

**Definition 1.5.1.**

Let  $A$  be a nonempty set of real numbers.

(a)  $M$  is the maximum of  $A$  if  $M$  is in  $A$  and  $x \leq M$  for all  $x$  in  $A$ .

(b)  $m$  is the minimum of  $A$  if  $m$  is in  $A$  and  $m \leq x$  for all  $x$  in  $A$ .

**Definition 1.5.2**

Let  $A$  be a set of real numbers.

(a) A number  $M$  is an upper bound for  $A$  if  $x \leq M$  for all  $x$  in  $A$ . We say that  $A$  is bounded above if it has an upper bound.

(b) A number  $m$  is a lower bound for  $A$  if  $x \geq m$  for all  $x$  in  $A$ . We say that  $A$  is bounded below if it has a lower bound.

(c)  $A$  is bounded if it is bounded both above and below.

**Definition 1.5.3**

Let  $A$  be a nonempty set of real numbers.

(a)  $\alpha$  in  $\mathbb{R}$  is a least upper bound for  $A$  if  $\alpha$  is an upper bound for  $A$  and  $\alpha \leq M$  for any upper bound  $M$  of  $A$ .

(b)  $B$  in  $\mathbb{R}$  is a greatest lower bound for  $A$  if  $B$  is a lower bound for  $A$  and  $B \geq m$  for any lower bound  $m$  of  $A$ .

**Theorem 1.5.4** If the set  $A \neq \emptyset$  has a least upper bound, it is unique, and similarly, if  $A$  has a greatest lower bound, it is unique.

**Notation:** We denote the least upper bound of a set  $A$  by  $\text{lub } A$ , and the greatest lower bound by  $\text{glb } A$ , whenever they exist. we will also use  $\text{lub}$  and  $\text{glb}$  as abbreviations for the phrases "least upper bound" and "greatest lower bound" respectively.

**Theorem 1.5.8.** For  $A \neq \emptyset$ ,  $\text{glb } A \leq \text{lub } A$  if they both exist.

**Theorem 1.5.9** If  $B \subset A$ , then  $\text{glb } A \leq \text{glb } B \leq \text{lub } B \leq \text{lub } A$ , provided all four exist.

**Theorem 1.5.10.**

(a) Let  $\alpha$  be an upper bound for  $A$ . Then  $\alpha = \text{lub } A$  if and only if for any  $u < \alpha$ , there is an  $x$  in  $A$  with  $u \leq x < \alpha$ .

(b) Let  $\beta$  be a lower bound for  $A$ . Then  $\beta = \text{glb } A$  if and only if for any  $\gamma > \beta$ , there is an  $x$  in  $A$  with  $\beta \leq x < \gamma$ .

**Least Upper Bound Axiom.** Every nonempty set of real numbers that is bounded above has a least upper bound.

**Corollary 1.5.11** Every nonempty set of real numbers that is bounded below has a greatest lower bound.

**Theorem 1.5.12**  $\mathbb{N}$  is not bounded above.

**Corollary 1.5.13** For every  $\varepsilon > 0$ , there is an  $n$  in  $\mathbb{N}$  with  $1/n < \varepsilon$ .

An equivalent result, provable in a similar way, states that if  $x$  and  $y$  are arbitrary real numbers With  $x > y$ , then there is an  $n$  from  $\mathbb{N}$  With  $y < nx$ . This is called the Archimedean property of the real numbers. we will not need this result; its proof is left for the exercises (Problem 15).

**Theorem 1.5.14** For every pair of numbers  $a$  and  $b$  with  $a < b$ , there is a rational number  $r$  such that  $a < r < b$ .

**Corollary 1.5.15** Every open interval  $(a, b)$  contains infinitely many rationals.

**Theorem 1.5.16** For every non-negative number  $a$ , there is a unique non-negative number  $b$  for which  $b^2 = a$ . (This justifies the definition of  $a$  in Section 1.3.)

**Theorem 1.5.17**  $\sqrt{2}$  is not rational.

**Corollary 1.5.18** Every open interval  $(a, b)$  contains infinitely many irrationals.

**Definition 1.5.19** Let  $A \subset \mathbb{R}$

(a) If  $A$  has no upper bound, then We say that  $\text{lub } A = \infty$ .

(b) If  $A$  has no lower bound, then we say that  $\text{glb } A = -\infty$

(c) We define  $\text{lub } \emptyset = -\infty$  and  $\text{glb } \emptyset = \infty$ .