Numerical solutions for G^2 Hermite interpolation problem with spirals

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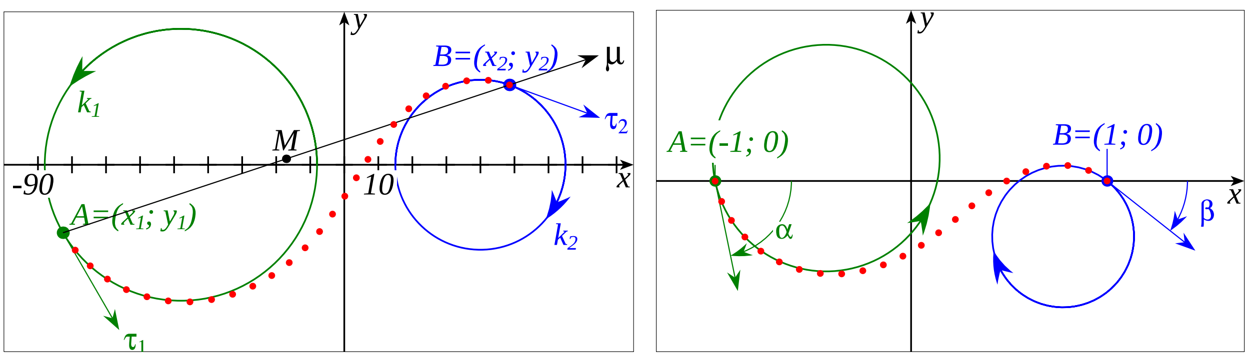
This project is intended to provide a numerical solution (or several solutions) of the **two-point G^2 Hermite interpolation problem with spirals**.
I. e., the *transition curve*, joining two given points A and B, is constructed, matching given tangents and curvatures at A and B (Figure 1). *Spirality* means the monotonicity of the Chesaro equation of the transition curve: function

$$k(s) \equiv \tau'_s(s)$$

is monotonous (k being curvature, s arc length, and τ the direction of the tangent to the curve).



VogtData

Figure 1. Example of 2-point G^2 Hermite data (boundary conditions).

The left picture in Figure 1 shows the example of 2-point G^2 Hermite data as:
point $A = (x_1, y_1)$, tangent τ_1 and curvature k_1 at A,
point $B = (x_2, y_2)$, tangent τ_2 and curvature k_2 at B. Namely,

$$\begin{array}{lllll} & x_1 = -82.64, & y_1 = -20., & \tau_1 = 300.0°, & k_1 = 0.025; \\ & x_2 = 48.55, & y_2 = 23.5, & \tau_2 = -20.0°, & k_2 - 0.04; \\ \text{let} & c = \frac{1}{2}\,|AB|\,, & \mu = \arg\left[x_2 - x_1 + \mathrm{i}(y_2 - y_1)\right]. & & \end{array}$$

Dotted curve is a an example of a spiral, matching this data. Its curvature is decreasing ($k_1 > k_2$), and has an inflection (due to $k_1 k_2 < 0$).

Normalization of given G^2 data. The right picture in Figure 1 shows the same data brought to *normalized position* by

- moving point $M = (A + B)/2$ to the origin;
- rotating by the angle $-\mu$ to align the chord $\overrightarrow{AB}$ with the x-axis;

- scaling (homothety) by the factor c^{-1} such that $A \to (-1;\,0)$, $B \to (1;\,0)$.

In this coordinate system the boundary conditions are transformed to:

$$\begin{array}{lll} (x_1,y_1) \to (-1;0), & \tau_1 \to \alpha = \tau_1 - \mu, & k_1 \to a = k_1 c; \\ (x_2,y_2) \to (+1;0), & \tau_2 \to \beta = \tau_2 - \mu, & k_2 \to b = k_2 c. \end{array}$$

The program shows solutions in both original (Page 1) and normalized coordinate systems (Page 2 and pages for every solution found). Additional transformation is used in calculations, transforming the case of decreasing curvature into increasing one by the the symmetry about the x-axis.

Existence of solutions. The above transformations (as well as Möbius maps) do no affect the value of

$$Q = (k_1 c + \sin\alpha)(k_2 c - \sin\beta) + \sin^2\frac{\alpha+\beta}{2}.$$

- A spiral arc (non-biarc), matching given 2-point G^2 data exists iff $Q < 0$.
- Let $\alpha, \beta \in (-\pi;\,\pi]$, if $k_1 < k_2$ (increasing curvature), or, otherwise, $\alpha, \beta \in [-\pi;\,\pi)$ $(k_1 > k_2)$. The proposed algorithm finds solutions if the following conditions are satisfied: $0 < |\alpha+\beta| < 2\pi$ or $0 < |\alpha+\beta \pm 2\pi| < 2\pi$ CHECK!!!.

Brief description of the method ...[1] (ref. to arxiv)

1. Running the program

The program is written in PostScrpit language. You need some PostScript (GhostScript) interpreter to be inslalled. Under Linux it is usually `gv` or `gs`. Under Windows it is `gsview` or console application `gswin32c`.

You have to edit a few lines in the file `G2spiral.ps` to include your own boundary conditions (user data), and run the program as

```
gv G2spiral.ps
```

Example of user input to construct the spiral in Figure 1 (red dotted curve):

```
<< /UserG2Data [/XYTK8 -82.64 -20. 300  0.025  48.55 23.5 -20.0 -0.04 ] >>
                /method   x1    y1  tau1    k1     x2   y2  tau2   k2
```

You can also keep the user data in a separate file, say, `UserData.ps`, and run the program as

```
gv     -arg="-sFname=/home/ak/G2spiral/UserData.ps" G2spiral.ps
gs -dNOSAFER -sFname=/home/ak/G2spiral/UserData.ps  G2spiral.ps
```

(comments to NOSAFER ...)

Conversion to pdf by ImageMagick `convert` utility (requires GhostScript):

```
convert G2spiral.ps G2spiral.pdf
```

(see `doc/Example-Converted_to_pdf.pdf`).

2. Setting user data

A few comments on PS syntax. Only data types, used in setting user data, are briefly commented below.

integer, real: as in all other languages: `1  1.  -3.14159`;

bool: `true  false`

string: string `"some text"` should be written in parenthesis as `(some text)`;

name: name, starts with `/`, e. g. `/OutputFile`;

array: `[el_0  el_1  el_2 ... ]`; array elements may be of different types;

dictionary: `<< /key1 value1   /key2 value2  /key3 value3 ... >>`.

Comment starts with `%`-sign:

```
% user data to approximate an arc of logarithmic spiral
% r(phi) = exp(phi*cot(nu)), 0 <= phi <= Phi
 [/LogSpiral2  80 180]         % nu = 80,  Phi = 180
```

The angles are given in degrees.

Example of a file with user data. User data is stored as a dictionary like

```
<<
  /UserG2Data  [/XYTK8 -82.64 -20. 300  0.025  48.55 23.5 -20.0 -0.04 ]
  /UserPhiData [...]
  /Margin      30
  /OutputFile  (D:/tmp/VogtSpiral.txt)    % ouputs the curve as X Y pairs
>>
```

Only the first key-value pair, `/UserG2Data [ array ]`, is obligatory. Its possible versions are described below. The 1-st element of the array is some `/method`, followed by 2–8 numeric arguments.

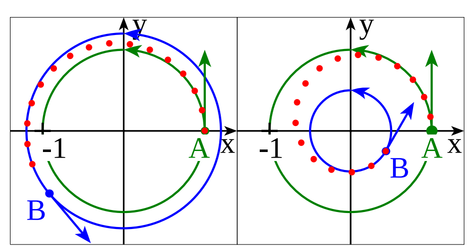
1. `[/XYTK8   x1  y1  tau1   k1    x2   y2  tau2   k2]`
   Direct setting of boundary conditions. 8 arguments are $x_1, y_1, \tau_1, k_1,\quad x_2, y_2, \tau_2, k_2$.

2. `[/Norm4 alpha k1   beta k2]`
   Data normalized to the unit chord $(x_1, y_1) = (-1, 0)$, $\tau_1 = \alpha$, $(x_2, y_2) = (1, 0)$, $\tau_2 = \beta$. 4 arguments, following the method, are α, k_1, β, k_2.

3. `[/Conc2 r2 phi2]`
   Concentric boundary conditions, $r_1 = 1$, $\varphi_1 = 0$, $\pi < \varphi_2 \leqslant 2\pi$, $r_2 \neq 1$:

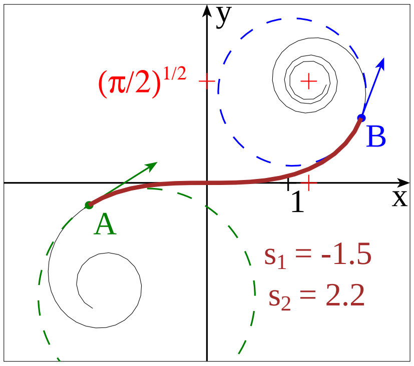
$$A = (x_1, y_1) = (1, 0), \qquad \tau_1 = \frac{\pi}{2}, \qquad k_1 = 1,$$

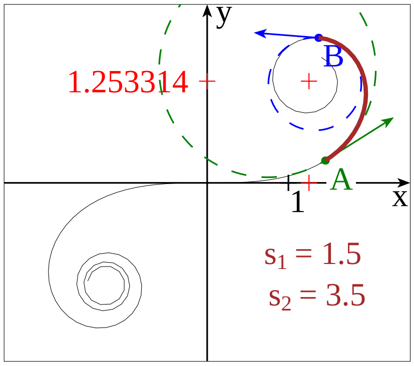
$$B = (x_2, y_2) = \begin{pmatrix} r_2 \cos\varphi_2 \\ r_2 \sin\varphi_2 \end{pmatrix}, \quad \tau_2 = \varphi_2 + \frac{\pi}{2}, \qquad k_2 = \frac{1}{r_2}.$$



4. [/Ell2 a b]
   Boundary coonditions are those of...

5. [/Cornu3 s1 s2 f]
   Boundary coonditions are borrowed from Cornu spiral with $k(s) = \frac{1}{2}s$, $s_1 \leqslant s \leqslant s_2$:





   Both the arc of the Cornu spiral and the approximating curve will be shown in the output. f parameter ?

6. [/LogSpir2 nu phi2]
   Boundary coonditions are borrowed from...

7. [/ABpp4 alpha beta p1 p2]
   (bilens parameters; for internal use).

The entry /UserPhiData [...] serves to select solutions from the whole family of solutions, thus controlling the number of solutions shown. If absent, default /UserPhiData [0.] is assumed. Possible values are:

/UserPhiData N, N integer. Shows solutions for $2N + 1$ possible values of the family parameter Φ ($2N + 3$?). For $N = 3$:
$\{-\Phi_{max}, -\Phi_2, -\Phi_1, \Phi_0 = 0., \Phi_1, \Phi_2, \Phi_{max}\}$.

/UserPhiData [ ] (empty array)– as the previous one with some automatic selection of N. This is the default behavior if this entry is absent in the user data dictionary.

/UserPhiData step, step is real, in degrees (e. g., 5. , not 5). Shows solutions for $\Phi \in \{0., \pm\text{step}, \pm 2 \cdot \text{step}, \pm 3 \cdot \text{step}, \ldots, \pm\Phi_{max}\}$.

/UserPhiData [Phi1 Phi2 N], N integer. Shows N solutions within the range $[\Phi_1; \Phi_2]$.

/UserPhiData [Phi1 Phi2 Phi3 ... PhiN] (reals) – list of desired family parameters.

Other options are listed below (add comments!):

```
/Margin  real    % margins in mm
/Margin  int     % margins in pt: /Margin 72 corresponds to 1 inch.
```

Note that A4 paper is 210 x 297 mm, or 595 x 842 pt.

```
/OutputFile    string  %  e.g., (/home/ak/tmp/mySpiral_XY.txt)
/BlackWhite    false
/BaseLineSkip  20
/ShowAll       true  % Show page for every solution found
/MaxLength     1000. % Reject too long curves (usually applied to curves with inflecti
```

About PostScript error messages. ...

Error messages due to PostScript syntax violation...

Errors due to forbidden output to disk...

Output. Output curve file is formatted as PS array for every solution found:
`[ x1 y1  x2 y2 ... xN yN ]`
Example:

```
% Phi: -45.0  Length: 4.08248
[
 -0.993767 -0.0829621 -0.987961 -0.0814893 -0.976391 -0.0786075 -0.953409 -0.0731016
 -0.908072 -0.0630687 -0.819742 -0.0465941 -0.651231 -0.0257036 -0.491357 -0.0172414
 .........................
 1.92118 0.795525 1.94762 0.903802 1.96055 1.00681 1.96188 1.10303
 1.95365 1.1915 1.93781 1.27174 1.91616 1.34367 1.86147 1.46366
 1.79837 1.55647
]
```

References

[1] *Kurnosenko A.I.* Two-point G^2 Hermite interpolation with spirals by inversion of hyperbola // Comp. Aided Geom. Design. 2010. V. 27. P. 474–481; ISSN 0167-8396, (http://www.sciencedirect.com/science/article/pii/S0167839610000270)