1 Solutions

54. As an exercise, I was asked to show $4x^2 + 6x + 3$ is a unit of $\mathbb{Z}_8[x]$, choosing our multiple as $ax + b$ we find

$$4\,a\,x^3 + 4\,b\,x^2 + 6\,a\,x^2 + 6\,b\,x + 3\,a\,x + 3\,b$$

which gives us the system of equations,

$$\begin{cases} 4\,a = 0 \\ 4\,b + 6\,a = 0 \\ 6\,b + 3\,a = 0 \\ 3\,b = 1 \end{cases}$$

and we can reduce the coefficients $a, b \mod {}_8$ to see $b = 3$ and $a = 2$, which shows $4\,x^2 + 6\,x + 3$ is a unit of $\mathbb{Z}_8[x]$

23. I was also asked to show determine the units of the Gaussian integers. to accomplish this I choose $a + bi, c + di \in \mathbb{Z}[i]$ and found,

$$a\,c - b\,d + (a\,d + b\,c)\;i$$

which implies that we have the following system of equations,

$$\begin{cases} a\,c - b\,d = 0 \\ a\,d + b\,c = 1 \end{cases}$$

These can be evaluated and parameterized as,

$$\begin{cases} a = \frac{t_1}{t_2^2 + t_1^2} \\ b = -\frac{t_2}{t_2^2 + t_1^2} \\ c = t_1 \\ d = t_2 \end{cases} \quad \begin{cases} a = \frac{1}{t_3} \\ b = 0 \\ c = t_3 \\ d = 0 \end{cases} \quad \begin{cases} a = 0 \\ b = -\frac{1}{t_4} \\ c = 0 \\ d = t_4 \end{cases}$$

Throwing out the first system, as $t_1^2 + t_2^2 \neq 1$ for any rational $t_{1,2}$, and so we wouldn't get an integer solution. We can determine that the units of $\mathbb{Z}[i]$ are $1, -1, i, -i$.