Comparison of two theorem provers: Isabelle & Coq

A. Yushkovskiy S. Tripakis

Department of Computer Science
School of Science
Aalto University

CS-E4000: Seminar in Computer Science
autumn 2017

Introduction

PRINCIPLES OF

MATHEMATICAL

LOGIC

BY

D. HILBERT AND W. ACKERMANN

TRANSLATED FROM THE GERMAN BY

LEWIS M. HAMMOND • GEORGE G. LECKIE • F. STEINHARDT

Professor of Philosophy, University of Virginia • Professor of Philosophy, Emory University • Columbia University

EDITED AND WITH NOTES BY

ROBERT E. LUCE

Assistant Professor of Mathematics

Rutgers University

PUBLISHED AND DISTRIBUTED IN THE PUBLIC INTEREST BY AUTHORITY OF THE ATTORNEY GENERAL UNDER LICENSE NUMBER A-1428

Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie [1]).

Von *L. E. J. Brouwer* in Amsterdam.

§ 1.

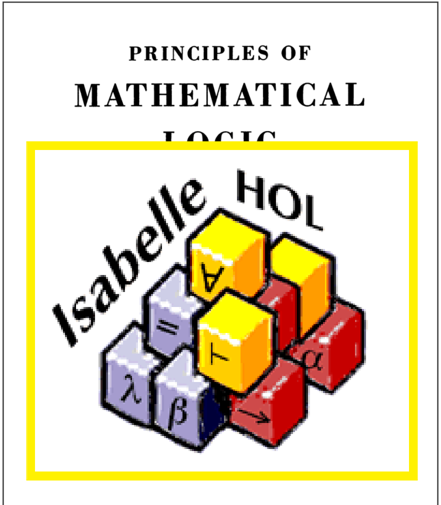
Innerhalb eines bestimmten endlichen „Hauptsystems" können Eigenschaften von Systemen, d. h. Abbildbarkeiten von Systemen auf andere Systeme mit vorgeschriebenen Elementkorrespondenzen, immer *geprüft* (d. h. entweder bewiesen oder ad absurdum geführt) werden; die durch die betreffende Eigenschaft angewiesene Abbildung besitzt nämlich auf jeden Fall nur eine endliche Anzahl von Ausführungsmöglichkeiten, von denen jede für sich unternommen und entweder bis zur Beendigung oder bis zur Hemmung fortgesetzt werden kann. (Hierbei liefert das Prinzip der mathematischen Induktion oft das Mittel, derartige Prüfungen ohne individuelle Betrachtung jedes an der Abbildung beteiligten Elementes bzw. jeder für die Abbildung bestehenden Ausführungsmöglichkeit durchzuführen; demzufolge kann die Prüfung auch für Systeme mit sehr großer Elementenzahl mitunter verhältnismäßig schnell verlaufen.)

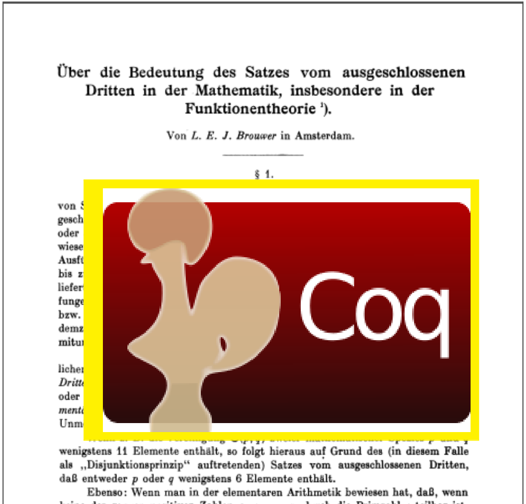
Auf Grund der obigen Prüfbarkeit gilt für innerhalb eines bestimmten endlichen Hauptsystems konzipierte Eigenschaften der *Satz vom ausgeschlossenen Dritten*, d. h. das Prinzip, daß jede Eigenschaft für jedes System entweder richtig oder unmöglich ist, und insbesondere der *Satz von der Reziprozität der Komplementärspezies*, d. h. das Prinzip, daß für jedes System aus der Unmöglichkeit der Unmöglichkeit einer Eigenschaft die Richtigkeit dieser Eigenschaft folgt.

Wenn z. B. die Vereinigung $\mathfrak{S}(p, q)$ zweier mathematischer Spezies p und q wenigstens 11 Elemente enthält, so folgt hieraus auf Grund des (in diesem Falle als „Disjunktionsprinzip" auftretenden) Satzes vom ausgeschlossenen Dritten, daß entweder p oder q wenigstens 6 Elemente enthält.

Ebenso: Wenn man in der elementaren Arithmetik bewiesen hat, daß, wenn

Introduction





Outline

Foundations of Formal Approach
- A formal system
- Classical and Intuitionistic logics

Two Theorem Provers
- Isabelle & Coq

Comparison of the theorem provers
- Comparison
- Proof examples

Elements of a Formal System

- A formula (judgement, statement) $\phi \in \Phi$:

$$\begin{aligned} \phi := p \mid q \mid \dots \\ \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \mid \neg\phi_1 \mid \phi_1 \to \phi_2 \\ \mid \mathit{true} \mid \mathit{false} \\ \mid \dots \end{aligned}$$

- Propositional variables: — $p, q, \dots \in V$
- An axiom — $\phi_A \in A$
- An inference rule τ — a transition function $\tau : \Phi \to \Phi$
- A formula ϕ provable from Φ — $\Phi \vdash \phi$
- A tautology $\top$ — $\vdash \phi$
- A contradiction $\bot$ — $\vdash \neg\phi$

Definition of the formal system

A *formal system* is a quadruple $\Gamma = < A, V, \Omega, R >$, where

- A – set of axioms
- V – set of propositional variables
- Ω – set of logical operators
- R – set of inference rules

A *formal proof* of the formula ϕ is a finite sequence of judgements

$$\psi_1 \xrightarrow{\tau_1} \psi_2 \xrightarrow{\tau_2} \dots \xrightarrow{\tau_n} \psi_n$$

where each ψ_i is either an axiom ϕ_{A_i}, or a formula inferred from the set of previously derived formulas according the rules of inference.

Properties of a formal system

A formal system Γ is called:

- *consistent*, if $\nexists\phi \in \Gamma : \ \Gamma \vdash \phi \land \Gamma \vdash \neg\phi \ \Leftrightarrow \ \Gamma \nvdash \bot$;
- *complete*, if $\forall\phi \in U : \ A \vdash \phi \lor A \vdash \neg\phi$;
- *independent*, if $\nexists a \in A : \ A \vdash a$.

Classical Logic

example: The Hilbert System

Set of axioms:

$$A \to (B \to A) \tag{A1}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{A2}$$

$$A \vee \neg A \tag{EM}$$

Single inference rule (*Modus Ponens*)

$$[\![A, A \to B]\!] \longrightarrow B \tag{MP}$$

Some provable tautologies:

$\neg\neg(A \vee \neg A)$	(nnEM)	$\neg(A \wedge B) \to \neg A \vee \neg B$	(DMdi)
$A \to \neg\neg A$	(DNi)	$\neg(A \vee B) \to \neg A \wedge \neg B$	(DMci)
$\neg\neg A \to A$	(DNe)	$\neg A \wedge \neg B \to \neg(A \vee B)$	(DMce)
$((A \to B) \to A) \to B$	(PL)	$\neg A \vee \neg B \to \neg(A \wedge B)$	(DMde)

Intuitionistic Logic

a.k.a. Constructive Logic

Set of axioms:

$$A \to (B \to A) \tag{A1}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{A2}$$

$$\cancel{A \vee \neg A} \tag{EM}$$

Single inference rule (*Modus Ponens*)

$$[\![A, A \to B]\!] \longrightarrow B \tag{MP}$$

Some provable tautologies:

$\neg\neg(A \vee \neg A)$	(nnEM)	$\cancel{\neg(A \wedge B) \to \neg A \vee \neg B}$	(DMdi)
$A \to \neg\neg A$	(DNi)	$\neg(A \vee B) \to \neg A \wedge \neg B$	(DMci)
$\cancel{\neg\neg A \to A}$	(DNe)	$\neg A \wedge \neg B \to \neg(A \vee B)$	(DMce)
$\cancel{((A \to B) \to A) \to B}$	(PL)	$\neg A \vee \neg B \to \neg(A \wedge B)$	(DMde)

Isabelle & Coq

First acquaintance

- based on **classical** higher-order logic
- created in 1986 at University of Cambridge and Technische Universität München
- uses functional language `HOL`
- has large collection of formalised theories

- based on **intuitionistic** logic (Calculus of Inductive Constructions)
- created in 1984 at INRIA (Paris, France)
- uses functional language `Gallina`
- has large collection of formalised theories

Isabelle & Coq

Definition of the basic datatypes

Isabelle & Coq

Definition of the basic datatypes

```
datatype bool =
  True | False

datatype nat =
  zero ("0") | Suc nat
```


A. Yushkovskiy, S. Tripakis

Isabelle & Coq

Definition of the basic datatypes

```
datatype bool =
  True | False

datatype nat =
  zero ("0") | Suc nat
```

```
Inductive False : Prop := .

Inductive True : Prop := I : True.

Inductive nat : Type :=
  | O : nat
  | S : nat -> nat.
```

Isabelle & Coq

Definition of a recursive function

A. Yushkovskiy, S. Tripakis

Isabelle & Coq

Definition of a recursive function

```
fun add ::
  "nat ⇒ nat ⇒ nat"
where
  "add 0 n = n"

 | "add (Suc m) n =
      Suc(add m n)"
```

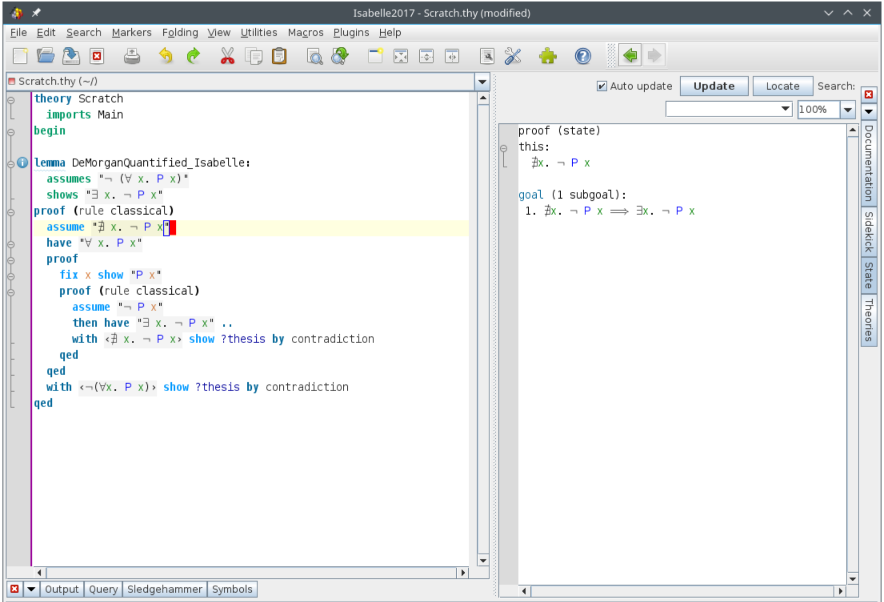
Isabelle & Coq

Definition of a recursive function

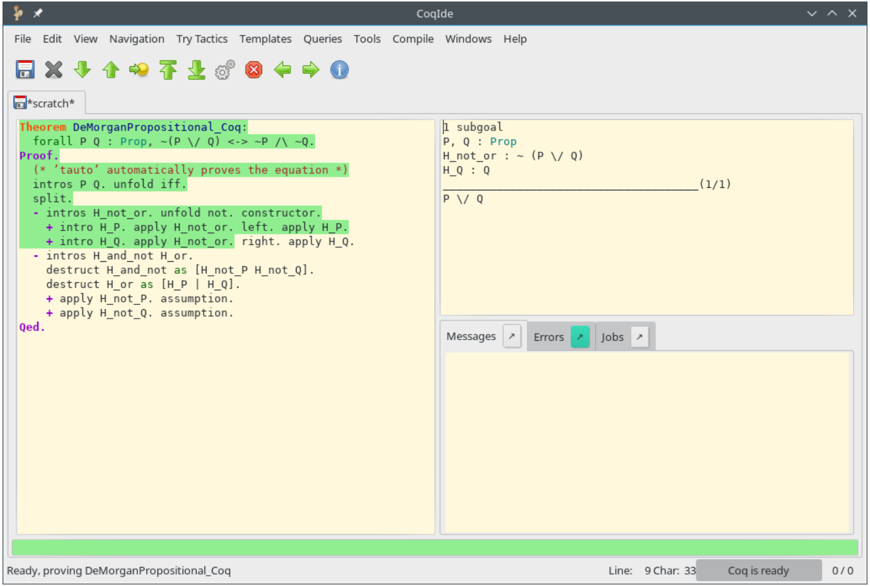
```
fun add ::
  "nat ⇒ nat ⇒ nat"
where
  "add 0 n = n"

  | "add (Suc m) n =
        Suc(add m n)"
```

```
Fixpoint add (n m: nat) : nat :=
  match n with
    | O => m
    | S n′ => S (n′ + m)
end

where "n + m" :=
  (add n m) : nat_scope.
```

Isabelle: Native graphical user interface



Coq: Native graphical user interface



Comparison

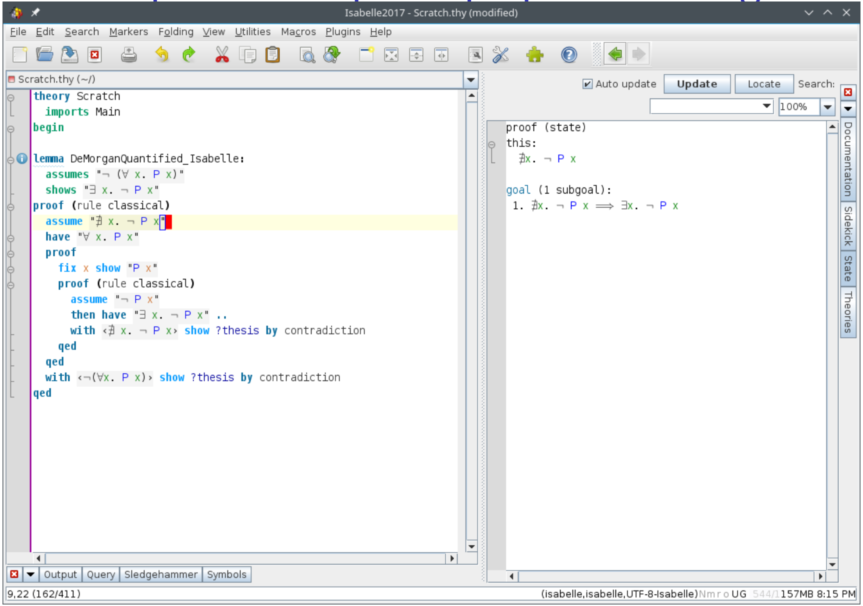
Major similarities:

- both work in a similar way of *verifying* the proof or *assisting* in creation of the new one
- *premises* $\xrightarrow{\textit{tactics}}$ *goals* (forward proof)
- *goals* $\xrightarrow{\textit{tactics}}$ *premises* (backward proof)
- both have large amount of libraries with formalised theories
- both dispose the set of highly automated tactics
- both are being actively developed these days

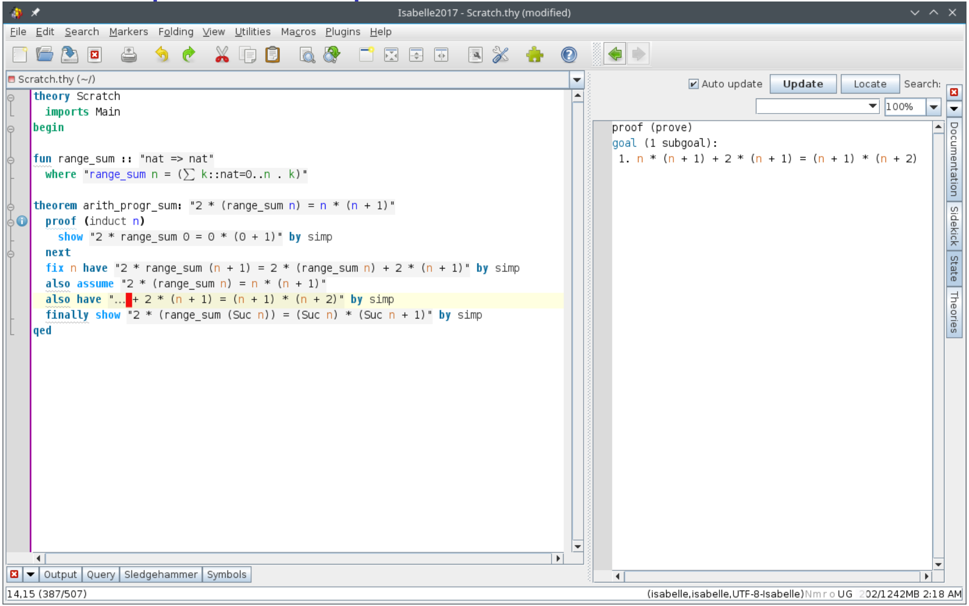
Major differences:

- based on different logics $\Rightarrow$
  - ⋆ unprovable statements and invalid proofs in Coq
  - ⋆ sometimes more complex proof in Coq
  - ⋆ *constructive* proof in Coq

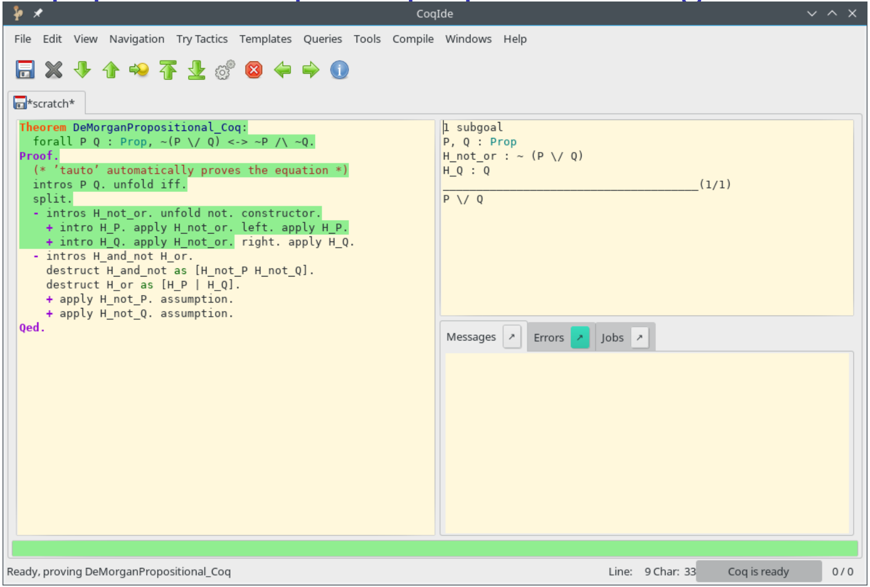
Isabelle: proof example in propositional logic



Isabelle: proof example over `nat`



Coq: proof example in propositional logic



Coq: proof example over `nat`

CoqIde

File Edit View Navigation Try Tactics Templates Queries Tools Compile Windows Help

scratch

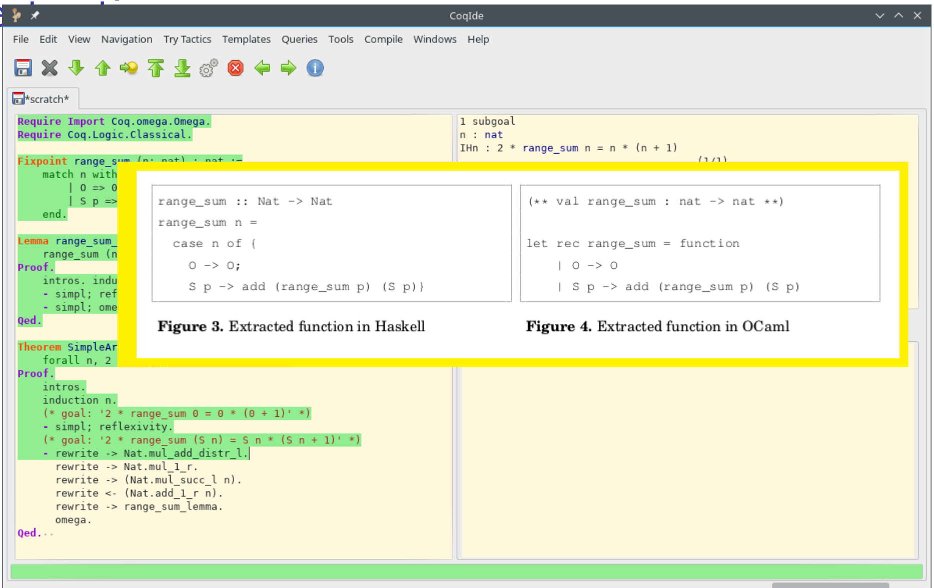
```
Require Import Coq.omega.Omega.
Require Coq.Logic.Classical.

Fixpoint range_sum (n: nat) : nat :=
    match n with
        | O => 0
        | S p => range_sum p + (S p)
    end.

Lemma range_sum_lemma: forall n: nat,
    range_sum (n + 1) = range_sum n + (n + 1).
Proof.
    intros. induction n.
    - simpl; reflexivity.
    - simpl; omega.
Qed.

Theorem SimpleArithProgressionSumFormula_Coq:
    forall n, 2 * range_sum n = n * (n + 1).
Proof.
    intros.
    induction n.
    (* goal: '2 * range_sum 0 = 0 * (0 + 1)' *)
    - simpl; reflexivity.
    (* goal: '2 * range_sum (S n) = S n * (S n + 1)' *)
    - rewrite -> Nat.mul_add_distr_l.
      rewrite -> Nat.mul_1_r.
      rewrite -> (Nat.mul_succ_l n).
      rewrite <- (Nat.add_1_r n).
      rewrite -> range_sum_lemma.
      omega.
Qed.
```

```
1 subgoal
n : nat
IHn : 2 * range_sum n = n * (n + 1)
______________________________________(1/1)
2 * range_sum (S n) = S n * S n + S n * 1
```

Messages Errors Jobs

Ready, proving SimpleArithProgressionSumFormula_Coq   Line: 26 Char: 38   Coq is ready   0 / 0

Coq: proof example over `nat` + verified code e




```
range_sum :: Nat -> Nat
range_sum n =
  case n of {
    O -> O;
    S p -> add (range_sum p) (S p)}
```

Figure 3. Extracted function in Haskell

```
(** val range_sum : nat -> nat **)

let rec range_sum = function
  | O -> O
  | S p -> add (range_sum p) (S p)
```

Figure 4. Extracted function in OCaml

Summary

- Two widespread theorem provers were considered: Isabelle and Coq
- The key difference between them lie in differences between logical theories they based on
- Nonetheless, they both may be used to solve applied problems, such as software testing and verification