

Definition 1. Zero-Divisors

A zero-divisor is a nonzero element a of a commutative ring R such that there is a nonzero element $b \in R$ with $ab = 0$.

Definition 2. Integral Domain

An integral domain is a commutative ring with unity and no zero-divisors.

EXAMPLE 1

The ring of integers is an integral domain.

EXAMPLE 2

The ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ is an integral domain.

EXAMPLE 3

The ring $\mathbb{Z}[x]$ of polynomials with integer coefficients is an integral domain.

EXAMPLE 4

The ring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is an integral domain.

EXAMPLE 5

The ring, \mathbb{Z}_p , of integers modulo a prime p is an integral domain.

EXAMPLE 6

The ring \mathbb{Z}_n of integers modulo n is not an integral domain when n is not prime.

EXAMPLE 7

The ring $M_2(\mathbb{Z})$ of 2×2 matrices over the integers is not an integral domain.

EXAMPLE 8

$\mathbb{Z}(G)$ is not an integral domain. Let a, b , and c belong to an integral domain. If $a \neq 0$ and $ab = ac$, then $b = c$.

Definition 3. Field

A field is a commutative ring with unity in which every nonzero element is a unit.

Theorem 1. A finite integral domain is a field.

Theorem 2. For every prime p , \mathbb{Z}_p , The ring of integers modulo p is a field.

Definition 4. Characteristic of a Ring

The characteristic of a ring R is the least positive integer n such that $nx = 0$ for all x in R . If no such integer exists, we say that R has characteristic 0. The characteristic of R is denoted by $\text{char}(R)$.

Theorem 3. Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then the characteristic of R is n .

Theorem 4. The characteristic of an integral domain is 0 or prime.