Chapter 4: Image Enhancement in the Frequency Domain

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4.1 Background

- The concept of frequency/spectrum analysis is originated from the French mathematician Jean Baptiste Joseph Fourier's work
  - Any function that periodically repeats itself can be expressed as the sum of a number of sinusoidal functions of different frequencies, each multiplied by a different coefficient.

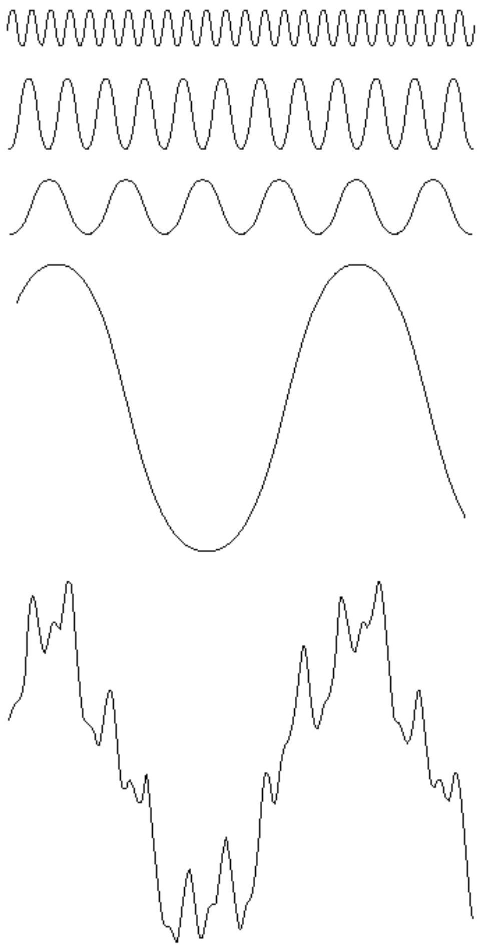


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Observations

- $g(t)=c_1f_1(t)+c_2f_2(t)+c_3f_3(t)+c_4f_4(t)$
- $f_i(t)$ has the form of either $\cos(2\pi f_i t+\Theta)$ or $\sin(2\pi f_i t+\Theta)$
- If the frequency of $g(t)$ is f_0, then f_i is a multiple of f_0.
  - They are harmonically related
  - Each component function repeats itself an integer number of times within one-period of the function $g(t)$

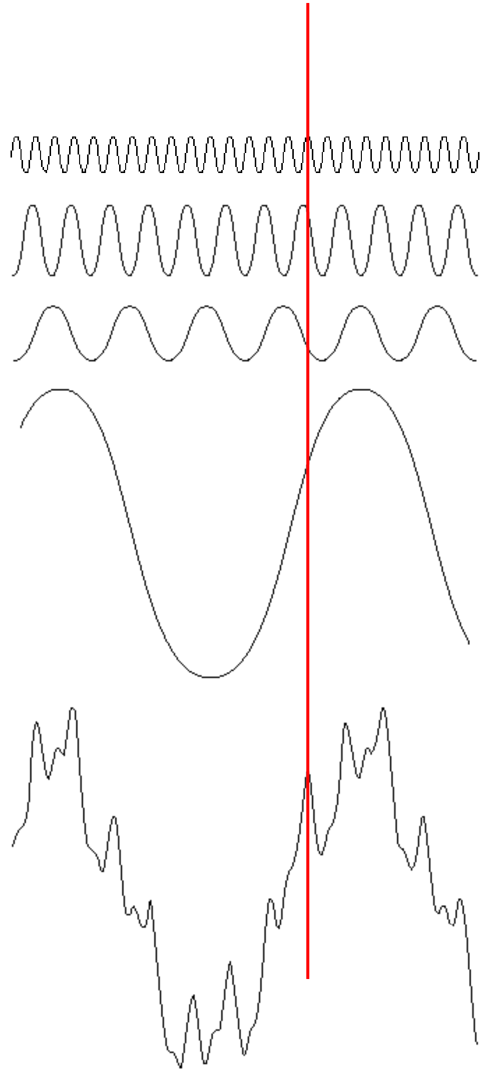


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

4.2.1 The one-Dimensional Fourier Transform and its Inverse

- The Fourier transform *F(u)* of a single variable continuous function *f(x)*, is defined

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx, \qquad \text{where } \mathrm{j} = \sqrt{-1}$$

- Obtain *f(x)* by inverse Fourier transform (IFT)

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} dx$$

- Extend to two variables, *u* and *v*

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

4.2.1 The one-Dimensional Fourier Transform and its Inverse

- The discrete Fourier transform (DFT) is defined

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}, \qquad \text{for u} = 0,1,2,\cdots,\text{M - 1}$$

- inverse discrete Fourier transform (IDFT)

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \qquad \text{for x} = 0,1,2,\cdots,\text{M - 1}$$

- Substitute this by Euler's formula:

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad (4.2\text{ - }7)$$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux/M - j\sin 2\pi ux/M], \qquad \text{for u} = 0,1,2,\cdots,\text{M - 1}$$

Interpretation

- $$f(x)=\sum_{u=0}^{M-1}F(u)e^{j2\pi ux/M}\ ,x=0,...,M-1$$
  - IDFT says that any sequence of M points can be decomposed as a weighted sum of M basic sequences of M points called basis functions
  - { $e^{j2\pi ux/M}, u=0,...,M-1$ } is called a basis of the space consisting of any sequence of M points
- $$F(u)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)e^{-j2\pi ux/M}, u=0,...,M-1$$
  - DFT says that the weight associated with the basis function $e^{j2\pi ux/M}$ is determined by the inner product of the sequence f(x) and the basis function

Useful Properties

- In the analysis of complex numbers, we express F(u) in polar coordinates:

$$F(u) = |F(u)| e^{-j\phi(u)}$$

$$|F(u)| = [R^2(u) + I^2(u)]^2 \qquad \text{(4.2-9) - magnitude (spectrum) of the FT}$$

- Phase angle or phase spectrum:

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

- Power spectrum:

$$F(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

Spectrum

- If we can express the complex valued F(u) (= R(u) + j I(u)) in polar form, i.e.,

$$F(u) = |F(u)| e^{j\phi(u)}$$

  - The term |F(u)| is called the magnitude or spectrum of the Fourier transform
  - The term $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$ is called the phase angle or phase spectrum of the Fourier transform
  - The square of the spectrum is called the power spectrum or spectral density of the Fourier transform

Frequency Domain Representation

- From $$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}, \; x = 0,\ldots,M-1$$
  $$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, u = 0,\ldots,M-1$$

  We know that the signal f(x) can be reconstructed from F(u) completely
  - A signal can be represented in time/space domain as well as in frequency domain
  - We are going to study image enhancement methods in frequency domain

4.2.2 The two-dimensional DFT and its Inverse (IDFT)

- 2D discrete Fourier transform (DFT) of an image f(x,y) of size M×N

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$u=0,\ldots,M-1 \text{ and } v=0,\ldots,N-1$$

- 2D inverse discrete Fourier transform (IDFT)

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$x=0,\ldots,M-1 \text{ and } y=0,\ldots,N-1$$

Spectrum, Phase (Angle), Power Spectrum

- $$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|\, e^{j\phi(u,v)}$$
  - Spectrum: $|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$
  - Phase:
    $$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$
  - Power spectrum:
    $$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$
  - For visualization purpose, spectrum or power spectrum is usually displayed

Properties

- Translation property
  - $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$
  - $F(u-u_0, v-v_0) \Leftrightarrow f(x,y)e^{i2\pi(u_0x/M+v_0y/N)}$
- Shifting f(x) by x_0, equivalently, each basis function $e^{j2\pi ux/M}$ has to shift by x_0.
  - f(x) ➔ f(x-x_0) ===> $e^{j2\pi ux/M}$ ➔ $e^{j2\pi u(x-x_0)/M}$

$$= e^{j(2\pi ux/M - 2\pi ux_0/M)}$$

$$= e^{j2\pi ux/M} \cdot e^{-j2\pi ux_0/M}$$

Shifting each basis function by x_0 is equal to adding a phase by the amount of $-2\pi ux_0/M$,

The added amount is linearly proportional to the frequency u.

Properties

- Linearity
  - $\Im[af_1 + bf_2] = a\Im[f_1] + b\Im[f_2]$
- Scaling
  - $f(ax, by) \Leftrightarrow \frac{1}{|ab|} F(u/a, v/b)$
- Rotation
  - Let $x = r\cos\theta, y = r\sin\theta, u = \omega\cos\varphi \text{ and } v = \omega\sin\varphi$

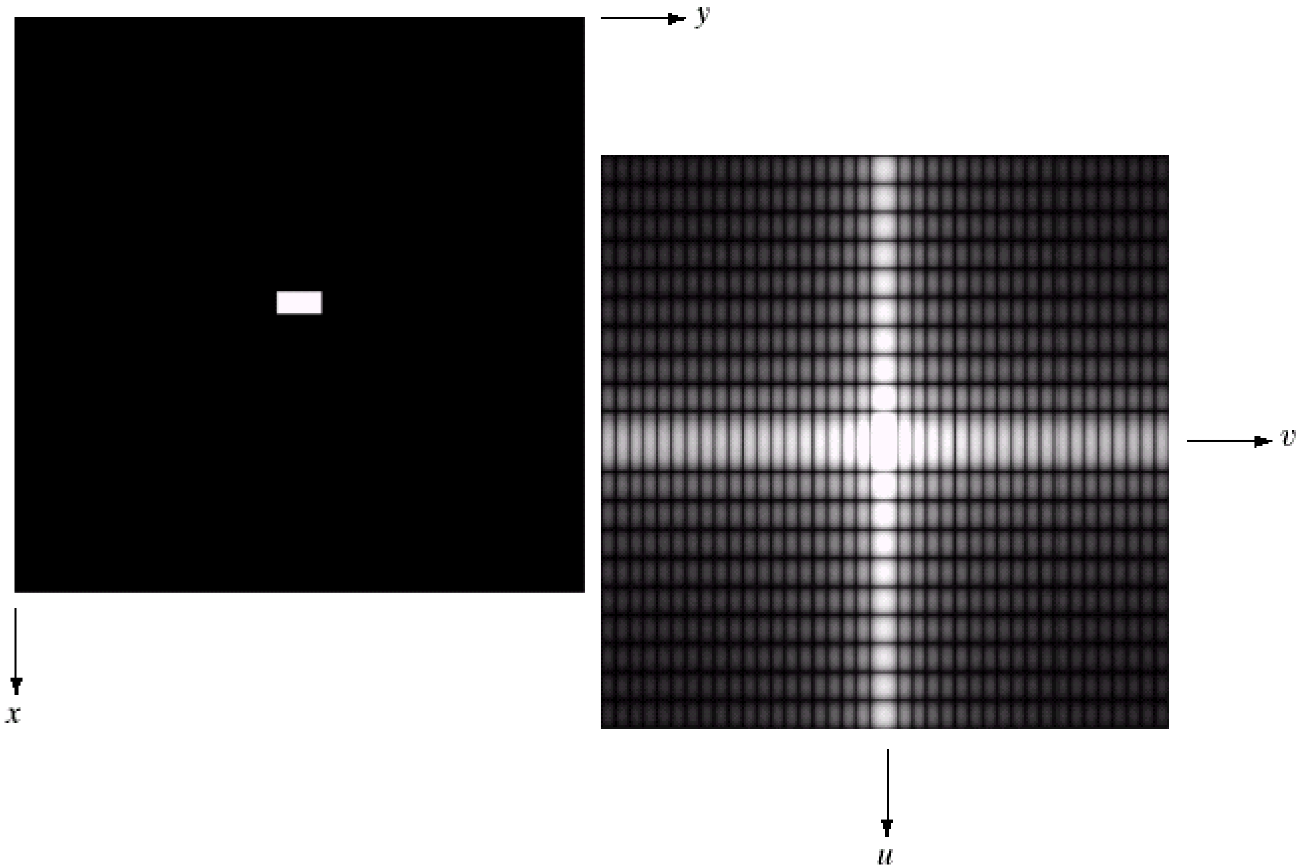
  If $f(r,\theta) \Leftrightarrow F(\omega,\varphi)$ ➔ $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\varphi_0)$

Periodicity and Conjugate Symmetry

- Both DFT and IDFT are periodic
  - $F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$
  - $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
- DFT is conjugate symmetric
  - $F(u,v) = F^*(-u,-v)$
  - Spectrum is symmetric
    - $|F(u,v)| = |F(-u,-v)|$

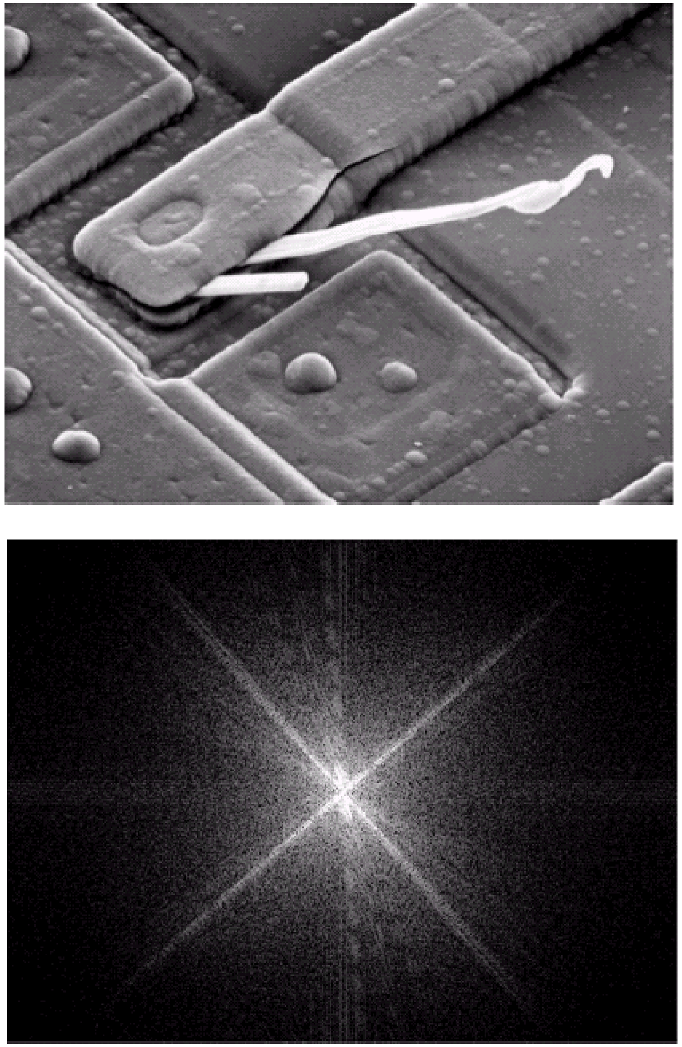
a b

FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



4.2.3 Filtering in the Frequency Domain

- Frequency is directly related to rate of change. The frequency of fast varying components in an image is higher than slowly varying components



a
b

FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster

4.2.3 Filtering in Frequency Domain

- From convolution theory
  - Multiplying the DFT of an image by the DFT of a spatial filter followed by IDFT.
  - Filtering means "altering the magnitudes and/or the phases of its constituent components (basis functions/images)"

Basics of filtering in the frequency domain

1. multiply the input image by $(-1)^{x+y}$ to center the transform to u = M/2 and v = N/2 (if M and N are even numbers, then the shifted coordinates will be integers)
2. computer F(u,v), the DFT of the image from (1)
3. multiply F(u,v) by a filter function H(u,v)
4. compute the inverse DFT of the result in (3)
5. obtain the real part of the result in (4)
6. multiply the result in (5) by $(-1)^{x+y}$ to cancel the multiplication of the input image.

Frequency Domain Filtering

- Let *f(x,y)* represent the input image in Step 1 and *F(u)* its Fourier transform

$$G(u,v)=H(u,v)F(u,v) \qquad (4.2\text{-}27)$$

- *H(u,v)* is called a filter (*filter transfer function*)
- The filtered image is obtained by taking the real part of the inverse Fourier transform of *G(u,v)*

$$\text{Filtered Image} = \Im^{-1}[G(u,v)] \qquad (4.2\text{-}28)$$

Basic steps for filtering in the frequency domain

Including multiplication the input/output image by $(-1)^{x+y}$.

zero-phase-shift filter: these filters do not change the phase of the transform

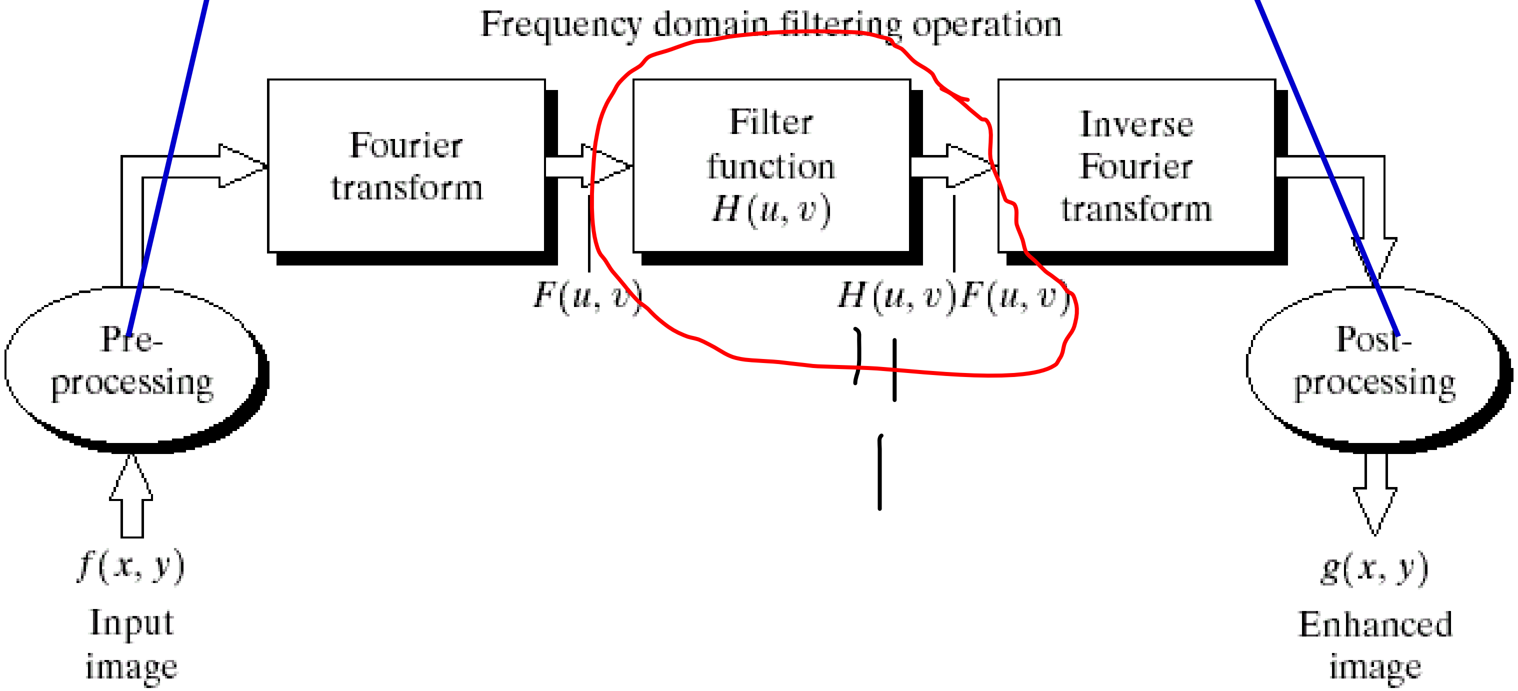
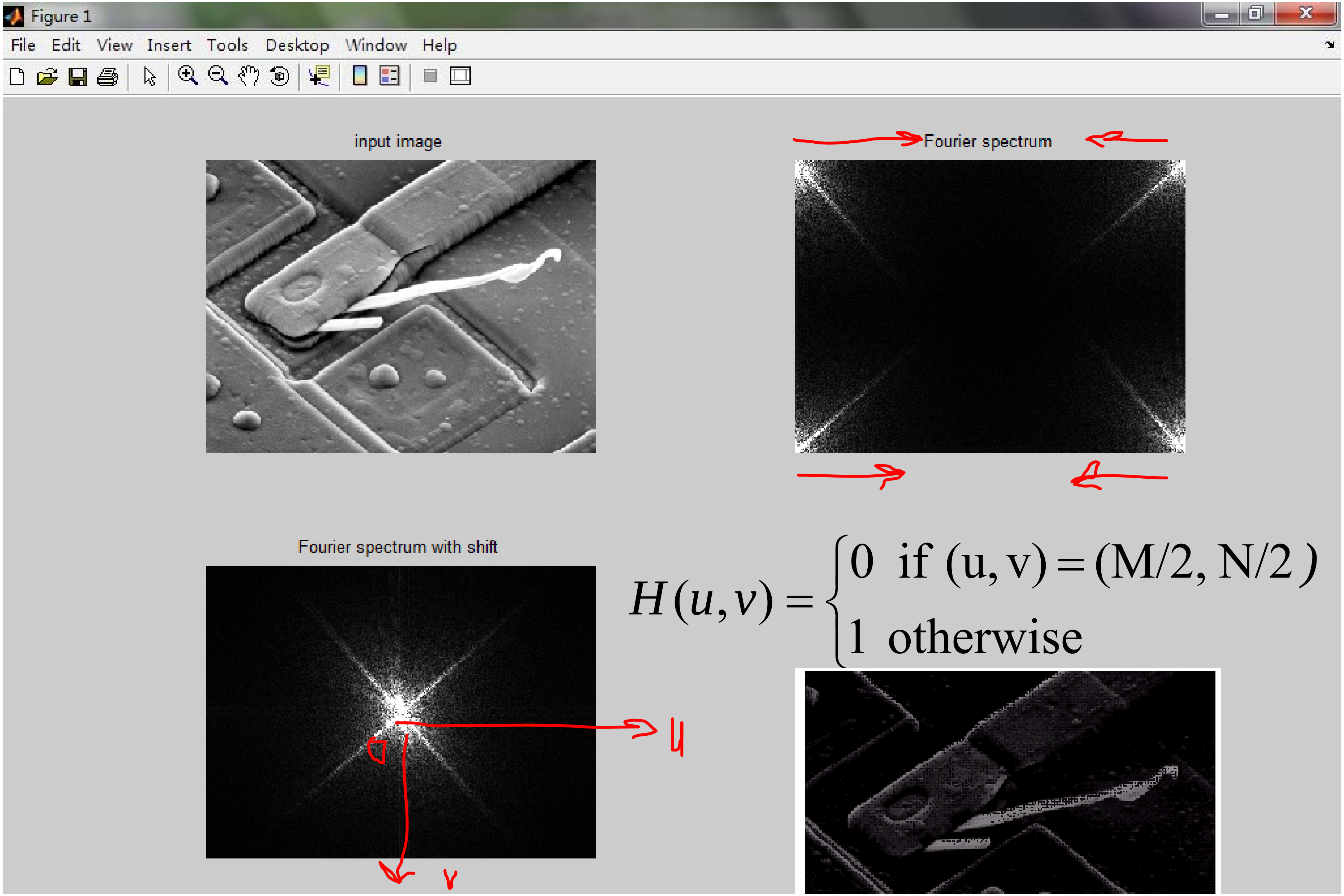
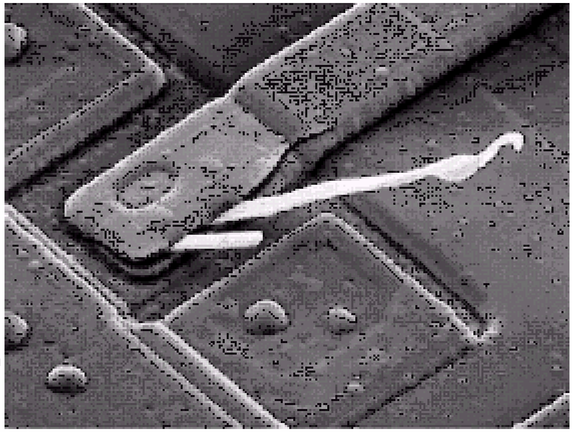


FIGURE 4.5 Basic steps for filtering in the frequency domain.



$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2,\ N/2) \\ 1 & \text{otherwise} \end{cases}$$

Notch filter



- this filter is to force the F(0,0) which is the average value of an image (dc component of the spectrum)
- the output has prominent edges
- in reality the average of the displayed image can't be zero as it needs to have negative gray levels. the output image needs to scale the gray level

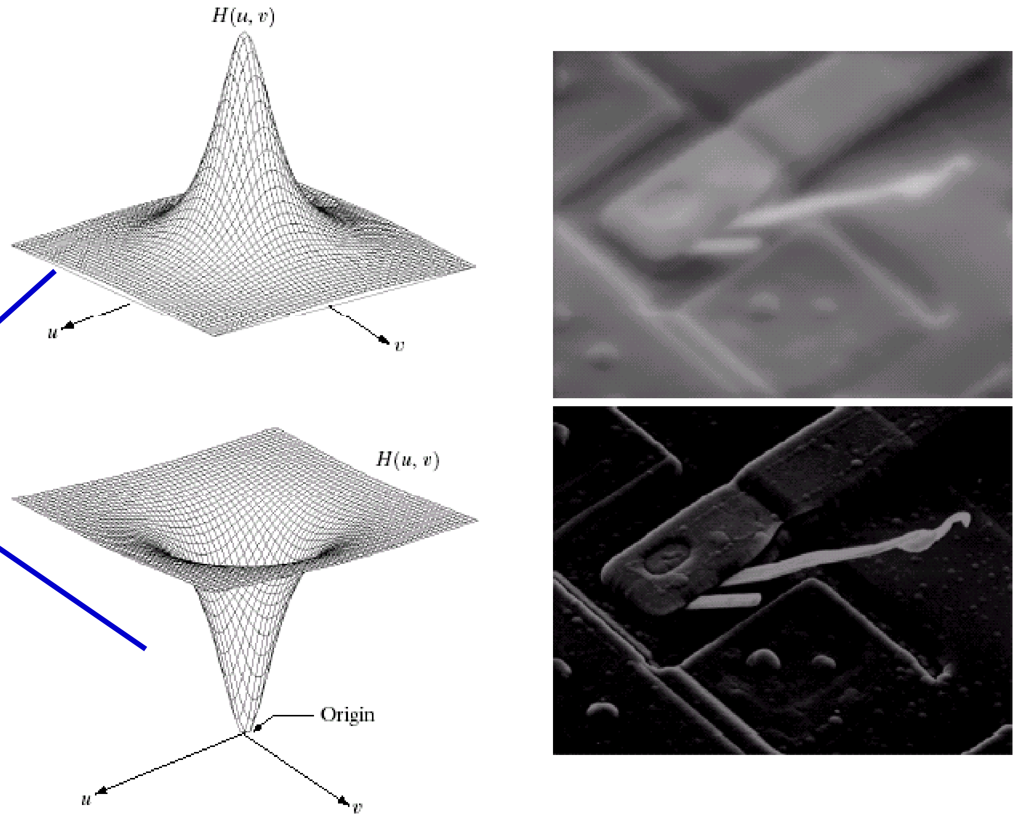
$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

Some Basic Filters and Their Functions

Low pass filter

Circular symmetry

high pass filter



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Add the ½ of filter height to F(0,0) of the high pass filter

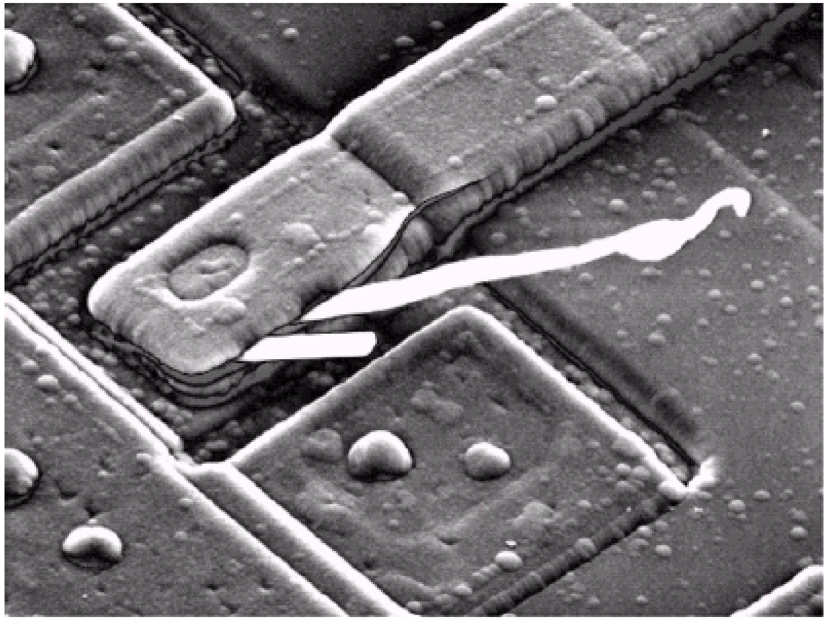


FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).

Low frequency filters: eliminate the gray-level detail and keep the general gray-level appearance. (blurring the image)
High frequency filters: have less gray-level variations in smooth areas and emphasized transitional (e.g., edge and noise) gray-level detail. (sharpening images)

4.2.4 Correspondence between filter in spatial and frequency domains

- Establish a direct link between the spatial filters and their frequency domain counterparts
- The most fundamental relationship the spatial filter and frequency domain is established by a well-known result called the *convolution theorem*

4.2.4 Correspondence between Filtering in the Spatial and Frequency Domain

- Convolution theorem:
  - The discrete convolution of two functions $f(x,y)$ and $h(x,y)$ of size $M \times N$ is defined as

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$
$$=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}h(m,n)f(x-m,y-n)$$

The process of implementation:

1) Flipping one function about the origin;
2) Shifting that function with respect to the other by changing the values of (x, y);
3) Computing a sum of products over all values of m and n, for each displacement.

–Let $F(u,v)$ and $H(u,v)$ denote the Fourier transforms of $f(x,y)$ and $h(x,y)$, then

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$ Eq. (4.2-31)

$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$ Eq. (4.2-32)

- an impulse function of strength A, located at coordinates (x_0,y_0): $A\delta(x-x_0, y-y_0)$ and is defined by

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)A\delta(x-x_0, y-y_0) = As(x_0, y_0)$$

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)\delta(x,y) = s(0,0)$$

The shifting property of impulse function

where $\delta(x,y)$: a unit impulse located at the origin

- The Fourier transform of a unit impulse at the origin (Eq4.2-35) :

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} \delta(x,y)e^{-j2\pi(ux/M+vy/N)} = \frac{1}{MN}$$

- Let $f(x,y)=\delta(x,y)$, then the convolution (Eq. (4.2-36))

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\delta(m,n)h(x-m,y-n)$$
$$=\frac{1}{MN}h(x,y)$$

- Combine Eqs. (4.2-35) (4.2-36) with Eq. (4.2-31), we obtain:

$$f(x,y)*h(x,y)\Leftrightarrow F(u,v)H(u,v)$$
$$\delta(x,y)*h(x,y)\Leftrightarrow \Im\left[\delta(x,y)\right]H(u,v)$$
$$\boxed{\frac{1}{MN}h(x,y)} \qquad \boxed{\frac{1}{MN}H(u,v)}$$
$$\Rightarrow \quad h(x,y)\Leftrightarrow H(u,v)$$

That is to say, the response of impulse input is the transfer function of filter.

The distinction and links between spatial and frequency filtering

- If the size of spatial and frequency filters is same, then the computation burden in spatial domain is larger than in frequency domain;
- However, whenever possible, it makes more sense to filter in the spatial domain using small filter masks.
- Filtering in frequency is more intuitive. We can specify filters in the frequency, take their inverse transform, and the use the resulting filter in spatial domain as a guide for constructing smaller spatial filter masks.
- Fourier transform and its inverse are linear process, so the following discussion is limited to linear processes.

- There is two reasons that filters based on Gaussian functions are of particular importance: 1) their shapes are easily specified; 2) both the forward and inverse Fourier transforms of a Gaussian are real Gaussian function.

- Let $H(u)$ denote a frequency domain, Gaussian filter function given the equation

$$H(u) = Ae^{-u^2/2\sigma^2}$$

where σ : the standard deviation of the Gaussian curve.

- The corresponding filter in the spatial domain is

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}, (A \geq B,\ \sigma_1 \geq \sigma_2)$$

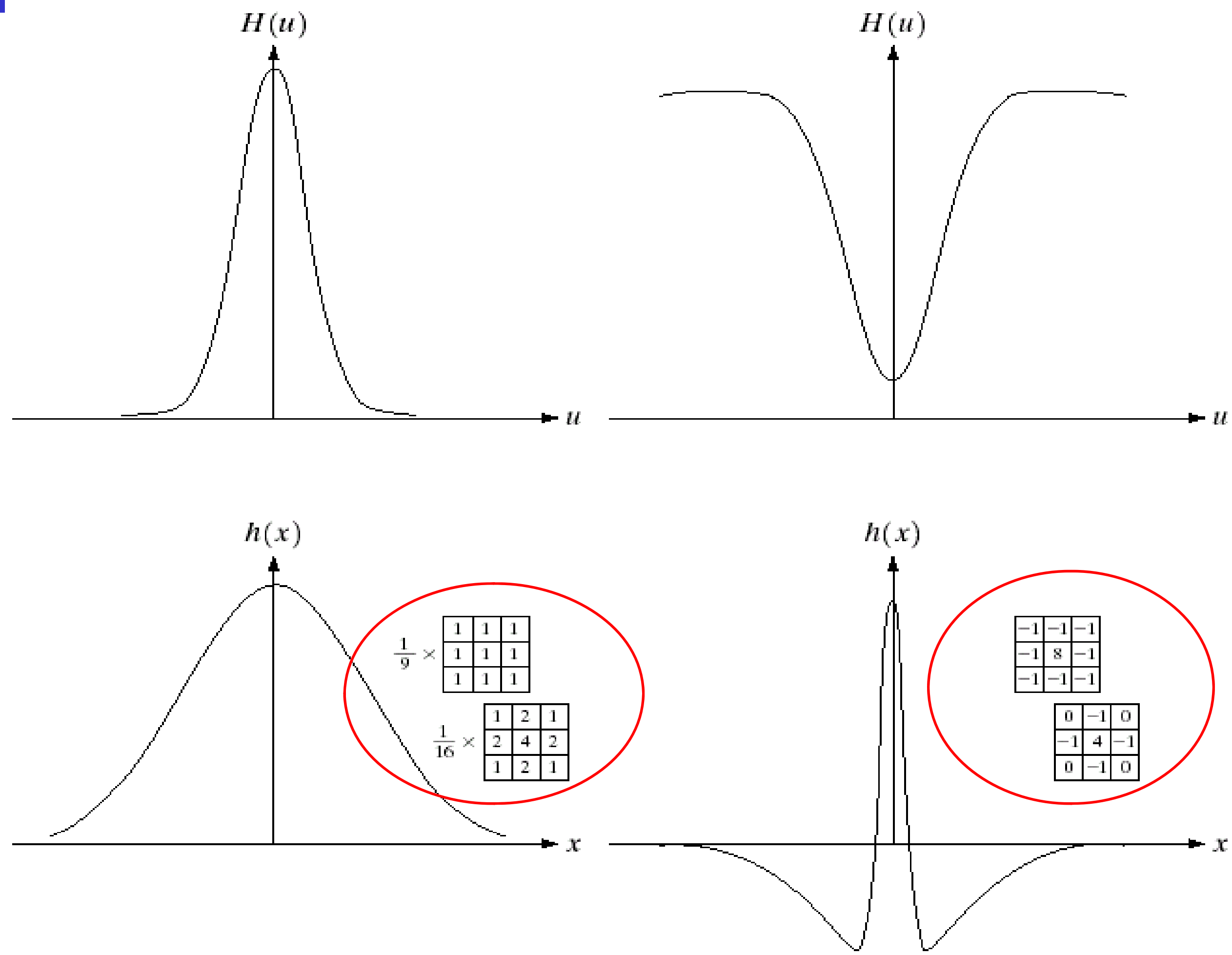
Fourier Transform and Convolution

Spatial Domain *(x,y)*		Frequency Domain *(u,v)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

So, we can find *g(x,y)* by Fourier transform

$$
\begin{tikzcd}
g & = & f \arrow[d, "\mathbf{FT}"] & * & h \arrow[d, "\mathbf{FT}"] \\
G \arrow[u, "\mathbf{IFT}"] & = & F & \times & H
\end{tikzcd}
$$

4.2.4 Correspondence between filter in spatial and frequency domains



a b
c d

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

- The corresponding filter in the spatial domain is

$$h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2}$$

- We can note that the value of this types of filter has both negative and positive values. Once the values turn negative, they never turn positive again.

- Filtering in frequency domain is usually used for the guides to design the filter masks in the spatial domain.

Some important properties of Gaussian filters functions

- One very useful property of the Gaussian function is that both it and its Fourier transform are real valued; there are no complex values associated with them.
- In addition, the values are always positive. So, if we convolve an image with a Gaussian function, there will never be any negative output values to deal with.
- There is also an important relationship between the widths of a Gaussian function and its Fourier transform. If we make the width of the function smaller, the width of the Fourier transform gets larger. This is controlled by the variance parameter σ^2 in the equations.

Some important properties of Gaussian filters functions

- These properties make the Gaussian filter very useful for lowpass filtering an image. The amount of blur is controlled by σ^2. It can be implemented in either the spatial or frequency domain.
- Other filters besides lowpass can also be implemented by using two different sized Gaussian functions.

4.2 Smoothing Frequency-Domain Filters

- The basic model for filtering in the frequency domain

$$G(u,v) = H(u,v)F(u,v)$$

where $F(u,v)$: the Fourier transform of the image to be smoothed

$H(u,v)$: a filter transfer function

4.2 Smoothing Frequency-Domain Filters

- Smoothing is fundamentally a lowpass operation in the frequency domain.
- There are several standard forms of lowpass filters (LPF).
  - Ideal lowpass filter
  - Butterworth lowpass filter
  - Gaussian lowpass filter

4.3 Smoothing Frequency-domain filters: Ideal Lowpass filter

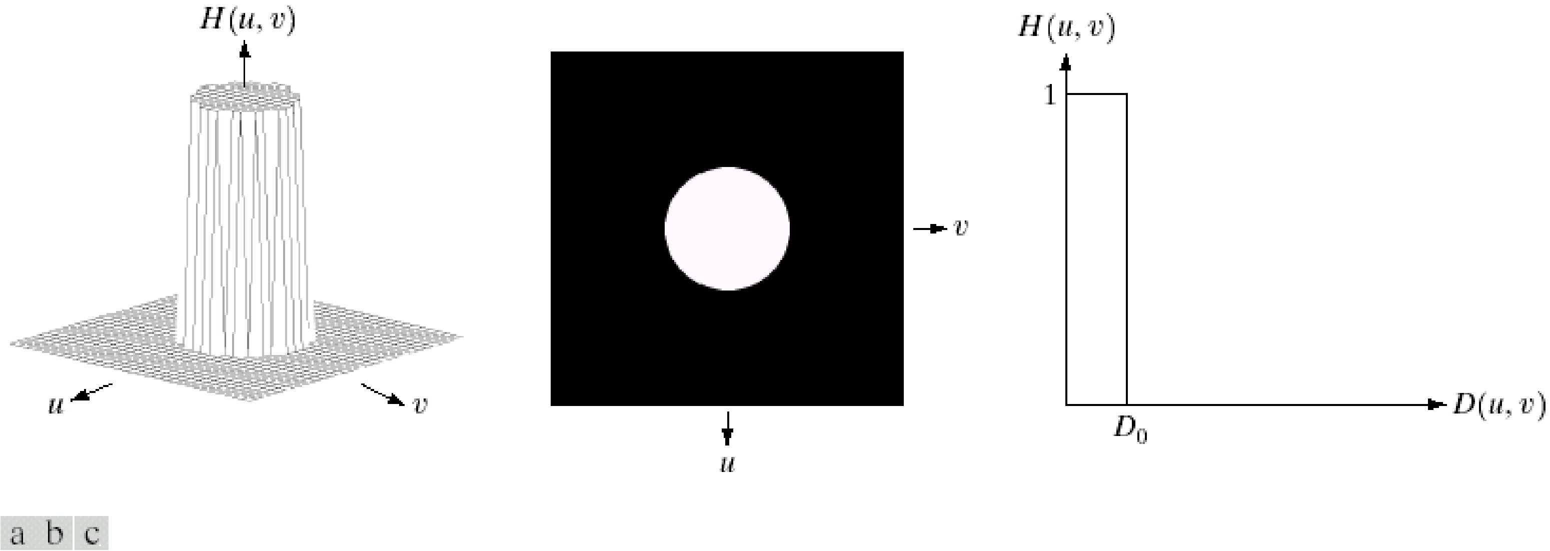


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

- How to determine the cutoff frequency D_0?
  - One way to do this is to compute circles that enclose specified amounts of total image power P_T.

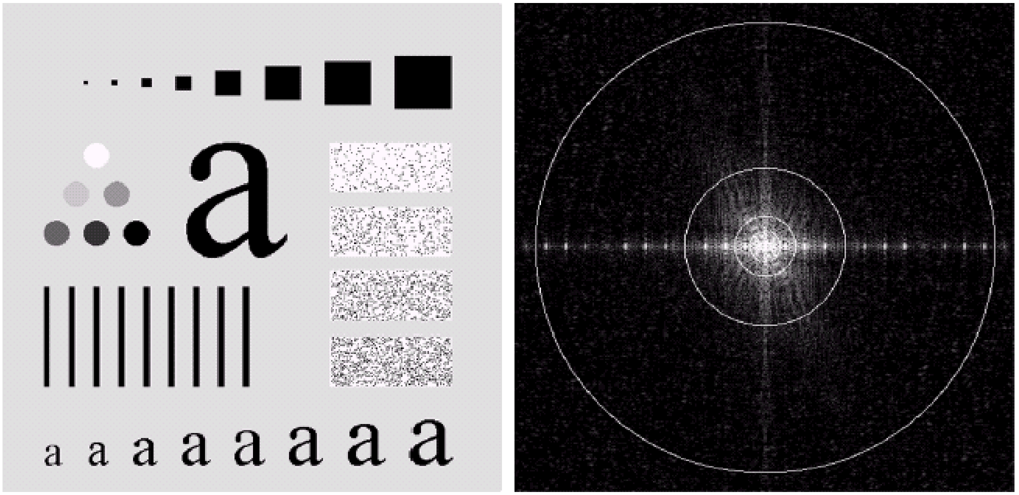
$$P_T = \sum_{u=0}^{u=M-1} \sum_{v=0}^{v=N-1} |F(u,v)|^2 \qquad \Big(\sum_{D(u,v)<D_0} |F(u,v)|^2 \Big) / P_T \le \alpha$$

Direct Construction of Frequency Domain Filters

- Ideal lowpass filters (ILPF)
  - Cut off all high-frequency components of the Fourier transform that are at a distance greater than a specified distance D_0 (cut off frequency) from the origin of the (centered) transform
  - The transfer function (frequency domain filter) is defined by
  - D(u,v) is the distance from point (u,v) to the origin (center) of the frequency domain filter

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

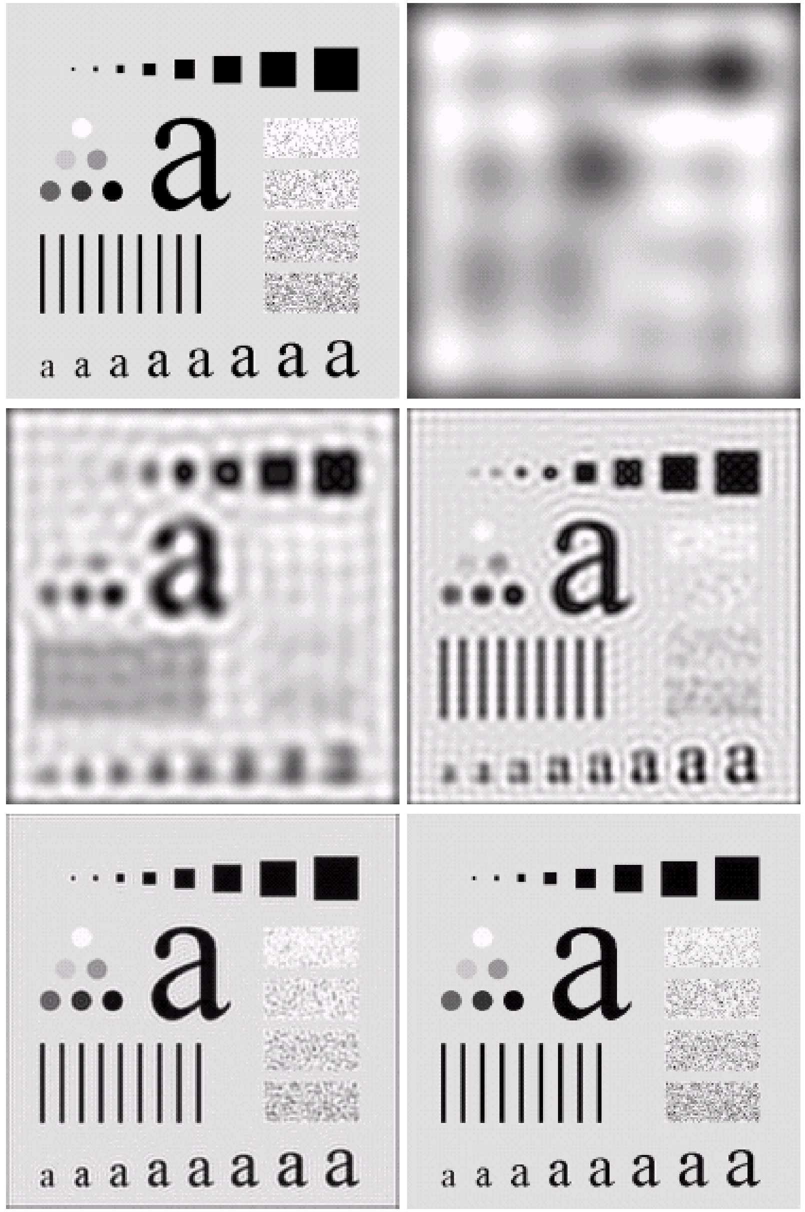
  - Usually, the image to be filtered is even-sized, in this case, the center of the filter is (M/2,N/2). Then the distance D(u,v) can be obtained by $D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$



a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

- As the filter radius increases, less and less power is removed/filtered out, more and more details are preserved.
- Ringing effect is clear in most cases except for the last one.
- Ringing effect is the consequence of applying ideal lowpass filters



Result of ILPF

Ringing Effect

- Ringing effect can be better explained in spatial domain
- Convolution of a function with an impulse "copies" the value of that function at the location of the impulse.
  - An impulse function is defined as

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = 0, y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Let } h(x, y) = \delta(x - x_0, y - y_0)$$

$$h(x, y) * f(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h(m, n) f(x - m, y - n)$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \delta(m - x_0, n - y_0) f(x - m, y - n)$$

$$= f(x - x_0, y - y_0)$$

- The transfer function of the ideal lowpass filter with radius 5 is ripple shaped
- Convolution of any image (consisting of groups of impulses of different strengths) with the ripple shaped function results in the ringing phenomenon.
- Lowpass filtering with less ringing will be discussed.

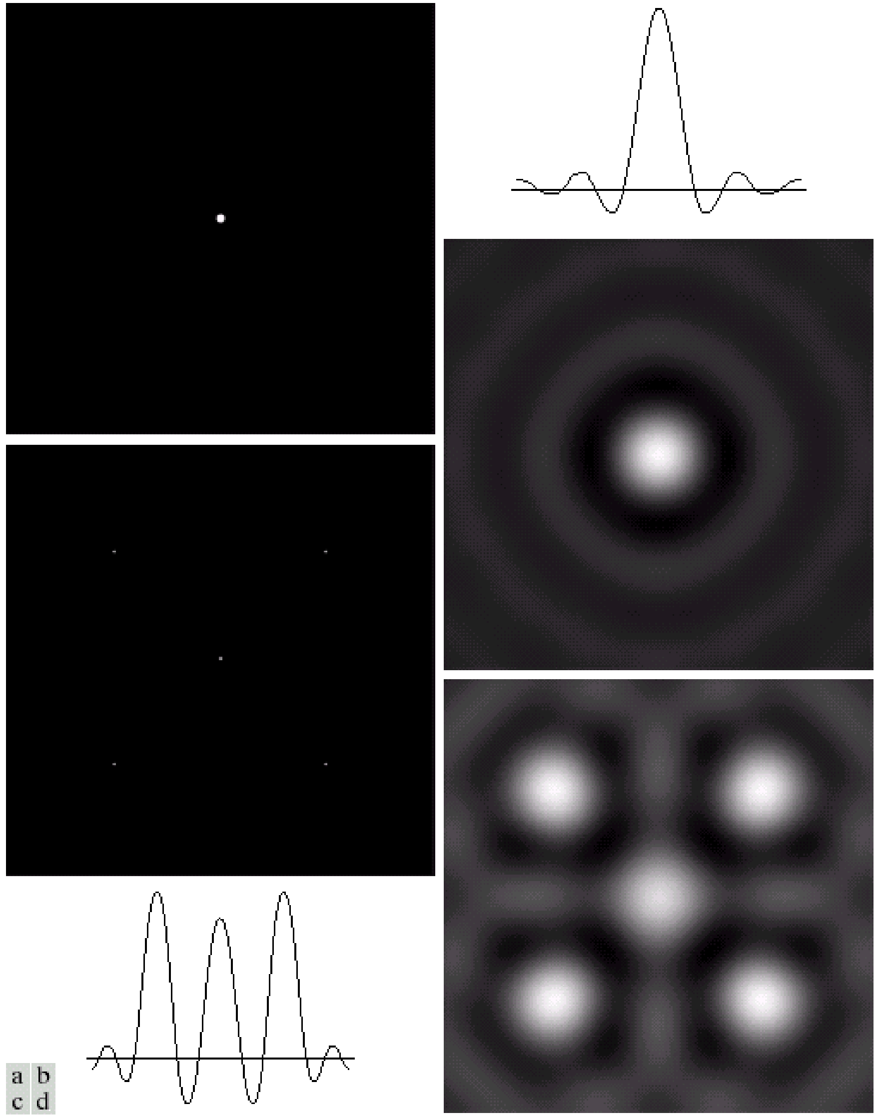


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

4.3.2 Butterworth Lowpass Filters

- A butterworth lowpass filter (BLPF) of order n with cutoff frequency at a distance D_0 from the origin is given by the following transfer function
  - $$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$$
  - BLPF does not have a sharp discontinuity
  - For BLPF, the cutoff frequency is defined as the frequency at which the transfer function has value which is half of the maximum

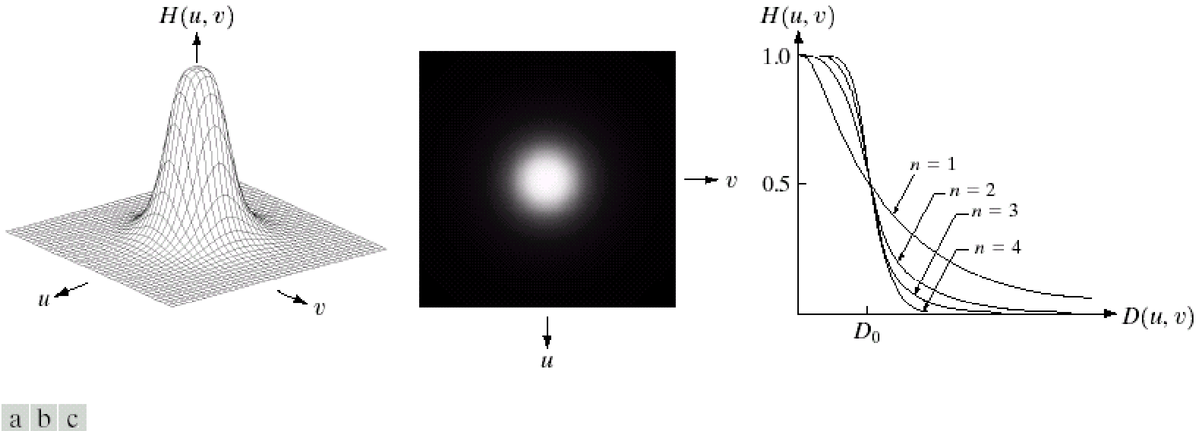
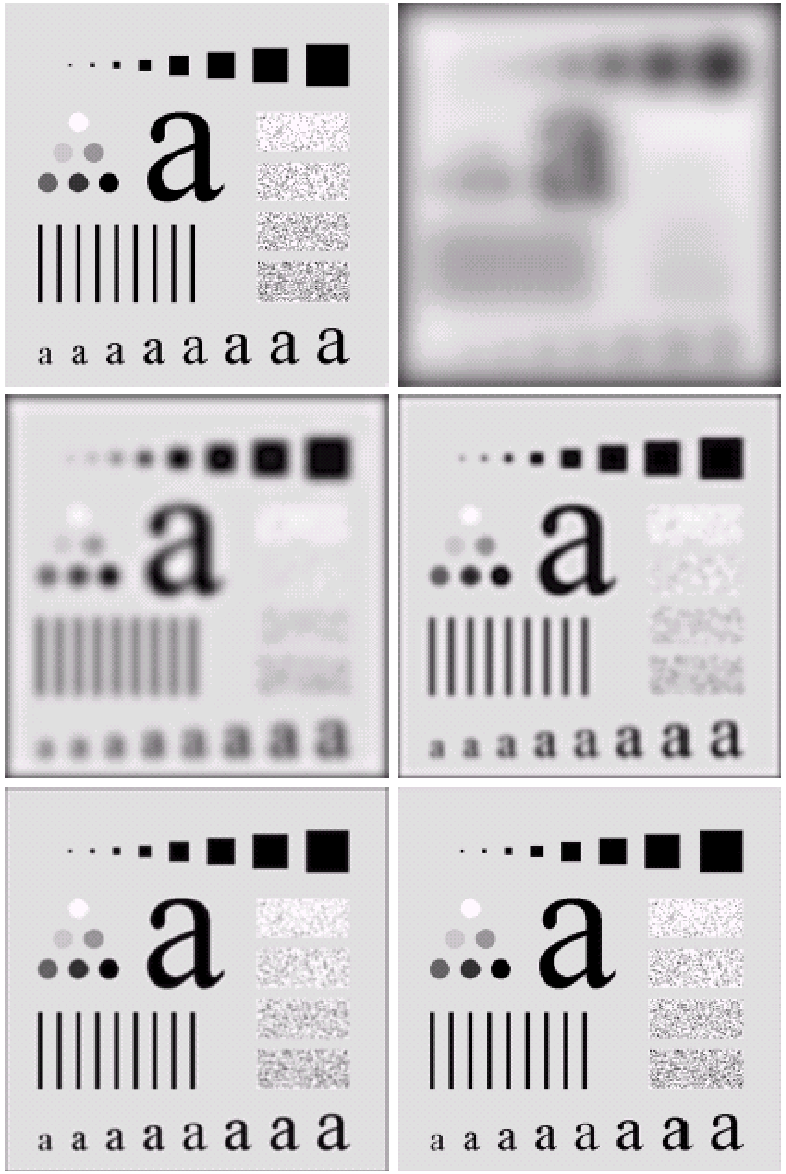


FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Example

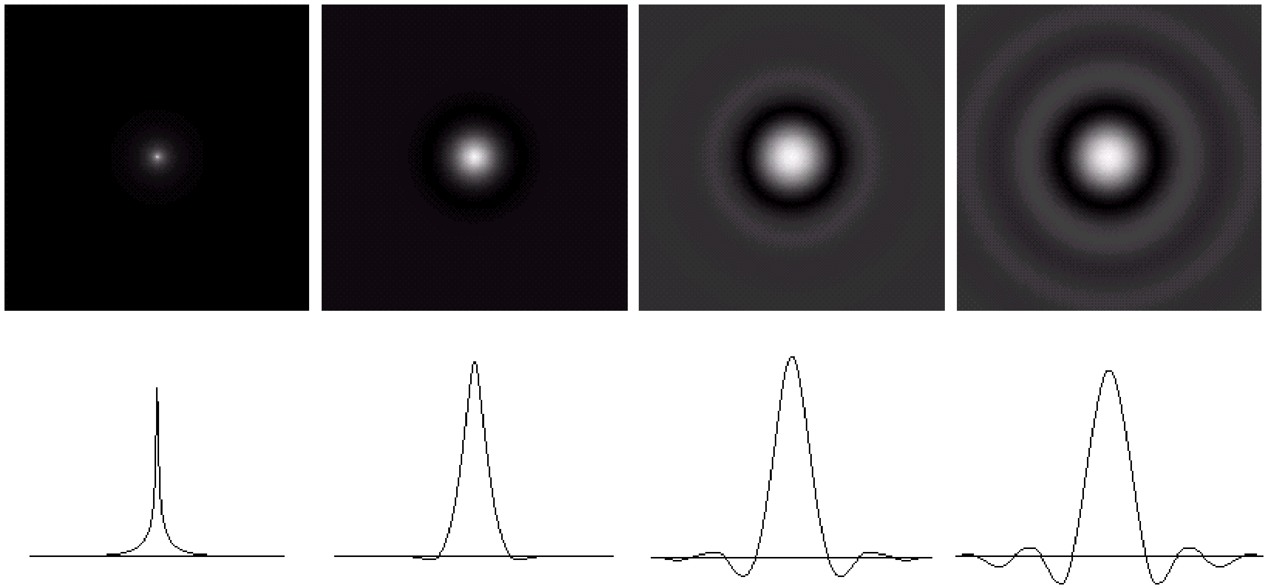
- Same order but with different cutoff frequencies
  - The larger the cutoff frequency, the more details are reserved



a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Spatial representation of BLPFs

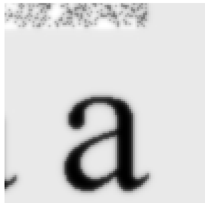


a b c d

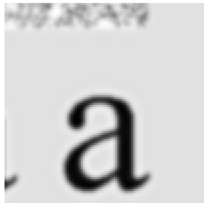
FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

- To check whether a Butterworth lowpass filter suffer the ringing effect as dose the ILPF, we need to examine the pattern of its equivalent spatial filter (How to obtain it?)

Original



D_0=80, n=1



D_0=80, n=2

D_0=80, n=3

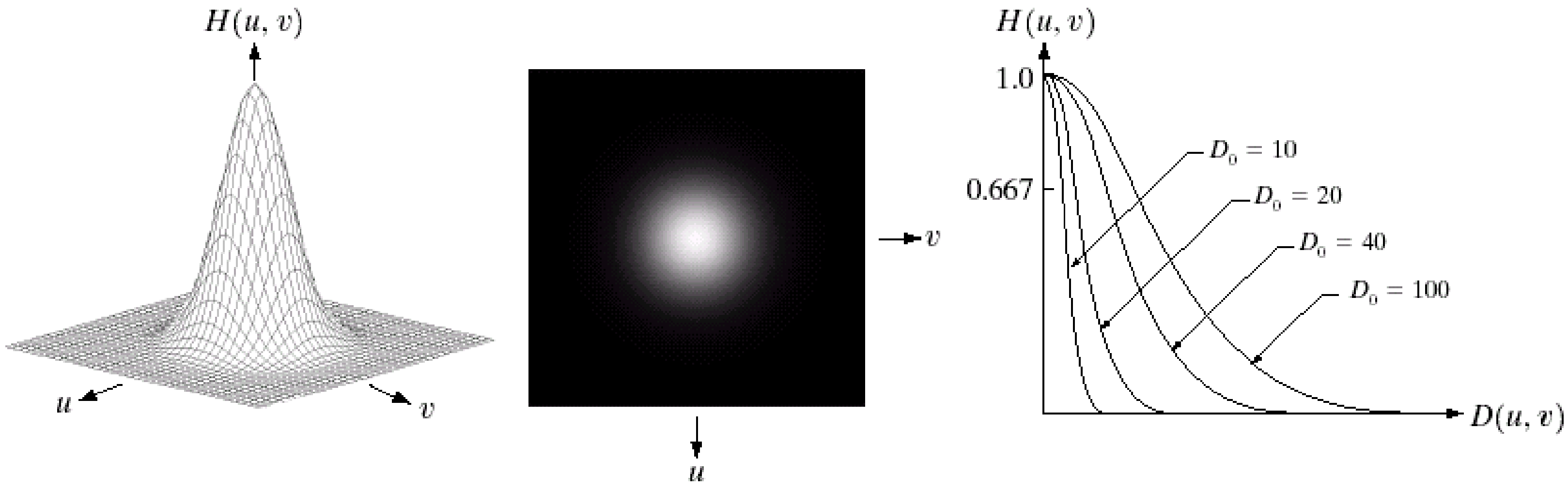
D_0=80, n=5

D_0=80, n=10

D_0=80, n=20

D_0=80, n=50

4.3.4 Gaussian Lowpass Filter: GLPF



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian Lowpass Filters

- 1D Gaussian distribution function is given by $$f(x) = Ae^{-(x-x_0)^2/2\sigma^2}$$
  - X_0 is the center of the distribution
  - σ is the standard deviation controlling the shape (width) of the curve
  - A is a normalization constant to ensure the area under the curve is one.
  - The Fourier transform of a Gaussian function is also a Gaussian function

Gaussian Lowpass Filters

- GLPF is given by the following (centered) transfer function

$$H(u,v) = e^{-[(u-u_0)^2+(v-v_0)^2]/2\sigma^2} = e^{-D^2(u,v)/2D_0{}^2}$$

  - (u_0,v_0) is the center of the transfer function
  - It is [M/2, N/2] if M,N are even and [(M+1)/2,(N+1)/2] if M,N are odd numbers
  - Does GLPF suffer from the ringing effect? (Answer: no)

Example

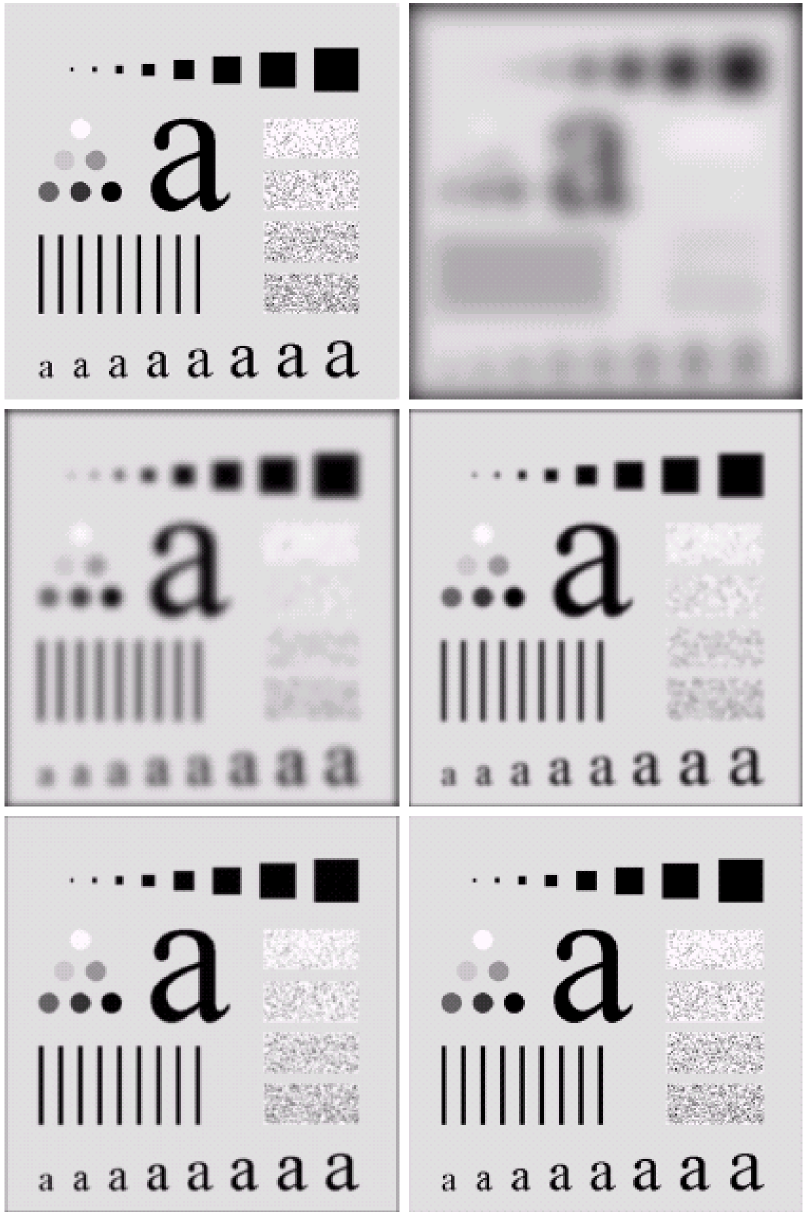
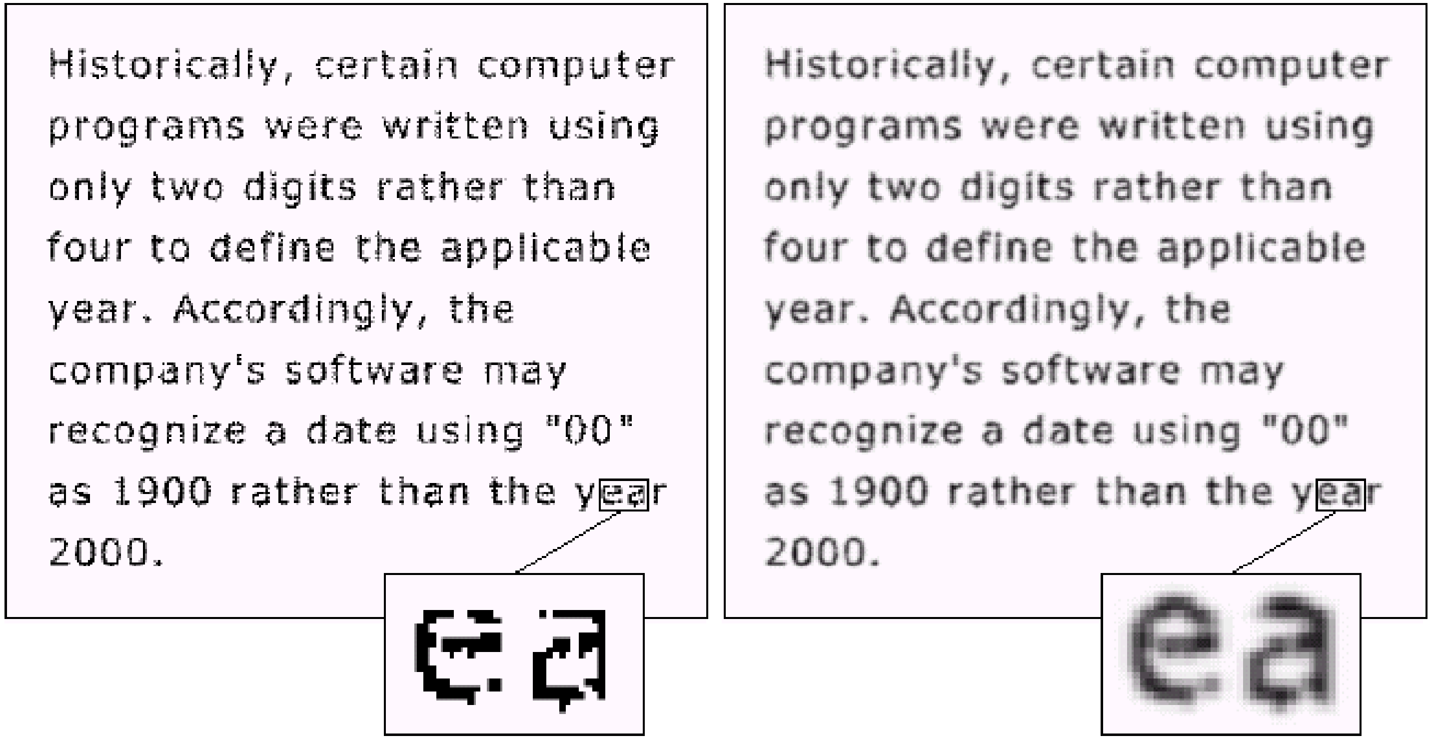


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

a b
c d
e f

Example



a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

Example

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Example

a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

4.4 Sharpening (Highpass) Frequency Domain Filters

- Used to attenuate low-frequency components without disturbing high-frequency information (preserving details)
- Highpass filters can be obtained from lowpass filters directly by the following manner:

$$H_{hp}(u,v)=1-H_{lp}(u,v)$$

- Ideal highpass filter (IHPF) :

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)>D_0\\ 0 & \text{if } D(u,v)\le D_0\end{cases}$$

- Butterworth highpass filter (BHPF) :

$$H(u,v)=1-\frac{1}{1+[D(u,v)/D_0]^{2n}}=\frac{1}{1+[D_0/D(u,v)]^{2n}}$$

- Gaussian highpass filter (GHPF) :

$$H(u,v)=1-e^{-D^2(u,v)/2D_0{}^2}$$

Sharpening Frequency Domain Filter:

Ideal highpass filter

$$H(u,v) = \begin{cases} 0 \text{ if } D(u,v) \le D_0 \\ 1 \text{ if } D(u,v) > D_0 \end{cases}$$

Butterworth highpass filter

$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$

Gaussian highpass filter

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

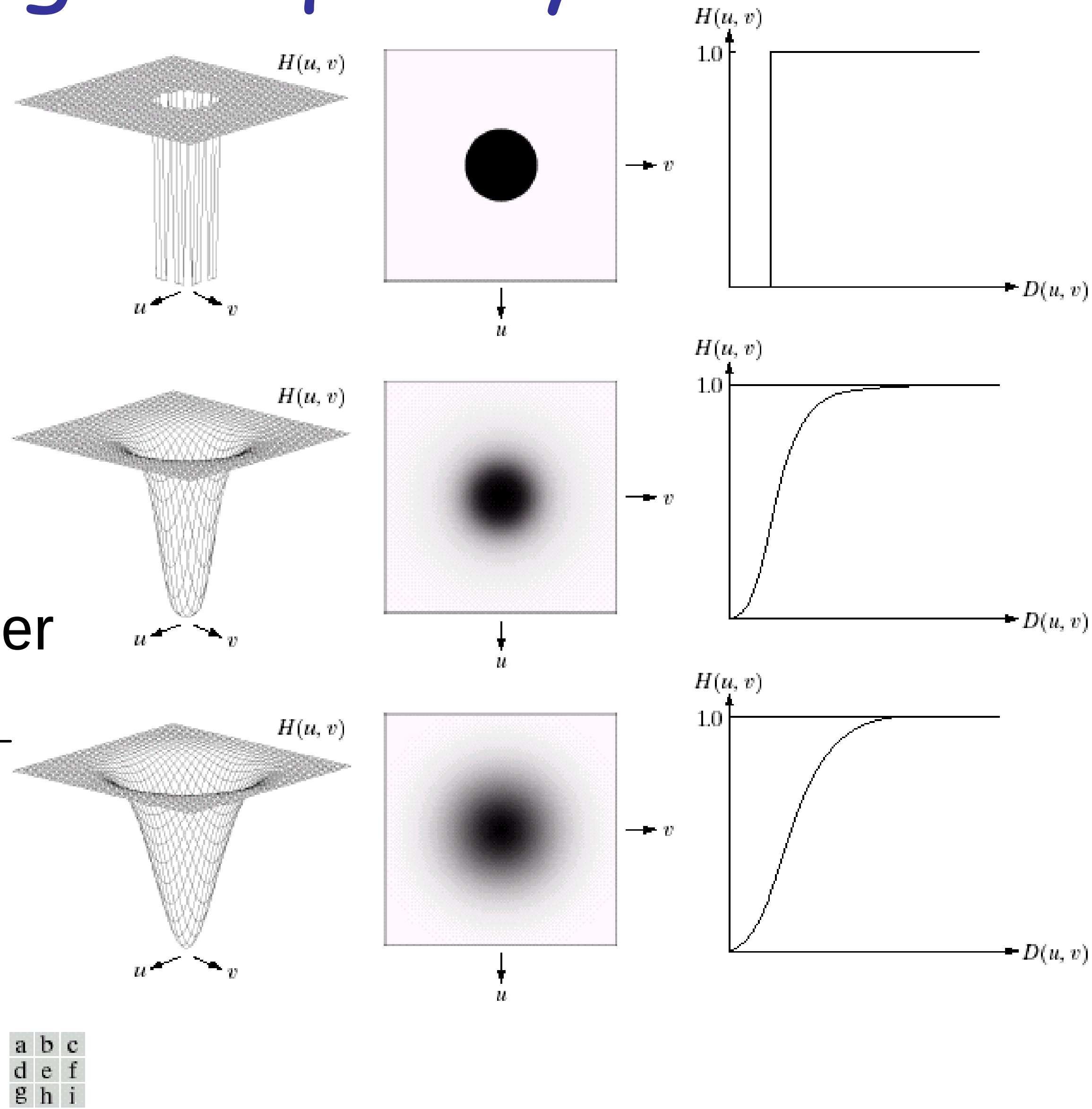
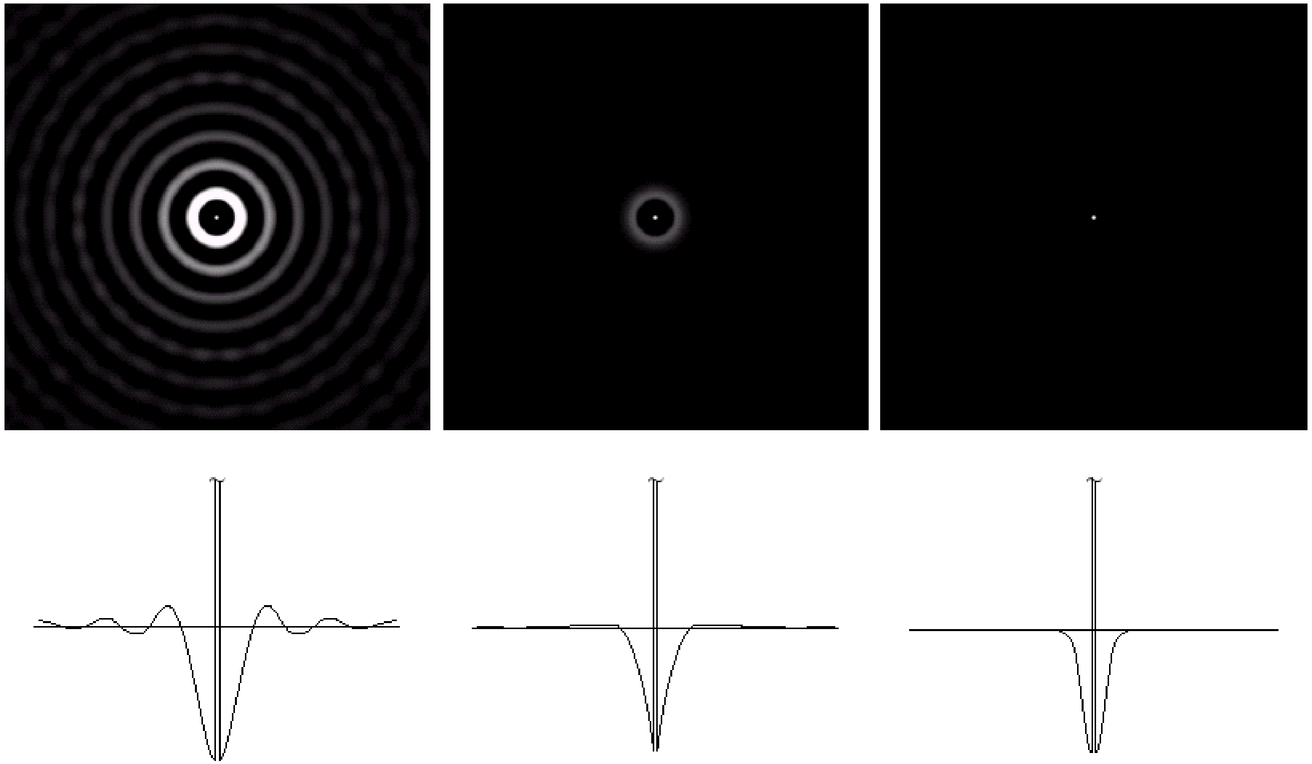


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

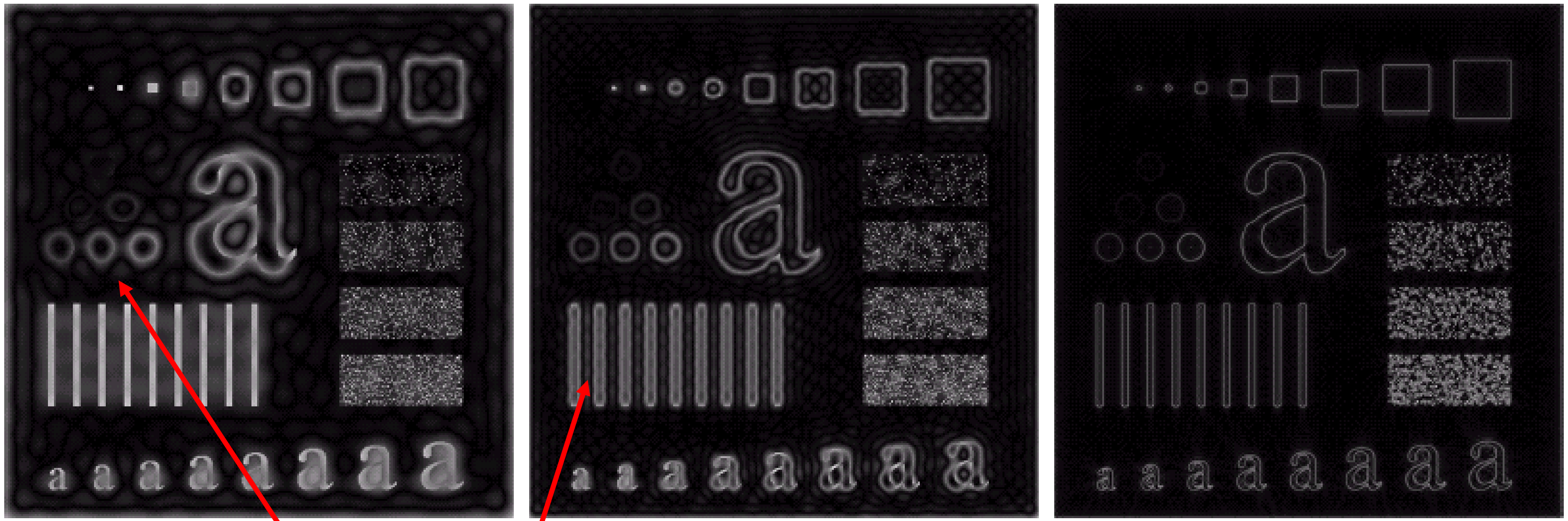
Spatial representation of Ideal, Butterworth and Gaussian highpass filters



a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Example: result of IHPF

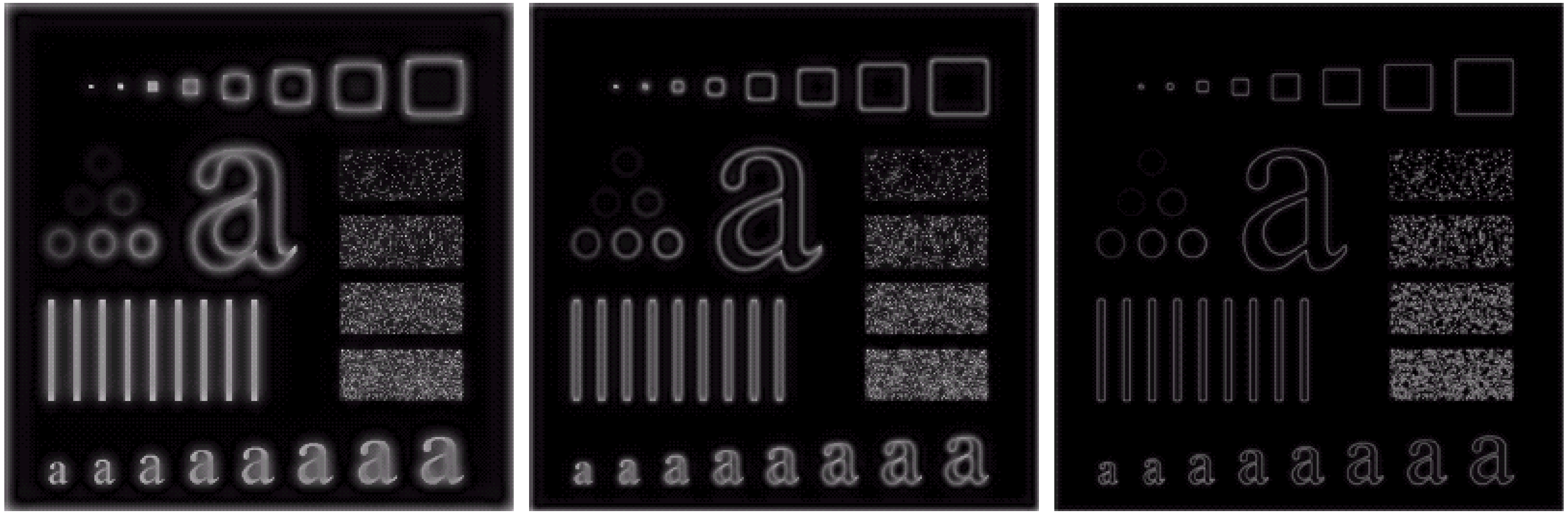


a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Ringing effect

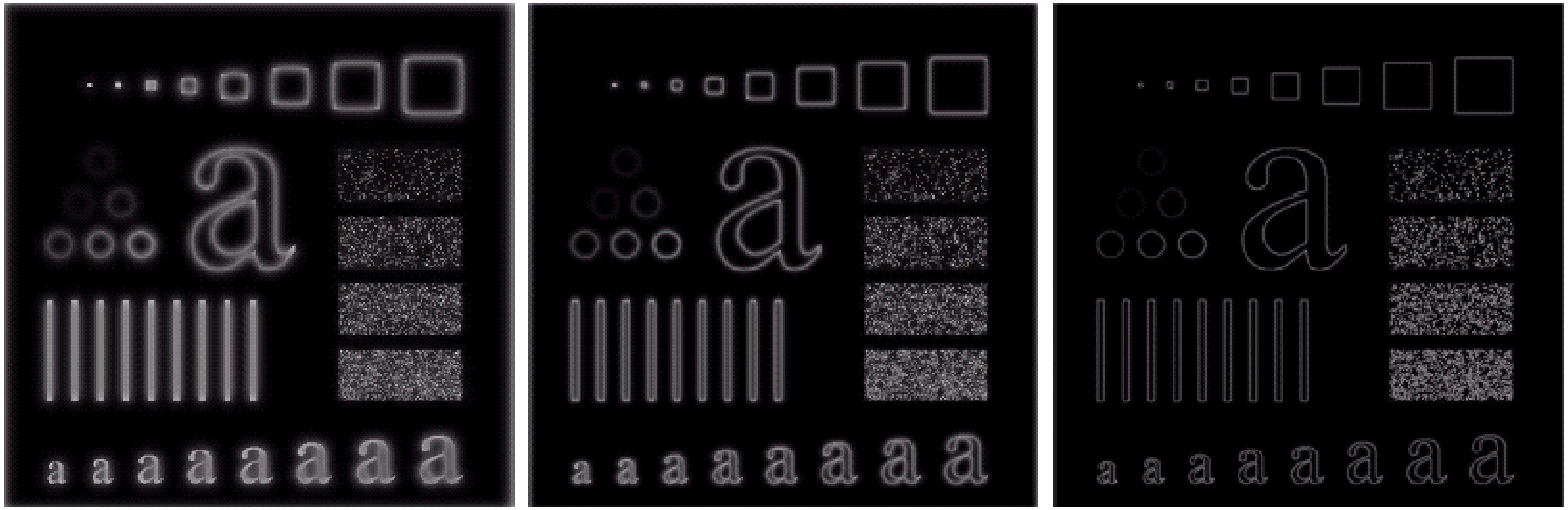
Example: result of BHPF



a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Example: result of GHPF



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

4.4.4 The Laplacian in the Frequency Domain

- Recall that the definition of a 2D continuous function is given by $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$
- The Laplacian operator can be approximated by linear filtering with masks shown below

(a)

0	1	0
1	−4	1
0	1	0

(b)

1	1	1
1	−8	1
1	1	1

(c)

0	−1	0
−1	4	−1
0	−1	0

(d)

−1	−1	−1
−1	8	−1
−1	−1	−1

a b
c d

FIGURE 3.39 (a) Filter mask used to implement the digital Laplacian, as defined in Eq. (3.7-4). (b) Mask used to implement an extension of this equation that includes the diagonal neighbors. (c) and (d) Two other implementations of the Laplacian.

The Laplacian in the Frequency Domain

- We want to find the equivalent operator in the frequency domain
- Continuous Fourier transform pair is defined as

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

- From the inverse FT, we can derive the following result

$$\Im\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$

The Laplacian in the Frequency Domain

- From the result of the previous slide, it follows that

$$\Im\left[\frac{\partial^2 f(x,y)}{\partial x^2}+\frac{\partial^2 f(x,y)}{\partial y^2}\right]=(ju)^2F(u,v)+(jv)^2F(u,v)$$

$$=-(u^2+v^2)F(u,v)$$

  - It means that the transfer function of the Laplacian operator is $H(u,v)=-(u^2+v^2)=-D^2(u,v)$
  - $\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$

The Laplacian in the Frequency Domain

- So, Laplacian can be implemented in the frequency domain by using the filter

$$H(u,v) = -(u^2 + v^2)$$

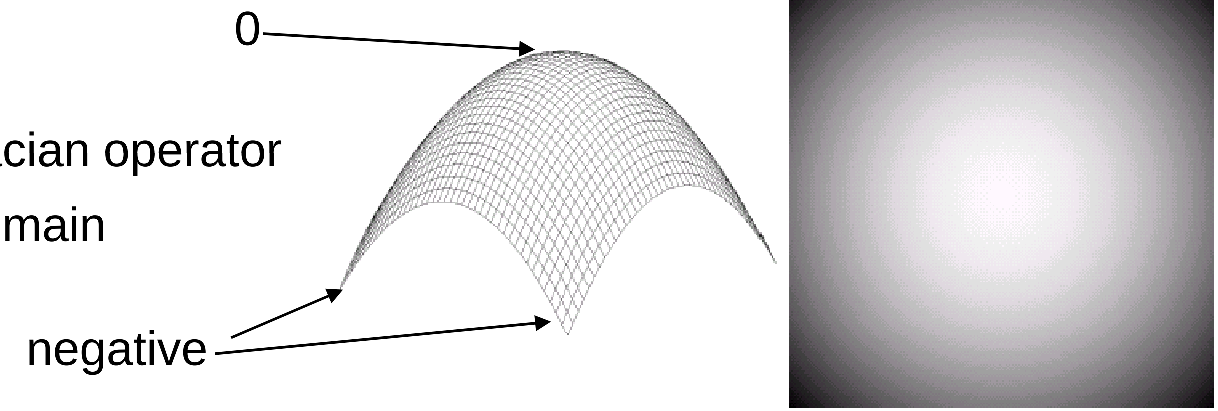
- Shift the center to (M/2, N/2) and obtain

$$H(u,v) = -\left[(u - M/2)^2 + (v - N/2)^2\right]$$

- We have the following Fourier transform pairs

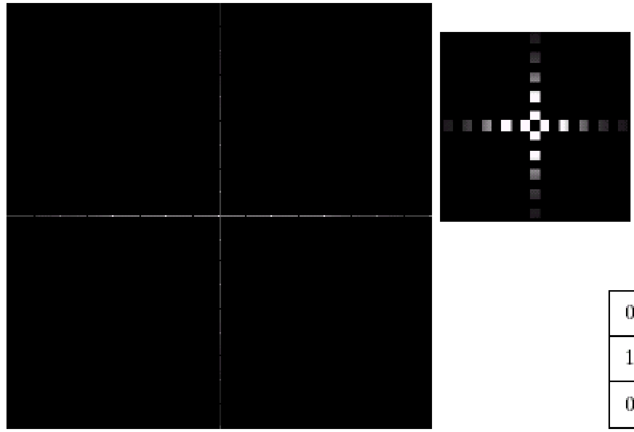
$$\nabla^2 f(x,y) \Leftrightarrow -\left[(u - M/2)^2 + (v - N/2)^2\right] F(u,v)$$

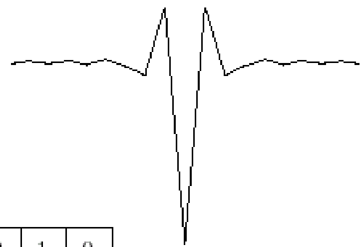
Centered Laplacian operator
in frequency domain



Frequency domain

Laplacian operator
in spatial domain





0	1	0
1	−4	1
0	1	0

Spatial domain

a b
c d e
f

FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

Q: Laplacian is lowpass or highpass? (A: highpass)

Operated in frequency domain

Operated in spatial domain

Implementation Issues

- Spatial domain vs frequency domain
  - Spatial domain is preferred whenever a small equivalent spatial mask can be obtained
  - (F) 4.34 sec vs (S)1.42 sec (100 runs)
- Computation of D(u,v)

(0,0)	(0,1)	(0,2)
(1,0)	(1,1)	(1,2)
(2,0)	(2,1)	(2,2)

D(0,0)	D(0,1)	D(0,2)
D(1,0)	D(1,1)	D(1,2)
D(2,0)	D(2,1)	D(2,2)

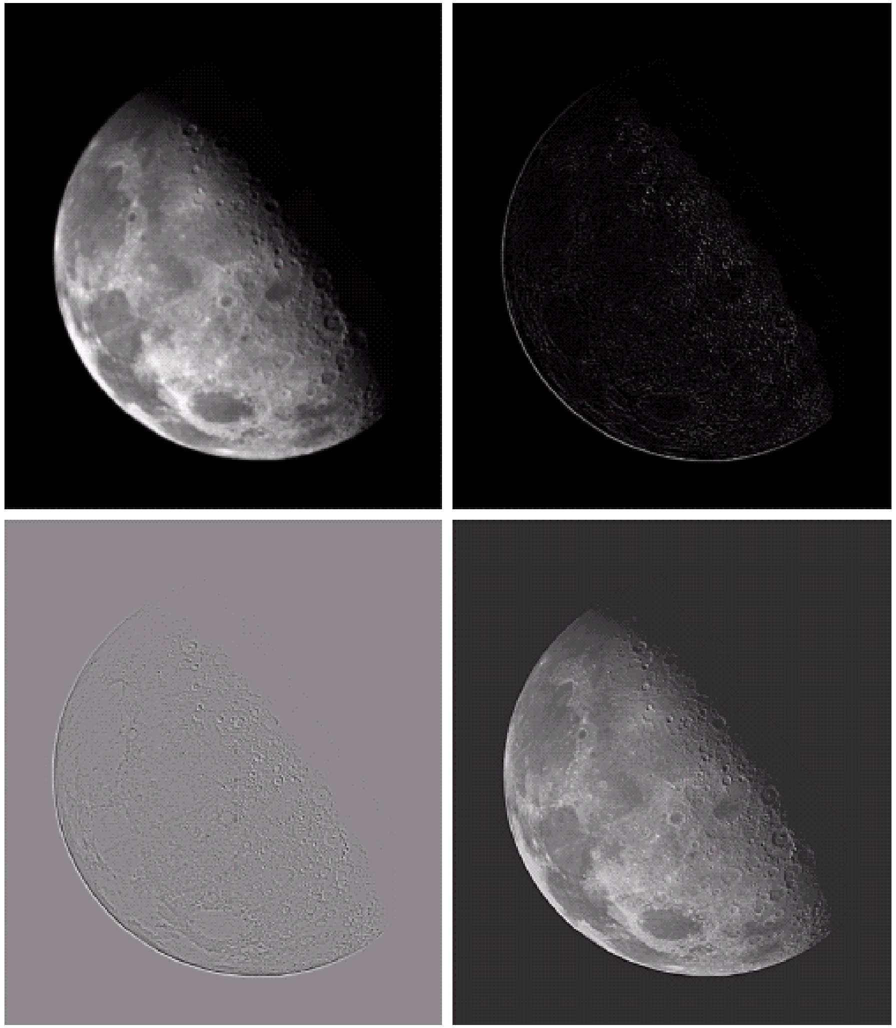
4.4.5 Unsharp Masking, High-Boost Filtering and High-Frequency Emphasis Filtering

- Unsharp masking
  - Generating a sharp image by subtracting a blurred version of an image from itself
  - $$f_{sh}(x,y) = f(x,y) - f_{bl}(x,y) = h_{id} * f(x,y) - h_{bl} * f(x,y)$$
    $$= (h_{id} - h_{bl}) * f(x,y) = h_{sh} * f(x,y)$$
  - Equivalently: $H_{sh/hp}(u,v) = 1 - H_{bl/lp}(u,v)$
- High-boost filtering
  - Certain amount of original image is included in the filtered image
  - $H_{hb}(u,v) = A - H_{lp}(u,v)$ A>=1
- High-frequency emphasis filtering
  - $H_{hfe}(u,v) = a + bH_{hp}(u,v),\ b > a \text{ and } a > 0$

Example: Laplacian filtered image

$$g(x,y) = f(x,y) - \nabla^2 f(x,y) = \Im^{-1}\left\{\left[1+\left(\left(u-\frac{M}{2}\right)^2+\left(v-\frac{N}{2}\right)^2\right)\right]F(u,v)\right\}$$

A integrated operation in frequency domain

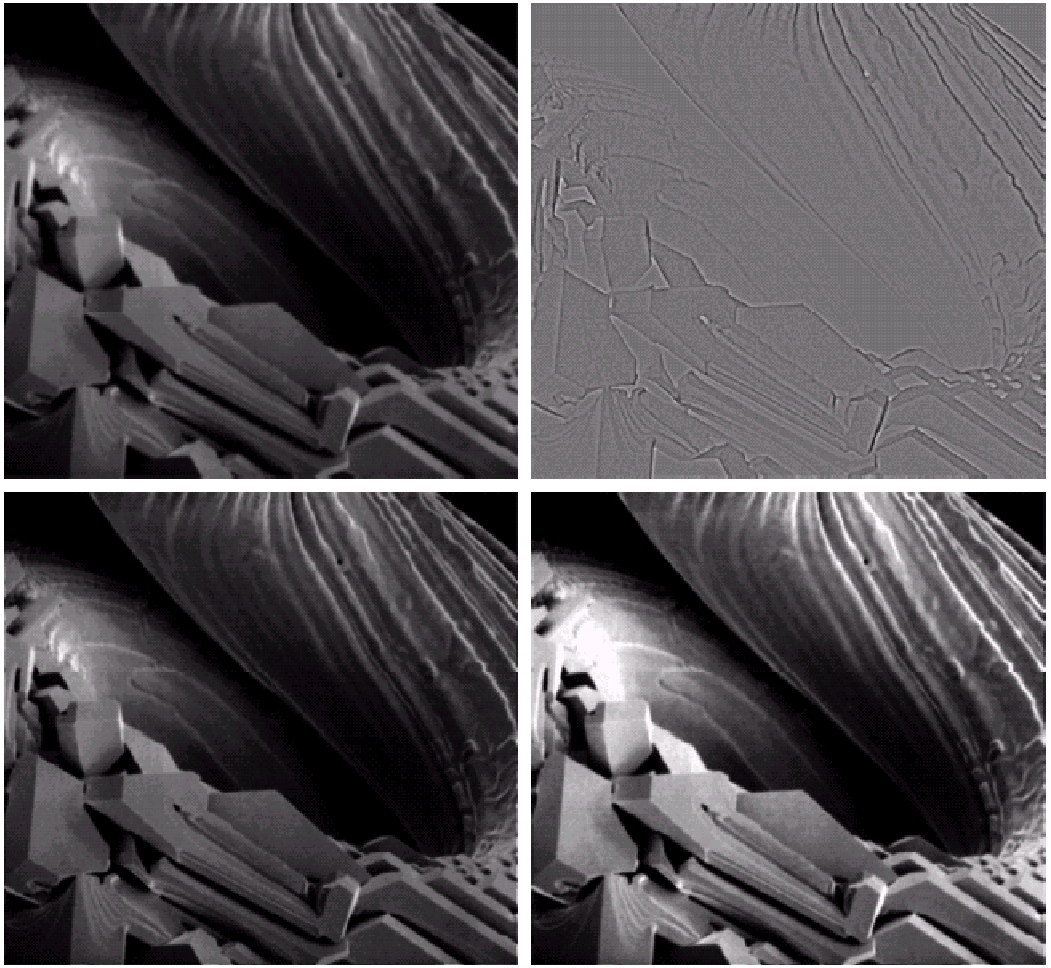


a b
c d

FIGURE 4.28 (a) Image of the North Pole of the moon. (b) Laplacian filtered image. (c) Laplacian image scaled. (d) Image enhanced by using Eq. (4.4-12). (Original image courtesy of NASA.)

For display purposes only

Example: high-boost filter



a b
c d

FIGURE 4.29 Same as Fig. 3.43, but using frequency domain filtering. (a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with $A = 2$. (d) Same as (c), but with $A = 2.7$. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

Examples

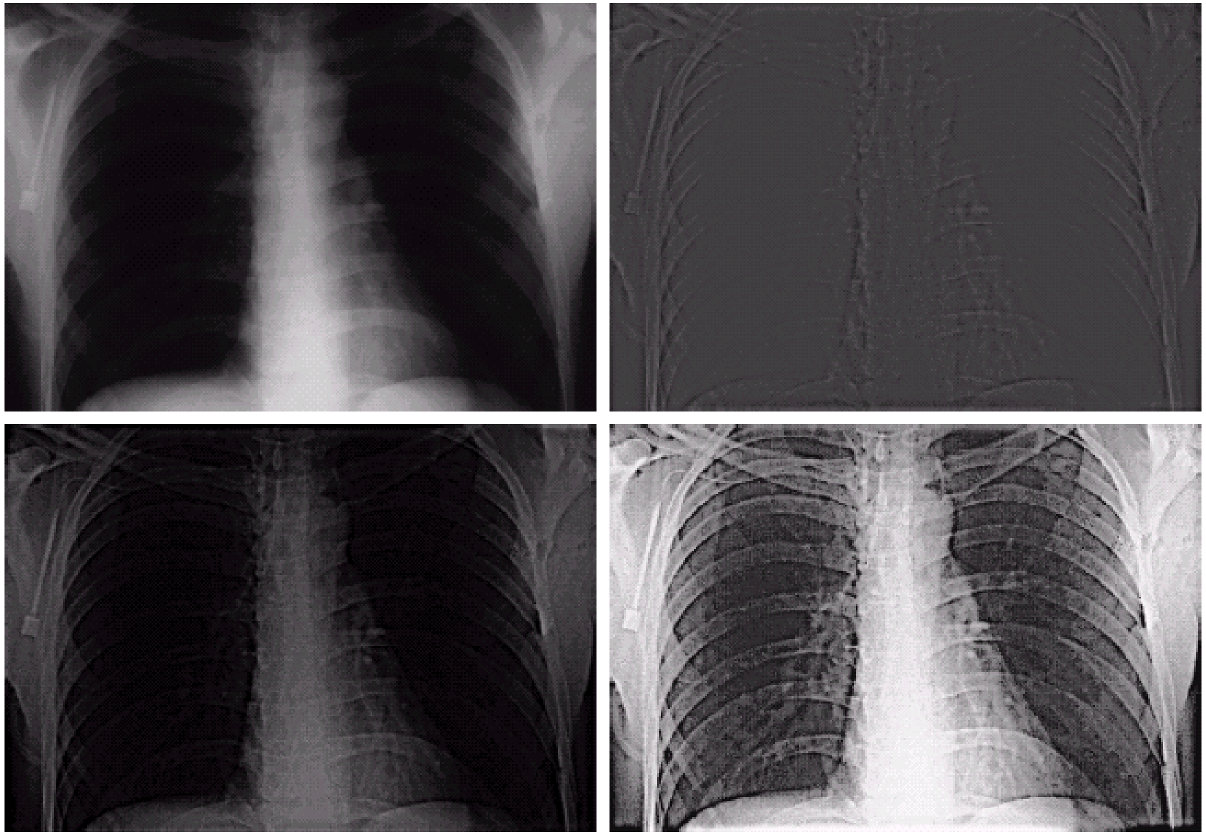


FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

4.5 Homomorphic Filtering

- Problems:
- When the illumination radiating to an object is non-uniform, the detail of the dark part in the image is more discernable.
- aims:
- Simultaneously compress the gray-level range and enhance contrast, eliminate the effect of non-uniform illumination, and emphasis the details.

4.5 Homomorphic Filtering

- Principal:
- Generally, the illumination component of an image is characterized by slow spatial variations, while the reflectance components tends to vary abruptly, particularly at the junctions of dissimilar objects. These characteristics lead to associating the low frequencies of the Fourier transform of the logarithm of an image with illumination and the high frequencies with reflectance.

4.5 Homomorphic Filtering

- For simultaneous dynamic range compression and contrast enhancement
  - Net effect is similar to the combination of high-boost filtering and log transformations
- Separation of illumination and reflectance components
  - $f(x,y)=i(x,y)\cdot r(x,y)$
  - $i(x,y)$ varies slowly ➔ low frequency components
  - $r(x,y)$ varies fast ➔ high frequency components
  - Attenuate components of $i(x,y)$ and accentuate components of $r(x,y)$➔ compress dynamic range and enhance details

Homomorphic Filtering

- For the foregoing purpose, we may desire a filter with a transfer function of the following shape

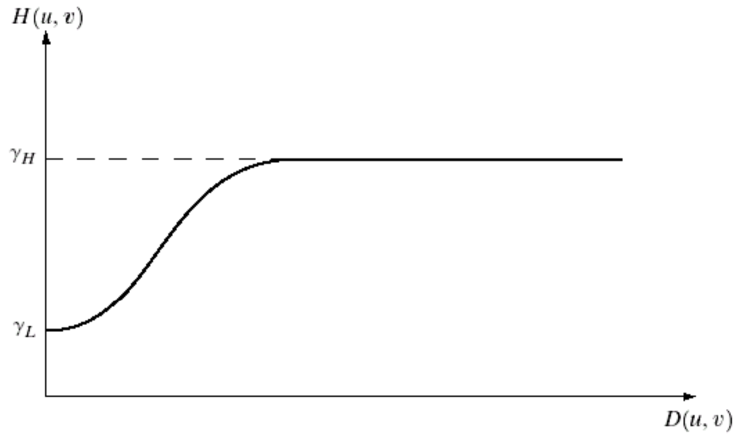


FIGURE 4.32 Cross section of a circularly symmetric filter function. $D(u, v)$ is the distance from the origin of the centered transform.

- However, we can not apply this filter on the FT of an image directly because the FT of an image is not equal to the sum of the FT of the illumination component and that of the reflection component

Homomorphic Filtering

- The curve shape shown in above figure can be approximated using basic form of the ideal highpass filters, for example, using a slightly modified form of the Gaussian highpass filter and can obtain

$$H(u,v) = (\gamma_H - \gamma_L)[1 - e^{-c(D^2(u,v)/D_0^2)}] + \gamma_L$$

The illumination-reflectance model of an image

- Illumination coefficient: $i(x, y)$
- reflectance coefficient: $r(x, y)$

$$f(x, y) = i(x, y)r(x, y)$$

Steps: 1) $z(x, y) = \ln f(x, y) = \ln i(x, y) + \ln r(x, y)$

2) $\mathcal{F}[z(x, y)] = \mathcal{F}[\ln i(x, y)] + \mathcal{F}[\ln r(x, y)]$

$$Z(u, v) = I(u, v) + R(u, v)$$

3) Determine the $H(u, v)$, which must compress the dynamic range of i(x,y), and enhance the contrast of r(x,y) component.

$$S(u, v) = H(u, v)I(u, v) + H(u, v)R(u, v)$$

The illumination-reflectance model of an image

Steps: 4) $$i'(x,y) = \mathcal{F}^{-1}[H(u,v)I(u,v)]$$

$$r'(x,y) = \mathcal{F}^{-1}[H(u,v)R(u,v)]$$

5) $$i_0(x,y) = \exp[i'(x,y)]$$

$$r_0(x,y) = \exp[r'(x,y)]$$

$$g(x,y) = i_0(x,y)r_0(x,y)$$

Homomorphic Filter

- Flow chart of the homomorphic filtering is given below

$s(x,y)$

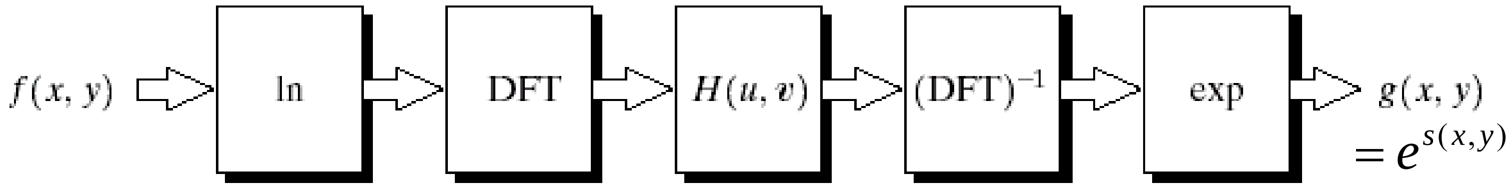


FIGURE 4.31 Homomorphic filtering approach for image enhancement.

$z(x,y)$ $Z(u,v)$ $S(u,v)$

$= Z(u,v) \cdot H(u,v)$

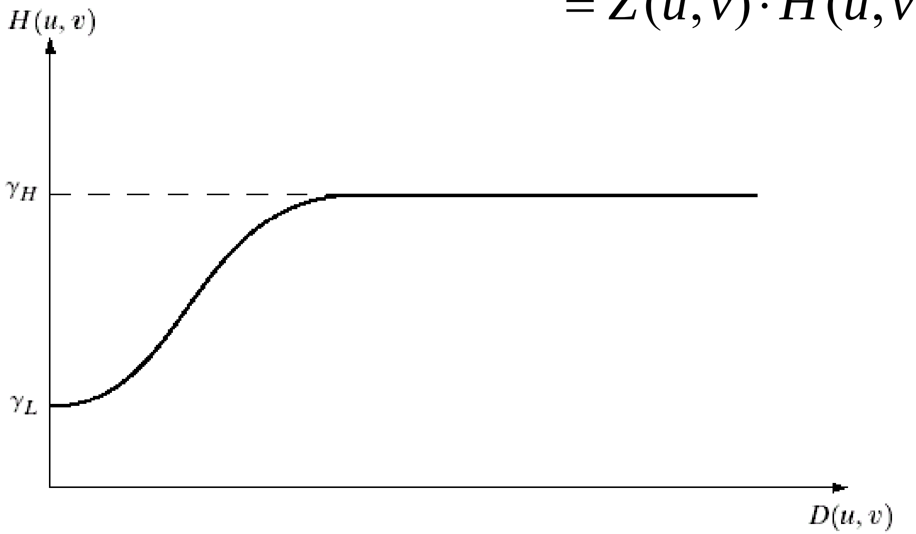


FIGURE 4.32 Cross section of a circularly symmetric filter function. $D(u,v)$ is the distance from the origin of the centered transform.

Result of Homomorphic filter

a b

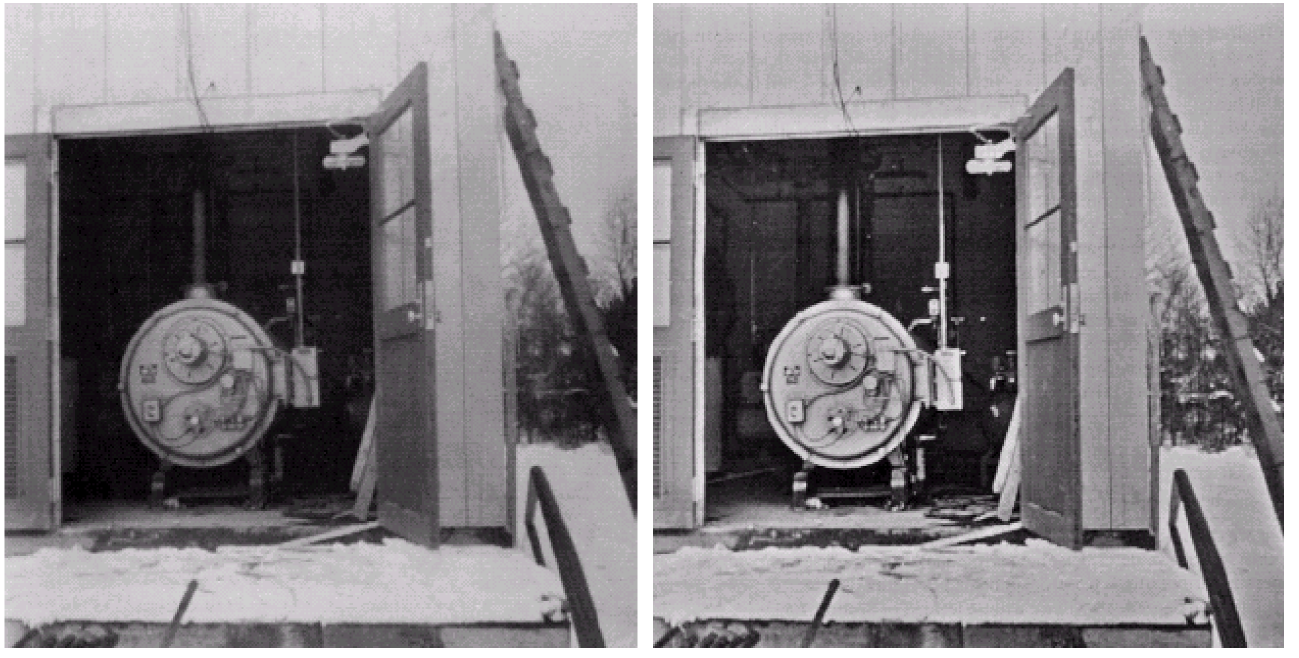


FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)

4.6 Implementation
4.6.1 Some Additional Properties of the 2D Fourier Transform

- Distributivity, scaling, and Translation:

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$$

$$f(x,y)e^{j2\pi(xu_0/M+yu_0/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

- Distributivity and scaling

$$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$$

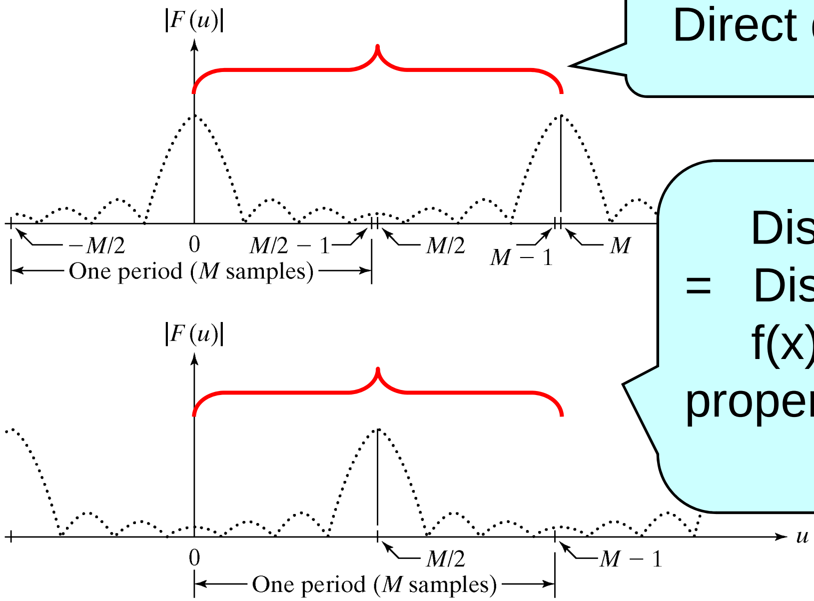
$$f(ax,by) \Leftrightarrow \frac{1}{ab}F(u/a, v/b)$$

- rotation

$$f(r,\theta+\theta_0) \Leftrightarrow F(w,\varphi+\theta_0)$$

Displaying a DFT

- DFT is periodic and is conjugate symmetric about the origin; therefore, it is usually displayed in the range [-(M-1)/2, (M-1)/2] if M is odd or [-M/2, M/2-1] if M is even, instead of being in the range of [0,M-1]



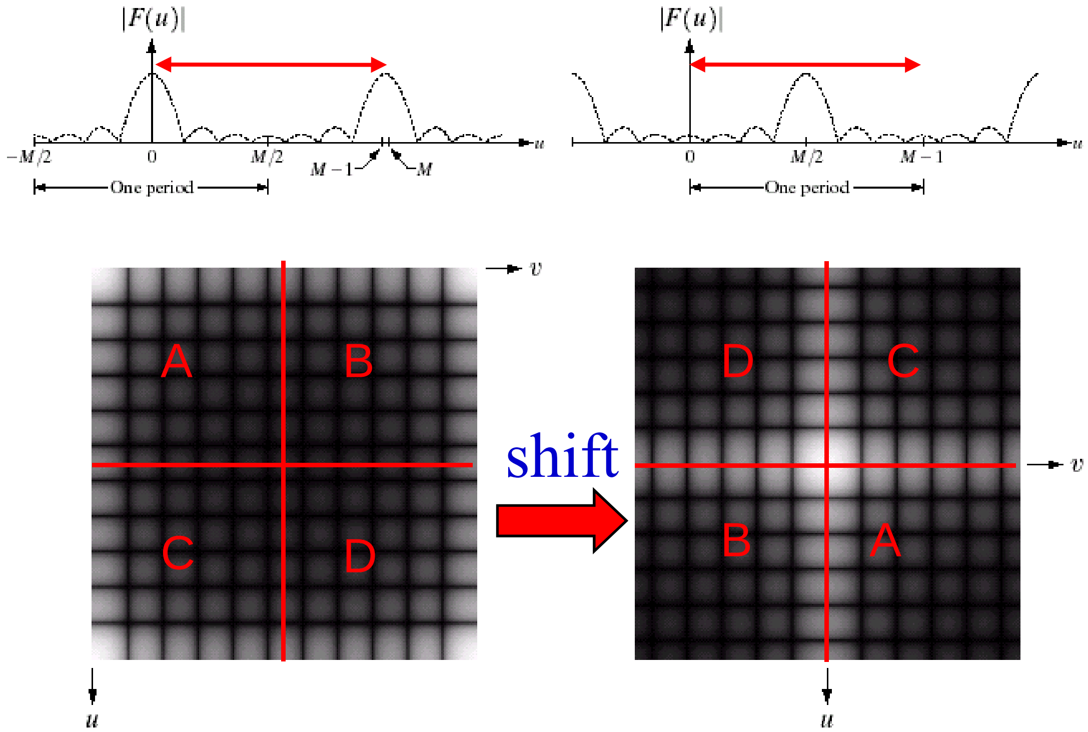
Direct display of |F(u)|

Display of the shifted DFT
= Display of the DFT of $f(x)(-1)^x$ (from the translation property)

Implementation

- Periodicity, conjugate symmetry, and back-to-back properties

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$
$$f(u,v) = f(u+M,v) = f(u,v+N) = f(u+M,v+N)$$
$$F(u,v) = F^*(-u,-v)$$



a b
c d

FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.

Separability

- 2D Fourier transform is separable
  - It implies that 2D Fourier transform can be implemented by a series of 1D Fourier transform.
  - $$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$
    $$= \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \sum_{y=0}^{N-1} f(x,y)e^{-2\pi vy/N}$$
    $$= \sum_{x=0}^{M-1} F(x,v)e^{-j2\pi ux/M}$$

1D Fourier transform of $F(x,v)$ for a fixed v. (column wise)

where

$$F(x,v) = \sum_{y=0}^{N-1} f(x,y)e^{-2\pi vy/N}$$

1D Fourier transform of $f(x,y)$ for a fixed x. (row wise)

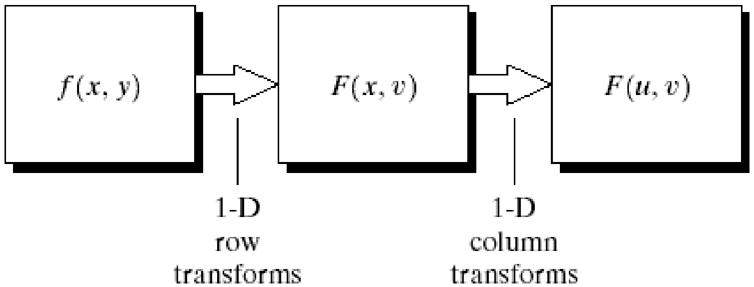


FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

4.6.2 computing the inverse FF using a forward transform algorithm

- Repeat the one-dimensional inverse FF:

$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}$$

- Take the complex conjugate of two side and multiply M

$$\frac{1}{M} f^*(x) = \frac{1}{M} \sum_{u=0}^{M-1} F^*(u) e^{-j2\pi ux/M}$$

4.6.2 computing the inverse FF using a forward transform algorithm

- Which is the form of forward FF. Take the complex conjugate of the result of the above equation and will get the inverse FF by forward transform.
- For two-dimensional case, similarly have

$$\frac{1}{MN} f^*(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u,v) e^{-j2\pi(ux/M + vy/N)}$$

- Here, we can treat F(u,v) as a simple function presenting on the forward transform equation

Convolution Theory

- Convolution theory is the bridge connecting spatial domain and frequency (Fourier) domain
  - Convolution of two functions f(x,y) and h(x,y) of size $M \times N$ is defined by $$g(x,y) = f(x,y) * h(x,y) = \sum_{m=0}^{M-1}\sum_{n}^{N-1} f(m,n)h(x-m, y-n)$$
  - If the size of h(x,y) is smaller than $M \times N$, it is first padded by zero to make the size $M \times N$
  - f(x,y) and h(x,y) are inherently periodic
  - Convolution theory says that the Fourier transform of g(x,y) is equal to the product of the Fourier transform f(x,y) and that of h(x,y) $$G(u,v) = F(u,v) \cdot H(u,v)$$
  - Similarly the Fourier transform of the product of f(x,y) and h(x,y) is equal to the convolution of the Fourier transform of f(x,y) and that of h(x,y) $$\Im(f(x,y) \cdot h(x,y)) = F(u,v) * H(u,v)$$

Convolution Theory

- Proof: (1D case)
  - Give f(x), h(x), and g(x) where

$$g(x) = f(x) * h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

  - We want to show that the DFT of g(x) is equal to F(u)·H(u)

$$G(u) = \Im(g(x)) \equiv \sum_{x=0}^{M-1}[\sum_{m=0}^{M-1} f(m)h(x-m)]e^{-j2\pi ux/M}$$

$$= \sum_{m=0}^{M-1} f(m)\sum_{x=0}^{M-1} h(x-m)e^{-j2\pi ux/M}$$

$$= \sum_{m=0}^{M-1} f(m)H(u)e^{-j2\pi um/M} \text{ (shift property)}$$

$$= H(u)F(u)$$

4.6.3 More on Periodicity: the need for padding

Convolution process

$$f(x) * h(x) = \frac{1}{M} \sum_{m=0}^{M-1} f(m) h(x-m)$$

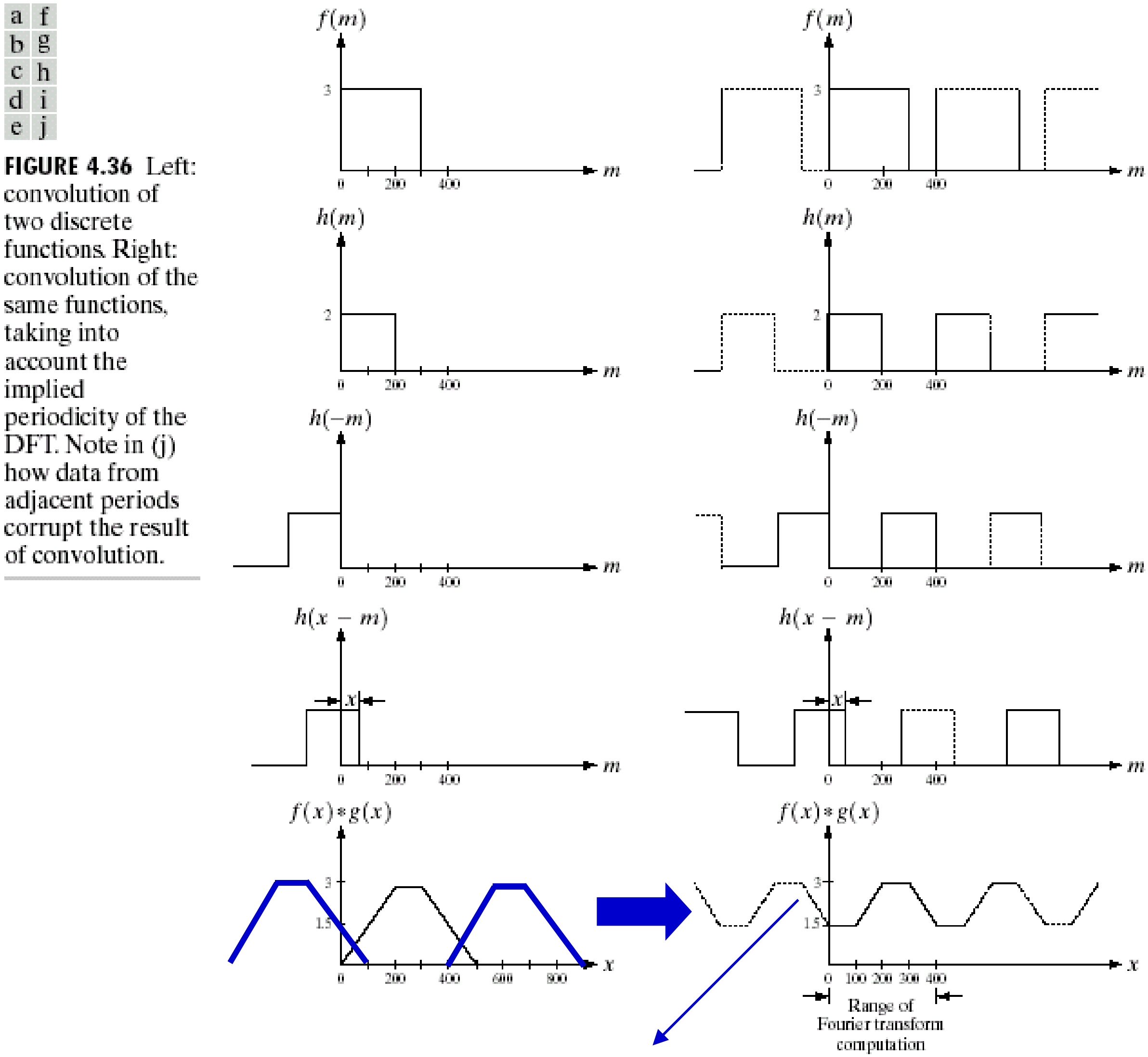


FIGURE 4.36 Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.

Aliasing or wraparound error

The methods of solving the aliasing problem is to extend and pad.

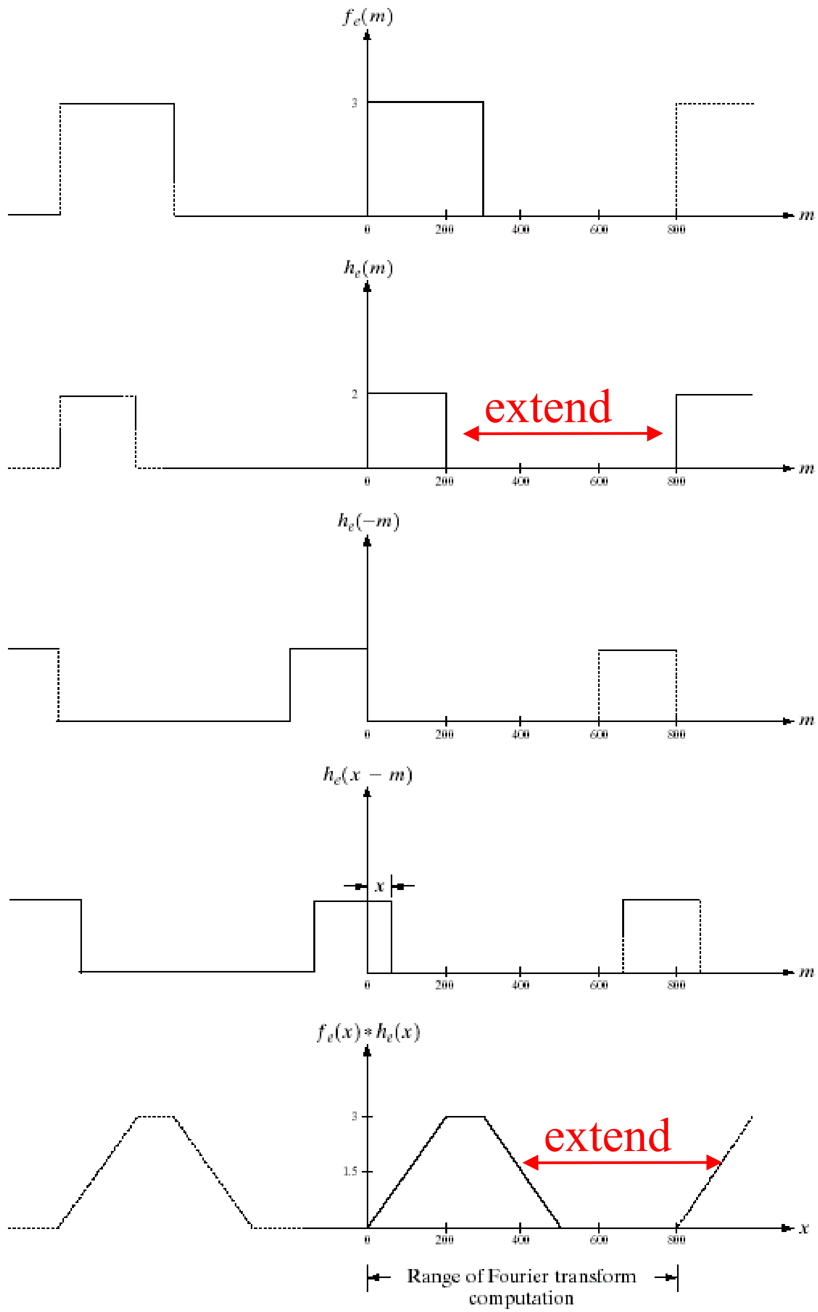
$$f_e(x) = \begin{cases} f(x) & 0 \le x \le A-1 \\ 0 & A \le x \le P-1 \end{cases}$$

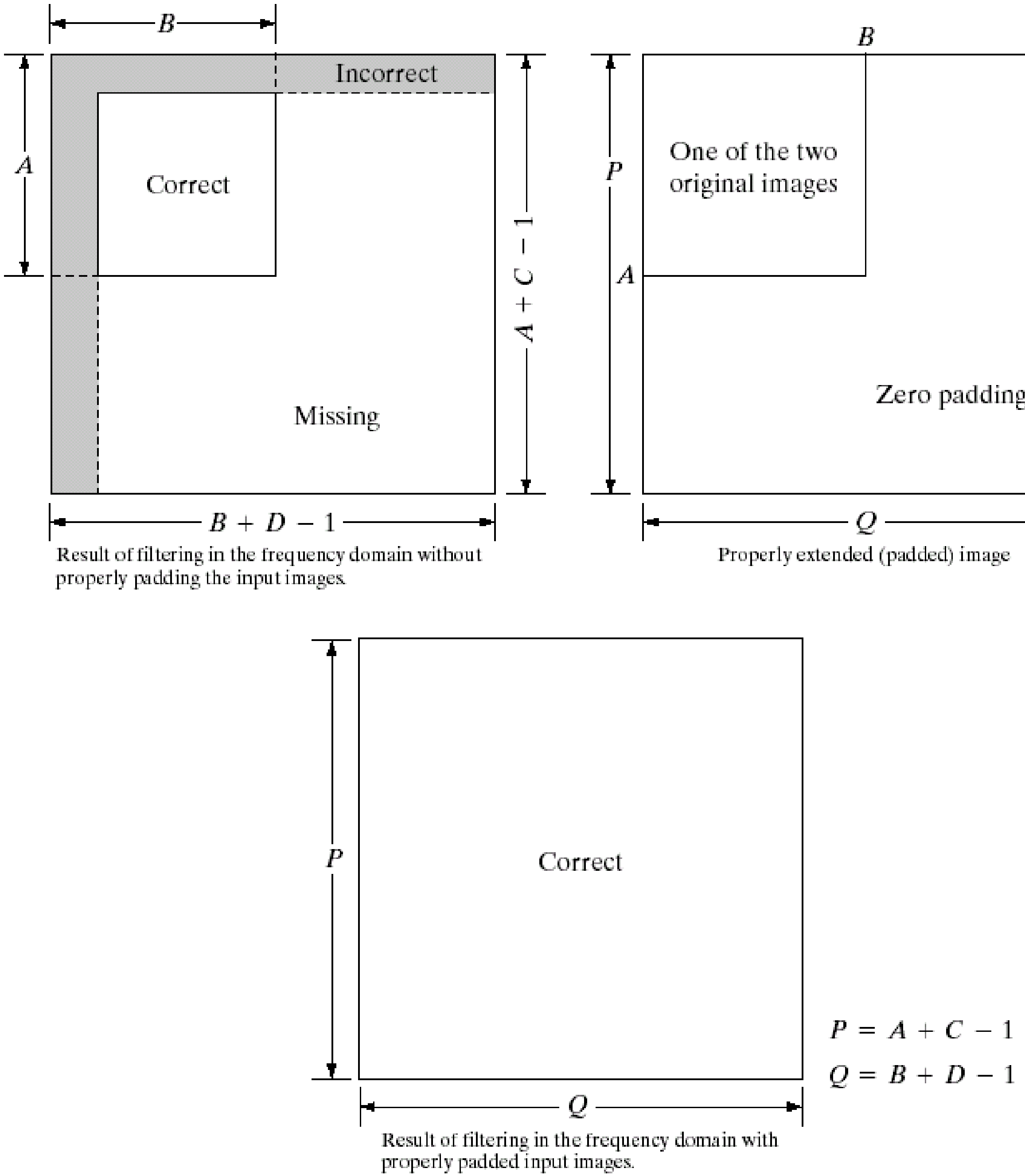
$$h_e(x) = \begin{cases} h(x) & 0 \le x \le B-1 \\ 0 & B \le x \le P-1 \end{cases}$$

where $P \ge A + B - 1$

a
b
c
d
e

FIGURE 4.37 Result of performing convolution with extended functions. Compare Figs. 4.37(e) and 4.36(e).

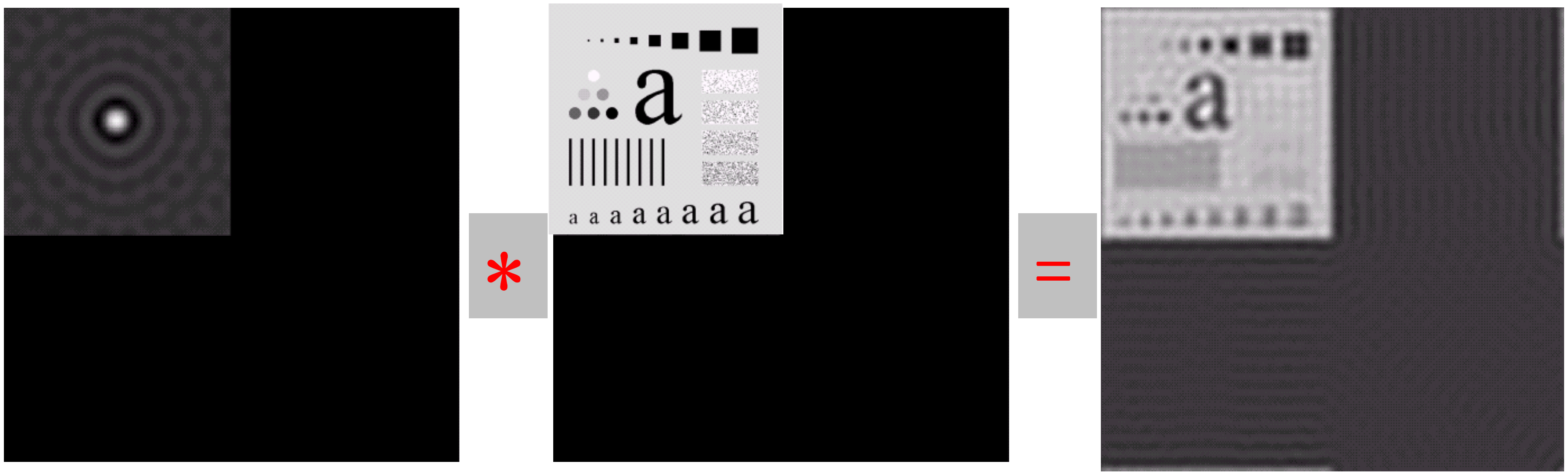




a b
c

FIGURE 4.38 Illustration of the need for function padding. (a) Result of performing 2-D convolution without padding. (b) Proper function padding. (c) Correct convolution result.

An example



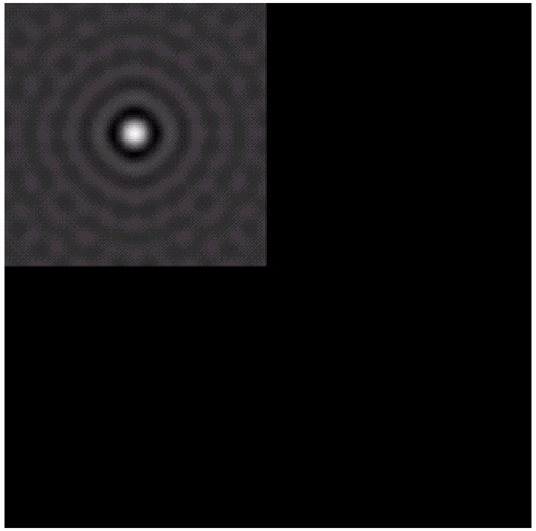
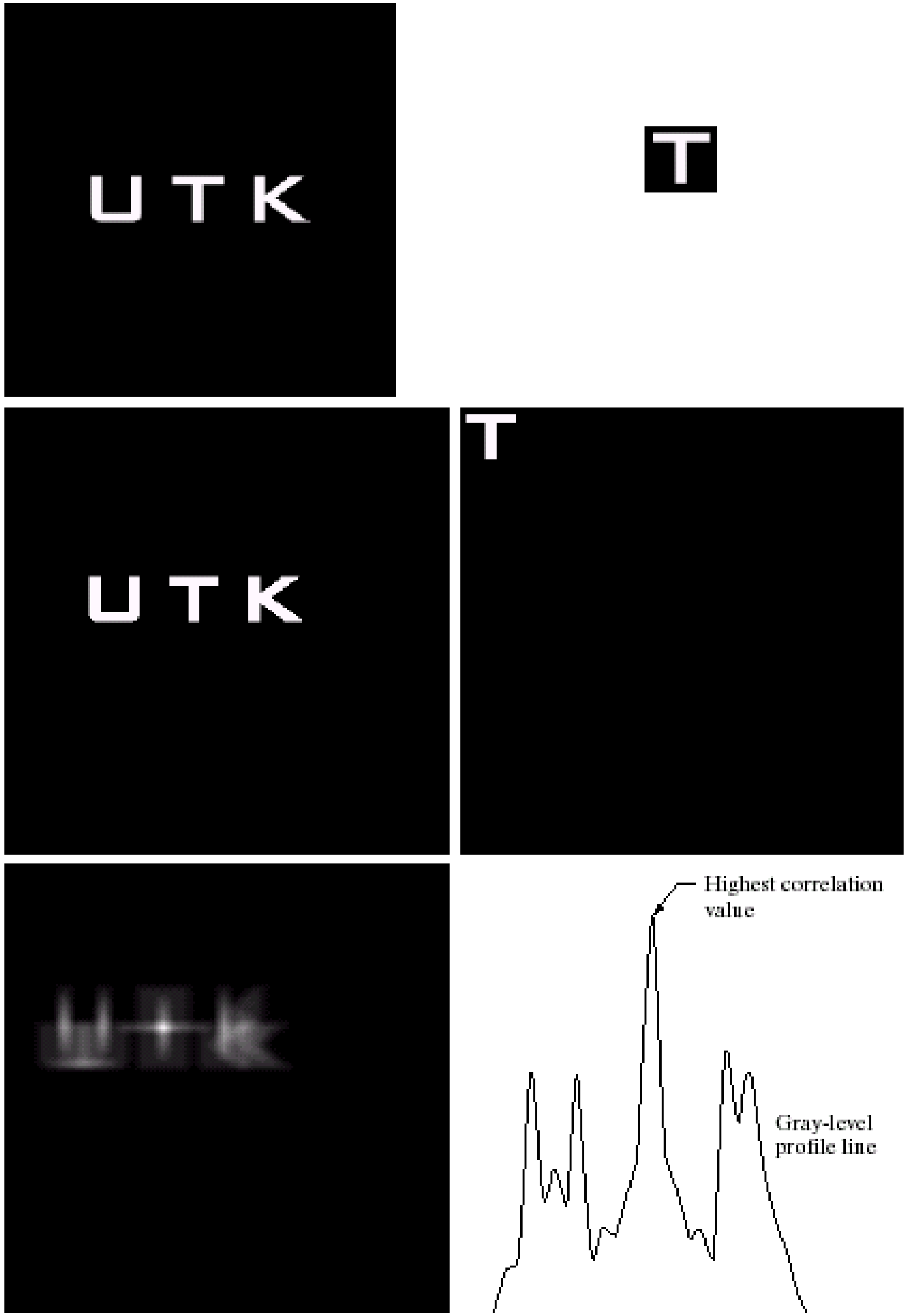


FIGURE 4.39 Padded lowpass filter is the spatial domain (only the real part is shown).

FIGURE 4.40 Result of filtering with padding. The image is usually cropped to its original size since there is little valuable information past the image boundaries.

4.6.4 convolution and correlation theorems

- Convolution $$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$
- Correlation $$f(x,y)\circ h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^*(m,n)h(x+m,y+n)$$
- Except for the complex of f and h not mirrored about the origin, everything else in the implementation of correlation is identical to convolution, including the need for padding.
- Correlation theorem
$$f(x,y)\circ h(x,y)=F^*(u,v)H(u,v)$$
$$f^*(x,y)h(x,y)=F(u,v)\circ H(u,v)$$
- Correlation includes across- and auto-correlation, and its main use is for matching and sure the location where h (template) finds a correspondence in f.



a b
c d
e f

FIGURE 4.41
(a) Image.
(b) Template.
(c) and
(d) Padded images.
(e) Correlation function displayed as an image.
(f) Horizontal profile line through the highest value in (e), showing the point at which the best match took place.

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \dfrac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $\quad R = \mathrm{Real}(F)$ and $I = \mathrm{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\dfrac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \dfrac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

TABLE 4.1 *(continued)*

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\lvert F(u, v)\rvert = \lvert F(-u, -v)\rvert$
Differentiation	$\dfrac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \dfrac{\partial^n F(u, v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$
Scaling	$af(x, y) \Leftrightarrow aF(u, v), f(ax, by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN}f^*(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting the function $F^*(u,v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x,y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution[†]	$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$
Correlation[†]	$f(x,y) \circ h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^*(m,n)h(x+m,y+n)$
Convolution theorem[†]	$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$; $f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$
Correlation theorem[†]	$f(x,y) \circ h(x,y) \Leftrightarrow F^*(u,v)H(u,v)$; $f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \circ H(u,v)$

TABLE 4.1
(continued)

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \dfrac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\dfrac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$

† Assumes that functions have been extended by zero padding.

TABLE 4.1
(continued)

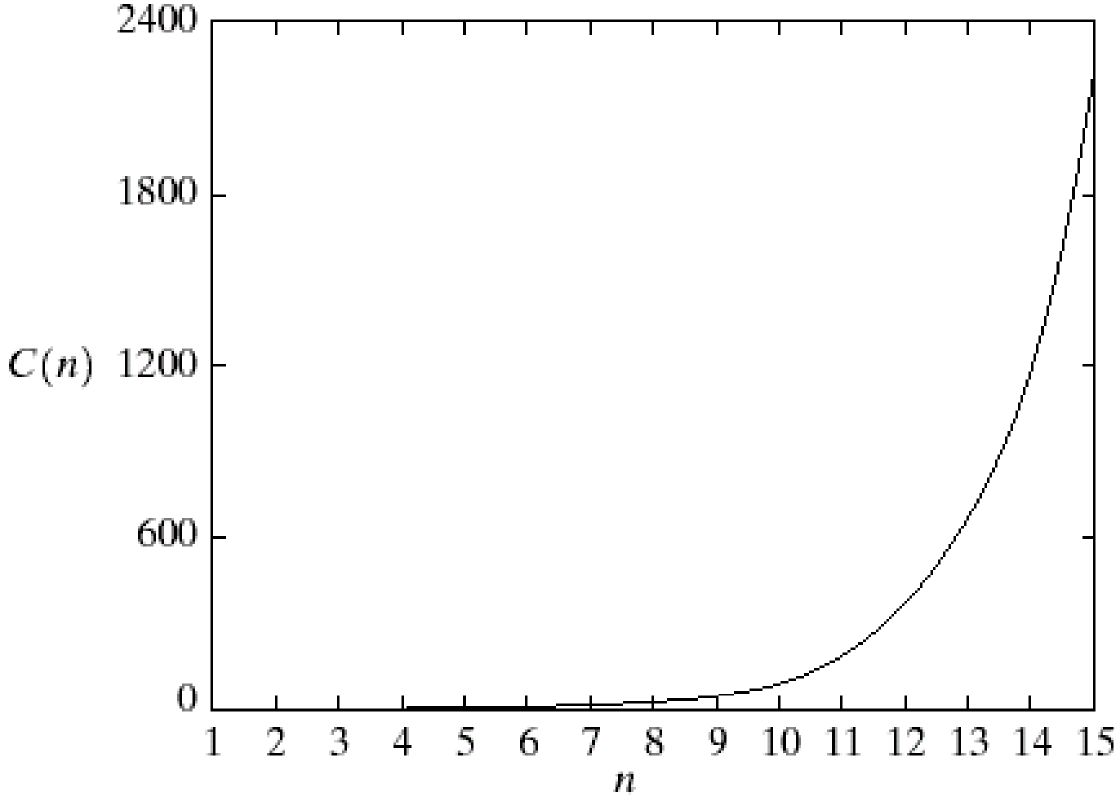
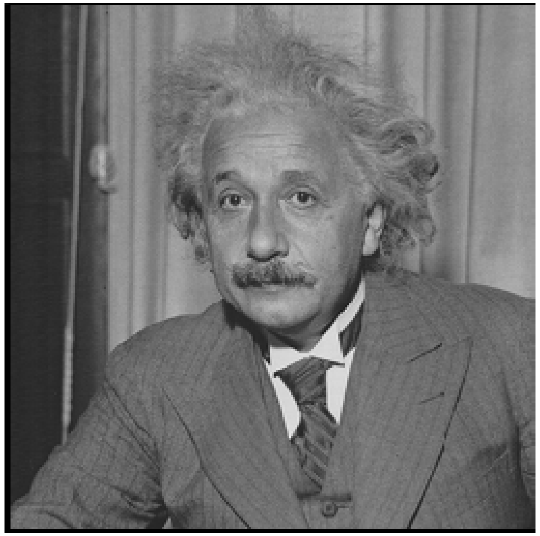


FIGURE 4.42 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of n.

The Fourier Transform Helps Us Analyze Convolutions

- Notation:

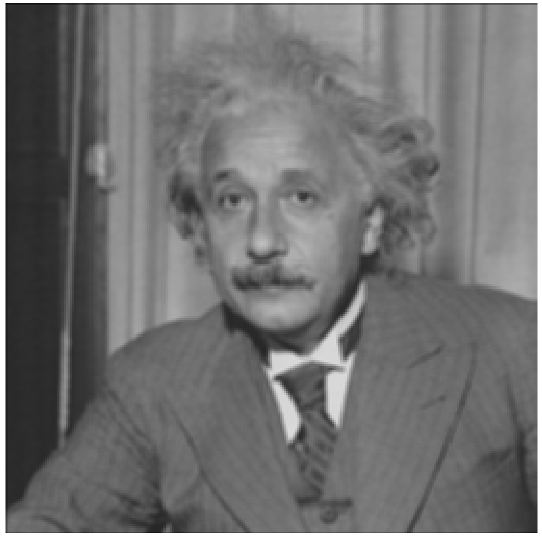
  $f(x,y)$ is the signal, $F(u,v)$ is the DFT
- If $h = f * g \leftarrow$ Convolution
  - Then $H(u,v) = F(u,v)G(u,v)$
- Convolution in the spatial domain is multiplication in the Fourier domain

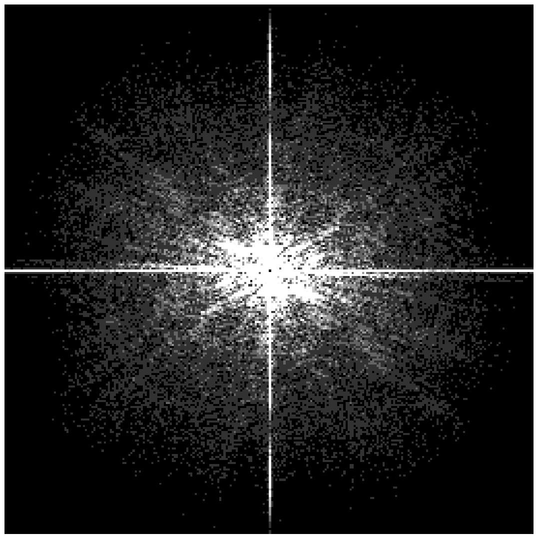


*

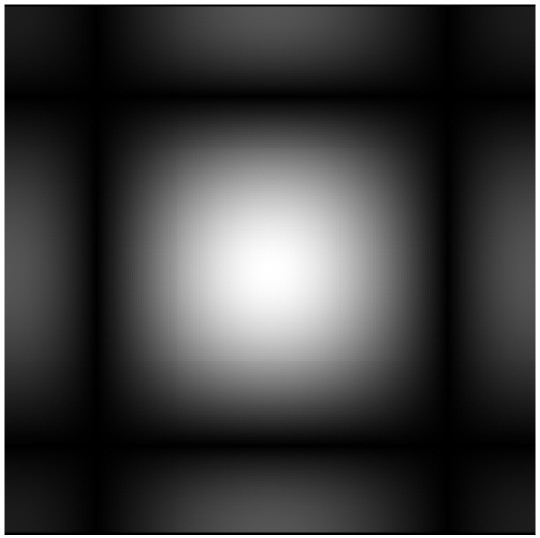
1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

=





X



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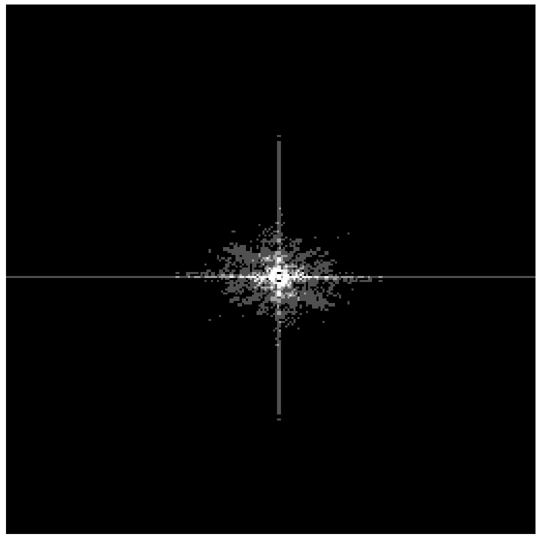

Summary

- Spatial and frequency filtering are both highly subjective processes
- The basic concept about Fourier Transform (FT)
- The algorithm of FT and the process of frequency filtering
- The physical meaning related to FT
- The relationship of resolution between spatial and frequency domain

Summary

- Correspondence between filtering in the spatial and frequency domains
- The general type of smoothing and sharpening filters and their main features
- Homomorphic filtering
- Some important properties of FT
- Convolution and correlation theorems