = 0$,
- $V_1(x, \phi) \geq 0$,
- $\mathbb{E}\left[V_1(x, \phi)\right] > 0$.

Proposition 2.1.4. *To rule out arbitrage in our model, we must have $d < 1 + r < u$.*

Proof. If this inequality is violated, consider the following strategies.

- If $d \geq 1 + r$, borrow S_0 from the money market.
- If $u \leq 1 + r$, short S_0 and invest in the money market.

These are both arbitrages and the proposition is proven. □

We have the following theorem, giving the converse of the above proposition

Theorem 2.1.5. *The condition $d < 1 + r < u$ is a necessary and sufficient no arbitrage condition. That is,*

$$\textit{No arbitrage} \iff d < 1 + r < u$$

Definition 2.1.6 (Replicating strategy or hedge)**.** A **replicating strategy** or **hedge** for the option $h(S_1)$ in our elementary single period market model is a trading strategy (x, ϕ) satisfying $V_1(x, \phi) = h(S_1)$. That is,

$$(x - \phi S_0)(1 + r) + \phi S_1(H) = h(S_1(H))$$
$$(x - \phi S_0)(1 + r) + \phi S_1(T) = h(S_1(T))$$

Theorem 2.1.7. *Let $h(S_1)$ be an option in our market model, and let (x, ϕ) be a replicating strategy for $h(S_1)$. Then x is the only price for the option at time $t = 0$ which does not allow arbitrage.*

To find a replicating strategy for an arbitrary option, define

$$\phi = \frac{h(S_1(H)) - h(S_1(T))}{S_1(H) - S_1(T)}$$
$$\tilde{p} = \frac{1 + r - d}{u - d}$$

Then by solving the above two equations for x, we have

$$x = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r} h(S_1)\right] = \frac{1}{1+r}\left[\tilde{p} h(S_1(H)) + (1 - \tilde{p}) h(S_1(T))\right]$$

2.2. **A general single period market model.**

Definition 2.2.1 (Trading strategy)**.** A **trading strategy** for an agent in our general single period market model is a pair (x, ϕ), where $\phi = (\phi^1, \ldots, \phi^n) \in \mathbb{R}^n$ specifying the initial investment in the i-th stock.

Definition 2.2.2 (Value and gains process)**.** Let the **value process** of the trading strategy (x, ϕ) is given $(V_0(x, \phi), V_1(x, \phi))$ where $V_0(x, \phi) = x$ and

$$V_1(x, \phi) = (x - \sum_{i=1}^{n} \phi^i S_0^i)(1 + r) + \sum_{i=1}^{n} \phi^i S_1^i$$

The **gains process** is defined as

$$G(x, \phi) = (x - \sum_{i=1}^{n} \phi^i S_0^i) r + \sum_{i=1}^{n} \phi^i \Delta S^i$$

where ΔS^i is defined as

$$\Delta S^i = S_1^i - S_0^i$$

We have the simple result

$$V_1(x, \phi) = V_0(x, \phi) + G(x, \phi)$$

To study the prices of the stocks in relation to the money market account, we introduce the **discounted stock prices** $\hat{S}_t^i$ defined as follows:

$$\hat{S}_0^i = S_0^i$$
$$\hat{S}_1^i = \frac{1}{1+r} S_1^i$$

Definition 2.2.3 (Discounted value and gains process)**.** We define the **discounted value process** $\hat{V}(x, \phi)$ by

$$\hat{V}_0(x, \phi) = x$$
$$\hat{V}_1(x, \phi) = (x - \sum_{i=1}^{n} \phi^i S_0^i) + \sum_{i=1}^{n} \phi^i \hat{S}_1^i$$

and the discounted ains process $\hat{G}(x, \phi)$ as

$$\hat{G}(x, \phi) = \sum_{i=1}^{n} \phi^i \Delta \hat{S}^i$$

with $\Delta \hat{S}^i = \hat{S}_1^i - \hat{S}_0^i$.

We then have the relation

$$\hat{V}_1(x, \phi) = \hat{V}_0(x, \phi) + \hat{G}(x, \phi)$$

Definition 2.2.4 (Arbitrage)**.** A trading strategy (x, ϕ) is an arbitrage in our general single period market model if

- $x = V_0(x, \phi) = 0$,
- $V_1(x, \phi) \geq 0$,
- $\mathbb{E}\left[V_1(x, \phi)\right] > 0$.

Alternatively, if a trading strategy satisfies the first two conditions above, it is an arbitrage if the following condition is satisfied:

$$\text{There exists } \omega \in \Omega \text{ with } V_1(x, \phi) > 0.$$

Alternatively, we can replace all references to V in the above definition with $\hat{V}$.

Definition 2.2.5 (Risk neutral measure)**.** A measure $\tilde{\mathbb{P}}$ on Ω is a **risk neutral measure** if

- $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$
- $\mathbb{E}_{\tilde{\mathbb{P}}}\left[\Delta \hat{S}^i\right] = 0$ for all i

Theorem 2.2.6 (Fundamental Theorem of Asset Pricing)**.** *In the general single period market model, there are no arbitrages if and only if there exists a risk neutral measure for the market model.*

Definition 2.2.7 (Alternative definition of arbitrage)**.** Define the set $\mathbb{W}$ by the following:

$$\mathbb{W} = \{X \in \mathbb{R}^k \mid X = \hat{G}(x, \phi) \text{ for some trading strategy } (x, \phi)\}$$

Then, letting $\mathbb{A}$ be given as

$$\mathbb{A} = \{X \in \mathbb{R}^k \mid X \geq 0, X \neq 0\}$$

Then we have a definition of arbitrage:

$$\text{no arbitrage} \iff \mathbb{W} \cap \mathbb{A} = \emptyset$$

Definition 2.2.8 (Set of risk neutral measures)**.** Now, consider the orthogonal component $\mathbb{W}^\perp$, defined as

$$\mathbb{W}^\perp = \{Y \in \mathbb{R}^k \mid \langle X, Y\rangle = 0 \text{ for all } X \in \mathbb{R}^k\},$$

the set of vectors in $\mathbb{R}^k$ perpendicular to all elements of $\mathbb{W}$. Furthermore, defining $\mathcal{P}^+$ as

$$\mathcal{P}^+ = \{X \in \mathbb{R}^k \mid \sum_{i=1}^{k} X_i = 1, X_i > 0\}$$

Then we have the following theorem.

Theorem 2.2.9. *A measure $\tilde{\mathbb{P}}$ is a risk neutral measure on Ω if and only if $\tilde{\mathbb{P}} \in \mathcal{P}^+ \cap \mathbb{W}^\perp$.*

We denote the set of risk neutral measures $\mathbb{M} = \mathcal{P}^+ \cap \mathbb{W}^\perp$.

Definition 2.2.10 (Contingent claim)**.** A **contingent claim** in our general single period market model is a random variable X on Ω representing a payoff at time $t = 1$.

Proposition 2.2.11. *Let X be a contingent claim in our general single period market model, and let (x, ϕ) be a hedging strategy for X, so that $V_1(x, \phi) = X$ then the only price of X which complies with the no arbitrage principle is $x = V_0(x, \phi)$.*

Definition 2.2.12 (Attainable contingent claim)**.** A contingent claim is **attainable** if there exists a trading strategy (x, ϕ) which replicates X, so that $V_1(x, \phi)$.

Theorem 2.2.13. *Let X be an attainable contingent claim and $\tilde{\mathbb{P}}$ be an arbitrary risk neutral measure. Then the price x of X at time $t = 0$ can be computed by the formula*

$$x = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r} X\right]$$

Corollary. *This theorem tells us that in particular, for any risk neutral measure in our model, we get the same value when taking the expectation above.*

Definition 2.2.14. We say that a price x for the contingent claim X **complies with the no arbitrage principle** if the extended model, which consists of the original assets $S^1, \ldots, S^n$ and an additional asset S^{n+1} which satisfies $S_0^{n+1} = x$ and $S_1^{n+1} = X$ is arbitrage free.

The following proposition shows that when using the risk neutral measure to price a contingent claim, one obtains a price which complies with the no-arbitrage principle.

Proposition 2.2.15. *Let X be a possibly unattainable contingent claim and $\tilde{\mathbb{P}}$ any risk neutral measure for our general single period market model. Then*

$$x = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}X\right]$$

defines a price for the contingent claim at time $t = 0$ which complies with the no-arbitrage principle.

Definition 2.2.16 (Complete market)**.** A financial market is called **complete** if for any contingent claim X there exists a replicating strategy (x, ϕ). A model which is not complete is called **incomplete.**

Proposition 2.2.17. *Assume a general single period market model consisting of stocks $S^1, \dots, S^n$ and a money market account modelled on the state space $\Omega = \{\omega_1, \dots, \omega_k$ is arbitrage free. Then this model is complete if and only if the $k \times (n+1)$ matrix A given by*

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) & \dots & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \dots & S_1^n(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1^1(\omega_k) & \dots & S_1^n(\omega_k) \end{pmatrix}$$

has full rank, that is, rank$(A) = k$.

Proof. First, a matrix A has full rank if and only if for every $X \in \mathbb{R}^k$, the equation $AZ = X$ has a solution $Z \in \mathbb{R}^{n+1}$.

Secondly, we have

$$\begin{pmatrix} 1+r & S_1^1(\omega_1) & \dots & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \dots & S_1^n(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1^1(\omega_k) & \dots & S_1^n(\omega_k) \end{pmatrix} \begin{pmatrix} x - \sum_{i=1}^n \phi^i S_0^i \\ \phi^1 \\ \vdots \\ \phi^n \end{pmatrix} = \begin{pmatrix} V_1(x,\phi)(\omega_1) \\ V_1(x,\phi)(\omega_2) \\ \vdots \\ V_1(x,\phi)(\omega_k) \end{pmatrix}$$

This shows that computing a replicating strategy for a contingent claim X is the same as to solve the equation $AZ = X$, and the proposition follows □

Proposition 2.2.18. *A contingent claim X is attainable, if and only if $\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}X\right]$ takes the same value for all $\tilde{\mathbb{P}} \in \mathbb{M}$.*

Theorem 2.2.19. *Under the assumption that the model is arbitrage free, it is complete, if and only if $\mathbb{M}$ consists of only one element - i.e., there is a unique risk measure.*

2.3. **Single Period Investment.**

Definition 2.3.1. A continuously differentialble function $u : \mathbb{R}^+ \to \mathbb{R}$ is called a risk averse utility function if it has the following two properties:

- u is strictly increasing - that is, $u'(x) > 0$ for all $x \in \mathbb{R}^+$
- u is strictly concave - that is, $u(\lambda x + (1-\lambda)y) > \lambda u(x) + (1-\lambda)u(y)$

In the case that $u''(x)$ exists, the second condition in the above definition is equivalent to $u''(x) < 0$ for all $x \in \mathbb{R}^+$. Sometimes one assumes in addition the condition:

- $\lim_{x\to 0} u'(x) = +\infty$ and $\lim_{x\to\infty} u'(x) = 0$

Example 2.3.2 (Utility functions)**.** The following are all risk averse utility functions:

(1) Logarithmic utility: $u(x) = \log(x)$
(2) Exponential utility: $u(x) = 1 - e^{-\lambda x}$
(3) Power utility: $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$ with $\gamma > 0, \gamma \neq 1$
(4) Square root utility: $u(x) = \sqrt{x}$

Proposition 2.3.3 (Principle of expected utility)**.** *We assume the following axiom of agents behaviour - that of maximising expected utility.*

$$X \text{ is preferred to } Y \iff \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

From Jensen's inequality, we have that for every risk averse utility function u and risky payoff X,

$$\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$$

Definition 2.3.4 (Certainty equivalent price)**.** The certain value $X_0 \in \mathbb{R}$ that makes an investor indifferent between X_0 and a risky payoff X is called the **certaintly equivalent price** of X, that is,

$$u(X_0) = \mathbb{E}[u(X)]$$

or equivalently,

$$X_0 = u^{-1}(\mathbb{E}[u(X)])$$

Lemma 2.3.5. *The certainty equivalent price of a risky payoff is invariant under a positive linear transformation of the utility function $u(x)$.*

Definition 2.3.6 (Risk premium)**.** We have that $X_0 < \mathbb{E}[X]$, and the difference between the two is called the **risk premium** ρ, that is,

$$\rho = \mathbb{E}[X] - u^{-1}(\mathbb{E}[u(X)])$$

We can write the equation above as

$$u(\mathbb{E}[X] - \rho) = \mathbb{E}[u(X)]$$

Definition 2.3.7 (Measures of risk aversion)**.** We define the following risk aversion coefficients, as a measure of how risk averse the investor is. The **absolute risk aversion** ρ_{abs}, given by

$$\rho_{abs} = -\frac{u''(x)}{u'(x)}$$

and the **relative risk aversion** ρ_{rel}, given by

$$\rho_{rel} = -\frac{xu''(x)}{u'(x)}$$

We now seek to find the optimal investment in a market, which can be translated as finding a trading strategy (x, ϕ) such that $\mathbb{E}\left[u(V_1(x,\phi))\right]$ achieves an optimal value.

Definition 2.3.8. A trading strategy $(x, \phi^\star)$ is a solution to the optimal portfolio problem with initial investment x and utility function u, if

$$\mathbb{E}\left[u(V_1(x,\phi^\star))\right] = \max_{\phi} \mathbb{E}\left[u(V_1(x,\phi))\right]$$

Proposition 2.3.9. *If there exists a solution to the optimal portfolio problem, then there can not exist an arbitrage in the market.*

Proposition 2.3.10. *Let (x, ϕ) be a solution to the optimal portfolio problem with intial wealth x and utility function u, then the measure $\mathbb{Q}$ defined by*

$$\mathbb{Q}(\omega) = \frac{\mathbb{P}(\omega)u'(V_1(x,\phi)(\omega_i))}{\mathbb{E}\left[u'(V_1(x,\phi))\right]}$$

Assume that our model is complete. In this case, there is a unique risk neutral measure which we denote by $\tilde{\mathbb{P}}$.

Definition 2.3.11. We define the set of attainable wealths from initial investment $x > 0$ by

$$\mathbb{W}_x = \left\{ W \in \mathbb{R}^k \,|\, \mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}W\right] = x \right\}$$

Our optimisation problem is hence:

$$\begin{aligned} \text{maximise} \quad & \mathbb{E}\left[u(W)\right] \\ \text{subject to} \quad & W \in \mathbb{W}_x \end{aligned}$$

To solve this problem, we use the Lagrange multiplier method. To do this, consider the Lagrange function

$$\mathcal{L}(W,\lambda) = \mathbb{E}\left[u(W)\right] - \lambda\left(\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}W\right] - x\right)$$

By introducing the **state price density**

$$L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)},$$

we can write the Lagrange function as

$$\mathcal{L}(W,\lambda) = \sum_{i=1}^{k} \mathbb{P}(\omega_i)\left[u(W(\omega_i)) - \lambda\left(L(\omega_i)\frac{1}{1+r}W(\omega_i) - x\right)\right]$$

Computing partial derivatives with respect to $W_i = W(\omega_i)$ and setting them equal to zero, multiplying with $\mathbb{P}(\omega_i)$ and summing over i, we deduce that

$$\lambda = \mathbb{E}\left[(1+r)u'(W)\right]$$

and (denoting the inverse function of $u'(x)$ by $I(x)$)

$$W(\omega) = I\left(\lambda\frac{L(\omega)}{1+r}\right)$$

Since we have

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}W\right] = x$$

and substituting the expression from the above equation into the last equation, we obtain

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r}I\left(\lambda\frac{L}{1+r}\right)\right] = x$$

3. Multi period Market Models

3.1. **The general model.**

Definition 3.1.1 (Specification of the general model)**.** The two most important new features of multi period market models are:

- Agents can buy and sell assets not only at the beginning of the trading period, but at any time t out of a discrete set of trading times $t \in \{0, 1, 2, \ldots, T\}$.
- Agents can gather information over time, since they can observe prices. Hence, they can make their investment decisions at time $t = 1$ dependent on the prices of the asset at time $t = 1$.

Throughout, we assume we are working on a finite state space Ω on which there is defined a probability measure $\mathbb{P}$.

Definition 3.1.2 (σ-algebra)**.** A collection $\mathcal{F}$ of subsets of the state space Ω is called a σ-algebra if the following conditions hold:

- $\Omega \in \mathcal{F}$
- If $F \in \mathcal{F}$, then $F^c \in \mathcal{F}$
- If $F_i \in \mathcal{F}$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

Definition 3.1.3 (Partition of a σ-algebra)**.** Let I denote some index set. A **partition** of a σ-algebra $\mathcal{F}$ is a collection of sets $\emptyset \neq A_i \in \mathcal{F}$ for $i \in I$, such that

- Every set $F \in \mathcal{F}$ can be written as a union of some of the A_i.
- The sets A_i are pairwise disjoint.

Definition 3.1.4. A random variable $X : \Omega \to \mathbb{R}$ is called $\mathcal{F}$-**measurable**, if for every closed interval $[a, b] \subset \mathbb{R}$, the preimage under X belongs to $\mathcal{F}$, that is,

$$X^{-1}([a, b]) \in \mathcal{F}$$

Proposition 3.1.5. *Let $X : \Omega \to \mathbb{R}$ be a random variable and (A_i) a partition of the σ-algebra $\mathcal{F}$, then X is $\mathcal{F}$-measurable if and only if X is constant on each of the sets of the partition, that is, there exist $c_j \in \mathbb{R}$ for all $j \in I$ such that*

$$X(\omega) = c_j \textit{ for all } \omega \in A_j$$

Definition 3.1.6. A sequence $(\mathcal{F}_t)_{0 \leq t \leq T}$ of σ-algebras on Ω is called a **filtration** if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$.

Definition 3.1.7. A family (X_t) with $0 \leq t \leq T$ consisting of random variables, is called a **stochastic process**. If $(\mathcal{F}_t)_{0 \leq t \leq T}$ is a filtration, the stochastic process (X_t) is called $(\mathcal{F}_t)$-**adapted** if for all t we have that X_t is $\mathcal{F}_t$-measurable.

Definition 3.1.8. Let $(X_t)_{0 \leq t \leq T}$ be a stochastic process on $(\Sigma, \mathcal{F}, \mathbb{P})$. Define

$$\mathcal{F}_t^X = \sigma\left(X_u^{-1}([a, b]) \,|\, 0 \leq u \leq t, a \leq b\right)$$

This is the smallest σ-algebra which contains all the sets $X_u^{-1}([a, b])$ where $0 \leq u \leq t$ and $a \leq b$. Clearly $(\mathcal{F}_s^X)$ is a filtration. It follows immediately from the definition that (X_t) is $(\mathcal{F}_t^X)$ adapted. (F_t^X) is called the filtration **generated** by the process X.

Definition 3.1.9 (Value process)**.** The **value process** corresponding to the trading strategy $\phi = (\phi_t)_{0 \leq t \leq T}$ is the stochastic process $(V_t(\phi))_{0 \leq t \leq T}$ where

$$V_t(\phi) = \phi_t^0 B_t + \sum_{i=1}^{n} \phi_t^i S_t^i$$

Definition 3.1.10. A trading strategy $\phi = (\phi_t)$ is called **self financing** if for all $t = 0, ;T - 1$,

$$\phi_t^0 B_{t+1} + \sum_{i=1}^{n} \phi_t^i S_{t+1}^i = \phi_{t+1}^0 B_{t+1} + \sum_{i=1}^{n} \phi_{t+1}^i S_{t+1}^i$$

Lemma 3.1.11. *For a self-financing trading strategy $\phi = (\phi_t)$, the value process can be alternatively computed via*

$$V_t(\phi) = \phi_{t-1}^0 B_t + \sum_{i=1}^{n} \phi_{t-1}^i S_t^i$$

Definition 3.1.12 (General multi period market model)**.** A general multi period market model is given by the following data.

- A probability space $(\Sigma, \mathcal{F}, \mathbb{P})$ together with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ of $\mathcal{F}$.
- A money market account (B_t) which evolvs according to $B_t = (1+r)^t$.
- A number of financial assets $(S_t^1), \dots, (S_t^n)$ which are assumed to be $(\mathcal{F}_t)$-adapted stochastic processes.
- A set $\mathcal{T}$ of self financing and $(\mathcal{F}_t)$-adapted trading strategies.

Definition 3.1.13. Assume we are given a general multi period market model as described above. The **increment process** (ΔS_t^i) is defined as

$$\Delta S_t^i = S_t^i - S_{t-1}^i$$

and

$$\Delta B_t = B_t - B_{t-1} = rB_{t-1}$$

Definition 3.1.14 (Gains process)**.** Given a trading strategy ϕ, the corresponding **gains process** $(G_t(\phi))_{0\leq t\leq T}$ is given by

$$G_t(\phi) = \sum_{s=0}^{t-1} \phi_s^0 \Delta B_{s+1} + \sum_{i=1}^{n} \sum_{s=0}^{t-1} \phi_s^i \Delta S_{s+1}^i$$

Proposition 3.1.15. *An adapted trading strategy $\phi = (\phi)_{0\leq t\leq T}$ is self financing, if and only if any of the two equivalent statements hold*

- $V_t(\phi) = V_0(\phi) + G_t(\phi)$
- $\hat{V}_t(\phi) = \hat{V}_0(\phi) + \hat{G}_t(\phi)$

for all $0 \leq t \leq T$. Here $\hat{V}_t$ and $\hat{G}_t$ denote the discounted value and gains process, as defined in the following definition.

Definition 3.1.16 (Discount processes)**.** The discounted prices are given by

$$\hat{S}_t^i = \frac{S_t^i}{B_t}$$

and discounted gains, discounted value process, and discounted gains process are all defined analogously.

3.2. **Properties of the general multi period market model.** Here, we redefine the general concepts of financial mathematics, such as arbitrage, hedging, in the context of a multi period market model.

Definition 3.2.1 (Arbitrage)**.** A (self-financing) trading strategy $\phi = (\phi)_{0\leq t\leq T}$ is called an arbitrage if

- $V_0(\phi) = 0$
- $V_T(\phi) \geq 0$
- $\mathbb{E}\left[V_T(\phi)\right] > 0$

Definition 3.2.2 (Contingent claim)**.** A **contingent claim** in a multi period market model is an $\mathcal{F}_T$-measurable random variable X on Ω representing a payoff at terminal time T. A **hedging strategy** for X in our model is a trading strategy $\phi \in \mathcal{T}$ such that

$$V_T(\phi) = X,$$

that is, the terminal value of the trading strategy is equal to the payoff of the contingent claim.

Proposition 3.2.3. *Let X be a contingent claim in a multi period market model, and let $\phi \in \mathcal{T}$ be a hedging strategy for X, then the only price of X at time t which complies with the no arbitrage principle is $V_t(\phi)$. In particular, the price at the beginning of the trading period at time $t = 0$ is the total initial investment in the hedge.*

Definition 3.2.4 (Attainable contingent claim)**.** A contingent claim X is called **attainableat** in $\mathcal{T}$, if there exists a trading strategy $\phi \in \mathcal{T}$ which replicates X, that is, $V_T(\phi) = X$.

Definition 3.2.5 (Complete market)**.** A general multi period market model is called **complete**, if and only if for any contingent claim X there exists a replicating strategy ϕ. A model which is not complete is called **incomplete**.

Definition 3.2.6 (Conditional expectation)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space and X an $\mathcal{F}$-measurable random variable. Assume that $\mathcal{G}$ is a σ-algebra which is contained in $\mathcal{F}$. Denoting the unique partition of $\mathcal{G}$ with $(A_i)_{i \in I}$, the **conditional expectation** $\mathbb{E}\left[X \mid \mathcal{G}\right]$ of X with respect to $\mathcal{G}$ is defined as the random variable which satisfies

$$\mathbb{E}\left[X \mid \mathcal{G}\right](\omega) = \sum_x x\mathbb{P}(X = x \mid A_i)$$

whenever $\omega \in A_i$.

We then have the following identity:

$$\int_G X \, d\mathbb{P} = \int_G \mathbb{E}\left[X \mid \mathcal{G}\right] \, d\mathbb{P}$$

for any $G \in \mathcal{G}$.

Proposition 3.2.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space and X an $\mathcal{F}$-measurable random variable. Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ be sub-σ-algebras. Assume furthermore that $\mathcal{G}_2 \subset \mathcal{G}_1$. Then*

- ***Tower property.***
$$\mathbb{E}\left[X \mid \mathcal{G}_2\right] = \mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{G}_1\right] \mid \mathcal{G}_2\right]$$
- ***Taking out what is known.*** *If $Y : \Omega \to \mathbb{R}$ is $\mathcal{G}$-measurable, then*
$$\mathbb{E}\left[YX \mid \mathcal{G}\right] = Y\mathbb{E}\left[X \mid \mathcal{G}\right]$$
- *If $\mathcal{G} = \{\emptyset, \Omega\}$ is the trivial σ-algebra, then*
$$\mathbb{E}\left[X \mid \mathcal{G}\right] = \mathbb{E}\left[X\right]$$

Definition 3.2.8 (Risk neutral measure)**.** A measure $\tilde{\mathbb{P}}$ on Ω is called a **risk neutral measure** for a general multi period market model if

- $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$
- $\mathbb{E}_{\tilde{\mathbb{P}}}\left[\Delta \hat{S}_t^i \,|\, \mathcal{F}_{t-1}\right] = 0$ for $i = 1, \ldots, n$ and for all $1 \leq t \leq T$.

An alternative formulation of the second condition in the previous definition is

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{1}{1+r} S_{t+1}^i \,|\, \mathcal{F}_t\right] = S_t^i$$

Definition 3.2.9 (Martingale)**.** A $\mathcal{F}_t$-adapted process (X_t) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **martingale** if for all $s < t$,

$$\mathbb{E}\left[X_t \,|\, \mathcal{F}_s\right] = X_s$$

Lemma 3.2.10. *Let $\tilde{\mathbb{P}}$ be a risk neutral measure. Then the discounted stock prices $(\hat{S}_t^i)$ for $i = 1, \ldots, n$ are martingales under $\tilde{\mathbb{P}}$.*

Proposition 3.2.11. *Let $\phi \in \mathcal{T}$ be a trading strategy. Then the discounted value process $(\hat{V}_t(\phi))$ and the discounted gains process $\hat{G}_t(\phi)$ are martingales under any risk neutral measure $\tilde{\mathbb{P}}$.*

Theorem 3.2.12 (Fundamental Theorem of Asset Pricing)**.** *Given a general multi period market model, if there is a risk neutral measure, then there are no arbitrage strategies $\phi \in \mathcal{T}$. Conversely, if there are no arbitrages among self financing and adapted trading strategies, then there exists a risk neutral measure.*

Definition 3.2.13. We say that an adapted stochastic process (X_t) is a price process for the contingent claim X which **complies with the no arbitrage principle**, if there is no adapted and self financing arbitrage strategy in the extended model, which consists of the original stocks $(S_t^1), \ldots, (S_t^n)$ and an additional asset given by $S_t^{n+1} = X_t$ for $0 \leq t \leq T-1$ and $S_T^{n+1} = X$.

Proposition 3.2.14. *Let X be a possibly unattainable contingent claim and $\tilde{\mathbb{P}}$ a risk neutral measure for a general multi period market model. Then*

$$X_t = B_t \mathbb{E}_{\tilde{\mathbb{P}}}\left[\frac{X}{B_T} \,|\, \mathcal{F}_t\right]$$

defines a price for the contingent claim consistent with the no arbitrage principle.

Theorem 3.2.15. *Under the assumption that a general multi period market model is arbitrage free, it is complete, if and only if there is a unique risk neutral measure.*