16 Chapter

Definition 16.1. *Ring of Polynomials over R*
Let R be a commutative ring. The set of formal symbols
$R[x] = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 | a_i \in R, n$ *is a nonnegative integer*$\}$
is called the ring of polynomials over R in the indeterminate x. Two elements
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *and*
$b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$
of $R[x]$ are considered equal if and only if $a_i = b_i$ for all nonnegative integers i. (Define $a_i = 0$ when $i > n$ and $b_i = 0$ when $i > m$.)

Definition 16.2. *Addition and Multiplication in $R[x]$*
Let R be a commutative ring and let
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *and*
$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ *belong to $R[x]$. Then*
$f(x) + g(x) = (a_s + b_s)x^s + (a_{s-1} + b_{s-1})x^{s-1} + \cdots + (a_1 + b_1)x + a_0 + b_0$,
where s is the maximum of m and n, $a_i = 0$ for $i > n$, and $b_i = 0$ for $i > m$. Also,
$f(x)g(x) = c_{m+n}x^{m+n} + c_{m+n-1}x^{m+n-1} + \cdots + c_1 x + c_0$, *where* $c_k = a_k b_0 + a_{k-1}b_1 + \cdots + a_1 b_{k-1} + a_0 b_k$ *for $k = 0, \ldots, m + n$.*

Theorem 16.1. *D an Integral Domain Implies $D[x]$ an Integral Domain*
If D is an integral domain, then $D[x]$ is an integral domain.

Theorem 16.2. *Division Algorithm for $F[x]$*
Let F be a field and let $f(x), g(x) \in F[x]$ with $g(x)$ at 0. Then there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $deg r(x) < deg g(x)$.

Corollary 16.3. *Remainder Theorem*
Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

Corollary 16.4. *Factor Theorem*
Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

Theorem 16.5. *Polynomials of Degree n Have at Most n Zeros*
A polynomial of degree n over a field has at most n zeros, counting multiplicity.

Definition 16.3. *Principal Ideal Domain (PID)*
A principal ideal domain is an integral domain R in which every ideal has the form $\langle a \rangle = \{ra | r \in R\}$ for some a in R.

Theorem 16.6. *$F[x]$ Is a PID*
Let F be a field. Then $F[x]$ is a principal ideal domain.

Theorem 16.7. *Criterion for $I = \langle g(x) \rangle$*
Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F[x]$. Then, $I = \langle g(x) \rangle$ if and only if $g(x)$ is a nonzero polynomial of minimum degree in I.