Comparison of two theorem provers: Isabelle & Coq

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Outline

Elements of a Formal System

- A formula (judgement, statement) $\phi \in \Phi$:

$$
\begin{aligned}
\phi := p \mid q \mid \ldots \\
&\mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \neg\phi_1 \mid \phi_1 \rightarrow \phi_2 \\
&\mid \mathit{true} \mid \mathit{false} \\
&\mid \ldots
\end{aligned}
$$

- Propositional variables: — $p, q, \ldots \in V$
- An axiom — $\phi_A \in A$
- An inference rule τ — a transition function $\tau : \Phi \rightarrow \Phi$
- A formula ϕ provable from Φ — $\Phi \vdash \phi$
- A tautology $\top$ — $\vdash \phi$
- A contradiction $\perp$ — $\vdash \neg\phi$

Definition of the Formal System

A *formal system* is a quadruple $\Gamma = < A, V, \Omega, R >$, where

- A – set of axioms
- V – set of propositional variables
- Ω – set of logical operators
- R – set of inference rules

A *formal proof* of the formula ϕ is a finite sequence of judgements

$$\psi_1 \xrightarrow{\tau_1} \psi_2 \xrightarrow{\tau_2} \dots \xrightarrow{\tau_n} \psi_n$$

where each ψ_i is either an axiom ϕ_{A_i}, or a formula inferred from the set of previously derived formulas according the rules of inference.

Classical Logic

example: The Hilbert System

Set of axioms:

$$A \rightarrow (B \rightarrow A) \tag{A1}$$

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \tag{A2}$$

$$A \vee \neg A \tag{EM}$$

Single inference rule (*Modus Ponens*)

$$[\![A, A \rightarrow B]\!] \longrightarrow B \tag{MP}$$

Some provable tautologies:

$\neg\neg(A \vee \neg A)$	(nnEM)	$\neg(A \wedge B) \rightarrow \neg A \vee \neg B$	(DMdi)
$A \rightarrow \neg\neg A$	(DNi)	$\neg(A \vee B) \rightarrow \neg A \wedge \neg B$	(DMci)
$\neg\neg A \rightarrow A$	(DNe)	$\neg A \wedge \neg B \rightarrow \neg(A \vee B)$	(DMce)
$((A \rightarrow B) \rightarrow A) \rightarrow B$	(PL)	$\neg A \vee \neg B \rightarrow \neg(A \wedge B)$	(DMde)

Intuitionistic Logic

a.k.a. Constructive Logic

Set of axioms:

$$A \to (B \to A) \tag{A1}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{A2}$$

$$\cancel{A \vee \neg A} \tag{EM}$$

Single inference rule (*Modus Ponens*)

$$[\![A, A \to B]\!] \longrightarrow B \tag{MP}$$

Some provable tautologies:

$\neg\neg(A \vee \neg A)$	(nnEM)	$\cancel{\neg(A \wedge B) \to \neg A \vee \neg B}$	(DMdi)
$\cancel{A \to \neg\neg A}$	(DNi)	$\neg(A \vee B) \to \neg A \wedge \neg B$	(DMci)
$\neg\neg A \to A$	(DNe)	$\neg A \wedge \neg B \to \neg(A \vee B)$	(DMce)
$\cancel{((A \to B) \to A) \to B}$	(PL)	$\neg A \vee \neg B \to \neg(A \wedge B)$	(DMde)

Curry-Howard Correspondence

The direct relationship between computer programs and mathematical proofs. [To be done]

Isabelle: First Acquaintance

- a generic proof assistant
- based on classical higher-order logic
- created in 1986 by
  - Larry Paulson @ University of Cambridge, and
  - Tobias Nipkow @ Technische Universität München
- uses powerful functional language HOL
- the proof system core *Isabelle* is extended by various theories: Isabelle/HOL, Isabelle/ZF, Isabelle/CCL, etc.

Example 1: Definition of basic datatypes

```
datatype bool =
  True | False

datatype nat =
  zero ("0") | Suc nat
```

Example 2: Definition of addition over nat

```
fun add :: "nat ⇒ nat ⇒ nat"
  where
    "add 0 n = n" |
    "add (Suc m) n = Suc(add m n)"
```

Coq: First Acquaintance

- a formal proof management system
- based intuitionistic logic (Calculus of Inductive Constructions)
- created at INRIA (Paris, France) in 1984
- uses powerful functional language Gallina
- has large collection of formalised theories
- widely used in software verification (proof code extraction)

Example 3: Definition of basic datatypes

```
Inductive False : Prop := .

Inductive True : Prop := I : True.

Inductive nat : Type :=
  | O : nat
  | S : nat -> nat.
```

Example 4: Definition of addition over nat

```
Fixpoint add (n m: nat) : nat :=
  match n with
    | O ⇒ m
    | S n′ ⇒ S (n′ + m)
  end
where "n + m" :=
  (add n m) : nat_scope.
```

Comparison

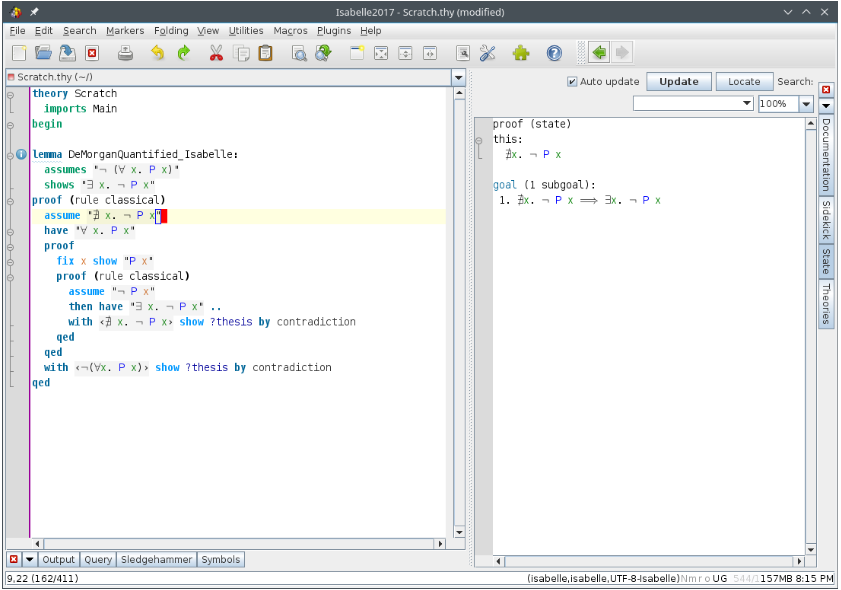
Major similarities:

- both work in a similar way of *verifying* the proof or *assisting* in creation of the new one
- $premises \xrightarrow{tactics} goals$ (forward proof)
- $goals \xrightarrow{tactics} premises$ (backward proof)
- both have large amount of libraries with formalised theories
- both dispose the set of highly automated tactics
- both are being actively developed these days

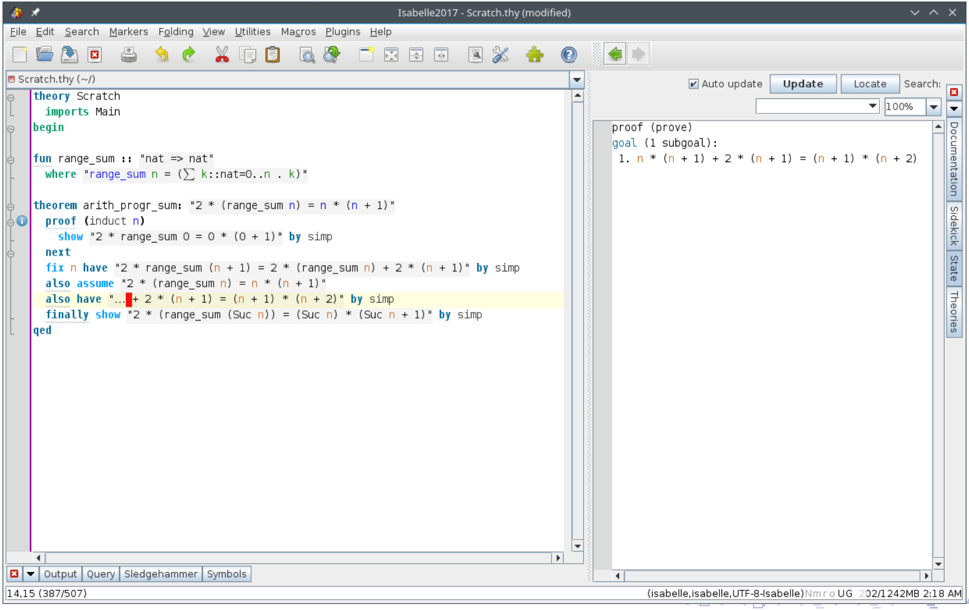
Major differences:

- based on different logics $\Rightarrow$
  - unprovable statements and invalid proofs in Coq
  - sometimes more complex proof in Coq
  - *constructive* proof in Coq

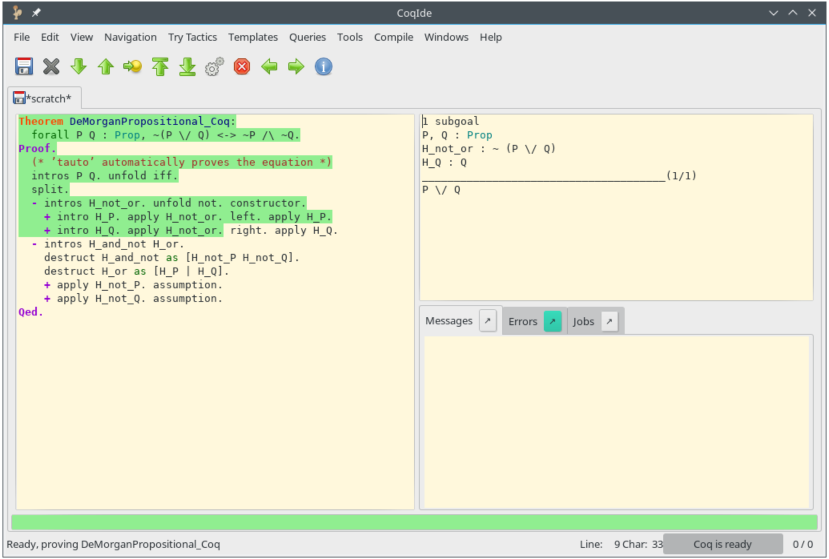
Isabelle: Proof Example in Propositional Logic



Isabelle: Proof Example over nat



Coq: Proof Example in Propositional Logic



Coq: Proof Example over `nat`

CoqIde

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scratch

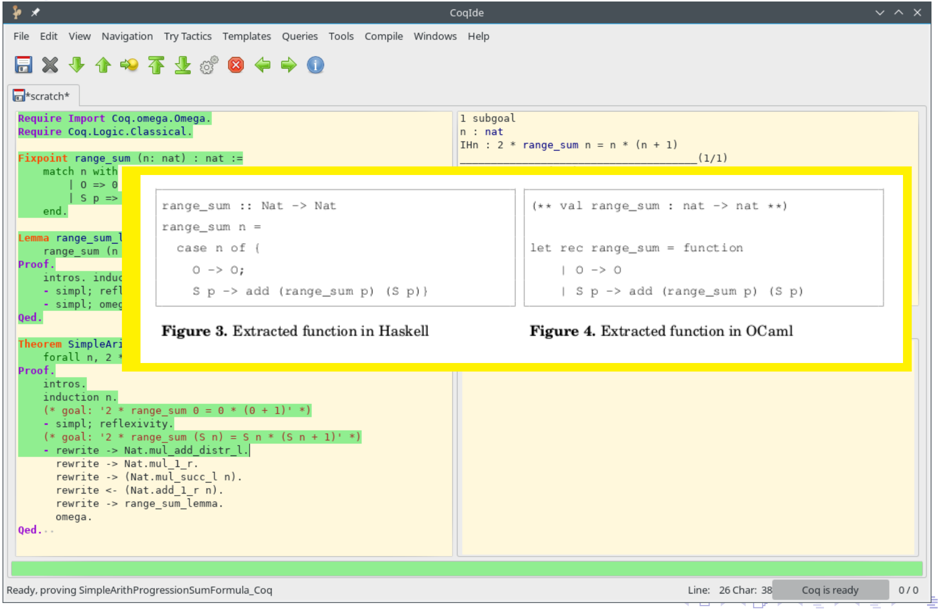
```
Require Import Coq.omega.Omega.
Require Coq.Logic.Classical.

Fixpoint range_sum (n: nat) : nat :=
    match n with
        | 0 => 0
        | S p => range_sum p + (S p)
    end.

Lemma range_sum_lemma: forall n: nat,
    range_sum (n + 1) = range_sum n + (n + 1).
Proof.
    intros. induction n.
    - simpl; reflexivity.
    - simpl; omega.
Qed.

Theorem SimpleArithProgressionSumFormula_Coq:
    forall n, 2 * range_sum n = n * (n + 1).
Proof.
    intros.
    induction n.
    (* goal: '2 * range_sum 0 = 0 * (0 + 1)' *)
    - simpl; reflexivity.
    (* goal: '2 * range_sum (S n) = S n * (S n + 1)' *)
    - rewrite -> Nat.mul_add_distr_l.
      rewrite -> Nat.mul_1_r.
      rewrite -> (Nat.mul_succ_l n).
      rewrite <- (Nat.add_1_r n).
      rewrite -> range_sum_lemma.
      omega.
Qed.
```

```
1 subgoal
n : nat
IHn : 2 * range_sum n = n * (n + 1)
______________________________________(1/1)
2 * range_sum (S n) = S n * S n + S n * 1
```

Messages Errors Jobs

Ready, proving SimpleArithProgressionSumFormula_Coq   Line: 26 Char: 38   Coq is ready   0 / 0

Coq: Proof Example over `nat` + Verified Code Extraction




```
range_sum :: Nat -> Nat
range_sum n =
  case n of {
    O -> O;
    S p -> add (range_sum p) (S p)}
```

Figure 3. Extracted function in Haskell

```
(** val range_sum : nat -> nat **)

let rec range_sum = function
    | O -> O
    | S p -> add (range_sum p) (S p)
```

Figure 4. Extracted function in OCaml

Summary

- Two widespread theorem provers were considered: Isabelle and Coq
- The key difference between them lie in differences between logical theories they based on
- Nonetheless, they both may be used to solve applied problems, such as software testing and verification