**Definition 1.5.1.**
Let $A$ be a nonempty set of real numbers.
(a) $M$ is the maximum of $A$ if $M$ is in $A$ and $x \leq M$ for all $x$ in $A$.
(b) $m$ is the minimum of $A$ if $m$ is in $A$ and $m \leq x$ for all $x$ in $A$.
**Definition 1.5.2**
Let $A$ be a set of real numbers.
(a) A number $M$ is an upper bound for $A$ if $x \leq M$ for all $x$ in $A$. We say that $A$ is bounded above if it has an upper bound.
(b) A number $m$ is a lower bound for $A$ if $x \geq m$ for all $x$ in $A$. We say that $A$ is bounded below if it has a lower bound.
(c) $A$ is bounded if it is bounded both above and below.
**Definition 1.5.3**
Let $A$ be a nonempty set of real numbers.
(a) $\alpha$ in $\mathbb{R}$ is a least upper bound for $A$ if $\alpha$ is an upper bound for $A$ and $\alpha \leq M$ for any upper bound $M$ of $A$.
(b) $B$ in $\mathbb{R}$ is a greatest lower bound for $A$ if $B$ is a lower bound for $A$ and $B \geq m$ for any lower bound $m$ of $A$.
**Theorem 1.5.4** If the set $A \neq \emptyset$ has a least upper bound, it is unique, and similarly, if $A$ has a greatest lower bound, it is unique.
**Notation:** We denote the least upper bound of a set $A$ by $lub\, A$, and the greatest lower bound by $glb$, whenever they exist. we will also use *lub* and *glb* as abbreviations for the phrases "least upper bound" and "greatest lower bound" respectively.
**Theorem 1.5.8.** For $A \neq \emptyset$, $glb\, A \leq lub\, A$ if they both exist.
**Theorem 1.5.9** If $B \subset A$, then $glb\, A \leq glb\, B \leq lub\, B \leq lub\, A$, provided all four exist.
**Theorem 1.5.10.**
(a) Let $\alpha$ be an upper bound for $A$. Then $\alpha = lub\, A$ if and only if for any $u < \alpha$, there is an $x$ in $A$ with $u \leq x < \alpha$.
(b) Let $\beta$ be a lower bound for $A$. Then $\beta = glb\, A$ if and only if for any $\gamma > \beta$, there is an $x$ in $A$ with $\beta \leq x < \gamma$.
**Least Upper Bound Axiom.** Every nonempty set of real numbers that is bounded above has a least upper bound.
**Corollary 1.5.11** Every nonempty set of real numbers that is bounded below has a greatest lower bound.
**Theorem 1.5.12** $\mathbb{N}$ is not bounded above.
**Corollary 1.5.13** For every $\varepsilon > 0$, there is an $n$ in $\mathbb{N}$ with $1/n < \varepsilon$.

An equivalent result, provable in a similar way, states that if $x$ and $y$ are arbitrary real numbers With $x > 0$, then there is an $n$ from $\mathbb{N}$ With $y < nx$. This is called the Archimedean property of the real numbers. we will not need this result; its proof is left for the exercises (Problem 15).
**Theorem 1.5.14** For every pair of numbers $a$ and $b$ with $a < b$, there is a rational number $r$ such that $a < r < b$.
**Corollary 1.5.15** Every open interval $(a, b)$ contains infinitely many rationals.
**Theorem 1.5.16** For every non-negative number $a$, there is a unique non-negative number $b$ for which $b^2 = a$. (This justifies the definition of $a$ in Section 1.3.)
**Theorem 1.5.17** $\sqrt{2}$ is not rational.
**Corollary 1.5.18** Every open interval $(a, b)$ contains infinitely many irrationals.
**Definition 1.5.19** Let $A \subset B$
(a) If $A$ has no upper bound, then We say that $lub\, A = \infty$.
(b) If $A$ has no lower bound, then we say that $glb\, A = -\infty$
(c) We define $lub\, \emptyset = -\infty$ and $glb\, \emptyset = \infty$.