Definition 1.5.1.
Let A be a nonempty set of real numbers.
(a) M is the maximum of A if M is in A and $x \leq M$ for all x in A.
(b) m is the minimum of A if m is in A and $m \leq x$ for all x in A.
Definition 1.5.2
Let A be a set of real numbers.
(a) A number M is an upper bound for A if $x \leq M$ for all x in A. We say that A is bounded above if it has an upper bound.
(b) A number m is a lower bound for A if $x \geq m$ for all x in A. We say that A is bounded below if it has a lower bound.
(c) A is bounded if it is bounded both above and below.
Definition 1.5.3
Let A be a nonempty set of real numbers.
(a) α in $\mathbb{R}$ is a least upper bound for A if α is an upper bound for A and $\alpha \leq M$ for any upper bound M of A.
(b) B in $\mathbb{R}$ is a greatest lower bound for A if B is a lower bound for A and $B \geq m$ for any lower bound m of A.
Theorem 1.5.4 If the set $A \neq \emptyset$ has a least upper bound, it is unique, and similarly, if A has a greatest lower bound, it is unique.
Notation: We denote the least upper bound of a set A by $lub\, A$, and the greatest lower bound by glb, whenever they exist. we will also use *lub* and *glb* as abbreviations for the phrases "least upper bound" and "greatest lower bound" respectively.
Theorem 1.5.8. For $A \neq \emptyset$, $glb\, A \leq lub\, A$ if they both exist.
Theorem 1.5.9 If $B \subset A$, then $glb\, A \leq glb\, B \leq lub\, B \leq lub\, A$, provided all four exist.
Theorem 1.5.10.
(a) Let α be an upper bound for A. Then $\alpha = lub\, A$ if and only if for any $u < \alpha$, there is an x in A with $u \leq x < \alpha$.
(b) Let β be a lower bound for A. Then $\beta = glb\, A$ if and only if for any $\gamma > \beta$, there is an x in A with $\beta \leq x < \gamma$.
Least Upper Bound Axiom. Every nonempty set of real numbers that is bounded above has a least upper bound.
Corollary 1.5.11 Every nonempty set of real numbers that is bounded below has a greatest lower bound.
Theorem 1.5.12 $\mathbb{N}$ is not bounded above.
Corollary 1.5.13 For every $\varepsilon > 0$, there is an n in $\mathbb{N}$ with $1/n < \varepsilon$.

An equivalent result, provable in a similar way, states that if x and y are arbitrary real numbers With $x > o$, then there is an n from N With $y < nx$. This is called the Archimedean property of the real numbers. we will not need this result; its proof is left for the exercises (Problem 15).
Theorem 1.5.14 For every pair of numbers a and b with $a < b$, there is a rational number r such that $a < r < b$.
Corollary 1.5.15 Every open interval (a, b) contains infinitely many rationals.
Theorem 1.5.16 For every non-negative number a, there is a unique non-negative number b for which $b^2 = a$. (This justifies the definition of a in Section 1.3.)
Theorem 1.5.17 $\sqrt{2}$ is not rational.
Corollary 1.5.18 Every open interval (a, b) contains infinitely many irrationals.
Definition 1.5.19 Let $A \subset B$
(a) If A has no upper bound, then We say that $lub\, A = \infty$.
(b) If A has no lower bound, then we say that $glb\, A = -\infty$
(c) We define $lub\, \emptyset = -\infty$ and $glb\, \emptyset = \infty$.