JEE ADVANCED

MATHEMATICS

16 YEARS TOPICWISE SOLVED PAPERS WITH SOLUTIONS

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SOLUTIONS

QUADRATIC EQUATION

1. For $x \in \mathbb{R}$, then number of real roots of the equation $3x^2 - 4|x^2 - 1| + x - 1 = 0$ is ____.

[JEE(Advanced) 2022]

2. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$ is **[JEE(Advanced) 2020]**

(A) 0 (B) 8000 (C) 8080 (D) 16000

Paragraph for Question No. 3 and 4

Let p,q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$.

For $n = 0, 1, 2, \ldots$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

3. If $a_4 = 28$, then $p + 2q =$ **[JEE(Advanced) 2017]**

(A) 14 (B) 7 (C) 12 (D) 21

4. $a_{12} =$ **[JEE(Advanced) 2017]**

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

5. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

[JEE(Advanced) 2016]

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

6. Let S be the set of all non-zero numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ? **[JEE(Advanced) 2015]**

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

7. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT-JEE 2011]**

(A) 1 (B) 2 (C) 3 (D) 4

8. A value of b for which the equations

$x^2 + bx - 1 = 0$

$x^2 + x + b = 0$,

have one root in common is - **[IIT-JEE 2011]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

9. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If a and b are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

[IIT-JEE 2010]

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$ (B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$ (D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

10. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is________. **[IIT-JEE 2009]**

11. Let a, b, c, p, q be real numbes. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and α, $\frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + x = 0$, where $\beta^2 \notin \{-1, 0, 1\}$

STATEMENT-1 : $(p^2 - q)(b^2 - ac) \geq 0$

and

STATEMENT-2 : $b \neq pa$ or $c \neq qa$ **[IIT-JEE 2008]**

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

LOGARITHM

1. The product of all positive real values of x satisfying the equation

$$x^{(16(\log_5 x)^3 - 68\log_5 x)} = 5^{-16}$$

is______. **[JEE(Advanced) 2022]**

2. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$ is ______. **[JEE(Advanced) 2018]**

3. If $3^x = 4^{x-1}$, then x = **[JEE(Advanced) 2013]**

(A) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (B) $\frac{2}{2 - \log_2 3}$ (C) $\frac{2}{1 - \log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3 - 1}$

4. The value of $6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots\cdots}}}\right)$ is **[IIT-JEE 2012]**

5. Let (x_0, y_0) be the solution of the following equations **[IIT-JEE 2011]**

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

$$3^{\ln x} = 2^{\ln y}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

SEQUENCE & SERIES

1. Let $7\overbrace{5...5}^{r}7$ denote the (r + 2) digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + ... + 7\overbrace{5...5}^{98}7$. If $S = \frac{7\overbrace{5...5}^{99}7 + m}{n}$, where m and n are natural numbers less than 3000, then the value of m + n is **[JEE(Advanced) 2023]**

2. Let $l_1, l_2, ..., l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1, and let $w_1, w_2, ..., w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2, where $d_1 d_2 = 10$. For each $i = 1, 2, ..., 100$, let R_i be a rectangle with length l_i, width w_i and area A_i. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ________. **[JEE(Advanced) 2022]**

3. Let $a_1, a_2, a_3, ...$ be an arithmetic progression with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, ...$ be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $T_{20} = 1604$ (B) $\sum_{k=1}^{20} T_k = 10510$

(C) $T_{30} = 3454$ (D) $\sum_{k=1}^{30} T_k = 35610$

4. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ______. **[JEE(Advanced) 2020]**

5. Let $a_1, a_2, a_3, ...$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, ...$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + ...+a_n) = b_1 + b_2 + ... + b_n$$

holds for some positive integer n, is ______ **[JEE(Advanced) 2020]**

6. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}, n \geq 2$.

Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$ (B) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(C) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$ (D) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

7. Let AP (a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then a + d equals ______. **[JEE(Advanced) 2019]**

8. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ..., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ______. **[JEE(Advanced) 2018]**

9. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? [JEE(Advanced) 2017]

10. Let $b_i > 1$ for $i = 1, 2, ... 101$. Suppose $\log_e b_1, \log_e b_2, ..., \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$ then [JEE(Advanced) 2016]

(A) $s > t$ and $a_{101} > b_{101}$
(B) $s > t$ and $a_{101} < b_{101}$
(C) $s < t$ and $a_{101} > b_{101}$
(D) $s < t$ and $a_{101} < b_{101}$

11. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is [JEE(Advanced) 2015]

12. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2+a-14}{a+1}$ is [JEE(Advanced) 2014]

13. Let $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) [JEE(Advanced) 2013]

(A) 1056
(B) 1088
(C) 1120
(D) 1332

14. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller to the numbers on the removed cards is k, then k – 20 = [JEE(Advanced) 2013]

15. Let $a_1, a_2, a_3, ...$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is [IIT-JEE 2012]

(A) 22
(B) 23
(C) 24
(D) 25

16. Let $a_1, a_2, a_3, ..., a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is [IIT-JEE 2011]

17. The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is [IIT-JEE 2011]

18. Let $a_1, a_2, a_3, ..., a_{11}$ be real numbers satisfying

$a_1 = 15, 27 - 2a_2 > 0$ and $ak = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, ..., 11$.

If $\frac{a_1^2 + a_2^2 + ... + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ... + a_{11}}{11}$ is equal to [IIT-JEE 2010]

19. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is [IIT-JEE 2009]

(A) $\frac{n(4n^2-1)c^2}{6}$
(B) $\frac{n(4n^2+1)c^2}{3}$
(C) $\frac{n(4n^2-1)c^2}{3}$
(D) $\frac{n(4n^2+1)c^2}{6}$

20. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

STATEMENT-1 : The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

and

STATEMENT-2 : The numbers b_1, b_2, b_3, b_4 are in H.P. [IIT-JEE 2008]

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

COMPOUND ANGLE

1. Let α and β be real numbers such that $-\frac{\pi}{4}<\beta<0<\alpha<\frac{\pi}{4}$. If $\sin(\alpha+\beta)=\frac{1}{3}$ and $\cos(\alpha-\beta)=\frac{2}{3}$, then the greatest integer less than or equal to

$$\left(\frac{\sin\alpha}{\cos\beta}+\frac{\cos\beta}{\sin\alpha}+\frac{\cos\alpha}{\sin\beta}+\frac{\sin\beta}{\cos\alpha}\right)^2$$

is ______. **[JEE(Advanced) 2022]**

2. Let α and β be nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha \cos\beta = 1$. Then which of the following is/are true ? **[JEE(Advanced) 2017]**

(A) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$ (B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

(C) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$ (D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

3. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to **[JEE(Advanced) 2016]**

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

4. Let $P=\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\}$ and $Q=\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\}$ be two sets. Then **[IIT-JEE 2011]**

(A) $P\subset Q$ and $Q-P\neq\varnothing$ (B) $Q\not\subset P$

(C) $P\not\subset Q$ (D) $P=Q$

5. The maximum value of the expression $\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$ is **[IIT-JEE 2010]**

6. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k>0$, then the value of $[k]$ is **[IIT-JEE 2010]**

[Note : [k] denotes the largest integer less than or equal to k]

7. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then :- **[IIT-JEE 2009]**

(A) $\tan^2 x=\frac{2}{3}$ (B) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$ (C) $\tan^2 x=\frac{1}{3}$ (D) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

TRIGONOMETRIC EQUATION

1. Consider the following lists:

List-I		List-II	
(I)	$\left\{x \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] : \cos x + \sin x = 1\right\}$	(P)	has two elements
(II)	$\left\{x \in \left[-\frac{5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$	(Q)	has three elements
(III)	$\left\{x \in \left[-\frac{6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos(2x) = \sqrt{3}\right\}$	(R)	has four elements
(IV)	$\left\{x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$	(S)	has five elements
		(T)	has six elements

The correct option is: **[JEE(Advanced) 2022]**

(A) (I) $\to$ (P); (II) $\to$ (S); (III) $\to$ (P); (IV) $\to$ (S)

(B) (I) $\to$ (P); (II) $\to$ (P); (III) $\to$ (T); (IV) $\to$ (R)

(C) (I) $\to$ (Q); (II) $\to$ (P); (III) $\to$ (T); (IV) $\to$ (S)

(D) (I) $\to$ (Q); (II) $\to$ (S); (III) $\to$ (P); (IV) $\to$ (R)

2. Let $f : [0, 2] \to \mathbb{R}$ be the function defined by **[JEE(Advanced) 2020]**

$$f(x) = (3 - \sin(2\pi x))\sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is ______.

3. Answer the following by appropriately matching the lists based on the information given in the paragraph. Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$$X = \{x : f(x) = 0\}, \qquad Y = \{x : f'(x) = 0\}$$
$$Z = \{x : g(x) = 0\}, \qquad W = \{x : g'(x) = 0\}.$$

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

[JEE(Advanced) 2019]

List-I		List-II	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination ?

Options :

(A) (II), (R), (S) (B) (I), (P), (R) (C) (II), (Q), (T) (D) (I), (Q), (U)

4. Answer the following by appropriately matching the lists based on the information given in the paragraph.
Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order : **[JEE(Advanced) 2019]**

$$X = \{x : f(x) = 0\}, \quad Y = \{x : f'(x) = 0\},$$
$$Z = \{x : g(x) = 0\}, \quad W = \{x : g'(x) = 0\}.$$

List-I contains the sets X, Y, Z and W. List-II contains some information regarding these sets.

List-I		List-II	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination ?

Options :

(A) (IV), (Q), (T)　(B) (IV), (P), (R), (S)　(C) (III), (R), (U)　(D) (III), (P), (Q), (U)

5. Let a, b, c be three non-zero real numbers such that the equation **[JEE(Advanced) 2018]**

$$\sqrt{3}a\cos x + 2b\sin x = c, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is ______.

6. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation

$\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to - **[JEE(Advanced) 2016]**

(A) $-\frac{7\pi}{9}$　(B) $-\frac{2\pi}{9}$　(C) 0　(D) $\frac{5\pi}{9}$

7. The number of distinct solutions of equation **[JEE(Advanced) 2015]**

$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$

in the interval $[0, 2\pi]$ is

8. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has **[JEE(Advanced) 2014]**

(A) infinitely many solutions　(B) three solutions

(C) one solution　(D) no solution

9. Let $\theta, \varphi \in [0, 2\pi]$ be such that **[IIT-JEE 2012]**

$$2\cos\theta(1 - \sin\varphi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\varphi - 1, \ \tan(2\pi - \theta) > 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2}$$

Then φ **cannot** satisfy-

(A) $0 < \varphi < \frac{\pi}{2}$　(B) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$　(C) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$　(D) $\frac{3\pi}{2} < \varphi < 2\pi$

10. The positive integer value of $n > 3$ satisfying the equation **[IIT-JEE 2011]**

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$ is

11. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is **[IIT-JEE 2010]**

SOLUTION OF TRIANGLE

Paragraph for Question No. 1 and 2

Consider on obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

1. Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is **[JEE(Advanced) 2023]**

2. Then the inradius of the triangle ABC is **[JEE(Advanced) 2023]**

3. Let PQRS be a quadrilateral in a plane, where $QR = 1$, $\angle PQR = \angle QRS = 70°$, $\angle PQS = 15°$ and $\angle PRS = 40°$. If $\angle RPS = \theta°$, $PQ = \alpha$ and $PS = \beta$, then the interval(s) that contain(s) the value of $4\alpha\beta\sin\theta°$ is/are **[JEE(Advanced) 2022]**

(A) $(0, \sqrt{2})$ (B) $(1, 2)$

(C) $(\sqrt{2}, 3)$ (D) $(2\sqrt{2}, 3\sqrt{2})$

4. In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$ and $CA = 4$. Then the value of $\frac{\cot A + \cot C}{\cot B}$ is______. **[JEE(Advanced) 2021]**

5. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$ (B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$ (D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

6. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE? **[JEE(Advanced) 2020]**

(A) $2Y = X + Z$ (B) $Y = X + Z$

(C) $\tan\frac{X}{2} = \frac{x}{y+z}$ (D) $x^2 + z^2 - y^2 = xz$

7. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, $q=1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$
(B) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2-\sqrt{3})$
(C) Length of $RS = \frac{\sqrt{7}}{2}$
(D) Length of $OE = \frac{1}{6}$

8. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018]**

(A) $\angle QPR = 45^\circ$
(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
(C) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$
(D) The area of the circumcircle of the triangle PQR is 100π.

9. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and $2s = x+y+z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

[JEE(Advanced) 2017]

(A) area of the triangle XYZ is $6\sqrt{6}$
(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$
(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

10. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is a third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is -

[JEE(Advanced) 2014]

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

11. In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are) **[JEE(Advanced) 2013]**

(A) 16 (B) 18 (C) 24 (D) 22

12. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

[IIT-JEE 2012]

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

13. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is **[IIT-JEE 2010]**

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

14. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are) **[IIT-JEE 2010]**

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

15. Consider a triangle ABC and let a,b and c denote the lengths of the sides opposite to vertices A,B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to **[IIT-JEE 2010]**

16. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then :- **[IIT-JEE 2009]**

(A) b + c = 4a (B) b + c = 2a
(C) locus of point A is an ellipse (D) locus of point A is a pair of straight lines

17. Let ABC and ABC' be two non–congruent triangles with sides AB = 4, AC = AC' = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is______. **[IIT-JEE 2009]**

DETERMINANT

1. Let α, β and γ be real numbers. consider the following system of linear equations **[JEE(Advanced) 2023]**

$x + 2y + z = 7$

$x + \alpha z = 11$

$2x - 3y + \beta z = \gamma$

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$, then the system has	(1) a unique solution
(Q) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$, then the system has	(2) no solution
(R) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma \neq 28$, then the system has	(3) infinitely many solutions
(S) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma = 28$, then the system has	(4) $x = 11$, $y = -2$ and $z = 0$ as a solution
	(5) $x = -15$, $y = 4$ and $z = 0$ as a solution

The correct option is :

(A) (P) → (3) (Q) → (2) (R) → (1) (S) → (4) (B) (P) → (3) (Q) → (2) (R) → (5) (S) → (4)
(C) (P) → (2) (Q) → (1) (R) → (4) (S) → (5) (D) (P) → (2) (Q) → (1) (R) → (1) (S) → (3)

2. Let p, q, r be nonzero real numbers that are, respectively, the 10^{th}, 100^{th} and 1000^{th} terms of a harmonic progression. Consider the system of linear equations **[JEE(Advanced) 2022]**

$$x + y + z = 1$$
$$10x + 100y + 1000z = 0$$
$$qr\,x + pr\,y + pq\,z = 0.$$

List-I		List-II	
(I)	If $\frac{q}{r} = 10$, then the system of linear equations has	(P)	$x = 0,\ y = \frac{10}{9},\ z = -\frac{1}{9}$ as a solution
(II)	If $\frac{p}{r} \neq 100$, then the system of linear equations has	(Q)	$x = \frac{10}{9},\ y = -\frac{1}{9},\ z = 0$ as a solution
(III)	If $\frac{p}{q} \neq 10$, then the system of linear equations has	(R)	infinitely many solutions
(IV)	If $\frac{p}{q} = 10$, then the system of linear equations has	(S)	no solution
		(T)	at least one solution

The correct option is:

(A) (I) → (T); (II) → (R); (III) → (S); (IV) → (T)

(B) (I) → (Q); (II) → (S); (III) → (S); (IV) → (R)

(C) (I) → (Q); (II) → (R); (III) → (P); (IV) → (R)

(D) (I) → (T); (II) → (S); (III) → (P); (IV) → (T)

3. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is **[JEE(Advanced) 2016]**

4. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$?

[JEE(Advanced) 2015]

(A) –4 (B) 9 (C) –9 (D) 4

5. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$(y + z)\cos 3\theta = (xyz)\sin 3\theta$

$x\sin 3\theta = \dfrac{2\cos 3\theta}{y} + \dfrac{2\sin 3\theta}{z}$

$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$ **[IIT-JEE 2010]**

6. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations

$3x - y - z = 0$

$-3x + z = 0$

$-3x + 2y + z = 0$

Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is ______. **[IIT-JEE 2009]**

7. Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1$

STATEMENT–1 : The system of equations has no solution for $k \neq 3$.

and

STATEMENT–2 : The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$. **[IIT-JEE 2008]**

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

STRAIGHT LINE

Question Stem for Question No. 1 and 2

Question Stem

Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2} + y - 1 = 0$ and $L_2 : x\sqrt{2} - y + 1 = 0$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'.

1. The value of λ^2 is ________. **[JEE(Advanced) 2021]**

2. The value of D is ________. **[JEE(Advanced) 2021]**

3. Let $a, \lambda, m \in \mathbb{R}$. Consider the system of linear equations

$ax + 2y = \lambda$

$3x - 2y = \mu$

Which of the following statement(s) is(are) correct ? **[JEE(Advanced) 2016]**

(A) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ .

(B) If $a \neq -3$, then the system has a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

4. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is **[JEE(Advanced) 2014]**

5. For $a > b > c > 0$, the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then **[JEE(Advanced) 2014]**

(A) $a + b - c > 0$ (B) $a - b + c < 0$ (C) $a - b + c > 0$ (D) $a + b - c < 0$

6. A straight line L through the point $(3, -2)$ is inclined at an angle $60°$ to the line $\sqrt{3}x + y = 1$. If L also intersect the x-axis, then the equation of L is **[IIT-JEE 2011]**

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

7. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}, 4\hat{i}, 3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a **[IIT-JEE 2010]**

(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square

8. The locus of the orthocentre of the triangle formed by the lines

$(1 + p) x - py + p (1 + p) = 0$,

$(1 + q)x - qy + q (1 + q) = 0$, and $y = 0$, where $p \neq q$, is **[IIT-JEE 2009]**

(A) a hyperbola (B) a parabola (C) an ellipse (D) a straight line

9. Consider three points $P = (-\sin(\beta - \alpha), -\cos\beta)$, $Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$. Then, **[IIT-JEE 2008]**

(A) P lies on the line segment RQ (B) Q lies on the segment PR

(C) R lies on the line segment QP (D) P, Q, R are non-collinear

10. Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

Match the Statement / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS. **[IIT-JEE 2008]**

	Column I		Column II
(A)	L_1, L_2, L_3 are concurrent, if	(p)	$k = -9$
(B)	One of L_1, L_2, L_3 is parallel to at least one p, q of the other two, if	(q)	$k = -\frac{6}{5}$
(C)	L_1, L_2, L_3 form a triangle, if	(r)	$k = \frac{5}{6}$
(D)	L_1, L_2, L_3 do not form a triangle, if	(s)	$k = 5$

CIRCLE

1. Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point $A = (4, 1)$, where $1 < r < 3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at P and C_2 at Q. The tangent ST touches C_1 at S and C_2 at T. Mid points of the line segments PQ and ST are joined to form a line which meets the x-axis at a point B. If $AB = \sqrt{5}$, then the value of r^2 is **[JEE(Advanced) 2023]**

2. Let ABC be the triangle with $AB = 1$, $AC = 3$ and $\angle BAC = \frac{\pi}{2}$. If a circle of radius $r > 0$ touches the sides AB, AC and also touches internally the circumcircle of the triangle ABC, then the value of r is ______.
[JEE(Advanced) 2022]

3. Let G be a circle of radius $R > 0$. Let $G_1, G_2, \ldots, G_n$ be n circles of equal radius $r > 0$. Suppose each of the n circles $G_1, G_2, \ldots, G_n$ touches the circle G externally. Also, for $i = 1, 2, \ldots, n-1$, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE ?
[JEE(Advanced) 2022]

(A) If $n = 4$, then $(\sqrt{2} - 1)r < R$
(B) If $n = 5$, then $r < R$
(C) If $n = 8$, then $(\sqrt{2} - 1)r < R$
(D) If $n = 12$, then $\sqrt{2}(\sqrt{3} + 1)r > R$

4. Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocenter of Δ is (1, 1), then the equation of the circle passing through the vertices of the triangle Δ is
[JEE(Advanced) 2021]

(A) $x^2 + y^2 - 3x + y = 0$
(B) $x^2 + y^2 + x + 3y = 0$
(C) $x^2 + y^2 + 2y - 1 = 0$
(D) $x^2 + y^2 + x + y = 0$

Paragraph for Question No. 5 and 6

Let

$$M = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \le r^2\},$$

where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, $n = 1, 2, 3, \ldots$. Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

5. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then
[JEE(Advanced) 2021]

(A) $k + 2l = 22$
(B) $2k + l = 26$
(C) $2k + 3l = 34$
(D) $3k + 2l = 40$

6. Consider M with $r = \frac{(2^{199} - 1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is
[JEE(Advanced) 2021]

(A) 198
(B) 199
(C) 200
(D) 201

7. Let O be the centre of the circle $x^2 + y^2 = r^2$, where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x + 2y = 4$, then the value of r is _____ **[JEE(Advanced) 2020]**

8. A line $y = mx + 1$ intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

[JEE(Advanced) 2019]

(A) $6 \le m < 8$ (B) $2 \le m < 4$ (C) $4 \le m < 6$ (D) $-3 \le m < -1$

9. Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____

[JEE(Advanced) 2019]

10. **Answer the following by appropriately matching the lists based on the information given in the paragraph**

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below:

[JEE(Advanced) 2019]

	List-I		**List-II**
(I)	$2h + k$	(P)	6
(II)	$\frac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\frac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\frac{5}{4}$
(IV)	α	(S)	$\frac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\frac{10}{3}$

Which of the following is the only INCORRECT combination ?

Options :

(A) (IV), (S) (B) (IV), (U) (C) (III), (R) (D) (I), (P)

11. Answer the following by appropriately matching the lists based on the information given in the paragraph.

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below:

[JEE(Advanced) 2019]

	List-I		List-II
(I)	$2h + k$	(P)	6
(II)	$\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\dfrac{5}{4}$
(IV)	α	(S)	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

Which of the following is the only CORRECT combination?

Options :

(A) (II), (T) (B) (I), (S) (C) (I), (U) (D) (II), (Q)

Paragraph for Question No. 12 and 13

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$

12. Let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop -1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3 and G_3 lie on the curve **[JEE(Advanced) 2018]**

(A) $x + y = 4$ (B) $(x-4)^2 + (y-4)^2 = 16$

(C) $(x-4)(y-4) = 4$ (D) $xy = 4$

13. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve-

[JEE(Advanced) 2018]

(A) $(x + y)^2 = 3xy$ (B) $x^{2/3} + y^{2/3} = 2^{4/3}$

(C) $x^2 + y^2 = 2xy$ (D) $x^2 + y^2 = x^2y^2$

14. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ? [JEE(Advanced) 2018]

(A) The point $(-2, 7)$ lies in E_1 (B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2 (D) The point $\left(0, \frac{3}{2}\right)$ does **NOT** lie in E_1

15. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points ? [JEE(Advanced) 2017]

16. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then- [JEE(Advanced) 2016]

(A) $Q_2Q_3 = 12$ (B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$ (D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

17. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)- [JEE(Advanced) 2016]

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

18. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then :- [JEE(Advanced) 2014]

(A) radius of S is 8 (B) radius of S is 7

(C) centre of S is (–7, 1) (D) centre is S is (–8, 1)

19. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are) [JEE(Advanced) 2013]

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$ (B) $x^2 + y^2 - 6x + 7y + 9 = 0$

(C) $x^2 + y^2 - 6x - 8y + 9 = 0$ (D) $x^2 + y^2 - 6x - 7y + 9 = 0$

20. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is- [IIT-JEE 2012]

(A) $20(x^2 + y^2) - 36x + 45y = 0$ (B) $20(x^2 + y^2) + 36x - 45y = 0$

(C) $36(x^2 + y^2) - 20x + 45y = 0$ (D) $36(x^2 + y^2) + 20x - 45y = 0$

Paragraph for Question No. 21 and 22

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$.

21. A common tangent of the two circles is **[IIT-JEE 2012]**

(A) $x = 4$ (B) $y = 2$ (C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

22. A possible equation of L is **[IIT-JEE 2012]**

(A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$ (C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

23. The circle passing through the point (–1,0) and touching the y-axis at (0,2) also passes through the point- **[IIT-JEE 2011]**

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$ (C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) (–4,0)

24. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\}.$$

then the number of point(s) in S lying inside the smaller part is **[IIT-JEE 2011]**

25. Tangents drawn from the point P(1, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is :- **[IIT-JEE 2009]**

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$ (B) $x^2 + y^2 - 4x - 10y + 19 = 0$

(C) $x^2 + y^2 - 2x + 6y - 29 = 0$ (D) $x^2 + y^2 - 6x - 4y + 19 = 0$

26. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is ________. **[IIT-JEE 2009]**

27. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then :- **[IIT-JEE 2008]**

(A) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ (B) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(C) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$ (D) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Paragraph for Question No. 28 to 30

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP and D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ.

28. The equation of circle C is :- **[IIT-JEE 2008]**

(A) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$ (B) $(x - 2\sqrt{3})^2 + (y + \frac{1}{2})^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$ (D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

29. Points E and F are given by – [IIT-JEE 2008]

(A) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),(\sqrt{3},0)$ (B) $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right),(\sqrt{3},0)$

(C) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ (D) $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

30. Equations of the sides QR, RP are :- [IIT-JEE 2008]

(A) $y=\frac{2}{\sqrt{3}}x+1,\ y=-\frac{2}{\sqrt{3}}x-1$ (B) $y=\frac{1}{\sqrt{3}}x,\ y=0$

(C) $y=\frac{2}{\sqrt{3}}x+1,\ y=\frac{\sqrt{3}}{2}x-1$ (D) $y=\sqrt{3}\ x,\ y=0$

31. Consider $L_1 : 2x + 3y + p - 3 = 0$

$L_2 = 2x + 3y + p + 3 = 0,$

where p is a real number, and $C : x^2 + y^2 + 6x - 10y + 30 = 0$.

STATEMENT–1 : If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

and

STATEMENT–2 : If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

[IIT-JEE 2008]

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

PARABOLA

1. Let P be a point on the parabola $y^2 = 4ax$, where $a > 0$. The normal to the parabola at P meets the x-axis at a point Q. The area of the triangle PFQ, where F is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a,m) is [JEE(Advanced) 2023]

(A) (2, 3) (B) (1, 3) (C) (2, 4) (D) (3, 4)

2. Consider the parabola $y^2 = 4x$. Let S be the focus of the parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE ? [JEE(Advanced) 2022]

(A) $SQ_1 = 2$ (B) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$ (C) $PQ_1 = 3$ (D) $SQ_2 = 1$

3. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) **TRUE**? [JEE(Advanced) 2021]

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Question Stem for Questions Nos. 4 and 5

Question Stem

Consider the region $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

4. The radius of the circle C is ______. **[JEE(Advanced) 2021]**

5. The value of α is ______. **[JEE(Advanced) 2021]**

6. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ? **[JEE(Advanced) 2017]**

(A) $p = 5, h = 4, k = -3$ (B) $p = -1, h = 1, k = -3$

(C) $p = -2, h = 2, k = -4$ (D) $p = 2, h = 3, k = -4$

7. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then-

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = (\sqrt{5}+1) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$ **[JEE(Advanced) 2016]**

8. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is **[JEE(Advanced) 2015]**

9. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is• **[JEE(Advanced) 2015]**

10. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ? **[JEE(Advanced) 2015]**

(A) $(4, 2\sqrt{2})$ (B) $(9, 3\sqrt{2})$ (C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $(1, \sqrt{2})$

11. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the point P, Q and the parabola at the points R,S. Then the area of the quadrilateral PQRS is - **[JEE(Advanced) 2014]**

(A) 3 (B) 6 (C) 9 (D) 15

Paragraph For Questions 12 and 13

Let a,r,s,t be nonzero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$.

12. The value of r is- **[JEE(Advanced) 2014]**

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

13. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is- **[JEE(Advanced) 2014]**

(A) $\frac{(t^2+1)^2}{2t^3}$ (B) $\frac{a(t^2+1)^2}{2t^3}$ (C) $\frac{a(t^2+1)^2}{t^3}$ (D) $\frac{a(t^2+2)^2}{t^3}$

Paragraph for Question 14 and 15

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

14. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$ **[JEE(Advanced) 2013]**

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

15. Length of chord PQ is **[JEE(Advanced) 2013]**

(A) 7a (B) 5a (C) 2a (D) 3a

16. A line L : $y = mx + 3$ meets y – axis at E(0,3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. **[JEE(Advanced) 2013]**

Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		List-II
P.	$m =$	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	$y_0 =$	3.	2
S.	$y_1 =$	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

17. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is **[IIT-JEE 2012]**

18. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[IIT-JEE 2011]**

19. Let (x,y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x,y) in the ratio 1 : 3. Then the locus of P is- **[IIT-JEE 2011]**

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

20. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9,6), then L is given by- **[IIT-JEE 2011]**

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

21. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be - **[IIT-JEE 2010]**

(A) $-\frac{1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $-\frac{2}{r}$

22. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose [IIT-JEE 2009]

(A) vertex is $\left(\frac{2a}{3}, 0\right)$
(B) directrix is $x = 0$
(C) latus rectum is $\frac{2a}{3}$
(D) focus is $(a, 0)$

23. Consider the two curves $C_1 : y^2 = 4x$ [IIT-JEE 2008]

$C_2 : x^2 + y^2 - 6x + 1 = 0$

Then,
(A) C_1 and C_2 touch each other only at one point
(B) C_1 and C_2 touch each other exactly at two points
(C) C_1 and C_2 intersect (but do not touch) at exactly two points
(D) C_1 and C_2 neither intersect nor touch each other

ELLIPSE

1. Let T_1 and T_2 be two distinct common tangents to the ellipse $E : \frac{x^2}{6} + \frac{y^2}{3} = 1$ and the parabola $P : y^2 = 12x$. Suppose that the tangent T_1 touches P and E at the point A_1 and A_2, respectively and the tangent T_2 touches P and E at the points A_4 and A_3, respectively. Then which of the following statements is(are) true? [JEE(Advanced) 2023]
(A) The area of the quadrilateral $A_1A_2A_3A_4$ is 35 square units
(B) The area of the quadrilateral $A_1A_2A_3A_4$ is 36 square units
(C) The tangents T_1 and T_2 meet the x-axis at the point $(-3, 0)$
(D) The tangents T_1 and T_2 meet the x-axis at the point $(-6, 0)$

2. Consider the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

Let $H(\alpha, 0)$, $0 < \alpha < 2$, be a point. A straight line drawn through H parallel to the y-axis crosses the ellipse and its auxiliary circle at points E and F respectively, in the first quadrant. The tangent to the ellipse at the point E intersects the positive x-axis at a point G. Suppose the straight line joining F and the origin makes an angle ϕ with the positive x-axis.

List-I		List-II	
(I)	If $\phi = \frac{\pi}{4}$, then the area of the triangle FGH is	(P)	$\frac{(\sqrt{3}-1)^4}{8}$
(II)	If $\phi = \frac{\pi}{3}$, then the area of the triangle FGH is	(Q)	1
(III)	If $\phi = \frac{\pi}{6}$, then the area of the triangle FGH is	(R)	$\frac{3}{4}$
(IV)	If $\phi = \frac{\pi}{12}$, then the area of the triangle FGH is	(S)	$\frac{1}{2\sqrt{3}}$
		(T)	$\frac{3\sqrt{3}}{2}$

The correct option is: [JEE(Advanced) 2022]

(A) (I) → (R); (II) → (S); (III) → (Q); (IV) → (P)

(B) (I) → (R); (II) → (T); (III) → (S); (IV) → (P)

(C) (I) → (Q); (II) → (T); (III) → (S); (IV) → (P)

(D) (I) → (Q); (II) → (S); (III) → (Q); (IV) → (P)

3. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. For any three distinct points P, Q and Q' on E, let M (P, Q) be the mid-point of the line segment joining P and Q, and M (P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M(P, Q) and M(P, Q'), as P, Q and Q' vary on E, is _____. [JEE(Advanced) 2021]

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is [JEE(Advanced) 2020]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

5. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct ? [JEE(Advanced) 2019]

(A) The eccentricities of E_{18} and E_{19} are NOT equal

(B) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(C) The length of latus rectum of E_9 is $\frac{1}{6}$

(D) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N

6. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE ? [JEE(Advanced) 2018]

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{4\sqrt{2}}(\pi-2)$

(D) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{16}(\pi-2)$

Paragraph for Question No. 7 and 8

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

7. The orthocentre of the triangle F_1MN is- **[JEE(Advanced) 2016]**

(A) $\left(-\frac{9}{10}, 0\right)$ (B) $\left(\frac{2}{3}, 0\right)$ (C) $\left(\frac{9}{10}, 0\right)$ (D) $\left(\frac{2}{3}, \sqrt{6}\right)$

8. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is- **[JEE(Advanced) 2016]**

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

9. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0,0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **[JEE(Advanced) 2015]**

10. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are) **[JEE(Advanced) 2015]**

(A) $e_1^2+e_2^2=\frac{43}{40}$ (B) $e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}$ (C) $\left|e_1^2-e_2^2\right|=\frac{5}{8}$ (D) $e_1e_2=\frac{\sqrt{3}}{4}$

11. A vertical line passing through the point (h,0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2\le h\le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2\le h\le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[JEE(Advanced) 2013]**

12. The ellipse E_1 : $\frac{x^2}{9}+\frac{x^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0,4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is - **[IIT-JEE 2012]**

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Paragraph for Question Nos. 13 to 15

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

13. The coordinates of A and B are **[IIT-JEE 2010]**

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

14. The orthocenter of the triangle PAB is **[IIT-JEE 2010]**

(A) $\left(5, \frac{8}{7}\right)$

(B) $\left(\frac{7}{5}, \frac{25}{8}\right)$

(C) $\left(\frac{11}{5}, \frac{8}{5}\right)$

(D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

15. The equation of the locus of the point whose distances from the point P and the line AB are equal, is **[IIT-JEE 2010]**

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

16. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is :- **[IIT-JEE 2009]**

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

17. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x–axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points **[IIT-JEE 2009]**

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$

(B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$

(D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

18. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are :- **[IIT-JEE 2008]**

(A) $x^2 + 2\sqrt{3}\,y = 3 + \sqrt{3}$

(B) $x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$

(D) $x^2 - 2\sqrt{3}\,y = 3 - \sqrt{3}$

HYPERBOLA

1. Consider the hyperbola

$$\frac{x^2}{100}-\frac{y^2}{64}=1$$

with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1=\alpha$, with $\alpha<\frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta=S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is______. **[JEE(Advanced) 2022]**

2. Let a and b be positive real numbers such that a > 1 and b < a. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Suppose the tangent to the hyperbola at P passes through the point (1,0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) $1<e<\sqrt{2}$ (B) $\sqrt{2}<e<2$ (C) $\Delta=a^4$ (D) $\Delta=b^4$

3. Let H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$. **[JEE(Advanced) 2018]**

LIST-I	LIST-II
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is :

(A) $P\to4$; $Q\to2$, $R\to1$; $S\to3$

(B) $P\to4$; $Q\to3$; $R\to1$; $S\to2$

(C) $P\to4$; $Q\to1$, $R\to3$; $S\to2$

(D) $P\to3$; $Q\to4$; $R\to2$; $S\to1$

4. If $2x-y+1=0$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE(Advanced) 2017]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2

MATCHING TYPE : (Q.5 to Q.7)

Column 1, 2 and 3 contain conics, equation of tangents to the conics and points of contact, respectively.

Column-1	Column-2	Column-3
(I) $x^2+y^2=a^2$	(i) $my=m^2x+a$	(P) $\left(\frac{a}{m^2},\frac{2a}{m}\right)$
(II) $x^2+a^2y^2=a^2$	(ii) $y=mx+a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}},\frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2=4ax$	(iii) $y=mx+\sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}},\frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2-a^2y^2=a^2$	(iv) $y=mx+\sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}},\frac{-1}{\sqrt{a^2m^2-1}}\right)$

5. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3},\frac{1}{2}\right)$ is found to be $\sqrt{3}x+2y=4$, then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (II) (iii) (R) (B) (IV) (iv) (S) (C) (IV) (iii) (S) (D) (II) (iv) (R)

6. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8,16), then which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**

(A) (III) (i) (P) (B) (III) (ii) (Q)

(C) (II) (iv) (R) (D) (I) (ii) (Q)

7. For $a=\sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1,1), then which of the following options is the only **CORRECT** combination for obtaining its equation ?

[JEE(Advanced) 2017]

(A) (II) (ii) (Q) (B) (III) (i) (P)

(C) (I) (i) (P) (D) (I) (ii) (Q)

8. Consider the hyperbola $H : x^2-y^2=1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1>1$ and $y_1>0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

[JEE(Advanced) 2015]

(A) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1>1$ (B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1>1$

(C) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1>1$ (D) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1>0$

9. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line 2x – y = 1. The points of contact of the tangents on the hyperbola are **[IIT-JEE 2012]**

(A) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3},-2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3},2\sqrt{2}\right)$

10. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2+4y^2=4$. If the hyperbola passes through a focus of the ellipse, then - **[IIT-JEE 2011]**

(A) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$

(B) a focus of the hyperbola is (2,0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2-3y^2=3$

11. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the x-axis at (9,0), then the eccentricity of the hyperbola is - **[IIT-JEE 2011]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{2}$

Paragraph for Question No. 12 and 13

The circle $x^2+y^2-8x=0$ and hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ intersect at the points A and B.

12. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is - **[IIT-JEE 2010]**

(A) $2x-\sqrt{5}y-20=0$ (B) $2x-\sqrt{5}y+4=0$

(C) $3x-4y+8=0$ (D) $4x-3y+4=0$

13. Equation of the circle with AB as its diameter is - **[IIT-JEE 2010]**

(A) $x^2+y^2-12x+24=0$ (B) $x^2+y^2+12x+24=0$

(C) $x^2+y^2+24x-12=0$ (D) $x^2+y^2-24x-12=0$

14. The line 2x + y =1 is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[IIT-JEE 2010]**

15. Let a and b be non-zero real numbers. Then, the equation

$$(ax^2+by^2+c)(x^2-5xy+6y^2)=0$$

represents **[IIT-JEE 2008]**

(A) four straight lines, when c = 0 and a, b are of the same sign

(B) two straight lines and a circle, when a = b, and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

16. Consider a branch of the hyperbola

$$x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$$

with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is :- **[IIT-JEE 2008]**

(A) $1-\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}}-1$ (C) $1+\sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}}+1$

PERMUTATION & COMBINATION

1. The number of 4-digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is _______. **[JEE(Advanced) 2022]**

2. Consider 4 boxes, where each box contains 3 red balls and 2 blue balls. Assume that all 20 balls are distinct. In how many different ways can 10 balls be chosen from these 4 boxes so that from each box at least one red ball and one blue ball are chosen? **[JEE(Advanced) 2022]**
(A) 21816 (B) 85536 (C) 12096 (D) 156816

3. Let
$S_1 = \{(i, j, k) : i, j, k \in \{1, 2, ..., 10\}\}$
$S_2 = \{(i, j) : 1 \le i < j + 2 \le 10,\ i, j \in \{1, 2, ..., 10\}\}$,
$S_3 = \{(i, j, k, l) : 1 \le i < j < k < l,\ i, j, k, l \in \{1, 2, ..., 10\}\}$,
and
$S_4 = \{(i, j, k, l) : i, j, k$ and l are distinct elements in $\{1, 2, ..., 10\}\}$.
If the total number of elements in the set S_r is n_r, $r = 1, 2, 3, 4$, then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**
(A) $n_1 = 1000$ (B) $n_2 = 44$ (C) $n_3 = 220$ (D) $\frac{n_4}{12} = 420$

4. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that no two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is _______. **[JEE(Advanced) 2020]**

5. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _______. **[JEE(Advanced) 2020]**

6. Five person A, B, C, D and E are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green, then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is **[JEE(Advanced) 2019]**

7. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is _______. **[JEE(Advanced) 2018]**

8. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta-\alpha)$ is _______. **[JEE(Advanced) 2018]**

9. In a high school, a committee has to be formed from a group of 6 boys M_1, M_2, M_3, M_4, M_5, M_6 and 5 girls G_1, G_2, G_3, G_4, G_5.
(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boy and 2 girls.
(ii) Let α_2 be the total number of ways in which the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.
(iii) Let α_3 be the total number of ways in which the committe can be formed such that the committee has 5 members, at least 2 of them being girls.
(iv) Let α_4 be the total number of ways in which the committee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.
[JEE(Advanced) 2018]

LIST-I	LIST-II
P. The value of α_1 is	1. 136
Q. The value of α_2 is	2. 189
R. The value of α_3 is	3. 192
S. The value of α_4 is	4. 200
	5. 381
	6. 461

The correct option is :-

(A) $P \to 4; Q \to 6, R \to 2; S \to 1$

(B) $P \to 1; Q \to 4; R \to 2; S \to 3$

(C) $P \to 4; Q \to 6, R \to 5; S \to 2$

(D) $P \to 4; Q \to 2; R \to 3; S \to 1$

10. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then. $\dfrac{y}{9x} =$ **[JEE(Advanced) 2017]**

11. Let $S = \{1, 2, 3,.....,9\}$. For $k = 1,2, ..., 5$, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ **[JEE(Advanced) 2017]**

(A) 125 (B) 252 (C) 210 (D) 126

12. A debate club consists of 6 girls and 4 boy. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 member) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is **[JEE(Advanced) 2016]**

(A) 380 (B) 320 (C) 260 (D) 95

13. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\dfrac{m}{n}$ is

[JEE(Advanced) 2015]

14. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. The the number of such distinct arrangements (n_1,n_2,n_3,n_4,n_5) is **[JEE(Advanced) 2014]**

15. Let $n \geq 2$ b an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is **[JEE(Advanced) 2014]**

16. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 in always placed in envelope numbered 2. Then the number of ways it can be done is - **[JEE(Advanced) 2014]**

(A) 264 (B) 265 (C) 53 (D) 67

17. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is - **[IIT-JEE 2012]**

(A) 75 (B) 150 (C) 210 (D) 243

Paragraph for Question No. 18 and 19

Let a_n denotes the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

18. The value of b_6 is **[IIT-JEE 2012]**

(A) 7 (B) 8 (C) 9 (D) 11

19. Which of the following is correct? **[IIT-JEE 2012]**

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$ (C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

20. Let S_k, k = 1,2,......,100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} \left|\left(k^2 - 3k + 1\right)S_k\right|$ is **[IIT-JEE 2010]**

21. Let S = {1,2,3,4}. The total number of unordered pairs of disjoint subsets of S is equal to - **[IIT-JEE 2010]**

(A) 25 (B) 34 (C) 42 (D) 41

22. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits, 1, 2 and 3 only, is :- **[IIT-JEE 2009]**

(A) 55 (B) 66 (C) 77 (D) 88

23. Consider all possible permutations of the letters of the word ENDEANOEL.

Match the Statement / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS. **[IIT-JEE 2008]**

	Column-I		Column-II
(A)	The number of permutations containing the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letter E occurs in the first and the last positions is	(q)	$2 \times 5!$
(C)	The number of permutations in which none of the letters D, L, N occurs in the last five positions is	(r)	$7 \times 5!$
(D)	The number of permutations in which the letters A, E, O occur only in odd positions is	(s)	$21 \times 5!$

BINOMIAL THEOREM

1. Let a and b be two nonzero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2+\frac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} is equal to the coefficient of $\left(ax-\frac{1}{bx^2}\right)^7$, then the value of 2b is

[JEE(Advanced) 2023]

2. For non-negative integers s and r, let $\binom{s}{r}=\begin{cases}\frac{s!}{r!(s-r)!} & \text{if } r\le s,\\ 0 & \text{if } r>s.\end{cases}$

For positive integers m and n, let $g(m,n)=\sum_{p=0}^{m+n}\frac{f(m,n,p)}{\binom{n+p}{p}}$

where for any nonnegative integer p, $f(m,n,p)=\sum_{i=0}^{p}\binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$

Then which of the following statements is/are TRUE? **[JEE(Advanced) 2020]**

(A) $g(m, n) = g(n, m)$ for all positive integers m, n

(B) $g(m, n+1) = g(m+1, n)$ for all positive integers m, n

(C) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n

(D) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

3. Suppose $\det\begin{bmatrix}\sum_{k=0}^{n}k & \sum_{k=0}^{n}{}^nC_k k^2\\ \sum_{k=0}^{n}{}^nC_k k & \sum_{k=0}^{n}{}^nC_k 3^k\end{bmatrix}=0$, holds for some positive integer n. Then $\sum_{k=0}^{n}\frac{{}^nC_k}{k+1}$ equals

[JEE(Advanced) 2019]

4. Let $X=\left({}^{10}C_1\right)^2+2\left({}^{10}C_2\right)^2+3\left({}^{10}C_3\right)^2+\ldots+10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r\in\{1, 2, ..., 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is ______. **[JEE(Advanced) 2018]**

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced) 2016]**

6. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3).....(1 + x^{100})$ is **[JEE(Advanced) 2015]**

7. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is - **[JEE(Advanced) 2014]**

(A) 1051 (B) 1106 (C) 1113 (D) 1120

8. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE(Advanced) 2013]

9. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10}A_r(B_{10}B_r-C_{10}A_r)$ is equal to- **[IIT-JEE 2010]**

(A) $B_{10}-C_{10}$ (B) $A_{10}\left(B_{10}^2-C_{10}A_{10}\right)$ (C) 0 (D) $C_{10}-B_{10}$

FUNCTION

1. Let $|M|$ denote the determinant of a square matrix M. Let $g : \left[0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ be the function defined by

$$g(\theta) = \sqrt{f(\theta) - 1} + \sqrt{f\left(\frac{\pi}{2} - \theta\right) - 1}$$

where

$$f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} + \begin{vmatrix} \sin\pi & \cos\left(\theta + \frac{\pi}{4}\right) & \tan\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta + \frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}.$$

Let $p(x)$ be a quadratic polynomial whose roots are the maximum and minimum values of the function $g(\theta)$, and $p(2) = 2 - \sqrt{2}$. Then, which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$ (B) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$

(C) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$ (D) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

2. If the function $f : R \rightarrow R$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE** ? **[JEE(Advanced) 2020]**

(A) f is one-one, but **NOT** onto (B) f is onto, but **NOT** one-one

(C) f is **BOTH** one-one and onto (D) f is **NEITHER** one-one **NOR** onto

3. Let the function $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{4^x}{4^x + 2}$

Then the value of $f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \ldots\ldots\ldots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$ is____. **[JEE(Advanced) 2020]**

4. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is _____. **[JEE(Advanced) 2018]**

5. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ be given by $f(x) = (\log(\sec x + \tan x))^3$ Then :- **[JEE(Advanced)-2014]**

(A) $f(x)$ is an odd function (B) $f(x)$ is a one-one function

(C) $f(x)$ is an onto function (D) $f(x)$ is an even function

6. The function $f: [0,3] \rightarrow [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is : **[IIT-JEE 2012]**

(A) one-one and onto (B) onto but not one-one

(C) one-one but not onto (D) neither one-one nor onto

7. Let $f : (-1,1) \to \mathbb{R}$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in \left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are) - **[IIT-JEE 2012]**

(A) $1-\sqrt{\frac{3}{2}}$ (B) $1+\sqrt{\frac{3}{2}}$ (C) $1-\sqrt{\frac{2}{3}}$ (D) $1+\sqrt{\frac{2}{3}}$

8. Let $f : (0, 1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then **[IIT-JEE 2011]**

(A) f is not invertible on (0, 1)

(B) $f \neq f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0,1)

9. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is- **[IIT-JEE 2011]**

(A) $\pm\sqrt{n\pi}, n \in \{0,1,2,....\}$ (B) $\pm\sqrt{n\pi}, n \in \{1,2,...\}$

(C) $\frac{\pi}{2}+2n\pi, n \in \{....,-2,-1,0,1,2,.....\}$ (D) $2n\pi, n \in \{.....,-2,-1,0,1,2,.....\}$

INVERSE TRIGONOMETRIC FUNCTION

1. Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, for $x \in \mathbb{R}$. Then the number of real solutions of the equation $\sqrt{1+\cos(2x)} = \sqrt{2}\tan^{-1}(\tan x)$ in the set $\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)\cup\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ is equal to **[JEE(Advanced) 2023]**

2. For any $y \in \mathbb{R}$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the sum of all the solutions of the equation $\tan^{-1}\left(\frac{6y}{9-y^2}\right)+\cot^{-1}\left(\frac{9-y^2}{6y}\right)=\frac{2\pi}{3}$ for $0<|y|<3$, is equal to **[JEE(Advanced) 2023]**

(A) $2\sqrt{3}-3$ (B) $3-2\sqrt{3}$ (C) $4\sqrt{3}-6$ (D) $6-4\sqrt{3}$

3. Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$$

is ________. **[JEE(Advanced) 2022]**

4. For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by $S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$, where for any $x \in \mathbb{R}$, $\cot^{-1}x \in (0,\pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) TRUE ? **[JEE(Advanced) 2021]**

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$ (B) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0,\infty)$ (D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

5. For non-negative integers n, let

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1} x$ takes values in $[0, \pi]$, which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $\sin(7\cos^{-1} f(5)) = 0$ (B) $f(4) = \frac{\sqrt{3}}{2}$

(C) $\lim_{n\to\infty} f(n) = \frac{1}{2}$ (D) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$

6. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10} \sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals **[JEE(Advanced) 2019]**

7. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is ______. **[JEE(Advanced) 2018]**

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

8. Let $E_1 = \left\{x \in \mathbb{R} : x \ne 1 \text{ and } \frac{x}{x-1} > 0\right\}$

and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].\right)$

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$ **[JEE(Advanced) 2018]**

LIST-I	LIST-II
P. The range of f is	1. $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	2. $(0, 1)$
R. The domain of f contains	3. $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	4. $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :

(A) $P \to 4; Q \to 2; R \to 1; S \to 1$
(B) $P \to 3; Q \to 3; R \to 6; S \to 5$
(C) $P \to 4; Q \to 2; R \to 1; S \to 6$
(D) $P \to 4; Q \to 3; R \to 6; S \to 5$

9. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$ where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE(Advanced) 2015]**

(A) $\cos\beta > 0$ (B) $\sin\beta < 0$ (C) $\cos(\alpha+\beta) > 0$ (D) $\cos\alpha < 0$

10. Let $f : [0,4\pi] \to [0,\pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0,4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is **[JEE(Advanced) 2014]**

11. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is **[JEE(Advanced) 2013]**

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

12. Match List-I with List-II and select the correct answer using the code given below the lists.

List-I	List-II
P. $\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)-\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$ takes value	1. $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q. If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos\frac{x-y}{2}$ is	2. $\sqrt{2}$
R. If $\cos\left(\frac{\pi}{4}-x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4}+x\right)\cos 2x$ then possible value of $\sec x$ is	3. $\frac{1}{2}$
S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right), x \neq 0$,	4. 1

Then possible value of x is [JEE(Advanced) 2013]

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

13. If $0 < x < 1$, then $\sqrt{1+x^2}\ [\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ [IIT-JEE 2008]

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

LIMITS

1. Let α be a positive real number. Let $f: \mathbb{R} \to \mathbb{R}$ and $g : (\alpha, \infty) \to \mathbb{R}$ be the functions defined by

$$f(x) = \sin\left(\frac{\pi x}{12}\right) \text{ and } g(x) = \frac{2\log_e\left(\sqrt{x}-\sqrt{\alpha}\right)}{\log_e\left(e^{\sqrt{x}} - e^{\sqrt{\alpha}}\right)}.$$

Then the value of $\lim\limits_{x\to\alpha^+} f(g(x))$ is ______. [JEE(Advanced) 2022]

2. If

$$\beta = \lim_{x\to 0}\frac{e^{x^3} - \left(1-x^3\right)^{\frac{1}{3}} + \left(\left(1-x^2\right)^{\frac{1}{2}} - 1\right)\sin x}{x\sin^2 x}$$

then the value of 6β is ______. [JEE(Advanced) 2022]

3. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x\to 0^+}\frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is ______. [JEE(Advanced) 2020]

4. The value of the limit

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2\sin 2x\sin\frac{3x}{2} + \cos\frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2}\cos 2x + \cos\frac{3x}{2}\right)}$$

is______. [JEE(Advanced) 2020]

5. Let $F : \mathbb{R} \to \mathbb{R}$ be a function. We say that f has

PROPERTY 1 if $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct ? [JEE(Advanced) 2019]

(A) $f(x) = x|x|$ has PROPERTY 2 (B) $f(x) = x^{2/3}$ has PROPERTY 1

(C) $f(x) = \sin x$ has PROPERTY 2 (D) $f(x) = |x|$ has PROPERTY 1

6. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) $\sum_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$

(B) $\sum_{j=1}^{10} \left(1 + f_j'(0)\right)\sec^2\left(f_j(0)\right) - 10$

(C) For any fixed positive integer n, $\lim\limits_{x\to\infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim\limits_{x\to\infty} \sec^2\left(f_n(x)\right) = 1$

7. Let $f(x) = \dfrac{1 - x\left(1 + |1 - x|\right)}{|1-x|}\cos\left(\dfrac{1}{1-x}\right)$ for $x \neq 1$. Then **[JEE(Advanced) 2017]**

(A) $\lim_{x\to1+} f(x)$ does not exist

(B) $\lim_{x\to1-} f(x)$ does not exist

(C) $\lim_{x\to1-} f(x) = 0$

(D) $\lim_{x\to1+} f(x) = 0$

8. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim\limits_{x\to0} \dfrac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals **[JEE(Advanced) 2017]**

9. Let $f(x) = \sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$ for all for $x \in \mathbb{R}$ and and $g(x) = \dfrac{\pi}{2}\sin x$ for all $x \in \mathbb{R}$. Let $(fog)(x)$ denotes $f(g(x))$ and $(gof)(x)$ denotes $g(f(x))$. Then which of the following is (are) true?

[JEE(Advanced) 2015]

(A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(B) Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\lim\limits_{x\to0} \dfrac{f(x)}{g(x)} = \dfrac{\pi}{6}$

(D) There is an $x \in \mathbb{R}$ such that $(gof)(x) = 1$

10. Let m and n be two positive integers greater than 1. If

$$\lim_{\alpha\to0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$$

then the value of $\dfrac{m}{n}$ is **[JEE(Advanced) 2015]**

11. The largest value of the non-negative integer a for which $\lim\limits_{x\to1}\left\{\dfrac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \dfrac{1}{4}$ is

[JEE(Advanced) 2014]

12. If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1} - ax - b\right) = 4$, then - **[IIT-JEE 2012]**

(A) a = 1, b = 4

(B) a = 1, b = –4

(C) a = 2, b = –3

(D) a = 2, b = 3

13. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where $a>-1$.

Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are **[IIT-JEE 2012]**

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and -1 (C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

14. If $\lim_{x\to 0}\left[1+x\ln(1+b^2)\right]^{\frac{1}{x}}=2b\sin^2\theta, b>0$ and $\theta\in(-\pi,\pi]$, then the value of θ is- **[IIT-JEE 2011]**

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

15. Let $L=\lim_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}, a>0$. If L is finite, then:- **[IIT-JEE 2009]**

(A) $a=2$ (B) $a=1$ (C) $L=\frac{1}{64}$ (D) $L=\frac{1}{32}$

16. Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right)=2$, then the value of p(2) is ________. **[IIT-JEE 2009]**

CONTINUITY

1. Let [x] be the greatest integer less than or equal to x. Then, at which of the following point(s) the function $f(x)=x\cos(\pi(x+[x]))$ is discontinuous ? **[JEE(Advanced) 2017]**

(A) $x=-1$ (B) $x=0$

(C) $x=2$ (D) $x=1$

2. For every pair of continuous function $f,g:[0,1]\to\mathbb{R}$ such that

$\max\{f(x):x\in[0,1]\}=\max\{g(x):x\in[0,1]\}$,

the correct statement(s) is(are) : **[JEE(Advanced) 2014]**

(A) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(B) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(C) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some $c\in[0,1]$

(D) $(f(c))^2=(g(c))^2$ for some $c\in[0,1]$

3. For every integer n, let a_n and b_n be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}a_n+\sin\pi x, & \text{for } x\in[2n,2n+1]\\ b_n+\cos\pi x, & \text{for } x\in(2n-1,2n)\end{cases}$$, for all integers n. **[IIT-JEE 2012]**

If f is continuous, then which of the following holds(s) for all n ?

(A) $a_{n-1}-b_{n-1}=0$ (B) $a_n-b_n=1$ (C) $a_n-b_{n+1}=1$ (D) $a_{n-1}-b_n=-1$

DIFFERENTIABILITY

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x - 1) \sin x$ and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \to \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)\, g(x)$. Then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

   (A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$

   (B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$

   (C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$

   (D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

2. Let the functions $f : (-1, 1) \to \mathbb{R}$ and $g : (-1,1) \to (-1, 1)$ be defined by

   $f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$,

   where $[x]$ denotes the greatest integer less than or equal to x. Let $fog:(-1, 1) \to \mathbb{R}$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval $(-1, 1)$ at which fog is **NOT** continuous, and suppose d is the number of points in the interval $(-1, 1)$ at which fog is **NOT** differentiable. Then the value of $c + d$ is _____. **[JEE(Advanced) 2020]**

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying $f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in \mathbb{R}$. If $\lim\limits_{x \to 0} g(x) = 1$, then which of the following statements is/are TRUE?

   [JEE(Advanced) 2020]

   (A) f is differentiable at every $x \in \mathbb{R}$

   (B) If $g(0) = 1$, then g is differentiable at every $x \in \mathbb{R}$

   (C) The derivative $f'(1)$ is equal to 1

   (D) The derivative $f'(0)$ is equal to 1

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

   $$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

   Then, then value of $\log_e(f(4))$ is _____ . **[JEE(Advanced) 2018]**

5. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by

   (i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

   (ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

   (iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

   (iv) $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   [JEE(Advanced) 2018]

List-I	List-II
P. The function f_1 is	1. NOT continuous at $x = 0$
Q. The function f_2 is	2. continuous at $x = 0$ and NOT differentiable at $x = 0$
R. The function f_3 is	3. differentiable at $x = 0$ and its derivative is NOT continuous at $x = 0$
S. The function f_4 is	4. differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is :

(A) $P \to 2; Q \to 3, R \to 1; S \to 4$

(B) $P \to 4; Q \to 1; R \to 2; S \to 3$

(C) $P \to 4; Q \to 2, R \to 1; S \to 3$

(D) $P \to 2; Q \to 1; R \to 4; S \to 3$

6. Let $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is - **[JEE(Advanced) 2016]**

(A) differentiable at $x = 0$ if $a = 0$ and $b = 1$

(B) differentiable at $x = 1$ if $a = 1$ and $b = 0$

(C) **NOT** differentiable at $x = 0$ if $a = 1$ and $b = 0$

(D) **NOT** differentiable at $x = 1$ if $a = 1$ and $b = 1$

7. Let $f:\left[-\frac{1}{2},2\right] \to \mathbb{R}$ and $g:\left[-\frac{1}{2},2\right] \to \mathbb{R}$ be function defined by $f(x)=[x^2-3]$ and $g(x) = |x|\,f(x) + |4x - 7|\,f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in \mathbb{R}$. Then **[JEE(Advanced) 2016]**

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

8. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable functions with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$. Let

$$f(x)=\begin{cases}\dfrac{x}{|x|}g(x) & , \quad x\neq 0\\ 0 & , \quad x=0\end{cases}$$

and $h(x) = e^{|x|}$ for all $x \in \mathbb{R}$. Let $(foh)(x)$ denote $f(h(x))$ and $(hof)(x)$ denote $h(f(x))$. Then which of the following is(are) true ? **[JEE(Advanced) 2015]**

(A) f is differentiable at $x = 0$

(B) h is differentiable at $x = 0$

(C) foh is differentiable at $x = 0$

(D) hof is differentiable at $x = 0$

9. Let $f: \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : \mathbb{R} \to \mathbb{R}$ by **[JEE(Advanced) 2014]**

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which h(x) is not differentiable is

10. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : [0, \infty) \to \mathbb{R}$, $f_3 : \mathbb{R} \to \mathbb{R}$ and $f_4 : \mathbb{R} \to [0, \infty)$ be defined by **[JEE(Advanced) 2014]**

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \ge 0; \end{cases}$$

$$f_2(x) = x^2;$$

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \ge 0 \end{cases}$$

and

$$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \ge 0. \end{cases}$$

List-I	**List-II**
P. f_4 is	1. onto but not one-one
Q. f_3 is	2. neither continuous nor one-one
R. $f_2 o f_1$ is	3. differentiable but not one-one
S. f_2 is	4. continuous and one-one

Codes :

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

11. Let $f(x) = \begin{cases} x^2 \left|\cos\dfrac{\pi}{x}\right| & , x \ne 0 \\ 0 & , x = 0 \end{cases}$, $x \in \mathbb{IR}$, then f is - **[IIT-JEE 2012]**

(A) differentiable both at x = 0 and at x = 2

(B) differentiable at x = 0 but not differentiable at x = 2

(C) not differentiable at x = 0 but differentiable at x = 2

(D) differentiable neither at x = 0 nor at x = 2

12. Let $f : R \to R$ be a function such that **[IIT-JEE 2011]**

$f(x + y) = f(x) + f(y), \forall x, y \in R$.

If $f(x)$ is differentiable at x = 0, then

(A) $f(x)$ is differentiable only in a finite interval containing zero

(B) $f(x)$ is continuous $\forall x \in R$

(C) $f'(x)$ is constant $\forall x \in R$

(D) $f(x)$ is differentiable except at finitely many points

13. If $f(x)=\begin{cases} -x-\frac{\pi}{2}, & x\le -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2}<x\le 0 \\ x-1, & 0<x\le 1 \\ \ell nx, & x>1 \end{cases}$ then - **[IIT-JEE 2011]**

(A) $f(x)$ is continuous at $x=-\frac{\pi}{2}$ (B) $f(x)$ is not differentiable at $x=0$

(C) $f(x)$ is differentiable at $x=1$ (D) $f(x)$ is differentiable at $x=-\frac{3}{2}$

14. Let $g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)}$; $0<x<2$, m and n are integers, $m\neq 0$, $n>0$, and let p be the left hand derivative of $|x-1|$ at $x=1$. If $\lim\limits_{x\to 1^-} g(x)=p$, then :- **[IIT-JEE 2008]**

(A) $n=1, m=1$ (B) $n=1, m=-1$

(C) $n=2, m=2$ (D) $n>2, m=n$

METHOD OF DIFFERENTIATION

1. Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x)=x^3+3x+2$, $g(f(x))=x$ and $h(g(g(x)))=x$ for all $x\in\mathbb{R}$. Then- **[JEE(Advanced) 2016]**

(A) $g'(2)=\frac{1}{15}$ (B) $h'(1)=666$ (C) $h(0)=16$ (D) $h(g(3))=36$

2. The slope of the tangent to the curve $(y-x^5)^2=x(1+x^2)^2$ at the point (1,3) is **[JEE(Advanced) 2014]**

3. Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[IIT-JEE 2011]

4. If the function $f(x)=x^3+e^{\frac{x}{2}}$ and $g(x)=f^{-1}(x)$, then the value of $g'(1)$ is ________.

[IIT-JEE 2009]

5. Let f and g be real valued functions defined on interval (–1, 1) such that $g''(x)$ is continuous, $g(0)\neq 0$, $g'(0)=0$, $g''(0)\neq 0$, and $f(x)=g(x)\sin x$.

STATEMENT–1 : $\lim\limits_{x\to 0}[g(x)\cot x-g(0)\operatorname{cosec}x]=f''(0)$ **[IIT-JEE 2008]**

and

STATEMENT–2 : $f'(0)=g(0)$

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

6. Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x f(x)$. Then, for N = 1, 2, 3, ..., **[IIT-JEE 2008]**

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

AOD (MONOTONICITY)

1. Let $S = (0, 1) \cup (1, 2) \cup (3, 4)$ and $T = \{0, 1, 2, 3\}$. Then which of the following statements is (are) true ? **[JEE(Advanced) 2023]**

(A) There are infinitely many functions from S to T

(B) There are infinitely many strictly increasing functions from S to T

(C) The number of continuous functions from S to T is at most 120

(D) Every continuous function from S to T is differentiable

2. Let S be the set of all twice differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $\frac{d^2f}{dx^2}(x) > 0$ for all $x \in (-1, 1)$. For $f \in S$, let X_f be the number of points $x \in (-1,1)$ for which $f(x) = x$. Then which of the following statements is(are) true? **[JEE(Advanced) 2023]**

(A) There exists a function $f \in S$ such that $X_f = 0$

(B) For every function $f \in S$, we have $X_f \le 2$

(C) There exists a function $f \in S$ such that $X_f = 2$

(D) There does **NOT** exist any function f in S such that $X_f = 1$

3. Let $f:\mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2-3x-6}{x^2+2x+4}.$$

Then which of the following statements is (are) **TRUE** ? **[JEE(Advanced) 2021]**

(A) f is decreasing in the interval $(-2,-1)$

(B) f is increasing in the interval $(1,2)$

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

4. For a polynomial g(x) with real coefficient, let m_g denote the number of distinct real roots of g(x). Suppose S is the set of polynomials with real coefficients defined by
$S = \{(x^2 - 1)^2 (a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}$.
For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _______ **[JEE(Advanced) 2020]**

5. Let $f:\mathbb{R} \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? **[JEE(Advanced) 2017]**

(A) $e^x - \int_0^x f(t)\sin t\,dt$

(B) $x^9 - f(x)$

(C) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

(D) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

Answer Q.6, Q.7 and Q.8 by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0,\infty)$.

* Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
* Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
* Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in $(0,1)$
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0,1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in $(0,1)$
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

6. Which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**
(A) (IV) (i) (S) (B) (I) (ii) (R) (C) (III) (iv) (P) (D) (II) (iii) (S)

7. Which of the following options is the only **CORRECT** combination ? **[JEE(Advanced) 2017]**
(A) (III) (iii) (R) (B) (I) (i) (P) (C) (IV) (iv) (S) (D) (II) (ii) (Q)

8. Which of the following options is the only **INCORRECT** combination ? **[JEE(Advanced) 2017]**
(A) (II) (iii) (P) (B) (II) (iv) (Q) (C) (I) (iii) (P) (D) (III) (i) (R)

9. If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then **[JEE(Advanced) 2017]**

(A) $0 < f'(1) \le \frac{1}{2}$ (B) $f'(1) \le 0$

(C) $f'(1) > 1$ (D) $\frac{1}{2} < f'(1) \le 1$

10. Let f, g : $[-1, 2] \to \mathbb{R}$ be continuous function which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table :

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are) **[JEE(Advanced) 2015]**

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$
(C) $f'(x) - 3g'(x) = 0$ has exactly one solutions in $(0, 2)$
(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

11. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$. Then **[JEE(Advanced) 2014]**

(A) $f(x)$ has three real roots if $a > 4$
(B) $f(x)$ has only one real roots if $a > 4$
(C) $f(x)$ has three real roots if $a < -4$
(D) $f(x)$ has three real roots if $-4 < a < 4$

12. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **[JEE(Advanced) 2013]**

(A) 6 (B) 4 (C) 2 (D) 0

13. Let $f(x) = x \sin \pi x,\ x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at- **[JEE(Advanced) 2013]**

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) a unique point in the interval $(n, n + 1)$

(D) two points in the interval $(n, n + 1)$

Paragraph for Question No. 14 and 15

Let $f(x) = (1 - x)^2\sin^2 x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$ for all $x \in (1,\infty)$.

14. Consider the statements : **[IIT-JEE 2012]**

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both **P** and **Q** are true (B) **P** is true and **Q** is false

(C) **P** is false and **Q** is true (D) both **P** and **Q** are false

15. Which of the following is true? **[IIT-JEE 2012]**

(A) g is increasing on $(1,\infty)$

(B) g is decreasing on $(1,\infty)$

(C) g is increasing on (1,2) and decreasing on $(2,\infty)$

(D) g is decreasing on (1,2) and increasing on $(2,\infty)$

16. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT-JEE 2011]**

17. For the function $f(x) = x\cos\frac{1}{x},\ x \geq 1$, **[IIT-JEE 2009]**

(A) for at least one x in the interval $[1,\infty)$, $f(x+2) - f(x) < 2$

(B) $\lim_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1,\infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

18. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is :-

[IIT-JEE 2008]

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

MAXIMA & MINIMA

1. Let $f:(0,1) \to \mathbb{R}$ be the function defined as $f(x)=[4x]\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x. Then which of the following statements is(are) true?

[JEE(Advanced) 2023]

(A) The function f is discontinuous exactly at one point in (0,1)

(B) There is exactly one point in (0,1) at which the function f is continuous but **NOT** differentiable

(C) The function f is **NOT** differentiable at more than three points in (0, 1)

(D) The minimum value of the function f is $-\frac{1}{512}$

2. Let $\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left(\frac{\pi}{6}\right)$.

Let $g:[0,1] \to \mathbb{R}$ be the function defined by

$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}$

Then, which of the following statements is/are TRUE? **[JEE(Advanced) 2022]**

(A) The minimum value of $g(x)$ is $2^{\frac{7}{6}}$

(B) The maximum value of $g(x)$ is $1 + 2^{\frac{1}{3}}$

(C) The function $g(x)$ attains its maximum at more than one point

(D) The function $g(x)$ attains its minimum at more than one point

Question Stem for Questions Nos. 3 and 4

Question Stem

Let $f_1:(0,\infty) \to \mathbb{R}$ and $f_2:(0,\infty) \to \mathbb{R}$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j \, dt, \; x > 0$$

and

$$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \; x > 0,$$

where, for any positive integer n and real numbers $a_1, a_2, \ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$

3. The value of $2m_1 + 3n_1 + m_1n_1$ is _____. **[JEE(Advanced) 2021]**

4. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____. **[JEE(Advanced) 2021]**

5. Consider all rectangles lying in the region

$$\left\{(x,y) \in \mathbb{R} \times \mathbb{R} : 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is **[JEE(Advanced) 2020]**

(A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

6. Let the function $f: (0, \pi) \to \mathbb{R}$ be defined by

$$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$$

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, \ldots, \lambda_r\pi\}$, where $0 < \lambda_1 < \ldots < \lambda_r < 1$. Then the value of $\lambda_1 + \ldots + \lambda_r$ is ______.

[JEE(Advanced) 2020]

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) f' has a local maximum at $x = 1$ (B) f is onto

(C) f is increasing on $(-\infty, 0)$ (D) f' is NOT differentiable at $x = 1$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x - 1)(x - 2)(x - 5)$. Define $F(x) = \int_0^x f(t)dt$, $x > 0$. Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) F has a local minimum at $x = 1$

(B) F has a local maximum at $x = 2$

(C) $F(x) \neq 0$ for all $x \in (0, 5)$

(D) F has two local maxima and one local minimum in $(0, \infty)$

9. Let $f(x) = \frac{\sin \pi x}{x^2}$, $x > 0$

Let $x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ be all the points of local maximum of f

and $y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f.

Then which of the following options is/are correct? **[JEE(Advanced) 2019]**

(A) $|x_n - y_n| > 1$ for every n (B) $x_1 < y_1$

(C) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n (D) $x_{n+1} - x_n > 2$ for every n

10. For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) There exist $r, s \in \mathbb{R}$, where $r < s$, such that f is one-one on the open interval (r, s)

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(C) $\lim_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

11. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then [JEE(Advanced) 2017]

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(B) $f(x)$ attains its maximum at $x = 0$

(C) $f(x)$ attains its minimum at $x = 0$

(D) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

12. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is - [JEE(Advanced) 2016]

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

13. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' ar continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced) 2016]

(A) f has a local minimum at $x = 2$ (B) f has a local maximum at $x = 2$

(C) $f''(2) > f(2)$ (D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

14. A cylindrical container is to be made from certain solid material with the following constraints. It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10mm, then the value of $\frac{V}{250\pi}$ is [JEE(Advanced) 2015]

15. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio of 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are [JEE(Advanced) 2013]

(A) 24 (B) 32 (C) 45 (D) 60

16. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x =

[JEE(Advanced) 2013]

(A) –2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

17. If $f(x) = \int_0^x e^{t^2} (t-2)(t-3)dt$ for all $x \in (0,\infty)$, then - [IIT-JEE -2012]

(A) f has a local maximum at $x = 2$

(B) f is decreasing on (2,3)

(C) there exists some $c \in (0,\infty)$ such that $f''(c) = 0$

(D) f has a local minimum at $x = 3$

18. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[IIT-JEE 2012]**

19. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is **[IIT-JEE 2012]**

20. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[IIT-JEE 2010]**

(A) a = b and c ≠ b
(B) a = c and a ≠ b
(C) a ≠ b and c ≠ b
(D) a = b = c

21. Let f be a function defined on **R** (the set of all real numbers) such that **[IIT-JEE 2010]**
$f'(x) = 2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$, for all $x \in R$.
If g is a function defined on **R** with values in the interval $(0,\infty)$ such that
$f(x) = \ell n\,(g(x))$, for all $x \in \mathbf{R}$,
then the number of points in **R** at which g has a local maximum is -

22. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set **[IIT-JEE 2009]**
$A = \{x \mid x^2 + 20 \le 9x\}$ is ______.

23. The total number of local maxima and local minima of the function **[IIT-JEE 2008]**

$$f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3} & -1 < x < 2 \end{cases} \text{ is :-}$$

(A) 0
(B) 1
(C) 2
(D) 3

INDEFINITE INTEGRATION

1. Let b be a nonzero real number. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(0) = 1$. If the derivative f' of f satisfies the equation $f'(x) = \dfrac{f(x)}{b^2 + x^2}$ for all $x \in \mathbb{R}$, then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) If b > 0, then f is an increasing function
(B) If b < 0, then f is a decreasing function
(C) $f(x)\,f(-x) = 1$ for all $x \in \mathbb{R}$
(D) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

2. The integral $\displaystyle\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}dx$ equals (for some arbitrary constant K) **[IIT-JEE 2012]**

(A) $-\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} - \dfrac{1}{7}(\sec x + \tan x)^2\right\} + K$

(B) $\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} - \dfrac{1}{7}(\sec x + \tan x)^2\right\} + K$

(C) $-\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} + \dfrac{1}{7}(\sec x + \tan x)^2\right\} + K$

(D) $\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} + \dfrac{1}{7}(\sec x + \tan x)^2\right\} + K$

3. Let $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$.

Then, for an arbitrary constant C, the value of J – I equals **[IIT-JEE 2008]**

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$

(B) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{2x}-e^{2x}+1}\right)+C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{2x}+1}{e^{2x}+e^{x}+1}\right)+C$

(D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

DEFINITE INTEGRATION

1. Let $f:(0,1)\to\mathbb{R}$ be the functions defined as $f(x)=\sqrt{n}$ if $x\in\left[\frac{1}{n+1},\frac{1}{n}\right)$ where $n\in N$. Let $g:(0,1)\to\mathbb{R}$ be a function such that $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt<g(x)<2\sqrt{x}$ for all $x\in(0,1)$. Then $\lim_{x\to 0}f(x)g(x)$

[JEE(Advanced) 2023]

(A) does **NOT** exist

(B) is equal to 1

(C) is equal to 2

(D) is equal to 3

2. For $x\in\mathbb{R}$, let $\tan^{-1}(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the minimum value of the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$f(x)=\int_0^{x\tan^{-1}x}\frac{e^{(t-\cos t)}}{1+t^{2023}}dt$ is **[JEE(Advanced) 2023]**

3. Consider the equation

$$\int_1^e\frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}dx=1,\quad a\in(-\infty,0)\cup(1,\infty).$$

Which of the following statements is/are TRUE ? **[JEE(Advanced) 2022]**

(A) No a satisfies the above equation

(B) An integer a satisfies the above equation

(C) An irrational number a satisfies the above equation

(D) More than one a satisfy the above equation

4. The greatest integer less than or equal to

$$\int_1^2\log_2(x^3+1)dx+\int_1^{\log_2 9}\left(2^x-1\right)^{\frac{1}{3}}dx$$

is______. **[JEE(Advanced) 2022]**

5. For positive integer n, define

$$f(n)=n+\frac{16+5n-3n^2}{4n+3n^2}+\frac{32+n-3n^2}{8n+3n^2}+\frac{48-3n-3n^2}{12n+3n^2}+\ldots+\frac{25n-7n^2}{7n^2}.$$

Then, the value of $\lim_{n\to\infty}f(n)$ is equal to **[JEE(Advanced) 2022]**

(A) $3+\frac{4}{3}\log_e 7$

(B) $4-\frac{3}{4}\log_e\left(\frac{7}{3}\right)$

(C) $4-\frac{4}{3}\log_e\left(\frac{7}{3}\right)$

(D) $3+\frac{3}{4}\log_e 7$

6. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$$f(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)\,dt=0$$

Then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Question Stem for Questions Nos. 7 and 8

Question Stem

Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$, $i=1,2$. and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that

$g_1(x)=1$, $g_2(x)=|4x-\pi|$ and $f(x)=\sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$

Define $S_i=\int\limits_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x)\cdot g_i(x)dx$, $i=1,2$

7. The value of $\frac{16S_1}{\pi}$ is ______. **[JEE(Advanced) 2021]**

8. The value of $\frac{48S_2}{\pi^2}$ is ______. **[JEE(Advanced) 2021]**

Paragraph for Question No. 9 and 10

Let $\psi_1:[0,\infty)\to\mathbb{R}$, $\psi_2:[0,\infty)\to\mathbb{R}$, $f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$ be functions such that

$f(0)=g(0)=0$,

$\psi_1(x)=e^{-x}+x,\ \ x\geq 0$,

$\psi_2(x)=x^2-2x-2e^{-x}+2,\ x\geq 0$,

$f(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}dt,\ \ x>0$

and

$g(x)=\int_0^{x^2}\sqrt{t}\,e^{-t}dt,\ x>0$

9. Which of the following statements is **TRUE** ? **[JEE(Advanced) 2021]**

(A) $f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=\frac{1}{3}$

(B) For every $x>1$, there exists an $\alpha\in(1,x)$ such that $\psi_1(x)=1+\alpha x$

(C) For every $x>0$, there exists a $\beta\in(0,x)$ such that $\psi_2(x)=2x(\psi_1(\beta)-1)$

(D) f is an increasing function on the interval $\left[0,\frac{3}{2}\right]$

10. Which of the following statements is TRUE ? **[JEE(Advanced) 2021]**

(A) $\psi_1(x) \le 1$, for all $x > 0$

(B) $\psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 - \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

11. For any real number x, let [x] denote the largest integer less than or equal to x. If then the value of 9I is ______. **[JEE(Advanced) 2021]**

12. Which of the following inequalities is/are TRUE? **[JEE(Advanced) 2020]**

(A) $\int_0^1 x\cos x\,dx \ge \frac{3}{8}$　　(B) $\int_0^1 x\sin x\,dx \ge \frac{3}{10}$

(C) $\int_0^1 x^2\cos x\,dx \ge \frac{1}{2}$　　(D) $\int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If $F : [0, \pi] \to \mathbb{R}$ is defined by $F(x) = \int_0^x f(t)dt$, and if

$$\int_0^{\pi} (f'(x) + F(x))\cos x\,dx = 2,$$

then the value of $f(0)$ is ______. **[JEE(Advanced) 2020]**

14. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals ______. **[JEE(Advanced) 2019]**

15. For $a \in \mathbb{R}$, $|a| > 1$, let $\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.......+\frac{1}{(an+n)^2}\right)}\right) = 54$. Then the possible value(s) of a is/are : **[JEE(Advanced) 2019]**

(A) 8　　(B) –9　　(C) –6　　(D) 7

16. The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals **[JEE(Advanced) 2019]**

17. For each positive integer n, let $y_n = \frac{1}{n}((n+1)(n+2)...(n+n))^{1/n}$

For $x \in \mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of [L] is ______. **[JEE(Advanced) 2018]**

18. The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}}dx$$

is ______. **[JEE(Advanced) 2018]**

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If

$$g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]\, dt$$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x) =$ **[JEE(Advanced) 2017]**

20. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then **[JEE(Advanced)-2017]**

(A) $I < \frac{49}{50}$ (B) $I < \log_e 99$ (C) $I > \frac{49}{50}$ (D) $I > \log_e 99$

21. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\, dt$, then **[JEE(Advanced) 2017]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$

(C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

22. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to **[JEE(Advanced) 2016]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

23. Let $f(x) = \lim_{n\to\infty} \left(\frac{n^n (x+n)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)} \right)^{x/n}$, for all $x > 0$. Then **[JEE(Advanced) 2016]**

(A) $f\left(\frac{1}{2}\right) \geq f(1)$ (B) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$ (C) $f'(2) \leq 0$ (D) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

24. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is

[JEE(Advanced) 2016]

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x] & , x \leq 2 \\ 0 & , x > 2 \end{cases}$,

where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$, then the value of $(4I - 1)$ is **[JEE(Advanced) 2015]**

26. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right) dx$, where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE(Advanced) 2015]**

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1,2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|dt$ for all $x \in [-1,2]$. If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[JEE(Advanced) 2015]**

28. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L ?$$ **[JEE(Advanced) 2015]**

(A) $a = 2, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (B) $a = 2, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

(C) $a = 4, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ (D) $a = 4, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

29. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) **[JEE(Advanced) 2015]**

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{12}$ (B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} x f(x)dx = \frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)dx = 1$

30. Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are **[JEE(Advanced) 2015]**

(A) $m = 13, M = 24$ (B) $m = \frac{1}{4}, M = \frac{1}{2}$

(C) $m = -11, M = 0$ (D) $m = 1, M = 12$

Paragraph For Questions 31 and 32

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$, $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

31. The correct statement(s) is(are) **[JEE(Advanced) 2015]**

(A) $f'(1) < 0$ (B) $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$ (D) $f'(x) = 0$ for some $x \in (1, 3)$

32. If $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$, then the correct expression(s) is(are) **[JEE(Advanced) 2015]**

(A) $9f'(3) + f'(1) - 32 = 0$ (B) $\int_1^3 f(x)dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$ (D) $\int_1^3 f(x)dx = -12$

33. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : \mathbb{R} \to \mathbb{R}$ be defined as [JEE(Advanced) 2014]

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)dt & \text{if } a \le x \le b. \\ \int_a^b f(t)dt & \text{if } x > b \end{cases}$$

Then

(A) $g(x)$ is continuous but not differentiable at a

(B) $g(x)$ is differentiable on $\mathbb{R}$

(C) $g(x)$ is continuous but not differentiable at b

(D) $g(x)$ is continuous and differentiable at either a or b but not both.

34. Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then [JEE(Advanced) 2014]

(A) $f(x)$ is monotonically increasing on $[1, \infty)$

(B) $f(x)$ is monotonically decreasing on $(0, 1)$

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of x on $\mathbb{R}$

35. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is [JEE(Advanced) 2014]

36. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec} x)^{17} dx$ is equal to - [JEE(Advanced) 2014]

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$ (B) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$

(C) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$ (D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

37. Let $f : [0, 2] \to \mathbb{R}$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then F(2) equals -

[JEE(Advanced) 2014]

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4

Paragraph For Questions Nos. 38 and 39

Given that for each $a \in (0,1)$, $\lim_{h\to 0^+} \int_h^{1-h} t^{-a}\left(1-t\right)^{a-1} dt$ exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0,1).

38. The value of $g\left(\frac{1}{2}\right)$ is - [JEE(Advanced) 2014]

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

39. The value of $g'\left(\frac{1}{2}\right)$ is- [JEE(Advanced) 2014]

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

40.

	List-I	List-II
P.	The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is	1. 8
Q.	The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	2. 2
R.	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	3. 4
S.	$\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_0^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ equals	4. 0

Codes : **[JEE(Advanced) 2014]**

	P	Q	R	S
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

41. For $a \in R$ (the set of all real numbers), $a \ne -1$.

$$\lim_{n\to\infty} \frac{(1^a + 2^a + \ldots\ldots + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\ldots\ldots+(na+n)]} = \frac{1}{60}$$

Then a = **[JEE(Advanced) 2013]**

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

42. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$. Then - **[IIT-JEE 2012]**

(A) $S \ge \frac{1}{e}$ (B) $S \ge 1 - \frac{1}{e}$ (C) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (D) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

43. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is [IIT-JEE 2012]

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

44. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}}\frac{x\sin x^2}{\sin x^2+\sin(\ln 6-x^2)}dx$ is [IIT-JEE 2011]

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

45. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\ell n(1+t)}{t^4+4}dt$ [IIT-JEE 2010]

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

46. The value(s) of $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$ is (are) [IIT-JEE 2010]

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

47. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x)=\ell nx+\int_0^x\sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true? [IIT-JEE 2010]

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $a > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (a, \infty)$

(D) there exists $b > 0$ such that $|f(x)|+|f'(x)| \le b$ for all $x \in (0, \infty)$

48. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by [IIT-JEE 2010]

$$f(x)=\begin{cases} x-[x] & \text{if } [x] \text{ is odd,}\\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$$ Then the value of $\frac{\pi^2}{10}\int_{-10}^{10}f(x)\cos\pi x\,dx$ is

49. Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)dt$, $0 \le x \le 1$, and $f(0) = 0$, then :- [IIT-JEE 2009]

(A) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$ (B) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(C) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$ (D) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

50. If $I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx$, n = 0, 1, 2, ..., then [IIT-JEE 2009]

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10}I_{2m+1}=10\pi$

(C) $\sum_{m=1}^{10}I_{2m}=0$ (D) $I_n = I_{n+1}$

51. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$

for n = 1, 2, 3, ... Then, [IIT-JEE 2008]

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

52. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'\left(\frac{1}{4}\right)=0$. Then, [IIT-JEE 2008]

(A) $f''(x)$ vanishes at least twice on [0, 1]

(B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x\, dx = 0$

(D) $\int_{0}^{1/2} f(t)e^{\sin \pi t}dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t}dt$

Paragraph for Question Nos. 53 to 55

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$, $0 < a < 2$.

53. Which of the following is true ? [IIT-JEE 2008]

(A) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(C) $f'(1) f'(-1) = (2 - a)^2$

(D) $f'(1) f'(-1) = -(2 + a)^2$

54. Which of the following is true ? [IIT-JEE 2008]

(A) $f(x)$ is decreasing on (–1, 1) and has a local minimum at x = 1

(B) $f(x)$ is increasing on (–1, 1) and has a local maximum at x = 1

(C) $f(x)$ is increasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1

(D) $f(x)$ is decreasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1

55. Let $g(x) = \int_{0}^{e^x} \frac{f'(t)}{1+t^2}dt$ [IIT-JEE 2008]

Which of the following is true ?

(A) g'(x) is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(B) g'(x) is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(C) g'(x) charges sign on both $(-\infty, 0)$ and $(0, \infty)$

(D) g'(x) does not change sign on $(-\infty, 0)$

AREA UNDER CURVE

1. Let $f : [0, 1] \to [0, 1]$ be the function defined by $f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$. Consider the square region $S = [0, 1] \times [0, 1]$. Let $G = \{(x, y) \in S : y > f(x)\}$ be called the green region and $R = \{(x, y) \in S : y < f(x)\}$ be called the red region. Let $L_h = \{(x, h) \in S : x \in [0, 1]\}$ be the horizontal line drawn at a height $h \in [0, 1]$. Then which of the following statements is(are) ture?

[JEE(Advanced) 2023]

(A) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h

(B) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h

(C) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h

(D) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h

2. Let $n \geq 2$ be a natural number and $f : [0,1] \to \mathbb{R}$ be the function defined by **[JEE(Advanced) 2023]**

$$f(x) = \begin{cases} n(1-2nx) & \text{if } 0 \leq x \leq \frac{1}{2n} \\ 2n(2nx-1) & \text{if } \frac{1}{2n} \leq x \leq \frac{3}{4n} \\ 4n(1-nx) & \text{if } \frac{3}{4n} \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

If n is such that the area of the region bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is 4, then the maximum value of the function f is

3. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2 + \frac{5}{12} \text{ and } g(x) = \begin{cases} 2\left(1 - \frac{4|x|}{3}\right), & |x| \leq \frac{3}{4}, \\ 0, & |x| > \frac{3}{4}. \end{cases}$$

If α is the area of the region

$$\left\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq \frac{3}{4}, 0 \leq y \leq \min\{f(x), g(x)\}\right\},$$

then the value of 9α is ______. **[JEE(Advanced) 2022]**

4. The area of the region $\left\{(x,y): 0 \le x \le \frac{9}{4},\ \ 0 \le y \le 1,\ \ x \ge 3y,\ \ x+y \ge 2\right\}$ is

[JEE(Advanced) 2021]

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$ (C) $\frac{37}{96}$ (D) $\frac{13}{32}$

5. Let the functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

[JEE(Advanced) 2020]

(A) $\left(2-\sqrt{3}\right)+\frac{1}{2}(e-e^{-1})$ (B) $\left(2+\sqrt{3}\right)+\frac{1}{2}(e-e^{-1})$

(C) $\left(2-\sqrt{3}\right)+\frac{1}{2}(e+e^{-1})$ (D) $\left(2+\sqrt{3}\right)+\frac{1}{2}(e+e^{-1})$

6. The area of the region $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ is [JEE(Advanced) 2019]

(A) $8\log_e 2 - \frac{14}{3}$ (B) $16\log_e 2 - \frac{14}{3}$

(C) $16\log_e 2 - 6$ (D) $8\log_e 2 - \frac{7}{3}$

7. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$

for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE ?

[JEE(Advanced) 2018]

(A) The curve $y = f(x)$ passes through the point (1, 2)

(B) The curve $y = f(x)$ passes through the point (2, –1)

(C) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(D) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

8. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ______. **[JEE(Advanced) 2018]**

9. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x,\ 0 \le x \le 1\}$ into two equal parts, then **[JEE(Advanced) 2017]**

(A) $\frac{1}{2} < \alpha < 1$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $0 < \alpha \le \frac{1}{2}$ (D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

10. Area of the region $\left\{(x,y) \in \mathbb{R}^2 : y \ge \sqrt{|x+3|}, 5y \le x+9 \le 15\right\}$ is equal to - **[JEE(Advanced) 2016]**

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

11. Let $F(x)=\int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in \mathbb{R}$ and $f:\left[0,\frac{1}{2}\right]\to[0,\infty)$ be a continuous function. For $a\in\left[0,\frac{1}{2}\right]$, if

$F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

[JEE(Advanced) 2015]

12. The area enclosed by the curve $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0,\frac{\pi}{2}\right]$ is

[JEE(Advanced) 2013]

(A) $4(\sqrt{2}-1)$ (B) $2\sqrt{2}(\sqrt{2}-1)$ (C) $2(\sqrt{2}+1)$ (D) $2\sqrt{2}(\sqrt{2}+1)$

13. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1(0 \le x \le b)$ d and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals **[IIT-JEE 2011]**

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

14. Let $f : [-1,2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. Let $R_1 = \int_{-1}^{2} x f(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x=-1$, $x=2$, and the x-axis. Then-

[IIT-JEE 2011]

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

Paragraph for Question Nos. 15 to 17

Consider the polynomial

$f(x) = 1 + 2x + 3x^2 + 4x^3$.

Let s be the sum of all distinct real roots of f(x) and let $t = |s|$.

15. The real number s lies in the interval **[IIT-JEE 2010]**

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

16. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

[IIT-JEE 2010]

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) $(9, 10)$ (D) $\left(0, \frac{21}{64}\right)$

17. The function $f'(x)$ is **[IIT-JEE 2010]**

(A) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in $(-t, t)$

(D) decreasing in $(-t, t)$

18. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is :- [IIT-JEE 2009]

(A) $e - 1$

(B) $\int_1^e \ln(e + 1 - y)dy$

(C) $e - \int_0^1 e^x dx$

(D) $\int_1^e \ln y\, dy$

Paragraph for Question Nos. 19 to 21

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

If $x \in (-\infty, -2)\cup(2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

19. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$ [IIT-JEE 2008]

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

20. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is [IIT-JEE 2008]

(A) $\int_a^b \frac{x}{3((f(x))^2 - 1)}dx + bf(b) - af(a)$

(B) $-\int_a^b \frac{x}{3((f(x))^2 - 1)}dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3((f(x))^2 - 1)}dx - bf(b) + af(a)$

(D) $-\int_a^b \frac{x}{3((f(x))^2 - 1)}dx - bf(b) + af(a)$

21. $\int_{-1}^{1} g'(x)\, dx =$ [IIT-JEE 2008]

(A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

22. The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is :- [IIT-JEE 2008]

(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

(B) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(C) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(D) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

DIFFERENTIAL EQUATION

1. Let $f:[(1,\infty)\to\mathbb{R}$ be a differentiable function such that $f(1)=\frac{1}{3}$ and $3\int_1^x f(t)dt = xf(x)-\frac{x^3}{3}, x\in[1,\infty)$.

Let e denote the base of the natural logarithm. Then the value of $f(e)$ is **[JEE(Advanced) 2023]**

(A) $\frac{e^2+4}{3}$ (B) $\frac{\log_e 4+e}{3}$

(C) $\frac{4e^2}{3}$ (D) $\frac{e^2-4}{3}$

2. For $x\in\mathbb{R}$, let $y(x)$ be a solution of the differential equation **[JEE(Advanced) 2023]**

$(x^2-5)\frac{dy}{dx}-2xy=-2x(x^2-5)^2$ such that $y(2)=7$.

Then the maximum value of the function $y(x)$ is

3. If $y(x)$ is the solution of the differential equation

$$xdy-(y^2-4y)dx=0 \text{ for } x>0,\ y(1)=2,$$

and the slope of the curve $y=y(x)$ is never zero, then the value of $10y\left(\sqrt{2}\right)$ is ______.

[JEE(Advanced) 2022]

4. For $x\in\mathbb{R}$, let the function $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx}+12y=\cos\left(\frac{\pi}{12}x\right),\ y(0)=0.$$

Then, which of the following statements is/are TRUE? **[JEE(Advanced) 2022]**

(A) $y(x)$ is an increasing function

(B) $y(x)$ is a decreasing function

(C) There exists a real number β such that the line $y=\beta$ intersects the curve $y=y(x)$ at infinitely many points

(D) $y(x)$ is a periodic function

5. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x\in\mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx}+\alpha y = xe^{\beta x},\ y(1)=1$$

Let $S=\{y_{\alpha,\beta}(x):\alpha,\beta\in\mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

[JEE(Advanced) 2021]

(A) $f(x)=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$ (B) $f(x)=-\frac{x^2}{2}e^{-x}+\left(e+\frac{1}{2}\right)e^{-x}$

(C) $f(x)=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$ (D) $f(x)=\frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{4}\right)e^{-x}$

6. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $y=\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)-\sqrt{1-x^2}$ (B) $xy'-\sqrt{1-x^2}=0$

(C) $y=-\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)+\sqrt{1-x^2}$ (D) $xy'+\sqrt{1-x^2}=0$

7. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two non-constant differentiable functions.

If $f'(x) = (e^{(f(x)-g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018]**

(A) $f(2) < 1 - \log_e 2$ (B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$ (D) $g(1) < 1 - \log_e 2$

8. Let $f:(0,\pi)\to\mathbb{R}$ be a twice differentiable function such that

$\lim\limits_{t\to x}\frac{f(x)\sin t - f(t)\sin x}{t-x}=\sin^2 x$ for all $x \in (0, \pi)$.

If $f\left(\frac{\pi}{6}\right)=-\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4\sqrt{2}}$

(B) $f(x)<\frac{x^4}{6}-x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=0$

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx}=(2+5y)(5y-2),$$

then the value of $\lim\limits_{x\to-\infty} f(x)$ is _____ . **[JEE(Advanced) 2018]**

10. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx,\quad x>0$$

and $y(0)=\sqrt{7}$, then $y(256) =$ **[JEE(Advanced) 2017]**

(A) 80 (B) 3 (C) 16 (D) 9

11. If $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then **[JEE(Advanced) 2017]**

(A) $f(x) > e^{2x}$ in $(0,\infty)$ (B) $f(x)$ is decreasing in $(0,\infty)$

(C) $f(x)$ is increasing in $(0,\infty)$ (D) $f'(x) < e^{2x}$ in $(0,\infty)$

12. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1,3). The the solution curve- **[JEE(Advanced) 2016]**

(A) intersects $y = x + 2$ exactly at one point

(B) intersects $y = x + 2$ exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does NOT intersect $y = (x + 3)^2$

13. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then **[JEE(Advanced) 2016]**

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$

(B) $\lim\limits_{x\to 0^+} xf\left(\frac{1}{x}\right) = 2$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0,2)$

14. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is(are) true ? **[JEE(Advanced) 2015]**

(A) $y(-4) = 0$

(B) $y(-2) = 0$

(C) y(x) has a critical point in the interval (–1,0)

(D) y(x) has no critical point in the interval (–1,0)

15. Consider the family of all circles whose centers lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P,Q are functions of x,y and y' (here $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true? **[JEE(Advanced) 2015]**

(A) $P = y + x$

(B) $P = y - x$

(C) $P + Q = 1 - x + y + y' + (y')^2$

(D) $P - Q = x + y - y' - (y')^2$

16. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in (–1, 1) satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is **[JEE(Advanced) 2014]**

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

(B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

(D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

17. Let $f:\left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval **[JEE(Advanced) 2013]**

(A) $(2e - 1, 2e)$

(B) $(e - 1, 2e - 1)$

(C) $\left(\frac{e-1}{2}, e-1\right)$

(D) $\left(0, \frac{e-1}{2}\right)$

18. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$. Then the equation of the curve is **[JEE(Advanced) 2013]**

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

Paragraph for Question No. 19 and 20

Let $f : [0,1] \to$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x, x \in [0,1]$.

19. If the function $e^{-x}f(x)$ assumes its minimum in the interval $[0,1]$ at $x = \frac{1}{4}$, which of the following is true? **[JEE(Advanced) 2013]**

(A) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ (B) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(C) $f'(x) < f(x), 0 < x < \frac{1}{4}$ (D) $f'(x) < f(x), \frac{3}{4} < x < 1$

20. Which of the following is true for $0 < x < 1$? **[JEE(Advanced) 2013]**

(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(C) $-\frac{1}{4} < f(x) < 1$ (D) $-\infty < f(x) < 0$

21. If y(x) satisfies the differential equation $y' - y\tan x = 2x \sec x$ and $y(0) = 0$, then **[IIT-JEE 2012]**

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

22. Let $f : [1,\infty) \to [2,\infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)dt = 3x f(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is **[IIT-JEE 2011]**

23. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of y(2) is **[IIT-JEE 2011]**

24. Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point P(x, y) on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to **[IIT-JEE 2010]**

25. Let f be a real-valued function defined on the interval (–1,1) such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$, for all $x \in (1,1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to- **[JEE(Advanced) 2010]**

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$

26. Match the statements/ expressions in **Column I** with the open intervals in **Column II.** **[IIT-JEE 2009]**

Column-I	**Column-II**
(A) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$	(P) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(Q) $\left(0, \frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(R) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(D) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(S) $\left(0, \frac{\pi}{8}\right)$
	(T) $(-\pi, \pi)$

27. Let a solution y = y(x) of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{x^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$.

STATEMENT–1 : $y(x) = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right)$

and

STATEMENT–2 : y(x) is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$ **[IIT-JEE 2008]**

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

MATRIX

1. Let $M = (a_{ij})$, $i, j \in \{1, 2, 3\}$, be the 3×3 matrix such that $a_{ij} = 1$ if $j+1$ is divisible by i, otherwise $a_{ij} = 0$. Then which of the following statements is (are) true ? **[JEE(Advanced) 2023]**

(A) M is invertible

(B) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$

(C) The set $\{X \in \mathbb{R}^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(D) The matrix $(M - 2I)$ is invertible, where I is the 3×3 identity matrix

2. Let $R = \left\{ \begin{pmatrix} a & 3 & b \\ c & 2 & d \\ 0 & 5 & 0 \end{pmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\} \right\}$. Then the number of invertible matrices in R is

[JEE(Advanced) 2023]

3. Let β be a real number. Consider the matrix

$$A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$$

If $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix, then the value of 9β is ______. **[JEE(Advanced) 2022]**

4. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal to M^{2022} ? **[JEE(Advanced) 2022]**

(A) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$ (B) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$

(C) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$ (D) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

5. For any 3×3 matrix M, let $|M|$ denote the determinant of M. Let **[JEE(Advanced) 2021]**

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3×3, then which of the following statements is (are) **TRUE** ?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $|(EF)^3| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

6. For any 3×3 matrix M, let |M| denote the determinant of M. Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE ?** **[JEE(Advanced) 2021]**

(A) $|FE| = |I - FE||FGE|$

(B) $(I - FE)(I + FGE) = I$

(C) $EFG = GEF$

(D) $(I - FE)(I - FGE) = I$

7. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statement is/are ALWAYS TRUE ? **[JEE(Advanced) 2020]**

(A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

8. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is -18, then the value of the determinant of A is ______ **[JEE(Advanced) 2020]**

9. Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$,

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real number, and I is the 2×2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

then the value of $\alpha^* + \beta^*$ is **[JEE(Advanced) 2019]**

(A) $-\frac{37}{16}$ (B) $-\frac{29}{16}$ (C) $-\frac{31}{16}$ (D) $-\frac{17}{16}$

10. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj}M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $a + b = 3$

(B) $\det(\text{adj}M^2) = 81$

(C) $(\text{adj}M)^{-1} + \text{adj}M^{-1} = -M$

(D) If $M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

11. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_K \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_K^T$

where P_K^T denotes the transpose of the matrix P_K. Then which of the following options is/are correct? **[JEE(Advanced) 2019]**

(A) $X - 30I$ is an invertible matrix

(B) The sum of diagonal entries of X is 18

(C) If $X\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30x$

(D) X is a symmetric matrix

12. Let $x \in \mathbb{R}$ and let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$.

Then which of the following options is/are correct? **[JEE(Advanced) 2019]**

(A) For $x = 1$, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(B) There exists a real number x such that $PQ = QP$

(C) $\det R = \det\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$, for all $x \in \mathbb{R}$

(D) For $x = 0$, if $R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then $a + b = 5$

13. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced) 2018]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

14. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is _____ . **[JEE(Advanced) 2018]**

15. Which of the following is(are) NOT the square of a 3×3 matrix with real entries? **[JEE(Advanced) 2017]**

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

16. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **[JEE(Advanced) 2017]**

17. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced) 2017]**

(A) 198 (B) 126

(C) 135 (D) 162

18. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then- **[JEE(Advanced) 2016]**

(A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$

(C) $\det(P\text{adj}(Q)) = 2^9$ (D) $\det(Q\text{adj}(P)) = 2^{13}$

19. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[JEE(Advanced) 2016]**

(A) 52 (B) 103 (C) 201 (D) 205

20. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ? **[JEE(Advanced) 2015]**

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$

(C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

21. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if **[JEE(Advanced) 2014]**

(A) the first column of M is the transpose of the second row of M

(B) the second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with nonzero entries in the main diagonal

(D) the product of entries in the main diagonal of M is not the square of an integer

22. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then **[JEE(Advanced) 2014]**

(A) determinant of $(M^2 + MN^2)$ is 0

(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix

(C) determinant of $(M^2 + MN^2) \geq 1$

(D) for a 3×3 matrix U, if $(M^2 + MN^2)$ U equals the zero matrix then U is the zero matrix

23. For 3×3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ? **[JEE(Advanced) 2013]**

(A) $N^TM\,N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) MN – NM is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M) (adj N) = adj (M N) for all invertible matrices M and N

24. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is - **[IIT-JEE 2012]**

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13}

25. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[IIT-JEE 2012]**

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$ (C) $PX = 2X$ (D) $PX = -X$

26. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 2 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are)- **[IIT-JEE 2012]**

(A) –2 (B) –1 (C) 1 (D) 2

27. Let M and N be two 3×3 non-singular skew-symmetric matrices such that MN = NM. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to - **[IIT-JEE 2011]**

(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

28. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a,b and c is either ω or ω^2. Then the number of distinct matrices in the set S is- **[IIT-JEE 2011]**

(A) 2 (B) 6 (C) 4 (D) 8

29. Let M be 3×3 matrix satisfying

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

Then the sum of the diagonal entries of M is **[IIT-JEE 2011]**

30. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is **[IIT-JEE 2010]**

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

Paragraph for Question No. 31 to 33

Let p be an odd prime number and T_p be the following set of 2×2 matrices:

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\}$$

31. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) divisible by p is - **[IIT-JEE 2010]**

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D) $2p-1$

32. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is -
[**Note :** The trace of a matrix is the sum of its diagonal entries.] **[IIT-JEE 2010]**

(A) $(p-1)(p^2-p+1)$ (B) $p^3-(p-1)^2$

(C) $(p-1)^2$ (D) $(p-1)(p^2-2)$

33. The number of A in T_p such that det (A) is not divisible by p is - **[IIT-JEE 2010]**

(A) $2p^2$ (B) p^3-5p (C) p^3-3p (D) p^3-p^2

34. Let k be a positive real number and let **[IIT-JEE 2010]**

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det(adj B) = 10^6, then [k] is equal to

[**Note :** adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k]

Paragraph for Questions Nos. 35 to 37

Let $\mathcal{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

35. The number of matrices in $\mathcal{A}$ is :- **[IIT-JEE 2009]**

(A) 12 (B) 6 (C) 9 (D) 3

36. The number of matrices A in $\mathcal{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, is :- **[IIT-JEE 2009]**

(A) less than 4 (B) at least 4 but less than 7

(C) at least 7 but less than 10 (D) at least 10

37. The number of matrices A in $\mathcal{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent, is :-

[IIT-JEE 2009]

(A) 0 (B) more than 2 (C) 2 (D) 1

38. Match the Statement / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

[IIT-JEE 2008]

Column-I		Column-II	
(A)	The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p)	0
(B)	Let A and B be 3 × 3 matrices of real numbers, where A is symmetirc, B is skew-symmetric, and $(A+B)(A-B) = (A-B)(A+B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible value of k are	(q)	1
(C)	Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(r)	2
(D)	If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}\left(\theta-\phi-\frac{\pi}{2}\right)$ are	(s)	3

VECTOR

1. Let P be the plane $\sqrt{3}x+2y+3z=16$ and let

$$S=\left\{\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k} : \alpha^2+\beta^2+\gamma^2=1 \text{ and the distance of } (\alpha,\beta,\gamma) \text{ from the plane P is } \frac{7}{2}\right\}.$$

Let $\vec{u},\vec{v}$ and $\vec{w}$ be three distinct vectors in S such that $|\vec{u}-\vec{v}|=|\vec{v}-\vec{w}|=|\vec{w}-\vec{u}|$.

Let V be the volume of the parallelepiped determined by vectors $\vec{u},\vec{v}$ and $\vec{w}$. Then the value of $\frac{80}{\sqrt{3}}V$ is

[JEE(Advanced) 2023]

2. Let the position vectors of the points P,Q,R and S be $\vec{a}=\hat{i}+2\hat{j}-5\hat{k}$, $\vec{b}=3\hat{i}+6\hat{j}+3\hat{k}$, $\vec{c}=\frac{17}{5}\hat{i}+\frac{16}{5}\hat{j}+7\hat{k}$ and $\vec{d}=2\hat{i}+\hat{j}+\hat{k}$, respectively. Then which of the following statements is true? **[JEE(Advanced) 2023]**

(A) The points P,Q,R and S are **NOT** coplanar

(B) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio 5 : 4

(C) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio 5 : 4

(D) The square of the magnitude of the vector $\vec{b}\times\vec{d}$ is 95

3. Let $\hat{i},\hat{j}$ and $\hat{k}$ be the unit vectors along the three positive coordinate axes. Let

$$\vec{a}=3\hat{i}+\hat{j}-\hat{k},$$

$$\vec{b}=\hat{i}+b_2\hat{j}+b_3\hat{k}, \qquad b_2, b_3 \in \mathbb{R},$$

$$\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

be three vectors such that $b_2b_3 > 0$, $\vec{a}\cdot\vec{b}=0$ and

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix}=\begin{pmatrix} 3-c_1 \\ 1-c_2 \\ -1-c_3 \end{pmatrix}.$$

Then, which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $\vec{a}\cdot\vec{c}=0$ (B) $\vec{b}\cdot\vec{c}=0$ (C) $|\vec{b}|>\sqrt{10}$ (D) $|\vec{c}|\le\sqrt{11}$

4. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and $\vec{u}.\vec{w} = 1$, $\vec{v}.\vec{w} = 1$, $\vec{w}.\vec{w} = 4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u} + 5\vec{v}|$ is____. **[JEE(Advanced) 2021]**

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$, then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

6. In a triangle PQR, let $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\frac{\vec{a}.(\vec{c} - \vec{b})}{\vec{c}.(\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$, then the value of $|\vec{a} \times \vec{b}|^2$ is ______

[JEE(Advanced) 2020]

7. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram PQRS is 8, then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) $a + b = 4$

(B) $a - b = 2$

(C) The length of the diagonal PR of the parallelogram PQRS is 4

(D) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

8. Three lines

$L_1 : \vec{r} = \lambda\hat{i},\ \lambda \in \mathbb{R}$,

$L_2 : \vec{r} = \hat{k} + \mu\hat{j},\ \mu \in \mathbb{R}$ and

$L_3 : \vec{r} = \hat{i} + \hat{j} + v\hat{k},\ v \in \mathbb{R}$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear? **[JEE(Advanced) 2020]**

(A) $\hat{k} + \hat{j}$ (B) $\hat{k}$ (C) $\hat{k} + \frac{1}{2}\hat{j}$ (D) $\hat{k} - \frac{1}{2}\hat{j}$

9. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, $\alpha, \beta \in \mathbb{R}$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $(\vec{c} - (\vec{a} \times \vec{b})).\vec{c}$ equals

[JEE(Advanced) 2019]

10. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b} = 0$. For some $x, y \in \mathbb{R}$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ______. **[JEE(Advanced) 2018]**

11. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$$\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$$

Then the triangle PQR has S as its **[JEE(Advanced) 2017]**

(A) incentre
(B) orthocenter
(C) circumcentre
(D) centroid

Paragraph for Question No. 12 and 13

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

12. $|\overrightarrow{OX} \times \overrightarrow{OY}| =$ **[JEE(Advanced) 2017]**

(A) sin(Q + R) (B) sin(P + R) (C) sin 2R (D) sin(P + Q)

13. If the triangle PQR varies, then the minimum value of cos(P + Q) + cos(Q + R) + cos(R + P) is

[JEE(Advanced) 2017]

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

14. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in $\mathbb{R}^3$ and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector in $\mathbb{R}^3$ such that $|\hat{u} \times \vec{v}| = 1$ and $\hat{w}.(\hat{u} \times \vec{v}) = 1$. Which of the following statement(s) is(are) correct?

[JEE(Advanced) 2016]

(A) There is exactly one choice for such $\vec{v}$
(B) There are infinitely many choice for such $\vec{v}$
(C) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$
(D) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

15. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$ and $\vec{b}.\vec{c} = 24$, then which of the following is (are) true ? **[JEE(Advanced) 2015]**

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$
(B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$
(C) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$
(D) $\vec{a}.\vec{b} = -72$

16.

Column-I	Column-II
(A) In a triangle ΔXYZ, let a,b and c be the lengths of the sides opposite to the angles X,Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	(P) 1
	(Q) 2
(B) In a triangle ΔXYZ, let a,b and c be the lengths of the sides opposite to the angles X,Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	
(C) In $\mathbb{R}^2$, Let $\sqrt{3}\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\vec{OX}$ and $\vec{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\|\beta\|$ is (are)	(R) 3
(D) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = \|\alpha x - 1\| + \|\alpha x - 2\| + \alpha x$, where $\alpha \in \{0,1\}$. Then the value(s) of $F(\alpha)+\frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is(are)	(S) 5
	(T) 6

[JEE(Advanced) 2015]

17. Suppose that $\vec{p},\vec{q}$ and $\vec{r}$ are three non-coplanar vectors in $\mathbb{R}^3$. Let the components of a vector $\vec{s}$ along $\vec{p},\vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{r}$ along $(-\vec{p}+\vec{q}+\vec{r})$, $(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x,y and z, respectively, then the value of $2x + y + z$ is

[JEE(Advanced) 2015]

18. Let $\vec{x},\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is nonzero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then **[JEE(Advanced) 2015]**

(A) $\vec{b}=(\vec{b}.\vec{z})(\vec{z}-\vec{x})$ (B) $\vec{a}=(\vec{a}.\vec{y})(\vec{y}-\vec{z})$

(C) $\vec{a}.\vec{b}=-(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a}=(\vec{a}.\vec{y})(\vec{z}-\vec{y})$

19. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is

[JEE(Advanced) 2014]

20.

	List-I		List-II
P.	Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ equals	1.	1
Q.	Let $A_1, A_2, \ldots\ldots A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a_k}$ be the position vector of the point A_k, $k = 1, 2, \ldots\ldots n$. If $\left\|\sum_{k=1}^{n-1}(\vec{a_k} \times \vec{a}_{k+1})\right\| = \left\|\sum_{k=1}^{n-1}(\vec{a_k} \cdot \vec{a}_{k+1})\right\|$, then the minimum value of n is	2.	2
R.	If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	3.	8
S.	Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is	4.	9

[JEE(Advanced) 2014]

Codes :

	P	Q	R	S
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

21. A line ℓ passing through the origin is perpendicular to the lines

$\ell_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, -\infty < t < \infty$

$\ell_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, -\infty < s < \infty$

Then, the coordinate(s) of the point(s) on ℓ_2 at a distance of $\sqrt{17}$ from the point of intersection of ℓ and ℓ_1 is(are) - **[JEE(Advanced) 2013]**

(A) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (B) $(-1,-1,0)$ (C) $(1,1,1)$ (D) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

22. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is **[JEE(Advanced) 2013]**

(A) 5 (B) 20 (C) 10 (D) 30

23. Consider the set of eight vectors $V=\{a\hat{i}+b\hat{j}+c\hat{k}: a,b,c \in \{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[JEE(Advanced) 2013]**

24. Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I	List-II
P.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	1. 100
Q.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	2. 30
R.	Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	3. 24
S.	Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	4. 60

Codes :

	P	Q	R	S
(A)	4	2	3	1
(B)	2	3	1	4
(C)	3	4	1	2
(D)	1	4	3	2

[JEE(Advanced) 2013]

25. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$, then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is **[IIT-JEE 2012]**

26. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}|=\sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b}).(-7\hat{i}+2\hat{j}+3\hat{k})$ is **[IIT-JEE 2012]**

(A) 0 (B) 3 (C) 4 (D) 8

27. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by **[IIT-JEE 2011]**

(A) $\hat{i}-3\hat{j}+3\hat{k}$ (B) $-3\hat{i}-3\hat{j}-\hat{k}$ (C) $3\hat{i}-\hat{j}+3\hat{k}$ (D) $\hat{i}+3\hat{j}-3\hat{k}$

28. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are **[IIT-JEE 2011]**

(A) $\hat{j}-\hat{k}$ (B) $-\hat{i}+\hat{j}$ (C) $\hat{i}-\hat{j}$ (D) $-\hat{j}+\hat{k}$

29. Let $\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r}.\vec{a} = 0$, then the value of $\vec{r}.\vec{b}$ is **[IIT-JEE 2011]**

30. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}}$, then the value of $\left(2\vec{a} + \vec{b}\right).\left[\left(\vec{a} \times \vec{b}\right) \times \left(\vec{a} - 2\vec{b}\right)\right]$ is. **[IIT-JEE 2010]**

31. Two adjacent sides of a parallelogram ABCD are given by
$\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$
The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by - **[IIT-JEE 2010]**

(A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

32. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $\left(\vec{a} \times \vec{b}\right).\left(\vec{c} \times \vec{d}\right) = 1$ and $\vec{a}.\vec{c} = \frac{1}{2}$, then :- **[IIT-JEE 2009]**

(A) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar (B) $\vec{b}, \vec{c}, \vec{d}$ are non-coplanar

(C) $\vec{b}, \vec{d}$ are non-parallel (D) $\vec{a}, \vec{d}$ are parallel and $\vec{b}, \vec{c}$ are parallel

33. The edges of a parallelopiped are of unit length and are parallel to non–coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that

$\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = \frac{1}{2}$.

Then, the volume of the parallelopiped is :- **[IIT-JEE 2008]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{3}}$

34. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overline{OP}$ (where O is the origin) is given by $\hat{a}\cos t + \hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overline{OP}$ and $\hat{u}$ be the unit vector along $\overline{OP}$. Then, **[IIT-JEE 2008]**

(A) $\hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}$ and $M = (1 + \hat{a}.\hat{b})^{1/2}$

(B) $\hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|}$ and $M = (1 + \hat{a}.\hat{b})^{1/2}$

(C) $\hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}$ and $M = (1 + 2\hat{a}.\hat{b})^{1/2}$

(D) $\hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|}$ and $M = (1 + 2\hat{a}.\hat{b})^{1/2}$

3D GEOMETRY

1. Let Q be the cube with the set of vertices $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3 \{0,1\}\}$. Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q. Let S be the set of all four lines containing the main diagonals of the cube Q; for instance, the line passing through the vertices (0, 0, 0) and (1, 1, 1) is in S. For lines ℓ_1 and ℓ_2, let $d(\ell_1, \ell_2)$ denote the shortest distance between them. Then the maximum value of $d(\ell_1, \ell_2)$, as ℓ_1 varies over F and ℓ_2 varies over S, is

[JEE(Advanced) 2023]

(A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{\sqrt{8}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{12}}$

2. Let ℓ_1 and ℓ_2 be the lines $\vec{r}_1 = \lambda(\hat{i}+\hat{j}+\hat{k})$ and $\vec{r}_2 = (\hat{j}-\hat{k})+\mu(\hat{i}+\hat{k})$, respectively. Let X be the set of all the planes H that contain the line ℓ_1. For a plane H, let d(H) denote the smallest possible distance between the points of ℓ_2 and H. Let H_0 be plane in X for which $d(H_0)$ is the maximum value of d(H) as H varies over all planes in X.

[JEE(Advanced) 2023]

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	List-II
(P) The value of $d(H_0)$ is	(1) $\sqrt{3}$
(Q) The distance of the point (0, 1, 2) from H_0 is	(2) $\frac{1}{\sqrt{3}}$
(R) The distance of origin from H_0 is	(3) 0
(S) The distance of origin from the point of intersection of planes $y = z$, $x = 1$ and H_0 is	(4) $\sqrt{2}$
	(5) $\frac{1}{\sqrt{2}}$

The correct option is :

(A) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)

(B) (P) → (5) (Q) → (4) (R) → (3) (S) → (1)

(C) (P) → (2) (Q) → (1) (R) → (3) (S) → (2)

(D) (P) → (5) (Q) → (1) (R) → (4) (S) → (2)

3. Let P_1 and P_2 be two planes given by

$$P_1: 10x + 15y + 12z - 60 = 0,$$
$$P_2: -2x + 5y + 4z - 20 = 0.$$

Which of the following straight lines can be an edge of some tetrahedron whose two faces lie on P_1 and P_2?

[JEE(Advanced) 2022]

(A) $\frac{x-1}{0}=\frac{y-1}{0}=\frac{z-1}{5}$ (B) $\frac{x-6}{-5}=\frac{y}{2}=\frac{z}{3}$

(C) $\frac{x}{-2}=\frac{y-4}{5}=\frac{z}{4}$ (D) $\frac{x}{1}=\frac{y-4}{-2}=\frac{z}{3}$

4. Let S be the reflection of a point Q with respect to the plane given by

$$\vec{r} = -(t+p)\hat{i} + t\hat{j} + (1+p)\hat{k}$$

where t, p are real parameters and $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the three positive coordinate axes. If the position vectors of Q and S are $10\hat{i}+15\hat{j}+20\hat{k}$ and $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ respectively, then which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $3(\alpha+\beta) = -101$ (B) $3(\beta+\gamma) = -71$

(C) $3(\gamma+\alpha) = -86$ (D) $3(\alpha+\beta+\gamma) = -121$

Question Stem for Question Nos. 5 and 6

Question Stem

Let α, β and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha$$

$$4x + 5y + 6z = \beta$$

$$7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point (0, 1, 0) from the plane P.

5. The value of $|M|$ is ________. **[JEE(Advanced) 2021]**

6. The value of D is ________. **[JEE(Advanced) 2021]**

7. Let L_1 and L_2 be the following straight lines.

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \text{ and } L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE? **[JEE(Advanced) 2020]**

(A) $\alpha - \gamma = 3$ (B) $\ell + m = 2$ (C) $\alpha - \gamma = 1$ (D) $\ell + m = 0$

8. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point (3,2,–1) is the mirror image of the point (1,0,–1) with respect to the plane $\alpha x + \beta y + \gamma z = \delta$. Then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) $\alpha + \beta = 2$ (B) $\delta - \gamma = 3$ (C) $\delta + \beta = 4$ (D) $\alpha + \beta + \gamma = \delta$

9. Let L_1 and L_2 denotes the lines

$\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in \mathbb{R}$ and

$\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in \mathbb{R}$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ? **[JEE(Advanced) 2019]**

(A) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$
(B) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$
(C) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$
(D) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$

10. Three lines are given by

$$\vec{r} = \lambda\hat{i}, \lambda \in \mathbb{R}$$

$$\vec{r} = \mu(\hat{i} + \hat{j}), \mu \in \mathbb{R} \text{ and}$$

$$\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k}), \nu \in \mathbb{R}.$$

Let the lines cut the plane $x + y + z = 1$ at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals _____. **[JEE(Advanced) 2019]**

11. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, –1

(B) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60°

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_2 is $\frac{2}{\sqrt{3}}$

12. Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the plane $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____. **[JEE(Advanced) 2018]**

13. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT.

If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is_____.

[JEE(Advanced) 2018]

14. The equation of the plane passing through the point (1,1,1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is- **[JEE(Advanced) 2017]**

(A) $14x + 2y + 15z = 31$
(B) $14x + 2y - 15z = 1$
(C) $-14x + 2y + 15z = 3$
(D) $14x - 2y + 15z = 27$

15. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then-

[JEE(Advanced) 2016]

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$

(B) the equaiton of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

16. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is [JEE(Advanced) 2016]

(A) $x + y - 3z = 0$ (B) $3x + z = 0$

(C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

17. In $\mathbb{R}^3$, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0,1,0) from P_3 is 1 and the distance of a point (α,β,γ) from P_3 is 2, then which of the following relations is (are) true ?

[JEE(Advanced) 2015]

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

18. In $\mathbb{R}^3$, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M ? [JEE(Advanced) 2015]

(A) $\left(0,-\frac{5}{6},-\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6},-\frac{1}{3},\frac{1}{6}\right)$ (C) $\left(-\frac{5}{6},0,\frac{1}{6}\right)$ (D) $\left(-\frac{1}{3},0,\frac{2}{3}\right)$

19. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

[JEE(Advanced) 2014]

(A) $\sqrt{2}$ (B) 1 (C) -1 (D) $-\sqrt{2}$

20. Perpendiculars are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of perpendiculars lie on the line [JEE(Advanced) 2013]

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

21. Two lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

[JEE(Advanced) 2013]

(A) 1 (B) 2 (C) 3 (D) 4

22. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}, L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to planes P_1 and P_2. **[JEE(Advanced) 2013]**

Match List-I with List-II and select the correct answer using the code given below the lists.

List-I	**List-II**
P. a =	1. 13
Q. b =	2. –3
R. c =	3. 1
S. d =	4. –2

Codes :

	P	Q	R	S
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3

23. The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1,–1,4) with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point T(2,1,4) to QR, then the length of the line segment PS is - **[IIT-JEE 2012]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

24. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$ is **[IIT-JEE 2012]**

(A) $5x - 11y + z = 17$ (B) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(C) $x + y + z = \sqrt{3}$ (D) $x - \sqrt{2}y = 1 - \sqrt{2}$

25. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are) **[IIT-JEE 2012]**

(A) $y + 2z = -1$ (B) $y + z = -1$ (C) $y - z = -1$ (D) $y - 2z = -1$

26. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is. **[IIT-JEE 2010]**

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$ (C) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

27. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is [IIT-JEE 2010]

28. If the distance of the point $P(1,-2,1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is- [IIT-JEE 2010]

(A) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (B) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$ (C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (D) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

29. Match the statements in Column-I with the values in Column-II. [IIT-JEE 2010]

Column-I		Column-II	
(A)	A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	(p)	–4
(B)	The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$ are	(q)	0
(C)	Non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b} = 0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$ and $2\left|(\vec{b}+\vec{c})\right| = \left|(\vec{b}-\vec{a})\right|$. If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the possible values of μ are	(r)	4
(D)	Let f be the function on $[-\pi, \pi]$ given by $f(0) = 9$ and $f(x) = \sin\left(\frac{9x}{2}\right) / \sin\left(\frac{x}{2}\right)$ for $x \neq 0$. The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is	(s)	5
		(t)	6

30. Let P(3, 2, 6) be a point in space and Q be a point on the line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is :- [IIT-JEE 2009]

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$

(C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

31. A line with positive direction cosines passes through the point P (2, –1, 2) and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals [IIT-JEE 2009]

(A) 1 (B) $\sqrt{2}$

(C) $\sqrt{3}$ (D) 2

32. Consider three planes

$$P_1 : x - y + z = 1$$

$$P_2 : x + y - z = -1$$

$$P_3 : x - 3y + 3z = 2$$

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2, respectively.

STATEMENT–1 : At least two of the lines L_1, L_2 and L_3 are non-parallel.

and

STATEMENT–2 : The three planes do not have a common point. **[IIT-JEE 2008]**

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

Paragraph for Question No. 33 to 35

Consider the lines $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$, $L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

33. The unit vector perpendicular to both L_1 and L_2 is :- **[IIT-JEE 2008]**

(A) $\frac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$ (B) $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$

(C) $\frac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$ (D) $\frac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$

34. The shortest distance between L_1 and L_2 is :- **[IIT-JEE 2008]**

(A) 0 (B) $\frac{17}{\sqrt{3}}$

(C) $\frac{41}{5\sqrt{3}}$ (D) $\frac{17}{5\sqrt{3}}$

35. The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is :- **[IIT-JEE 2008]**

(A) $\frac{2}{\sqrt{75}}$ (B) $\frac{7}{\sqrt{75}}$

(C) $\frac{13}{\sqrt{75}}$ (D) $\frac{23}{\sqrt{75}}$

COMPLEX NUMBER

1. Let $A = \left\{\dfrac{1967+1686i\sin\theta}{7-3i\cos\theta} : \theta \in \mathbb{R}\right\}$. If A contains exactly one positive integer n, then the value of n is

[JEE(Advanced) 2023]

2. Let z be complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be nonzero. [JEE(Advanced) 2023]

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	List-II
(P) $|z|^2$ is equal to	(1) 12
(Q) $|z-\bar{z}|^2$ is equal to	(2) 4
(R) $|z|^2 + |z+\bar{z}|^2$ is equal to	(3) 8
(S) $|z+1|^2$ is equal to	(4) 10
	(5) 7

The correct option is :

(A) (P) → (1) (Q) → (3) (R) → (5) (S) → (4)

(B) (P) → (2) (Q) → (1) (R) → (3) (S) → (5)

(C) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)

(D) (P) → (2) (Q) → (3) (R) → (5) (S) → (4)

3. Let $A_1, A_2, A_3, \ldots\ldots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle and let PA_i denote the distance between the points P and A_i for $i = 1,2,\ldots,8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \ldots\ldots PA_8$, is

[JEE(Advanced) 2023]

4. Let z be a complex number with non-zero imaginary part. If

$$\frac{2+3z+4z^2}{2-3z+4z^2}$$

is a real number, then the value of $|z|^2$ is ______. [JEE(Advanced) 2022]

5. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i = \sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation

$$\bar{z} - z^2 = i(\bar{z} + z^2)$$

is ______. [JEE(Advanced) 2022]

6. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of

$$(\bar{z})^2 + \frac{1}{z^2}$$

are integers, then which of the following is/are possible value(s) of $|z|$? [JEE(Advanced) 2022]

(A) $\left(\dfrac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ (B) $\left(\dfrac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$

(C) $\left(\dfrac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ (D) $\left(\dfrac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

7. Let $\theta_1, \theta_2, ..., \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + ... + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for k = 2, 3, ..., 10, where $i = \sqrt{-1}$. Consider the statements P and Q given below : **[JEE(Advanced) 2021]**

$P : |z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$

$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$

Then,

(A) P is **TRUE** and Q is **FALSE** (B) Q is **TRUE** and P is **FALSE**

(C) both P and Q are **TRUE** (D) both P and Q are **FALSE**

8. For any complex number w = c + id, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers z = x + iy satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x,y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0.$$

Then which of the following statements is (are) **TRUE** ? **[JEE(Advanced) 2021]**

(A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$ (B) $|z| \le 2$ for all $z \in S$

(C) $\left|z + \frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$ (D) The set S has exactly four elements

10. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $Re(z_1) > 0$ and $Re(z_2) < 0$, is _____. **[JEE(Advanced) 2020]**

11. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

[JEE(Advanced) 2019]

(A) $\frac{\pi}{4}$ (B) $-\frac{\pi}{2}$ (C) $\frac{3\pi}{4}$ (D) $\frac{\pi}{2}$

12. Let $\omega \ne 1$ be a cube root of unity. Then the minimum of the set

$$\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$$

equals _____. **[JEE(Advanced) 2019]**

13. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) **FALSE**? **[JEE(Advanced) 2018]**

(A) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

14. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in \mathbb{R}, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

15. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ? **[JEE(Advanced) 2017]**

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

16. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is **[JEE(Advanced) 2016]**

17. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a + ibt}, t \in \mathbb{R}, t \neq 0\right\}$, where.

If $z = x + iy$ and $z \in S$, then (x,y) lies on **[JEE(Advanced) 2016]**

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the x-axis for $a \neq 0, b = 0$

(D) the y-axis for $a = 0, b \neq 0$

18. **[JEE(Advanced) 2015]**

	Column-I		Column-II
(A)	In $\mathbb{R}^2$, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\ge 1\end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$ then possible value(s) of n is (are)	(R)	3
(D)	Let the harmonic mean of two positive real number a and b be 4. If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	(S)	4
		(T)	5

19. For any integer k, let $\alpha_k=\cos\left(\frac{k\pi}{7}\right)+i\sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}\lvert\alpha_{k+1}-\alpha_k\rvert}{\sum_{k=1}^{3}\lvert\alpha_{4k-1}-\alpha_{4k-2}\rvert} \text{ is}$$

[JEE(Advanced) 2015]

20. The quadratic equation p(x) = 0 with real coefficients has purely imaginary roots. Then the equation p(p(x)) = 0 has **[JEE(Advanced) 2014]**

(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots.

21. Let $z_k=\cos\left(\frac{2k\pi}{10}\right)+i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ... 9. **[JEE(Advanced) 2014]**

	List-I		**List-II**
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1.	True
Q.	There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2.	False
R.	$\frac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert\ldots\lvert 1-z_9\rvert}{10}$ equals	3.	1
S.	$1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	4.	2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

22. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2+2$, then $|\alpha| =$

[JEE(Advanced) 2013]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

23. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when $n =$ [JEE(Advanced) 2013]

(A) 57 (B) 55 (C) 58 (D) 56

24. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ [JEE(Advanced) 2013]

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Paragraph for Question No. 25 and 26

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and

$S_3 = \{z \in C : \text{Re}\, z > 0\}$.

25. $\min\limits_{z \in S} |1-3i-z| =$ [JEE(Advanced) 2013]

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

26. Area of S = [JEE(Advanced) 2013]

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

27. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value - [IIT-JEE 2012]

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Paragraph for Question No. 28 to 30

Let a,b and c be three real numbers satisfying

$$[a \quad b \quad c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0 \quad 0 \quad 0] \qquad ...(E)$$

28. If the point P(a,b,c), with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

[IIT-JEE 2011]

(A) 0 (B) 12 (C) 7 (D) 6

29. Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of

$\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to - **[IIT-JEE 2011]**

(A) -2 (B) 2 (C) 3 (D) -3

30. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation

$ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is- **[IIT-JEE 2011]**

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

31. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is

[IIT-JEE 2011]

32. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a + b + c = x$$

$$a + b\omega + c\omega^2 = y$$

$$a + b\omega^2 + c\omega = z.$$

Then the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is **[IIT-JEE 2011]**

33. Match the statements given in **Column-I** with the values given in **Column-II.** **[IIT-JEE 2011]**

Column-I	Column-II
(A) If $\vec{a}=\hat{j}+\sqrt{3}\hat{k}$, $\vec{b}=-\hat{j}+\sqrt{3}\hat{k}$ and $\vec{c}=2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x)-3x)dx = a^2-b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ell n3}\int_{7/6}^{5/6}\sec(\pi x)dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1, z \ne 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

34. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II.** **[IIT-JEE 2011]**

Column-I	Column-II
(A) The set $\left\{\text{Re}\left(\dfrac{2iz}{1-z^2}\right): z \text{ is a complex number}, \lvert z\rvert=1, z \neq \pm 1\right\}$ is	(p) $(-\infty,-1)\cup(1,\infty)$
(B) The domain of the function $f(x)=\sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty,0)\cup(0,\infty)$
(C) If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1\end{vmatrix}$, then the set	(r) $[2,\infty)$
$\left\{f(\theta): 0 \le \theta < \dfrac{\pi}{2}\right\}$ is	(s) $(-\infty,-1]\cup[1,\infty)$
(D) If $f(x)=x^{3/2}(3x-10)$, $x \ge 0$, then $f(x)$ is increasing in	(t) $(-\infty,0]\cup[2,\infty)$

35. Let z_1 and z_2 be two distinct complex numbers and let $z=(1-t)z_1+tz_2$ for some real number t with $0<t<1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then **[IIT-JEE 2010]**

(A) $\lvert z-z_1\rvert+\lvert z-z_2\rvert=\lvert z_1-z_2\rvert$ (B) $\text{Arg}(z-z_1)=\text{Arg}(z-z_2)$

(C) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1\end{vmatrix}=0$ (D) $\text{Arg}(z-z_1)=\text{Arg}(z_2-z_1)$

36. Let ω be the complex number $\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega\end{vmatrix}=0$ is equal to **[IIT-JEE 2010]**

37. Match the statements in Column-I with those in Column-II.

[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.] **[IIT-JEE 2010]**

	Column-I		**Column-II**
(A)	The set of points z satisfying $\lvert z-i\lvert z\rvert\rvert=\lvert z+i\lvert z\rvert\rvert$ is contained in or equal to	(p)	an ellipse with eccentricity $\dfrac{4}{5}$
(B)	The set of points z satisfying $\lvert z+4\rvert+\lvert z-4\rvert=10$ is contained in or equal to	(q)	the set of points z satisfying Im z = 0
(C)	If $\lvert w\rvert=2$, then the set of points $z=w-\dfrac{1}{w}$ is contained in or equal to	(r)	the set of points z satisfying $\lvert \text{Im } z\rvert<1$
(D)	If $\lvert w\rvert=1$, then the set of points $z=w+\dfrac{1}{w}$ is contained in or equal to	(s)	the set of points z satisfying $\lvert \text{Re } z\rvert\le 2$
		(t)	the set of points z satisfying $\lvert z\rvert\le 3$

38. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is :- [IIT-JEE 2009]

(A) $\frac{1}{\sin 2°}$ (B) $\frac{1}{3\sin 2°}$ (C) $\frac{1}{2\sin 2°}$ (D) $\frac{1}{4\sin 2°}$

39. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is :- [IIT-JEE 2009]

(A) 48 (B) 32 (C) 40 (D) 80

40. Match the conics in **Column I** with the statements/ expressions in **Column II.** [IIT-JEE 2009]

Column-I

(A) Circle

(B) Parabola

(C) Ellipse

(D) Hyperbola

Column-II

(P) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$

(Q) Points z in the complex plane satisfying $|z+2|-|z-2| = \pm 3$

(R) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$

(S) The eccentricity of the conic lies in the interval $1 \le x < \infty$

(T) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = |z|^2 + 1$

Paragraph for Question Nos. 41 to 43

Let A, B, C be three sets of complex numbers as defined below

$$A = \{z : \text{Im}\, z \ge 1\}$$
$$B = \{z : |z-2-i| = 3\}$$
$$C = \{z : \text{Re}((1-i)z) = \sqrt{2}\}$$

41. The number of elements in the set $A \cap B \cap C$ is :- [IIT-JEE 2008]

(A) 0 (B) 1 (C) 2 (D) ∞

42. Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between :- [IIT-JEE 2008]

(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

43. Let z be any point in $A \cap B \cap C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z|-|\omega| + 3$ lies between :- [IIT-JEE 2008]

(A) –6 and 3 (B) –3 and 6 (C) –6 and 6 (D) –3 and 9

44. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by :- [IIT-JEE 2008]

(A) 6 + 7i (B) –7 + 6i (C) 7 + 6i (D) –6 + 7i

PROBABILITY

1. Let $X = \left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : \frac{x^2}{8} + \frac{y^2}{20} < 1 \text{ and } y^2 < 5x\right\}$. Three distinct points P, Q and R are randomly chosen from X. Then the probability that P, Q and R form a triangle whose area is a positive integer, is **[JEE(Advanced) 2023]**

(A) $\frac{71}{220}$ (B) $\frac{73}{220}$ (C) $\frac{79}{220}$ (D) $\frac{83}{220}$

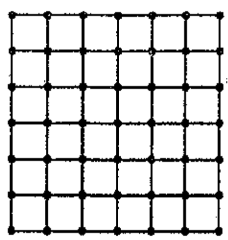
2. Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are same. If the probability of a random toss resulting in head is $\frac{1}{3}$, then the probability that the experiment stops with head is. **[JEE(Advanced) 2023]**

(A) $\frac{1}{3}$ (B) $\frac{5}{21}$ (C) $\frac{4}{21}$ (D) $\frac{2}{7}$

3. Let X be the set of all five digit numbers formed using 1,2,2,2,4,4,0. For example, 22240 is in X while 02244 and 44422 are not in X. Suppose that each element of X has an equal chance of being chosen. Let p be the conditional probability that an element chosen at random is a multiple of 20 given that it is a multiple of 5. Then the value of 38p is equal to- **[JEE(Advanced) 2023]**

Paragraph for Question No. 4 and 5

Consider the 6×6 square in the figure. Let $A_1, A_2, \ldots, A_{49}$ be the points of intersections (dots in the picture) in some order. We say that A_i and A_j are friends if they are adjacent along a row or along a column. Assume that each point A_i has an equal chance of being chosen.



4. Let p_i be the probability that a randomly chosen point has i many friends, $i = 0,1,2,3,4$. Let X be a random variable such that for $i = 0,1,2,3,4$, the probability $P(X = i) = p_i$. Then the value of $7E(X)$ is **[JEE(Advanced) 2023]**

5. Two distinct points are chosen randomly out of the points $A_1, A_2, \ldots, A_{49}$. Let p be the probability that they are friends. Then the value of $7p$ is **[JEE(Advanced) 2023]**

6. In a study about a pandemic, data of 900 persons was collected. It was found that

190 persons had symptom of fever,
220 persons had symptom of cough,
220 persons had symptom of breathing problem,
330 persons had symptom of fever or cough or both,
350 persons had symptom of cough or breathing problem or both,
340 persons had symptom of fever or breathing problem or both,
30 persons had all three symptoms (fever, cough and breathing problem).

If a person is chosen randomly from these 900 persons, then the probability that the person has at most one symptom is ________. **[JEE(Advanced) 2022]**

7. Two players, P_1 and P_2, play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If $x > y$, then P_1 scores 5 points and P_2 scores 0 point. If $x = y$, then each player scores 2 points. If $x < y$, then P_1 scores 0 point and P_2 scores 5 points. Let X_i and Y_i be the total scores of P_1 and P_2, respectively, after playing the i^{th} round. **[JEE(Advanced) 2022]**

List-I		List-II	
(I)	Probability of $(X_2 \geq Y_2)$ is	(P)	$\frac{3}{8}$
(II)	Probability of $(X_2 > Y_2)$ is	(Q)	$\frac{11}{16}$
(III)	Probability of $(X_3 = Y_3)$ is	(R)	$\frac{5}{16}$
(IV)	Probability of $(X_3 > Y_3)$ is	(S)	$\frac{355}{864}$
		(T)	$\frac{77}{432}$

The correct option is:

(A) (I) → (Q); (II) → (R); (III) → (T); (IV) → (S)

(B) (I) → (Q); (II) → (R); (III) → (T); (IV) → (T)

(C) (I) → (P); (II) → (R); (III) → (Q); (IV) → (S)

(D) (I) → (P); (II) → (R); (III) → (Q); (IV) → (T)

8. Suppose that

Box-I contains 8 red, 3 blue and 5 green balls,
Box-II contains 24 red, 9 blue and 15 green balls,
Box-III contains 1 blue, 12 green and 3 yellow balls,
Box-IV contains 10 green, 16 orange and 6 white balls.

A ball is chosen randomly from Box-I ; call this ball b. If b is red then a ball is chosen randomly from Box-II, if b is blue then a ball is chosen randomly from Box-III, and if b is green then a ball is chosen randomly from Box-IV. The conditional probability of the event 'one of the chosen balls is white' given that the event 'at least one of the chosen balls is green' has happened, is equal to

[JEE(Advanced) 2022]

(A) $\frac{15}{256}$ (B) $\frac{3}{16}$ (C) $\frac{5}{52}$ (D) $\frac{1}{8}$

9. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements.

Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements.

Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is **[JEE(Advanced) 2021]**

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Question Stem for Question Nos. 10 and 11

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1, 2, 3, ..., 100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

10. The value of $\frac{625}{4}p_1$ is ________. **[JEE(Advanced) 2021]**

11. The value of $\frac{125}{4}p_2$ is ________. **[JEE(Advanced) 2021]**

12. Let E, F and G be three events having probabilities

$P(E)=\frac{1}{8}, P(F)=\frac{1}{6}$ and $P(G)=\frac{1}{4}$, and let $P(E\cap F\cap G)=\frac{1}{10}$.

For any event H, if H^C denotes its complement, then which of the following statements is(are) **TRUE ?**

[JEE(Advanced) 2021]

(A) $P(E\cap F\cap G^C)\le\frac{1}{40}$ (B) $P(E^C\cap F\cap G)\le\frac{1}{15}$

(C) $P(E\cup F\cup G)\le\frac{13}{24}$ (D) $P(E^C\cap F^C\cap G^C)\le\frac{5}{12}$

13. A number is chosen at random from the set $\{1, 2, 3, ..., 2000\}$. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of 500p is ________.

[JEE(Advanced) 2021]

14. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently, Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is **[JEE(Advanced) 2020]**

(A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

15. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is NOT less than 0.95, is ________.

[JEE(Advanced) 2020]

16. Two fair dice, each with faces numbered 1,2,3,4,5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of 14p is ________.

[JEE(Advanced) 2020]

17. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls, Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(B) Probability that the chosen ball is green equals $\frac{39}{80}$

(C) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(D) Probability that the selected bag is B_3, given that the chosen balls is green, equals $\frac{5}{13}$

18. Let S be the sample space of all 3×3 matrices with entries from the set $\{0, 1\}$. Let the events E_1 and E_2 be given by

$E_1 = \{A \in S : \det A = 0\}$ and

$E_2 = \{A \in S : \text{sum of entries of } A \text{ is } 7\}$.

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ____

[JEE(Advanced) 2019]

19. Let $|X|$ denote the number of elements in set X. Let $S = \{1,2,3,4,5,6\}$ be a sample space, where each element is equally likely to occur. If A and B are indepenent events associated with S, then the number of ordered pairs (A, B) such that $1 \leq |B| < |A|$, equals **[JEE(Advanced) 2019]**

PARAGRAPH "A"

There are five students S_1, S_2, S_3, S_4 and S_5 in a music class and for them there are five seats R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to the student S_i, $i = 1, 2, 3, 4, 5$. But, on the examination day, the five students are randomly allotted the five seats.

(There are two questions based on Paragraph "A". the question given below is one of them)

20. The probability that, on the examination day, the student S_1 gets the previously allotted seat R_1 and **NONE** of the remaining students gets the seat previously allotted to him/her is - **[JEE(Advanced) 2018]**

(A) $\frac{3}{40}$ (B) $\frac{1}{8}$ (C) $\frac{7}{40}$ (D) $\frac{1}{5}$

PARAGRAPH "A"

There are five students S_1, S_2, S_3, S_4 and S_5 in a music class and for them there are five seats R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to the student S_i, $i = 1, 2, 3, 4, 5$. But, on the examination day, the five students are randomly allotted the five seats.

(There are two questions based on Paragraph "A", the question given below is one of them)

21. For $i = 1, 2, 3, 4$, let T_i denote the event that the students S_i and S_{i+1} do **NOT** sit adjacent to each other on the day of the examination. Then the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is-

[JEE(Advanced) 2018]

(A) $\frac{1}{15}$ (B) $\frac{1}{10}$ (C) $\frac{7}{60}$ (D) $\frac{1}{5}$

22. Let X and Y be two events such that $P(X) = \frac{1}{3}$, $P(X|Y) = \frac{1}{2}$ and $P(Y|X) = \frac{2}{5}$. Then

[JEE(Advanced) 2017]

(A) $P(X'|Y) = \frac{1}{2}$ (B) $P(X \cap Y) = \frac{1}{5}$

(C) $P(X \cup Y) = \frac{2}{5}$ (D) $P(Y) = \frac{4}{15}$

23. Three randomly chosen non negative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is [JEE(Advanced) 2017]

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$ (C) $\frac{5}{11}$ (D) $\frac{1}{2}$

24. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that [JEE(Advanced) 2016]

P(computer turns out to be defective given that is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probabality that it is produced in plant T_2 is

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Paragraph For Questions No. 25 and 26

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2, respectively, after two games

25. $P(X > Y)$ is- [JEE(Advanced) 2016]

(A) $\frac{1}{4}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{7}{12}$

26. $P(X = Y)$ is- [JEE(Advanced) 2016]

(A) $\frac{11}{36}$ (B) $\frac{1}{3}$ (C) $\frac{13}{36}$ (D) $\frac{1}{2}$

27. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is [JEE(Advanced) 2015]

Paragraph For Questions Nos. 28 and 29

Let n_1 and n_2 be the number of red and black balls respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

28. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

[JEE(Advanced) 2015]

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$
(B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$
(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$
(D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

29. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are)

[JEE(Advanced) 2015]

(A) $n_1 = 4$ and $n_2 = 6$
(B) $n_1 = 2$ and $n_2 = 3$
(C) $n_1 = 10$ and $n_2 = 20$
(D) $n_1 = 3$ and $n_2 = 6$

30. Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is - **[JEE(Advanced) 2014]**

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Paragraph For Questions No. 31 and 32

Box 1 contains three cards bearing numbers, 1,2,3 ; box 2 contains five cards bearing numbers 1,2,3,4,5; and box 3 contains seven cards bearing numbers 1,2,3,4,5,6,7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the i^{th} box, $i = 1,2,3$.

31. The probability that $x_1 + x_2 + x_3$ is odd, is **[JEE(Advanced) 2014]**

(A) $\frac{29}{105}$ (B) $\frac{53}{105}$ (C) $\frac{57}{105}$ (D) $\frac{1}{2}$

32. The probability that x_1, x_2, x_3 are in an arithmetic progression, is- **[JEE(Advanced) 2014]**

(A) $\frac{9}{105}$ (B) $\frac{10}{105}$ (C) $\frac{11}{105}$ (D) $\frac{7}{105}$

33. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$.

Then the probability that the problem is solved correctly by at least one of them is

[JEE(Advanced) 2013]

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$ (C) $\frac{3}{256}$ (D) $\frac{253}{256}$

34. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\, p = \alpha\beta$ and $(\beta - 3\gamma)\, p = 2\beta\gamma$. All the given probabilities are assumed of lie in the interval (0,1).

Then $\dfrac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$ [JEE(Advanced) 2013]

Paragraph for Question No. 35 and 36

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

35. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is **[JEE(Advanced) 2013]**

(A) $\frac{116}{181}$ (B) $\frac{126}{181}$ (C) $\frac{65}{181}$ (D) $\frac{55}{181}$

36. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, the probability that all 3 drawn balls are of the same colour is **[JEE(Advanced) 2013]**

(A) $\frac{82}{648}$ (B) $\frac{90}{648}$ (C) $\frac{558}{648}$ (D) $\frac{566}{648}$

37. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and X_1, X_2, X_3 denotes respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true?

[IIT-JEE 2012]

(A) $P\left[X_1^c \mid X\right] = \frac{3}{16}$

(B) P[Exactly two engines of ship are functioning $\mid X] = \frac{7}{8}$

(C) $P[X \mid X_2] = \frac{5}{16}$

(D) $P[X \mid X_1] = \frac{7}{16}$

38. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is -

[IIT-JEE 2012]

(A) $\frac{91}{216}$ (B) $\frac{108}{216}$ (C) $\frac{125}{216}$ (D) $\frac{127}{216}$

39. Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}, P(Y \mid X) = \frac{1}{3}$ and, $P(X \cap Y) = \frac{1}{6}$. Which of the following is(are) correct ? **[IIT-JEE 2012]**

(A) $P(X \cup Y) = \frac{2}{3}$ (B) X and Y are independent

(C) X and Y are not independent (D) $P(X^C \cap Y) = \frac{1}{3}$

Paragraph for Question No. 40 and 41

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls. and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

40. The probability of the drawn ball from U_2 being white is - **[IIT-JEE 2011]**

(A) $\frac{13}{30}$ (B) $\frac{23}{30}$ (C) $\frac{19}{30}$ (D) $\frac{11}{30}$

41. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is - **[IIT-JEE 2011]**

(A) $\frac{17}{23}$ (B) $\frac{11}{23}$ (C) $\frac{15}{23}$ (D) $\frac{12}{23}$

42. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If P(T) denotes the probability of occurrence of the event T, then - **[IIT-JEE 2011]**

(A) $P(E)=\frac{4}{5}$, $P(F)=\frac{3}{5}$ (B) $P(E)=\frac{1}{5}, P(F)=\frac{2}{5}$

(C) $P(E)=\frac{2}{5}, P(F)=\frac{1}{5}$ (D) $P(E)=\frac{3}{5}, P(F)=\frac{4}{5}$

43. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1}+\omega^{r_2}+\omega^{r_3}=0$ is **[IIT-JEE 2010]**

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$ (C) $\frac{2}{9}$ (D) $\frac{1}{36}$

44. A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is - **[IIT-JEE 2010]**

(A) $\frac{3}{5}$ (B) $\frac{6}{7}$ (C) $\frac{20}{30}$ (D) $\frac{9}{20}$

Paragraph for Questions Nos. 45 to 47

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

45. The probability that $X = 3$ equals :- **[IIT-JEE 2009]**

(A) $\frac{25}{216}$ (B) $\frac{25}{36}$

(C) $\frac{5}{36}$ (D) $\frac{125}{216}$

46. The probability that $X \geq 3$ equals :- **[IIT-JEE 2009]**

(A) $\frac{125}{216}$ (B) $\frac{25}{36}$

(C) $\frac{5}{36}$ (D) $\frac{25}{216}$

47. The conditional probability that $X \geq 6$ given $X > 3$ equals :- **[IIT-JEE 2009]**

(A) $\frac{125}{216}$ (B) $\frac{25}{216}$

(C) $\frac{5}{36}$ (D) $\frac{25}{36}$

48. Consider the system of equations

$ax + by = 0$, $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$

STATEMENT–1 : The probability that the system of equations has a unique solution is $\frac{3}{8}$.

and

STATEMENT–2 : The probability that the system of equations has a solution is 1.

[IIT-JEE 2008]

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

49. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is :- **[IIT-JEE 2008]**

(A) 2, 4, or 8 (B) 3, 6 or 9

(C) 4 or 8 (D) 5 or 10

STATISTICS

1. Consider the given data with frequency distribution **[JEE(Advanced) 2023]**

x_i	3	8	11	10	5	4
f_i	5	2	3	2	4	4

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) The mean of the above data is	(1) 2.5
(Q) The median of the above data is	(2) 5
(R) The mean deviation about the mean of the above data is	(3) 6
(S) The mean deviation about the median of the above data is	(4) 2.7
	(5) 2.4

The correct option is :

(A) (P) $\rightarrow$ (3) (Q) $\rightarrow$ (2) (R) $\rightarrow$ (4) (S) $\rightarrow$ (5)

(B) (P) $\rightarrow$ (3) (Q) $\rightarrow$ (2) (R) $\rightarrow$ (1) (S) $\rightarrow$ (5)

(C) (P) $\rightarrow$ (2) (Q) $\rightarrow$ (3) (R) $\rightarrow$ (4) (S) $\rightarrow$ (1)

(D) (P) $\rightarrow$ (3) (Q) $\rightarrow$ (3) (R) $\rightarrow$ (5) (S) $\rightarrow$ (5)

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IMPORTANT NOTE

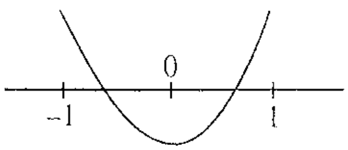
QUADRATIC EQUATION

1. Ans. (4)

Sol. $3x^2 + x - 1 = 4|x^2 - 1|$

If $x \in [-1, 1]$,

$3x^2 + x - 1 = -4x^2 + 4 \Rightarrow 7x^2 + x - 5 = 0$

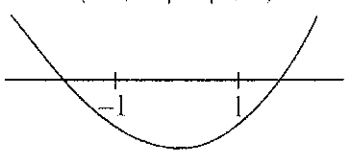


say $f(x) = 7x^2 + x - 5$

$f(1) = 3$; $f(-1) = 1$; $f(0) = -1$

[Two Roots]

If $x \in (-\infty, -1] \cup [1, \infty)$



$3x^2 + x - 1 = 4x^2 - 4 \Rightarrow x^2 - x - 3 = 0$

Say $g(x) = x^2 - x - 3$

$g(-1) = -1$; $g(1) = -3$

[Two Roots]

So total 4 roots.

2. Ans. (D)

Sol. $x^2 + 20x - 2020 = 0$ has two roots $a, b \in R$

$x^2 - 20x + 2020 = 0$ has two roots $c, d \in$ complex

$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$

$= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$

$= a^2(c + d) + b^2(c + d) - c^2(a + b) - d^2(a + b)$

$= (c + d)(a^2 + b^2) - (a + b)(c^2 + d^2)$

$= (c + d)((a + b)^2 - 2ab) - (a + b)((c + d)^2 - 2cd)$

$= 20[(20)^2 + 4040] + 20[(20)^2 - 4040]$

$= 20[(20)^2 + 4040 + (20)^2 - 4040]$

$= 20 \times 800 = 16000$

3. Ans. (C)

Sol. $\alpha^2 = \alpha + 1 \Rightarrow \alpha^4 = 3\alpha + 2$

$\therefore a_4 = 28 \Rightarrow p\alpha^4 + q\beta^4 = p(3\alpha + 2) + q(3\beta + 2) = 28$

$\Rightarrow \quad p(3\alpha + 2) + q(3 - 3\alpha + 2) = 28$

$\Rightarrow \quad \alpha(3p - 3q) + 2p + 5q = 28$

(as $\alpha \in Q^c$)

$\Rightarrow \quad p = q, 2p + 5q = 28 \Rightarrow p = q = 4$

$\therefore \quad p + 2q = 12$

4. Ans. (C)

Sol. $\alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\Rightarrow p\alpha^n + q\beta^n = p(\alpha^{n-1} + \alpha^{n-2}) + q(\beta^{n-1} + \beta^{n-2})$

$a_n = a_{n-1} + a_{n-2}$

$\Rightarrow a_{12} = a_{11} + a_{10}$

5. Ans. (C)

Sol. $\alpha_1 = \dfrac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2}$

$\beta_2 = \dfrac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$ $\quad \{\because \alpha_2 > \beta_2\}$

$\alpha_1 = \sec\theta + |\tan\theta|$ $\quad \{\because \alpha_1 > \beta_1\}$

$\beta_2 = -\tan\theta - \sec\theta$

$\alpha_1 = \sec\theta - \tan\theta \quad \left(\because \theta \in \left(-\dfrac{\pi}{6}, -\dfrac{\pi}{12}\right)\right)$

$\alpha_1 + \beta_2 = -2\tan\theta$

6. Ans. (A, D)

Sol. $\alpha x^2 - x + \alpha = 0$

$D = 1 - 4\alpha^2$

distinct real roots $D > 0$

$\Rightarrow \alpha \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ ...(i)

given $|x_1 - x_2| < 1$

$\Rightarrow \dfrac{\sqrt{1 - 4\alpha^2}}{|\alpha|} < 1$

$\Rightarrow 1 - 4\alpha^2 < \alpha^2$

$\Rightarrow \alpha \in \left(-\infty, \dfrac{-1}{\sqrt{5}}\right) \cup \left(\dfrac{1}{\sqrt{5}}, \infty\right)$ ...(ii)

from (i) & (ii)

$\alpha \in \left(\dfrac{-1}{2}, \dfrac{-1}{\sqrt{5}}\right) \cup \left(\dfrac{1}{\sqrt{5}}, \dfrac{1}{2}\right)$

7. Ans. (C)

Sol. α, β are roots of $x^2 - 6x - 2 = 0$

$\Rightarrow \quad \alpha^2 - 6\alpha - 2 = 0$ & $\beta^2 - 6\beta - 2 = 0$

$\dfrac{a_0 - 2a_8}{2a_9} = \dfrac{\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)}$

$= \dfrac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)}$

$= \dfrac{\alpha^8.6\alpha - \beta^8.6\beta}{2(\alpha^9 - \beta^9)} = 3$

8. Ans.(B)

Sol. $\dfrac{x^2}{b^2+1} = \dfrac{-x}{b+1} = \dfrac{1}{1-b}$

$\Rightarrow \quad x = \dfrac{b+1}{b-1} \qquad ...(i)$

$\& \quad x^2 = \dfrac{b^2+1}{1-b} \qquad ...(ii)$

from (i) & (ii)

$$\left(\frac{b+1}{b-1}\right)^2 = \frac{b^2+1}{1-b}$$

$\Rightarrow \quad (b^2+1)(1-b) = (b+1)^2$

$\Rightarrow \quad -b^3 + 1 + b^2 - b = b^2 + 1 + 2b$

$\Rightarrow \quad -b^3 - 3b = 0 \Rightarrow \quad b(b^2+3) = 0$

$\Rightarrow \quad b = 0,\ b = \pm\sqrt{3}i$

9. Ans. (B)

Sol. $\alpha^3 + \beta^3 = q$

$\Rightarrow (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = q$

$\Rightarrow \alpha\beta = \dfrac{p^3+q}{3p}$

sum of the roots $= \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta}$

$$\frac{p^2 - 2\left(\dfrac{p^3+q}{3p}\right)}{\dfrac{p^3+q}{3p}} = \frac{p^3-2q}{p^3+q}$$

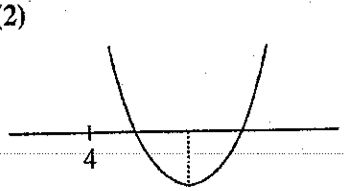
Product of the roots = 1.

Required equation is

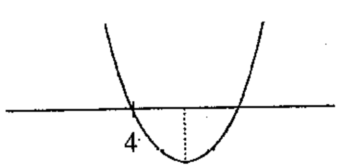
$(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

10. Ans. (2)

Sol.



or



$f(x) = x^2 - 8kx + 16(k^2 - k + 1) = 0$

$\because$ roots are real, $D > 0$

$64k^2 - 64(k^2 - k + 1) > 0$

$k > 1 \Rightarrow k \in (1, \infty)$

at both the roots ≥ 4

$$\Rightarrow \begin{cases} f(4) \geq 0 \\ \dfrac{-b}{2a} > 4 \end{cases} \Rightarrow \begin{cases} k^2 - 3k + 2 \geq 0 \\ k > 1 \end{cases}$$

$\Rightarrow k \in [2, \infty) \Rightarrow$ least value of $k = 2$

11. Ans. (B)

Sol. $x^2 + 2px + q = 0$

then $\alpha + \beta = -2p$ & $\alpha\beta = q$

and $ax^2 + 2bx + c = 0$

$\alpha + \dfrac{1}{\beta} = -\dfrac{2b}{a}$ & $\dfrac{\alpha}{\beta} = \dfrac{c}{a}$

$\dfrac{\alpha\beta+1}{\beta} = \dfrac{-2b}{a} \Rightarrow \dfrac{q+1}{\beta} = -\dfrac{2b}{a}$

$\Rightarrow \beta$ is real $\Rightarrow \alpha$ is real

so $(p^2 - q)(b^2 - ac) \geq 0$

hence S(I) is true

Let $\dfrac{b}{a} = p$ and $\dfrac{c}{a} = q$

$x^2 + \dfrac{2b}{a}x + \dfrac{c}{a} = 0 \Rightarrow x^2 + 2px + q = 0 \Rightarrow \beta = \dfrac{1}{\beta}$

$\Rightarrow \beta = \pm 1$ (not possible)

Hence S(II) is True but S(II) is not the correct explaination of S(I)

LOGARITHM

1. Ans. (1)

Sol. $x^{16(\log_5 x)^3 - 68\log_5 x} = 5^{-16}$

Take log to the base 5 on both sides and put $\log_5 x = t$

$16t^4 - 68t^2 + 16 = 0$

$$\Rightarrow 4t^4 - 17t^2 + 4 = 0 \begin{cases} t_1 \\ t_2 \\ t_3 \\ t_4 \end{cases}$$

$t_1 + t_2 + t_3 + t_4 = 0$

$\log_5 x_1 + \log_5 x_2 + \log_5 x_3 + \log_5 x_4 = 0$

$x_1 x_2 x_3 x_4 = 1$

2. **Ans. (8)**

Sol. $\log_2 9^{\frac{2}{\log_2(\log_2 9)}} \times 7^{\frac{1/2}{\log_4 7}}$

$= (\log_2 9)^{2\log_{\log_2 9}2} \times 7^{\frac{1}{2}\log_7 4}$

$= 4 \times 2 = 8$

3. **Ans. (A, B, C)**

Sol. $3^x = 4^{x-1}$

taking log on both sides with base a

$x\log_a 3 = (x-1)\log_a 4 \Rightarrow x = \frac{\log_a 4}{\log_a 4 - \log_a 3}$

If $a = 3: \ x = \frac{2\log_3 2}{2\log_3 2 - 1}$

If $a = 2: \ x = \frac{2}{2 - \log_2 3}$

If $a = 4: \ x = \frac{1}{1 - \log_4 3}$

4. **Ans. (4)**

Sol. Let $y = \sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots\ldots}}}$

$\Rightarrow \quad y^2 = 4 - \frac{1}{3\sqrt{2}}\,y \Rightarrow 3\sqrt{2}\,y^2 = 12\sqrt{2} - y$

$\Rightarrow \quad 3\sqrt{2}y^2 + y - 12\sqrt{2} = 0$

$\Rightarrow \quad (3y - 4\sqrt{2})(\sqrt{2}y + 3) = 0$

$\Rightarrow \quad y = \frac{4\sqrt{2}}{3}; \ y = -\frac{3}{\sqrt{2}}$ (reject)

$\therefore \quad V = 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}y\right)$

$= 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}} \cdot \frac{4\sqrt{2}}{3}\right) = 6 + \log_{3/2}\left(\frac{2}{3}\right)^2$

$= 6 - 2 = 4$

5. **Ans. (C)**

Sol. $(2x)^{\ell n 2} = (3y)^{\ell n 3}$

$\Rightarrow \quad \ell n2\,(\ell n2 + \ell nx) = \ell n3(\ell n3 + \ell ny) \ \ldots(i)$

$3^{\ell nx} = 2^{\ell ny}$

$\Rightarrow \quad (\ell nx)(\ell n3) = (\ell ny)(\ell n2) \quad \ldots(ii)$

using (ii) in (i)

$\Rightarrow \quad \ell n2(\ell n2 + \ell nx) = \ell n3\left(\ell n3 + \frac{(\ell nx)(\ell n3)}{\ell n2}\right)$

$\Rightarrow \quad \ell n^2 2 - \ell n^2 3 = \ell nx\left\{\frac{\ell n^2 3}{\ell n2} - \ell n2\right\}$

$\Rightarrow \quad \ell nx = -\ell n2$

$\Rightarrow \quad x = \frac{1}{2}$

SEQUENCE & SERIES

1. **Ans. (1219)**

Sol. $S = 77 + 757 + 7557 + \ldots + 7\underbrace{5\ldots\ldots5}_{98}7$

$10S = 770 + 7570 + \ldots + 75\ldots570 + 755\ldots\ldots570$

$9S = -77 + \underbrace{13 + 13 + \ldots\ldots + 13}_{98 \text{ times}} + 7\underbrace{5\ldots\ldots5}_{98}70$

$= -77 + 13 \times 98 + 7\underbrace{5\ldots\ldots5}_{99}7 + 13$

$S = \frac{7\underbrace{5\ldots\ldots5}_{99}7 + 1210}{9}$

$m = 1210$

$n = 9$

$m + n = 1219$

2. **Ans. (18900.00)**

Sol. Given

$A_{51} - A_{50} = 1000 \Rightarrow \ell_{51}w_{51} - \ell_{50}w_{50} = 1000$

$\Rightarrow (\ell_1 + 50d_1)(w_1 + 50d_2) - (\ell_1 + 49d_1)(w_1 + 49d_2) = 1000$

$\Rightarrow (\ell_1 d_2 + w_1 d_1) = 10 \quad \ldots(1)$

(As $d_1d_2 = 10$)

$\therefore \ A_{100} - A_{90} = \ell_{100}w_{100} - \ell_{90}w_{90}$

$= (\ell_1 + 99d_1)(w_1 + 99d_2) - (\ell_1 + 89d_1)(w_1 + 89d_2)$

$= 10(\ell_1 d_2 + w_1 d_1) + (99^2 - 89^2)d_1d_2$

$= 10(10) + \underbrace{(99 - 89)}_{=10}(99 + 89)(10)$

(As, $d_1d_2 = 10$)

$= 100(1 + 188) = 100(189) = 18900$

3. Ans. (B, C)

Sol. $a_1 = 7, d = 8$

$T_{n+1} - T_n = a_n \forall n \geq 1$

$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$

$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$

on subtraction

$T_n = T_1 + a_1 + a_2 + \ldots + a_{n-1}$

$T_n = 3 + (n-1)(4n-1)$

$T_n = 4n^2 - 5n + 4$

$$\sum_{k=1}^{n} T_k = 4\sum n^2 - 5\sum n + 4n$$

$T_{20} = 1504$

$T_{30} = 3454$

$$\sum_{k=1}^{30} T_k = 35615$$

$$\sum_{k=1}^{20} T_k = 10510$$

4. Ans. (8.00)

Sol. $$\frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq \left[3^{(y_1+y_2+y_3)}\right]^{\frac{1}{3}}$$

$\Rightarrow \quad 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3^4$

$\Rightarrow \quad \log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq 41$

$\Rightarrow \quad m = 4$

Also, $\dfrac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 x_2 x_3}$

$\Rightarrow \quad x_1 x_2 x_3 \leq 27$

$\Rightarrow \quad \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3$

$\Rightarrow \quad M = 3$

Thus, $\log_2(m^3) + \log_3(M^2) = 6 + 2 = 8$

5. Ans. (1.00)

Sol. Given $2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$

$$\Rightarrow \quad 2 \times \frac{n}{2}(2c + (n-2)2) = c\left(\frac{2^n - 1}{2-1}\right)$$

$\Rightarrow \quad 2n^2 - 2n = c(2^n - 1 - 2n)$

$$\Rightarrow \quad c = \frac{2n^2 - 2n}{2^n - 1 - 2n} \in \mathbb{N}$$

So, $\quad 2n^2 - 2n \geq 2^n - 1 - 2n$

$\Rightarrow \quad 2n^2 + 1 \geq 2^n \Rightarrow n < 7$

$\Rightarrow \quad$ n can be 1,2,3,....,

Checking c against these values of n

we get c = 12 (when n = 3)

Hence number of such c = 1

6. Ans. (A, B, D)

Sol. α, β are roots of $x^2 - x - 1$

$$a_{r+2} - a_r = \frac{\left(\alpha^{r+2} - \beta^{r+2}\right) - \left(\alpha^r - \beta^r\right)}{\alpha - \beta}$$

$$= \frac{\left(\alpha^{r+2} - \alpha^r\right) - \left(\beta^{r+2} - \beta^r\right)}{\alpha - \beta}$$

$$= \frac{\alpha^r\left(\alpha^2 - 1\right) - \beta^r\left(\beta^2 - 1\right)}{\alpha - \beta}$$

$$= \frac{\alpha^r\alpha - \beta^r\beta}{\alpha - \beta} = \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = a_{r+1}$$

$\Rightarrow a_{r+2} - a_{r+1} = a_r$

$$\Rightarrow \sum_{r=1}^{n} a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta}$$

$$= a_{n+2} - (\alpha + \beta) = a_{n+2} - 1$$

Now $$\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{\sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n - \sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n}{\alpha - \beta}$$

$$= \frac{\dfrac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \dfrac{\frac{\beta}{10}}{1 - \frac{\beta}{10}}}{\alpha - \beta} = \frac{\dfrac{\alpha}{10 - \alpha} - \dfrac{\beta}{10 - \beta}}{(\alpha - \beta)}$$

$$= \frac{10}{(10 - \alpha)(10 - \beta)} = \frac{10}{89}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \frac{a_{n-1} + a_{n+1}}{10^n} = \frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} = \frac{12}{89}$$

Further, $b_n = a_{n-1} + a_{n+1}$

$$= \frac{\left(\alpha^{n-1} - \beta^{n-1}\right) + \left(\alpha^{n+1} - \beta^{n+1}\right)}{\alpha - \beta}$$

(as $\alpha\beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n\beta$ & $\beta^{n-1} = -\alpha\beta^n$)

$$= \frac{\alpha^n(\alpha - \beta) + (\alpha - \beta)\beta^n}{\alpha - \beta} = \alpha^n + \beta^n$$

7. Ans. (157.00)

Sol. We equate the general terms of three respective A.P.'s as $1 + 3a = 2 + 5b = 3 + 7c$

$\Rightarrow \quad$ 3 divides $1 + 2b$ and 5 divides $1 + 2c$

$\Rightarrow \quad 1 + 2c = 5, 15, 25$ etc.

So, first such terms are possible when

$1 + 2c = 15$ i.e. $c = 7$

Hence, first term = a = 52

$d = \text{lcm}(3, 5, 7) = 105 \quad \Rightarrow a + d = 157$

8. Ans. (3748)

Sol. $X : 1, 6, 11, \ldots\ldots, 10086$

$Y : 9, 16, 23, \ldots\ldots, 14128$

$X \cap Y : 16, 51, 86, \ldots\ldots$

Let $m = n(X \cap Y)$

$\therefore \quad 16 + (m-1) \times 35 \leq 10086$

$\Rightarrow \quad m \leq 288.71$

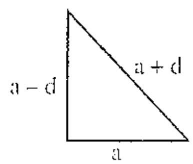
$\Rightarrow \quad m = 288$

$\therefore \quad n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= 2018 + 2018 - 288 = 3748$

9. Ans. (6)

Sol.



where $d > 0, a > 0$

$\Rightarrow$ length of smallest side $= a - d$

Now $(a+d)^2 = a^2 + (a-d)^2$

$\Rightarrow a(a - 4d) = 0$

$\therefore a = 4d \quad \ldots(1)$

(As $a = 0$ is rejected)

Also, $\frac{1}{2}a.(a-d) = 24$

$\Rightarrow a(a-d) = 48 \quad \ldots(2)$

$\therefore$ From (1) and (2), we get $a = 8, d = 2$

Hence, length of smallest side

$\Rightarrow (a-d) = (8-2) = 6$

10. Ans. (B)

Sol. If $\log_e b_1, \log_e b_2 \ldots\ldots \log_e b_{101} \rightarrow$ AP ; $D = \log_e 2$

$\Rightarrow b_1\, b_2\, b_3 \ldots. b_{101} \rightarrow$ GP ; $\quad r = 2$

$\therefore \quad b_1, 2b_1, 2^2b_1, \ldots\ldots\ldots, 2^{100}b_1 \ldots\ldots$ GP

$a_1\, a_2\, a_3 \ldots\ldots. a_{101} \ldots..$ AP

Given, $a_1 = b_1 \quad \& \quad a_{51} = b_{51}$

$\Rightarrow \quad a_1 + 50D = 2^{50}b_1$

$\therefore \quad a_1 + 50D = 2^{50}a_1 \quad$ (As $b_1 = a_1$)

Now, $t = b_1(2^{51} - 1)$; $s = \frac{51}{2}(2a_1 + 50D)$

$\Rightarrow t < a_1.2^{51} \ldots(i) \quad \because s = \frac{51}{2}(a_1 + a_1 + 50D)$

$s = \frac{51}{2}(a_1 + 2^{50}a_1)$

$s = \frac{51a_1}{2} + \frac{51}{2}.2^{50}a_1$

$\Rightarrow \quad s > a_1.2^{51} \quad \ldots.(ii)$

clearly $s > t$ (from equation (i) and (ii))

Also $a_{101} = a_1 + 100D$; $b_{101} = b_1.2^{100}$

$\therefore\ a_{101} = a_1 + 100\left(\frac{2^{50}a_1 - a_1}{50}\right)$; b_{101}

$= 2^{100}a_1 \quad \ldots.(iii)$

$a_{101} = a_1 + 2^{51}a_1 - 2a_1 \Rightarrow a_{101} = 2^{51}a_1 - a_1$

$\Rightarrow a_{101} < 2^{51}a_1 \quad \ldots.(iv)$

clearly $b_{101} > a_{101}$ (from equation (iii) and (iv))

11. Ans. (9)

Sol. $\dfrac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \dfrac{6}{11} \Rightarrow \dfrac{7(2a+6d)}{(2a+10d)} = 6$

$\Rightarrow \quad 2a = 18d$

$a = 9d$

also $130 < a + 6d < 140$

$\frac{26}{3} < d < \frac{28}{3} \Rightarrow \quad d = 9$

12. Ans. (4)

Sol. Let a,b,c are a, ar, ar^2 where $r \in N$

also $\frac{a+b+c}{3} = b + 2$

$\Rightarrow a + ar + ar^2 = 3(ar) + 6$

$\Rightarrow ar^2 - 2ar + a = 6$

$\Rightarrow (r-1)^2 = \frac{6}{a}$

$\because \frac{6}{a}$ must be perfect square & $a \in N$

$\therefore$ a can be 6 only.

$\Rightarrow r - 1 = \pm 1 \Rightarrow r = 2$

& $\frac{a^2 + a - 14}{a+1} = \frac{36 + 6 - 14}{7} = 4$

13. Ans. (A, D)

Sol. $S_n = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 \ldots\ldots$

$S_n = (3^2 - 1^2) + (4^2 - 2^2) + \ldots\ldots$

$S_n = 2(1 + 2 + 3 + \ldots\ldots + 4n)$

$= \frac{2(4n)(4n+1)}{2}$

$S_n = 4n(4n + 1) = 1056$ is possible when $n = 8$

$4n(4n + 1) = 1088$ not possible

$4n(4n + 1) = 1120$ not possible

$4n(4n + 1) = 1332$ possible when $n = 9$

14. Ans. (5)

Sol. When 1 and 2 are removed from numbers 1 to n thenwe get maximum possible sum of remaining numbers and when $n-1, n$ are removed then we get minimum possible sum of remaining numbers.

$$\Rightarrow \frac{n(n+1)}{2} - (2n-1) \le 1224 \le \frac{n(n+1)}{2} - 3$$

$$\Rightarrow \begin{cases} n^2 + n - 2454 \ge 0 \\ n^2 - 3n - 2446 \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} n \ge 50 \\ n \le 50 \end{cases} \Rightarrow n = 50$$

Now let x and x + 1 be two consecutive numbers

$$\Rightarrow \frac{50(50+1)}{2} - x - x - 1 - 1224$$

$\Rightarrow x = 25$

$\Rightarrow 25^{th}$ and 26^{th} cards are removed from pack

$\Rightarrow k = 25 \quad \Rightarrow k - 20 = 5$

15. Ans. (D)

Sol. $a_1, a_2, a_3 \ldots$ be in H.P $\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots$ be in A.P.

in A.P. $T_1 = \frac{1}{a_1} = \frac{1}{5}$ and $T_{20} = \frac{1}{a_{20}} = \frac{1}{25}$

$\therefore \quad T_{20} = T_1 + 19d$

$\frac{1}{25} = \frac{1}{5} + 19d \qquad \Rightarrow d = -\frac{4}{19 \times 25}$

$T_n = T_1 + (n-1)d < 0$

$$\Rightarrow \frac{1}{5} - \frac{(n-1).4}{19 \times 25} < 0 \Rightarrow \frac{1}{5} < \frac{4(n-1)}{25 \times 19}$$

$$\Rightarrow \frac{5 \times 19}{4} + 1 < n$$

$$\Rightarrow \frac{99}{4} < n$$

$\Rightarrow$ least positive integer n is 25.

16. Ans. (9 or 3)

Sol. (Comment : The information about the common difference i.e. <u>zero or non-zero</u> is not given in the question. Hence there are two possible answers)

Consider $d \neq 0$ the solution is

$a_1, a_2, a_3, \ldots\ldots, a_{100} \rightarrow$ AP

$a_1 = 3 \; ; \; S_p = \sum_{i=1}^{p} a_i \quad 1 \le n \le 20$

$m = 5n$

$$\frac{S_m}{S_n} = \frac{\frac{m}{2}[2a_1 + (m-1)d]}{\frac{n}{2}[2a_1 + (n-1)d]}$$

$$\frac{S_m}{S_n} = \frac{5[(2a_1 - d) + 5nd]}{[(2a_1 - d) + nd]}$$

for $\frac{S_m}{S_n}$ to be independent of n

$\therefore 2a_1 - d = 0 \Rightarrow d = 2a_1 \Rightarrow d = 6 \Rightarrow a_2 = 9$

If $d = 0 \qquad \Rightarrow a_2 = a_1 = 3$

17. Ans. (8)

Sol. As $a > 0$

and all the given terms are positive

hence considering A.M. $\ge$ G.M. for given numbers:

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10}}{7} \ge$$

$$\left(a^{-5}.a^{-4}.a^{-3}.a^{-3}.a^{-3}.a^8.a^{10}\right)^{\frac{1}{7}}$$

$$\Rightarrow \frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10}}{7} \ge 1$$

$$\Rightarrow \left(a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10}\right)_{min} = 7$$

where $a^{-5} = a^{-4} = a^{-3} = a^8 = a^{10}$ i.e. $a = 1$

$\Rightarrow (a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1)_{min} = 8$

when $a = 1$

18. Ans. (0)

Sol. $a_1 = 15$

$27 - 2a_2 > 0$

$a_k = 2a_{k-1} - a_{k-2} \ \forall \ k = 3, 4, 5 \ldots 11$

$$\frac{a_1^2 + a_2^2 + \ldots\ldots a_{11}^2}{11} = 90$$

$$a_{k-1} = \frac{a_k + a_{k-2}}{2}$$

all $a_i (i = 1, 2, \ldots\ldots 11)$ are in A.P.

Let the numbers are

$(a_6+5d), (a_6+4d), \ldots\ldots a_6, \ldots\ldots (a_6-4d), (a_6-5d)$

$11a_6^2 + 110d^2 = 990$

$a_6 = 15 - 5d$

$a_6^2 + 10d^2 = 90$

$(15 - 5d)^2 + 10d^2 = 90 \quad \Rightarrow 7d^2 - 30d + 27 = 0$

$\Rightarrow d = 3, \dfrac{9}{7}$

for $d = 3 \Rightarrow a_2 = 12$ (possible)

for $d = 9/7 \Rightarrow a_2 = 13.7$

(not possible since $a_2 < 13.5$)

$$a_6 = 0 \Rightarrow \frac{a_1 + a_2 + \ldots + a_{11}}{11} = a_6 = 0$$

19. Ans.(C)

Sol. $S_n = cn^2$

$S_{n-1} = c(n-1)^2$

$T_n = S_n - S_{n-1} = c(2n - 1)$

$T'_n = T_n^2 = c^2(4n^2 - 4n + 1)$

$$\sum T'_n = nc^2\left(\frac{4n^2-1}{3}\right)$$

20. Ans. (C)

Sol. Let $a_1 = a, a_2 = ar, a_3 = ar^2, a_4 = ar^3$

Now $b_1 = a, b_2 = a + ar, b_3 = a + ar + ar^2$

$b_4 = a + ar + ar^2 + ar^3$

so b, b_2, b_3, b_4 are neither in A.P. nor in G.P. & nor in H.P.

so S(I) is true & S(II) is False.

COMPOUND ANGLE

1. Ans. (1)

Sol. $\alpha \in \left(0, \frac{\pi}{4}\right), \beta \in \left(-\frac{\pi}{4}, 0\right) \Rightarrow \alpha + \beta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\sin(\alpha+\beta) = \frac{1}{3}, \cos(\alpha-\beta) = \frac{2}{3}$

$$\left(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\alpha}{\sin\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\sin\beta}{\cos\alpha}\right)^2$$

$$\left(\frac{\cos(\alpha-\beta)}{\cos\beta\sin\beta} + \frac{\cos(\beta-\alpha)}{\sin\alpha\cos\alpha}\right)^2$$

$$= 4\cos^2(\alpha-\beta)\left(\frac{1}{\sin 2\beta} + \frac{1}{\sin 2\alpha}\right)^2$$

$$= 4\cos^2(\alpha-\beta)\left(\frac{2\sin(\alpha+\beta)\cos(\alpha-\beta)}{\sin 2\alpha \sin 2\beta}\right)^2 \ldots(1)$$

$$= \frac{16\cos^4(\alpha-\beta)\sin^2(\alpha+\beta)\times 4}{(\cos 2(\alpha-\beta) - \cos 2(\alpha+\beta))^2}$$

$$= \frac{64\cos^4(\alpha-\beta)\sin^2(\alpha+\beta)}{(2\cos^2(\alpha-\beta) - 1 - 1 + 2\sin^2(\alpha+\beta))^2}$$

$$= 64 \times \frac{16}{81} \times \frac{1}{9} \frac{1}{\left(2\times\frac{4}{9} - 1 - 1 + \frac{2}{9}\right)^2}$$

$$= \frac{64\times 16}{81\times 9} \cdot \frac{81}{64} = \frac{16}{9}$$

$$\left[\frac{16}{9}\right] = 1$$

2. Ans. (Bonus)

3. Ans. (C)

Sol. We have,

$$= 2.\sum_{k=1}^{13} \frac{\sin\left(\left(\frac{k\pi}{6} + \frac{\pi}{4}\right) - \left((k-1)\frac{\pi}{6} + \frac{\pi}{4}\right)\right)}{\sin\left(\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right) . \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

$$= 2\sum_{k=1}^{13}\left(\cot\left((k-1)\frac{\pi}{6} + \frac{\pi}{4}\right) - \cot\left(\frac{k\pi}{6} + \frac{\pi}{4}\right)\right)$$

$$= 2\left[\cot\frac{\pi}{4} - \cot\left(\frac{13\pi}{6} + \frac{\pi}{4}\right)\right]$$

$$= 2\left(1 - \cot\left(\frac{5\pi}{12}\right)\right)$$

$$= 2(1 - (2 - \sqrt{3})) = 2(\sqrt{3} - 1)$$

4. Ans. (D)

Sol. $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$

$\Rightarrow \quad \tan\theta = \sqrt{2}+1 \quad ...(i)$

$Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{2}-1} = \sqrt{2}+1 \quad ...(ii)$

from (i) & (ii)

$\Rightarrow \quad P = Q$

5. Ans. (2)

Sol. $y = \dfrac{1}{\dfrac{(1-\cos 2\theta)}{2} + \dfrac{3\sin 2\theta}{2} + 5\dfrac{(1+\cos 2\theta)}{2}}$

$= \dfrac{1}{3 + \left(2\cos 2\theta + \dfrac{3\sin 2\theta}{2}\right)}$

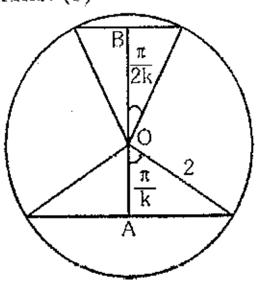
$y = \dfrac{2}{6 + (4\cos 2\theta + 3\sin 2\theta)}$

$y_{max} = \dfrac{2}{6-5} = 2$

(Since $(-5 \le 4\cos 2\theta + 3\sin 2\theta \le 5)$)

6. Ans. (3)

Sol.



$OA = 2\cos\dfrac{\pi}{k}$

$OB = 2\cos\dfrac{\pi}{2k}$

$2\cos\dfrac{\pi}{k} + 2\cos\dfrac{\pi}{2k} = \sqrt{3}+1$

$2\cos^2\dfrac{\pi}{2k} - 1 + \cos\dfrac{\pi}{2k} = \dfrac{\sqrt{3}+1}{2}$

Let $\cos\dfrac{\pi}{2k} = t$

$2t^2 + t - 1 - \dfrac{\sqrt{3}+1}{2} = 0 \Rightarrow 4t^2 + 2t - (3+\sqrt{3}) = 0$

$\Rightarrow t = \dfrac{\sqrt{3}}{2}, -\dfrac{1+\sqrt{3}}{2}$

$t = -\dfrac{1+\sqrt{3}}{2}$ (not possible)

$t = \dfrac{\sqrt{3}}{2} = \cos 30° = \cos\dfrac{\pi}{6} \Rightarrow \cos\dfrac{\pi}{2k} = \cos\dfrac{\pi}{6}$

$k = 3$

7. Ans. (A, B)

Sol. $\dfrac{\tan^4 x}{2} + \dfrac{1}{3} = \dfrac{\sec^4 x}{5}$

put $\tan^2 x = t \Rightarrow t = 2/3 \quad \Rightarrow \sin^2 x = \dfrac{2}{5}$

$\Rightarrow \cos^2 x = \dfrac{3}{5}$

TRIGONOMETRIC EQUATION

1. Ans. (B)

Sol.

(I) $\left\{x \in \left[\dfrac{-2\pi}{3}, \dfrac{2\pi}{3}\right] : \cos x + \sin x = 1\right\}$

$\cos x + \sin x = 1$

$\Rightarrow \dfrac{1}{\sqrt{2}}\cos x + \dfrac{1}{\sqrt{2}}\sin x = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4}$

$\Rightarrow x - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{\pi}{4}$; $n \in Z$

$\Rightarrow x = 2n\pi$; $x = 2n\pi + \dfrac{\pi}{2}$; $n \in Z$

$\Rightarrow x \in \left\{0, \dfrac{\pi}{2}\right\}$ in given range has two solutions

(II) $\left\{x \in \left[\dfrac{-5\pi}{18}, \dfrac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$

$\sqrt{3}\tan 3x = 1 \Rightarrow \tan 3x = \dfrac{1}{\sqrt{3}}$

$\Rightarrow 3x = n\pi + \dfrac{\pi}{6}$

$\Rightarrow x = (6n+1)\dfrac{\pi}{18}$; $n \in Z$

$\Rightarrow x \in \left\{\dfrac{\pi}{18}, \dfrac{-5\pi}{18}\right\}$ in given range has two solutions

(III) $\left\{x \in \left[-\dfrac{6\pi}{5}, \dfrac{6\pi}{5}\right] ; 2\cos(2x) = \sqrt{3}\right\}$

$2\cos 2x = \sqrt{3}$

$\Rightarrow \cos 2x = \dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6}$

$\Rightarrow 2x = 2n\pi \pm \dfrac{\pi}{6}$; $n \in Z$

$\Rightarrow x = n\pi \pm \dfrac{\pi}{12}$; $n \in Z$

$x \in \left\{\pm\dfrac{\pi}{12}, \pi \pm \dfrac{\pi}{12}, -\pi \pm \dfrac{\pi}{12}\right\}$

Six solutions in given range

(IV) $\left\{x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$

$\cos x - \sin x = -1$

$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} = \cos\frac{3\pi}{4}$

$\Rightarrow x + \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$; $n \in Z$

$\Rightarrow x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi - \pi$; $n \in Z$

$\Rightarrow x \in \left\{\frac{\pi}{2}, \frac{-3\pi}{2}, \pi, -\pi\right\}$

four solutions in given range

2. **Ans. (1.00)**

Sol. Let $\pi x - \frac{\pi}{4} = \theta \in \left[\frac{-\pi}{4}, \frac{7\pi}{4}\right]$

So, $\left(3 - \sin\left(\frac{\pi}{2} + 2\theta\right)\right)\sin\theta \geq \sin(\pi + 3\theta)$

$\Rightarrow$ $(3 - \cos 2\theta)\sin\theta \geq -\sin 3\theta$

$\Rightarrow$ $\sin\theta[3 - 4\sin^2\theta + 3 - \cos 2\theta] \geq 0$

$\Rightarrow$ $\sin\theta(6 - 2(1 - \cos 2\theta) - \cos 2\theta) \geq 0$

$\Rightarrow$ $\sin\theta(4 + \cos 2\theta) \geq 0$

$\Rightarrow$ $\sin\theta \geq 0$

$\Rightarrow$ $\theta \in [0, \pi] \Rightarrow 0 \leq \pi x - \frac{\pi}{4} \leq \pi$

$\Rightarrow$ $x \in \left[\frac{1}{4}, \frac{5}{4}\right]$

$\Rightarrow$ $\beta - \alpha = 1$

3. **Ans. (C)**

4. **Ans. (B)**

Sol. $f(x) = \sin(\pi \cos x)$

$X : \{x : f(x) = 0\}$

$f(x) = 0 \Rightarrow \sin(\pi \cos x) = 0 \Rightarrow \cos x = n \Rightarrow \cos x = 1, -1, 0 \Rightarrow x = \frac{n\pi}{2}$

$X = \left\{\frac{n\pi}{2} : n \in N\right\} = \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, ...\right\}$

$g(x) = \cos(2\pi \sin x)$

$Z = \{x : g(x) = 0\}$

$\cos(2\pi \sin x) = 0 \Rightarrow 2\pi \sin x = (2n + 1)\frac{\pi}{2}$

$\Rightarrow \sin x = \frac{(2n+1)}{4}$

$\sin x = -\frac{1}{4}, \frac{1}{4}, \frac{-3}{4}, \frac{3}{4}$

$Z = \left\{n\pi \pm \sin^{-1}\left(\frac{1}{4}\right), n\pi \pm \sin^{-1}\left(\frac{3}{4}\right), n \in I\right\}$

$Y = \{x : f'(x) = 0\}$

$f(x) = \sin(\pi \cos x) \Rightarrow f'(x)$

$= \cos(\pi \cos x) . (-\pi \sin x) = 0$

$\sin x = 0 \Rightarrow x = n\pi.$

$\cos(\pi \cos x) = 0 \Rightarrow \pi \cos x = (2n + 1)\frac{\pi}{2}$

$\Rightarrow \cos x = \frac{(2n+1)}{2} \Rightarrow \cos x = -\frac{1}{2}, \frac{1}{2}$

$Y = \left\{n\pi, n\pi \pm \frac{\pi}{3}\right\} = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi......\right\}$

$W = \{x : g'(x) = 0\}$

$g(x) = \cos(2\pi \sin x) \Rightarrow g'(x)$

$= -\sin(2\pi \sin x).(2\pi \cos x) = 0$

$\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$

$\sin(2\pi \sin x) = 0 \Rightarrow 2\pi \sin x = n\pi \Rightarrow \sin x = \frac{n}{2}$

$= -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$

$W = \left\{\frac{n\pi}{2}, n\pi \pm \frac{\pi}{6}, n \in I\right\}$

$= \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, ...\right\}$

Now check the options

5. **Ans. (0.5)**

Sol. $\sqrt{3}\cos x + \frac{2b}{a}\sin x = \frac{c}{a}$

Now, $\sqrt{3}\cos\alpha + \frac{2b}{a}\sin\alpha = \frac{c}{a}$ .....(1)

$\sqrt{3}\cos\beta + \frac{2b}{a}\sin\beta = \frac{c}{a}$ .....(2)

$\sqrt{3}[\cos\alpha - \cos\beta] + \frac{2b}{a}(\sin\alpha - \sin\beta) = 0$

$\sqrt{3}\left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right] +$

$\frac{2b}{a}\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right] = 0$

$-\sqrt{3} + 2\sqrt{3}.\frac{b}{a} = 0$

$\frac{b}{a} = \frac{1}{2} = 0.5$

6. Ans. (C)

Sol. $\sqrt{3}\sin x + \cos x = 2\cos 2x$

$\Rightarrow \cos 2x = \cos\left(x - \frac{\pi}{3}\right)$

$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right)$

$x = (6n-1)\frac{\pi}{3}$ or $(6n+1)\frac{\pi}{9}$

$\Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{9}, \frac{7\pi}{9}$ and $-\frac{5\pi}{9}$ in $(-\pi, \pi)$

$\Rightarrow$ sum $= 0$

7. Ans. (8)

Sol. Given equation

$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

$\frac{5}{4}\cos^2 2x + 1 - 2\sin^2 x\cos^2 x + 1 - 3\sin^2 x\cos^2 x = 2$

$\frac{5}{4}\cos^2 2x - \frac{1}{2}(1-\cos^2 2x) - \frac{3}{4}(1-\cos^2 2x) = 0$

$\frac{5}{2}\cos^2 2x = \frac{5}{4} \quad \Rightarrow \quad \cos^2 2x = \frac{1}{2} \quad \Rightarrow \quad \cos 4x = 0$

Clearly having 8 solutions in $[0, 2\pi]$

8. Ans. (D)

Sol. $\sin x - \sin 3x + 2\sin 2x = 3$

$-2\sin x \cos 2x + 4\sin x \cos x = 3$

$2\sin x\{-\cos 2x + 2\cos x\} = 3$

$2\sin x\{-(2\cos^2 x - 1) + 2\cos x\} = 3$

$2\sin x\{-2\cos^2 x + 2\cos x + 1\} = 3$

$$\underbrace{2\sin x}_{\le 2}\underbrace{\left\{\frac{3}{2} - 2\left(\cos x - \frac{1}{2}\right)^2\right\}}_{\le \frac{3}{2}} = 3$$

for equality to hold

$\sin x = 1$ & $\cos x = \frac{1}{2}$

which is not possible

hence no solution

9. Ans. (A, C, D)

Sol. $\tan(2\pi - \theta) > 0$

$\Rightarrow \quad 2\pi - \theta$ lies in I or III quadrant

$\Rightarrow \quad \because$ lies in II or IV Quadrant ....(i)

$\because -1 < \sin\theta < -\frac{\sqrt{3}}{2} \Rightarrow \theta \in \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)$ ....(ii)

$\therefore$ By (i) and (ii) : $\theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$

Also, given

$2\cos\theta(1 - \sin\phi) = \sin^2\theta\left(\tan\frac{0}{2} + \cot\frac{0}{2}\right)\cos\phi - 1$

$\Rightarrow 2\cos\theta - 2\cos\theta\sin\phi = \frac{\sin^2\theta\cos\phi}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}} - 1$

$\Rightarrow 2\cos 0 - 2\cos 0\sin\phi = 2\sin 0\cos\phi - 1$

$\Rightarrow \sin(\theta + \phi) = \frac{1 + 2\cos\theta}{2}$

$\Rightarrow \frac{1}{2} < \sin(\theta + \phi) < 1$

$\Rightarrow \frac{\pi}{6} < \theta + \phi < \frac{5\pi}{6}$ or $\frac{13\pi}{6} < \theta + \phi < \frac{17\pi}{6}$

$\because \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right) \Rightarrow \frac{13\pi}{6} < \theta + \phi < \frac{17\pi}{6}$

$\Rightarrow \frac{\pi}{2} < \phi < \frac{4\pi}{3}$

10. Ans. (7)

Sol. $\frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$

$\Rightarrow \frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow \frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow 2\cos\frac{2\pi}{n}\sin\frac{2\pi}{n} = \sin\frac{3\pi}{n}$

$\Rightarrow \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \frac{4\pi}{n} = K\pi + (-1)^K \frac{3\pi}{n}$

If $K = 2m \Rightarrow \frac{\pi}{n} = 2m\pi$

$\Rightarrow n = \frac{1}{2m} \Rightarrow n = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots$

If $K = 2m+1 \Rightarrow \frac{7\pi}{n} = (2m+1)\pi$

$\Rightarrow n = \frac{7}{2m+1} \Rightarrow n = 7, \frac{7}{3}, \frac{7}{5}, \ldots\ldots$

Possible value of n is 7

11. Ans. (3)

Sol. $\sin 2\theta = \cos 4\theta$

$\sin 2\theta = 1 - 2\sin^2 2\theta$

$2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow \sin 2\theta = -1, \frac{1}{2}$

$\therefore \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

Now $\tan\theta = \cot 5\theta$ (Given)

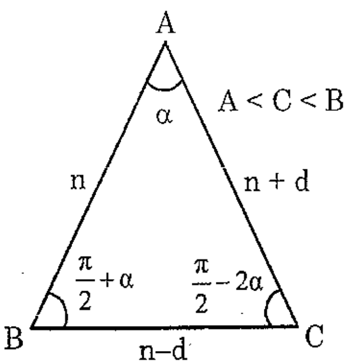
All three obtained values of θ satisy the given equation.

Hence, number of values of θ are 3.

SOLUTION OF TRIANGLE

1. Ans. (1008.00)

Sol.



$n - d = 2\sin\alpha \quad \ldots.(1)$

$n + d = 2\sin\left(\frac{\pi}{2}+\alpha\right)$

$\Rightarrow n + d = 2\cos\alpha \quad \ldots.(2)$

$n = 2\sin\left(\frac{\pi}{2}-2\alpha\right)$

$\Rightarrow n = 2\cos 2\alpha \quad \ldots.(3)$

$\Rightarrow 2\cos 2\alpha = \sin\alpha + \cos\alpha$

$\Rightarrow 2(\cos\alpha - \sin\alpha) = 1$

$\Rightarrow \sin 2\alpha = \frac{3}{4}$

Then,

$a = \frac{1}{2}.n.(n+d).\sin\alpha = \frac{1}{2}.2\cos 2\alpha.2\cos\alpha.\sin\alpha$

$= \sin 2\alpha.\cos 2\alpha$

$= \frac{3}{4} \times \frac{\sqrt{7}}{4} = \frac{3\sqrt{7}}{16}$

$(64a)^2 = \left(64 \times \frac{3\sqrt{7}}{16}\right)^2 = 16 \times 9 \times 7 = 1008$

2. **Ans. (0.25)**

Sol. From above equation in Ques. 1

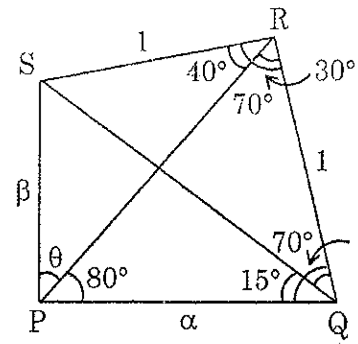
$r = \frac{\Delta}{s} = \frac{1}{2}\frac{n(n+d)\sin\alpha}{\left(\frac{3n}{2}\right)}$

$= \frac{(n+d).\sin\alpha}{3} = \frac{2\cos\alpha.\sin\alpha}{3}$ (from (2))

$r = \frac{\sin 2\alpha}{3} = \frac{1}{4}$

3. **Ans. (A, B)**

Sol.



$\angle PRQ = 70° - 40° = 30°$

$\angle RQS = 70° - 15° = 55°$

$\angle QSR = 180° - 55° - 70° = 55$

$\therefore QR = RS = 1$

$\angle QPR = 180° - 70° - 30° = 80°$

Apply sine-rule in ΔPRQ :

$\frac{\alpha}{\sin 30°} = \frac{1}{\sin 80°} \Rightarrow \alpha = \frac{1}{2\sin 80°} \quad \ldots.(1)$

Apply sine-rule in ΔPRS

$\frac{\beta}{\sin 40°} = \frac{1}{\sin\theta} \Rightarrow \beta\sin\theta = \sin 40° \quad \ldots.(2)$

$4\alpha\beta\sin\theta = \frac{4\sin 40°}{2\sin 80°} = \frac{4\sin 40°}{2(2\sin 40°\cos 40°)}$

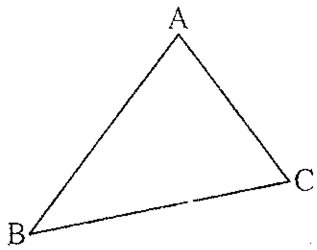
$= \sec 40°$

Now $\sec 30° < \sec 40° < \sec 45°$

$\Rightarrow \frac{2}{\sqrt{3}} < \sec 40° < \sqrt{2}$

4. Ans. (2)

Sol.



Given $c = \sqrt{23}$; $a = 3$; $b = 4$

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc \sin A}$$

$$= \frac{b^2 + c^2 - a^2}{2.2\Delta} \quad \left\{\Delta = \frac{1}{2} bc \sin A\right\}$$

$$\cot A = \frac{b^2 + c^2 - a^2}{4\Delta}$$

Similarly, $\cot B = \dfrac{a^2 + c^2 - b^2}{4\Delta}$

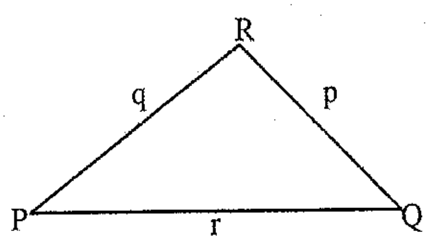
& $\cot C = \dfrac{a^2 + b^2 - c^2}{4\Delta}$

$$\therefore \frac{\cot A + \cot C}{\cot B} = \frac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$$

$$= \frac{2b^2}{a^2 + c^2 - b^2} = \frac{32}{16} = 2$$

5. Ans. (A, B)

Sol.



(A)

$$\text{Cos}P = \frac{q^2 + r^2 - p^2}{2qr} = \frac{q^2 + r^2}{2qr} - \frac{p^2}{2qr} \geq 1 - \frac{p^2}{2qr}$$

(as $p^2 + q^2 \geq 2qr$ (AM $\geq$ GM)),

So (A) is correct

(B) $(p + q) \cos R \geq (q - r) \cos P + (p - r) \cos Q$

$\Rightarrow$ $(p \cos R + r \cos P) + (q \cos R + r \cos Q)$

$\geq q\cos P + p \cos Q$

$\Rightarrow$ $q + p \geq r$

So (B) is correct

(C) $\dfrac{q + r}{p} = \dfrac{\sin Q + \sin R}{\sin P} \geq \dfrac{2\sqrt{\sin Q \times \sin R}}{\sin P}$

So (C) is incorrect

(D) $\cos Q > \dfrac{p}{r}$ $\Rightarrow \sin R \cos Q > \sin P$

$\Rightarrow \sin P + \sin (R - Q) > 2 \sin P$

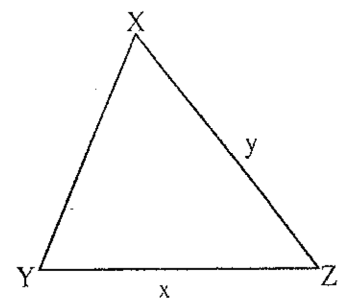
$\Rightarrow \sin (R - Q) > \sin P$

need not necessarily hold true if $R < Q$

Hence (A), (B)

6. Ans. (B, C)

Sol.



$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x + y + z}$$

$$\frac{\Delta}{S(S - x)} + \frac{\Delta}{S(S - z)} = \frac{2y}{2S}$$

$$\frac{\Delta}{S}\left(\frac{2S - (x + z)}{(S - x)(S - z)}\right) = \frac{y}{S}$$

$$\Rightarrow \quad \frac{\Delta y}{S(S - x)(S - z)} = \frac{y}{S}$$

$\Rightarrow$ $\Delta^2 = (S - x)^2 (S - z)^2$

$\Rightarrow$ $S(S - y) = (S - x)(S - z)$

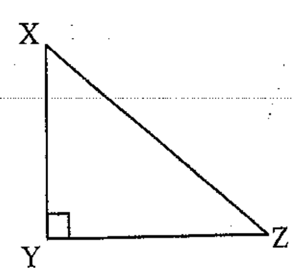
$\Rightarrow$ $(x + y + z)(x + z - y) = (y + z - x)(x + y - z)$

$\Rightarrow$ $(x + z)^2 - y^2 = y^2 - (z - x)^2$

$\Rightarrow$ $(x + z)^2 + (x - z)^2 = 2y^2$

$\Rightarrow$ $x^2 + z^2 = y^2 \Rightarrow \angle Y = \dfrac{\pi}{2}$

$\Rightarrow$ $\angle Y = \angle X + \angle Z$



$$\tan \frac{X}{2} = \frac{\Delta}{S(S - x)}$$

$$\tan\frac{X}{2}=\frac{\frac{1}{2}xz}{\frac{(y+z)^2-x^2}{4}}$$

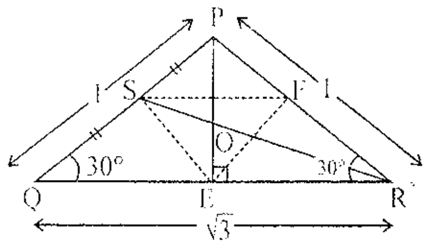
$$\tan\frac{X}{2}=\frac{2xz}{y^2+z^2+2yz-x^2}$$

$$\tan\frac{X}{2}=\frac{2xz}{2z^2+2yz}\qquad(\text{using } y^2=x^2+z^2)$$

$$\tan\frac{X}{2}=\frac{x}{y+z}$$

7. Ans. (B, C, D)

Sol.



$$\frac{\sin P}{\sqrt{3}}=\frac{\sin Q}{1}=\frac{1}{2R}=\frac{1}{2}$$

$$\Rightarrow P=\frac{\pi}{3}\text{ or }\frac{2\pi}{3}\text{ and }Q=\frac{\pi}{6}\text{ or }\frac{5\pi}{6}$$

Since $p>q \Rightarrow P>Q$

So, if $P=\frac{\pi}{3}$ and $Q=\frac{\pi}{6} \Rightarrow R=\frac{\pi}{2}$

(not possible)

Hence, $P=\frac{2\pi}{3}$ and $Q=R=\frac{\pi}{6}$

$$r=\frac{\Delta}{s}=\frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)}=\frac{\sqrt{3}}{2}(2-\sqrt{3})$$

Now, area of $\Delta SEF=\frac{1}{4}$ area of ΔPQR

$\Rightarrow$ area of $\Delta SOE=\frac{1}{3}$ area of $\Delta SEF=\frac{1}{12}$

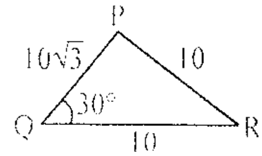
area of $\Delta PQR=\frac{1}{12}.\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{48}$

$$RS=\frac{1}{2}\sqrt{2(3)+2(1)-1}=\frac{\sqrt{7}}{2}$$

$$OE=\frac{1}{3}PE=\frac{1}{3}.\frac{1}{2}\sqrt{2(1)^2+2(1)^2-3}=\frac{1}{6}$$

8. Ans. (B, C, D)

Sol.



$$\cos 30^\circ=\frac{(10\sqrt{3})^2+(10)^2-(PR)^2}{2\times 10\sqrt{3}\times 10}$$

$\Rightarrow$ PR = 10

$\because$ QR = PR $\Rightarrow$ $\angle PQR=\angle QPR$

$\angle QPR=30^\circ$

(B) area of ΔPQR

$$=\frac{1}{2}\times 10\sqrt{3}\times 10\times\sin 30^\circ=\frac{1}{2}\times 10\times 10\sqrt{3}\times\frac{1}{2}$$

$=25\sqrt{3}$

$\angle QRP=180^\circ-(30^\circ+30^\circ)=120^\circ$

(C) $$r=\frac{\Delta}{s}=\frac{25\sqrt{3}}{\left(\frac{10+10+10\sqrt{3}}{2}\right)}=\frac{25\sqrt{3}}{10+5\sqrt{3}}$$

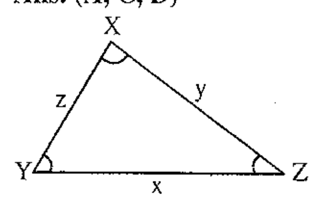
$$=5\sqrt{3}.(2-\sqrt{3})=10\sqrt{3}-15$$

(D) $$R=\frac{a}{2\sin A}=\frac{10}{2\sin 30^\circ}=10$$

$\therefore$ Area $=\pi R^2=100\pi$

9. **Ans. (A, C, D)**

Sol.



Let $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}=k$

$s-x=4k \Rightarrow y+z-x=8k$

$s-y=3k \Rightarrow x+z-y=6k$

$s-z=2k \Rightarrow x+y-z=4k$

$\Rightarrow x=5k, y=6k, z=7k$

$\Rightarrow 3s-(x+y+z)=9k$

$s=9k$

Let r be inradius

$$\Rightarrow \pi r^2=\frac{8\pi}{3}$$

$$\pi\left(\frac{\Delta}{s}\right)^2=\frac{8\pi}{3}$$

$$\Delta=\sqrt{\frac{8}{3}}.s$$

$\sqrt{s(s-x)(s-y)(s-z)} = \sqrt{\frac{8}{3}}.s$

$\sqrt{9k.4k.3k.2k} = \sqrt{\frac{8}{3}}9k$

$\sqrt{24.9k^2} = \sqrt{\frac{8}{3}}.9k$

$k = 1$

$\Rightarrow \quad x = 5, y = 6, z = 7$

$\Delta = \sqrt{\frac{8}{3}}.9k = \sqrt{\frac{8}{3}}.9 = 6\sqrt{6}$

$R = \text{circumradius} = \frac{xyz}{4\Delta} = \frac{5.6.7}{4.6\sqrt{6}} = \frac{35}{4\sqrt{6}}$

$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{r}{4R} = \frac{\frac{\Delta}{s}}{\frac{xyz}{\Delta}} = \frac{\Delta^2}{s.xyz}$

$= \frac{36.6}{9.5.6.7} = \frac{4}{35}$

$\cos Z = \frac{x^2+y^2-z^2}{2xy} = \frac{25+36-49}{2.5.6} = \frac{1}{5}$

$\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{1+\cos Z}{2} = \frac{1+\frac{1}{5}}{2} = \frac{3}{5}$

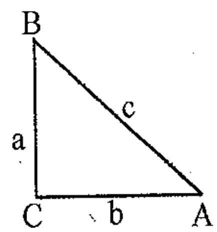
10. Ans. (B)

Sol. $a + b = x$

$ab = y$

$(a+b)^2 - c^2 = ab$

$\Rightarrow a^2 + b^2 - c^2 = -ab$



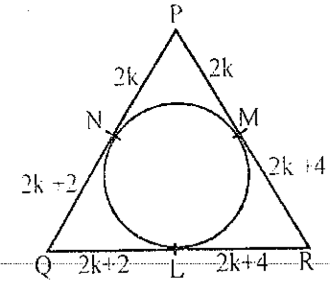
$\cos c = -\frac{1}{2} \Rightarrow c = \frac{2\pi}{3}$

Now, $\frac{r}{R} = \frac{\frac{\Delta}{S}}{\frac{abc}{4\Delta}} = \frac{4\Delta^2}{\frac{(a+b+c)}{2}abc}$

$= \frac{8\left(\frac{1}{2}ab\frac{\sqrt{3}}{2}\right)^2}{(x+c)(yc)} = \frac{\frac{1}{2}y^2.3}{(x+c)cy} = \frac{3y}{2c(x+c)}$

11. Ans. (B, D)

Sol.



$PQ = 4k + 2$

$QR = 4k + 6$

$PR = 4k + 4$

$\cos P = \frac{1}{3} = \frac{(4k+2)^2 + (4k+4)^2 - (4k+6)^2}{2(4k+2)(4k+4)}$

$\Rightarrow \quad k^2 - 3k - 4 = 0$

$\Rightarrow \quad k = 4, k = -1$ (reject)

side lengths 18, 22, 20

12. Ans. (C)

Sol. $\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{(1-\cos P)}{(1+\cos P)}$

$= \tan^2\frac{A}{2} = \frac{\Delta^2}{s^2(s-a)^2} = \frac{((s-b)(s-c))^2}{\Delta^2}$

$s = 4$

$= \left(\frac{\left(4-\frac{7}{2}\right)\left(4-\frac{5}{2}\right)}{\Delta}\right)^2 = \left(\frac{3}{4\Delta}\right)^2$

13. Ans. (D)

Sol. A, B, C are in AP

$\Rightarrow 2B = A + C$

$\Rightarrow A + B + C = \pi$

$\Rightarrow 3B = \pi \Rightarrow B = \frac{\pi}{3}$

Now, $\frac{a}{c}\sin 2C \; \frac{c}{a} + \sin 2A$

$= \frac{a}{c}\, 2\sin C\cos C + \frac{c}{a}\, 2\sin A\cos A$

$= 2\left[\frac{a}{c}\frac{c}{2R}\cos C + \frac{c}{a}\frac{a}{2R}\cos A\right]$

$= \frac{1}{R}[a\cos C + c\cos A] = \frac{b}{R}$

(By projection formula) $= 2\sin B$

$= 2\sin\frac{\pi}{3} = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$

14. Ans. (B)

Sol. $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$\frac{\sqrt{3}}{2} = \frac{(x^2+x+1)^2 + (x^2-1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$$

$\Rightarrow x^4(2-\sqrt{3}) + x^3(2-\sqrt{3}) - 3x^2 - x(2-\sqrt{3}) + (\sqrt{3}+1) = 0$

$\Rightarrow (x^2-1)[(2-\sqrt{3})x^2 + (2-\sqrt{3})x - (\sqrt{3}+1)] = 0$

$$\text{Now }\begin{cases} x^2+x+1+x^2-1 > 2x+1 \\ x^2+x+1+2x+1 > x^2-1 \\ 2x+1+x^2-1 > x^2+x+1 \end{cases}$$

($\because$ sum of two sides is greater than third side)

$\Rightarrow x > 1$

$\Rightarrow x = 1 + \sqrt{3}$

Alternate :

$$\tan\frac{\pi}{12} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}$$

$$s = x^2 + \frac{3x+1}{2}, s-b = \frac{3(x+1)}{2}, s-a = \frac{x-1}{2},$$

$$s - c = x^2 - \frac{(x+1)}{2}$$

$$\tan\frac{\pi}{12} = \sqrt{\frac{\frac{3}{2}(x+1)\frac{(x-1)}{2}}{\left(x^2+\frac{3x+1}{2}\right)\left(x^2-\frac{x+1}{2}\right)}}$$

Simplifying

$$2 - \sqrt{3} = \frac{\sqrt{3}}{2x+1}$$

$$x = \sqrt{3} + 1$$

15. Ans. (3)

Sol. $\Delta = 15\sqrt{3}$

$\angle ACB > 90°$

$r^2 = ?$

$$\Delta = \frac{1}{2}\ 10.6.\sin C = 15\sqrt{3}$$

$$\sin C = \frac{\sqrt{3}}{2}$$

$C = 120°$

$$\cos C = \frac{100 + 36 - c^2}{2.10.6}$$

$c = 14$

$$r = \frac{\Delta}{s}$$

$$s = \frac{a+b+c}{2} = \frac{10+6+14}{2} = 15$$

$\therefore r = \sqrt{3} \quad \Rightarrow r^2 = 3$

16. Ans. (B, C)

Sol. $$2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$$

$$\Rightarrow \cos\left(\frac{B-C}{2}\right) = 2\sin\frac{A}{2}$$

$$\Rightarrow 2\cos\frac{A}{2}\ \cos\left(\frac{B-C}{2}\right) = 2\ .\ 2\sin\frac{A}{2}.\ \cos\frac{A}{2}$$

$$\Rightarrow 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 2\sin A$$

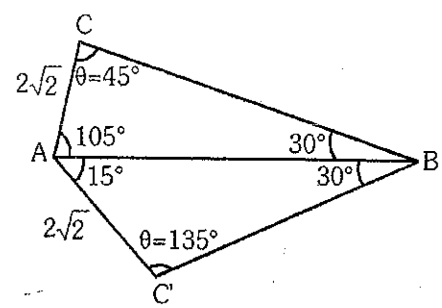
$\Rightarrow \sin B + \sin C = 2\sin A$

$\Rightarrow b + c = 2a$

$\Rightarrow$ locus is ellipse.

17. Ans. (4)

Sol. Applying sine-law in ΔABC



$$\frac{4}{\sin\theta} = \frac{2\sqrt{2}}{\sin 30°} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$\Rightarrow \theta = 45°, 135°$

Area of ΔABC – Area $\Delta AC'B$

$$= \frac{1}{2} \times 2\sqrt{2} \times 4\ (\sin 105° - \sin 15°)$$

$$= 4\sqrt{2} \times 2\cos 60° \sin 45°$$

$$= 4\sqrt{2} \times 2 \times \frac{1}{2} \times \frac{1}{\sqrt{2}} = 4$$

DETERMINANT

1. Ans. (A)

Sol. Given $x + 2y + z = 7$ .... (1)

$x + \alpha z = 11$ .... (2)

$2x - 3y + \beta z = \gamma$ .... (3)

$$\text{Now, } \Delta = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & \alpha \\ 2 & -3 & \beta \end{vmatrix} = 7\alpha - 2\beta - 3$$

$\therefore$ if $\beta = \frac{1}{2}(7\alpha - 3)$

$\Rightarrow \boxed{\Delta = 0}$

$$\text{Now, } \Delta_x = \begin{vmatrix} 7 & 2 & 1 \\ 11 & 0 & \alpha \\ \gamma & -3 & \beta \end{vmatrix}$$

$= 21\alpha - 22\beta + 2\alpha\gamma - 33$

$\therefore$ if $\gamma = 28$

$\Rightarrow \Delta_x = 0$

$$\Delta_y = \begin{vmatrix} 1 & 7 & 1 \\ 1 & 11 & \alpha \\ 2 & \gamma & \beta \end{vmatrix}$$

$\Delta_y = 4\beta + 14\alpha - \alpha\gamma + \gamma - 22$

$\therefore$ if $\gamma = 28$

$\Rightarrow \Delta_y = 0$

$$\text{Now, } \Delta_z = \begin{vmatrix} 1 & 2 & 7 \\ 1 & 0 & 11 \\ 2 & -3 & \gamma \end{vmatrix} = 56 - 2\gamma$$

If $\gamma = 28$

$\Rightarrow \boxed{\Delta_z = 0}$

$\therefore$ if $\gamma = 28$ and $\beta = \frac{1}{2}(7\alpha - 3)$

$\Rightarrow$ system has infinite solution

and if $\gamma \neq 28$

$\Rightarrow$ system has no solution

$\Rightarrow P \to (3)$; $Q \to (2)$

Now if $\beta \neq \frac{1}{2}(7\alpha - 3)$

$\Rightarrow \Delta \neq 0$

and for $\alpha = 1$ clearly

$y = -2$ is always be the solution

$\therefore$ if $\gamma \neq 28$

System has a unique solution

if $\gamma = 28$

$\Rightarrow$ $x = 11$, $y = -2$ and

$z = 0$ will be one of the solution

$\therefore R \to 1$; $S \to 4$

$\therefore$ option 'A' is correct

2. Ans. (B)

Sol. If $\frac{q}{r} = 10 \to A = D \Rightarrow D_x = D_y = D_z = 0$

So, there are infinitely many solutions

Look of infinitely many solutions can be given as

$x + y + z = 1$

& $10x + 100y + 1000z = 0$

$\Rightarrow$ $x + 10y + 100z = 0$

Let $z = \lambda$

then $x + y = 1 - \lambda$

and $x + 10y = -100\lambda$

$\Rightarrow x = \frac{10}{9} + 10\lambda$; $y = \frac{-1}{9} - 11\lambda$

i.e., $(x, y, z) \equiv \left(\frac{10}{9} + 10\lambda, \frac{-1}{9} - 11\lambda, \lambda\right)$

$Q\left(\frac{10}{9}, \frac{-1}{9}, 0\right)$ valid for $\lambda = 0$

$P\left(0, \frac{10}{9}, \frac{-1}{9}\right)$ not valid for any λ.

(I) $\to$ Q,R,T

(II) If $\frac{p}{r} \neq 100$, then $D_y \neq 0$

So no solution

(II) $\to$ (S)

(III) If $\frac{p}{q} \neq 10$, then $D_z \neq 0$ so, no solution

(III) $\to$ (S)

(IV) If $\frac{p}{q} = 10 \Rightarrow D_z = 0 \Rightarrow D_x = D_y = 0$

so infinitely many solution

(IV) $\to$ Q,R,T

3. Ans. (2)

Sol. $x^3\begin{vmatrix}1 & 1 & 1+x^3\\2 & 4 & 1+8x^3\\3 & 9 & 1+27x^3\end{vmatrix}=10$

$\Rightarrow \quad x^3\begin{vmatrix}1 & 1 & 1\\1 & 2 & 4\\1 & 3 & 9\end{vmatrix}+x^3.x^3\begin{vmatrix}1 & 1 & 1\\2 & 4 & 8\\3 & 9 & 27\end{vmatrix}=0$

$\Rightarrow \quad x^3(25-23)+6x^6.2=10$

$\Rightarrow \quad 6x^6+x^3-5=0$

$\Rightarrow \quad x^3=\frac{5}{6},-1$

two real solutions

4. Ans. (B, C)

Sol. $\begin{vmatrix}1 & \alpha & \alpha^2\\4 & 2\alpha & \alpha^2\\9 & 3\alpha & \alpha^2\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\2 & 4 & 6\\1 & 4 & 9\end{vmatrix}=-648\alpha$

$\alpha^3\begin{vmatrix}1 & 1 & 1\\4 & 2 & 1\\9 & 3 & 1\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\2 & 4 & 6\\1 & 4 & 9\end{vmatrix}=-648\alpha$

$-8\alpha^3=-648\alpha$

$\Rightarrow \alpha^3=81\alpha$

$\therefore \alpha=0,9,-9$

5. **Ans. (3)**

Sol. $(y+z)\cos 3\theta=(xyz)\sin 3\theta$ ...(i)

$x\sin 3\theta=\frac{2\cos 3\theta}{y}+\frac{2\sin 3\theta}{z}$ ...(ii)

$(xyz)\sin 3\theta=(y+2z)\cos 3\theta+y\sin 3\theta$ ...(iii)

Where $yz\neq 0$ and $0<\theta<\pi$

from (i) & (iii)

$(y+z)\cos 3\theta=(y+2z)\cos 3\theta+y\sin 3\theta$

$\Rightarrow z\cos 3\theta+y\sin 3\theta=0$ ...(iv)

from eq^n (ii)

$2z\cos 3\theta+2y\sin 3\theta=xyz\sin 3\theta$ ...(v)

from equation (iv) & (v)

$\Rightarrow xyz\sin 3\theta=0$

$\Rightarrow x\sin 3\theta=0$ as $yz\neq 0$

Possible cases are either $x=0$ or $\sin 3\theta=0$

Case (1) : if $x=0$

$\Rightarrow y+z=0 \Rightarrow y=-z$

from eq^n (iv) $\cos 3\theta=\sin 3\theta$

$\Rightarrow 3\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$

$\Rightarrow \theta=\frac{\pi}{12},\frac{5\pi}{12},\frac{9\pi}{12}$

Case (2) : if $\sin 3\theta=0$

$\Rightarrow \theta=\frac{\pi}{3},\frac{2\pi}{3}$

But these values does not satisfy given equations.

Hence, total number of possible values of θ are 3.

6. **Ans. (7)**

Sol. $3x-y-z=0$

$-3x+z=0\Rightarrow y=0$ and $z=3x$

$-3x+2y+z=0$

Now, put $x=k$, $k+k$

$\Rightarrow z=3k$

$\Rightarrow$ Solution of system are $(k, 0, 3k)$

Now, $\because x^2+y^2+z^2\leq 100$

$\Rightarrow k^2\leq 100$

$\Rightarrow k\in\left[-\sqrt{10},\sqrt{10}\right]$

$\Rightarrow$ No. of integral values of k is 7.

No. of such points is 7.

7. **Ans. (A)**

Sol. $x-2y+3z=-1, -x+y-2z$

$=k$ & $x-3y+4z=1$

$\Delta=\begin{vmatrix}1 & -2 & 3\\-1 & 1 & -2\\1 & -3 & 4\end{vmatrix}=0$ & $\begin{vmatrix}1 & 3 & -1\\-1 & -2 & k\\1 & 4 & 1\end{vmatrix}=3-k$

Hence if $k=3$ then system will have infinite solutions and $k\neq 3$ then system will have no solution. so S(I) & S(II) both are true & (II) is correct explaination for (I).

STRAIGHT LINE

1. Ans. (9.00)

Sol. $\left|\frac{\sqrt{2}x+y-1}{\sqrt{3}}\right|\left|\frac{\sqrt{2}x-y+1}{\sqrt{3}}\right|=\lambda^2$

$\left|\frac{2x^2-(y-1)^2}{3}\right|=\lambda^2, C:\left|2x^2-(y-1)^2\right|=3\lambda^2$

line $y = 2x + 1$, $RS=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$R(x_1,y_1)$ and $S(x_2,y_2)$

$y_1 = 2x_1 + 1$ and $y_2 = 2x_2 + 1$

$\Rightarrow (y_1 - y_2) = 2(x_1 - x_2)$

$RS=\sqrt{5(x_1-x_2)^2}=\sqrt{5}|x_1-x_2|$

solve curve C and line $y = 2x + 1$ we get

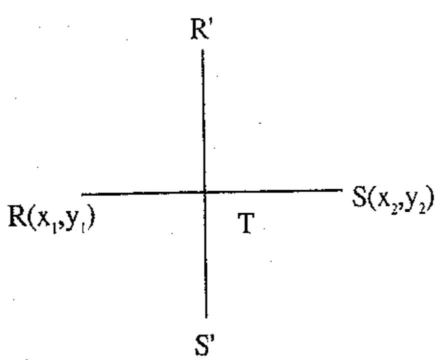
$\left|2x^2-(2x)^2\right|=3\lambda^2 \Rightarrow x^2=\frac{3\lambda^2}{2}$

$RS=\sqrt{5}\left|\frac{2\sqrt{3}\lambda}{\sqrt{2}}\right|=\sqrt{30}\lambda=\sqrt{270}$

$\Rightarrow 30\lambda^2=270\Rightarrow\lambda^2=9$

2. Ans. (77.14)

Sol.



$\perp$ bisector of RS

$T\equiv\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$

Here $x_1 + x_2 = 0$

$T = (0,1)$

Equation of

$R'S' : (y-1)=-\frac{1}{2}(x-0)\Rightarrow x+2y=2$

$R'(a_1,b_1)$ $S'(a_2,b_2)$

$D=(a_1-a_2)^2+(b_1-b_2)^2=5(b_1-b_2)^2$

solve $x+2y=2$ and $\left|2x^2-(y-1)^2\right|=3\lambda^2$

$\left|8(y-1)^2-(y-1)^2\right|=3\lambda^2\Rightarrow(y-1)^2=\left(\frac{\sqrt{3}\lambda}{\sqrt{7}}\right)^2$

$y-1=\pm\frac{\sqrt{3}\lambda}{\sqrt{7}}\Rightarrow b_1=1+\frac{\sqrt{3}\lambda}{\sqrt{7}}, b_2=1-\frac{\sqrt{3}\lambda}{\sqrt{17}}$

$D=5\left(\frac{2\sqrt{3}\lambda}{\sqrt{7}}\right)^2=\frac{5\times4\times3\lambda^2}{7}$

$=\frac{5\times4\times27}{7}=77.14$

3. Ans. (B, C, D)

Sol. $ax + 2y = \lambda$

$3x - 2y = \mu$

for $a = -3$ above lies will be parallel or coincident

parallel for $\lambda + \mu \neq 0$ and coincident if $\lambda + \mu = 0$

and if $a \neq -3$ lies are intersecting

$\Rightarrow$ unique solution.

4. Ans. (6)

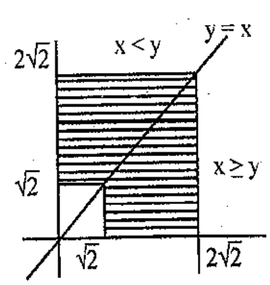
Sol. Let P(x,y) is the point in I quad.

Now $2\le\left|\frac{x-y}{\sqrt{2}}\right|+\left|\frac{x+y}{\sqrt{2}}\right|\le4$

$2\sqrt{2}\le|x-y|+|x+y|\le4\sqrt{2}$

Case-I : $x\ge y$

$2\sqrt{2}\le(x-y)+(x+y)\le4\sqrt{2}$



$\Rightarrow \quad x\in[\sqrt{2},2\sqrt{2}]$

Case-II : $x<y$

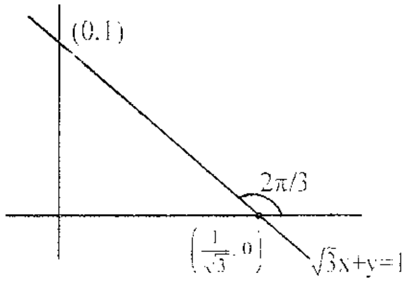
$2\sqrt{2}\le y-x+(x+y)\le4\sqrt{2}$

$y\in[\sqrt{2},2\sqrt{2}]$

$A=\left(2\sqrt{2}\right)^2-\left(\sqrt{2}\right)^2=6$

5. Ans. (A or C or A, C)

Sol.



Point of intersection of both lines is

$$\left(-\frac{c}{(a+b)},-\frac{c}{(a+b)}\right)$$

Distance between $\left(-\frac{c}{(a+b)},-\frac{c}{(a+b)}\right)$

& (1,1) is

$$\text{Distance}=\sqrt{\frac{(a+b+c)^2}{(a+b)^2}\times 2}<2\sqrt{2}$$

$a+b+c<2(a+b)$

$a+b-c>0$

According to given condition option (C) also correct.

6. Ans. (B)

Sol. Line L has two possible slopes with inclination;

$\theta=\frac{\pi}{3},\ \theta=0$

$\therefore$ equation of line L when $\theta=\frac{\pi}{3}$,

$y+2=\sqrt{3}(x-3)$

$\Rightarrow\ y-\sqrt{3}x+2+3\sqrt{3}=0$

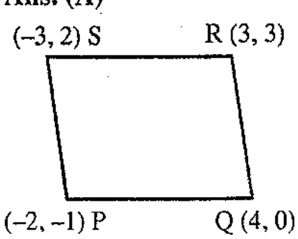
equation of line L when $\theta=0$,

$y=-2$ (rejected)

$\therefore$ required line L is $y-\sqrt{3}x+2+3\sqrt{3}=0$

7. Ans. (A)

Sol.



$\overrightarrow{PQ}=6\hat{i}+\hat{j}$

$\overrightarrow{SR}=6\hat{i}+\hat{j}$

$\therefore \overrightarrow{PQ}=\overrightarrow{SR}$

$\overrightarrow{PS}=-\hat{i}+3\hat{j}$

$\overrightarrow{QR}=-\hat{i}+3\hat{j}$

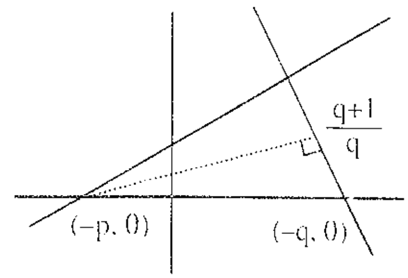
$\therefore \overrightarrow{PS}=\overrightarrow{QR}$

But $\overrightarrow{PQ}.\overrightarrow{PS}=-6+3=-3\neq 0$ & $\left|\overrightarrow{PQ}\right|\neq\left|\overrightarrow{PS}\right|$

$\Rightarrow$ PQRS is a parallelogram but neither a rhombus nor a rectangle.

8. Ans. (D)

Sol.



$x-\frac{p\cdot y}{p+1}+p=0$

$x-\frac{q\cdot y}{1+q}+q=0$

$(q+1)y=-q(x+p)$ ...(i)

$(p+1)y=-p\,(x+a)$ ...(ii)

(i)–(ii)

$\Rightarrow (q-p)y=(p-q)x$

$y=-x$

9. Ans. (D)

Sol. Method : 1

$P\equiv(-\sin(\beta-\alpha),-\cos\beta)\equiv(x_1,y_1)$,

$Q\equiv(\cos(\beta-\alpha),\sin\beta)\equiv(x_2,y_2)$

and $R\equiv(x_2\cos\theta+x_1\sin\theta,\ y_2\cos\theta+y_1\sin\theta)$

We see that

$T\equiv\left(\frac{x_2\cos\theta+x_1\sin\theta}{\cos\theta+\sin\theta},\frac{y_2\cos\theta+y_1\sin\theta}{\cos\theta+\sin\theta}\right)$ and P, Q, T are collinear $\Rightarrow$ P, Q, R are non–collinear

Method : 2

$$\begin{vmatrix}-\sin(\beta-\alpha) & -\cos\beta & 1\\ \cos(\beta-\alpha) & \sin\beta & 1\\ \cos(\beta-\alpha+\theta) & \sin(\beta-\theta) & 1\end{vmatrix}$$

Applying $R_3\to R_3+R_1\sin\theta-R_2\cos\theta$

$$\begin{vmatrix}-\sin(\beta-\alpha) & -\cos\beta & 1\\ \cos(\beta-\alpha) & \sin\beta & 1\\ 0 & 0 & 1+\sin\theta-\cos\theta\end{vmatrix}$$

$=(1+\sin\theta-\cos\theta)\,[-\sin\beta\sin(\beta-\alpha)+\cos\beta\cos(\beta-\alpha)]$

$=(1+\sin\theta-\cos\theta)\cos(2\beta-\alpha)\neq 0$

Hence P, Q, R are non collinear.

10. **Ans.** (A) – s, (B) – p, q, (C) – r, (D) – p, q, s

Sol. $x + 3y - 5 = 0,\ 3x - xy - 1 = 0,\ 5x + 2y - 12 = 0$

(A) For concurrency

$$\begin{vmatrix} 1 & 3 & -5 \\ 3 & -k & -1 \\ 5 & 2 & -12 \end{vmatrix} = 0$$

$\Rightarrow \quad (12k + 2) - 3(-36 + 5) - 5(6 + 5k) = 0$

$\therefore \quad k = 5$ **(A) ⟶ (s)**

(B) For parallel

either $-\dfrac{1}{3} = \dfrac{3}{k}$ $\quad \therefore k = -9$

or $\quad -\dfrac{5}{2} = \dfrac{3}{k}$ $\quad \therefore k = \dfrac{-6}{5}$ **(B) ⟶ (p, q)**

(C) They will form triangle

when $k \neq 5, -9, \dfrac{-6}{5}$ **(C) ⟶ (r)**

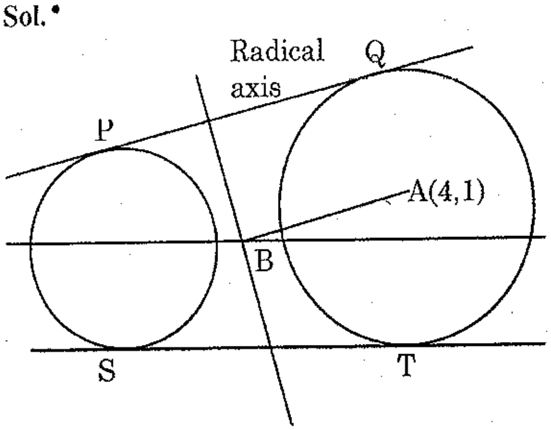
(D) They will not form triangle

when $k = 5, -9, \dfrac{-6}{5}$ **(D) ⟶ (p, q, s)**

CIRCLE

1. **Ans. (2)**

Sol.



Let $C_2\ (x-4)^2 + (y-1)^2 = r^2$

radical axis $8x + 2y - 17 = 1 - r^2$

$8x + 2y = 18 - r^2$

$B\left(\dfrac{18 - r^2}{8}, 0\right)$ $A(4, 1)$

$AB = \sqrt{5}$

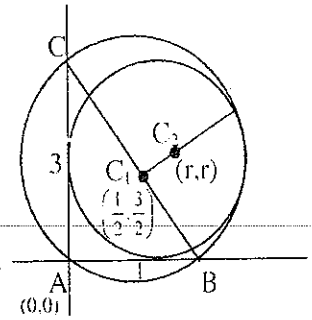
$$\sqrt{\left(\frac{18 - r^2}{8} - 4\right)^2 + 1} = \sqrt{5}$$

$r^2 = 2$

$\Rightarrow n = \sin\alpha + \cos\alpha$

2. **Ans. (0.83 or 0.84)**

Sol. $4 - \sqrt{10} = 0.83$ or 0.84



$C_1\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$ and $r_1 = \dfrac{\sqrt{10}}{2}$

$C_2 = (r, r)$

$\therefore$ circle C_2 touches C_1 internally

$$\Rightarrow C_1C_2 = \left| r - \frac{\sqrt{10}}{2} \right|$$

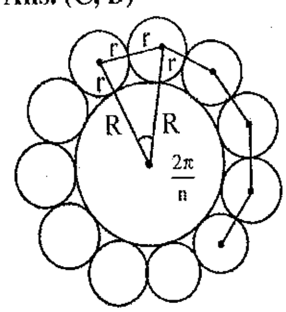
$$\Rightarrow \left(r - \frac{1}{2}\right)^2 + \left(r - \frac{3}{2}\right)^2 = \left(r - \frac{\sqrt{10}}{2}\right)^2$$

$r^2 - 4r + \sqrt{10}r = 0$

$r = 0$ (reject) or $r = 4 - \sqrt{10}$

3. **Ans. (C, D)**

Sol.



$$2(R + r)\sin\frac{\pi}{n} = 2r$$

$$\frac{R + r}{r} = \operatorname{cosec}\frac{\pi}{n}$$

(A) $n = 4,\ R + r = \sqrt{2}\,r$

(B) $n = 5,\ \dfrac{R + r}{r} = \operatorname{cosec}\dfrac{\pi}{5} < \operatorname{cosec}\dfrac{\pi}{6}$

$R + r < 2r \Rightarrow r > R$

(C) $n = 8,\ \dfrac{R + r}{r} = \operatorname{cosec}\dfrac{\pi}{8} > \operatorname{cosec}\dfrac{\pi}{4}$

$R + r > \sqrt{2}\,r$

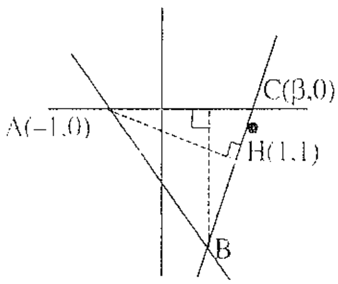
(D) $n = 12,\ \dfrac{R + r}{r} = \operatorname{cosec}\dfrac{\pi}{12} = \sqrt{2}\left(\sqrt{3} + 1\right)$

$R + r = \sqrt{2}\left(\sqrt{3} + 1\right)r$

$\sqrt{2}\left(\sqrt{3} + 1\right)r > R$

4. Ans. (B)

Sol.



$(1,-2)=(\alpha,-\alpha-1)$

$\Rightarrow \alpha = 1$

one of the vertex is intersection of x-axis and

$x + y + 1 = 0 \Rightarrow A(-1,0)$

Let vertex B be $(\alpha,-\alpha-1)$

Line $AC \perp BH \Rightarrow \alpha = 1 \Rightarrow B(1,-2)$

Let vertex C be$(\beta,0)$

Line $AH \perp BC$

$m_{AH}.m_{BC} = -1$

$\frac{1}{2}.\frac{2}{\beta-1} = -1 \Rightarrow \beta = 0$

Centroid of ΔABC is $\left(0,-\frac{2}{3}\right)$

Now G(centroid) divides line joining circum centre (O) and ortho centre (H) in the ratio 1: 2

$\Rightarrow$ (h,k) $\left(0,-\frac{2}{3}\right)$ (1,1)

O 1 G 2 H

$2h + 1 = 0 \quad 2k + 1 = -2$

$h = -\frac{1}{2} \quad k = -\frac{3}{2}$

$\Rightarrow$ circum centre is $\left(-\frac{1}{2},-\frac{3}{2}\right)$

Equation of circum circle is

(passing through C(0,0)) is

$x^2 + y^2 + x + 3y = 0$

5. Ans. (D)

Sol. $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^{n-1}}$

$= 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}$

Centre of C_n is $\left(2-\frac{1}{2^{n-2}},0\right)$

and radius of C_n is $\frac{1}{2^{n-1}}$

when $r = \frac{1025}{513} < 2$

C_n will lie inside m

when $2-\frac{1}{2^{n-2}}+\frac{1}{2^{n-1}} < \frac{1025}{513}$

$\Rightarrow k = 10$

Also $\ell = 5$

$3k + 2\ell = 30 + 10 = 40$

6. Ans. (B)

Sol. Center of D_n is (S_{n-1}, S_{n-1})

$r = \frac{1}{2^{n-1}}$

D_n will lie inside

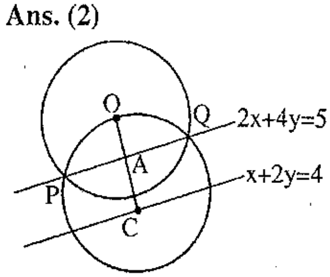
when $\sqrt{2}(S_{n-1})+a_n < \frac{2^{199}-1}{2^{198}}\sqrt{2}$

$\Rightarrow \frac{\sqrt{2}}{2^{n-2}} > \frac{\sqrt{2}}{2^{198}}+\frac{1}{2^{n-1}}$

$\Rightarrow n = 199$

7. Ans. (2)

Sol.



M-I

$OA = \frac{\sqrt{5}}{2} \quad OC = \frac{4}{\sqrt{5}}$

$CQ = OC = \frac{4}{\sqrt{5}}$ and $CA = \frac{3}{2\sqrt{5}}$

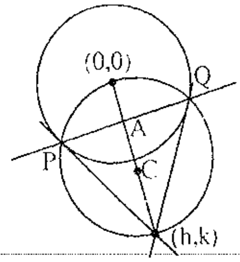
$\therefore \quad OQ = \sqrt{OA^2 + AQ^2}$

$= \sqrt{OA^2 + (CQ^2 - CA^2)}$

$\Rightarrow \sqrt{\frac{5}{4}+\frac{16}{5}-\frac{9}{20}} = \sqrt{4}$

$\Rightarrow \quad 2 = r$

M-II



PQ : $hx + ky = r^2$

Given PQ $2x + 4y = 5$

$$\Rightarrow \quad \frac{h}{2} = \frac{k}{4} = \frac{r^2}{5} \Rightarrow h = \frac{2r^2}{5} \quad k = \frac{4r^2}{5}$$

$$\therefore \quad C = \left(\frac{r^2}{5}, \frac{2r^2}{5}\right)$$

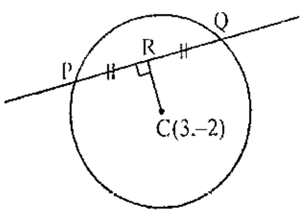
$\therefore$ C lies on $x + 2y = 4$

$$\Rightarrow \quad \frac{r^2}{5} + 2\left(\frac{2r^2}{5}\right) = 4$$

$$\Rightarrow \quad r^2 = 4 \qquad \Rightarrow r = 2$$

8. Ans. (B)

Sol.



$$R \equiv \left(-\frac{3}{5}, \frac{-3m}{5} + 1\right)$$

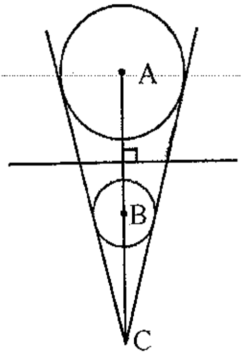
$$\text{So,} \quad m\left(\frac{-\frac{3m}{5} + 3}{-\frac{3}{5} - 3}\right) = -1$$

$$\Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow m = 2, 3$$

9. Ans. (10.00)

Sol.



Distance of point A from given line $= \frac{5}{2}$

$$\frac{CA}{CB} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{2}{1}$$

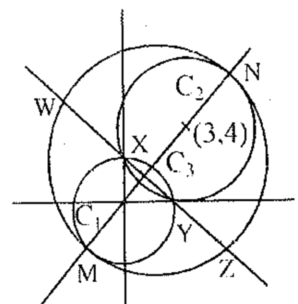
$$\Rightarrow \quad AC = 2 \times 5 = 10$$

Q.10 Ans. (A)

Q.11 Ans. (D)

Solution for Q.10 and Q.11

Sol.



$MC_1 + C_1C_2 + C_2N = 2r$

$\Rightarrow 3 + 5 + 4 = 2r \Rightarrow r = 6 \Rightarrow$ Radius of $C_3 = 6$

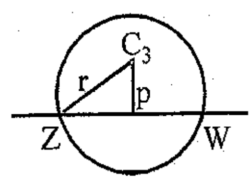
Suppose centre of C_3 be

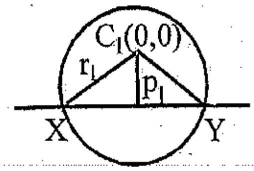
$$(0 + r_4\cos\theta, 0 + r_4\sin\theta), \begin{Bmatrix} r_4 = C_1C_3 = 3 \\ \tan\theta = \frac{4}{3} \end{Bmatrix}$$

$$C_3 = \left(\frac{9}{5}, \frac{12}{5}\right) = (h, k) \Rightarrow 2h + k = 6$$

Equation of ZW and XY is $3x + 4y - 9 = 0$

(common chord of circle $C_1 = 0$ and $C_2 = 0$)





$$ZW = 2\sqrt{r^2 - p^2} = \frac{24\sqrt{6}}{5}$$

(where $r = 6$ and $p = \frac{6}{5}$)

$$XY = 2\sqrt{r_1^2 - p_1^2} = \frac{24}{5}$$

(where $r_1 = 3$ and $p_1 = \frac{9}{5}$)

$$\frac{\text{Length of } ZW}{\text{Length of } XY} = \sqrt{6}$$

Let length of perpendicular from M to ZW be λ, $\lambda = 3 + \frac{9}{5} = \frac{24}{5}$

$$\frac{\text{Area of } \Delta MZN}{\text{Area of } \Delta ZMW} = \frac{\frac{1}{2}(MN) \times \frac{1}{2}(ZW)}{\frac{1}{2} \times ZW \times \lambda}$$

$$= \frac{1}{2}\frac{MN}{\lambda} = \frac{5}{4}$$

$$C_3 : \left(x - \frac{9}{5}\right)^2 + \left(y - \frac{12}{5}\right)^2 = 6^2$$

$C_1 : x^2 + y^2 - 9 = 0$

common tangent to C_1 and C_3 is common chord of C_1 and C_3 is $3x + 4y + 15 = 0$.

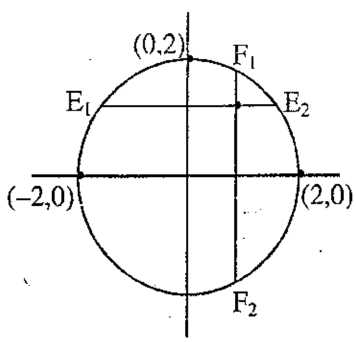
Now $3x + 4y + 15 = 0$ is tangent to parabola $x^2 = 8\alpha y$.

$$x^2 = 8\alpha\left(\frac{-3x - 15}{4}\right) \Rightarrow 4x^2 + 24\alpha x + 120\alpha = 0$$

$$D = 0 \Rightarrow \alpha = \frac{10}{3}$$

12. Ans. (A)

Sol.



co-ordinates of E_1 and E_2 are obtained by solving $y = 1$ and $x^2 + y^2 = 4$

$\therefore \quad E_1\left(-\sqrt{3}, 1\right)$ and $E_2\left(\sqrt{3}, 1\right)$

co-ordinates of F_1 and F_2 are obtained by solving $x = 1$ and $x^2 + y^2 = 4$

$F_1\left(1, \sqrt{3}\right)$ and $F_2\left(1, -\sqrt{3}\right)$

Tangent at E_1 : $-\sqrt{3}x + y = 4$

Tangent at E_2 : $\sqrt{3}x + y = 4$

$\therefore \quad E_3(0, 4)$

Tangent at $F_1 : x + \sqrt{3}y = 4$

Tangent at $F_2 : x - \sqrt{3}y = 4$

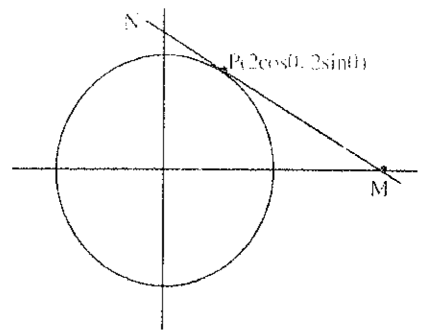
$\therefore \quad F_3(4, 0)$

and similarly $G_3(2, 2)$

(0, 4), (4, 0) and (2, 2) lies on $x + y = 4$

13. Ans. (D)

Sol.



Tangent at P(2cosθ, 2sinθ) is $x\cos\theta + y\sin\theta = 2$

M(2secθ, 0) and N(0, 2cosecθ)

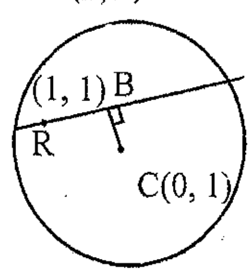
Let midpoint be (h, k)

$h = \sec\theta, \; k = \operatorname{cosec}\theta$

$$\frac{1}{h^2} + \frac{1}{k^2} = 1$$

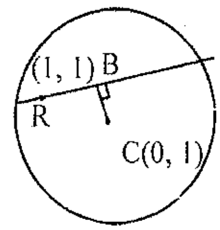
$$\frac{1}{x^2} + \frac{1}{y^2} = 1$$

14. Ans. (B, D)



AP = AQ = AM

Locus of M is a circle having PQ as its diameter



Hence, $E_1 : (x-2)(x+2)+(y-7)(y+5)=0$

and $x \neq \pm 2$

Locus of B (midpoint)

is a circle having RC as its diameter

$E_2 : x(x-1)+(y-1)^2=0$

Now, after checking the options, we get (D)

The points $\left(\frac{4}{5},\frac{7}{5}\right)$, (1, 1), (–2, 7) are collinear

therefore (B) option is also correct.

15. **Ans. (2)**

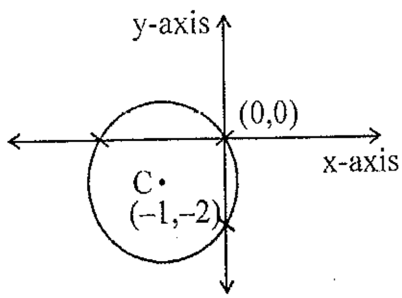
Sol. We shall consider 3 cases.

Case I : When $p=0$

(i.e. circle passes through origin)

Now, equation of circle becomes

$x^2+y^2+2x+4y=0$



Case II : When circle intersects x-axis at 2 distinct points and touches y-axis

Now $(g^2-c)>0$ & $f^2-c=0$

$\Rightarrow \quad 1-(-p)>0$ & $4-(-p)=0$

$\Rightarrow \quad p=-4$

$\Rightarrow \quad p>-1$

$\therefore$ Not possible.

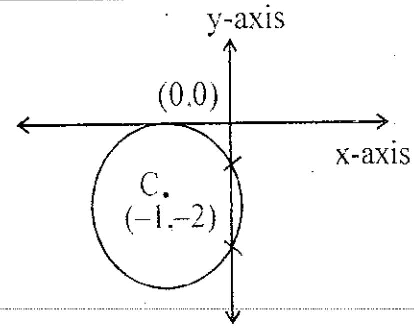
Case III : When circle intersects y-axis at 2 distinct points & touches x-axis.

Now, $g^2-c=0$ & $f^2-c>0$

$\Rightarrow \quad 1-(-p)=0$ & $4-(-p)>0$

$\Rightarrow \quad p=-1$ $\Rightarrow \quad p>-4$

$\therefore \quad p=-1$ is possible.



$\therefore$ Finally we conclude that $p=0,-1$

$\Rightarrow$ Two possible values of p.

16. **Ans. (A, B, C)**

Sol. On solving $x^2+y^2=3$ and

$x^2=2y$ we get point $P(\sqrt{2},1)$

Equation of tangent at P

$\sqrt{2}.x+y=3$

Let Q_2 be $(0,k)$ and radius is $2\sqrt{3}$

$\therefore \left|\frac{\sqrt{2}(0)+k-3}{\sqrt{2+1}}\right|=2\sqrt{3}$

$\therefore k=9,-3$

$Q_2(0,9)$ and $Q_3(0,-3)$

hence $Q_2Q_3=12$

R_2R_3 is internal common tangent of circle C_2 and C_3

$\therefore R_2R_3=\sqrt{(Q_2Q_3)^2-(2\sqrt{3}+2\sqrt{3})^2}$

$=\sqrt{12^2-48}=\sqrt{96}=4\sqrt{6}$

Perpendicular distance of origin O from R_2R_3 is equal to radius of circle $C_1=\sqrt{3}$

Hence area of ΔOR_2R_3

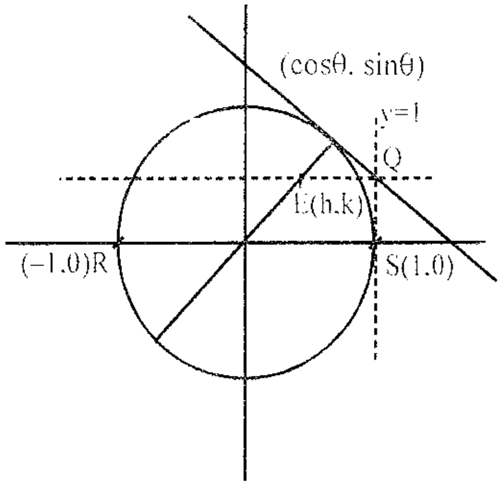
$=\frac{1}{2}\times(R_2R_3)\sqrt{3}=\frac{1}{2}.4\sqrt{6}.\sqrt{3}=6\sqrt{2}$

Perpendicular Distance of P from $Q_2Q_3=\sqrt{2}$

$\therefore$ Area of $\Delta PQ_2Q_3=\frac{1}{2}\times 12\times\sqrt{2}=6\sqrt{2}$

17. Ans. (A, C)

Sol.



Tangent at P : $x\cos\theta + y\sin\theta = 1$ ....(i)

Tangent at S : $x = 1$ ....(ii)

$\therefore$ By (i) & (ii) : $Q\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$

Line through Q parallel to RS :

$y = \frac{1-\cos\theta}{\sin\theta} \Rightarrow y = \tan\frac{\theta}{2}$ ....(iii)

Normal at P :

$y = \frac{\sin\theta}{\cos\theta}x \Rightarrow y = \tan\theta.x$ ....(iv)

Point of intersection of equation (iii) and (iv),

$E:\ h = \frac{1-\tan^2\frac{\theta}{2}}{2};\ k = \tan\frac{\theta}{2}$

eliminating θ : $h = \frac{1-k^2}{2} \Rightarrow y^2 = 1-2x$

Options (A) and (C) satisfies the locus.

18. Ans. (B,C)

Sol. Let circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Put (0,1) $\quad 1 + 2f + c = 0$ ....(1)

orthogonal with

$x^2 + y^2 - 2x - 15 = 0$

$2g(-1) = c - 15 \Rightarrow c = 15 - 2g$ ....(2)

orthogonal with

$x^2 + y^2 - 1 = 0$

$c = 1$ ....(3)

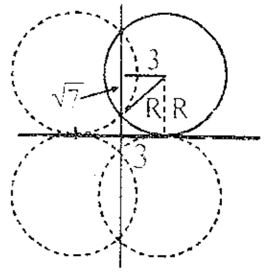
$\Rightarrow g = 7\ \&\ f = -1$

centre is $(-g, -f) \equiv (-7, 1)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{49+1-1} = 7$

19. Ans. (A, C)

Sol. As per figure,



$R^2 = 3^2 + \left(\sqrt{7}\right)^2$

$\Rightarrow R = 4$

$\therefore$ centre $\equiv (3,4)$

radius 4

$\therefore$ equation $x^2 + y^2 - 6x - 8y + 9 = 0$

such a circle can lie in all 4 quadrants as shown in figure.

$\therefore$ equation can be $x^2 + y^2 \pm 6x \pm 8y + 9 = 0$

20. Ans. (A)

Sol. Let mid point be (h, k).

then chord of contact :

$hx + ky = h^2 + k^2$ ....(i)

Let any point on the line $4x - 5y = 20$ be

$\left(x_1, \frac{4x_1 - 20}{5}\right)$

$\therefore$ Chord of contact :

$5x_1x + (4x_1 - 20)y = 45$ ....(ii)

(i) and (ii) are same

$\therefore \quad \frac{5x_1}{h} = \frac{4x_1 - 20}{k} = \frac{45}{h^2+k^2}$

$\Rightarrow \quad x_1 = \frac{9h}{h^2+k^2}$ and

$x_1 = \frac{45k + 20(h^2+k^2)}{4(h^2+k^2)}$

$\Rightarrow \quad \frac{9h}{h^2+k^2} = \frac{45k+20(h^2+k^2)}{4(h^2+k^2)}$

$\Rightarrow \quad 20(h^2+k^2) - 36h + 45k = 0$

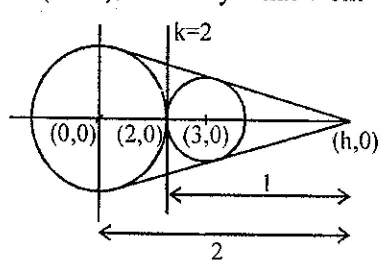
$\therefore$ Locus is $20(x^2+y^2) - 36x + 45y = 0$

21. Ans. (D)

Sol. $h = \frac{2\times 3 - 1\times 0}{2-1} = 6$

equation of tangents from (6, 0) :

$y - 0 = m(x-6) \Rightarrow y - mx + 6m = 0$



use p = r

$$\left|\frac{6m}{\sqrt{1+m^2}}\right| = 2 \Rightarrow 36m^2 = 4 + 4m^2$$

$$32m^2 = 4$$

$$m^2 = 1/8 \quad \Rightarrow m = \pm\frac{1}{2\sqrt{2}}$$

at $m = -\frac{1}{2\sqrt{2}}$

equation of tangent will be $x + 2\sqrt{2}y = 6$.

22. Ans. (A)

Sol. Equation of tangent at P will be $\sqrt{3}x + y = 4$

Slope of line L will be $\frac{1}{\sqrt{3}}$

Let equation of L be : $y = \frac{x}{\sqrt{3}} + c$

$\Rightarrow x - \sqrt{3}y + \sqrt{3}c = 0$

Now this L is tangent to 2nd circle

So $\frac{3+\sqrt{3}c}{2} = \pm 1 \quad \Rightarrow c = -\frac{1}{\sqrt{3}}$ or $c = -\frac{5}{\sqrt{3}}$

using $c = -\frac{1}{\sqrt{3}}$

$y = \frac{x}{\sqrt{3}} - \frac{1}{\sqrt{3}} \qquad \Rightarrow x - \sqrt{3}y = 1$

23. Ans. (D)

Sol. Family of circle which touches y-axis at (0, 2) is

$x^2 + (y-2)^2 + \lambda x = 0$

Passing through (–1,0)

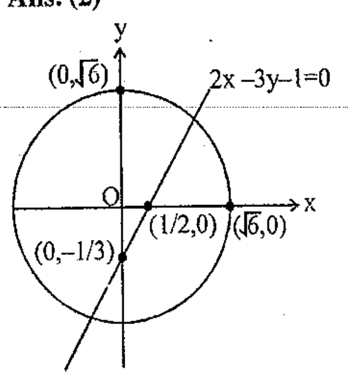
$\Rightarrow \quad 1 + 4 - \lambda = 0 \qquad \Rightarrow \quad \lambda = 5$

$\therefore \quad x^2 + y^2 + 5x - 4y + 4 = 0$

which satisfy the point (–4, 0)

24. Ans. (2)

Sol.



If the point lies inside the smaller part, then origin and point should give opposite signs w.r.t. line & point should lie inside the circle.

for origin : $2 \times 0 - 3 \times 0 - 1 = -1$ (–ve)

for $(2, \frac{3}{4})$: $2 \times 2 - 3 \times \frac{3}{4} - 1 = \frac{3}{4}$ (+ve);

point lies inside the circle

for $(\frac{5}{2}, \frac{3}{4})$: $2 \times \frac{5}{2} - 3 \times \frac{3}{4} - 1 = \frac{7}{4}$ (+ve) ;

point lies outside the circle

For $\left(\frac{1}{4}, -\frac{1}{4}\right)$: $2 \times \frac{1}{4} - 3\left(-\frac{1}{4}\right) - 1 = \frac{1}{4}$ (+ve) ;

point lies inside the circle

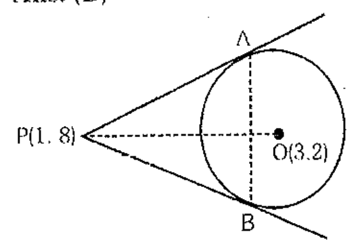
For $\left(\frac{1}{8}, \frac{1}{4}\right)$: $2 \times \frac{1}{8} - 3\left(\frac{1}{4}\right) - 1 = \frac{-3}{2}$ (–ve) ;

point lies inside the circle.

$\therefore$ 2 points lie inside smaller part.

25. Ans. (B)

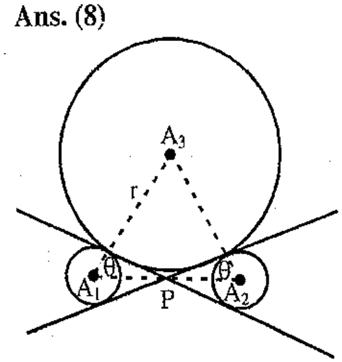
Sol.



The required circle is a circle described on OP as diameter.

26. Ans. (8)

Sol.



In triangle $A_1A_2A_3$

$A_1A_3 = A_3A_2$

Let angle $A_3A_1A_2 = \theta$, $\cos\theta = \frac{1}{3}$,

$\sin\theta = \frac{2\sqrt{2}}{3}$

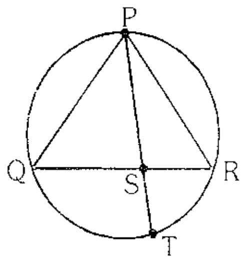
Apply sine rule in triangle $A_1A_2A_3$

$$\frac{6}{\sin(\pi - 2\theta)} = \frac{r+1}{\sin\theta}$$

$\Rightarrow r = 8$

27. Ans. (B, D)

Sol.



PS. ST. = SQ.SR

Now G.M. $\geq$ H.M

$\sqrt{PS.ST} \geq \frac{2PS \cdot ST}{PS + ST}$

$\frac{PS + ST}{PS \cdot ST} \geq \frac{2}{\sqrt{PS \cdot ST}}$

$\frac{1}{PS} + \frac{1}{ST} \geq \frac{2}{\sqrt{QS.SR}}$

Now $SQ + SR = QR$

$SQ + SR \geq 2\sqrt{SQ \cdot SR}$

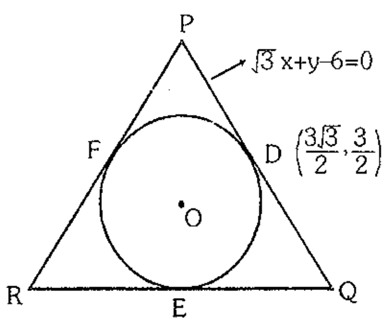
$QR \geq 2\sqrt{SQ \cdot SR}$

$\frac{2}{\sqrt{SQ \cdot SR}} \geq \frac{4}{QR}$

$\therefore \quad \frac{1}{PS} + \frac{1}{ST} \geq \frac{4}{QR}$

28. Ans. (D)

Sol.



$m = -\sqrt{3}$

so slope of OD $= \frac{1}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\therefore \frac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{y - \frac{3}{2}}{\frac{1}{2}} = \pm 1$

$(2\sqrt{3}, 2)$(not possible) & $(\sqrt{3}, 1)$

hence circle is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

29. Ans. (A)

Sol. For point E $\frac{x - \sqrt{3}}{-\frac{\sqrt{3}}{2}} = \frac{y - 1}{\frac{1}{2}} = 1$

$\therefore \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

For point F $\frac{x - \sqrt{3}}{0} = \frac{y - 1}{-1} = 1$ $(\sqrt{3}, 0)$

30. Ans. (D)

Sol. Eq. of line RP $\quad y = 0$

Eq. of line QR $\quad y - \frac{3}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)$

$y = \sqrt{3}\,x$

31. Ans. (C)

Sol. Eq. of circle is $(x + 3)^2 + (y - 5)^2 = 4$

Distance between the given lines

$= \frac{6}{\sqrt{5}}$ < radius

So, S(II) is false & S(I) is true

PARABOLA

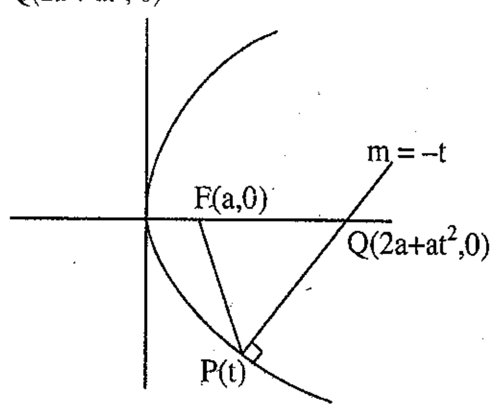
1. Ans. (A)

Sol. Let point P $(at^2, 2at)$

normal at P is $y = -tx + 2at + at^3$

$y = 0,\ x = 2a + at^2$

$Q(2a + at^2, 0)$



Area of $\Delta PFQ = \left|\frac{1}{2}(a + at^2)(2at)\right| = 120$

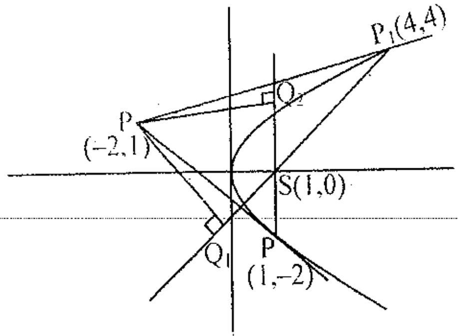
$\because m = -t$

$\because \ a^2[1 + m^2]\, m = 120$

$(a, m) = (2, 3)$ will satisfy

2. Ans. (B, C, D)

Sol. Let equation of tangent with slope 'm' be



$T : y = mx + \dfrac{1}{m}$

T : passes through (–2, 1) so

$1 = -2m + \dfrac{1}{m}$

$\Rightarrow m = -1$ or $m = \dfrac{1}{2}$

Points are given by $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

So, one point will be (1, –2) & (4, 4)

Let $P_1(4, 4)$ & $P_2(1, -2)$

$P_1S: 4x - 3y - 4 = 0$

$P_2S : x - 1 = 0$

$PQ_1 = \left|\dfrac{4(-2)-3(1)-4}{5}\right| = 3$

$SP = \sqrt{10}$; $PQ_2 = 3$; $SQ_1 = 1 = SQ_2$

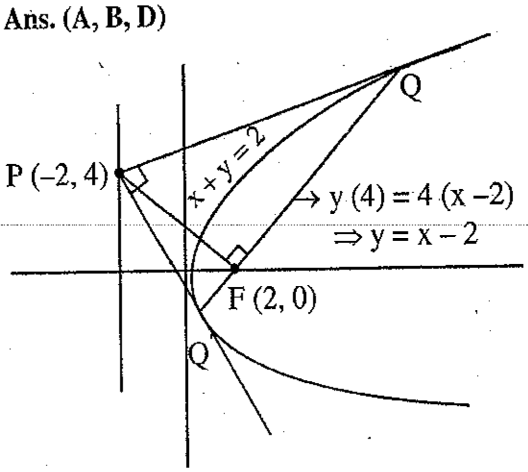
$\dfrac{1}{2}\left(\dfrac{Q_1Q_2}{2}\right) \times \sqrt{10} = \dfrac{1}{2} \times 3 \times 1$

(comparing Areas)

$\Rightarrow Q_1Q_2 = \dfrac{2\times 3}{\sqrt{10}} = \dfrac{3\sqrt{10}}{5}$

3. Ans. (A, B, D)

Sol.



Note that P lies on directrix so triangle PQQ' is right angled, hence QQ' passes through focus F.

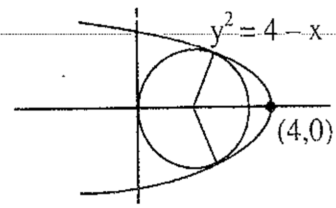
$PF = 4\sqrt{2}$

Equation of QF is $y = x - 2$ & PF is $x + y = 2$

Hence A, B, D.

4. Ans. (1.50)

Sol.



Let the circle be

$x^2 + y^2 + \lambda x = 0$

For point of intersection of circle & parabola

$y^2 = 4 - x.$

$x^2 + 4 - x + \lambda x = 0 \Rightarrow x^2 + x(\lambda - 1) + 4 = 0$

For tangency : $\Delta = 0 \Rightarrow (\lambda - 1)^2 - 16 = 0$

$\Rightarrow \lambda = 5$ (rejected) or $\lambda = -3$

Circle : $x^2 + y^2 - 3x = 0$

Radius $= \dfrac{3}{2} = 1.5$

5. Ans. (2.00)

Sol. For point of intersection :

$x^2 - 4x + 4 = 0 \Rightarrow x = 2$ so $\alpha = 2$

6. Ans. (D)

Sol. Equation of chord with mid point (h, k) :

$$k.y - 16\left(\frac{x+h}{2}\right) = k^2 - 16h$$

$\Rightarrow \quad 8x - ky + k^2 - 8h = 0$

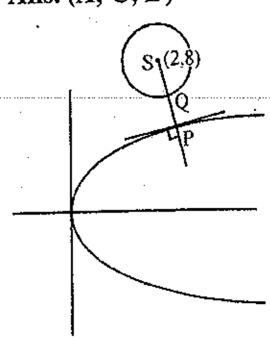
Comparing with $2x + y - p = 0$, we get

$k = -4$; $2h - p = 4$

only (D) satisfies above relation.

7. Ans. (A, C, D)

Sol.



$y^2 = 4x$

point P lies on normal to parabola passing through centre of circle

$y + tx = 2t + t^3$ ....(i)

$8 + 2t = 2t + t^3$

$t = 2$

$P(4, 4)$

$SP = \sqrt{(4-2)^2 + (4-8)^2}$

$SP = 2\sqrt{5}$

$SQ = 2$

$\Rightarrow \quad PQ = 2\sqrt{5} - 2$

$\frac{SQ}{QP} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$

To find x intercept

put $y = 0$ in (i)

$\Rightarrow \quad x = 2 + t^2$

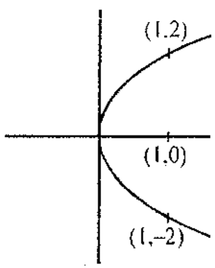
$x = 6$

$\because \quad$ Slope of common normal $= -t = -2$

$\therefore \quad$ Slope of tangent $= \frac{1}{2}$

8. **Ans. (2)**

Sol.



The co-ordinates of latus rectum are (1,2) and (1,–2)

clearly slope of tangent is given by $\frac{dy}{dx} = \frac{2}{y}$

$\therefore$ At $y = 2$ slope of normal $= -1$

and At $y = -2$ slope of normal $= 1$

$\therefore$ Equation of normal at (1,2)

$(y - 2) = -1(x - 1) \Rightarrow x + y = 3$

Now, this line is tangent to circle

$(x - 3)^2 + (y + 2)^2 = r^2$

$\therefore$ perpendicular distance from centre to line

= Radius of circle

$\therefore \frac{|3-2-3|}{\sqrt{2}} = r \Rightarrow r^2 = 2$

9. **Ans. (4)**

Sol. Let there be a point $(t^2, 2t)$ on $y^2 = 4x$

Clearly its reflection in $x + y + 4 = 0$ is given by

$\frac{x-t^2}{1} = \frac{y-2t}{1} = \frac{-2(t^2+2t+4)}{2}$

$\therefore \quad x = -(2t + 4)$ & $y = -(t^2 + 4)$

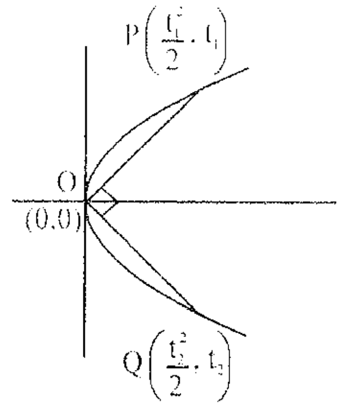
Now, $y = -5 \quad \Rightarrow \quad t = \pm 1$

$\therefore \quad x = -6 \quad$ or $\quad x = -2$

$\therefore \quad$ Distance between A & B = 4

10. **Ans. (A, D)**

Sol.



$\because \angle POQ = \frac{\pi}{2} \qquad \Rightarrow \qquad t_1t_2 = -4$

$$\therefore \quad \left| \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ \frac{t_1^2}{2} & t_1 & 1 \\ \frac{t_2^2}{2} & t_2 & 1 \end{vmatrix} \right| = 3\sqrt{2}$$

$$\Rightarrow \quad \left| \frac{t_1^2t_2 - t_1t_2^2}{2} \right| = 6\sqrt{2}$$

$$\Rightarrow \quad |t_1 - t_2| = 3\sqrt{2}$$

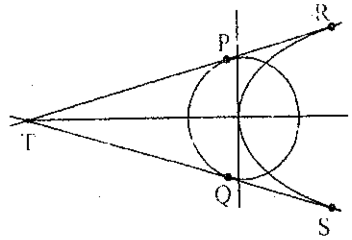
$$\Rightarrow \quad t_1 + \frac{4}{t_1} = 3\sqrt{2} \qquad (\because \quad t_1 > 0)$$

We get $t_1 = 2\sqrt{2}, \sqrt{2}$

$P(4, 2\sqrt{2})$ or $(1, \sqrt{2})$

11. Ans. (D)

Sol.



$$y = mx + \frac{2}{m}$$

$$\frac{\left|0 - 0 + \frac{2}{m}\right|}{\sqrt{1+m^2}} = \sqrt{2} \Rightarrow 2 = m^2(1+m^2)$$

$\Rightarrow m = \pm 1$

TP : $-x + y = 2$

So $P(-1, 1)$ & $Q(-1, -1)$

& $R\left(\frac{2}{m}, \frac{4}{m}\right) = R(2,4)$ & $S(2,-4)$

So $\Delta = \frac{1}{2}10.3 = 15$

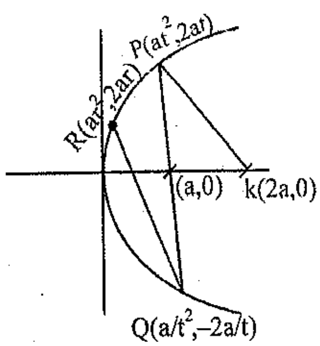
So $P(-1, 1)$ & $Q(-1, -1)$

& $R\left(\frac{2}{m}, \frac{4}{m}\right) \equiv R(2,4)$ & $S(2,-4)$

So $\Delta = \frac{1}{2}10.3 = 15$

12. Ans. (D)

Sol.



$\because$ PQ is a focal chord

$\therefore$ co-ordinates of point Q are $= \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

$$m_{QR} = \frac{2a\left(r + \frac{1}{t}\right)}{a\left(r^2 - \frac{1}{t^2}\right)} = \frac{2}{\left(r - \frac{1}{t}\right)}$$

$$m_{PK} = \frac{2at - 0}{a\left(t^2 - 2\right)} = \frac{2t}{t^2 - 2}$$

Given $m_{QR} = m_{PK}$

$$\Rightarrow \frac{2}{r - \frac{1}{t}} = \frac{2t}{t^2-2} \Rightarrow r = \frac{t^2-2}{t} + \frac{1}{t}$$

$$\Rightarrow r = t - \frac{2}{t} + \frac{1}{t} \Rightarrow r = \frac{t^2-1}{t}$$

13. Ans. (B)

Sol. Equation of tangent at point P is

$ty = x + at^2$ ....(i)

Equation of normal at point S is

$$\frac{1}{t}x + y = \frac{2a}{t} + \frac{a}{t^3}$$

$\Rightarrow \quad t^2x + t^3y = 2at^2 + a$ ....(ii)

Multiply equation (i) by t^2 and then subtract from equation (ii),

we get,

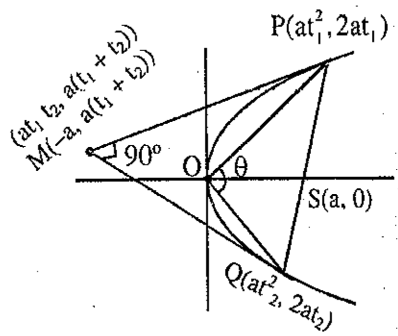
$2t^3y = 2at^2 + at^4 + a$

$\Rightarrow 2t^3y = a(1 + t^4 + 2t^2)$

$$\Rightarrow \boxed{y = \frac{a(1+t^2)^2}{2t^3}}$$

14. Ans. (D)

Sol.



Single tangent at the extrimities of a focal chord will intersect on directrix.

$\therefore \quad M(-a, a(t_1 + t_2))$

lies on $y = 2x + a$

$a(t_1 + t_2) = -2a + a \quad \Rightarrow \quad t_1 + t_2 = -1$

& $\quad t_1t_2 = -1$

$$\tan\theta = \left(\frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1t_2}}\right) = \left(\frac{2(t_2 - t_1)}{3}\right)$$

$\because \quad (t_2 - t_1)^2 = (t_2 + t_1)^2 - 4t_1t_2 = 5$

$t_2 - t_1 = \pm\sqrt{5}$

$\therefore \quad \tan\theta = \pm\frac{2\sqrt{5}}{3}$

but θ is obtuse because O is the interior point of the circle for which PQ is diameter.

$\therefore \quad \tan\theta = \frac{-2\sqrt{5}}{3}$

15. Ans. (B)

Sol. Length of focal chord

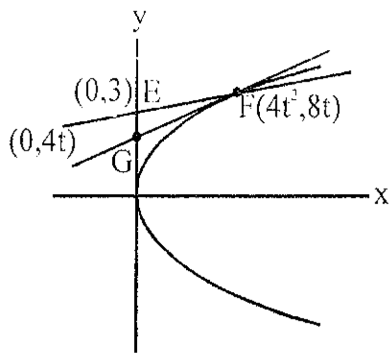
$PQ = a(t_1 - t_2)^2$

$= a|(t_1 + t_2)^2 - 4t_1t_2|$

$= a|1 + 4| = 5a$

16. Ans. (A)

Sol.



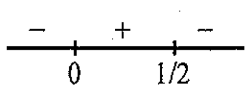
Let $F(4t^2, 8t)$

where $0 \le 8t \le 6 \Rightarrow \quad 0 \le t \le \frac{3}{4}$

$\Delta EFG = \frac{1}{2}(3-4t)4t^2$

$\Delta = (6t^2 - 8t^3)$

$$\frac{d\Delta}{dt} = 12t - 24t^2 = 0 \begin{cases} t = 0 \text{ (minima)} \\ t = \frac{1}{2} \text{ (maxima)} \end{cases}$$



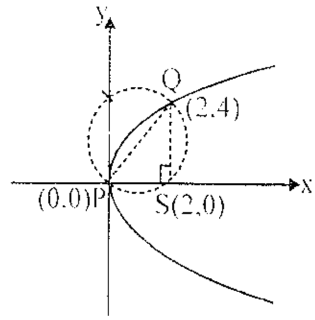
$\Rightarrow \quad m = \frac{8t-3}{4t^2-0} = \frac{4-3}{1} = 1$

$(\Delta EFG)_{max} = \frac{6}{4} - 1 = \frac{1}{2}$

$y_0 = 8t = 4$ & $y_1 = 4t = 2$

17. Ans. (4)

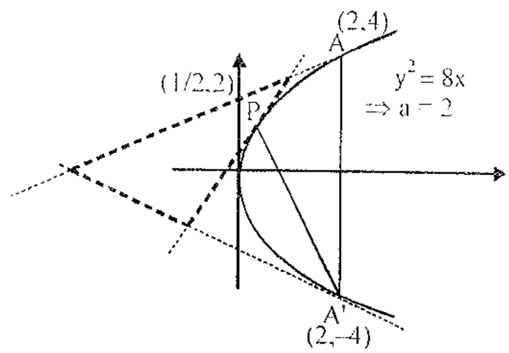
Sol. Focus of parabola S(2,0) points of intersection of given curves : (0,0) and (2,4).



Area $(\Delta PSQ) = \frac{1}{2}.2.4 = 4$ sq. units

18. Ans. (2)

Sol.



Δ_1 = area of Δ PAA'

$= \frac{1}{2}.8.\frac{3}{2} = 6$

$\Delta_2 = \frac{1}{2}(\Delta_1)$

(Using property : Area of triangle formed by tangents is always half of original triangle)

$\Rightarrow \quad \frac{\Delta_1}{\Delta_2} = 2$

19. Ans. (C)

Sol. Let P be (h, k)

on using section formula $P\left(\frac{x}{4}, \frac{y}{4}\right)$

$\therefore \quad h = \frac{x}{4}$ and $k = \frac{y}{4}$

$\Rightarrow \quad x = 4h$ and $y = 4k$

$\because \quad (x, y)$ lies on $y^2 = 4x$

$\therefore \quad 16k^2 = 16h \Rightarrow \quad k^2 = h$

Locus of point P is $y^2 = x$.

20. Ans. (A, B, D)

Sol. Equation of normal is $y = mx - 2m - m^3$

It passes through the point (9, 6) then

$6 = 9m - 2m - m^3$

$\Rightarrow \quad m^3 - 7m + 6 = 0$

$\Rightarrow \quad (m-1)(m-2)(m+3) = 0$

$\Rightarrow \quad m = 1, 2, -3$

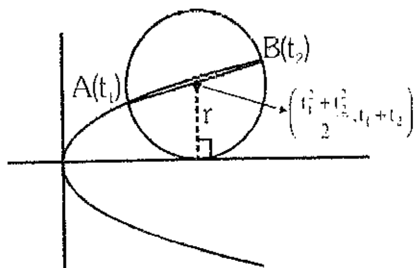
Equations of normals are

$y - x + 3 = 0,\ y + 3x - 33 = 0$ &

$y - 2x + 12 = 0$

21. Ans. (C, D)

Sol.



$t_1 + t_2 = r$

$\frac{2}{r} = \frac{2}{t_1 + t_2}$

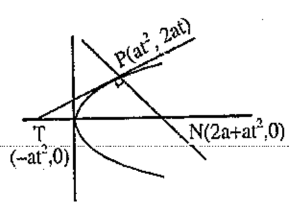
similarly $-\frac{2}{r}$ is also possible

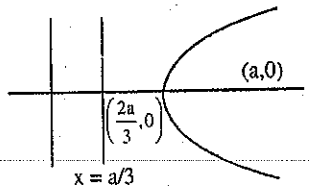
22. Ans. (A, D)

Sol. $3h = 2a + at^2$

$3k = 2at$

$3h = 2a + \frac{a \cdot 9k^2}{4a^2}$



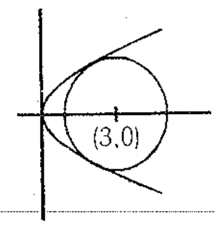


$y^2 = \frac{4a}{9}(3x - 2a)$

$y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

23. Ans. (B)

Sol. $C_1 : y^2 = 4x$ $C_2 : x^2 + y^2 - 6x + 1 = 0$



$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0 \qquad \Rightarrow x = 1$

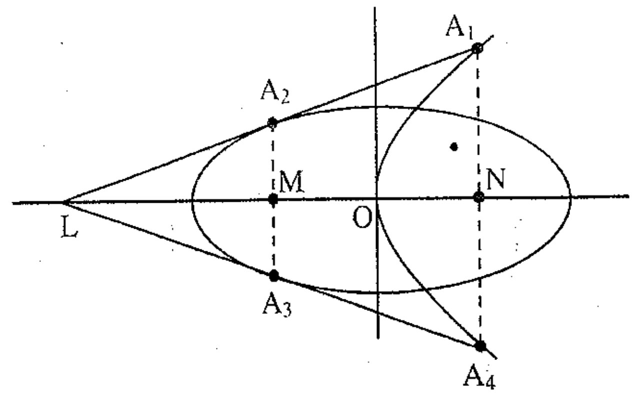
$y = \pm 2$

so the curves touches each other at two points (1, 2) & (1, –2)

ELLIPSE

1. Ans. (A, C)

Sol



$y = mx + \frac{3}{m}$

$C^2 = a^2m^2 + b^2$

$\frac{9}{m^2} = 6m^2 + 3 \qquad \Rightarrow m^2 = 1$

T_1 & T_2

$y = x + 3,\ y = -x - 3$

Cuts x-axis at (–3, 0)

$A_1(3, 6) \qquad A_4(3, -6)$

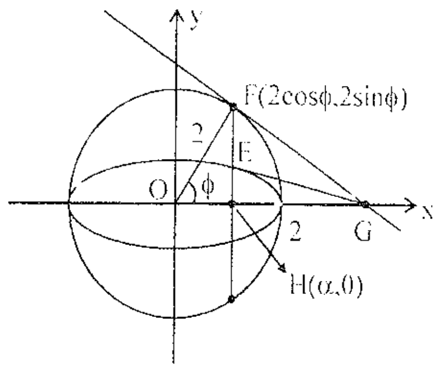
$A_2(-2, 1) \qquad A_3(-2, -1)$

$A_1A_4 = 12,\quad A_2A_3 = 2,\quad MN = 5$

Area $= \frac{1}{2}(12 + 2) \times 5 = 35$ sq.unit

2. Ans. (C)

Sol.



Let $F(2\cos\phi, 2\sin\phi)$

& $E(2\cos\phi, \sqrt{3}\sin\phi)$

$$EG : \frac{x}{2}\cos\phi + \frac{y}{\sqrt{3}}\sin\phi = 1$$

$$\therefore G\left(\frac{2}{\cos\phi}, 0\right) \text{ and } \alpha = 2\cos\phi$$

$$ar(\Delta FGH) = \frac{1}{2}\ HG \times FH$$

$$= \frac{1}{2}\left(\frac{2}{\cos\phi} - 2\cos\phi\right) \times 2\sin\phi$$

$f(\phi) = 2\tan\phi\sin^2\phi$

$$\therefore \text{(I) } f\left(\frac{\pi}{4}\right) = 1 \quad \text{(II) } f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$

$$\text{(III) } f\left(\frac{\pi}{6}\right) = \frac{1}{2\sqrt{3}}$$

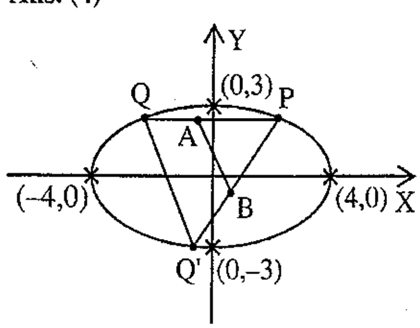
$$\text{(IV) } f\left(\frac{\pi}{12}\right) = 2\left(2-\sqrt{3}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$$

$$= \left(4-2\sqrt{3}\right)\frac{\left(\sqrt{3}-1\right)^2}{8} = \frac{\left(\sqrt{3}-1\right)^4}{8}$$

$\therefore$ (I) $\rightarrow$ (Q) ; (II) $\rightarrow$ (T) ; (III) $\rightarrow$ (S) ; (IV) $\rightarrow$ (P)

3. Ans. (4)

Sol.



A and B be midpoints of segment PQ and PQ' respectively

AB = distance between M(P, Q) and

$M(P, Q') = \frac{1}{2}.QQ'$

Since, Q, Q' must be on E, so,

maximum of QQ' = 8

$\therefore$ Maximum of AB $= \frac{8}{2} = 4$

4. Ans. (A)

Sol. $y^2 = 4\lambda x$, $P(\lambda, 2\lambda)$

Slope of the tangent to the parabola at point P

$$\frac{dy}{dx} = \frac{4\lambda}{2y} = \frac{4\lambda}{2 \times 2\lambda} = 1$$

Slope of the tangent to the ellipse at P

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

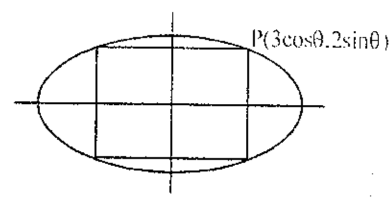
As tangents are perpendicular $y' = -1$

$$\Rightarrow \frac{2\lambda}{a^2} - \frac{4\lambda}{b^2} = 0 \Rightarrow \frac{a^2}{b^2} = \frac{1}{2}$$

$$\Rightarrow e = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

5. Ans. (C, D)

Sol.



Area of $R_1 = 3\sin 2\theta$; for this to be maximum

$$\Rightarrow \theta = \frac{\pi}{4} \Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$$

Hence for subsequent areas of rectangles R_n to be maximum the coordinates will be in GP with common ratio

$$r = \frac{1}{\sqrt{2}} \Rightarrow a_n = \frac{3}{\left(\sqrt{2}\right)^{n-1}} \ ; \ b_n = \frac{3}{\left(\sqrt{2}\right)^{n-1}}$$

Eccentricity of all the ellipses will be same

Distance of a focus from the centre in $E_9 = a_9e_9$

$$= \sqrt{a_9^2 - b_9^2} = \frac{\sqrt{5}}{16}$$

$$\text{Length of latus rectum of } E_9 = \frac{2b_9^2}{a_9} = \frac{1}{6}$$

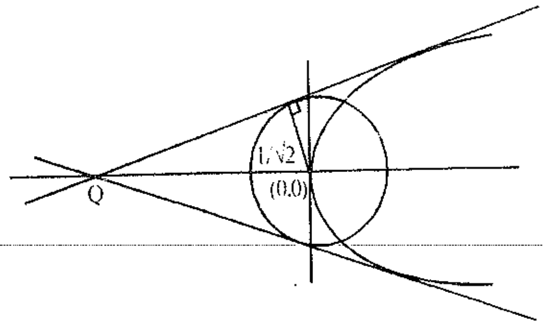
$$\because \sum_{n=1}^{\infty} \text{Area of } R_n = 12 + \frac{12}{2} + \frac{12}{4} + \ldots\ldots\infty = 24$$

$$\Rightarrow \sum_{n=1}^{N}\left(\text{area of } R_n\right) < 24,$$

for each positive integer N

6. Ans. (A, C)

Sol.



Let equation of common tangent is

$$y = mx + \frac{1}{m}$$

$$\therefore \left|\frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

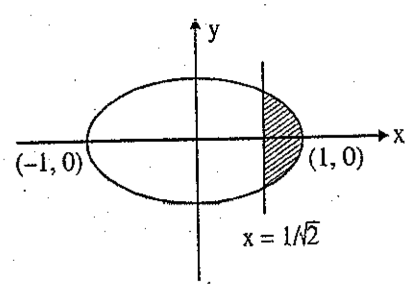
Equation of common tangents are $y = x + 1$ and $y = -x - 1$

point Q is $(-1, 0)$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $LR = \frac{2b^2}{a} = 1$

(C)



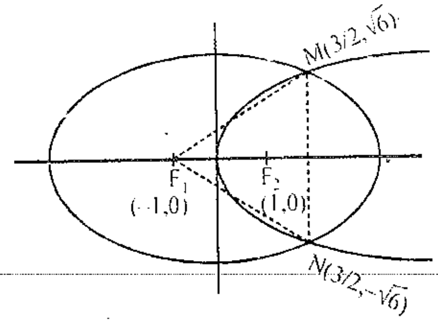
$$\text{Area} = 2.\int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}.\sqrt{1-x^2}\,dx$$

$$= \sqrt{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi - 2}{4\sqrt{2}}$$

7. Ans. (A)

Sol.



Orthocentre lies on x-axis

Equation of altitude through M :

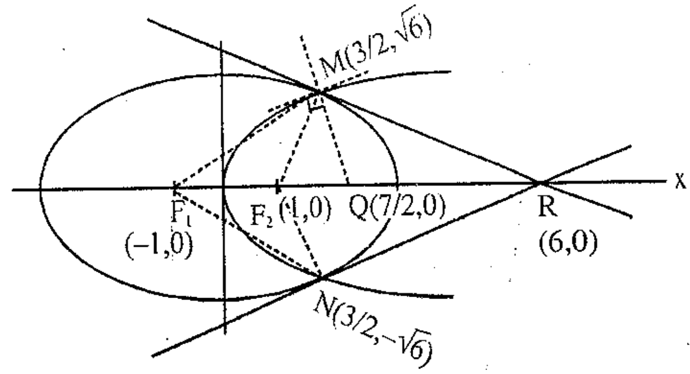
$$y - \sqrt{6} = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right)$$

Equation of altitude through F_1 : $y = 0$

solving, we get orthocentre $\left(-\frac{9}{10}, 0\right)$

8. Ans. (C)

Sol.



Normal to parabola at M :

$$y - \sqrt{6} = -\frac{\sqrt{6}}{2.1}\left(x - \frac{3}{2}\right)$$

Solving it with $y = 0$, we get $Q \equiv \left(\frac{7}{2}, 0\right)$

Tangent to ellipse at M : $\frac{x.\frac{3}{2}}{9} + \frac{y(\sqrt{6})}{8} = 1$

Solving it with $y = 0$, we get $R \equiv (6, 0)$

$\therefore$ Area of triangle MQR

$$= \frac{1}{2}.\left(6 - \frac{7}{2}\right).\sqrt{6} = \frac{5\sqrt{6}}{4}$$

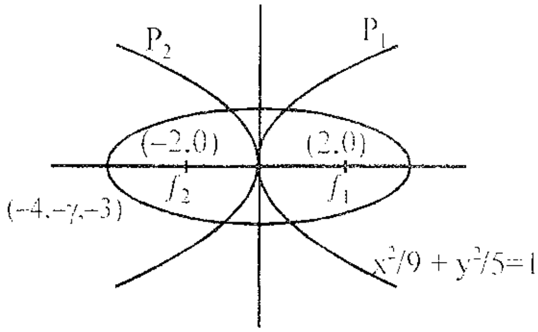
Area of quadrilateral MF_1NF_2

$$= 2.\frac{1}{2}.(1-(-1)).\sqrt{6} = 2\sqrt{6}$$

Required ratio = 5 : 8

9. Ans. (4)

Sol.



$\therefore$ P_1 is $y^2 = 8x$

P_2 is $y^2 = -16x$

Let equation of tangent at $(2t^2, 4t)$

$\therefore y = m_1x + \frac{2}{m_1}$

If passes through $(-4,0)$

$\therefore -4m_1 + \frac{2}{m_1} = 0$

$\therefore m_1^2 = \frac{1}{2}$

equation of tangent to P_2

$$y = m_2x + \frac{(-4)}{m_2}$$

It passes through (2, 0), $2m_2 - \frac{4}{m_2} = 0$

$\Rightarrow m_2^2 = 2$

$\therefore \frac{1}{m_1^2} + m_2^2 = 4$

10. **Ans. (A, B)**

Sol. Let $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$

& $E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$ $(c < d)$

& $S : x^2 + (y-1)^2 = 2$

& tangent to E_1, E_2 & S is $x + y = 3$

Now, point of contact of S & tangent is (x_1, y_1)

Let $x = X$ & $y - 1 = Y$

$\therefore$ $X^2 + Y^2 = 2$

& $X + Y = 2$

Let (X_1, Y_1) be point of contact.

$\therefore$ $XX_1 + YY_1 = 2$

$\therefore$ $X_1 = 1$ & $Y_1 = 1$

$\therefore$ $x_1 = 1$ & $y_1 = 2$

Now, parametric equation of $x + y = 3$

is $\frac{x-1}{-\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm\frac{2\sqrt{2}}{3}$ $\Rightarrow x = \frac{5}{3}$, $y = \frac{4}{3}$

$\Rightarrow$ $x = \frac{1}{3}$ & $y = \frac{8}{3}$

$\therefore$ $P = (1, 2)$, $Q = \left(\frac{5}{3}, \frac{4}{3}\right)$ & $R = \left(\frac{1}{3}, \frac{8}{3}\right)$

Now, equation tangent at Q on ellipse E_1

$\frac{x.5}{3a^2} + \frac{y.4}{3b^2} = 1$ Comparing it with $x + y = 3$

$\therefore$ $a^2 = 5$ & $b^2 = 4$ Now, $e_1^2 = 1 - \frac{4}{5} = \frac{1}{5}$

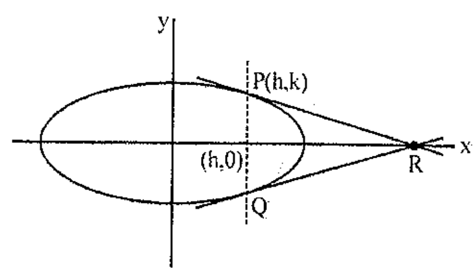
Similarly, $e_2^2 = \frac{7}{8}$

$\therefore$ $e_1^2e_2^2 = \frac{7}{40}$ $\Rightarrow$ $e_1e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

$e_1^2 + e_2^2 = \frac{1}{5} + \frac{7}{8} = \frac{43}{40}$; $|e_1^2 - e_2^2| = \left|\frac{1}{5} - \frac{7}{8}\right| = \frac{27}{40}$

11. **Ans. (9)**

Sol.



Tangent at P(h, k) is $\frac{xh}{4} + \frac{ky}{3} = 1$

$\Rightarrow R\left(\frac{4}{h}, 0\right)$

$\Delta PQR = k\left(\frac{4}{h} - h\right)$

$= \sqrt{3\left(1 - \frac{h^2}{4}\right)}\left(\frac{4}{h} - h\right)$

which is a decreasing function in $\left[\frac{1}{2}, 1\right]$

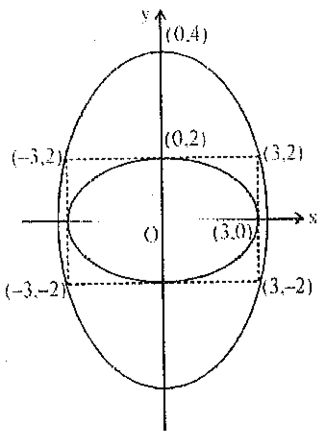
$$\Rightarrow \Delta_1 = \sqrt{3\left(1-\frac{1}{16}\right)}\left(8-\frac{1}{2}\right) = \frac{45\sqrt{5}}{8}$$

$$\& \ \Delta_2 = \sqrt{3\left(1-\frac{1}{4}\right)}(4-1) = \frac{9}{2}$$

$$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$$

12. Ans. (C)

Sol.



Let equation of E_2 be

$\frac{x^2}{a^2} + \frac{y^2}{16} = 1$ ($\because$ E_2 passes through (0, 4))

$\because$ E_2 passes through (3,2)

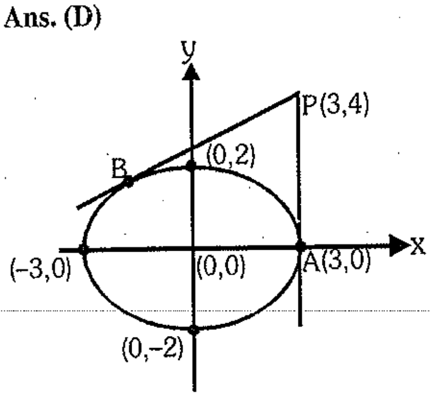
$\therefore \quad \frac{9}{a^2} + \frac{4}{16} = 1$

$\Rightarrow \quad a^2 = 12$

$\therefore \quad e^2 = 1 - \frac{a^2}{16} = 1 - \frac{3}{4} \Rightarrow e = \frac{1}{2}$

13. Ans. (D)

Sol.



As shown in figure, one of the point of contact is (3,0)

Let equation of other tangent,

$y = mx + \sqrt{9m^2+4}$ as $c > 0$

It passes through (3,4), so

$$4 = 3m + \sqrt{9m^2+4}$$

$$(4-3m)^2 = 9m^2 + 4$$

Solving, $m = \frac{1}{2}$

As we know that point of contact for the tangent given by

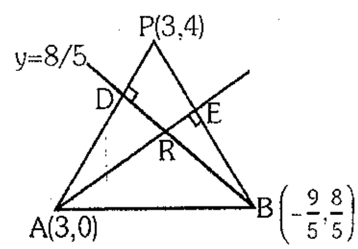
$\therefore$ Point of contact is

$$\left(-\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

$\therefore$ Point of contact is $\left(-\frac{9}{5}, \frac{8}{5}\right)$

14. Ans. (A)

Sol. Equation of line BD : $y = \frac{8}{5}$



Equation of line AE : $2x + y = 6$

Now orthocentre R of ΔPAB will be intersection of line BD and line AE.

Solving for R, we get $R \equiv \left(\frac{11}{5}, \frac{8}{5}\right)$

15. Ans. (A)

Sol. Equation of line AB is $x + 3y = 3$

Now let the point be (h,k)

According to question,

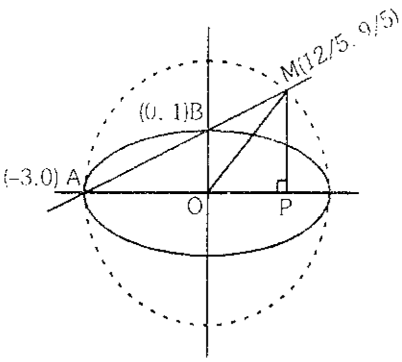
$$\left|\frac{h+3k-3}{\sqrt{1^2+3^2}}\right| = \sqrt{(h-3)^2+(4-k)^2}$$

After solving, we get

$$9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$$

16. Ans. (D)

Sol.



Area of triangle AOM $= \frac{1}{2}$ AO. PM

$\Rightarrow$ Equation of AM is

$y = \frac{1}{3}(x+3)$

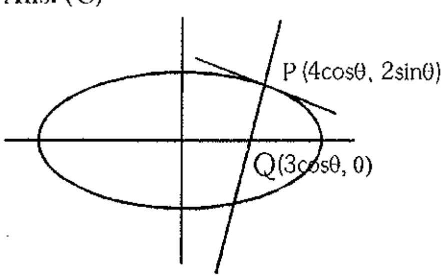
$x - 3y + 3 = 0$ which is chord of auxiliary circle $x^2 + y^2 = 9$, and PM is ordinate of point M

$\Rightarrow (3y-3)^2 + y^2 = 9 \Rightarrow y = \frac{9}{5} = PM$

$\Rightarrow$ Area of triangle $= \frac{1}{2} \cdot 3 \cdot \frac{9}{5} = \frac{27}{10}$

17. Ans. (C)

Sol.



$\frac{x^2}{16} + \frac{y^2}{4} = 1$

$4x\sec\theta - 2y\,\text{cosec}\theta = 12$

$x = 3\cos\theta$

$Q \equiv (3\cos\theta, 0)$

$2h = 7\cos\theta$

$2k = 2\sin\theta$

$\frac{4x^2}{49} + \frac{y^2}{1} = 1 \qquad ...(i)$

L.R $\Rightarrow x = 2\sqrt{3}$

Putting in (i)

$y = \pm\frac{1}{7}$

$\therefore \left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$

18. Ans. (B, C)

Sol. Eq. of ellipse is $\frac{x^2}{4} + y^2 = 1$

eccentricity $e = \frac{\sqrt{3}}{2}$

so focus are $(\sqrt{3}, 0) \& (-\sqrt{3}, 0)$

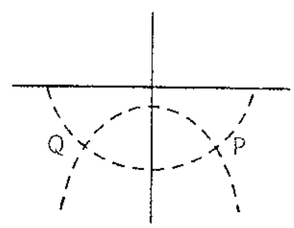
so end points of latus rectum will be

$\left(\sqrt{3}, \frac{1}{2}\right)\left(\sqrt{3}, -\frac{1}{2}\right), \left(-\sqrt{3}, \frac{1}{2}\right) \& \left(-\sqrt{3}, -\frac{1}{2}\right)$

$\because y_1 < 0 \;\; \& \; y_2 < 0$

Hence coordinates of P & Q will be

$P\left(\sqrt{3}, -\frac{1}{2}\right) \;\&\; Q\left(-\sqrt{3}, -\frac{1}{2}\right)$



So now equation of parabola taking these points as end points of latus rectum.

Focus will be $(0, -1/2)$

$4a = 2\sqrt{3} \Rightarrow \qquad a = \frac{\sqrt{3}}{2}$

Hence vertex of the parabolas will be

$\left(0, -\frac{1}{2} + \frac{\sqrt{3}}{2}\right), \left(0, -\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$

so eq. of parabolas will be

$x^2 = -2\sqrt{3}\left(y + \frac{1}{2} - \frac{\sqrt{3}}{2}\right)$ &

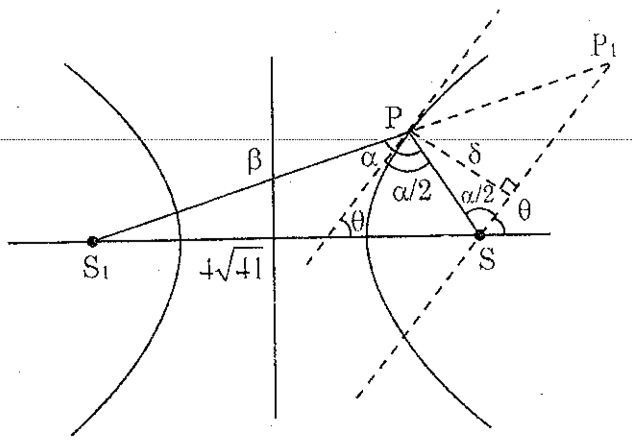
$x^2 = 2\sqrt{3}\left(y + \frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

$x^2 + 2\sqrt{3}y = 3 - \sqrt{3} \qquad x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

HYPERBOLA

1. Ans. (7)

Sol.



$S_1P - SP = 20$

$$\beta - \frac{\delta}{\sin\frac{\alpha}{2}} = 20$$

$$\beta^2 + \frac{\delta^2}{\sin^2\frac{\alpha}{2}} - 400 = \frac{2\beta\delta}{\sin\frac{\alpha}{2}}$$

$$\frac{1}{SP} = \frac{\sin\frac{\alpha}{2}}{\delta}$$

$$\cos\alpha = \frac{SP^2 + \beta^2 - 656}{2\beta\frac{\delta}{\sin\frac{\alpha}{2}}}$$

$$= \frac{\frac{2\beta\delta}{\sin\frac{\alpha}{2}} - 256}{\frac{2\beta S}{\sin\frac{\alpha}{2}}} = \cos\alpha$$

$$\frac{\lambda - 128}{\lambda} = \cos\alpha$$

$\lambda(1 - \cos\alpha) = 128$

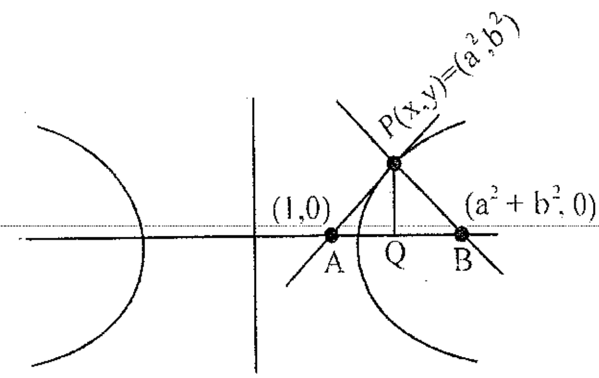
$$\frac{\beta\delta}{\sin\frac{\alpha}{2}}.2\sin^2\frac{\alpha}{2} = 128$$

$$\frac{\beta\delta}{9}\sin\frac{\alpha}{2} = \frac{64}{9} \Rightarrow \left[\frac{\beta\delta}{9}\sin\frac{\alpha}{2}\right] = 7$$

where [.] denotes greatest integer function

2. Ans. (A, D)

Sol.



Since Normal at point P makes equal intercept on co-ordinate axes, therefore slope of Normal $= -1$

Hence slope of tangent = 1

Equation of tangent

$y - 0 = 1(x - 1)$

$y = x - 1$

Equation of tangent at $(x_1 y_1)$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$x - y = 1$ (equation of Tangent)

on comparing $x_1 = a^2$, $y_1 = b^2$

Also $a^2 - b^2 = 1$ ...(1)

Now equation of normal at $(x_1, y_1) = (a^2, b^2)$

$y - b^2 = -1(x - a^2)$

$x + y = a^2 + b^2$ ...(Normal)

point of intersection with x-axis is $(a^2 + b^2)$

Now $e = \sqrt{1 + \frac{b^2}{a^2}}$

$e = \sqrt{1 + \frac{b^2}{b^2+1}}$ $\left[\text{from (1)} \frac{b^2}{b^2+1} < 1\right]$

$1 < e < \sqrt{2}$ option (A)

$\Delta = \frac{1}{2}.AB.PQ$

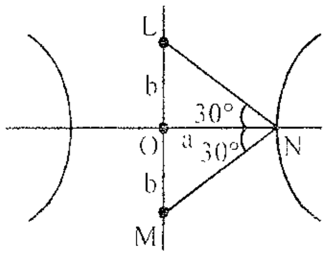
and $\Delta = \frac{1}{2}\left(a^2 + b^2 - 1\right).b^2$

$\Delta = \frac{1}{2}\left(2b^2\right)b^2$ (from (1) $a^2 - 1 = b^2$)

$\Delta = b^4$ so option (D)

3. Ans. (B)

Sol.



$\tan 30° = \frac{b}{a}$

$\Rightarrow \quad a = b\sqrt{3}$

Now area of $\Delta LMN = \frac{1}{2}.2b.b\sqrt{3}$

$4\sqrt{3} = \sqrt{3}b^2$

$\Rightarrow \quad b = 2 \;\&\; a = 2\sqrt{3}$

$\Rightarrow \quad e = \sqrt{1+\frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$

P. Length of conjugate axis = 2b = 4

So $P \to 4$

Q. Eccentricity $e = \frac{2}{\sqrt{3}}$

So $Q \to 3$

R. Distance between foci = 2ae

$= 2\left(2\sqrt{3}\right)\left(\frac{2}{\sqrt{3}}\right) = 8$

So $R \to 1$

S. Length of latus rectum =

$\frac{2b^2}{a} = \frac{2(2)^2}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

So $S \to 2$

4. **Ans. (B, C, D)**

Sol. The line y = mx + c is tangent to hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 - b^2$

$\therefore (1)^2 = 4a^2 - 16 \Rightarrow a^2 = \frac{17}{4}$

$\Rightarrow a = \frac{\sqrt{17}}{2}$

For option (A), sides are $\sqrt{17}, 4, 1$

($\Rightarrow$ Right angled triangle)

For option (B), sides are $\sqrt{17}, 8, 1$

($\Rightarrow$ Triangle is not possible)

For option (C), sides are $\frac{\sqrt{17}}{2}, 4, 1$

($\Rightarrow$ Triangle is not possible)

For option (D), sides are $\frac{\sqrt{17}}{2}, 4, 2$

($\Rightarrow$ Triangle exist but not right angled)

5. Ans. (D)

Sol. $P\left(\sqrt{3}, \frac{1}{2}\right)$; tangent $\sqrt{3}x + 2y = 4$

$\Rightarrow \left(\sqrt{3}\right)x + 4\left(\frac{1}{2}\right)y = 4$ comparing with (II)

$\Rightarrow a = 2 \quad \therefore y = mx + \sqrt{a^2m^2+1}$ is tangent

for $m = -\frac{\sqrt{3}}{2}$ i.e (ii)

$\therefore$ point of contact for a = 2, $m = -\frac{\sqrt{3}}{2}$ is R

6. **Ans. (A)**

Sol. y = x + 8 is tangent $\Rightarrow$ m = 1; P(8, 16)

Comparing tangent with (i) of column 2, m = 1 satisfied and a = 8 obtained which matches for point of contact (P) of column 3 and (III) of column 1.

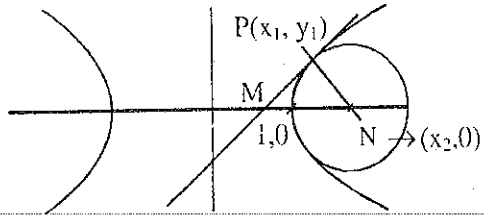
7. **Ans. (D)**

Sol. For $a = \sqrt{2}$ and point (–1,1) only I of column-1 satisfies. Hence equaiton of tangent is –x + y = 2 or y = x + 2 $\Rightarrow$ m = 1 which matches with (ii) of column-2 and also with Q of column-3

Let $f(x) = x + \log_e x - x\log_e x$, $x \in (0,\infty)$.

8. Ans. (A, B, D)

Sol. Given H : $x^2 - y^2 = 1$



Now, equation of family of circle touching hyperbola at (x_1, y_1) is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(xx_1 - yy_1 - 1) = 0$

Now, its centre is $(x_2, 0)$

$\therefore \quad 2y_1 + \lambda y_1 = 0 \quad \Rightarrow \quad \lambda = -2$

$$\therefore \quad x_2 = \frac{2x_1 - \lambda x_1}{2} = \frac{4x_1}{2} = 2x_1$$

$\therefore \quad P \equiv (x_1, \sqrt{x_1^2 - 1})$

$N = (2x_1, 0)$

$$\& \quad M \equiv \left(\frac{1}{x_1}, 0\right)$$

$$\therefore \quad \ell = \frac{3x_1 + \frac{1}{x_1}}{3} = x_1 + \frac{1}{3x_1}$$

$$\Rightarrow \quad \frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2} \qquad x_1 > 1$$

$$m = \frac{\sqrt{x_1^2 - 1}}{3} \quad \Rightarrow \quad \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

$$\text{Also } m = \frac{y_1}{3} \quad \Rightarrow \quad \frac{dm}{dy_1} = \frac{1}{3} \qquad y_1 > 0$$

9. Ans. (A, B)

Sol. Let parametric coordinates be $P(3\sec\theta, 2\tan\theta)$

Equation of tangent at point P will be

$$\frac{x\sec\theta}{3} - \frac{y\tan\theta}{2} = 1$$

$\because$ tangent is parallel to $2x - y = 1$

$$\Rightarrow \frac{2\sec\theta}{3\tan\theta} = 2 \quad \Rightarrow \sin\theta = \frac{1}{3}$$

$\therefore$ coordinates are

$$\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{9}{2\sqrt{2}}, -\frac{1}{2}\right)$$

10. Ans. (B, D)

Sol. Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

ellipse is $\frac{x^2}{2^2} + \frac{y^2}{1} = 1$

$$\text{eccentricity of ellipse} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\text{eccentricity of hyperbola} = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{4}{3}}$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{1}{3} \quad \Rightarrow 3b^2 = a^2 \qquad ....(1)$$

also hyperbola passes through foci of ellipse

$(\pm\sqrt{3}, 0)$

$$\frac{3}{a^2} = 1 \qquad \Rightarrow a^2 = 3 \qquad ....(2)$$

from (1) & (2)

$b^2 = 1$

equation of hyperbola is

$$\frac{x^2}{3} - \frac{y^2}{1} = 1 \quad \Rightarrow \quad x^2 - 3y^2 = 3$$

$$\text{eccentricity of hyperbola} = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}}$$

$$\text{focus of hyperbola} = \left(\pm\sqrt{3}.\frac{2}{\sqrt{3}}, 0\right) \equiv (\pm 2, 0)$$

11. Ans. (B)

Sol. Equation of normal at P(6, 3) on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

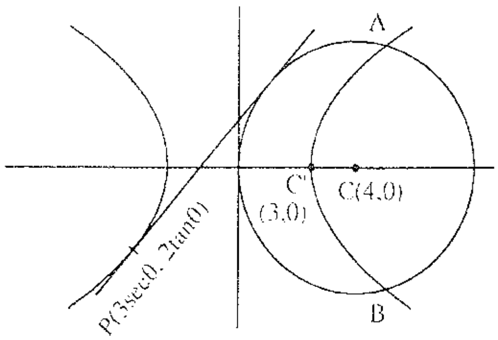
$$\frac{a^2x}{6} + \frac{b^2y}{3} = a^2e^2$$

It intersects x-axis at (9, 0)

$$\Rightarrow \quad a^2\frac{9}{6} = a^2e^2 \quad \Rightarrow \quad e = \sqrt{\frac{3}{2}}$$

12. Ans. (B)

Sol.



Let the point on the hyperbola P(3secθ, 2tanθ)

Equation of tangent $\frac{x\sec\theta}{3} - \frac{y\tan\theta}{2} = 1$

$|p| = r$

$$\frac{\left|\frac{4}{3}\sec\theta - 1\right|}{\sqrt{\frac{\sec^2\theta}{9} + \frac{\tan^2\theta}{4}}} = 4$$

$$\Rightarrow \frac{16}{9}\sec^2\theta + 1 - \frac{8}{3}\sec\theta = 16\left(\frac{4\sec^2\theta + 9\tan^2\theta}{4\times 9}\right)$$

$16\sec^2\theta + 9 - 24\sec\theta = 52\sec^2\theta - 36$

$\Rightarrow 36\sec^2\theta + 24\sec\theta - 45 = 0$

$\Rightarrow 12\sec^2\theta + 8\sec\theta - 15 = 0$

$\Rightarrow 12\sec^2\theta + 18\sec\theta - 10\sec\theta - 15 = 0$

$\Rightarrow (6\sec\theta - 5)(2\sec\theta + 3) = 0$

$\sec\theta = \frac{5}{6}$ (not possible), $\sec\theta = -\frac{3}{2}$

$$\tan\theta = \pm\sqrt{\frac{9}{4} - 1} = \pm\frac{\sqrt{5}}{2}$$

($\because$ slope is positive $\Rightarrow \tan\theta = -\frac{\sqrt{5}}{2}$)

Hence the required equation be

$$-\frac{3x}{2\times 3} + \frac{y\sqrt{5}}{2\times 2} = 1 \Rightarrow 2x - \sqrt{5}y + 4 = 0$$

13. Ans. (A)

Sol. Solving (a) & (b) for x, we get

$x = 6$

$y = \pm 2\sqrt{3}$

$(x-6)^2 + y^2 - 12 = 0$

$x^2 + y^2 - 12x + 24 = 0$

14. Ans. (2)

Sol. As directrix cut the x–axis at $(\pm a/e, 0)$

Hence. $\frac{2a}{e} + 0 = 1$ (for nearer directrix)

$\Rightarrow 2a = e$ ...(i)

Now, $b^2 = a^2(e^2 - 1) = a^2(4a^2 - 1)$

$\Rightarrow \frac{b^2}{a^2} = 4a^2 - 1$ ...(ii)

Given line $y = -2x + 1$ is a tangent to the hyperbola condition of tangency is

$c^2 = a^2m^2 - b^2$

$\Rightarrow 1 = 4a^2 - b^2$

$\Rightarrow 4a^2 - 1 = b^2$ ...(iii)

from (ii) & (iii), $a^2 = 1$

$\Rightarrow$ from (ii). $b^2 = 3$

$\Rightarrow e = \sqrt{\frac{1+3}{1}} = 2$

15. Ans. (B)

Sol. $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

either $x^2 - 5xy + 6y^2 = 0$

$\Rightarrow$ two st. lines passing through origin.

or $ax^2 + by^2 + c = 0$

(A) If c = 0, and a and b are of same sign then it will represent a point.

(B) If a = b, c is of sign opposite to a then will represent circle.

(C) When a & b are of same sign and c is of sign opposite to a then it will represent ellipse.

(D) This is clearly incorrect.

16. Ans. (B)

Sol. The given equation is

$$(x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 4$$

$$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$$

$a = 2$, $b = \sqrt{2}$

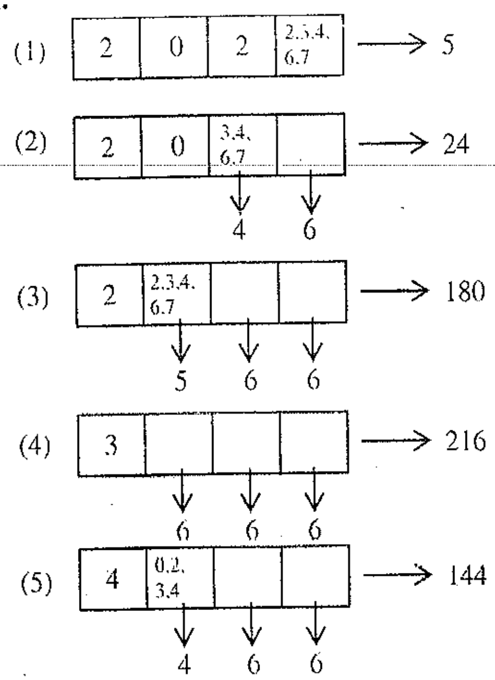
hence eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$

Area $= \frac{1}{2}a(e-1)\times\frac{b^2}{a} = \left(\sqrt{\frac{3}{2}} - 1\right)$ sq. units.

PERMUTATION & COMBINATION

1. Ans. (569.00)

Sol.



Number of 4 digit integers in [2022,4482]

$= 5 + 24 + 180 + 216 + 144 = 569$

2. Ans. (A)

Sol.

B-1	B-2	B-3	B-4
3R 2B	3R 2B	3R 2B	3R 2B

Case-I : when exactly one box provides four balls (3R 1B or 2R 2B)

Number of ways in this case

$^5C_4\,(^3C_1 \times {}^2C_1)^3 \times 4$

Case-II : when exactly two boxes provide three balls (2R 1B or 1R 2B) each

Number of ways in this case

$(^5C_3 - 1)^2\,(^3C_1 \times {}^2C_1)^2 \times 6$

Required number of ways = 21816

Language ambiguity : If we consider at least one red ball and exactly one blue ball, then required number of ways is 9504. None of the option is correct.

3. Ans. (A, B, D)

Sol. (A) $n_1 = 10 \times 10 \times 10 = 1000$

(B) As per given condition

$1 \le i < j + 2 \le 10 \Rightarrow j \le 8$ & $i \ge 1$

for $i = 1, 2,$

$j = 1, 2, 3, \ldots, 8 \rightarrow (8 + 8)$ possibilities

for $i = 3$, $\quad j = 2, 3, \ldots, 8 \rightarrow 7$ possibilities

$i = 4$, $\quad j = 3, \ldots, 8 \rightarrow 6$ possibilities

$i = 9$, $\quad j = 1 \quad \rightarrow 1$ possibility

So $n_2 = (1 + 2 + 3 + \ldots + 8) + 8 = 44$

(C) $n_3 = {}^{10}C_4$ (Choose any four)

$= 210$

(D) $n_4 = {}^{10}C_4 . 4! = (210)(24)$

$\Rightarrow \dfrac{n_4}{12} = 420$

So correct Ans. (A), (B), (D)

4. Ans. (495.00)

Sol. Selection of 4 days out of 15 days such that no two of them are consecutive

$= {}^{15-4+1}C_4 = {}^{12}C_4$

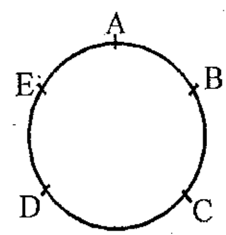
$= \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2} = 11 \times 5 \times 9 = 495$

5. Ans. (1080.00)

Sol. required ways $= \dfrac{6!}{2!\,2!\,1!\,1!\,2!\,2!} \times 4! = 1080$

6. Ans. (30.00)

Sol.



When 1R, 2B, 2G

$5C_1 \times 2 = 10$

Other possibilities

1B, 2R, 2G

or 1G, 2R, 2B

So total no. of ways $= 3 \times 10 = 30$

7. Ans. (625)

Sol. Option for last two digits are (12), (24), (32), (44) are (52).

$\therefore$ Total No. of digits

$= 5 \times 5 \times 5 \times 5 = 625$

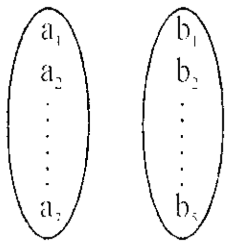
8. Ans. (119)

Sol. $n(X) = 5$

$n(Y) = 7$

$\alpha \to$ Number of one-one function $= {}^7C_5 \times 5!$

$\beta \to$ Number of onto function Y to X



1, 1, 1, 1, 3   1, 1, 1, 2, 2

$$\frac{7!}{3!4!}\times 5! + \frac{7!}{(2!)^3 3!}\times 5!$$

$$= \left({}^7C_3 + 3.{}^7C_3\right)5! = 4\times{}^7C_3 \times 5!$$

$$\frac{\beta-\alpha}{5!} = 4\times{}^7C_3 - {}^7C_5 = 4\times 35 - 21 = 119$$

9. Ans. (C)

Sol. (1) $\alpha_1 = \binom{6}{3}\binom{5}{2} = 200$

So $P \to 4$

(2) $\alpha_2 = \binom{6}{1}\binom{5}{1} + \binom{6}{2}\binom{5}{2} +$

$\binom{6}{3}\binom{5}{3} + \binom{6}{4}\binom{5}{4} + \binom{6}{5}\binom{5}{5}$

$= \binom{11}{5} - 1$

$= 461$

So $Q \to 6$

(3) $\alpha_3 = \binom{5}{2}\binom{6}{3} + \binom{5}{3}\binom{6}{2} + \binom{5}{4}\binom{6}{1} + \binom{5}{5}\binom{6}{0}$

$= \binom{11}{5} - \binom{5}{0}\binom{6}{5} - \binom{5}{1}\binom{6}{4}$

$= 381$

So $R \to 5$

(4) $\alpha_4 = \binom{5}{2}\binom{6}{2} - \binom{4}{1}\binom{5}{1} +$

$\binom{5}{3}\binom{6}{1} - \binom{4}{2}\binom{1}{1} + \binom{5}{4} = 189$

So $S \to 2$

10. Ans. (5)

Sol. $x = 10!$

$$y = {}^{10}C_1\,{}^9C_8\,\frac{10!}{2!}$$

$$\frac{y}{9x} = \frac{5.9.10!}{9.10!} = 5$$

11. Ans. (D)

Sol. $N_1 + N_2 + N_3 + N_4 + N_5$

= Total ways – {when no odd}

Total ways $= {}^9C_5$

Number of ways when no odd, is zero

($\because$ only available even are 2, 4, 6, 8)

$\therefore$ **Ans :** ${}^9C_5$ – zero = 126

12. Ans. (A)

Sol. $({}^6C_4 + {}^6C_3.{}^4C_1).{}^4C_1 = 380$

13. Ans. (5)

Sol. $n = 5!6!$

$m = 5!\,{}^6C_2.{}^5C_4.{}^2C_1.4!$

$\therefore \frac{m}{n} = 5$

14. Ans. (7)

Sol. as $n_1 \ge 1, n_2 \ge 2, n_3 \ge 3, n_4 \ge 4, n_5 \ge 5$

Let $n_1 - 1 = x_1 \ge 0$,

$n_2 - 2 = x_2 \ge 0$ ....

$n_5 - 5 = x_5 \ge 0$

$\Rightarrow$ New equation will be

$x_1 + 1 + x_2 + 2 + \ldots\ldots x_5 + 5 = 20$

$\Rightarrow$ $x_1 + x_2 + x_3 + x_4 + x_5 = 20 - 15 = 5$

Now $x_1 \le x_2 \le x_3 \le x_4 \le x_5$

x_1	x_2	x_3	x_4	x_5
0	0	0	0	5
0	0	0	1	4
0	0	0	2	3
0	0	1	1	3
0	0	1	2	2
0	1	1	1	2
1	1	1	1	1

So, 7 possible cases will be there.

15. **Ans. (5)**

Sol. Number of red line segments $= {}^nC_2 - n$

Number of blue line segments $= n$

$\therefore {}^nC_2 - n = n$

$$\frac{n(n-1)}{2} = 2n \Rightarrow n = 5 \text{ Ans.}$$

16. **Ans. (C)**

Sol Total number of dearrangement

$$6!\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right]$$

$= 360 - 120 + 30 - 6 + 1$

$= 240 + 25 = 265$

There are equal chances that card 1 goes into any envelope from 2 to 6

$$\therefore \frac{1}{5}(265) = 53$$

17. **Ans. (B)**

Sol. Balls can be distributed as 1, 1, 3 or 1, 2, 2 to each person.

When 1, 1, 3 balls are distributed to each person, then total number of ways :

$$= \frac{5!}{1!\,1!\,3!} \cdot \frac{1}{2!} \cdot 3! = 60$$

When 1, 2, 2 balls are distributed to each person, then total number of ways :

$$= \frac{5!}{1!\,2!\,2!} \cdot \frac{1}{2!} \cdot 3! = 90$$

$\therefore$ total $= 60 + 90 = 150$

Solution for Q.18 & Q.19

For a_n

The first digit should be 1

For b_n

$$\underbrace{1\_\,\_\,\_\,\_\,\_ \cdots \_\,\_1}_{(n-2 \text{ Places})}$$

Last digit is 1. so b_n is equal to number of ways of a_{n-1} (i.e. remaining $(n-1)$ places)

$b_n = a_{n-1}$

For c_n

Last digit is 0 so second last digit must be 1

So $c_n = a_{n-2}$

$b_n + c_n = a_n$

So $a_n = a_{n-1} + a_{n-2}$

Similarly $b_n = b_{n-1} + b_{n-2}$

18. **Ans.(B)**

Sol. $a_1 = 1, a_2 = 2$

So $a_3 = 3, \quad a_4 = 5, a_5 = 8$

$\Rightarrow b_6 = a_5 = 8$

19. **Ans.(A)**

Sol. $a_n = a_{n-1} + a_{n-2}$

put $n = 17$

$a_{17} = a_{16} + a_{15}$ (A) is correct

$c_n = c_{n-1} + c_{n-2}$

So put $n = 17$

$c_{17} = c_{16} + c_{15}$ (B) is incorrect

$b_n = b_{n-1} + b_{n-2}$

put $n = 17$

$b_{17} = b_{16} + b_{15}$ (C) is incorrect

$a_{17} = a_{16} + a_{15}$

while (D) says $a_{17} = a_{15} + a_{15}$

(D) is incorrect

20. **Ans. (3)**

Sol. $$S_k = \frac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \frac{1}{(k-1)!} \text{ for } k \geq 2, S_1 = 0$$

$$\text{Now, } \frac{100^2}{100!} + \sum_{k=2}^{100}\left|(k^2-3k+1)\cdot\frac{1}{(k-1)!}\right| + S_1$$

$$\frac{100^2}{100!} + \frac{1}{1!} + \sum_{k=3}^{100}\left|\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right| + 0$$

$$\left(\because S_2 = \frac{1}{1!}\right)$$

$$= \frac{100^2}{100!} + 1 + \left|\frac{1}{0!} - \frac{1}{2!}\right| + \left|\frac{1}{1!} - \frac{1}{3!}\right| + \left|\frac{1}{2!} - \frac{1}{4!}\right| + \ldots$$

$$+ \left|\frac{1}{97!} - \frac{1}{99!}\right|$$

$$= \frac{100^2}{100!} + 3 - \frac{1}{98!} - \frac{1}{99!}$$

$$= \frac{100^2}{100!} + 3 - \frac{100}{99!} = 3$$

21. Ans. (D)

Sol. Case- I : The number of elements in the pairs can be 1,1; 1,2; 1,3,; 2,2

$= {}^4C_2 + {}^4C_1 \times {}^3C_2 + {}^4C_1 \times {}^3C_3 + \frac{{}^4C_2 \cdot {}^2C_2}{2}$

$= 25$

Case- II : Number of pairs with f as one of subsets = 24 = 16

$\therefore$ Total pairs = 25 + 16 = 41

22. Ans. (C)

Sol. Case I : When 1, 1, 1, 1, 1, 2, 3 are used

number of such numbers $= \frac{7!}{5!} = 42$

Case II : When 1, 1, 1, 1, 2, 2, 2 are used

number of such numbers $= \frac{7!}{4!3!} = 35$

Total numbers of numbers = 42 + 35 = 77.

23. Ans. (A) – p, (B) – s, (C) – q, (D) – q

Sol. (A) ENDEA, N, O, E, L

No. of permutation $= \underline{|5}$

(A) $\longrightarrow$ (p)

(B) E is at first & last position

so no. of permutation $= \frac{\underline{|7}}{\underline{|2}} = 21 \times \underline{|5}$

(B) $\longrightarrow$ (s)

(C) For first four letters $= \frac{\underline{|4}}{\underline{|2}}$

for last five letters $= \frac{\underline{|5}}{\underline{|3}}$

Number of permutation

$= \frac{\underline{|4}}{\underline{|2}} \times \frac{\underline{|5}}{\underline{|3}} = 2 \times \underline{|5}$

(C) $\longrightarrow$ (q)

(D) For A, E and O $= \frac{\underline{|5}}{\underline{|3}}$

and for others $\frac{\underline{|4}}{\underline{|2}}$

hence no. of permutation

$= \frac{\underline{|5}}{\underline{|3}} \times \frac{\underline{|4}}{\underline{|2}} = 2 \times \underline{|5}$

(D) $\longrightarrow$ (q)

BINOMIAL THEOREM

1. Ans. (3)

Sol. $T_{r+1} = {}^4C_r\,(a.x^2)^{4-r}\cdot\left(\frac{70}{27bx}\right)^r$

$= {}^4C_r.a^{4-r}.\frac{70^r}{(27b)^r}.x^{8-3r}$

here $8 - 3r = 5$

$8 - 5 = 3r \Rightarrow r = 1$

$\therefore$ coeff. $= 4.a^3.\frac{70}{27b}$

$T_{r+1} = {}^7C_r(ax)^{7-r}\left(\frac{-1}{bx^2}\right)^r$

$= {}^7C_r.a^{7-r}\left(\frac{-1}{b}\right)^r.x^{7-3r}$

$7 - 3r = -5 \Rightarrow 12 = 3r \Rightarrow r = 4$

coeff. : ${}^7C_4.a^3.\left(\frac{-1}{b}\right)^4 = \frac{35a^3}{b^4}$

now $\frac{35a^3}{b^4} = \frac{280a^3}{27b}$

$b^3 = \frac{35\times 27}{280} = b = \frac{3}{2} \Rightarrow 2b = 3$

2. Ans. (A, B, D)

Sol. Solving

$f(m,n,p) = \sum_{i=0}^{p} {}^mC_i\,.{}^{n+i}C_p\,.{}^{p+n}C_{p-i}$

${}^mC_i.{}^{n+i}C_p.{}^{p+n}C_{p-i}$

${}^mC_i.\frac{(n+i)!}{p!(n-p+i)!}\times\frac{(n+p)!}{(p-i)!(n+i)!}$

${}^mC_i\times\frac{(n+p)!}{p!}\times\frac{1}{(n-p+i)!(p-i)!}$

${}^mC_i\times\frac{(n+p)!}{p!n!}\times\frac{n!}{(n-p+i)!(p-i)!}$

${}^mC_i.{}^{n+p}C_p.{}^nC_{p-i}\quad\left\{{}^mC_i.{}^nC_{p-i} = {}^{m+n}C_p\right\}$

$f(m,n,p) = {}^{n+p}C_p.{}^{m+n}C_p$

$\frac{f(m,n,p)}{{}^{n+p}C_p} = {}^{m+n}C_p$

Now, $g(m,n) = \sum_{p=0}^{m+n}\frac{f(m,n,p)}{{}^{n+p}C_p}$

$g(m,n) = \sum_{p=0}^{m+n}{}^{m+n}C_p$

$g(m, n) = 2^{m+n}$

(A) $g(m, n) = g(n,m)$

(B) $g(m, n+1) = 2^{m+n+1}$

$g(m+n, n) = 2^{m+1+n}$

(D) $g(2m, 2n) = 2^{2m+2n}$

$= (2^{m+n})^2$

$= (g(m,n))^2$

3. Ans. (6.20)

Sol. Suppose

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(n-1).2^{n-2}+n.2^{n-1} \\ n.2^{n-1} & 4^n \end{vmatrix} = 0$$

$$\frac{n(n+1)}{2}.4^n - n^2(n-1).2^{2n-3} - n^2 2^{2n-2} = 0$$

$$\frac{n(n+1)}{2} - \frac{n^2(n-1)}{8} - \frac{n^2}{4} = 0$$

$n^2 - 3n - 4 = 0$

$n = 4$

$$\text{Now } \sum_{k=0}^{4}\frac{^4C_k}{k+1} = \sum_{k=0}^{4}\frac{k+1}{5}.{}^5C_{k+1}\frac{1}{k+1}$$

$$= \frac{1}{5}.\left[{}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5\right]$$

$$= \frac{1}{5}\left[2^5 - 1\right] = \frac{31}{5} = 6.20$$

4. Ans. (646)

Sol. $X = \sum_{r=0}^{n} r.\left({}^nC_r\right)^2; n = 10$

$$X = n.\sum_{r=0}^{n} {}^nC_r.{}^{n-1}C_{r-1}$$

$$X = n.\sum_{r=1}^{n} {}^nC_{n-r}.{}^{n-1}C_{r-1}$$

$$X = n.{}^{2n-1}C_{n-1};\ n = 10$$

$$X = 10.{}^{19}C_9$$

$$\frac{X}{1430} = \frac{1}{143}.{}^{19}C_9$$

$= 646$

5. Ans. (5)

Sol. Coefficient of x^2 in the expansion of

$(1+x)^2 + (1+x)^3 + ...(1+x)^{49} + (1+mx)^{50}$ is

${}^2C_2 + {}^3C_2 + ... {}^{49}C_2 + {}^{50}C_2m^2 = (3n+1)\,{}^{51}C_3$

${}^3C_3 + {}^3C_2 + ... {}^{49}C_2 + {}^{50}C_2m^2 = (3n+1)\,{}^{51}C_3$

${}^{50}C_3 + {}^{50}C_2m^2 = (3n+1)\,{}^{51}C_3$

$$\frac{50.49.48}{6} + \frac{50.49}{2}m^2 = (3n+1)\frac{51.50.49}{6}$$

$m^2 = 51n + 1$

must be a perfect quuared

$\Rightarrow n = 5$ and $m = 16$

6. Ans. (8)

Sol. There are 8 product

$1^{99}x^9, 1^{98}x\,x^8, 1^{98}x^2x^7, 1^{98}x^3x^6, 1^{98}x^4x^5$

$1^{97}x\,x^2x^6, 1^{97}x\,x^3x^5, 1^{97}x^2x^3x^4$

which generate x^9 so coeff. is 8

7. Ans. (C)

Sol. Coefficient of x^{11} in

$(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$

$= {}^4C_0.{}^7C_1.{}^{12}C_2 + {}^4C_1.{}^7C_3.{}^{12}C_0 + {}^4C_2.{}^7C_1.{}^{12}C_1 + {}^4C_4.{}^7C_1.{}^{12}C_0$

$= 462 + 140 + 504 + 7 = 1113$

8. Ans. (6)

Sol. Let the three consecutive terms be

${}^{n+5}C_{r-1}, {}^{n+5}C_r, {}^{n+5}C_{r+1}$

$$\therefore \frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r} = \frac{1}{2} \Rightarrow \frac{r}{n-r+6} = \frac{1}{2}$$

$\Rightarrow \quad n = 3r - 6 \qquad ....(1)$

$$\text{Also, } \frac{{}^{n+5}C_r}{{}^{n+5}C_{r+1}} = \frac{5}{7} \Rightarrow \frac{r+1}{n-r+5} = \frac{5}{7}$$

$\Rightarrow \quad 12r = 5n + 18 \qquad ....(2)$

Solving (1) and (2), we get n = 6

9. Ans. (D)

Sol. $A_r = {}^{10}C_r, B_r = {}^{20}C_r, C_r = {}^{30}C_r$

$$\sum_{r=1}^{10}\left({}^{20}C_{10}\,{}^{10}C_r\,{}^{20}C_r - {}^{30}C_{10}\left({}^{10}C_r\right)^2\right)$$

$$= {}^{20}C_{10}\left({}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + ... + {}^{10}C_{10}\,{}^{20}C_{10}\right)$$

$$- {}^{30}C_{10}\left({}^{10}C_1^2 + {}^{10}C_2^2 + ... + {}^{10}C_{10}^2\right)$$

$$= {}^{20}C_{10}\left({}^{30}C_{10} - 1\right) - {}^{30}C_{10}\left({}^{20}C_{10} - 1\right)$$

$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$

FUNCTION

1. Ans. (A, C)

Sol. $f(\theta)=\frac{1}{2}\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}+$

$\begin{vmatrix}\sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right)\\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\frac{\pi}{4} & \tan\pi\end{vmatrix}$.

$f(\theta)=\frac{1}{2}\begin{vmatrix}2 & \sin\theta & 1\\ 0 & 1 & \sin\theta\\ 0 & -\sin\theta & 1\end{vmatrix}+$

$\begin{vmatrix}0 & -\sin\left(\theta-\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & 0 & \log_e\left(\frac{4}{\pi}\right)\\ -\tan\left(\theta-\frac{\pi}{4}\right) & -\log_e\left(\frac{4}{\pi}\right) & 0\end{vmatrix}$

$f(\theta)=(1+\sin^2\theta)+0$ (skew symmetric)

$g(\theta)=\sqrt{f(\theta)-1}+\sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$

$=|\sin\theta|+|\cos\theta|$ for $\theta\in\left[0,\frac{\pi}{2}\right]$

$g(\theta)\in\left[1,\sqrt{2}\right]$

Again let $P(x)=k\left(x-\sqrt{2}\right)(x-1)$

$2-\sqrt{2}=k\left(2-\sqrt{2}\right)(2-1)$

$\Rightarrow k=1 \quad (P(2)=2-\sqrt{2}$ given)

$\therefore\ P(x)=\left(x-\sqrt{2}\right)(x-1)$

for option (A) $P\left(\frac{3+\sqrt{2}}{4}\right)<0$ correct

option (B) $P\left(\frac{1+3\sqrt{2}}{4}\right)<0$ incorrect

option (C) $P\left(\frac{5\sqrt{2}-1}{4}\right)>0$ correct

option (D) $P\left(\frac{5-\sqrt{2}}{4}\right)>0$ incorrect

2. Ans. (C)

Sol. $f(x)$ is a non-periodic, continuous and odd function

$f(x)=\begin{cases}-x^2+x\sin x, & x<0\\ x^2-x\sin x, & x\geq 0\end{cases}$

$f(-\infty)=\lim_{x\to-\infty}(-x^2)\left(1-\frac{\sin x}{x}\right)=-\infty$

$f(\infty)=\lim_{x\to\infty}x^2\left(1-\frac{\sin x}{x}\right)=\infty$

$\Rightarrow$ Range of $f(x)=R$

$\Rightarrow$ $f(x)$ is an onto function ...(1)

$f'(x)=\begin{cases}-2x+\sin x+x\cos x, & x<0\\ 2x-\sin x-x\cos x, & x\geq 0\end{cases}$

For $(0,\infty)$

$f'(x)=(x-\sin x)+x(1-\cos x)$

always +ve or 0   always +ve or 0

$\Rightarrow$ $f'(x)>0$

$\Rightarrow$ $f'(x)\geq 0,\ \forall\ x\in(-\infty,\infty)$

equality at $x=0$

$\Rightarrow$ $f(x)$ is one-one function ... (2)

From (1) & (2), $f(x)$ is both one-one & onto.

3. Ans. (19.00)

Sol. $f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2}$

$=\frac{4^x}{4^x+2}+\frac{4/4^x}{\frac{4}{4^x}+2}$

$=\frac{4^x}{4^x+2}+\frac{4}{4+2.4^x}$

$=\frac{4^x}{4^x+2}+\frac{2}{2+4^x}=1$

so, $f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+...+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$

$=19+f\left(\frac{1}{2}\right)-f\left(\frac{1}{2}\right)=19$

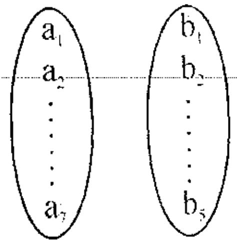
4. Ans. (119)

Sol. $n(X) = 5$

$n(Y) = 7$

$\alpha \to$ Number of one-one function $= {}^7C_5 \times 5!$

$\beta \to$ Number of onto function Y to X



1, 1, 1, 1, 3   1, 1, 1, 2, 2

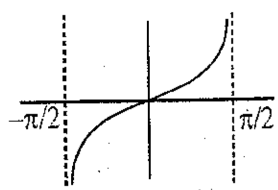
$$\frac{7!}{3!4!}\times 5! + \frac{7!}{(2!)^3 3!}\times 5!$$

$$= \left({}^7C_3 + 3.{}^7C_3\right)5! = 4\times{}^7C_3 \times 5!$$

$$\frac{\beta-\alpha}{5!} = 4\times{}^7C_3 - {}^7C_5 = 4\times 35 - 21 = 119$$

5. Ans. (A, B, C)

Sol. $f(x) = (\ell n(\sec x + \tan x))^3$



$$f'(x) = \frac{3(\ell n(\sec x + \tan x))^2(\sec x\tan x + \sec^2 x)}{(\sec x + \tan x)} > 0$$

$f(x)$ is an increasing function

$\lim\limits_{x\to -\frac{\pi}{2}} f(x) \to -\infty$ & $\lim\limits_{x\to\frac{\pi}{2}} f(x)\to\infty$

Range of $f(x)$ is R and onto function

$$f(-x) = (\ell n(\sec x - \tan x))^3 = \left(\ell n\left(\frac{1}{\sec x + \tan x}\right)\right)^3$$

$f(-x) = -(\ell n(\sec x + \tan x))^3$

$f(x) + f(-x) = 0 \Rightarrow f(x)$ is an odd function.

6. Ans. (B)

Sol. $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\Rightarrow f'(x) = 6(x^2 - 5x + 6)$

$= 6(x-2)(x-3)$

$\because f(x)$ is non monotonic in $x \in [0,3]$

$\Rightarrow$ $f(x)$ is not one-one

$f(x)$ is increasing in $x \in [0,2)$ and decreasing in $x \in (2,3]$

$f(0) = 1$, $f(2) = 29$ & $f(3) = 28$

$\therefore$ Range of $f(x)$ is $[1,29]$

$\Rightarrow f(x)$ is onto.

7. Ans. (A, B)

Sol. $\because$ $\theta \in \left(0, \frac{\pi}{4}\right)\cup\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\Rightarrow$ $2\theta \in \left(0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right)$

Now,

$$f(\cos 4\theta) = \frac{2}{2-\sec^2\theta} = \frac{1+\cos 2\theta}{\cos 2\theta} - 1 + \frac{1}{\cos 2\theta} \quad \text{....(i)}$$

Let $\cos 4\theta = \frac{1}{3}$

$\Rightarrow$ $2\cos^2 2\theta - 1 = \frac{1}{3}$

$\Rightarrow$ $\cos 2\theta = \pm\sqrt{\frac{2}{3}}$

$\Rightarrow$ From (i)

$$f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}}$$

$\Rightarrow$ (A, B) are correct

8. Ans. (A)

Sol. $f : (0,1) \to R$

$$f(x) = \frac{b-x}{1-bx} \qquad b \in (0,1)$$

$\Rightarrow$ $f'(x) = \frac{b^2-1}{(bx-1)^2}$

$\Rightarrow$ $f'(x) < 0\ \forall\ x \in (0, 1)$

hence $f(x)$ is decreasing function

hence its range $(-1, b)$

$\Rightarrow$ co-domain $\neq$ range

$\Rightarrow$ $f(x)$ is non-invertible function

9. Ans. (A)

Sol. Given $f(x) = x^2$; $g(x) = \sin x$

$fogogof(x) = \sin^2(\sin x^2)$ and

$gogof(x) = \sin(\sin x^2)$

given $fogogof(x) = gogof(x)$

$\Rightarrow$ $\sin^2(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow$ $\sin(\sin x^2) = 0$ or 1 (rejected)

$\sin(\sin x^2) = 0 \Rightarrow x^2 = n\pi$

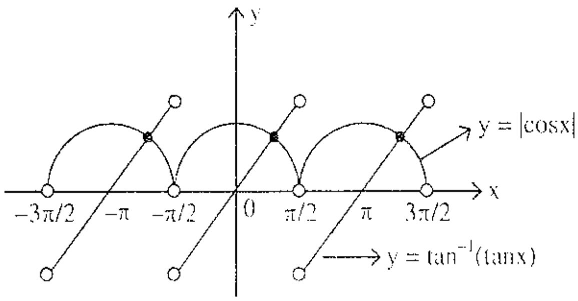
$\Rightarrow x = \pm\sqrt{n\pi};\ x \in \{0, 1, 2, 3, ...\}$

INVERSE TRIGONOMETRIC FUNCTION

1. Ans. (3)

Sol. $\sqrt{2}|\cos x| = \sqrt{2}.\tan^{-1}(\tan x)$

$|\cos x| = \tan^{-1}\tan x$



No. of solutions = 3

2. Ans. (C)

Sol. Case-I : $y \in (-3,0)$

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right)+\pi+\tan^{-1}\left(\frac{6y}{9-y^2}\right)=\frac{2\pi}{3}$$

$$2\tan^{-1}\left(\frac{6y}{9-y^2}\right)=-\frac{\pi}{3}$$

$$y^2-6\sqrt{3}y-9=0 \Rightarrow y=3\sqrt{3}-6\ (\because\ y\in(-3,0))$$

Case-I : $y \in (0,3)$

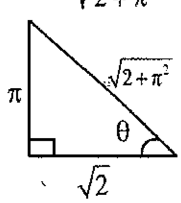
$$2\tan^{-1}\left(\frac{6y}{9-y^2}\right)=\frac{2\pi}{3} \Rightarrow \sqrt{3}y^2+6y-9\sqrt{3}=0$$

$y=\sqrt{3}$ or $y=-3\sqrt{3}$ (rejected)

sum $=\sqrt{3}+3\sqrt{3}-6=4\sqrt{3}-6$

3. Ans. (2.35 or 2.36)

Sol. $\cos^{-1}\sqrt{\dfrac{2}{2+\pi^2}}=\tan^{-1}\dfrac{\pi}{\sqrt{2}}$



$$\sin^{-1}\left(\frac{2\sqrt{2}\pi}{2+\pi^2}\right)=\sin^{-1}\left(\frac{2\times\frac{\pi}{\sqrt{2}}}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$$

$$=\pi-2\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

$$\left(\text{As, }\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi-2\tan^{-1}x, x\ge 1\right)$$

and $\tan^{-1}\dfrac{\sqrt{2}}{\pi}=\cot^{-1}\left(\dfrac{\pi}{\sqrt{2}}\right)$

$\therefore$ Expression

$$=\frac{3}{2}\left(\tan^{-1}\frac{\pi}{\sqrt{2}}\right)+\frac{1}{4}\left(\pi-2\tan^{-1}\frac{\pi}{\sqrt{2}}\right)+\cot^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

$$=\left(\frac{3}{2}-\frac{2}{4}\right)\tan^{-1}\frac{\pi}{\sqrt{2}}+\frac{\pi}{4}+\cot^{-1}\frac{\pi}{\sqrt{2}}$$

$$=\left(\tan^{-1}\frac{\pi}{\sqrt{2}}+\cot^{-1}\frac{\pi}{\sqrt{2}}\right)+\frac{\pi}{4}$$

$$=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}=2.35 \text{ or } 2.36$$

4. Ans. (A, B)

Sol. $S_n(x)=\displaystyle\sum_{k=1}^{n}\tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$

$$=\sum_{k=1}^{n}\tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$$

$S_n(x)=\tan^{-1}(nx+x)-\tan^{-1}x$

$$=\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

(A) $S_{10}(x)=\tan^{-1}\dfrac{10x}{1+11x^2}$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{1+11x^2}{10x}\right)\ (x>0)$$

(B) $\lim\limits_{n\to\infty}\cot(S_n(x))=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}+\left(1+\frac{1}{n}\right)x^2}{x}$

$=x\ (x>0)$

(C) $S_3(x)=\tan^{-1}\dfrac{3x}{1+4x^2}=\dfrac{\pi}{4}$

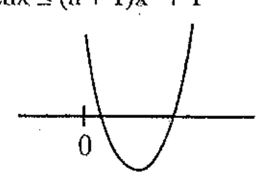
$\Rightarrow 4x^2-3x+1=0 \Rightarrow x\notin\mathbb{R}$

(D) $\tan(S_n(x))=\dfrac{nx}{1+(n+1)x^2}$; $\forall\ n\ge 1$; $x>0$

We need to check the validity of

$$\frac{nx}{1+(n+1)x^2}\le\frac{1}{2}\ \forall\ n\ge 1 ; x>0 ; n\in\mathbb{N}$$

$\Rightarrow 2nx \le (n+1)x^2+1$



$\Rightarrow (n+1)x^2-2nx+1\ge 0\ \forall\ n\ge 1 ; x>0 ; n\in\mathbb{N}$

Discriminant of $y = (n+1)x^2 - 2nx + 1$ is

$D = 4n^2 - 4(n+1)$ and $n \in N$

$D < 0$ for $n = 1$; true for $x > 0$

$D > 0$ for $n \geq 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of

$y = 0$ will be positive.

$y = (n+1)x^2 - 2nx + 1, n \geq 2$

So, $y \geq 0\ \forall\ n \geq 1\ ;\ \forall\ x > 0\ ;\ n \in N$ is false.

5. **Ans. (A, B, D)**

Sol. $$f(n) = \frac{\sum_{k=0}^{n}\left(\cos\left(\frac{\pi}{n+2}\right) - \cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{\sum_{k=0}^{n}\left(1 - \cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \left(\sum_{k=0}^{n}\cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{(n+1) - \left(\sum_{k=0}^{n}\cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n) = \frac{(n+1)\cos\frac{\pi}{n+2} - \left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\left(\frac{n+3}{n+2}\right)\pi\right)}{(n+1)\left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\left(\frac{2(n+2)\pi}{2(n+2)}\right)\right)}$$

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{(n+1)+1}$$

$$\Rightarrow g(x) = \cos\left(\frac{\pi}{n+2}\right)$$

(A) $\sin\left(7\cos^{-1}\cos\frac{\pi}{7}\right) = \sin\pi = 0$

(B) $f(4) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(C) $\lim_{n\to\infty}\cos\left(\frac{\pi}{n+2}\right) = 1$

(D) $\alpha = \tan\left(\cos^{-1}\cos\frac{\pi}{8}\right) = \sqrt{2} - 1 \Rightarrow \alpha + 1 = \sqrt{2}$

$\alpha^2 + 2\alpha - 1 = 0$

6. **Ans. (0.00)**

Sol. $$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{1}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{\sin\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2} - \left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right).\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(\sum_{k=0}^{10}\tan\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right) - \tan\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(\tan\left(\frac{11\pi}{2}+\frac{7\pi}{12}\right) - \tan\left(\frac{7\pi}{12}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(-\cot\frac{7\pi}{12} - \tan\frac{7\pi}{12}\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(-\frac{1}{\sin\frac{7\pi}{12}\cos\frac{7\pi}{12}}\right)\right)$$

$$= \sec^{-1}\left(-\frac{1}{2}\times\frac{1}{\sin\frac{7\pi}{6}}\right) = \sec^{-1}(1) = 0.00$$

7. **Ans. (2)**

Sol. $$\sum_{i=1}^{\infty}x^{i+1} = \frac{x^2}{1-x}$$

$$\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \frac{x}{2-x}$$

$$\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i = \frac{-x}{2+x}$$

$$\sum_{i=1}^{\infty}(-x)^i = \frac{-x}{1+x}$$

To have real solutions

$$\sum_{i=1}^{\infty}x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$x(x^3 + 2x^2 + 5x - 2) = 0$

$\therefore\ x = 0$ and let $f(x) = x^3 + 2x^2 + 5x - 2$

$f'(x) > 0 \Rightarrow f$ is $\uparrow$

$$f\left(\frac{1}{2}\right).f\left(-\frac{1}{2}\right) < 0$$

Hence two solutions exist

8. **Ans. (A)**

Sol. $E_1 : \dfrac{x}{x-1} > 0$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 0 & & 1 & \end{array}$$

$\Rightarrow \quad E_1 : x \in (-\infty, 0) \cup (1, \infty)$

$E_2 : -1 \le \ell n\left(\dfrac{x}{x+1}\right) \le 1$

$\dfrac{1}{e} \le \dfrac{x}{x-1} \le e$

Now $\dfrac{x}{x-1} - \dfrac{1}{e} \ge 0$

$\Rightarrow \quad \dfrac{(e-1)x+1}{e(x-1)} \ge 0$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -1/(e-1) & & 1 & \end{array}$$

$\Rightarrow \quad x \in \left(-\infty, \dfrac{1}{1-e}\right] \cup (1, \infty)$

also $\dfrac{x}{x-1} - e \le 0$

$\dfrac{(e-1)x - e}{x-1} \ge 0$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 1 & & e/(e-1) & \end{array}$$

$\Rightarrow \quad x \in (-\infty, 1) \cup \left[\dfrac{e}{e-1}, \infty\right]$

So $E_2 : \left(-\infty, \dfrac{1}{1-e}\right] \cup \left[\dfrac{e}{e-1}, \infty\right]$

as Range $\dfrac{x}{x-1}$ of is $R^+ - \{1\}$

$\Rightarrow$ Range of f is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range of g is $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \setminus \{0\}$ or

$\left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$

Now $P \to 4, Q \to 2, R \to 1, S \to 1$

9. **Ans. (B, C, D)**

Sol. $\alpha = 3\sin^{-1}\dfrac{6}{11}$ & $\beta = 3\cos^{-1}\dfrac{4}{9}$

$\because \dfrac{6}{11} > \dfrac{1}{2} \Rightarrow \sin^{-1}\dfrac{6}{11} > \sin^{-1}\dfrac{1}{2}$

$\Rightarrow 3\sin^{-1}\dfrac{6}{11} > 3\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$

$\therefore \alpha > \dfrac{\pi}{2}$

$\therefore \cos\alpha < 0$

Now, $\beta = 3\cos^{-1}\dfrac{4}{9}$

$\because \dfrac{4}{9} < \dfrac{1}{2} \Rightarrow 3\cos^{-1}\dfrac{4}{9} > 3\cos^{-1}\dfrac{1}{2}$

$\therefore \beta > \pi$

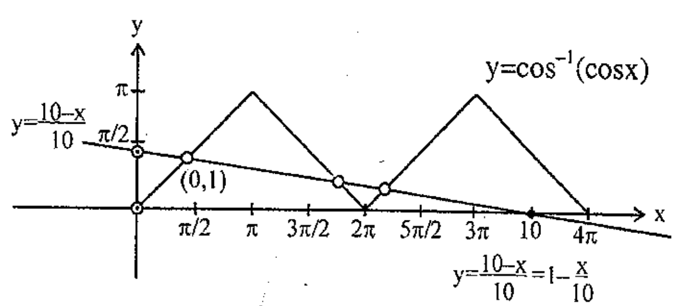
$\therefore \cos\beta < 0$ & $\sin\beta < 0$

Now, α is slightly greater than $\dfrac{\pi}{2}$ & β is slightly greater than π

$\therefore \cos(\alpha + \beta) > 0$

10. **Ans. (3)**

Sol.



from above figure it is clear that $y = \dfrac{10-x}{10}$ and $y = \cos^{-1}(\cos x)$ intersect at 3 distinct points, so number of solutions = 3

11. **Ans. (B)**

Sol. $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$

$= \cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\frac{2n(n+1)}{2}\right)\right)$

$= \cot\left(\sum_{n=1}^{23}\tan^{-1}\frac{1}{1+n(n+1)}\right)$

$= \cot\left(\sum_{n=1}^{23}\tan^{-1}\frac{(n+1)-n}{1+n(n+1)}\right)$

$= \cot\left(\sum_{n=1}^{23}\left(\tan^{-1}(n+1)-\tan^{-1}n\right)\right)$

$= \cot\left(\tan^{-1}24-\tan^{-1}1\right)$

$= \cot\left(\tan^{-1}\frac{23}{25}\right) = \cot\left(\cot^{-1}\left(\frac{25}{23}\right)\right) = \frac{25}{23}$

12. **Ans. (B)**

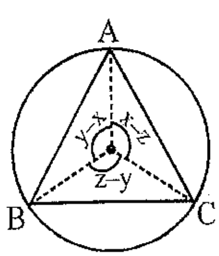
Sol. (P) $\left(\frac{1}{y^2}\left(\frac{\frac{1}{\sqrt{1+y^2}}+\frac{y^2}{\sqrt{1+y^2}}}{\frac{\sqrt{1-y^2}}{y}+\frac{y}{\sqrt{1-y^2}}}\right)^2+y^4\right)^{\frac{1}{2}}$

$\Rightarrow \left(\left(\frac{(1+y^2)y^2(1-y^2)}{y^2}\right)+y^4\right)^{\frac{1}{2}}$

$\Rightarrow \left((1-y^4)+y^4\right)^{\frac{1}{2}} = 1$

(Q) A(cosx, sinx), B(cosy, siny) & C(cosz, sinz) lie on circle $x^2+y^2=1$

$\because$ (0,0) is circumcentre as well as centroid of ΔABC



$\Rightarrow \Delta ABC$ is an equilateral triangle

$y-x=\frac{2\pi}{3}$

$\cos\frac{y-x}{2}=\cos\frac{\pi}{3}=\frac{1}{2}$

(R) $\cos 2x\left(\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{\pi}{4}-x\right)\right)$

$=\sin 2x(1-\tan x)$

$\sqrt{2}\sin x\cos 2x=\sin 2x(1-\tan x)$

$\sin x\left(\sqrt{2}\cos 2x-2(\sin x-\cos x)\right)=0$

$\Rightarrow \sin x=0$ or $\sin x$

$=\cos x$ or $\sin x+\cos x=\sqrt{2}$

$\Rightarrow \sec x=\pm 1$ or $\sec x=\pm\sqrt{2}$

(S) $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$

$\frac{|x|}{\sqrt{1-x^2}}=\frac{\sqrt{6}x}{\sqrt{1+6x^2}}$ $(x>0)$

$\Rightarrow 1-6x^2=6+6x^2$

$\Rightarrow x=\sqrt{\frac{5}{12}}=\frac{\sqrt{5}}{2\sqrt{3}}$

13. **Ans. (C)**

Sol. $\sqrt{1+x^2}\left[\left\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\right\}^2-1\right]^{1/2}$

Let $x=\cot\theta$

then

$\operatorname{cosec}\theta\left[\{\cot\theta\cos\theta+\sin\theta\}^2-1\right]^{1/2}$

$=\operatorname{cosec}\theta\left[\operatorname{cosec}^2\theta-1\right]^{1/2}$

$=\sqrt{1+\cot^2\theta}\,\cot\theta=x\sqrt{1+x^2}$

LIMITS

1. Ans. (0.50)

Sol. $\lim\limits_{x\to\alpha^+}\dfrac{2\ln\left(\sqrt{x}-\sqrt{\alpha}\right)}{\ln\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}\quad\left(\dfrac{0}{0}\text{ form}\right)$

∴ Using Lopital rule.

$$=2\lim_{x\to\alpha^+}\frac{\left(\dfrac{1}{\sqrt{x}-\sqrt{\alpha}}\right)\cdot\dfrac{1}{2\sqrt{x}}}{\left(\dfrac{1}{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}\right)\cdot e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}}$$

$$=\frac{2}{e^{\sqrt{\alpha}}}\lim_{x\to\alpha^+}\frac{\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}{\left(\sqrt{x}-\sqrt{\alpha}\right)}\quad\left(\frac{0}{0}\right)$$

$$=\frac{2}{e^{\sqrt{\alpha}}}\lim_{x\to\alpha^+}\frac{\left(e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}-0\right)}{\left(\dfrac{1}{2\sqrt{x}}-0\right)}=2$$

so, $\lim\limits_{x\to\alpha^+}f\left(g(x)\right)=\lim\limits_{x\to\alpha^+}f(2)$

$$=f(2)=\sin\frac{\pi}{6}=\frac{1}{2}=0.50$$

2. Ans. (5)

Sol. $\beta=\lim\limits_{x\to0}\dfrac{e^{x^3}-\left(1-x^3\right)^{1/3}}{\dfrac{x\sin^2x}{x^2}x^2}+\dfrac{\left(\left(1-x^2\right)^{1/2}-1\right)\sin x}{x\dfrac{\sin^2x}{x^2}x^2}$

use expansion

$$\beta=\lim_{x\to0}\frac{\left(1+x^3\right)-\left(1-\dfrac{x^3}{3}\right)}{x^3}+\lim_{x\to0}\frac{\left(\left(1-\dfrac{x^2}{2}\right)-1\right)}{x^2}\frac{\sin x}{x}$$

$$\beta=\lim_{x\to0}\frac{4x^3}{3x^3}+\lim_{x\to0}\frac{-x^2}{2x^2}$$

$$\beta=\frac{4}{3}-\frac{1}{2}=\frac{5}{6}$$

$6\beta=5$

3. Ans. (1.00)

Sol. $\lim\limits_{x\to0^+}\dfrac{e^{\left(\frac{\ln(1-x)}{x}\right)}-\dfrac{1}{e}}{x^a}$

$$=\lim_{x\to0^+}\frac{1}{e}\frac{e^{\left(1+\frac{\ln(1-x)}{x}\right)}-1}{x^a}$$

$$=\frac{1}{e}\lim_{x\to0^+}\frac{1+\dfrac{\ln(1-x)}{x}}{x^a}$$

$$=\frac{1}{e}\lim_{x\to0^+}\frac{\ln(1-x)+x}{x^{(a+1)}}$$

$$=\frac{1}{e}\lim_{x\to0^+}\frac{\left(-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\dots\dots\right)+x}{x^{a+1}}$$

Thus, $a=1$

4. Ans. (8)

Sol.

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}\cdot2\sin2x\cos x}{2\sin2x\sin\dfrac{3x}{2}+\left(\cos\dfrac{5x}{2}-\cos\dfrac{3x}{2}\right)-\sqrt{2}(1+\cos2x)}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2x}{2\sin2x\left(\sin\dfrac{3x}{2}-\sin\dfrac{x}{2}\right)-2\sqrt{2}\cos^2x}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2x}{4\sin x\cos x\left(2\cos x\cdot\sin\dfrac{x}{2}\right)-2\sqrt{2}\cos^2x}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x}{8\sin x\cdot\sin\dfrac{x}{2}-2\sqrt{2}}=8$$

5. Ans. (B, D)

Sol. P-1 :

$\lim\limits_{h\to0}\dfrac{f(h)-f(0)}{\sqrt{|h|}}$ = exist and finite

(B) $f(x)=x^{2/3},\lim\limits_{h\to0}\dfrac{h^{2/3}-0}{\sqrt{|h|}}=\lim\limits_{h\to0}\dfrac{|h|^{2/3}}{\sqrt{|h|}}=0$

(D) $f(x)=|x|,\lim\limits_{h\to0}\dfrac{|h|-0}{\sqrt{|h|}}\Rightarrow\lim\limits_{h\to0}\sqrt{|h|}=0$

P-2 :

$\lim\limits_{h\to0}\dfrac{f(h)-f(0)}{h^2}$ = exist and finite

(A) $f(x)=x|x|$,

$$\lim_{h\to0}\frac{h|h|-0}{h^2}=\begin{cases}\text{RHL}=\lim\limits_{h\to0}\dfrac{h^2}{h^2}=1\\[2ex]\text{LHL}=\lim\limits_{h\to0}\dfrac{-h^2}{h^2}=-1\end{cases}$$

(C) $f(x)=\sin x\ \lim\limits_{h\to0}\dfrac{\sin h-0}{h^2}$ = DNE

6. **Ans. (D)**

Sol. $f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$

$f_n(x)=\sum_{j=1}^{n}[\tan^{-1}(x+j)-\tan^{-1}(x+j-1)]$

$f_n(x)=\tan^{-1}(x+n)-\tan^{-1}x$

$f_n'(x)=\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$

$\therefore\ \tan(f_n(x))=\tan[\tan^{-1}(x+n)-\tan^{-1}x]$

$\tan(f_n(x))=\frac{(x+n)-x}{1+x(x+n)}$

$\tan(f_n(x))=\frac{n}{1+x^2+nx}$

$\therefore\ \sec^2(f_n(x))=1+\tan^2(f_n(x))$

$\sec^2(f_n(x))=1+\left(\frac{n}{1+x^2+nx}\right)^2$

$\because$ 0 is not in the domain of f_n

so, no menaning of $f_j'(0)$ and $f_j(0)$

$\therefore$ option (A) and (B) are wrong

(C) For any fixed positive integer n,

$\lim_{x\to\infty}\tan(f_n(x))=\lim_{x\to\infty}\frac{n}{1+x^2+nx}=0$

(D) For any fixed positive integer n,

$\lim_{x\to\infty}\sec^2(f_n(x))=\lim_{x\to\infty}\left(1+\left(\frac{n}{1+x^2+nx}\right)^2\right)$

$=1$

7. **Ans. (A,C)**

Sol. $f(x)=\begin{cases}(1-x)\cos\frac{1}{1-x}, & x<1\\ -(1+x)\cos\frac{1}{1-x}, & x>1\end{cases}$

$\lim_{x\to1^+}f(x)=\text{d.n.e},\ \lim_{x\to1^-}f(x)=0$

8. **Ans. (7)**

Sol. If $\alpha\neq1$, then $\lim_{x\to0}\frac{x\sin\beta x}{\frac{\alpha x-\sin x}{x}}=0$

$\therefore\quad \alpha=1 \quad\Rightarrow \lim_{x\to0}\frac{\beta x^3\frac{\sin\beta x}{\beta x}}{x^3\left(\frac{x-\sin x}{x^3}\right)}=\frac{\beta}{1/6}$

$\Rightarrow\quad 6\beta=1 \quad\Rightarrow \beta=\frac{1}{6}$

$6(\alpha+\beta)=7$

9. **Ans. (A, B, C)**

Sol. (A) $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right),\ x\in R$

$=\sin\left(\frac{\pi}{6}\sin\theta\right),\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$=\sin\alpha,\ \alpha\in\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$

(B) $f(g(x))=f(t),t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow f(t)\in\left[-\frac{1}{2},\frac{1}{2}\right]$

(C) $\lim_{x\to0}\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\cdot\frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}$

$=1.\frac{\pi}{6}$

(D) $g(f(x))=1\Rightarrow \sin f(x)=\frac{2}{\pi}$

but $f(x)\in\left[-\frac{1}{2},\frac{1}{2}\right]\subset\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$

$\Rightarrow \sin f(x)\in\left[-\frac{1}{2},\frac{1}{2}\right]$ so no solutions

10. **Ans. (2)**

Sol. $\lim_{\alpha\to0}\frac{e\left(e^{\cos\alpha^n-1}-1\right)}{\left(\cos\alpha^n-1\right)}\frac{\left(\cos\alpha^n-1\right)}{\left(\alpha^n\right)^2}\alpha^{2n-m}$

$=-\frac{e}{2}\qquad \therefore 2n=m\Rightarrow\frac{m}{n}=2$

11. **Ans. (0)**

Sol. $\lim_{x\to1}\left\{\frac{\sin(x-1)+a(1-x)}{(x-1)+\sin(x-1)}\right\}^{\frac{(1+\sqrt{x})(1-\sqrt{x})}{1-\sqrt{x}}}=\frac{1}{4}$

$\lim_{x\to1}\left\{\frac{\frac{\sin(x-1)}{(x-1)}-a}{1+\frac{\sin(x-1)}{(x-1)}}\right\}^{1+\sqrt{x}}=\frac{1}{4}\Rightarrow\left(\frac{1-a}{2}\right)^2=\frac{1}{4}$

$\Rightarrow(a-1)^2=1\Rightarrow a=2$ or 0

but for a = 2 base of above limit approaches $-\frac{1}{2}$ and exponent approaches to 2 and since base cannot be negative hence limit does not exist.

12. Ans. (B)

Sol. $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$

$\Rightarrow \lim_{x\to\infty}\left(\frac{(1-a)x^2+x(1-a-b)+1-b}{x+1}\right)=4$

$\Rightarrow \lim_{x\to\infty}\frac{\left((1-a).x+1-a-b+\left(\frac{1-b}{x}\right)\right)}{1+\frac{1}{x}}=4$

for limit to exist finitely

$1-a=0$ and $1-a-b=4$

$\Rightarrow a=1$ and $b=-4$

13. Ans. (B)

Sol. $\left(\left(1+\frac{a}{3}\right)-1\right)x^2+\left(\left(1+\frac{a}{2}\right)-1\right)x+\left(1+\frac{a}{6}-1\right)=0$

$a\left(\frac{x^2}{3}+\frac{x}{2}+\frac{1}{6}\right)=0 \Rightarrow 2x^2+3x+1=0$

$\Rightarrow x=-\frac{1}{2},-1 \Rightarrow \lim_{a\to0^+}\alpha(a)$ and

$\lim_{a\to0^+}\beta(a)$ are $-\frac{1}{2}$ and -1

14. Ans. (D)

Sol. $\lim_{x\to0}\left[1+x\ell n(1+b^2)\right]^{\frac{1}{x}}=e^{\lim_{x\to0}\frac{x\ell n(1+b^2)}{x}}=1+b^2$

Hence $1+b^2=2b\sin^2\theta$

$\Rightarrow \quad \sin^2\theta=\frac{1}{2}\left(b+\frac{1}{b}\right)\geq1$

$\therefore \quad \sin^2\theta=1 \quad \Rightarrow \quad \sin\theta=\pm1$

$\Rightarrow \quad \theta=\pm\frac{\pi}{2}$

15. Ans. (A, C)

Sol. $\frac{a-a\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}-\frac{x^2}{4}}{x^4}$

$=\frac{a-a\left(1-\frac{x^2}{2a^2}-\frac{1}{8}\frac{x^4}{a^4}\right)-\frac{x^2}{4}}{x^4}$

$a=2$, (coefficient of $x^2=0$)

$\therefore L=\frac{1}{64}$

16. Ans. (0)

Sol. $\operatorname{Lt}_{x\to0}\left(1+\frac{p(x)}{x^2}\right)=2$

$\Rightarrow \operatorname{Lt}_{x\to0}\frac{p(x)}{x^2}=1$

Let $p(x)=ax^4+bx^3+cx^2$

$\because \quad \operatorname{Lt}_{x\to0}\frac{p(x)}{x^2}=1 \Rightarrow c=1$

$p(x)=ax^4+bx^3+x^2$

Now, $p'(x)=4ax^3+3bx^2+2x$

$\because p'(1)=0,\ p'(2)=0$

$\Rightarrow 4a+3b+2=0$

$32a+12b+4=0 \Rightarrow a=\frac{1}{4},\ b=-1$

$\Rightarrow \ p(x)=\frac{1}{4}x^4-x^3+x^2 \Rightarrow p(2)=4-8+4=0$

CONTINUITY

1. Ans. (A, C, D)

Sol. $f(x)=x\cos(\pi x+[x]\pi)$

$\Rightarrow f(x)=(-1)^{[x]}x\cos\pi x.$

Discontinuous at all integers except zero.

2. Ans. (A, D)

Sol. $f, g\ [0,1]\to R$

we take two cases.

Let f & g attain their common maximum value at p.

$\Rightarrow f(p)=g(p)$ where $p\in[0,1]$

let f & g attain their common maximum value at different points.

$\Rightarrow f(a)=M$ & $g(b)=M$

$\Rightarrow f(a)-g(a)>0$ & $f(b)-g(b)<0$

$\Rightarrow f(c)-g(c)=0$ for some $c\in[0,1]$ as 'f' & 'g' are continuous functions.

$\Rightarrow f(c)-g(c)=0$ for some $c\in[0,1]$ for all cases. ...(1)

Option (A) $\Rightarrow f^2(c)-g^2(c)+3(f(c)-g(c))=0$ which is true from (1)

Option (D) $\Rightarrow f^2(c)-g^2(c)=0$ which is true from (1)

Now, if we take $f(x)=1$ & $g(x)=1\ \forall\ x\in[0,1]$ options (B) & (C) does not hold. Hence option (A) & (D) are correct.

3. Ans. (B, D)

Sol. For f to be continuous :

$f(2n^-) = f(2n^+)$

$\Rightarrow \quad b_n + \cos 2n\pi = a_n + \sin 2n\pi$

$\Rightarrow \quad b_n + 1 = a_n$

$\Rightarrow \quad a_n - b_n = 1$

($\therefore$ B is correct)

$$\text{Also } f(x) = \begin{cases} b_n + \cos \pi x & (2n-1, 2n) \\ a_n + \sin \pi x & [2n, 2n+1] \\ b_{n+1} + \cos \pi x & (2n+1, 2n+2) \\ a_n + \sin \pi x & [2n+2, 2n+3] \end{cases}$$

Again $f((2n+1)^-) = f((2n+1)^+)$

$\Rightarrow \quad a_n = b_{n+1} - 1$

$\Rightarrow \quad a_n - b_{n+1} = -1$

$\Rightarrow \quad a_{n-1} - b_n = -1$ ($\therefore$ D is correct)

DIFFERENTIABILITY

1. Ans. (A,C)

Sol. $f : R \to R \quad f(x) = (x^2 + \sin x)(x-1)\ f(1^+)$

$= f(1^-) = f(1) = 0$

$fg(x) : f(x).g(x) \quad fg : R \to R$

let $fg(x) = h(x) = f(x).g(x) \qquad h : R \to R$

option (c) $\quad h'(x) = f'(x)g(x) + f(x)\,g'(x)$

$h'(1) = f'(1)\,g(1) + 0,$

(as $f(1) = 0$, $g'(x)$ exists}

$\Rightarrow$ if g(x) is differentiable then h(x) is also differentiable (true)

option (A) If g(x) is continuous at x = 1 then $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h\to 0^+} \frac{h(1+h) - h(1)}{h}$$

$$h'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)g(1+h) - 0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)g(1-h) - 0}{-h} = f'(1)g(1)$$

So $\quad h(x) = f(x).g(x)$ is differentiable

at x = 1 $\qquad$ (True)

option (B) $\quad$ (D)

$$h'(1^+) = \lim_{h\to 0^+} \frac{h(1+h) - h(1)}{-h}$$

$$h'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$

$$h'(1^-) = \lim_{h\to 0^-} \frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$$

$\Rightarrow g(1^+) = g(1^-)$

So we cannot comment on the continuity and differentiability of the function.

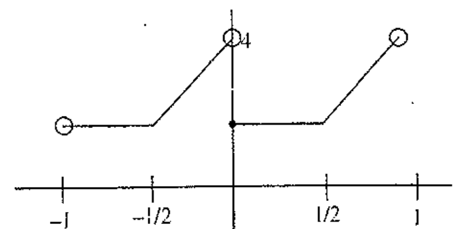
2. Ans. (4)

Sol. $f(x) = |2x-1| + |2x+1|$

$g(x) = \{x\}$

$f(g(x)) = |2\{x\} - 1| + |2\{x\} + 1|$

$$= \begin{cases} 2 & \{x\} \le \frac{1}{2} \\ 4\{x\} & \{x\} > \frac{1}{2} \end{cases}$$



discontinuous at $x = 0 \Rightarrow c = 1$

Non differential at $x = -\frac{1}{2},\ 0,\ \frac{1}{2} \Rightarrow d = 3$

$\therefore \quad c + d = 4$

3. Ans. (A, B, D)

Sol. since $f(x) = xg(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} xg(x)$$

$$\lim_{x\to 0} f(x) = \left(\lim_{x\to 0} x\right).\left(\lim_{x\to 0} g(x)\right)$$

$$\lim_{x\to 0} f(x) = 0 \times 1 = 0 \qquad ...(1)$$

$f(x+y) = f(x) + f(y) + f(x)\,f(y)$

Now we check continuity of $f(x)$

at x = a

$$\lim_{h\to 0} f(a+h) = f(a) + f(h) + f(a)f(h)$$

$$\lim_{h\to 0}\left(f(a) + f(h)\left(1 + f(a)\right)\right)$$

$$\lim_{h\to 0} f(a+h) = f(a)$$

$\therefore f(x)$ is continuous $\forall x \in R$

$$\lim_{x\to 0} f(x) = f(0) = 0 \quad \left(\lim_{x\to 0} f(x) = 0\right)$$

$\therefore f(0) = 0$

and $\lim_{x\to 0} \frac{f'(x)}{1} = 1$

$\therefore f'(0) = 1$

Now

$f(x + y) = f(x) + f(y) + f(x) f(y)$

using partial derivative (w.r.t. y)

$f'(x + y) = f'(y) + f(x) f'(y)$

put $y = 0$

$f'(x) = f'(0) + f(x) f'(0)$

$f'(x) = 1 + f(x)$

$$\int \frac{f'(x)}{1+f(x)} dx = \int 1 dx$$

$$\ell n\left|\left(1+f(x)\right)\right| = x + C$$

$f(0) = 0$; $c = 0$ $\quad \therefore \left|1+f(x)\right| = e^x$

$1 + f(x) = \pm e^x$ or $f(x) = \pm e^x - 1$

Now $f(0) = 0 \therefore f(x) = e^x - 1$

$\therefore f(x) = e^x - 1$

option (A) is correct

and $f'(x) = e^x$

$f'(0) = 1$ option(D) is correct

$$g(x) = \frac{f(x)}{x} = \begin{cases} \frac{e^x - 1}{x} & ; \; x \neq 0 \\ 1 & ; \; x = 0 \end{cases}$$

$$g'(0+h) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{e^h - 1}{h} - 1}{h} = \frac{1}{2}$$

option (B) is correct

4. **Ans. (2)**

Sol. $P(x, y) : f(x + y) = f(x)f'(y)+f'(x) f(y) \; \forall \; x, y \in R$

$P(0, 0) : f(0) = f(0)f'(0) + f'(0) f(0)$

$\Rightarrow 1 = 2f'(0)$

$\Rightarrow f'(0) = \frac{1}{2}$

$P(x, 0) : f(x) = f(x). f'(0) + f'(x).f(0)$

$\Rightarrow f(x) = \frac{1}{2} f(x) + f'(x)$

$\Rightarrow f'(x) = \frac{1}{2} f(x)$

$\Rightarrow f(x) = e^{\frac{1}{2}x}$

$\Rightarrow \ln(f(4)) = 2$

5. **Ans. (D)**

Sol. (i) $f(x) = \sin\sqrt{1 - e^{-x^2}}$

$$f_1'(x) = \cos\sqrt{1-e^{-x^2}} \cdot \frac{1}{2\sqrt{1-e^{-x^2}}}\left(0 - e^{-x^2}.(-2x)\right)$$

$f_1'(x)$ does not exist at $x = 0$

So. $P \rightarrow 2$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$\lim_{x\to 0^+} \frac{\sin x}{x} \frac{x}{\tan^{-1} x} = 1$$

$\Rightarrow f_2(x)$ does not continuous at $x = 0$

So $Q \rightarrow 1$

(iii) $f_3(x) = [\sin \ell n(x + 2)] = 0$

$1 < x + 2 < e^{\pi/2}$

$\Rightarrow 0 < \ell n(x + 2) < \frac{\pi}{2}$

$\Rightarrow 0 < \sin(\ell n(x + 2)) < 1$

$\Rightarrow f_3(x) = 0$

So $R \rightarrow 4$

(iv) $f_4(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 \;, & x = 0 \end{cases}$

So $S \rightarrow 3$

6. **Ans. (A, B)**

Sol. If $x^3 - x \ge 0 \Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$

$x^3 - x < 0 \Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$

Similarly

$b|x|\sin|x^3 + x| = bx\sin(x^3 + x)$ for all $x \in R$

$\therefore \quad f(x) = a\cos(x^3 - x) + bx\sin(x^3 + x)$

which is composition and sum of differentiable functions

therefore always continuous and differentiable.

7. **Ans. (B, C)**

Sol. $f(x) = [x^2] - 3$

$g(x) = (|x| + |4x - 7|)([x^2] - 3)$

$\because \quad f$ is discontinuous at in $\left[-\frac{1}{2}, 2\right]$

and $|x| + |4x - 7| \neq 0$ at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$\Rightarrow$ g(x) is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}$ in $\left(-\frac{1}{2}, 2\right)$

In $(0 - \delta, 0 + \delta)$

$g(x) = (|x| + |4x - 7|).(-3)$

$\Rightarrow$ 'g' is non derivable at $x = 0$.

In $\left(\frac{7}{4} - \delta, \frac{7}{4} + \delta\right)$

$g(x) = 0$ as $f(x) = 0$

$\Rightarrow$ Derivable at $x = \frac{7}{4}$

$\therefore$ 'g' is non-derivable at $0, 1, \sqrt{2}, \frac{7}{4}$

8. **Ans. (A, D)**

Sol. (A) $f(x) = \begin{cases} g(x) & , \ x > 0 \\ 0 & , \ x = 0 \\ -g(x) & , \ x < 0 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} g'(x) & , \ x \ge 0 \\ -g'(x) & , \ x < 0 \end{cases}$

so 'A' is right

(B) $h(x) = \begin{cases} e^x & , \ x \ge 0 \\ e^{-x} & , \ x < 0 \end{cases}$

$\Rightarrow h'(x) = \begin{cases} e^x & , \ x > 0 \\ -e^{-x} & , \ x < 0 \end{cases}$

$h'(0^+) = 1$, $h'(0^-) = -1$, $\therefore$ B is wrong

(C) $f(h(x)) = g(h(x))$ as $h(x) > 0$

$z = g\left(e^{|x|}\right) = \begin{cases} g\left(e^x\right) & , \ x \ge 0 \\ g\left(e^{-x}\right) & , \ x < 0 \end{cases}$

$z' = \begin{cases} g'\left(e^x\right)e^x & , \ x > 0 \\ -g'\left(e^{-x}\right)e^{-x} & , \ x < 0 \end{cases}$

$z'\left(0^+\right) = g'(1), z'\left(0^-\right) = -g'(1)$ and

$g'(1) \# -g'(1)$

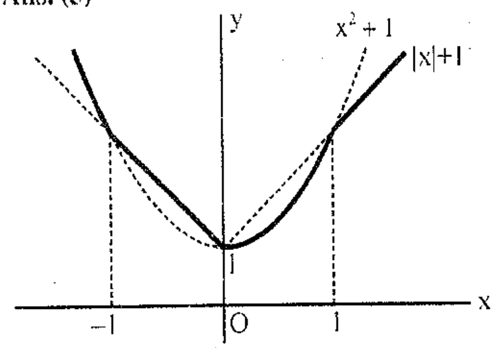
so C is wrong

(D) $\lim\limits_{x \to 0} \frac{e^{|g(x)|} - 1}{|g(x)|} \cdot \frac{|g(x)|}{x}$

$\lim\limits_{x \to 0} \frac{e^{|g(x)|} - 1}{|g(x)|} \cdot \frac{|g(x) - 0|}{x} \cdot \frac{|X|}{X} = 0$

9. **Ans. (3)**

Sol.

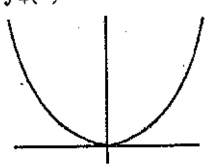


h(x) is not differentiable at $x = \pm 1$ & 0

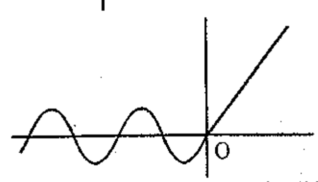
10. **Ans. (D)**

Sol. (P) $f_4(x) = \begin{cases} |x|^2 & ; \ x < 0 \\ e^{2x} - 1 & ; \ x \ge 0 \end{cases}$

$f_4(x)$ is onto and one-one



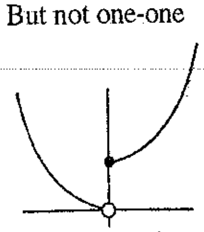
(Q)



RHD = LHD = 1, $f_3(x)$ is differentiable

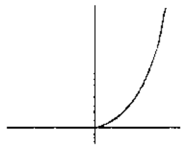
But not one-one

(R)



$f_2(f_1(x)) = \begin{cases} x^2 & ; \ x < 0 \\ e^{2x} & ; \ x \ge 0 \end{cases}$

Neither continuous nor one-one

(S) 

Continuous and one-one function

11. Ans. (B)

Sol. At x = 0

R.H.D

$$= \lim_{h\to 0}\frac{(0+h)-(0)}{h} = \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|-0}{h}$$

$$= \lim_{h\to 0} h\left|\cos\frac{\pi}{h}\right| = 0\times\cos(\infty)$$

$= 0 \times \text{finite} = 0$

$$\text{LHD} : = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{h^2\cos\left(\frac{\pi}{-h}\right)-0}{-h} = \lim_{h\to 0} -h\cos\left(\frac{\pi}{h}\right)$$

$= 0$

$\because$ LHD = RHD at x = 0

$\Rightarrow$ $f(x)$ is differentiable at x = 0

At x = 2

$$\text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{(2+h)^2.\cos\left(\frac{\pi}{2+h}\right)-0}{h}$$

$$= 4\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= -4\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2+h}\right).\left(-\frac{\pi}{(2+h)^2}\right)}{1} = \pi$$

$$\text{LHD} : \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{(2-h)^2\left(-\cos\left(\frac{\pi}{2-h}\right)\right)}{-h}$$

$$= \lim_{h\to 0}\frac{4\left(\sin\frac{\pi}{(2-h)}\right)\left(\frac{\pi}{(2-h)^2}\right)}{-1} = -\pi$$

LHD $\neq$ RHD at x = 2

$\therefore$ Not differentiable at x = 2

12. Ans. (B, C) or (B, C,D)

Sol. $f(x+y) = f(x)+f(y)$

$f(0) = 0$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$f'(x) = f'(0) = k$ (k is constant)

$\Rightarrow$ $f(x) = kx$, hence $f(x)$ is continuous and $f'(x)$ is constant $\forall\ x \in R$

13. Ans.(A, B, C, D)

Sol. $f\left(-\frac{\pi}{2}\right) = 0,\ f\left(-\frac{\pi}{2}^+\right) = 0$

$$f'(x) = \begin{cases} -1 & x \le \frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} < x \le 0 \\ 1 & 0 < x \le 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

$f'(0^-) = 0,\ f'(0^+) = 1$

$\therefore$ not differentiable at x = 0

$f'(1^-) = 1,\ f'(1^+) = 1$

$\therefore$ differentiable at x = 1

as $-\frac{3}{2} \in \left(-\frac{\pi}{2}, 0\right)$

$f'(x) = \sin x$ which is differentiable at $x = -\frac{3}{2}$

14. Ans. (C)

Sol. P = –1

Now,

$$\lim_{x\to 1^+}\frac{(x-1)^n}{\log\cos^m(x-1)} = \lim_{x\to 1^+}\frac{(x-1)^n}{\log\left[\cos^m(x-1)-1+1\right]}$$

$$= \lim_{x\to 1^+}\frac{(x-1)^n}{\cos^m(x-1)-1} = \lim_{x\to 1^+}\frac{n(x-1)^{n-1}}{m\cos^{m-1}(x-1)\sin(x-1)}$$

$$= \frac{-n}{m}\lim_{x\to 1^+}\frac{(x-1)}{\sin(x-1)}\cdot\frac{1}{\cos^{m-1}(x-1)}\times(x-1)^{n-2}$$

$$= \frac{-n}{m}\lim_{x\to 1^+}(x-1)^{n-2} = -1 \text{ (Given)}$$

$\Rightarrow$ n = 2, and m = 2

METHOD OF DIFFERENTIATION

1. Ans. (B, C)

Sol. (A) $f'(x)=3x^2+3$

so, $g'(2)=\dfrac{1}{f'(0)}$ (Given $g(x)=f^{-1}(x)$)

$\Rightarrow g'(2)=\dfrac{1}{3}$

(B) $h(g(g(x)))=x$

$h'(g(g(x)))=\dfrac{1}{g'(g(x)).g'(x)}$

Now, $g(g(x))=1$

$g(x)=f(1)=6$

$\therefore x=f(6)=236$

so $h'(1)=\dfrac{1}{g'(6).g'(236)}=\dfrac{1}{\frac{1}{6}\cdot\frac{1}{111}}$

$\Rightarrow \quad h'(1)=666$

(C) $g(g(x))=0$

$\therefore g(x)=g^{-1}(0)\Rightarrow g(x)=f(0)\Rightarrow g(x)=2$

$\Rightarrow x=g^{-1}(2)\Rightarrow x=f(2)\Rightarrow x=16$

so $h(0)=16$

(D) $g(x)=3 \Rightarrow x=g^{-1}(3)\Rightarrow x=f(3)$

$\Rightarrow x=38$ so $h(g(3))=38$

2. Ans. (8)

Sol. $(y-x^5)^2=x(1+x^2)^2$

$\Rightarrow 2(y-x^5)$

$\left(\dfrac{dy}{dx}-5x^4\right)=(1+x^2)^2+2x(1+x^2).2x$

Put $x=1, y=3$

$\dfrac{dy}{dx}=8$

3. Ans. (1)

Sol. Let $f(\theta)=\sin\alpha$ where $\alpha=\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)$

$\Rightarrow \quad \tan\alpha=\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}$

$\Rightarrow \quad \sin\alpha=\dfrac{\sin\theta}{\cos\theta}=\tan\theta\left(\because \theta\in\left(-\dfrac{\pi}{4},\dfrac{\pi}{4}\right)\right)$

$\Rightarrow \quad f(\theta)=\tan\theta$

$\Rightarrow \quad \dfrac{d(f(\theta))}{d(\tan\theta)}=1$

4. Ans. (2)

Sol. $f(x)=x^3+e^{x/2}, g(x)=f^{-1}(x)$

$\Rightarrow g'(f(x)).f'(x)=1$

Put $f(x)=1\Rightarrow x^3+e^{x/2}=1$

$\Rightarrow x=0$

$\Rightarrow g'(1).f'(0)=1, f'(x)=3x^2+e^{x/2}.\dfrac{1}{2}$

$\Rightarrow \quad g'(1)=2$

5. Ans. (A)

Sol. $\lim\limits_{x\to 0}\dfrac{[g(x)\cos x-g(0)]}{\sin x}$

$=\lim\limits_{x\to 0}\dfrac{g'(x)\cos x-g(x)\sin x}{\cos x}=0$

Now $f(x)=g(x)\sin x$

$f'(x)=g'(x)\sin x+g(x)\cos x \quad \therefore f'(0)=0$

$f''(x)=g''(x)\sin x+g'(x)\cos x-g(x)\sin x+g'(x)\cos x$

$f'0)=0$

$\therefore$ Given limit $=f''(0)$ & also $f'(0)=g(0)$

So, S(I) & S(II) both are correct but S(II) is not correct explaination of S(I)

6. Ans. (A)

Sol. $g(x+1)=\log(f(x+1))=\log x+\log f(x)$

$\Rightarrow \quad g(x+1)=\log x+g(x)$

$\Rightarrow g(x+1)-g(x)=\log x$

$\Rightarrow g'(x+1)-g'(x)=\dfrac{1}{x}$

$\Rightarrow \quad g''(x+1)-g''(x)=\dfrac{1}{x^2}$

$\Rightarrow g''\left(1+\dfrac{1}{2}\right)-g''\left(\dfrac{1}{2}\right)=-4$

$\Rightarrow g''\left(2+\dfrac{1}{2}\right)-g''\,\underline{|5}=-\dfrac{4}{9}$

....

....

$g''\left(N+\dfrac{1}{2}\right)-g''\left(N-\dfrac{1}{2}\right)=\dfrac{-4}{(2N-1)^2}$

by adding

Hence $g''\left(N=\dfrac{1}{2}\right)-g''\dfrac{1}{2}$

$=-4\left(1+\dfrac{1}{9}+........+\dfrac{1}{(2N-1)^2}\right)$

AOD (MONOTONICITY)

1. **Ans. (A, C, D)**

Sol. $S = (0, 1) \cup (1, 2) \cup (3, 4)$

$T = \{0, 1, 2, 3\}$

Number of functions :

Each element of S have 4 choice

Let n be the number of element in set S.

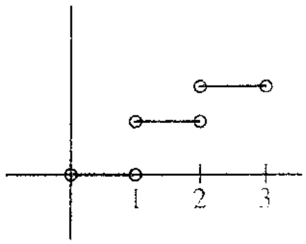
Number of function $= 4^n$

Here $n \to \infty$

$\Rightarrow$ Option (A) is correct.

Option (B) is incorrect (obvious)

(C)



For continuous function

Each interval will have 4 choices.

$\Rightarrow$ Number of continuous functions

$= 4 \times 4 \times 4 = 64$

$\Rightarrow$ Option (C) is correct.

(D) Every continuous function is piecewise constant functions

$\Rightarrow$ Differentiable.

Option (D) is correct.

2. **Ans. (A, B, C)**

Sol. S = Set of all twice differentiable functions

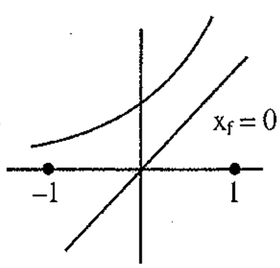
$f : R \to R$

$\frac{d^2f}{dx^2} > 0$ in $(-1, 1)$

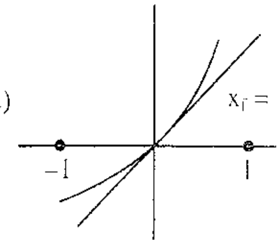
Graph 'f' is Concave upward.

Number of solutions of $f(x) = x \to x_f$

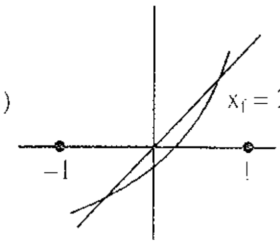
(1)



(2)



(3)



$\Rightarrow$ Graph of $y = f(x)$ can intersect graph of $y = x$ at atmost two points $\Rightarrow 0 \le x_f \le 2$

3. **Ans. (A, B)**

Sol. $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$$f'(x) = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = \frac{5x(x+4)}{(x^2 + 2x + 4)^2}$$

f '(x) : + (−4) − (0) +

$f(-4) = \frac{11}{6}, \quad f(0) = -\frac{3}{2}, \quad \lim_{x \to \pm\infty} f(x) = 1$

Range : $\left[-\frac{3}{2}, \frac{11}{6}\right]$, clearly f(x) is into

4. **Ans. (5.00)**

Sol. $f(x) = (x^2 - 1)^2 h(x)$;

$h(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

Now, $f(1) = f(-1) = 0$

$\Rightarrow f'(\alpha) = 0, \alpha \in (-1, 1)$ [Rolle's Theorem]

Also, $f'(1) = f'(-1) = 0 \Rightarrow f'(x) = 0$ has at least 3 root, $-1, \alpha, 1$ with $-1 < \alpha < 1$

$\Rightarrow f''(x) = 0$ will have at least 2 root, say β, γ such that

$-1 < \beta < \alpha < \gamma < 1$ [Rolle's Theorem]

So, $\min(m_{f''}) = 2$ and

we find $(m_{f'} + m_{f''}) = 5$ for $f(x) = (x^2 - 1)^2 h(x)$

5. **Ans. (B, D)**

Sol. For option (A),

Let $g(x) = e^x - \int_0^x f(t)\sin t\,dt$

$\therefore g'(x) = e^x - (f(x).\sin x) > 0\ \forall\ x \in (0,1)$

$\Rightarrow g(x)$ is strictly increasing function.

Also, $g(0) = 1$

$\Rightarrow g(x) > 1\ \forall\ x \in (0,1)$

$\therefore$ option (A) is not possible.

For option (B), let

$k(x) = x^9 - f(x)$

Now, $k(0) = -f(0) < 0$ (As $f \in (0,1)$)

Also, $k(1) = 1 - f(1) > 0$ (As $f \in (0,1)$)

$\Rightarrow k(0).\ k(1) < 0$

So, option(B) is correct.

For option (C), let

$$T(x) = f(x) + \int_0^{\frac{\pi}{2}} f(t).\sin t\,dt$$

$\Rightarrow T(x) > 0\ \forall x \in (0,1)$ (As $f \in (0,1)$)

so, option(C) is not possible.

For option (D),

$$\text{Let } M(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$$

$$\therefore M(0) = 0 - \int_0^{\pi/2} f(t).\cos t\,dt < 0$$

$$\text{Also, } M(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t).\cos t\,dt > 0$$

$\Rightarrow M(0).\ M(1) < 0$

$\therefore$ option (D) is correct.

6. **Ans. (D)**

7. **Ans. (D)**

8. **Ans. (D)**

Sol. **6 to 8**

$f(x) = x + \ell nx - x\ell nx,\ x > 0$

$$f'(x) = \cancel{1} + \frac{1}{x} - \ell n\,x - \cancel{1}$$

$$f''(x) = -\frac{1}{x^2} - \frac{1}{x} = \frac{-(x+1)}{x^2}$$

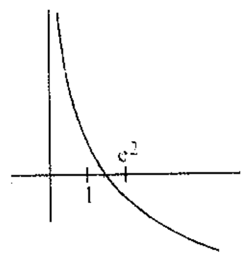
(I) $f(1)\,f(e^2) < 0$ so true

(II) $f'(1)\,f'(e) < 0$ so true

(III) Graph of $f'(x)$ so (III) is false

(IV) Is false

$$\text{As } \lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\left[1 + \frac{\ell n\,x}{x} - \ell n\,x\right] = -\infty$$



$\therefore$ (i) is false (ii) is true

$\lim_{x\to\infty} f'(x) = -\infty$ so (iii) is true

$\lim_{x\to\infty} f''(x) = 0$ so (iv) is true.

(P) $f'(x)$ is positive in $(0,1)$ so true

(Q) $f'(x) < 0$ for in (e,e^2) so true

As $f'(x) < 0\ \forall\ x > 0$ therefor R is false, S is true.

Alternate :

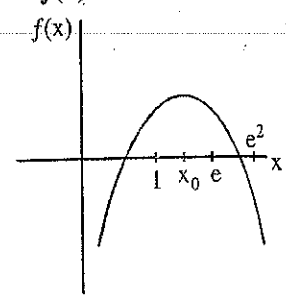
$f(x) = x + \ell nx - x\ell nx$

$f'(x) = \frac{1}{x} - \ell nx = 0 =$ at $x = x_0$ where

$x_0 \in (1,e)$

$$f''(x) = -\frac{1}{x^2} - \frac{1}{x} < 0\ \forall\ x > 0$$

$\Rightarrow f(x)$ concave down



9. Ans. (C)

Sol. Using LMVT on $f(x)$ for $x \in \left[\frac{1}{2}, 1\right]$

$$\frac{f(1)-f\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = f'(c), \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$$\frac{1-\frac{1}{2}}{\frac{1}{2}} = f'(c) \Rightarrow f'(c) = 1, \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$\because$ $f'(x)$ is an increasing function $\forall\ x \in R$

$\therefore$ $f'(1) > 1$

10. Ans. (B, C)

Sol. Let $F(x) = f(x) - 3g(x)$

$\therefore F(-1) = 3;\ F(0) = 3\ \&\ F(2) = 3$

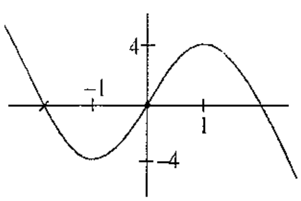
$\therefore F'(x)$ will vanish at least twice in $(-1,0) \cup (0,2)$

$\because F''(x) > 0$ or $< 0\ \forall\ x \in (-1,0) \cup (0,2)$

So there will be exactly 9 solution in $(-1,0)$ and one in $(0,2)$

11. Ans. (B, D)

Sol.



$f(x) = x^5 - 5x + a$

$\therefore a = 5x - x^5$

$\therefore f(x)$ has only one real root if $a > 4$ or $a < -4$

$f(x)$ has three real roots if $-4 < a < 4$

12. Ans. (C)

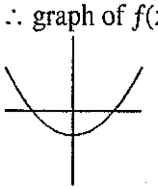
Sol. $f(x) = x^2 - x\sin x - \cos x$

$f'(x) = 2x - x\cos x - \sin x + \sin x$

$= x(2 - \cos x)$

$\underset{0}{\overset{-\quad\quad +}{\longleftrightarrow}}$

$\therefore$ graph of $f(x)$ will be



$\therefore f(x)$ is zero for 2 values of x

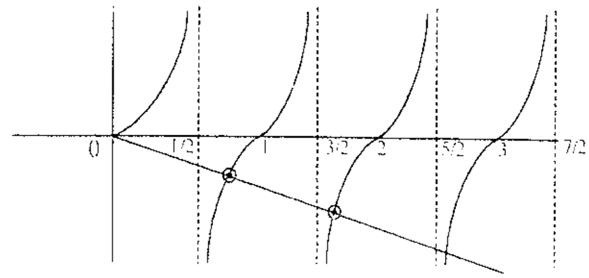
$\therefore$ (c)

13. Ans. (B, C)

Sol. $f'(x) = \sin\pi x + \pi x\cos\pi x = 0$

$\tan\pi x = -\pi x$

$y = \tan x\pi\ \&\ y = -\pi x$



intersection point lies in ...

$$\left(\frac{1}{2},1\right) \cup \left(\frac{3}{2},2\right) \cup \left(\frac{5}{2},3\right)$$

option (B) is correct for $\left(n+\frac{1}{2}, n\right)$

as well $(n, (n+1))$ because root lies in

$$(0,1) \cup (1,2) \cup (2,3)$$

14. Ans (C)

Sol. $f(x) = (1-x)^2\sin^2 x + x^2$

P : $f(x) + 2x = 2(1 + x^2)$

$\Rightarrow (1-x)^2\sin^2 x + x^2 + 2x = 2 + 2x^2$

$\Rightarrow (1-x)^2\sin^2 x - x^2 + 2x - 2 = 0$

$(1-x)^2\cos^2 x + 1 = 0$

which is not possible.

$\therefore$ P is false.

Q : $2f(x) + 1 = 2x(1 + x)$

$2x^2 + 2(1-x)^2\sin^2 x + 1 = 2x^2 + 2x$

$2(1-x)^2\sin^2 x - 2x + 1 = 0.$

Let $h(x) = 2(1-x)^2\sin^2 x - 2x + 1$, clearly $h(1) = -1$

and $h(x) = 2(x^2 - 2x + 1)\sin^2 x - 2x + 1$

$$= x^2\left[2\left(1-\frac{2}{x}+\frac{1}{x^2}\right).\sin^2 x - \frac{2}{x} + \frac{1}{x^2}\right]$$

$\therefore h(x) \to \infty$ as $x \to \infty$.

$\therefore$ By intermediate value theorem

$h(x) = 0$ has a root which is greater than 1.

Hence Q is true.

15. Ans. (B)

Sol. $g(x)=\int_1^x\left(\frac{2(t-1)}{(t+1)}-\ell nt\right)f(t)dt$

$g'(x)=\left(\frac{2(x-1)}{x+1}-\ell nx\right)f(x)$

$f(x)>0\ \forall\ x\in R$

Suppose

$h(x)=\frac{2(x-1)}{x+1}-\ell nx$

$h(x)=2-\left(\frac{4}{x+1}+\ell nx\right)$

$h'(x)=\frac{4}{(x+1)^2}-\frac{1}{x}$

$h'(x)=-\frac{(x-1)^2}{x(x+1)^2}$

$h'(x)<0$

So $h(x)$ is decreasing

so $h(x)<h(1)$. $\forall\ x>1$

$h(x)<0$ $\forall\ x>1$

So $g'(x)=h(x)\ f(x)$

$g'(x)<0$ $\forall\ x>1$

$g(x)$ is decreasing in $(1,\infty)$

16. Ans. 2

Sol. Let $f(x)=x^4-4x^3+12x^2+x-1$

$f'(x)=4x^3-12x^2+24x$

$f''(x)=12x^2-24x+24$

$=12(x^2-2x+2)>0$

$\Rightarrow f'(x)$ is strictly increasing function

$\because f'(x)$ is cubic polynomial

hence number of roots of $f'(x)=0$ is 1

$\Rightarrow$ Number of maximum roots of $f(x)=0$ are 2

Now $f(0)=-1,\ f(1)=9,\ f(-1)=15$

$\Rightarrow f(x)$ has exactly 2 distinct real roots.

17. Ans. (B, C, D)

Sol. Let $x\geq 1$

Consider $[x, x+2]$

by LMVT

$\frac{f(x+2)-f(x)}{2}$

$=f'(c)$, for some $c\in(x, x+2)$

$f'(x)=\cos\frac{1}{x}+\frac{1}{x}\sin\frac{1}{x}$

$f''(x)=-\frac{1}{x^3}\cos\frac{1}{x}$

$\Rightarrow f'(x)$ decreasing function

since $\lim\limits_{x\to\infty}f'(x)=1$ & $f'(1)=\cos 1+\sin 1>1$

$\Rightarrow f'(x)>1$

$\Rightarrow \frac{f(x+2)-f(x)}{2}>1$ for all x.(by LMVT)

$f''(x)=-\frac{1}{x^3}\cos\frac{1}{x}$

($f'(x)$ is strictly decreasing)

18. Ans. (C)

Sol. $g(u)=2\tan^{-1}(e^u)-\frac{\pi}{2}$

$g(u)=2\tan^{-1}(e^u)-\tan^{-1}(e^u)-\cot^{-1}(e^u)$

$=\tan^{-1}(e^u)-\cot^{-1}(e^u)$

$g(-u)=\tan^{-1}(e^u)-\cot^{-1}(e^u)$

$=\cot^{-1}(e^u)-\tan^{-1}(e^u)$

$g(-u)=-g(u)$

Hence it is an odd functions. and

$g'(u)=\frac{2e^u}{1+e^{2u}}>0$

so it is an increasing function.

MAXIMA & MINIMA

1. Ans. (A, B)

Sol.
$$f(x)=\begin{cases}0 & : 0<x<\frac{1}{4}\\ \left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & : \frac{1}{4}\le x<\frac{1}{2}\\ 2\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & : \frac{1}{2}\le x<\frac{3}{4}\\ 3\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & : \frac{3}{4}\le x<1\end{cases}$$

f(x) is discontinuous at $x=\frac{3}{4}$ only

$$f'(x)=\begin{cases}0 & : 0<x<\frac{1}{4}\\ 2\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+\left(x-\frac{1}{4}\right)^2 & : \frac{1}{4}<x<\frac{1}{2}\\ 4\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+2\left(x-\frac{1}{4}\right)^2 & : \frac{1}{2}<x<\frac{3}{4}\\ 6\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+3\left(x-\frac{1}{4}\right)^2. & : \frac{3}{4}<x<1\end{cases}$$

f(x) is non-differentiable at $x=\frac{1}{2}$ and $\frac{3}{4}$

minimum values of f(x) occur at $x=\frac{5}{12}$ whose value is $-\frac{1}{432}$

2. **Ans. (A, B, C)**

Sol. $\alpha=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^6+\ldots$

$$\alpha=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}$$

$\therefore g(x)=2^{x/3}+2^{1/3(1-x)}$

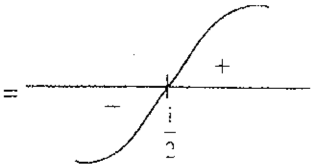
$\therefore g(x)=2^{x/3}+\dfrac{2^{1/3}}{2^{x/3}}$

where $g(0)=1+2^{1/3}$ & $g(1)=1+2^{1/3}$

$\therefore g'(x)=\frac{1}{3}\left(2^{x/3}-\frac{2^{1/3}}{2^{x/3}}\right)=0$

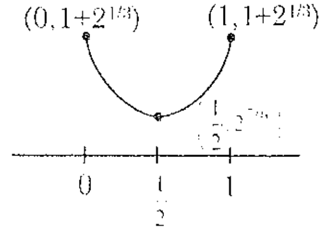
$\Rightarrow 2^{2x/3}=2^{1/3}\Rightarrow x=\frac{1}{2}$ = critical point

$\therefore$ graph of $g'(x)$ =



& $g\left(\frac{1}{2}\right)=2^{\frac{7}{6}}$

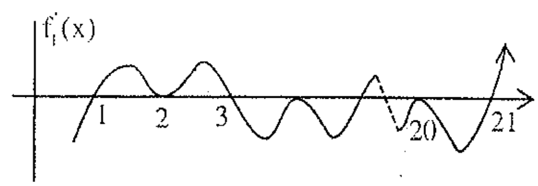
$\therefore$ graph of g(x) in [0, 1]



3. **Ans. (57.00)**

Sol. $f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j dt$

$f_1'(x)=\prod_{j=1}^{21}(x-j)^j=(x-1)(x-2)^2(x-3)^3\ldots(x-21)^{21}$



So points of minima one 4m + 1 where

$m=0,1,\ldots..5\Rightarrow m_1=6$

Points of maxima are 4m – 1 where

$m=1,2,\ldots..5\Rightarrow n_1=5$

$\Rightarrow 2m_1+3n_1+m_1n_1=57$

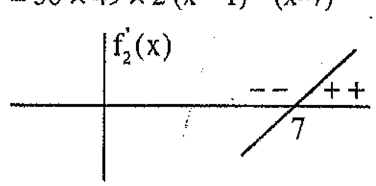
4. **Ans. (6.00)**

Sol. $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450$

$\Rightarrow f_2'(x)=2\times49\times50(x-1)^{49}-50\times12\times49(x-1)^{48}$

$=50\times49\times2(x-1)^{48}(x-1-6)$

$=50\times49\times2\,(x-1)^{48}\,(x-7)$



Point of minima = 7

$\Rightarrow m_2=1$

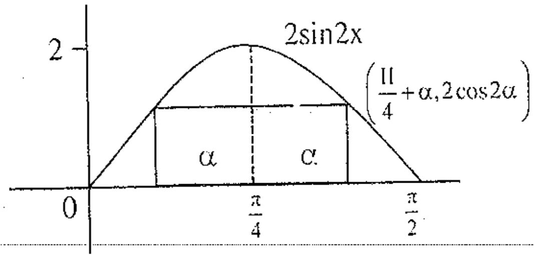
No point of maxima

$\Rightarrow n_2=0$

$6m_2+4n_2+8m_2n_2=6$

5. **Ans. (C)**

Sol.



Perimeter = $2(2\alpha + 2\cos 2\alpha)$

$P = 4(\alpha + \cos 2\alpha)$

$\frac{dP}{d\alpha} = 4(1 - 2\sin 2\alpha) = 0$

$\sin 2\alpha = \frac{1}{2}$

$2\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{d^2P}{d\alpha^2} = -4\cos 2\alpha$

for maximum $\alpha = \frac{\pi}{12}$

Area = $(2\alpha)(2\cos 2\alpha)$

$= \frac{\pi}{6} \times 2 \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$

6. **Ans. (0.50)**

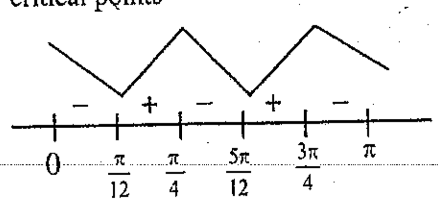
Sol. $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$

$f(\theta) = \sin^2 2\theta - \sin 2\theta + 2$

$f'(\theta) = 2(\sin 2\theta).(2\cos 2\theta) - 2\cos 2\theta$

$= 2\cos 2\theta(2\sin 2\theta - 1)$

critical points



so, minimum at $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$\lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$

7. **Ans. (A, B, D)**

Sol.

$$f(x) = \begin{cases} (x+1)^5 - 2x, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

for $x < 0$, $f(x)$ is continuous

& $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to 0} f(x) = 1$

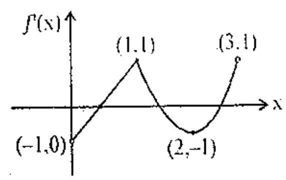
Hence, $(-\infty, 1) \subset$ Range of $f(x)$ in $(-\infty, 0)$

$f'(x) = 5(x+1)^4 - 2$, which changes sign in $(-\infty, 0)$

$\Rightarrow f(x)$ is non-monotonic in $(-\infty, 0)$

For $x \ge 3$, $f(x)$ is again continuous and

$\lim\limits_{x \to \infty} f(x) = \infty$ and $f(3) = \frac{1}{3}$



$\Rightarrow \left[\frac{1}{3}, \infty\right) \subset$ Range of $f(x)$ in $[3, \infty)$

Hence, range of $f(x)$ is $\mathbb{R}$

$$f'(x) = \begin{cases} 2x - 1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \end{cases}$$

Hence f' has a local maximum at x = 1 and f' is NOT differentiable at x = 1.

8. **Ans. (A,B,C)**

Sol. $f(x) = (x-1)(x-2)(x-5)$

$F(x) = \int_0^x f(t)\,dt, x > 0$

$F'(x) = f(x) = (x-1)(x-2)(x-5), x > 0$

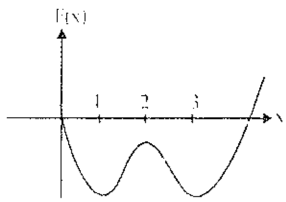
clearly F(x) has local minimum at x = 1,5

F(x) has local maximum at x = 2

$f(x) = x^3 - 8x^2 + 17x - 10$

$$\Rightarrow F(x) = \int_0^x (t^3 - 8t^2 + 17t - 10)\,dt$$

$$F(x) = \frac{x^4}{4} - \frac{8x^3}{3} + \frac{17x^2}{2} - 10x$$

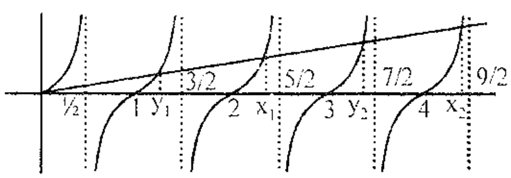


from the graph of $y = F(x)$.

clearly $F(x) \neq 0\ \forall\ x \in (0, 5)$

9. Ans. (A, C, D)

Sol. $f(x) = \dfrac{\sin \pi x}{x^2}$



$$f'(x) = \frac{2x\cos \pi x\left(\frac{\pi x}{2} - \tan \pi x\right)}{x^4}$$

$\Rightarrow \quad |x_n - y_n| > 1$ for every n

$x_1 > y_1$

$x_n \in (2n, 2n + 1/2)$

$x_{n+1} - x_n > 2.$

10. Ans. (A, B, D)

Sol. $f(x)$ can't be constant throughout the domain. Hence we can find $x \in (r, s)$ such that $f(x)$ is one-one

option (A) is true.

Option (B) :

$$|f'(x_0)| = \left|\frac{f(0) - f(-4)}{4}\right| \leq 1 \text{ (LMVT)}$$

Option (C) :

$f(x) = \sin(\sqrt{85}x)$ satisfies given condition

but $\quad \lim\limits_{x\to\infty} \sin(\sqrt{85})$ D.N.E.

$\Rightarrow \quad$ Incorrect

Option (D) : $g(x) = f^2(x) + (f'(x))^2$

By LMVT $\exists\ x_1 \in (-4, 0)$ such that $|f'(x_1)| \leq 1$

$|f(x_1)| \leq 2 \quad$ (given)

$\Rightarrow \quad g(x_1) \leq 5$

Similarly, we can find some $x_2 \in (0, 4)$ such that $g(x_2) \leq 5$

$g(0) = 85 \Rightarrow g(x)$ has maxima in (x_1, x_2) say at α.

$g'(\alpha) = 0$ & $g(\alpha) \geq 85$

$2f'(\alpha)(f(\alpha) + f''(\alpha)) = 0$

If $f'(\alpha) = 0 \Rightarrow g(\alpha) = f^2(\alpha) \geq 85$ Not possible

$\Rightarrow \quad f(\alpha) + f''(\alpha) = 0$

$\exists \alpha \in (x_1, x_2) \in (-4, 4)$

option (D) correct.

11. Ans. (B, D)

Sol. Expansion of determinant

$f(x) = \cos 2x + \cos 4x$

$f'(x) = -2\sin 2x - 4\sin 4x = -2\sin x(1 + 4\cos 2x)$

$$\underset{0}{\overset{+ \qquad -}{\longleftarrow\!\!+\!\!\longrightarrow}}$$

$\therefore$ maxima at $x = 0$

$f'(x) = 0 \Rightarrow$

$$x = \frac{n\pi}{2},\ \cos 2x = -\frac{1}{4}$$

$\Rightarrow$ more than two solutions

12. Ans. (C)

Sol. $f(x) = 4\alpha x^2 + \dfrac{1}{x};\ x > 0$

$$f'(x) = 8\alpha x - \frac{1}{x^2}$$

$$= \frac{8\alpha x^3 - 1}{x^2}$$

$f(x)$ attains its minimum at $x = \left(\dfrac{1}{8\alpha}\right)^{1/3}$

$$f\left(\left(\frac{1}{8\alpha}\right)^{1/3}\right) = 1$$

$$\Rightarrow 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} = 1$$

$$\Rightarrow 3\alpha^{1/3} = 1 \Rightarrow \alpha = \frac{1}{27}$$

13. Ans. (A, D)

Sol. Using L'Hôpital's Rule

$$\lim_{x\to 2}\frac{f'(x)g(x)+f(x)g'(x)}{f''(x)g'(x)+f'(x)g''(x)}=1$$

$$\Rightarrow \frac{f(2)g'(2)}{f''(2)g'(2)}=1 \Rightarrow \quad f'(2)=f(2)>0$$

option (D) is right and option (C) is wrong

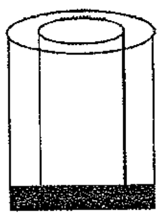
also $f'(2) = 0$ and $f''(2) > 0$

$\therefore$ x = 2 is local minima.

14. Ans. (4)

Sol. Let the inner radius of cylinderical container be r then radius of outer cylinder is (r + 2).

Now, $V = \pi r^2 h$ where h is height of cylinder



Now, Let volume of total material used be T

$$\therefore\ T=\pi\left((r+2)^2-r^2\right).\frac{V}{\pi r^2}+\pi(r+2)^2.2$$

$$\therefore\ T=V\left(\frac{r+2}{r}\right)^2+2\pi(r+2)^2-V$$

$$\text{Now}\ \frac{dT}{dr}=2V\left(\frac{r+2}{r}\right)\times\left(-\frac{2}{r^2}\right)+4\pi(r+2)$$

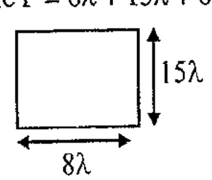
$$\text{Now At } r=10\text{mm}\ \frac{dT}{dr}=0$$

$$\therefore 0=(r+2).4\left(\pi-\frac{V}{r^3}\right)$$

$$\Rightarrow \frac{V}{\pi}=1000\Rightarrow\frac{V}{250\pi}=4$$

15. Ans. (A, C)

Sol. Where $P = 8\lambda + 15\lambda + 8\lambda + 15\lambda$ & λ is constant



Let removed length from each sides is x

Removed area is $4x^2 = 100 \Rightarrow x = 5$

$V = (8\lambda - 2x)(15\lambda - 2x)x$

$V = 120\lambda^2 x - 46\lambda x^2 + 4x^3$

$$\frac{dv}{dx}=120\lambda^2-92\lambda x+12x^2=0$$

Put $x = 5 \Rightarrow 120\lambda^2 - 460\lambda + 300 = 0$

$12\lambda^2 - 40\lambda + 30 = 0$

$6\lambda^2 - 23\lambda + 15 = 0$

$(\lambda - 3)(6\lambda - 5) = 0$

$\lambda = 3$ & $\lambda = \dfrac{5}{6}$

$$\frac{d^2v}{dx^2}=-92\lambda+24x=120-92\lambda$$

$$\text{at } \lambda=3\Rightarrow\frac{d^2v}{dx^2}<0$$

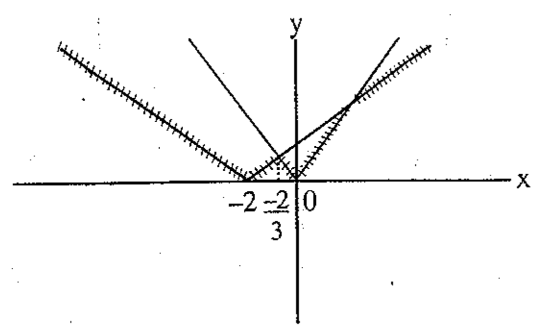
$$\text{at } \lambda=\frac{5}{6}\Rightarrow\frac{d^2v}{dx^2}>0\ \text{(rejected)}$$

16. Ans. (A, B)

Sol. $f(x) = (a + b) - |b - a|$

$$=\begin{cases}2a\ , & a\le b\\ 2b\ , & a>b\end{cases}=2\min(a,b)$$

where $a = 2|x|$, $b = |x + 2|$

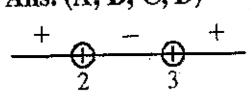


$\therefore$ Local maxima and minima at

$x=-2,\ -\dfrac{2}{3}\ \&\ 0$

17. Ans. (A, B, C, D)

Sol.



$$f(x)=\int_0^x e^{t^2}(t-2)(t-3)dt$$

$\Rightarrow \quad f'(x) = e^{x^2}(x-2)(x-3)$

$\therefore \quad f'(2) = f'(3) = 0$

$\Rightarrow \quad f''(c) = 0$ for same $c \in (2, 3)$

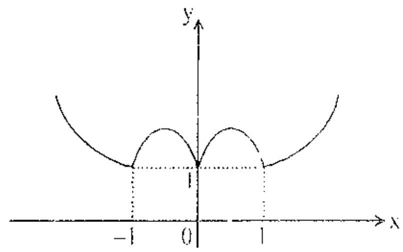
(by Rolle's theorem)

18. Ans. (5)

Sol. $f(x) = |x| + |(x+1)(x-1)|$

$\Rightarrow \quad f(x)$

$$= \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \le x < 0 \\ -x^2 + x + 1 & 0 \le x < 1 \\ x^2 + x - 1 & x \ge 1 \end{cases}$$



$\therefore$ f has 5 points where it attains either a local maximum or local minimum.

19. Ans. (9)

Sol. Let $P'(x) = k(x-1)(x-3)$

$= \quad k(x^2 - 4x + 3)$

$\Rightarrow \quad P(x) = k\left(\dfrac{x^3}{3} - 2x^2 + 3x\right) + c$

$\because \quad P(1) = 6$

$\Rightarrow \quad \dfrac{4k}{3} + c = 6 \quad$ ....(1)

$P(3) = 2$

$\Rightarrow \quad c = 2 \quad$ ....(2)

by (i) and (ii)

$k = 3$

$\therefore \quad P'(x) = 3(x-1)(x-3)$

$\Rightarrow \quad P'(0) = 9$

20. **Ans. (D)**

Sol. If $x \in [0, 1]$

then $x^2 \le x \le 1$

$x^2 e^{x^2} \le x e^{x^2} \le e^{x^2}$

Add e^{-x^2} to all sides

$x^2 e^{x^2} + e^{-x^2} \le x e^{x^2} + e^{-x^2} \le e^{x^2} + e^{-x^2}$

$\Rightarrow h(x) \le g(x) \le f(x) \quad$ ....(i)

where, $f(x) = e^{x^2} + e^{-x^2}$

$f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) > 0$

$\Rightarrow$ f(x) has a maxima at x = 1

$\Rightarrow a = e + \dfrac{1}{e}$

$h(x) = x^2 e^{x^2} + e^{-x^2}$

$h'(x) = 2x^3 e^{x^2} + 2x e^{x^2} - 2x e^{-x^2}$

$= 2x^3 e^{x^2} + 2x\left(e^{x^2} - x e^{-x^2}\right) > 0$

$\Rightarrow$ h(x) has a maxima at x = 1

$\Rightarrow c = e + \dfrac{1}{e}$

Now $\because h(x) \le g(x) \le f(x)$

$\Rightarrow$ g(x) also has a maximum value at x = 1

$\Rightarrow a = b = c$

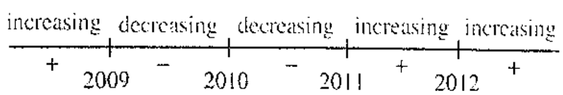
21. Ans. (1)

Sol. $f'(x) = 2010(x - 2009)(x - 2010)^2(x - 2011)^3 (x - 2012)^4 \ \forall x \in R$

$f(x) = \ln(g(x)) \ \forall x \in R$

$g(x) = e^{f(x)}$

$g'(x) = 0 \Rightarrow e^{f(x)}.f'(x) = 0 \Rightarrow f'(x) = 0$



local maximum at x = 2009, hence only 1 point.

22. Ans. (7)

Sol. $f(x) = 2x^3 - 15x^2 + 36x - 48$

Set $A = \{x \mid x^2 + 20 \le 9x\}$

$\because x^2 - 9x + 20 \le 0$

$(x-5)(x-4) \le 0$

$\Rightarrow x \in [4, 5]$

Now, $f'(x) = 6x^2 - 30x + 36 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$x = 2, 3$ and $f(x) \uparrow$ in $x \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow$ In the set A, f(x) is increasing

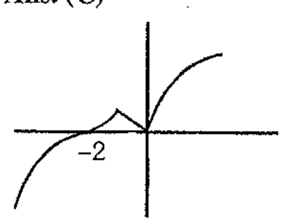
$\Rightarrow f(x)_{max} = f(5)$

$= 2.125 - 15.25 + 36.5 - 48$

$= 7$

23. **Ans. (C)**

Sol.



one maxima & one minima

INDEFINITE INTEGRATION

1. Ans. (A, C)

Sol. $f'(x) = \dfrac{f(x)}{b^2 + x^2}$

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{dx}{x^2 + b^2}$$

$$\Rightarrow \ell n|f(x)| = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + c$$

Now $f(0) = 1$

$\therefore c = 0$

$$\therefore |f(x)| = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$$\Rightarrow f(x) = \pm e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

since $f(0) = 1$

$$\therefore f(x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$x \to -x$

$$f(-x) = e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\therefore f(x).f(-x) = e^0 = 1$ (option C)

and for $b > 0$

$$f(x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\Rightarrow f(x)$ is increasing for all $x \in R$ (option A)

2. **Ans. (C)**

Sol. Let $I = \displaystyle\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}dx$

$$= \int \frac{\sec x(\sec x + \tan x)\sec x}{(\sec x + \tan x)^{11/2}}dx$$

Put $\sec x + \tan x = t$

$\Rightarrow \quad (\sec x \tan x + \sec^2 x)\, dx = dt$

Also $\because \sec^2 x - \tan^2 x = 1$

$$\Rightarrow \quad \sec x - \tan x = \frac{1}{t}$$

$$\therefore \quad \sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)$$

$$\therefore \quad I = \frac{1}{2}\int \frac{\left(1 + \frac{1}{t}\right)dt}{t^{11/2}}$$

$$\Rightarrow \quad I = \frac{1}{2}.\int \left(t^{-9/2} + t^{-13/2}\right)dt$$

$$\Rightarrow I = \frac{1}{2}\left(-\frac{2t^{-7/2}}{7} - \frac{2t^{-11/2}}{11}\right) + K$$

$$\Rightarrow \quad I = -\frac{1}{t^{11/2}}\left(\frac{1}{11} + \frac{t^2}{7}\right) + K$$

$$\Rightarrow \quad I = -\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$

3. Ans. (C)

Sol. $J - I = \displaystyle\int \left(\frac{e^{3x}}{e^{4x} + e^{2x} + 1} - \frac{e^x}{e^{4x} + e^{2x} + 1}\right)dx$

$$= \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1}dx = \int \frac{e^x(e^{2x} - 1)}{e^{4x} + e^{2x} + 1}dx$$

Let $e^x = t \quad e^x dx = dt$

$$= \int \frac{t^2 - 1}{t^4 + t^2 + 1}dt = \int \frac{1 - 1/t^2}{(t + 1/t)^2 - 1}dt$$

$$= \frac{1}{2}\ell n\left|\frac{t + \frac{1}{t} - 1}{t + \frac{1}{t} + 1}\right| + C = \frac{1}{2}\ell n\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + C$$

DEFINITE INTEGRATION

1. **Ans. (C)**

Sol. $\displaystyle\int_{x^2}^{x} \sqrt{\frac{1-t}{t}}dt.\sqrt{n} \le f(x)g(x) \le 2\sqrt{x}\sqrt{n}$

$$\because \int_{x^2}^{x} \sqrt{\frac{1-t}{t}}dt = \sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}x - x\sqrt{1-x^2}$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}x - x\sqrt{1-x^2}}{\sqrt{x}} \le f(x)g(x) \le \frac{2\sqrt{x}}{\sqrt{x}}\right)$$

$$\Rightarrow 2 \le \lim_{x\to 0} f(x)g(x) \le 2$$

$$\Rightarrow \lim_{x\to 0} f(x)g(x) = 2$$

2. Ans. (0)

Sol. $f(x) = \int_0^{x\tan^{-1}x} \frac{e^{(t-\cos t)}}{1+t^{2023}}dt$

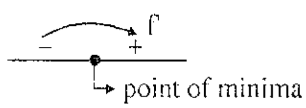
$f'(x) = \frac{e^{(x\tan^{-1}x - \cos(x\tan^{-1}x))}}{1+(x\tan^{-1}x)^{2023}} \cdot \left(\frac{x}{1+x^2} + \tan^{-1}x\right)$

For $x < 0$, $\tan^{-1}x \in \left(-\frac{\pi}{2}, 0\right)$

For $x \geq 0$, $\tan^{-1}x \in \left[0, \frac{\pi}{2}\right)$

$\Rightarrow x\tan^{-1}x \geq 0 \ \forall\ x \in R$

And $\frac{x}{1+x^2} + \tan^{-1}x = \begin{cases} >0 & \text{For } x>0 \\ <0 & \text{For } x<0 \\ 0 & \text{For } x=0 \end{cases}$



Hence minimum value is $f(0) = \int_0^0 = 0$

3. Ans. (C, D)

Sol. $\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2} = 1$

Let $a - (\log_e x)^{3/2} = t$

$\frac{(\log_e x)^{1/2}}{x}dx = -\frac{2}{3}dt$

$= \frac{2}{3}\int_a^{a-1}\frac{-dt}{t^2} = \frac{2}{3}\left(\frac{1}{t}\right)_a^{a-1} = 1$

$\frac{2}{3a(a-1)} = 1$

$3a^2 - 3a - 2 = 0$

$a = \frac{3 \pm \sqrt{33}}{6}$

4. Ans. (5)

Sol. $f(x) = \log_2(x^3+1) = y$

$x^3 + 1 = 2^y \Rightarrow x = (2^y - 1)^{1/3} = f^{-1}(y)$

$f^{-1}(x) = (2^x - 1)^{1/3}$

$= \int_1^2 \log_2(x^3+1)dx + \int_1^{\log_2 9}(2^x-1)^{1/3}dx$

$= \int_1^2 f(x)dx + \int_1^{\log_2 9} f^{-1}(x)dx = 2\log_2 9 - 1$

$= 8 < 9 < 2^{7/2} \Rightarrow 3 < \log_2 9 < \frac{7}{2}$

$= 5 < 2\log_2 9 - 1 < 6$

$[2\log_2 9 - 1] = 5$

5. Ans. (B)

Sol. $f(n) = n + \sum_{r=1}^{n}\frac{16r+(9-4r)n-3n^2}{4rn+3n^2}$

$f(n) = n + \sum_{r=1}^{n}\frac{(16r+9n)-(4rn+3n^2)}{4rn+3n^2}$

$f(n) = n + \left(\sum_{r=1}^{n}\frac{16r+9n}{4rn+3n^2}\right) - n$

$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\sum\frac{16r+9n}{4rn+3n^2}$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\left(16\left(\frac{r}{n}\right)+9\right)\frac{1}{n}}{4\left(\frac{r}{n}\right)+3}$

$= \int_0^1\frac{16x+9}{4x+3}dx = \int_0^1 4\,dx - \int_0^1\frac{3\,dx}{4x+3}$

$= 4 - \frac{3}{4}(\ell n|4x+3|)_0^1$

$= 4 - \frac{3}{4}\ell n\frac{7}{3}$

6. Ans. (A, B, C)

Sol. (A) Let $g(x) = f(x) - 3\cos 3x$

Now,

$$\int_0^{\pi/3} g(x)dx = \int_0^{\pi/3} f(x)dx - 3\int_0^{\pi/3} \cos 3x\,dx = 0$$

Hence $g(x) = 0$ has a root in $\left(0, \frac{\pi}{3}\right)$

(B) Let $h(x) = f(x) - 3\sin 3x + \frac{6}{\pi}$

Now,

$$\int_0^{\pi/3} h(x)dx = \int_0^{\pi/3} f(x)dx - 3\int_0^{\pi/3} \sin 3x\,dx + \int_0^{\pi/3} \frac{6}{\pi}dx$$

$= 0 - 2 + 2 = 0$

Hence $h(x) = 0$ has a root in $\left(0, \frac{\pi}{3}\right)$

(C) $$\lim_{x\to 0} \frac{x\int_0^x f(t)\,dt}{1-e^{x^2}} = \lim_{x\to 0} \underbrace{\left(\frac{x^2}{1-e^{x^2}}\right)}_{-1} \underbrace{\frac{\int_0^x f(t)\,dt}{x}}_{\text{Apply L' Hopital's Rule}}$$

$$= -1 \lim_{x\to 0} \frac{f(x)}{1} = -1$$

(D) $$\lim_{x\to 0} \frac{(\sin x)\int_0^x f(t)\,dt}{x^2}$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin x}{x}\right)}_{1} \underbrace{\frac{\int_0^x f(t)\,dt}{x}}_{\text{Apply L' Hopitals Rule}}$$

$$= 1 \lim_{x\to 0} \frac{f(x)}{1} = 1$$

7. Ans. (2.00)

Sol.

$$S_1 = \int_{\pi/8}^{3\pi} f(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\,dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)dx$$

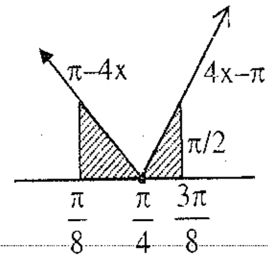
$$= \int_{\pi/8}^{3\pi/8} \cos^2 x\,dx$$

$$2S_1 = \int_{\pi/8}^{3\pi/8} (\sin^2 x + \cos^2 x)dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

8. Ans. (1.50)

Sol.



$$S_2 = \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\,|4x-\pi|\,dx$$

$$= \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{2}-x\right)\left|4\left(\frac{\pi}{2}-x\right)-\pi\right|dx$$

$$= \int_{\pi/8}^{3\pi/8} (\cos^2 x)|\pi - 4x|\,dx$$

$$\Rightarrow 2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|\,(\sin^2 x + \cos^2 x)dx$$

$$= \int_{\pi/8}^{3\pi/8} |4x-\pi|\,dx$$

$$= 2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$$

9. Ans. (C)

Sol. $f'(x) = (|x| - x^2)e^{-x^2} + (|x| - x^2)e^{-x^2},\ x \geq 0$

$f' = 2(x - x^2)\,e^{-x^2}$

$$\begin{array}{ccc} & +++ & --- \\ 0 & & 1 \end{array}$$

hence option (D) is wrong

$g'(x) = xe^{-x^2}\,2x$

$f'(x) + g'(x) = 2\times e^{-x^2}$

$f(x) + g(x) = -e^{-x^2} + c$

$f(x) + g(x) = -e^{-x^2} + 1$

$f(\ell n3) + g\left(\sqrt{\ell n3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$

(option (A) is wrong)

$H(x) = \psi_1(x) - 1 - \alpha x = e^{-x} + x - 1 - \alpha x$.

$x \geq 1$ & $\alpha \in (1, x)$

$H(1) = e^{-1} + 1 - 1 - \alpha < 0$

$H'(x) = -e^{-x} + 1 - \alpha > 0 \Rightarrow H(x)$ is $\downarrow$

$\Rightarrow$ option (B) is wrong

(C) $\psi_2(x) = 2(\psi_1(\beta) - 1)$

Applying L.M.V.T to $\psi_2(x)$ in $[0, x]$

$$\psi_2'(\beta) = \frac{\psi_2(x) - \psi_2(0)}{x}$$

$$2\beta - 2 + 2e^{-\beta} = \frac{\psi_2(x) - 0}{x}$$

$\Rightarrow \psi_2(x) = 2x(\psi_1(\beta) - 1)$ has one solution

option (C) is correct.

10. Ans. (D)

Sol. (A) $\psi_1(x) = e^{-x} + x, \quad x \geq 0$

$\psi_1'(x) = 1 - e^{-x} > 0 \quad \Rightarrow \psi_1(x)$ is $\uparrow$

$\psi_1(x) \geq \psi_1(0) \quad \forall x \geq 0 \Rightarrow \psi_1(x) \geq 1$

(B) $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x} \quad x \geq 0$

$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \geq 0 \quad \forall x \geq 0$

$\Rightarrow \psi_2(x)$ is $\uparrow$

$\Rightarrow \psi_2(x) \geq \psi_2(0) \Rightarrow \psi_2(x) \geq 0$

(C) $f(x) = 2\int_0^x (t - t^2)e^{-t^2} dt$ & $x \in \left(0, \frac{1}{2}\right)$

Let, $H(x) = f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5, \ x \in \left(0, \frac{1}{2}\right)$

$H(0) = 0$

$H'(x) = 2(x - x^2)e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$

$= -2x^2e^{-x^2} + 2x^2 - 2x^4$

$= 2x^2(1 - x^2 - e^{-x^2})$

$\because e^{-x} \geq 1 - x \ \forall x \geq 0$

$\Rightarrow H'(x) \leq 0$

$\Rightarrow H(x)$ is $\downarrow \Rightarrow H(x) \leq 0 \ \forall x \in \left(0, \frac{1}{2}\right)$

$f(x) \leq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5; \forall x \in \left(0, \frac{1}{2}\right)$

Let $P(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7 x \in \left(0, \frac{1}{2}\right)$

$P'(x) = 2x^2e^{-x^2} - 2x^2 + 2x^4 - x^6$

$= 2x^2\left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3} + \ldots\right) - 2x^2 + 2x^4 - x^6$

$= -\frac{x^8}{3} + \frac{x^{10}}{12} \ldots\ldots\ldots$

$\Rightarrow P'(x) \leq 0$

$\Rightarrow P(x)$ is $\downarrow$

$\Rightarrow P(x) \leq 0$

option (D) is correct

11. Ans. (182)

Sol. Let $f(x) = \left(\frac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\frac{(x+1) - x}{(x+1)^2}\right) = \frac{10}{(x+1)^2} > 0$

$\forall x \in [0, 10]$,

So, $f(x)$ is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$I = \int_0^{1/9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx +$$

$$\int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$= 0 + \int_{1/9}^{2/3} dx + 2\int_{2/3}^{9} dx + 3\int_{9}^{10} dx$$

$$= \frac{2}{3} - \frac{1}{9} + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3$$

$$= \frac{5 + 150 + 27}{9} = \frac{182}{9} \Rightarrow 9I = 182$$

12. Ans. (A, B, D)

Sol. (A) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$

$\cos x \geq 1 - \frac{x^2}{2}$

$\int_0^1 x\cos x \geq \int_0^1 x\left(1 - \frac{x^2}{2}\right) - \frac{1}{2} - \frac{1}{8}$

$\int_0^1 x\cos x \geq \frac{3}{8}$ (True)

(B) $\sin x \geq x - \frac{x^3}{6}$

$\int_0^1 x\sin x \geq \int_0^1 x\left(x - \frac{x^3}{6}\right)dx$

$\int_0^1 x\sin x \geq \frac{1}{3} - \frac{1}{30} \Rightarrow \int_0^1 x\sin x\ dx \geq \frac{3}{10}$

(True)

(D) $\int_0^1 x^2\sin x\ dx \geq \int_0^1 x^2\left(x - \frac{x^3}{6}\right)dx$

$\int_0^1 x^2\sin x\ dx \geq \frac{1}{4} - \frac{1}{36}$

$\int_0^1 x^2\sin x\ dx \geq \frac{2}{9}$ (True)

(C) $\cos x < 1$

$x^2\cos x < x^2$

$\int_0^1 x^2\cos x\ dx < \int_0^1 x^2 dx$

$\int_0^1 x^2\cos x\ dx < \frac{1}{3}$

So, option (C) is incorrect.

13. Ans. (1080.00)

Sol. $F(x) = \int_0^x f(t).\ dt$

$\Rightarrow F'(x) = f(x)$

$I = \int_0^\pi f'(x).\cos x\ dx + \int_0^\pi F(x)\cos(x)\ dx = 2 ...(1)$

$I_1 = \int_0^\pi f'(x).\cos x\ dx$ (Let)

Using by parts

$I_1 = (\cos x.f(x))_0^\pi + \int_0^\pi \sin x.f(x)dx$

$I_1 = 6 - f(0) + \int_0^\pi \sin x.F'(x)dx$

$I_1 = 6 - f(0) + I_2$ ...(2)

$I_2 = \int_0^\pi \sin x.F'(x).dx$

Using by part we get

$I_2 = (\sin x.F(x))_0^\pi - \int_0^\pi \cos x.F(x)\ dx$

$I_2 = -\int_0^\pi \cos x.F(x)\ dx$

$(2) \Rightarrow I_1 = 6 - f(0) - \int_0^\pi \cos x.F(x)\ dx$

$(1) \Rightarrow I = 6 - f(0) = 2 \Rightarrow f(0) = 4$

14. Ans. (4.00)

Sol.

$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[\frac{1}{(1+e^{\sin x})(2-\cos 2x)} + \frac{1}{(1+e^{-\sin x})(2-\cos 2x)}\right]dx$

(using King's Rule)

$\Rightarrow I = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x}$

$\Rightarrow I = \frac{2}{\pi}\int_0^{\pi/4}\frac{dx}{2-\cos 2x} = \frac{2}{\pi}\int_0^{\pi/4}\frac{\sec^2 x dx}{1+3\tan^2 x}$

$= \frac{2}{\sqrt{3}\pi}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/4} = \frac{2}{3\sqrt{3}}$

$\Rightarrow 27I^2 = 27 \times \frac{4}{27} = 4$

15. **Ans. (A, B)**

Sol. $\lim\limits_{n\to\infty} \dfrac{n^{1/3}\left(\sum\limits_{r=1}^{n}\left(\dfrac{r}{n}\right)^{1/3}\right)}{n^{7/3}\left(\sum\limits_{r=1}^{n}\dfrac{1}{(an+r)^2}\right)} = 54$

$\lim\limits_{n\to\infty}\left(\dfrac{\frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\sum\limits_{r=1}^{n}\frac{1}{(a+r/n)^2}}\right) = 54$

$\Rightarrow \dfrac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{1}{(a+x)^2}dx} = 54$

$\Rightarrow \dfrac{\frac{3}{4}}{\frac{1}{a(a+1)}} = 54$

$\Rightarrow \quad a(a+1) = 72$

$\Rightarrow \quad a^2 + a - 72 = 0 \Rightarrow a = -9, 8$

16. **Ans. (0.50)**

Sol. $I = \int_0^{\pi/2} \dfrac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$

$= \int_0^{\pi/2} \dfrac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$

$2I = \int_0^{\pi/2} \dfrac{3d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4}$

$= 3\int_0^{\pi/2} \dfrac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$

Let $1+\sqrt{\tan\theta} = t$

$\dfrac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta = dt$

$\sec^2\theta d\theta = 2(t-1)dt$

$= 3\int_1^{\infty}\dfrac{2(t-1)dt}{t^4}$

$= 6\int_1^{\infty}\left(t^{-3}-t^{-4}\right)dt$

$2I = 6\left(\dfrac{t^{-2}}{-2}-\dfrac{t^{-3}}{-3}\right)_1^{\infty} = 6\left[0-0-\left\{-\dfrac{1}{2}+\dfrac{1}{3}\right\}\right]$

$I = 0.50$

17. **Ans. (1)**

Sol. $y_n = \left\{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)\ldots\left(1+\dfrac{n}{n}\right)\right\}^{\frac{1}{n}}$

$y_n = \prod\limits_{r=1}^{n}\left(1+\dfrac{r}{n}\right)^{1/n}$

$\log y_n = \dfrac{1}{n}\sum\limits_{r=1}^{n}\ln\left(1+\dfrac{r}{n}\right)$

$\Rightarrow \quad \lim\limits_{n\to\infty}\log y_n = \lim\limits_{x\to\infty}\sum\limits_{r=1}^{n}\dfrac{1}{n}\ln\left(1+\dfrac{r}{n}\right)$

$\Rightarrow \quad \log L = \int_0^1 \ln(1+x)dx$

$\Rightarrow \quad \log L = \log\dfrac{4}{e}$

$\Rightarrow \quad L = \dfrac{4}{e}$

$\Rightarrow \quad [L] = 1$

18. **Ans. (2)**

Sol. $\int_0^{\frac{1}{2}}\dfrac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$

$\int_0^{\frac{1}{2}}\dfrac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\dfrac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$

Put $\dfrac{1-x}{1+x} = t \Rightarrow = \dfrac{-2dx}{(1+x)^2}dt$

$I = \int_1^{1/3}\dfrac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \dfrac{-(1+\sqrt{3})}{2}\times\left|\dfrac{-2}{\sqrt{t}}\right|_1^{1/3}$

$= \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) = 2$

19. Ans. (2)

Sol. $g(x) = \int_x^{\pi/2} (f'(t)\text{cosec}t - f(t)\text{cosec}t \cot t)dt$

$= \int_x^{\pi/2} (f(t)\text{cosec}t)'dt$

$= f\left(\frac{\pi}{2}\right)\text{cosec}\left(\frac{\pi}{2}\right) - \frac{f(x)}{\sin x} = 3 - \frac{f(x)}{\sin x}$

$\therefore \quad \lim_{x\to 0} g(x) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$; as $f'(0) = 1$

$\Rightarrow \quad \lim_{x\to 0} g(x) = 3 - 1 = 2$

20. Ans. (B,C)

Sol. $S_k = \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx = \sum_{k=1}^{98} I_k$

$1 \le k \le x \le k+1 \le 99$

$2 \le k+1 \le x+1 \le k+2 \le 100$

$\frac{k+1}{(k+1)(100)} \le \frac{k+1}{x(x+1)} \le \frac{k+1}{x(x+1)}$

$\int_k^{k+1}\frac{1}{100} \le \int_k^{k+1}\frac{k+1}{x(x+1)} \le \int_k^{k+1}\frac{1}{x}$

$\frac{1}{100} \le I_k \le \ell n\left(\frac{k+1}{k}\right)$

$\frac{98}{100} \le \sum_{4\alpha=1}^{98} I_k \le \ell n 99$

$\frac{49}{50} \le S_k \le \ell n 99$

Aliter

$I = \sum_{k=1}^{98}\left(\int_k^{k+1}\frac{(k+1)}{x(x+1)}dx\right)$

$= \sum_{k=1}^{98}(k+1)\left(\int_k^{k+1}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx\right)$

$= \sum_{k=1}^{98}(k+1)\left((\ell n\, x - \ell n\,(x+1))_k^{k+1}\right)$

$= \sum_{k=1}^{98}(k+1)(\ell n\,(k+1) - \ell n\,(k+2) - \ell n\, k + \ell n\,(k+1))$

$= \sum_{k=1}^{98}((k+1)\ell n\,(k+1) - k.\ell n\, k) -$

$\sum_{k=1}^{98}((k+1).\ell n\,(k+2) - k.\ell n\,(k+1)) + \sum_{k=1}^{98}(\ell n\,(k+1) - \ell n\, k)$

(Difference series)

$\therefore \quad I = (99\,\ell n\, 99) + (-99\,\ell n\, 100 + \ell n\, 2) + (\ell n\, 99)$

$= \ell n\left(\frac{2\times(99)^{100}}{(100)^{99}}\right) \quad ......(1)$

For option (B) :

Now, consider $(100)^{99} = (1 + 99)^{99}$

$= {}^{99}C_0 + {}^{99}C_1(99) + {}^{99}C_2(99)^2 + ... +$

${}^{99}C_{97}(99)^{97} + \underbrace{{}^{99}C_{98}(99)^{98}}_{(\text{value}=(99)^{99})} + \underbrace{{}^{99}C_{99}(99)^{99}}_{(\text{value}=(99)^{99})}$

$\Rightarrow (100)^{99} > 2.(99)^{99} \quad \Rightarrow \frac{2\times(99)^{99}}{(100)^{99}} < 1$

$\therefore \quad \frac{2\times(99)^{100}}{(100)^{99}} < 99$ (on multiplying by 99)

$\Rightarrow \quad I < \ell n 99$

For option (C) :

Since, $\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{(x+1)^2}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{(k+1)dx}{x(x+1)}$

$\Rightarrow \quad \sum_{k=1}^{98}\left(\frac{1}{k+2}\right) < I$

(on integration)

$\Rightarrow \quad \underbrace{\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... + \frac{1}{100}\right)}_{98\text{ terms}} < I$

$\Rightarrow \quad \frac{98}{100} < \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... + \frac{1}{100} < I$

$\therefore \quad I > \frac{49}{50}$

Hence option (C) is correct.

21. Ans. (BONUS)

Sol. $g(x) = \int_{\sin x}^{\sin 2x}\sin^{-1} t\, dt$

$\Rightarrow \quad g'(x) = 2\sin^{-1}(\sin 2x) \times \cos 2x - \sin^{-1}(\sin x)\cos x$

$\Rightarrow \quad g'\left(\frac{\pi}{2}\right) = 0\ \&\ g'\left(-\frac{\pi}{2}\right) = 0$

No option matches the result

22. Ans. (A)

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$

$= \int_0^{\pi/2} \left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}} \right) x^2 \cos x \, dx$

$= \int_0^{\pi/2} x^2 \cos x \, dx = \left(x^2 \sin x \right)_0^{\pi/2} - 2 \int_0^{\pi/2} x . \sin x \, dx$

(I) (II) (I) (II)

$= \frac{\pi^2}{4} - 2\left[-(x.\cos x)_0^{\pi/2} + \int_0^{\pi/2} 1.\cos x \right]$

$= \frac{\pi^2}{4} - 2[0+1] = \left(\frac{\pi^2}{4} - 2 \right)$

23. Ans. (B, C)

Sol.

$$\ell n f(x) = \lim_{n\to\infty} \frac{x}{n} \ell n \left[\frac{\prod_{r=1}^{n} \left(x + \frac{1}{r/n} \right)}{\prod_{r=1}^{n} \left(x^2 + \frac{1}{(r/n)^2} \right)} \frac{1}{\prod_{r=1}^{n} (r/n)} \right]$$

$$= x \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \ell n \left(\frac{x\frac{r}{n}+1}{\left(x\frac{r}{n} \right)^2 + 1} \right)$$

$$= x \int_0^1 \ell n \left(\frac{1+tx}{1+t^2x^2} \right) dt \qquad \text{(put } tx = z\text{)}$$

$$\ell n f(x) = \int_0^x \ell n \left(\frac{1+z}{1+z^2} \right) dz$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ell n \left(\frac{1+x}{1+x^2} \right)$$

sign scheme of $f'(x)$: $+$ before 1, $-$ after 1

also $f'(1) = 0$

$$\Rightarrow f\left(\frac{1}{2}\right) < f(1),\ f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right),\ f'(2) < 0$$

Also $\frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \ell n \left(\frac{4}{10} \right) - \ell n \left(\frac{3}{5} \right)$

$= \ell n \left(\frac{4}{6} \right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$

24. Ans. (1)

Sol. Let $f(x) = \int_0^x \frac{t^2}{1+t^4} dt - 2x + 1$

$\Rightarrow f'(x) = \frac{x^2}{1+x^4} - 2$

as $\frac{1+x^4}{x^2} \geq 2 \Rightarrow \frac{x^2}{1+x^4} \leq \frac{1}{2}$

$\Rightarrow f'(x) \leq -\frac{3}{2} \Rightarrow f(x)$ is continuous and decreasing

$f(0) = 1$ and $f(1) = \int_0^1 \frac{t^2}{1+t^4} dt - 2 \leq -\frac{3}{2}$

by IVT $f(x) = 0$ possesses exactly one solution in [0, 1]

25. Ans. (0)

Sol. Given $f(x) = \begin{cases} [x] & x \leq 2 \\ 0 & x > 2 \end{cases}$

where [x] denotes greatest integer function.

Now $I = \int_{-1}^{2} \frac{x f(x^2)}{2+f(x+1)} dx$

$$I = \int_{-1}^{0} \frac{x f(x^2)}{2+f(x+1)} dx + \int_{0}^{1} \frac{x f(x^2)}{2+f(x+1)} dx + \int_{1}^{\sqrt{2}} \frac{x f(x^2)}{2+f(x+1)} dx + \int_{\sqrt{2}}^{\sqrt{3}} \frac{x f(x^2)}{2+f(x+1)} dx + \int_{\sqrt{3}}^{2} \frac{x f(x^2)}{2+f(x+1)} dx$$

$\therefore \quad I = I_1 + I_2 + I_3 + I_4 + I_5$

Clearly I_1, I_2, I_4 & I_5 are zero using definition of $f(x)$

$\therefore \quad I = I_3 = \int_1^{\sqrt{2}} \frac{x f(x^2)}{2+f(x+1)} dx$

$= \int_1^{\sqrt{2}} \frac{x.1}{2+0} dx = \left.\frac{x^2}{4}\right|_1^{\sqrt{2}} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$

$\therefore \quad 4I - 1 = 0$

26. Ans. (9)

Sol. $\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)} \left(9 + \frac{3}{1+x^2} \right) dx$

$\alpha = \left(e^{9x+3\tan^{-1}x} \right)_0^1$

$= e^{9+\frac{3\pi}{4}} - 1$

$\log|\alpha + 1| = 9 + \frac{3\pi}{4} \Rightarrow \text{Ans.} = 9$

27. Ans. (7)

Sol. $\lim_{x\to 1}\frac{F(x)}{G(x)}=\lim_{x\to 1}\frac{F'(x)}{G'(x)}$

$=\lim_{x\to 1}\frac{f(x)}{x\left|f(f(x))\right|}=\frac{1}{14}$

$\Rightarrow \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|}=\frac{1}{14}\Rightarrow\left|f\left(\frac{1}{2}\right)\right|=7$

$f\left(\frac{1}{2}\right)=7$

$f\left(\frac{1}{2}\right)\neq -7$ as $f(x)$ vanishes exactly at one point.

28. Ans. (A, C)

Sol. Let $I_1=\int_0^{4\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt$

$=\int_0^{\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt+\int_{\pi}^{2\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt$

$+\int_{2\pi}^{3\pi}e^t\left(\sin^6 at+\cos^6 at\right)dt+\int_{3\pi}^{4\pi}e^t\left(\sin^6 at+\cos^6 at\right)dt$

$\therefore\ I_1=\int_0^{\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt+$

$\int_0^{\pi}e^{\pi+t}\left(\sin^6 at+\cos^4 at\right)dt$

$+\int_0^{\pi}e^{2\pi+t}\left(\sin^6 at+\cos^4 at\right)dt+$

$\int_0^{\pi}e^{t+3\pi}\left(\sin^6 at+\cos^4 at\right)dt$

$=\left(1+e^{\pi}+e^{2\pi}+e^{3\pi}\right)\int_0^{\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt$

$$\frac{\int_0^{4\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt}{\int_0^{\pi}e^t\left(\sin^6 at+\cos^4 at\right)dt}=1+e^{\pi}+e^{2\pi}+e^{3\pi}=\frac{e^{4\pi}-1}{e^{\pi}-1}$$

29. Ans. (A, B)

Sol. Given $f(x)=\left(7\tan^6 x-3\tan^2 x\right)\sec^2 x$

$\therefore\ \int_0^{\pi/4}\underbrace{x}_{\mathrm{I}}\underbrace{\left(7\tan^6 x-3\tan^2 x\right)\sec^2 x}_{\mathrm{II}}\,dx$

Using I.B.P.

$=x.\left(\tan^7 x-\tan^3 x\right)_0^{\pi/4}-\int_0^{\pi/4}\left(\tan^7 x-\tan^3 x\right)dx$

$=-\int_0^{\pi/4}\tan^3 x\left(\tan^2 x-1\right)\sec^2 x\,dx$

Put $\tan x=t$

$=\int_0^1\left(t^3-t^5\right)dt=\frac{1}{4}-\frac{1}{6}=\frac{3-2}{12}=\frac{1}{12}$

Also,

$\int_0^{\pi/4}\left(7\tan^6 x-3\tan^2 x\right)\sec^2 x\,dx$

$=\tan^7 x-\tan^3 x\Big|_0^{\pi/4}=0$

30. Ans. (D)

Sol. $f'(x)=\frac{192x^3}{2+\sin^4\pi x}$

$\frac{192x^3}{3}\le f'(x)\le\frac{192x^3}{2}$

$\frac{192}{12}\left(x^4-\frac{1}{16}\right)\le\int_{1/2}^{x}f'(x)dx\le\frac{192}{8}\left(x^4-\frac{1}{16}\right)$

$16x^4-1\le f(x)\le 24x^4-\frac{3}{2}$

$2.6\le\int_{1/2}^{1}f(x)\,dx\le 3.9$

out of given options only option (D) is correct.

31. Ans. (A, B, C)

Sol. According to given data

$F(x)<0\ \forall\ x\in(1,3)$

$f(x)=x\,F(x)$

$f'(x)=F(x)+x\,F'(x)$ ...(i)

$f'(1)=F(1)+F'(1)<0$

(use $F(1)=0$ & $F'(x)<0$)

$f(2)=2F(2)<0$

(use $F(x)<0\ \forall\ x\in(1,3)$)

$f'(x)=F(x)+x\,F'(x)<0$

(use $F(x)<0\ \forall\ x\in(1,3)$)

$F'(x)<0$

32. **Ans. (C, D)**

Sol. Given

$$\int_1^3 x^2F'(x)\,dx = -12$$

$$\Rightarrow [x^2F(x)]_1^3 - 2\int_1^3 x\,F(x)\,dx = -12$$

$$\Rightarrow \int_1^3 f(x)\,dx = -12 \quad \text{Use } x\,F(x) = f(x)$$

Given

$$\int_1^3 x^3\,F''(x)\,dx = 40$$

$$\Rightarrow [x^3F'(x)]_1^3 - 3\int_1^3 x^2F'(x)\,dx = 40$$

$$\Rightarrow \left[x^2(f'(x) - F(x)\right]_1^3 = 4$$

$$9(f'(3) - F(3)) - \left(f'(1) - F(1)\right) = 4$$

$$9f'(3) + 36 - f'(1) = 4$$

$$9f'(3) - f'(1) + 32 = 0$$

33. **Ans. (A, C)**

Sol. Given that $f : [a, b] \to [1, \infty)$

$$g(x) = \begin{cases} 0 & x < a \\ \int_a^x f(t)dt, & a \le x \le b \\ \int_a^b f(t)dt, & x > b \end{cases}$$

Now $g(a^-) = 0 = g(a^+) = g(a)$

$g(b^-) = g(b^+) = g(b) = \int_a^b f(t)dt$

$\Rightarrow$ g is continuous $\forall\ x \in R$

Now $g'(x) = \begin{cases} 0 & : x < a \\ f(x) & : a < x < b \\ 0 & : x > b \end{cases}$

$g'(a^-) = 0$ but $g'(a^+) = f(a) \ge 1$

$\Rightarrow$ g is non differentiable at $x = a$

and $g'(b^+) = 0$ but $g'(b^-) = f(b) \ge 1$

$\Rightarrow$ g is non differentiable at $x = b$

34. **Ans. (A, C, D)**

Sol. $f(x) = \int_{1/x}^{x} \dfrac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$

$$f'(x) = 1.\frac{e^{-\left(x+\frac{1}{x}\right)}}{x} - \left(\frac{-1}{x^2}\right)\frac{e^{-\left(\frac{1}{x}+x\right)}}{1/x}$$

$$= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$$

$\therefore$ $f(x)$ is monotonically increasing on $(0, \infty)$

$\Rightarrow$ A is correct & B is wrong.

Now $f(x) + f\left(\dfrac{1}{x}\right) = \int_{1/x}^{x} \dfrac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt + \int_{x}^{1/x} \dfrac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$

$= 0\ \forall\ x \in (0, \infty)$

Now let $g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \dfrac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$

$g(-x) = f(2^{-x}) \qquad = \int_{2^x}^{2^{-x}} \dfrac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt = -g(x)$

$\therefore$ $f(2^x)$ is an odd function.

35. **Ans. (2)**

Sol. using integration by part

$$\int_0^1 4x^3\left(\left(1-x^2\right)^5\right)''dx$$

$$= 4x^3\left(\left(1-x^2\right)^5\right)'\Big]_0^1 - \int_0^1 12x^2\left(\left(1-x^2\right)^5\right)'dx$$

using integration by part

$$= -12\left[x^2\left(\left(1-x^2\right)^5\right)\Big|_0^1 - \int_0^1 2x\left(1-x^2\right)^5dx\right]$$

$$= 12.2.\int_0^1 x(1-x^2)^5dx$$

Let $1 - x^2 = t \Rightarrow\ xdx = -\dfrac{dt}{2}$

$$= 24\int_1^0 t^5\left(-\frac{dt}{2}\right)$$

$$= 12\int_0^1 t^5dt = 2$$

36. Ans. (A)

Sol. Let $\operatorname{cosec}x + \cot x = e^u$

$\operatorname{cosec}x - \cot x = e^{-u}$

$\operatorname{cosec}x = \frac{1}{2}(e^u + e^{-u})$ & $\cot x = \frac{1}{2}(e^u - e^{-u})$

$\operatorname{cosec}^2 x\,dx = -\frac{1}{2}(e^u + e^{-u})du$

$$\Rightarrow \int_{\ln(\sqrt{2}+1)}^{0} 2^{17}\left(\frac{1}{2}(e^u+e^{-u})\right)^{15}\left\{-\frac{1}{2}(e^u+e^{-u})\right\}du$$

$$= \int_0^{\ln(\sqrt{2}-1)} 2(e^u+e^{-u})^{16}\,du.$$

37. Ans. (B)

Sol. $f(x) = \int_0^{x^2} f(\sqrt{t})dt$

$f'(x) = 2x\,f(x) \qquad \because x \in [0, 2]$

$\Rightarrow f'(x) = 2x\,f(x) \qquad \because F'(x) = f'(x)$

$\Rightarrow \frac{f'(x)}{f(x)} = 2x \Rightarrow \ell n(f(x)) = x^2 + c$

$c = 0 \ (\because f(0) = 1)$

$\Rightarrow f(x) = e^{x^2}$

$f(x) = \int_0^{x^2} f(\sqrt{t})dt = \int_0^{x^2} e^t.dt$

$\therefore f(2) = \int_0^{x^2} e^t\,dt = e^4 - 1.$

38. Ans. (A)

Sol.

$$g\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h}\frac{dt}{\sqrt{t(1-t)}}$$

$$= \lim_{h\to 0^+}\int_h^{1-h}\frac{dt}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}$$

$$= \sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\Bigg|_0^1 = \sin^{-1}(1) - \sin^{-1}(-1) = \pi$$

39. Ans. (D)

Sol. Given,

$g(a) = \lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$

$g'(a) = \lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}(-\ln t + \ln(1-t))dt$

$$g'\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h}\frac{\ell n\left(\frac{1-t}{t}\right)dt}{\sqrt{t(1-t)}} \qquad \ldots(i)$$

$$g'\left(\frac{1}{2}\right) \lim_{h\to 0^+}\int_h^{1-h}\frac{\ln\left(\frac{1-(1-t)}{1-t}\right)}{\sqrt{(1-t)t}}dt \qquad \ldots(ii)$$

(Apply $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$)

$\Rightarrow 2g'\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h} 0\,dt \Rightarrow g'\left(\frac{1}{2}\right) = 0$

40. Ans. (D)

Sol. (P) let $f(x) = ax^2 + bx + c$ where a,b,c are non negative integers

$f(0) = c = 0 \qquad \ldots(1)$

and $\int_0^1 (ax^2 + bx)dx = 1$

$= \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = \frac{a}{3} + \frac{b}{2} = 1$

$\Rightarrow 2a + 3b = 6$

$\Rightarrow a = 3$ & $b = 0$ OR $a = 0$ & $b = 2$

(Q) maximum of $\sin x^2 + \cos x^2 = \sqrt{2}$

$\Rightarrow \sin\left(\frac{\pi}{4} + x^2\right) = 1$ but $x^2 \in [0,13]$

$\Rightarrow \frac{\pi}{4} + x^2 = (4n+1)\frac{\pi}{2}$

$\Rightarrow$ which is satisfied for $n = 0$ & 1

$\Rightarrow$ 4 solutions

(R) $I = \int_{-2}^{2}\frac{3x^2}{1+e^x}dx$ put x

$= -t \quad I = -\int_2^{-2}\frac{3t^2e^t dt}{1+e^t}$

$\Rightarrow 2I = \int_{-2}^{2}\frac{3x^2(1+e^x)}{1+e^x}dx = 2\int_0^2 3x^2 dx$

$\Rightarrow I = \left[x^3\right]_0^2 = 8$

(S) $$\frac{\int_{-\frac{1}{2}}^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}{\int_{0}^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}$$

$$=\frac{\int_{-1/2}^{1/2}(\text{odd function})dx}{\int_{0}^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}=0$$

41. Ans. (B)

Sol.

$$L=\lim_{n\to\infty}\frac{1^a+2^a+\ldots\ldots+n^a}{(n+1)^{a-1}\left[\underbrace{na+na+na+\ldots\ldots+na}_{n\text{ times}}+1+2+3+\ldots..+n\right]}$$

$$=\lim_{n\to\infty}\frac{\sum_{r=1}^{n}r^a}{(n+1)^{a-1}\left[n^2a+\frac{n(n+1)}{2}\right]}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)n^{a+1}}{(n+1)^{a-1}\left[n^2a+\frac{n(n+1)}{2}\right]}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)}{\left(\frac{n+1}{n}\right)^{a-1}\left[\frac{n^2a+\frac{n(n+1)}{2}}{n^2}\right]}$$

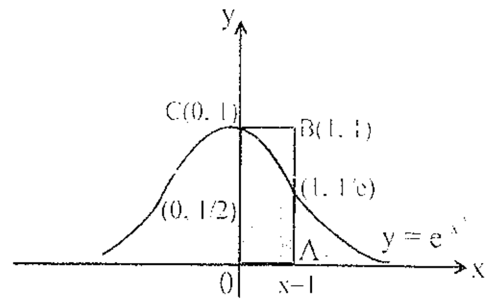
$$=\frac{\int_0^1 x^a dx}{\left(a+\frac{1}{2}\right)}=\frac{1}{60}\Rightarrow\frac{2}{(a+1)(2a+1)}=\frac{1}{60}$$

$$\Rightarrow 2a^2+3a-119=0\qquad\Rightarrow a=7\ \&\ -\frac{17}{2}$$

$a=-\frac{17}{2}$ will be rejected as $\int_0^1 x^{-\frac{17}{2}}dx$ is not defined.

42. Ans.(A, B, D)

Sol.



Area (OABC) = 1

Shaded area is S.

Clearly S < 1

and $\int_0^1 e^{-x^2}dx>\int_0^1 e^{-x}dx$

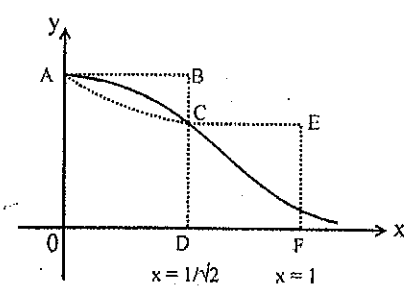
$\Rightarrow\quad S>1-\frac{1}{e}$ ($\therefore$ (B) is correct

Again S $\geq$ Area (trapezium ACDO)

$$\Rightarrow\quad S\geq\frac{1}{2}\left(1+\frac{1}{\sqrt{e}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow\quad S\geq\frac{1}{2\sqrt{2}}\left(1+\frac{1}{\sqrt{e}}\right)$$

$\therefore\quad$ C is wrong



Also S $\leq$ Sum of areas of rectangles ABDO and CEFD

$$\Rightarrow\quad S\leq\frac{1}{\sqrt{2}}\times 1+\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$$

$$\Rightarrow\quad S\leq\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$$

($\therefore$ (D) is correct)

43. Ans. (B)

Sol. $$\int_{-\pi/2}^{\pi/2} x^2\cos x\,dx + \int_{-\pi/2}^{\pi/2} \ell n\left(\frac{\pi+x}{\pi-x}\right)\cos x dx$$

$$= \int_{-\pi/2}^{\pi/2} x^2\cos x\,dx = 2\int_0^{\pi/2} \underset{\text{I}}{x^2}\,\underset{\text{II}}{\cos x}\,dx$$

$$= 2\left((x^2\sin x)_0^{\pi/2} - 2\int_0^{\pi/2} x\sin x dx\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\left(-(x\cos x)_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx\right)\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\int_0^{\pi/2}\cos x\,dx\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^2}{2} - 4$$

44. Ans. (A)

Sol. $$I = \int_{\sqrt{\ell n 2}}^{\sqrt{\ell n 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\ell n 6 - x^2)}dx\ ;$$

put $x^2 = t \Rightarrow 2xdx = dt$

$$\Rightarrow \quad I = \frac{1}{2}\int_{\ell n 2}^{\ell n 3} \frac{\sin t}{\sin t + \sin(\ell n6 - t)}dt \quad ....(i)$$

$$\Rightarrow \quad I = \frac{1}{2}\int_{\ell n 2}^{\ell n 3} \frac{\sin(\ell n6 - t)}{\sin(\ell n6 - t) + \sin t}dt \quad ....(ii)$$

Adding equation (i) & (ii)

$$\Rightarrow \quad 2I = \frac{1}{2}\int_{\ell n2}^{\ell n3} dt \ \Rightarrow\ I = \frac{1}{4}\ell n\left(\frac{3}{2}\right)$$

45. Ans. (B)

Sol. Applying L-Hospital rule,

$$\lim_{x\to 0}\frac{\displaystyle\int_0^x \frac{t\ell n(1+t)}{t^4+4}dt}{x^3} = \lim_{x\to 0}\frac{\dfrac{x\ell n(1+x)}{x^4+4}}{3x^2}$$

$$= \lim_{x\to 0}\frac{\ell n(1+x)}{3x(x^4+4)} = \frac{1}{12}$$

46. Ans. (A)

Sol. $$I = \int_0^1 \frac{x^4(1-2x+x^2)^2}{1+x^2}dx$$

$$I = \int_0^1 \frac{x^4\{(1+x^2)^2 - 4x(1+x^2) + 4x^2\}}{1+x^2}$$

$$\int_0^1 (1+x^2)x^4dx - \int_0^1 4x^5dx + 4\int_0^1 \frac{(x^6+1)-1}{1+x^2}dx$$

$$= \frac{1}{5} + \frac{1}{7} - 4.\frac{1}{6} + 4\int_0^1 \frac{(x^2+1)^3 - 3x^2(1+x^2)}{1+x^2}dx - 4\int_0^1\frac{dx}{1+x^2}$$

$$= \frac{12}{35} - \frac{2}{3} + 4\int_0^1 (x^4+2x^2+1)dx - 12\int_0^1 x^2dx - \pi$$

$$= \frac{12}{35} - \frac{2}{3} + 4\left(\frac{1}{5} + \frac{2}{3} + 1\right) - 4 - \pi$$

$$= \frac{12}{35} - \frac{2}{3} + \frac{52}{15} - \pi = \frac{22}{7} - \pi$$

47. Ans. (B, C)

Sol. $$f(x) = \ell nx + \int_0^x \sqrt{1+\sin t}\,dt$$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$$

$$f'(x) = \frac{1}{x} + \sqrt{2}\left|\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right|$$

$\because \left|\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right|$ is non-derivable

$\therefore$ f '(x) is non-derivable but continuous.

hence option (A) is incorrect & option (B) is correct.

For option C

$$f(x) = (\ell nx) + \int_0^x \left(\sqrt{1+\sin x}\right)dx$$

since f(x) is positive increasing function for all $x > 1$

$\Rightarrow |f(x)| = f(x)$ & $|f'(x)| = f'(x)$

Let f(x) = y

$$f'(x) - f(x) = \frac{1}{x} - \ell nx + \sqrt{1+\sin x} - \int_0^x \sqrt{1+\sin t}\,dt$$

$$f'(x) - f(x) = \frac{1}{x} - \ell nx + \sqrt{1+\sin x} - \sqrt{2}\int_0^x \left|\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)\right|dt$$

$\frac{1}{x} - \ell nx < 0$, when $a > e$

$0 \le \sqrt{1+\sin x} \le \sqrt{2}$

$\int_0^x \left|\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)\right| dt > \sqrt{2}\ \forall a > \frac{3\pi}{2}$

$\Rightarrow f'(x) - f(x) < 0\ \forall a > \frac{3\pi}{2} > 1$

Hence option (C) is correct.

For option (D) $|f(x)| + |f'(x)| \to \infty$ when $x \to \infty$.

Therefore option (D) is incorrect.

Alternate :

$f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$

$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$ ....(i)

for $x > 1$

$\frac{1}{x} + \sqrt{1+\sin x} < 1 + \sqrt{2}$

but $\ell nx + \int_0^x \sqrt{1+\sin t}\,dt$ will always be more than $1+\sqrt{2}$ for some $a > 1$

$\because \int_0^x \sqrt{1+\sin t} > 0$ & $\ln x$ is increasing in $(1, \infty)$

$\Rightarrow f(x) > f'(x)$ " $a > 1$

$\therefore$ (C) is correct

$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$

$\Rightarrow f'$ is not derivable on $(0, \infty)$

at $\frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore$ (B) is also correct

$f(x)$ is unbounded near $x = 0$ in $(0, 1)$ hence $|f(x)|$ can never be made less than a finite number hence $|f(x)|+|f'(x)|$ can never be less than b.

48. **Ans. (4)**

Sol.

$$f(x) = \begin{cases} \{x\} & \text{when } -9 \le x < -8; -7 \le x < -6 \ldots\ldots \\ 1-\{x\} & \text{when } -10 \le x \le -9; -8 \le x < -7, \ldots\ldots \end{cases}$$

Since $f(x)$ & $\cos\pi x$ both are periodic functions having period 2.

$$I = \frac{10\times\pi^2}{10}\left(\int_0^1 (1-\{x\})\cos\pi x\,dx + \int_1^2 \{x\}\cos\pi x\,dx\right)$$

$$= \pi^2\left(\int_0^1 (1-x)\cos\pi x\,dx + \int_1^2 (x-1)\cos\pi x\,dx\right)$$

$$= \pi^2\left(\int_0^1 \cos\pi x\,dx - \int_1^2 \cos\pi x\,dx + \int_1^2 x\cos\pi x\,dx - \int_0^1 x\cos\pi x\,dx\right)$$

$\Rightarrow I = 4$

49. **Ans. (C)**

Sol. $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)dt$, $0 \le x \le 1$

differentiating both the sides & squaring

$\Rightarrow \quad 1 - (f'(x))^2 = f^2(x) \Rightarrow \frac{f'(x)}{\sqrt{1-f^2(x)}} = 1$

$\Rightarrow \quad \sin^{-1} f(x) = x + c$

$f(0) = 0$

$\Rightarrow \quad f(x) = \sin x \Rightarrow \because \sin x \le x$ for $x \in [0, 1]$

$\Rightarrow \quad f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

50. **Ans. (A, B, C)**

Sol. $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$

$I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}dx$

$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}dx$ ...(i)

$2\,I_{n+2} = \int_{-\pi}^{\pi} \frac{\sin(n+2)x}{\sin x}dx$ ...(ii)

(ii) – (i)

$\Rightarrow 2(I_{n+2} - I_n) = \int_{-\pi}^{\pi} \cos(n+1)x = 0$

$\Rightarrow I_{n+2} = I_n$

$$\sum_{m=1}^{10} I_{2m} = 10 \sum_{m=1}^{10} I_2$$

$$= \frac{10}{2}\int_{-\pi}^{\pi}\frac{\sin 2x}{\sin x}dx = 0$$

Put n = 1 in equation (i)

$$2I_1 = \int_{-\pi}^{\pi}\frac{\sin x dx}{\sin x} = 2\pi$$

$$I_1 = \pi$$

$$\sum_{m=1}^{10} I_{2m+1} = 10\pi$$

51. Ans. (A, D)

Sol. $S_n = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k/n+k^2/n^2}$

$$S_n \le \int_0^1 \frac{dx}{x^2+x+1}$$

$$S_n \le \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$S_n \le \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1$$

$$S_n \le \frac{2}{\sqrt{3}}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]$$

$$S_n \le \frac{\pi}{3\sqrt{3}}$$

Now $T_n - S_n = 1 - \frac{1}{3n} \Rightarrow T_n - S_n > \frac{2}{3}$

$\Rightarrow T_n > S_n + \frac{2}{3}$

as $S_n \le \frac{\pi}{3\sqrt{3}}$ so $T_n \ge \frac{\pi}{3\sqrt{3}}$

52. Ans. (A, B, C, D)

Sol. Given f(x) = f(1–x)

$\Rightarrow f(0) = f(1) \Rightarrow f'(c) = 0$ for $c \in (0, 1)$ and

$f\left(\frac{1}{2}+x\right) = f\left(\frac{1}{2}-x\right)$

$f'\left(\frac{1}{2}+x\right) = -f'\left(\frac{1}{2}-x\right)$

so $f'\left(\frac{1}{2}\right) = -f'\left(\frac{1}{2}\right) \Rightarrow \boxed{f'\left(\frac{1}{2}\right)=0}$

and also $f'(3/4) = -f'(1/4) = 0$

so $f'(c) = 0$ atleast two times.

Now $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x dx$

$$= \int_{-1/2}^{0} f\left(x+\frac{1}{2}\right)\sin x dx + \int_{0}^{1/2} f\left(x+\frac{1}{2}\right)\sin x dx$$

put $x + \frac{1}{2} = t - \frac{1}{2}$

dx = dt

$$\int_{1/2}^{1} f\left(t-\frac{1}{2}\right)\sin(t-1)dt + \int_0^{1/2} f\left(x+\frac{1}{2}\right)\sin x dx$$

put $t - 1 = -x$

$dt = -dx$

$$= -\int_{1/2}^{0} f\left(\frac{1}{2}-x\right)\sin(-x)dx + \int_0^{1/2} f\left(x+\frac{1}{2}\right)\sin x dx$$

$$= -\int_0^{1/2} f\left(\frac{1}{2}+x\right)\sin x\, dx + \int_0^{1/2} f\left(\frac{1}{2}+x\right)\sin x\, dx = 0$$

Now $\int_{1/2}^{1} f(1-t)e^{\sin \pi t}dt$ Put $1 - t = z$

$$= -\int_{1/2}^{0} f(z)e^{\sin \pi z}dz = \int_0^{1/2} f(t)e^{\sin \pi t}dt$$

53. Ans. (A)

Sol. $f(x) = \frac{x^2-ax+1}{x^2+ax+1}$

$f'(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}$ and $f''(x) = \frac{4a(-x^3+3x+a)}{(x^2+ax+1)^3}$

$f''(1) = \frac{4a}{(a+2)^2}$ and $f''(-1) = \frac{-4a}{(a-2)^2}$

$\therefore (a+2)^2 f''(1) + (2-a)^2 f''(-1) = 0$

54. Ans. (A)

Sol. As when $x \notin (-1, 1)$, $f'(x) < 0$

so f(x) is decreasing on (–1, 1) at x = 1

$f''(1) = \frac{4a}{(a+2)^2} > 0$

so local minima at x = 1.

55. Ans. (B)

Sol. $g(x) = \int_0^{e^x}\frac{f'(t)}{1+t^2}dt$

$g'(x) = \frac{f'(e^x)}{1+e^{2x}}e^x = \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})}$

$g'(x) > 0$ when $x > 0$

$g'(x) < 0$ when $x < 0$

AREA UNDER CURVE

1. Ans. (B, C, D)

Sol. $f(x)=\frac{x^3}{3}-x^2+\frac{5x}{9}+\frac{17}{36}$

$f'(x)=x^2-2x+\frac{5}{9}$

$f'(x)=0$ at $x=\frac{1}{3}$ in $[0,1]$

A_R = Area of Red region

A_G = Area of Green region

$A_R=\int_0^1 f(x)dx=\frac{1}{2}$

Total area = 1

$\Rightarrow A_G=\frac{1}{2}$

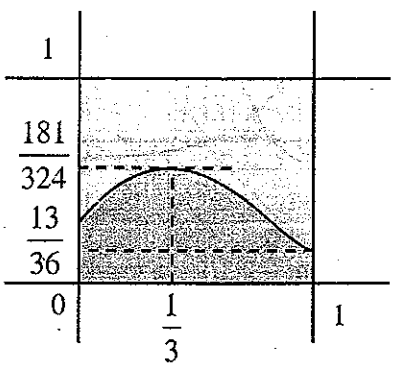
$\int_0^1 f(x)dx=\frac{1}{2}$

$A_G=A_R$

$f(0)=\frac{17}{36}$

$f(1)=\frac{13}{36}\approx 0.36$

$f\left(\frac{1}{3}\right)=\frac{181}{324}\approx 0.558$



(A) Correct when $h=\frac{3}{4}$ but $h\in\left[\frac{1}{4},\frac{2}{3}\right]$

$\Rightarrow$ (A) is incorrect

(B) Correct when $h=\frac{1}{4}$

$\Rightarrow$ (B) is correct

(C) When $h=\frac{181}{324}, A_R=\frac{1}{2}, A_G<\frac{1}{2}$

$h=\frac{13}{36}, A_R<\frac{1}{2}, A_G=\frac{1}{2}$

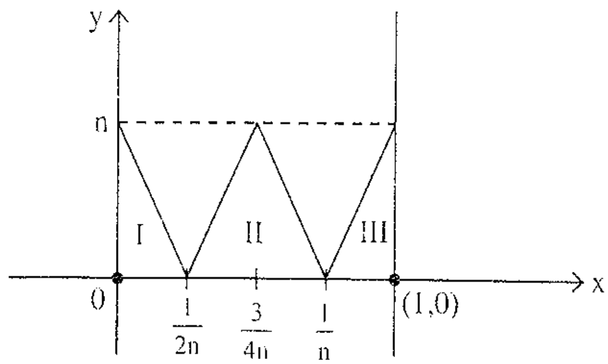
$\Rightarrow A_R=A_G$ for some $h\in\left(\frac{13}{36},\frac{181}{324}\right)$

$\Rightarrow$ (C) is correct

Option (D) is remaining coloured part of option (C), hence option (D) is also correct.

2. Ans. (8)

Sol.



Area = Area of (I + II + III) = 4

$=\frac{1}{2}\times\frac{1}{2n}\times n+\frac{1}{2}\times\frac{1}{2n}\times n+\frac{1}{2}\left(1-\frac{1}{n}\right)\times n$

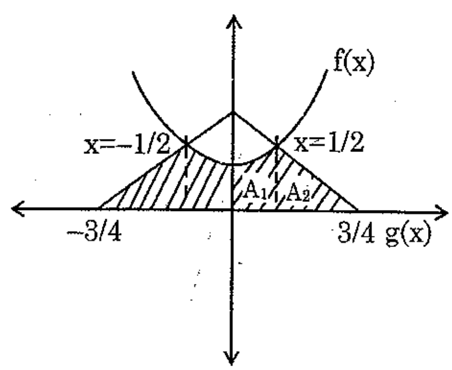
$=\frac{1}{4}+\frac{1}{4}+\frac{n-1}{2}=4$

$\boxed{n=8}$

$\therefore$ maximum value of $f(x)=8$

3. Ans. (6)

Sol.



$x^2+\frac{5}{12}=\frac{2-8x}{3}$

$x^2+\frac{8x}{3}+\frac{5}{12}-2=0$

$12x^2 + 32x - 19 = 0$

$12x^2 + 38x - 6x - 19 = 0$

$2x(6x + 19) - 1(6x + 19) = 0$

$(6x + 19)(2x - 1) = 0$

$\boxed{x = \frac{1}{2}}$

$\alpha = 2(A_1 + A_2)$

$$\alpha = 2\left(\int_0^{1/2} x^2 + \frac{5}{12}dx + \frac{1}{2}\times\frac{1}{4}\times\frac{2}{3}\right)$$

$$\Rightarrow \alpha = 2\left[\left(\frac{x^3}{3}+\frac{5x}{12}\right)_0^{1/2} + \frac{1}{12}\right]$$

$$\Rightarrow \alpha = 2\left[\frac{1}{24}+\frac{5}{24}+\frac{1}{12}\right]$$

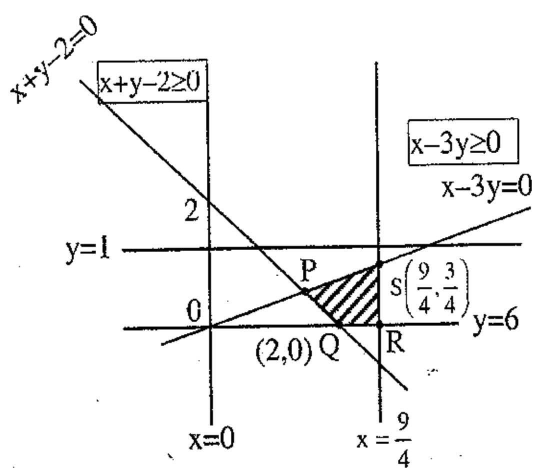
$$\Rightarrow \alpha = 2\left[\frac{1+5+2}{24}\right] \Rightarrow \alpha = 2\times\frac{8}{24} \Rightarrow 9\alpha = 9\times\frac{8}{12}$$

$$\Rightarrow 9\alpha = 6$$

4. Ans. (A)

Sol. $x + y - 2 = 0$

$P\left(\frac{3}{2},\frac{1}{2}\right)$; $Q(2, 0)$; $R\left(\frac{9}{4},0\right)$; $S\left(\frac{9}{4},\frac{3}{4}\right)$



$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix}\frac{3}{2} & \frac{1}{2}\\ 2 & 0\end{vmatrix} + \begin{vmatrix}2 & 0\\ \frac{9}{4} & 0\end{vmatrix} + \begin{vmatrix}\frac{9}{4} & 0\\ \frac{9}{4} & \frac{3}{4}\end{vmatrix} + \begin{vmatrix}\frac{9}{4} & \frac{3}{4}\\ \frac{3}{2} & \frac{1}{2}\end{vmatrix}\right|$$

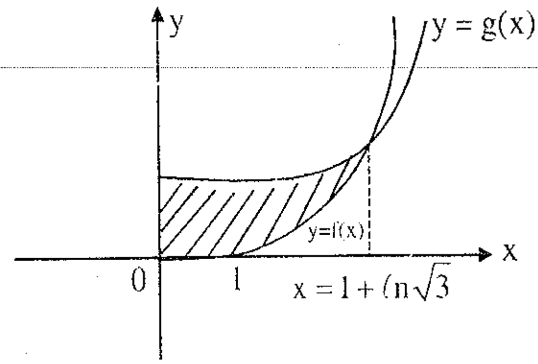
$$= \frac{1}{2}\left|(0-1)+(0-0)+\left(\frac{27}{16}-0\right)+\left(\frac{9}{8}-\frac{9}{8}\right)\right|$$

$$= \frac{11}{32}$$

5. Ans. (A)

Sol. Here, $f(x) = \begin{cases} 0 & x \le 1 \\ e^{x-1} - e^{1-x} & x \ge 1 \end{cases}$

& $g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$



solve $f(x)$ & $g(x) \Rightarrow x = 1 + \ell n\sqrt{3}$

So bounded area = $\int_0^1 \frac{1}{2}\left(e^{x-1}+e^{1-x}\right)dx +$

$$\int_1^{1+\ell n\sqrt{3}}\left[\frac{1}{2}\left(e^{x-1}+e^{1-x}\right)-\left(e^{x-1}-e^{1-x}\right)\right]dx$$

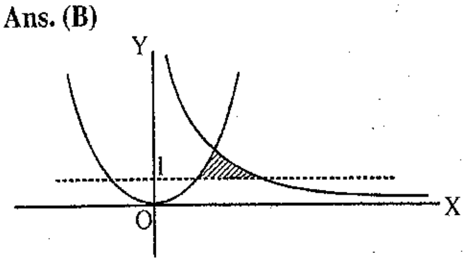
$$= \frac{1}{2}\left[e^{x-1}-e^{1-x}\right]_0^1 + \left[-\frac{1}{2}e^{x-1}-\frac{3}{2}e^{1-x}\right]_1^{1+\ell n\sqrt{3}}$$

$$= \frac{1}{2}\left[e-\frac{1}{e}\right]+\left[\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)+2\right]$$

$$= 2-\sqrt{3}+\frac{1}{2}\left(e-\frac{1}{e}\right)$$

6. Ans. (B)

Sol.



For intersection, $\frac{8}{y} = \sqrt{y} \Rightarrow y = 4$

Hence, required area = $\int_1^4\left(\frac{8}{y}-\sqrt{y}\right)dy$

$$= \left[8\ell ny - \frac{2}{3}y^{3/2}\right]_1^4 = 16\ell n2 - \frac{14}{3}$$

Remark : The question should contain the phrase "area of the bounded region in the first quadrant". Because, in the 2^{nd} quadrant, the region above the line y = 1 and below $y = x^2$, satisfies the region, which is unbounded.

7. **Ans. (B, C)**

Sol. $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$

$\Rightarrow \quad e^{-x} f(x) = e^{-x}(1-2x) + \int_0^x e^{-t} f(t)\,dt$

Differentiate w.r.t. x.

$-e^{-x} f(x) + e^{-x} f'(x)$

$= -e^{-x}(1-2x) + e^{-x}(-2) + e^{-x} f(x)$

$\Rightarrow \quad -f(x) + f'(x) = -(1-2x) - 2 + f(x).$

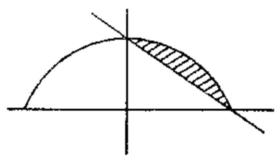
$\Rightarrow \quad f'(x) - 2f(x) = 2x - 3$

Integrating factor $= e^{-2x}$.

$f(x).e^{-2x} = \int e^{-2x}(2x-3)\,dx$

$= (2x-3)\int e^{-2x}dx - \int\left((2)\int e^{-2x}dx\right)dx$

$= \dfrac{(2x-3)e^{-2x}}{-2} - \dfrac{e^{-2x}}{2} + c$



$f(x) = \dfrac{2x-3}{-2} - \dfrac{1}{2} + ce^{2x}$

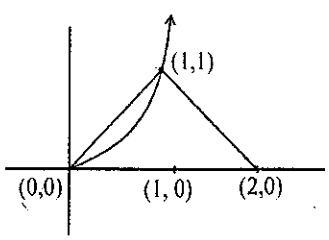
$f(0) = \dfrac{3}{2} - \dfrac{1}{2} + c = 1 \Rightarrow c = 0$

$\therefore \quad f(x) = 1 - x$

Area $= \dfrac{\pi}{4} - \dfrac{1}{2} = \dfrac{\pi - 2}{4}$

8. **Ans. (4)**

Sol.



Area $= \int_0^1 (x - x^n)\,dx = \dfrac{3}{10}$

$\left[\dfrac{x^2}{2} - \dfrac{x^{n+1}}{n+1}\right]_0^1 = \dfrac{3}{10}$

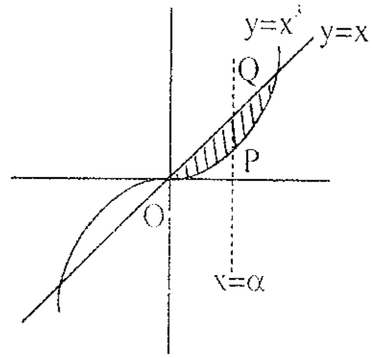
$\dfrac{1}{2} - \dfrac{1}{n+1} = \dfrac{3}{10} \qquad \therefore\ n + 1 = 5$

$\Rightarrow n = 4$

9. **Ans. (A, D)**

Sol. Area between $y = x^3$ and $y = x$ in $x \in (0,1)$ is

$A = \int_0^1 (x - x^3)\,dx = \dfrac{1}{4}$



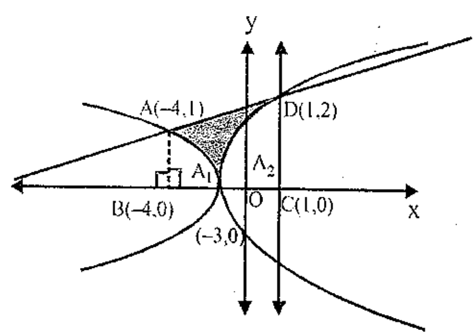
Area of curve linear triangle OPQ $= \dfrac{A}{2} = \dfrac{1}{8}$

$\Rightarrow \int_0^\alpha (x - x^3)\,dx = \dfrac{1}{8} \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$

$\Rightarrow (\alpha^2 - 1)^2 = \dfrac{1}{2} \Rightarrow \alpha^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}}$

10. **Ans. (C)**

Sol.



Clearly required area

= area (trapezium ABCD) – ($A_1 + A_2$) ....(i)

area (trapezium ABCD) $= \dfrac{1}{2}(1+2)(5) = \dfrac{15}{2}$

$A_1 = \int_{-4}^{-3} \sqrt{-(x+3)}\,dx = \dfrac{2}{3}$

and $A_2 = \int_{-3}^{1} (x+3)^{1/2}\,dx = \dfrac{16}{3}$

$\therefore$ From equation (1), we get required area

$= \dfrac{15}{2} - \left(\dfrac{2}{3} + \dfrac{16}{3}\right) = \dfrac{3}{2}$

11. Ans. (3)

Sol. From the question

$$\int_0^a f(x) = F'(a) + 2$$

Differentiating, we get

$$f(a) = F''(a) \Rightarrow f(0) = F''(0)$$

Now, $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$,

$$\therefore\ F'(x) = 2\cos^2\left(x^2+\frac{\pi}{6}\right)\times 2x - 2\left(\cos^2 x\right)$$

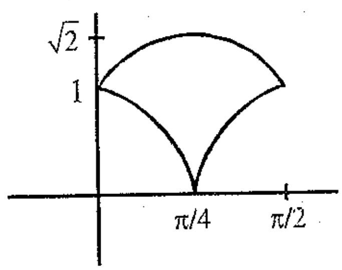
$$F''(x) = 4\left(\cos^2\left(x^2+\frac{\pi}{6}\right) - 4x^2\cos\left(x^2+\frac{\pi}{6}\right)\sin\left(x^2+\frac{\pi}{6}\right)\right) + 4\cos x\sin x$$

$$F''(0) = 4\cos^2\frac{\pi}{6} = 3$$

$\therefore f(0) = 3$

12. Ans. (B)

Sol.



$$y = \sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

$$y = |\cos x - \sin x| = \sqrt{2}\left(\cos\left(x+\frac{\pi}{4}\right)\right)$$

Area

$$= \int_0^{\pi/4}\left[\left(\sin x+\cos x\right)-\left(\cos x-\sin x\right)\right]dx + \int_{\pi/4}^{\pi/2}\left[\left(\sin x+\cos x\right)-\left(\sin x-\cos x\right)\right]dx$$

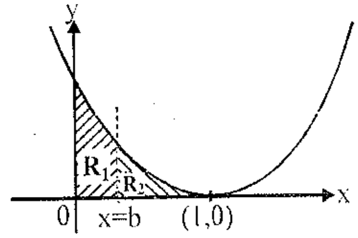
$$= \int_0^{\pi/4} 2\sin x\,dx + \int_{\pi/4}^{\pi/2} 2\cos x\,dx$$

$$= \left[-2\cos x\right]_0^{\pi/4} + \left[2\sin x\right]_{\pi/4}^{\pi/2}$$

$$= 2\sqrt{2}\left(\sqrt{2}-1\right)$$

13. Ans. (B)

Sol.



$$\because\quad R_1 - R_2 = \frac{1}{4}$$

$$\Rightarrow \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4}$$

$$\Rightarrow -\left(\frac{(1-x)^3}{3}\right)_0^b + \left(\frac{(1-x)^3}{3}\right)_b^1 = \frac{1}{4}$$

$$\Rightarrow -\left\{\frac{(1-b)^3}{3} - \frac{1}{3}\right\} - \frac{(1-b)^3}{3} = \frac{1}{4}$$

$$\Rightarrow \frac{1}{3} - \frac{2}{3}(1-b)^3 = \frac{1}{4}$$

$$\Rightarrow \frac{2}{3}(1-b)^3 = \frac{1}{12}$$

$$\Rightarrow (1-b)^3 = \frac{1}{8} \quad \Rightarrow 1-b = \frac{1}{2} \Rightarrow b = \frac{1}{2}$$

14. Ans. (C)

Sol. $R_2 = \int_{-1}^{2} f(x)dx$, $\quad R_1 = \int_{-1}^{2} x f(x)dx$

$$= \int_{-1}^{2} (1-x) f(1-x)dx$$

$$\left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right) \pm\frac{\pi}{4}$$

(given $f(x) = f(1-x)$)

$$= \int_{-1}^{2} f(x)dx - \int_{-1}^{2} x f(x)dx$$

or $\quad R_1 = R_2 - R_1 \qquad \Rightarrow \quad 2R_1 = R_2$

15. Ans. (C)

Sol. $\because f'(x) = 2 + 6x + 12x^2 > 0\ \forall\ x \in R$

$\therefore$ f(x) is strictly increasing in R

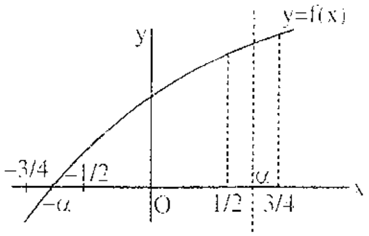
$\because f(0) = 1,\ f(-1) = -2 f\left(-\frac{1}{2}\right) = \frac{1}{4}$,

$\&\ f\left(-\frac{3}{4}\right) = -\frac{1}{2}$

$\therefore$ f(x) = 0 has only one real root lying in $\left(-\frac{3}{4}, -\frac{1}{2}\right)$

16. Ans. (A)

Sol.



Let real root is $-\alpha$ $\Rightarrow t = |s| = \alpha$

Required area

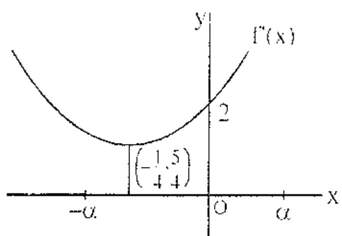
$$A = \int_0^{\alpha} f(x)dx \ \& \int_0^{1/2} f(x)dx < A < \int_0^{3/4} f(x)dx$$

$$\left|x + x^2 + x^3 + x^4\right|_0^{1/2} < A < \left|x + x^2 + x^3 + x^4\right|_0^{3/4} < |4x|_0^{3/4}$$

$$\Rightarrow \frac{15}{16} < A < 3$$

17. Ans. (B)

Sol.



$f'(x) = 2(6x^2 + 3x + 1)$

$\Rightarrow f'(x)$ is decreasing in $\left(-\alpha, -\frac{1}{4}\right)$

increasing in $\left(-\frac{1}{4}, \alpha\right)$

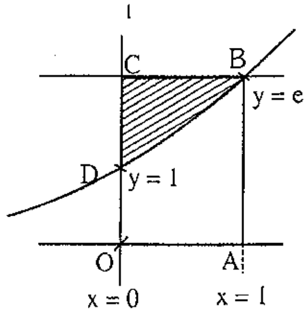
or $f'(x)$ is decreasing in $\left(-t, -\frac{1}{4}\right)$ and

increasing in $\left(-\frac{1}{4}, t\right)$

18. Ans. (B, C, D)

Sol. $A = \int_1^e \ln y \, dy$

Apply $= \int_1^e \ln(e + 1 - y)dy$



$A = ar(OABC) - ar(OABD) = e - \int_1^e e^x dx$

19. Ans. (B)

Sol. $y^3 - 3y + x = 0$

$3y^2y' - 3y' + 1 = 0 \ y' = y_x = \frac{-1}{3(y^2-1)}$

$f(-10\sqrt{2}) = 2\sqrt{2}$

$f'(-10\sqrt{2}) = -\frac{1}{3(7)} = -\frac{1}{21}$

$6y(y')^2 + 3y^2y'' - 3y'' = 0$

$y'' = -\frac{2y(y')^2}{y^2-1}$

$f''(-10\sqrt{2}) = \frac{2(2\sqrt{2})}{441\times 7} = \frac{4\sqrt{2}}{7^3 3^2}$

20. Ans. (A)

Sol. $\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$

$= bf(b) - af(a) + \int_a^b \frac{x}{3\left[(f(x))^2 - 1\right]}dx$

$= \int_a^b \frac{x}{3\left[(f(x))^2 - 1\right]}dx + bf(b) - af(a)$

21. Ans. (D)

Sol. $\int_{-1}^{1} g'(x)dx = g(1) - g(-1)$

Now $g(1) = -(g(-1))$

(as $g'(x)$ is an even function)

so $\int_{-1}^{1} g'(x)dx = 2g(1) = -2g(-1)$

22. Ans. (B)

Sol. Area $= \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$= \int_0^{\pi/4} \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\sqrt{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}}dx$$

$$= \int_0^{\pi/4} \frac{2\sin\frac{x}{2}}{\sqrt{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}}dx = \int_0^{\pi/4} \frac{2\tan\frac{x}{2}}{\sqrt{1 - \tan^2\frac{x}{2}}}dx$$

Let $\tan\frac{x}{2} = t$ $\quad \sec^2 dx = 2dt \Rightarrow dx = \frac{2dt}{(1+t^2)}$

$\therefore$ Area $= \int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

DIFFERENTIAL EQUATION

1. Ans. (C)

Sol. Diff. w.r.t 'x'

$3f(x) = f(x) + xf'(x) - x^2$

$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$

$IF = e^{-2\ell nx} = \frac{1}{x^2}$

$y\left(\frac{1}{x^2}\right) = \int x.\frac{1}{x^2}dx$

$y = x^2\ell nx + cx^2$

$\therefore\ y(1) = \frac{1}{3} \Rightarrow c = \frac{1}{3}$

$y(e) = \frac{4e^2}{3}$

2. **Ans. (16)**

Sol. $\frac{dy}{dx} - \frac{2x}{x^2-5}y = -2x(x^2-5)$

$IF = e^{-\int\frac{2x}{x^2-5}dx} = \frac{1}{(x^2-5)}$

$y.\frac{1}{x^2-5} = \int -2x.dx + c$

$\Rightarrow \frac{y}{x^2-5} = -x^2 + c$

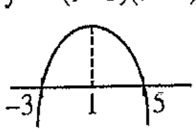
$x = 2, y = 7$

$\frac{7}{-1} = -4 + c \ \Rightarrow c = -3$

$y = -(x^2-5)(x^2+3)$

put $x^2 = t \geq 0$

$y = -(t-5)(t+3)$



$y_{max} = 16$ when $x^2 = 1$

$y_{max} = 16$

3. Ans. (8)

Sol. $xdy - (y^2 - 4y)dx = 0, x > 0$

$\int\frac{dy}{y^2-4y} = \int\frac{dx}{x}$

$\int\left(\frac{1}{y-4} - \frac{1}{y}\right)dy = 4\int\frac{dx}{x}$

$\log_e|y-4| - \log_e|y| = 4\log_e x + \log_e c$

$\frac{|y-4|}{|y|} = cx^4 \xrightarrow{(1,2)} c = 1$

$|y-4| = |y|x^4$

C-1 and C-2

$y - 4 = yx^4 \qquad y - 4 = -yx^4$

$y = \frac{4}{1-x^4} \qquad y = \frac{4}{1+x^4}$

$y(1) =$ ND (rejected) $\qquad y(1) = 2$

$y(\sqrt{2}) = \frac{4}{5} \Rightarrow 10y(\sqrt{2}) = 8$

4. **Ans. (C)**

Sol. $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right)$

Linear D.E.

$I.F. = e^{\int 12.dx} = e^{12x}$

Solution of DE

$y.e^{12x} = \int e^{12x}.\cos\left(\frac{\pi}{12}x\right)dx$

$y.e^{12x} = \frac{e^{12x}}{(12)^2 + \left(\frac{\pi}{12}\right)^2}\left(12\cos\frac{\pi}{12}x + \frac{\pi}{12}\sin\frac{\pi}{12}x\right) + C$

$\Rightarrow y = \frac{(12)}{(12)^4+\pi^2}\left((12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right)\right) + \frac{C}{e^{12x}}$

$y = \frac{(12)}{(12)^4+\pi^2}\left((12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right)\right) + \frac{C}{e^{12x}}$

Given $y(0) = 0$

$\Rightarrow 0 = \frac{12}{12^4+\pi^2}(12^2 + 0) + C$

$\Rightarrow C = \frac{-12^3}{12^4+\pi^2}$

$\therefore\ y = \frac{12}{12^4+\pi^2}\left[(12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right) - 12^2.e^{-12x}\right]$

Now,

$$\frac{dy}{dx} = \frac{12}{12^4+\pi^2}\left[\underbrace{-12\pi\sin\left(\frac{\pi x}{12}\right)+\frac{\pi^2}{12}\cos\left(\frac{\pi x}{12}\right)}_{\text{min value}}+12^3e^{-12x}\right]$$

$$\left(-\sqrt{144\pi^2+\frac{\pi^4}{144}} = -12\pi\sqrt{1+\frac{\pi^2}{12^4}}\right)$$

$\Rightarrow \frac{dy}{dx} > 0 \ \forall\ x \le 0$ & may be negative/positive for $x > 0$

So. $f(x)$ is neither increasing nor decreasing

For some $\beta \in R$. $y = \beta$ intersects $y = f(x)$ at infinitely many points

So, option (C) is correct

5. Ans. (A, C)

Sol. Integrating factor $= e^{\alpha x}$

So $ye^{\alpha x} = \int xe^{(\alpha+\beta)x}dx$

Case-I

$$\text{If } \alpha+\beta = 0 \quad ye^{\alpha x} = \frac{x^2}{2}+c$$

It passes through $(1, 1) \Rightarrow C = e^{\alpha} - \frac{1}{2}$

$$\text{So } ye^{\alpha x} = \frac{x^2-1}{2}+e^{\alpha}$$

for $\alpha = 1$

$$y = \frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x} \rightarrow (A)$$

Case-II

If $\alpha+\beta \neq 0$

$$ye^{\alpha x} = \int \frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{1}{\alpha+\beta}e^{(\alpha+\beta)x}\,dx$$

$$\Rightarrow ye^{\alpha x} = \frac{xe^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2}+c$$

$$\Rightarrow \text{So } c = e^{\alpha} - \frac{e^{\alpha+\beta}}{\alpha+\beta}+\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)^2}\left((\alpha+\beta)x-1\right)+e^{-\alpha x}\left(e^{\alpha} - \frac{e^{\alpha+\beta}}{\alpha+\beta}+\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$$

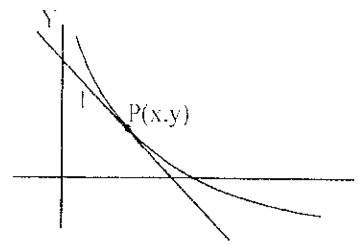
If $\alpha = \beta = 1$

$$y = \frac{e^x}{4}(2x-1)+e^{-x}\left(e-\frac{e^2}{2}+\frac{e^2}{4}\right)$$

$$y = \frac{e^x}{2}\left(x-\frac{1}{2}\right)+e^{-x}\left(e-\frac{e^2}{4}\right) \rightarrow (c)$$

6. Ans. (A, D)

Sol.



$$Y - y = y'(X - x)$$

So, $Y_P = (0, y - xy')$

So, $x^2+(xy')^2 = 1 \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-x^2}{x^2}}$

[$\frac{dy}{dx}$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x \in (0, 1)$]

Hence, $\int dy = -\int \frac{\sqrt{1-x^2}}{x}\,dx$

$\Rightarrow y = -\int \frac{\cos^2\theta\, d\theta}{\sin\theta}$; put $x = \sin\theta$

$\Rightarrow y = -\int \text{cosec}\theta\, d\theta + \int \sin\theta\, d\theta$

$\Rightarrow y = \ln(\text{cosec}\theta + \cot\theta) - \cos\theta + C$

$$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}+C$$

$$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$$

(as $y(1) = 0$)

7. **Ans. (B, C)**

Sol. $f'(x) = e^{(f(x)-g(x))}\, g'(x)\ \forall\ x \in \mathbb{R}$

$\Rightarrow\quad e^{-f(x)}.\, f'(x) - e^{-g(x)} g'(x) = 0$

$\Rightarrow\quad \int\left(e^{-f(x)} f'(x) - e^{-g(x)}.g'(x)\right)dx = C$

$\Rightarrow\quad -e^{-f(x)} + e^{-g(x)} = C$

$\Rightarrow\quad -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$

$\Rightarrow\quad -\dfrac{1}{e} + e^{-g(1)} = -e^{-f(2)} + \dfrac{1}{e}$

$\Rightarrow\quad e^{-f(2)} + e^{-g(1)} = \dfrac{2}{e}$

$\therefore\quad e^{-f(2)} < \dfrac{2}{e}$ and $e^{-g(1)} < \dfrac{2}{e}$

$\Rightarrow\quad -f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow\quad f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

8. **Ans. (B, C, D)**

Sol. $\lim\limits_{t\to x} \dfrac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$

by using L'Hopital

$\lim\limits_{t\to x} \dfrac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$

$\Rightarrow$ f(x)cosx – f'(x)sinx = $\sin^2 x$

$\Rightarrow\ -\left(\dfrac{f'(x)\sin x - f(x)\cos x}{\sin^2 x}\right) = 1$

$\Rightarrow\ -d\left(\dfrac{f(x)}{\sin x}\right) = 1$

$\Rightarrow\ \dfrac{f(x)}{\sin x} = -x + c$

Put $x = \dfrac{\pi}{6}$ & $f\left(\dfrac{\pi}{6}\right) = -\dfrac{\pi}{12}$

$\therefore\ c = 0 \Rightarrow f(x) = -x\sin x$

(A) $f\left(\dfrac{\pi}{4}\right) = \dfrac{-\pi}{4}\dfrac{1}{\sqrt{2}}$

(B) f(x) = –x sinx

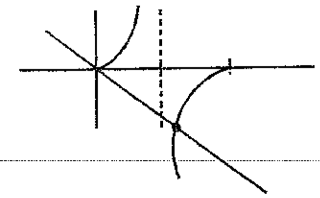
as $\sin x > x - \dfrac{x^3}{6}$, $-x\sin x < -x^2 + \dfrac{x^4}{6}$

$\therefore\ f(x) < -x^2 + \dfrac{x^4}{6}\ \forall\ x \in (0, \pi)$

(C) f'(x) = –sinx – x cosx

f'(x) = 0 $\Rightarrow$ tanx = –x

$\Rightarrow$ there exist $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$



(D) f"(x) = –2cosx + xsinx

$f''\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$, $f\left(\dfrac{\pi}{2}\right) = -\dfrac{\pi}{2}$

$f''\left(\dfrac{\pi}{2}\right) + f\left(\dfrac{\pi}{2}\right) = 0$

9. **Ans. (0.4)**

Sol. $\dfrac{dy}{dx} = 25y^2 - 4$

So, $\dfrac{dy}{25y^2 - 4} = dx$

Integrating, $\dfrac{1}{25} \times \dfrac{1}{2\times\frac{2}{5}} \ln\left|\dfrac{y - \frac{2}{5}}{y + \frac{2}{5}}\right| = x + c$

$\Rightarrow\ \ln\left|\dfrac{5y-2}{5y+2}\right| = 20(x + c)$

Now, c = 0 as f(0) = 0

Hence $\left|\dfrac{5y-2}{5y+2}\right| = e^{(20x)}$

$\underset{x\to-\infty}{\text{let}} \left|\dfrac{5f(x)-2}{5f(x)+2}\right| = \underset{x\to-\infty}{\text{let}}\ e^{(20x)}$

Now, RHS = 0 $\Rightarrow\ \underset{x\to-\infty}{\text{let}}\ (5f(x) - 2) = 0$

$\Rightarrow\ \underset{x\to-\infty}{\text{let}}\ f(x) = \dfrac{2}{5}$

10. **Ans. (B)**

Sol. $y = \dfrac{1}{8}\int \dfrac{dx}{\sqrt{4+\sqrt{9+x}}.\sqrt{x}.\sqrt{9+\sqrt{x}}}$

put $\sqrt{9+\sqrt{x}} = t \Rightarrow \dfrac{dx}{\sqrt{x}.\sqrt{9+\sqrt{x}}} = 4dt$

$\therefore y = \frac{4}{8}\int \frac{dt}{\sqrt{4+t}}$

$\Rightarrow y = \sqrt{4+t} + C$

$\Rightarrow y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}} + C$

at $x = 0$: $y(0) = \sqrt{7}$ $\Rightarrow$ $C = 0$

$\therefore y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}}$

$\Rightarrow$ $y(256) = 3$

11. Ans. (A, C)

Sol. Given that,

$f'(x) > 2f(x)\ \forall\ x \in R$

$\Rightarrow$ $f'(x) - 2f(x) > 0\ \ \forall\ x \in R$

$\therefore$ $e^{-2x}(f'(x) - 2f(x)) > 0\ \forall\ x \in R$

$\Rightarrow$ $\frac{d}{dx}\left(e^{-2x}f(x)\right) > 0\ \forall\ x \in R$

Let $g(x) = e^{-2x}f(x)$

Now, $g'(x) > 0\ \forall\ x \in R$

$\Rightarrow$ $g(x)$ is strictly increasing $\forall\ x \in R$

Also, $g(0) = 1$

$\therefore$ $\forall\ x > 0$

$\Rightarrow$ $g(x) > g(0) = 1$

$\therefore$ $e^{-2x}.f(x) > 1\ \forall\ x \in (0, \infty) \Rightarrow f(x) > e^{2x}\ \forall$ $x \in (0,\infty)$

$\therefore$ option (A) is correct

As, $f'(x) > 2\ f(x) > 2e^{2x} > 2\ \forall\ x \in (0, \infty)$

$\Rightarrow$ $f(x)$ is strictly increasing on $x \in (0, \infty)$

$\Rightarrow$ option (C) is correct

As, we have proved above that

$f'(x) > 2.e^{2x}\ \forall\ x \in (0, \infty)$

$\Rightarrow$ option (D) is incorrect

$\therefore$ options (A) and (C) are correct

12. Ans. (A, D)

Sol. $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$

$((x+2)^2 + y(x+2))\frac{dy}{dx} = y^2$

Let $x + 2 = X$, $y = Y$

$(X)(X+Y)\frac{dY}{dX} = Y^2$

$-X^2dY = XYdY - Y^2dX$

$-X^2dY = Y(XdY - YdX)$

$-\frac{dY}{Y} = \frac{XdY - YdX}{X^2}$

$-\ell n|Y| = \left(\frac{Y}{X}\right) + C$

$-\ell n|y| = \frac{y}{x+2} + C$

$\because$ it is passing through $(1, 3)$

$-\ell n3 = 1 + C$

$C = -1 - \ell n3$

$\therefore$ curve $\frac{y}{x+2} + \ell n|y| - 1 - \ell n3 = 0, x > 0$ ....(i)

put $y = x + 2$ in equation (i)

then $\frac{x+2}{x+2} + \ell n|x+2| - 1 - \ell n3 = 0$

$x = 1, -5$(reject)

$\therefore$ curve intersect $y = x + 2$ at point $(1, 3)$

for option (C), put $y = (x + 2)^2$, we will get

$x + 2 + 2\ell n(x + 2) = 1 + \ell n3$

Clearly left hand side is an increasing function

Hence, it is always greater than $2 + 2\ell n2$

therefore no solution

for option (C) put $y = (x + 3)^2$ in equation (i)

$\frac{(x+3)^2}{x+2} + \ell n(x+3)^2 - 1 - \ell n3 = 0$

$\frac{(x+3)^2}{x+2} + \ell n\frac{(x+3)^2}{3} - 1 = 0$

$\because$ $x > 0 \Rightarrow x + 3 > x + 2$ and $x + 3 > 3$

So $\frac{(x+3)^2}{x+2} + \ell n\frac{(x+3)^2}{3} > 1$

$\therefore$ $\frac{(x+3)^2}{x+2} + \ell n\frac{(x+3)^2}{3} - 1 = 0$

has no solution

$\Rightarrow$ curve $y = (x + 3)^2$ does not intersect

13. Ans. (A)

Sol. Let $y = f(x)$

$$\frac{dy}{dx}+\frac{y}{x}=2 \text{ (linear differential equation)}$$

$$\therefore\ y.e^{\int\frac{dx}{x}} = 2\int e^{\int\frac{dx}{x}} = 2\int e^{\int\frac{dx}{x}}dx + c$$

$$\Rightarrow yx = 2\int x\,dx + c$$

$$\therefore\ yx = x^2 + c$$

$$\Rightarrow f(x) = x + \frac{c}{x}\ ;\ \text{As } f(1)\neq 1 \Rightarrow c \neq 0$$

$$\Rightarrow\ f'(x) = 1-\frac{c}{x^2}, c\neq 0$$

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim\limits_{x\to 0^+}\left(1-cx^2\right)=1$

(B) $\lim\limits_{x\to 0^+} xf\left(\frac{1}{x}\right) = \lim\limits_{x\to 0^+} x\left(\frac{1}{x}+cx\right) = \lim\limits_{x\to 0^+}\left(1+cx^2\right)=1$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x) = \lim\limits_{x\to 0^+} x^2\left(1-\frac{c}{x^2}\right) = \lim\limits_{x\to 0^+}\left(x^2-c\right)=-c$

(D) $f(x) = x+\frac{c}{x}, c\neq 0$

for $c > 0$

$\therefore\ \lim\limits_{x\to 0^+} f(x) = \infty$

$\Rightarrow$ function is not bounded in $(0, 2)$

14. Ans. (A, C)

Sol. $y' + e^x y' + ye^x = 1$

$\Rightarrow\quad dy + d(e^x y) = dx$

$\Rightarrow\quad y + e^x y = x + c$

$\because\quad y(0) = 2 \quad\Rightarrow\quad c = 4$

$\Rightarrow\quad y = \dfrac{x+4}{1+e^x}$

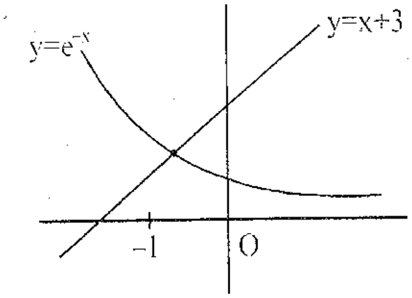
$\therefore\quad y(-4) = 0$

for critical point given

$$\frac{dy}{dx} = \frac{1-ye^x}{1+e^x} = \frac{1-\left(\frac{x+4}{1+e^x}\right)e^x}{1+e^x} = \frac{1-(x+3)e^x}{(1+e^x)^2}$$

$\because\quad \dfrac{dy}{dx} = 0 \quad\Rightarrow\quad e^x(x+3) - 1 = 0$

$\Rightarrow\quad x + 3 = e^{-x}$



$y(x)$ has a critical point in the interval $(-1, 0)$

15. Ans. (B, C)

Sol. Let Circle

$$x^2 + y^2 - 2ax - 2ay + c = 0$$

On differentiation

$$2x + 2yy' - 2a - 2ay' = 0$$

$\Rightarrow\quad x + yy' - a(1 + y') = 0$

$\Rightarrow\quad a = \dfrac{x+yy'}{1+y'}$

again differentiation

$$\frac{(1+(y')^2+yy'')(1+y')-(x+yy')(y'')}{(1+y')^2} = 0$$

$\Rightarrow\quad 1 + y'((y')^2 + y' + 1) + y''(y - x) = 0$

$\therefore\quad P = y - x$

$\quad\quad Q = 1 + y' + (y')^2$

16. Ans. (B)

Sol. $\left(\dfrac{dy}{dx}-\dfrac{xy}{(1-x^2)}\right)\sqrt{1-x^2} = x^4 + 2x$

$\Rightarrow\quad \sqrt{1-x^2}\,dy + d\left(\sqrt{1-x^2}\right)y = (x^4+2x)dx$

$\Rightarrow\quad y\sqrt{1-x^2} = \dfrac{x^5}{5} + x^2 + c$

by $(0, 0)$ $c = 0$

$$y = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left(\frac{x^5}{5\sqrt{1-x^2}}+\frac{x^2}{\sqrt{1-x^2}}\right)dx$$

$$= 2\int_0^{\sqrt{3}/2}\frac{x^2}{\sqrt{1-x^2}}dx \quad (\text{put } x = \sin\theta)$$

$$= 2\int_0^{\pi/3}\sin^2\theta\, d\theta = \int_0^{\pi/3}(1-\cos 2\theta)d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

17. Ans. (D)

Sol. $f'(x) - 2f(x) < 0$

Multiply both side by e^{-2x}

$e^{-2x} f'(x) - 2e^{-2x} f(x) < 0$

$\frac{d}{dx}\left(e^{-2x} f(x)\right) < 0$

Now, $g(x) = e^{-2x} f(x)$

$\therefore$ $g(x)$ is a decreasing function.

$x > \frac{1}{2}$

$g(x) < g\left(\frac{1}{2}\right)$

$\Rightarrow e^{-2x} f(x) < \frac{1}{e}$

$\Rightarrow f(x) < e^{2x-1}$

$\Rightarrow \int_{1/2}^{1} f(x)dx < \frac{1}{e}\int_{1/2}^{1} e^{2x}dx$

$= \left[\frac{1}{2e}e^{2x}\right]_{1/2}^{1} = \frac{1}{2e}\left(e^2 - e\right) = \frac{1}{2}(e-1)$

$\Rightarrow \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$

obviously $f(x)$ is positive

$\therefore \int_{1/2}^{1} f(x)dx > 0$

18. Ans. (A)

Sol. $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$

Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + \frac{xdv}{dx} = v + \sec v$

$\cos v\, dv = \frac{dx}{x}$

$\sin v = \ell nx + c$

$\sin\left(\frac{y}{x}\right) = \ell nx + c$

$\because$ passing through

$\left(1, \frac{\pi}{6}\right) \Rightarrow \sin\frac{\pi}{6} = c \Rightarrow c = \frac{1}{2}$

$\therefore \sin\frac{y}{x} = \ell nx + \frac{1}{2}$

19. Ans. (C)

Sol. $e^{-x}(f''(x) - 2f'(x) + f(x)) \geq 1$

$D((f'(x) - f(x))e^{-x}) \geq 1$

$\Rightarrow D((f'(x) - f(x))e^{-x}) \geq 0$

$\Rightarrow (f'(x) - f(x))e^{-x}$ is an increasing function.

As we know that $e^{-x}f(x)$ has local minima at $x = \frac{1}{4}$

$e^{-x}(f'(x) - f(x)) = 0$ at $x = \frac{1}{4}$

Let $F(x) = e^{-x}(f'(x) - f(x))$

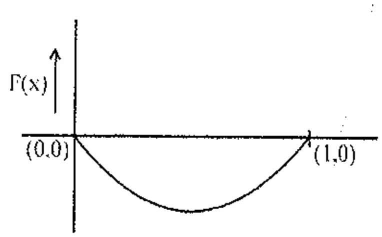
$F(x) < 0$ in $\left(0, \frac{1}{4}\right)$

$e^{-x}(f'(x) - f(x) < 0$ in $\left(0, \frac{1}{4}\right)$

$f'(x) < f(x)$ in $\left(0, \frac{1}{4}\right)$

20. Ans. (D)

Sol.



$D(e^{-x}(f'(x) - f(x)) \geq 0 \ \forall\ x \in (0, 1)$

$D(D(e^{-x}f(x)) \geq 0 \ \forall\ x \in (0, 1)$

$D^2(e^{-x}f(x)) \geq 0$

Let $F(x) = e^{-x}f(x)$

$F''(x) > 0$ means it is concave upward.

$F(0) = F(1) = 0$

$F(x) < 0 \ \forall\ x \in (0, 1)$

$e^{-x}f(x) < 0 \ \forall\ x \in (0, 1)$

$f(x) < 0$

Option D is possible

21. Ans. (A, D)

Sol. $\frac{dy}{dx} - y\tan x = 2x\sec x$

I.F. $= e^{\int -\tan x\,dx} = \cos x$

$\therefore$ Equation reduces to

$y.\cos x = \int 2x.\sec x.\cos x\,dx$

$\Rightarrow y\cos x = x^2 + C$

$\because y(0) = 0 \Rightarrow 0 = 0 + C$

$\therefore y\cos x = x^2$

$\Rightarrow y(x) = x^2\sec x$

$\therefore y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$ ($\therefore$ (A) is correct)

$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}.2 = \frac{2\pi^2}{9}$ ($\therefore$ (C) is wrong)

Also $y'(x) = 2x\sec x + x^2\sec x\tan x$

$\Rightarrow y'\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.\sqrt{2} + \frac{\pi^2\sqrt{2}}{16}$ ($\therefore$ (B) is wrong)

and $y'\left(\frac{\pi}{3}\right) = 2.\frac{\pi}{3}.2 + \frac{\pi^2}{9}.2.\sqrt{3}$

$= \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$ ($\therefore$ (D) is correct)

22. Ans. (6) (Bonus)

Sol. ***(Comment : The given relation does not hold for x = 1, therefore it is not an identity. Hence there is an error in given question. The correct identity must be-)***

$6\int_1^x f(t)\,dt = 3xf(x) - x^3 - 5,\ \forall x \geq 1$

Now applying Newton Leibnitz theorem

$6f(x) = 3xf'(x) - 3x^2 + 3f(x)$

$\Rightarrow 3f(x) = 3xf'(x) - 3x^2$

Let $y = f(x)$

$\Rightarrow x\frac{dy}{dx} - y = x^2 \Rightarrow \frac{xdy - ydx}{x^2} = dx$

$\Rightarrow \int d\left(\frac{y}{x}\right) = \int dx$

$\Rightarrow \frac{y}{x} = x + C$ (where C is constant)

$\Rightarrow y = x^2 + Cx$

$\therefore f(x) = x^2 + Cx$

Given $f(1) = 2 \Rightarrow C = 1$

$\therefore f(2) = 2^2 + 2 = 6$

23. Ans. (0)

Sol. Given $y(0) = 0$, $g(0) = g(2) = 0$

Let $y'(x) + y(x).g'(x) = g(x)g'(x)$

$\Rightarrow y'(x) + (y(x) - g(x))g'(x) = 0$

$\Rightarrow \frac{y'(x)}{g'(x)} + y(x) = g(x)$

$\Rightarrow \frac{dy(x)}{dg(x)} + y(x) = g(x)$

$\Rightarrow$ I.F. $= e^{\int d(g(x))} = e^{g(x)}$

$\Rightarrow y(x).e^{g(x)} = \int e^{g(x)}g(x).dg(x)$

$y(x).e^{g(x)} = g(x).e^{g(x)} - e^{g(x)} + c$

put $x = 0$

$\Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$

$\Rightarrow y(2).e^{g(2)} = g(2)e^{g(2)} - e^{g(2)} + 1$

$\Rightarrow y(2) = 0 - e^0 + 1 \Rightarrow y(2) = 0$

24. Ans. (9)

Sol. Given $y = f(x)$

Tangent at point P(x, y)

$Y - y = \left(\frac{dy}{dx}\right)_{(x,y)}(X - x)$

Now y – intercept $\Rightarrow Y = y - x\frac{dy}{dx}$

Given that, $y - x\frac{dy}{dx} = x^3$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -x^2$ is a linear differential equation

with I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$

Hence, solution is $\frac{y}{x} = \int -x^2.\frac{1}{x}dx + C$

or $\frac{y}{x} = -\frac{x^2}{2} + C$

Given $f(1) = 1$

Substituting we get. $C = \frac{3}{2}$

So $y = -\frac{x^3}{2} + \frac{3}{2}x$

Now $f(-3) = \frac{27}{2} - \frac{9}{2} = 9$

25. Ans. (B)

Sol. $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$

$e^{-x}f'(x) - e^{-x}f(x) = \sqrt{x^4+1}$

$\Rightarrow f'(x) - f(x) = e^x\sqrt{x^4+1}$

$\Rightarrow \frac{dy}{dx} = y + e^x\sqrt{x^4+1}$ (say) ....(i)

considering $y = f(x)$. so that $x = f'(y)$

$f^{-1}{}'(2) = \left(\frac{dx}{dy}\right)_{y=2}$ ....(ii)

for $x = 0 \Rightarrow f(x) = 2$ i.e. $y = 2$

$\Rightarrow f^{-1}(2) = 0$

$\frac{dy}{dx} = 2 + 1\sqrt{1} = 3$

from (2), $f{-}1'(2) = \frac{1}{3}$

26. Ans. $A \to (P, Q, S)$; $B \to (P, T)$;

$C \to (P, Q, R, T)$; $D \to (S)$

Sol.

(A) $\frac{dy}{dx} = -\frac{y}{(x-3)^2} \Rightarrow \ln y = \frac{1}{x-3} + c$

$\Rightarrow y = e^{\frac{1}{x-3}+c}, x \neq 3.$

(B) $I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$

Applying $x \to 6 - x$

$I = \int_1^5 (5-x)(4-x)(3-x)(2-x)(1-x)\,dx$

$= -I \Rightarrow I = 0.$

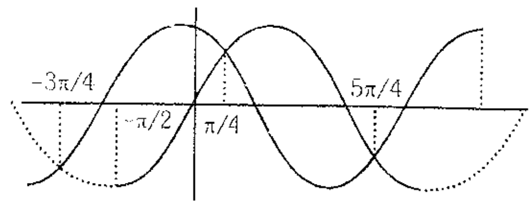
(C) $f(x) = \cos^2 x + \sin x$

$f'(x) = -2\cos x \sin x + \cos x$

$\Rightarrow \cos x(-2\sin x + 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

sign of $f'(x)$ changes from –ve to +ve while $f(x)$ passes through $x = \frac{\pi}{6}, \frac{5\pi}{6}$.



(D) $f(x) = \tan^{-1}(\sin x + \cos x)$

$f(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} > 0$

$x \in (-3\pi/4, \pi/4)$

27. Ans. (C)

Sol. $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{y\sqrt{y^2-1}}$

$\sec^{-1}x = \sec^{-1}y + c$

$\because y(2) = \frac{2}{\sqrt{3}} \therefore c = \frac{\pi}{6}$

$\sec^{-1}x = \sec^{-1}y + \frac{\pi}{6} \Rightarrow y = \sec(\sec^{-1}x - \frac{\pi}{6})$

Now $\cos^{-1}\frac{1}{x} = \cos^{-1}\frac{1}{y} + \frac{\pi}{6}$

$\Rightarrow \cos\frac{1}{y} = \cos^{-1}\frac{1}{x} - \cos^{-1}\sqrt{\frac{3}{2}}$

$\frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\left(\frac{1}{2}\right)$

$\frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$

Hence S(I) is true and S(II) is false.

MATRIX

1. **Ans. (B, C)**

Sol. $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$|M| = -1 + 1 = 0$

$\Rightarrow$ M is singular so non-invertible

Option (B) :

$$M\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}$$

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= -a_1 \\ a_1 + a_3 &= -a_2 \\ a_2 &= -a_3 \end{aligned}\right\} \Rightarrow a_1 = 0 \text{ and } a_2 + a_3 = 0$$

infinite solutions exists [B] is correct.

Option (D) :

$$M - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$|M - 2I| = 0 \Rightarrow$ [D] is wrong

Option (C) :

$$MX = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

$x + z = 0$

$y = 0$

$\therefore$ Infinite solution

[C] is correct

2. **Ans. (3780)**

Sol. Let us calculate when $|R| = 0$

Case-I $ad = bc = 0$

Now $ad = 0$

$\Rightarrow$ Total – (When none of a & d is 0)

$= 8^2 - 7^2 = 15$ ways

Similarly $bc = 0 \Rightarrow 15$ ways

$\therefore 15 \times 15 = 225$ ways of $ad = bc = 0$

Case-II $ad = bc \neq 0$

either $a = d = b = c$

OR $a \neq d,\ b \neq d$ but $ad = bc$

${}^7C_1 = 7$ ways

${}^7C_2 \times 2 \times 2 = 84$ ways

Total 91 ways

$\therefore |R| = 0$ in $225 + 91 = 316$ ways

$|R| \neq 0$ in $8^4 - 316 = 3780$

3. **Ans. (3)**

Sol. $A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$ $|A| = -1$

$\Rightarrow |A^7 - (\beta - 1)A^6 - \beta A^5| = 0$

$\Rightarrow |A|^5 |A^2 - (\beta - 1)A - \beta I| = 0$

$\Rightarrow |A|^5 |(A^2 - \beta A) + A - \beta I| = 0$

$\Rightarrow |A|^5 |A(A - \beta I) + I(A - \beta I)| = 0$

$|A|^5 |(A + I)(A - \beta I)| = 0$

$$A + I = \begin{pmatrix} \beta+1 & 0 & 1 \\ 2 & 2 & -2 \\ 3 & 1 & -1 \end{pmatrix} \Rightarrow |A + I| = -4, \text{ Here}$$

$|A| \neq 0$ & $|A + I| \neq 0$

$$A - \beta I = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1-\beta & -2 \\ 3 & 1 & -2-\beta \end{pmatrix}$$

$$|A - \beta I| = 2 - 3(1-\beta) = 3\beta - 1 = 0 \Rightarrow \beta = \frac{1}{3}$$

$9\beta = 3$

4. **Ans. (A)**

Sol. $M = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$

$$M = \begin{bmatrix} \frac{3}{2}+1 & \frac{3}{2} \\ \frac{-3}{2} & \frac{-3}{2}+1 \end{bmatrix}$$

$$M = I + \frac{3}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M^{2022} = \left(I + \frac{3}{2}A\right)^{2022}$$

$$= I + 3033A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3033\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$$

5. **Ans.(A, B, D)**

Sol. $PEP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

$|E| = 0$ and $|F| = 0$ and $|Q| \neq 0$

$|EQ| = |E||Q| = 0$, $|PFQ^{-1}| = \dfrac{|P||F|}{|Q|} = 0$

$T = EQ + PFQ^{-1}$

$TQ = EQ^2 + PF = EQ^2 + P^2EP = EQ^2 + EP$

$= E(Q^2 + P)$

$|TQ| = |E(Q^2 + P)| \Rightarrow |T||Q|$

$= |E||Q^2 + P| = 0 \Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(C) $|(EF)^3| > |EF|^2$

Here $0 > 0$ (false)

(D) as $P^2 = I \Rightarrow P^{-1} = P$ so $P^{-1}FP = PFP$

$= PPEPP = E$

so $E + P^{-1}FP = E + E = 2E$

$P^{-1}EP + F \Rightarrow PEP + F = 2PEP$

$Tr(2PEP) = 2Tr(PEP) = 2Tr(EPP) = 2Tr(E)$

6. **Ans. (A, B, C)**

Sol. $|I - EF| \neq 0$; $G = (I - EF)^{-1} \Rightarrow G^{-1} = I - EF$

Now, $G.G^{-1} = I = G^{-1}G$

$\Rightarrow G(I - EF) = I = (I - EF)G$

$\Rightarrow G - GEF = I = G - EFG$

$\Rightarrow GEF = EFG$ [C is Correct]

$(I - FE)(I + FGE) = I + FGE - FE - FEFGE$

$= I + FGE - FE - F(G - I)E$

$= I + FGE - FE - FGE + FE$

$= I$ [(B) is Correct]

(So 'D' is Incorrect)

We have

$(I - FE)(I + FGE) = I$ .....(I)

Now

$FE(I + FGE)$

$= FE + FEFGE$

$= FE + F(G - I)E$

$= FE + FGE - FE$

$= FGE$

$\Rightarrow |FE||I + FGE| = |FGE|$

$\Rightarrow |FE| \times \dfrac{1}{|I - FE|} = |FGE|$ (from (1))

$\Rightarrow |FE| = |I - FE||FGE|$

(option (A) is Correct)

7. **Ans. (B, C, D)**

Sol. $\det(M) \neq 0$

$M^{-1} = adj(adj\ M)$

$M^{-1} = \det(M).M$

$M^{-1}M = \det(M).M^2$

$I = \det(M).M^2$ .... (i)

$\det(I) = (\det(M))^5$

$1 = \det(M)$ .... (ii)

From (i) $I = M^2$

$(adj\ M)^2 = adj\ (M^2) = adj\ I = I$

8. Ans. (5)

Sol. M-I

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$

$$A^3 = \begin{bmatrix} a^3+2abc+bdc & a^2b+abd+b^2c+bd^2 \\ a^2c+adc+bc^2+d^2c & abc+2bcd+d^3 \end{bmatrix}$$

Given trace(A) = a + d = 3

and trace$(A^3) = a^3 + d^3 + 3abc + 3bcd = -18$

$\Rightarrow \quad a^3 + d^3 + 3bc(a + d) = -18$

$\Rightarrow \quad a^3 + d^3 + 9bc = -18$

$\Rightarrow \quad (a + d)((a + d)^2 - 3ad) + 9bc = -18$

$\Rightarrow \quad 3(9 - 3ad) + 9bc = -18$

$\Rightarrow \quad ad - bc = 5 =$ determinant of A

M-II

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ; \quad \Delta = ad - bc$

$|A - \lambda I| = (a - \lambda)(d - \lambda) - bc$

$= \lambda^2 - (a + d)\lambda + ad - bc$

$= \lambda^2 - 3\lambda + \Delta$

$\Rightarrow \quad O = A^2 - 3A + \Delta I$

$\Rightarrow \quad A^2 = 3A - \Delta I$

$\Rightarrow \quad A^3 = 3A^2 - \Delta A$

$= 3(3A - \Delta I) - \Delta A$

$= (9 - \Delta)A - 3\Delta I$

$= (9-\Delta)\begin{bmatrix} a & b \\ c & d \end{bmatrix} - 3\Delta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \quad$ trace $A^3 = (9 - \Delta)(a + d) - 6\Delta$

$\Rightarrow \quad -18 = (9 - \Delta)(3) - 6\Delta$

$= 27 - 9\Delta$

$\Rightarrow \quad 9\Delta = 45 \Rightarrow \Delta = 5$

9. Ans. (B)

Sol. Given $M = \alpha I + \beta M^{-1}$

$\Rightarrow M^2 - \alpha M - \beta I = O$

By putting values of M and M^2, we get

$\alpha(\theta) = 1 - 2\sin^2\theta\cos^2\theta = 1 - \dfrac{\sin^2 2\theta}{2} \geq \dfrac{1}{2}$

Also. $\beta(\theta) = -(\sin^4\theta\cos^4\theta + (1+\cos^2\theta)(1+\sin^2\theta))$

$= -(\sin^4\theta\cos^4\theta + 1 + \cos^2\theta + \sin^2\theta + \sin^2\theta\cos^2\theta)$

$= -(t^2 + t + 2), t = \dfrac{\sin^2 2\theta}{4} \in \left[0, \dfrac{1}{4}\right]$

$\Rightarrow \beta(\theta) \geq -\dfrac{37}{16}$

10. Ans. (A, C, D)

Sol. $(\text{adj}M)_{11} = 2 - 3b = -1 \Rightarrow b = 1$

Also, $(\text{adj}M)_{22} = -3a = -6 \Rightarrow a = 2$

Now, $\det M = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2$

$\Rightarrow \det(\text{adj}M^2) = (\det M^2)^2$

$= (\det M)^4 = 16$

Also $M^{-1} = \dfrac{\text{adj}M}{\det M}$

$\Rightarrow \text{adj}M = -2M^{-1}$

$\Rightarrow (\text{adj}M)^{-1} = \dfrac{-1}{2}M$

And, $\text{adj}(M^{-1}) = (M^{-1})^{-1}\det(M^{-1})$

$= \dfrac{1}{\det M}M = \dfrac{-M}{2}$

Hence, $(\text{adj}M)^{-1} + \text{adj}(M^{-1}) = -M$

Further, $\quad MX = b$

$\Rightarrow \quad X = M^{-1}b = \dfrac{-\text{adj}M}{2}b$

$= \dfrac{-1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \dfrac{-1}{2}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow (\alpha, \beta, \gamma) = (1, -1, 1)$

11. Ans. (B, C, D)

Sol. Let $Q = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$$X = \sum_{K=1}^{6}\left(P_K Q P_K^T\right)$$

$$X^T = \sum_{K=1}^{6}\left(P_K Q P_K^T\right)^T = X$$

X is symmetric

Let $R = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$XR = \sum_{K=1}^{6} P_K Q P_K^T R . \; [\because P_K^T R = R]$$

$$= \sum_{K=1}^{6} P_K QR. = \left(\sum_{K=1}^{6} P_K\right) QR$$

$$\sum_{K=1}^{6} P_K = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad QR = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow XR = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = 30R$$

$\Rightarrow \alpha = 30.$

$$\text{Trace } X = \text{Trace}\left(\sum_{K=1}^{6} P_K Q P_K^T\right)$$

$$= \sum_{K=1}^{6} \text{Trace}\left(P_K Q P_K^T\right) = 6\left(\text{Trace}\, Q\right) = 18$$

$$X\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 30\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(X - 30I\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = O \Rightarrow |X - 30I| = 0$$

$\Rightarrow X - 30I$ is non-invertible

12. Ans. (C, D)

Sol. $\det(R) = \det(PQP^{-1}) = (\det P)(\det Q)\left(\dfrac{1}{\det P}\right)$

$= \det Q$

$= 48 - 4x^2$

Option (A) :

for $x = 1$ $\det(R) = 44 \neq 0$

$\therefore$ for equation $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We will have trivial solution

$\alpha = \beta = \gamma = 0$

Option (B) :

$PQ = QP$

$PQP^{-1} = Q$

$R = Q$

No value of x.

Option (C) :

$$\det\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$$

$= (40 - 4x^2) + 8 = 48 - 4x^2 = \det R \ \forall\ x \in R$

Option (D):

$$R = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\left(R - 6I\right)\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = O$$

$$\Rightarrow -4 + a + \frac{2b}{3} = 0$$

$$-2a + \frac{4b}{3} = 0$$

$\Rightarrow a = 2 \qquad b = 3$

$a + b = 5$

13. Ans. (A, D)

Sol. We find D = 0 & since no pair of planes are parallel, so there are infinite number of solutions.

Let $\alpha P_1 + \lambda P_2 = P_3$

$\Rightarrow P_1 + 7P_2 = 13P_3$

$\Rightarrow b_1 + 7b_2 = 13b_3$

(A) $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3

(B) $D = 0$ but $P_1 + 7P_2 \neq 13P_3$

(C) As planes are parallel and there exist infinite ordered triplet for which they will be non coincident although satisfying $b_1 + 7b_2 = 13b_3$.

$\therefore$ rejected.

(D) $D \neq 0$

14. Ans. (4)

Sol. $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=\underbrace{(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)}_{x} - \underbrace{(a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)}_{y}$$

Now if $x \le 3$ and $y \ge -3$

the Δ can be maximum 6

But it is not possible

as $x = 3 \Rightarrow$ each term of $x = 1$

and $y = 3 \Rightarrow$ each term of $y = -1$

$$\Rightarrow \prod_{i=1}^{3} a_ib_ic_i = 1 \text{ and } \prod_{i=1}^{3} a_ib_ic_i = -1$$

which is contradiction

so now next possibility is 4

which is obtained as

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1+1) - 1(-1-1) + 1(1-1) = 4$$

15. Ans. (A, B)

16. Ans. (1)

Sol. $\Delta = 0 \Rightarrow 1(1-\alpha^2) - \alpha(\alpha - \alpha^3) + \alpha^2(\alpha^2 - \alpha^2) = 0$

$(1-\alpha^2) - \alpha^2 + \alpha^4 = 0$

$(\alpha^2 - 1)^2 = 0 \Rightarrow \alpha = \pm 1$

but at $\alpha = 1$ No solution so rejected

at $\alpha = -1$ all three equation become

$x - y + z = 1$ (coincident planes)

$\therefore \quad 1 + \alpha + \alpha^2 = 1$

17. Ans. (A)

Sol. Let $M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\therefore tr(M^TM) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2$

$= 5$, where entries are $\{0,1,2\}$

Only two cases are possible.

(I) five entries 1 and other four zero

$\therefore {}^9C_5 \times 1$

(II) One entry is 2, one entry is 1 and others are 0.

$\therefore {}^9C_2 \times 2!$

Total = 126 + 72 = 198

18. Ans. (B, C)

Sol. $PQ = kI$

$|P|.|Q| = k^3$

$\Rightarrow |P| = 2k \neq 0 \Rightarrow P$ is an invertible matrix

$\because PQ = kI$

$\therefore Q = kP^{-1}I$

$\therefore Q = \dfrac{adj.P}{2}$

$\because q_{23} = -\dfrac{k}{8}$

$\therefore \dfrac{-(3\alpha+4)}{2} = -\dfrac{k}{8} \Rightarrow k = 4$

$\therefore |P| = 2k \Rightarrow k = 10 + 6\alpha$ ...(i)

Put value of k in (i).. we get $\alpha = -1$

$\therefore 4\alpha - k + 8 = 0$

& det (P(adj.Q)) = $|P|\ |adj.Q| = 2k.\left(\dfrac{k^2}{2}\right)^2$

$= \dfrac{k^5}{2} = 2^9$

19. Ans. (B)

Sol. $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16+32 & 8 & 1 \end{bmatrix}$

so, $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16+32+48 & 12 & 1 \end{bmatrix}$

(from the symmetry)

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ \frac{16.50.51}{2} & 200 & 1 \end{bmatrix}$$

As. $P^{50} - Q = I \Rightarrow q_{31} = \dfrac{16.50.51}{2}$

$q_{32} = 200$ and $q_{21} = 200$

$\therefore \dfrac{q_{31}+q_{32}}{q_{21}} = \dfrac{16.50.51}{2.200} + 1$

$= 102 + 1 = 103$

20. Ans. (C, D)

Sol. $x^T = -x,\ y^T = -y,\ z^T = z$

(A) Let $P = y^3z^4 - z^4y^3$

$P^T = (y^3z^4)^T - (z^4y^3)^T$

$= -z^4y^3 + y^3 z^4 = P \Rightarrow$ symmetric

(B) Let $P = x^{44} + y^{44}$

$P^T = (X^{44})^T + (y^{44})^T = P \Rightarrow$ symmetric

(C) Let $P = x^4z^3 - z^3x^4$

$P^T = (z^3)^T(x^4)^T - (x^4)^T(z^3)^T$

$= z^3x^4 - x^4z^3 = -P \Rightarrow$ skew symmetric

(D) Let $P = x^{23} + y^{23}$

$P^T = -x^{23} - y^{23} = -P \Rightarrow$ skew symmetric

21. Ans. (C, D)

Sol. Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(A) Given that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow a = b = c = \alpha(\text{let})$

$\Rightarrow M = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \Rightarrow |M| = 0 \Rightarrow$ Non-invertible

(B) Given that $[b\ c] = [a\ b] \Rightarrow a = b = c = \alpha(\text{let})$

again $|M| = 0 \Rightarrow$ Non-invertible

(C) As given $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$

($\because$ a & c are non zero)

$\Rightarrow$ M is invertible

(D) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow |M| = ac - b^2 \neq 0$

$\because$ ac is not equal to square of an integer

$\therefore$ M is invertible

22. Ans. (A, B)

Sol. (A) $(M - N^2)(M + N^2) = \mathbf{O}$ ...(1)

($\therefore MN^2 = N^2M$)

$\Rightarrow |M - N^2|\,|M + N^2| = 0$

Case I : If $|M + N^2| = 0$

$\therefore |M^2 + MN^2| = 0$

Case II : If $|M + N^2| \neq 0 \Rightarrow M + N^2$ is invertible

from (1)

$(M - N^2)(M + N^2)(M + N^2)^{-1} = \mathbf{O}$

$\Rightarrow M - N^2 = \mathbf{O}$ which is wrong

(B) $(M + N^2)(M - N^2) = \mathbf{O}$

pre-multiply by M

$\Rightarrow (M^2 + MN^2)(M - N^2) = \mathbf{O}$ ...(2)

Let $M - N^2 = U$

$\Rightarrow$ from equation (2) there exist same non zero 'U'

$(M^2 + MN^2)U = \mathbf{O}$

23. Ans. (C, D)

Sol. (A) Let $A = N^T M N$

$A^T = N^T M^T N$

$A = A^T$ when $M = M^T$

$A = -A^T$ when $M = -M^T$

(B) Let $B = MN - NM$ & $M^T = M$ & $N^T = N$

$B^T = N^T M^T - M^T N^T$

$B^T = NM - MN$

$B + B^T = 0$

(C) $M = M^T$ & $M = N^T$

$A = MN$

$A^T - N^T M^T \Rightarrow A^T = NM$

$A \neq A^T$

(D) $\text{adj}(MN) = (\text{adj } N)(\text{adj } M)$

Hence correct answer are C and D

24. Ans. (D)

Sol. $$|Q| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$\Rightarrow \quad |Q| = 2^2.2^3.2^4.\begin{vmatrix} a_{11} & 2a_{12} & 2^2 a_{13} \\ a_{21} & 2a_{22} & 2^2 a_{23} \\ a_{31} & 2a_{32} & 2^2 a_{33} \end{vmatrix}$$

$= 2^2.2^3.2^4|P|.2^3$

$= 2^2.2^3.2^4.2.2^3 = 2^{13}$

25. Ans. (D)

Sol. $P^T = 2P + I$

$\Rightarrow P = 2P^T + I$

$\Rightarrow P = 2(2P + I) + I$

$\Rightarrow P = 4P + 3I$

$\Rightarrow P = -I$

$\Rightarrow PX = -X$

26. Ans. (A, D)

Sol. $$|\text{adj}P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix}$$

$\Rightarrow |P|^2 = 4 \Rightarrow |P| = \pm 2$

(Bonus)

27. Ans. (C) (Bonus)

Sol. (Comment : Although 3×3 skew symmetric matrices can never be non-singular. Therefore the information given in question is wrong. Now if we consider only non singular skew symmetric matrices M & N, then the solution is-)

Given $M^T = -M$

$N^T = -N$

$MN = NM$

according to question $M^2N^2(M^TN)^{-1}(MN^{-1})^T$

$= M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^T M^T$

$= M^2N^2N^{-1}(-M)^{-1}(N^T)^{-1}(-M)$

$= -M^2 N M^{-1} N^{-1} M$

$= -M^2 N N^{-1} M^{-1} M = -M^2$

$$\left[\begin{array}{l} MN = NM \\ (MN)^{-1} = (NM)^{-1} \\ N^{-1}M^{-1} = M^{-1}N^{-1} \end{array}\right.$$

28. Ans.(A)

Sol. $$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix}$$

$= 1 - c\omega - a(\omega - \omega^2 c) = (1 - c\omega) - a\omega(1 - c\omega)$

$= (1 - c\omega)(1 - a\omega)$

for non singular matrix

$$c \neq \frac{1}{\omega} \ \& \ a \neq \frac{1}{\omega}$$

$\Rightarrow \quad c \neq \omega^2, \quad a \neq \omega^2$

$\Rightarrow \quad$ a & c must be ω & b can be ω or ω^2

$\therefore \quad$ total matrices = 2

29. Ans. (9)

Sol. Let $M = \begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix}$

according to question

$$\begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad b = -1, y = 2, m = 3 \quad \ldots(1)$

$$\Rightarrow \begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$a - b = 1$

$x - y = 1$

$\ell - m = -1$

from (1) $\quad a = 0$

$x = 3$

$\ell = 2$

$$\begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$\ell + m + n = 12$

$\Rightarrow \quad 2 + 3 + n = 12 \quad \Rightarrow \quad n = 7$

Now $a + y + n = 0 + 2 + 7 = 9$

30. **Ans. (A)**

Sol. The given matrix system is a linear system in x, y, z, hence it can have either a unique solution or no-solution or infinitely many solutions. It can never have exactly two distinct solutions.

31. **Ans. (D)**

Sol. If A is symmetric, AT = A

$$\Rightarrow \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & a \end{bmatrix}$$

$\Rightarrow b = c$

If A is skew symmetric, AT = – A

$$\Rightarrow \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -a \end{bmatrix}$$

$\Rightarrow a = 0, b + c = 0$

$\because b, c \geq 0 \Rightarrow a = 0, b = 0, c = 0$

Now, $\det(A) = a^2 - bc$

$= a^2 - b^2$ ($\because$ b = c for A being symmetric or skew symmetric or both)

$= (a - b)(a + b)$ is divisible by p.

Let $(a - b)(a + b) = \lambda p, \lambda \in I$

Range of $(a + b)$ is 0 to $2p - 2$ which includes only one multiple of p i.e. p

$\therefore a + b = p \ \& \ a - b \in I$

$\Rightarrow$ possible number of pairs of a & b will be $p-1$.

Also, range of $(a - b)$ is $1 - p$ to $p - 1$ which includes only one multiple of p i.e. 0

$\therefore a - b = 0 \ \& \ a + b \in I$

$\Rightarrow$ Possible number of pairs of a & b will be p.

Hence total number of A in T_p will be $p + p - 1 = 2p - 1$

32. **Ans. (C)**

33. **Ans. (D)**

Sol. Total number of A in $T_P = p^3$

when $a \neq 0$ & det(A) is divisible by p, then number of A will be $(p - 1)^2$

When $a = 0$ & det(A) is divisible by p, then number of A will be $2p - 1$.

So, total number of A for which det(A) is divisible by p

$= (p - 1)^2 + 2p - 1$

$= p^2$

So number of A for which det(A) is not divisible by p

$= p^3 - p^2$

34. **Ans. (4)**

Sol. $|\text{adj}A| = |A|^{n-1} = |A|^2$

$$|A| = (1 + 2k)\begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1 & -1 \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$|A| = (1 + 2k)\begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 0 & -1 \\ -2\sqrt{k} & 2k-1 & -1 \end{vmatrix}$$

$C_2 \to C_2 + C_3$

$|A| = (2k + 1)\{(2k - 1)^2 + 8k\} = (2k + 1)^3$

$\because$ B is a skew symmetric matrix

$\Rightarrow |B| = 0$

$\det(\text{adj}A) = |A|^2 = (2k + 1)^6$

$\Rightarrow (2k + 1)^6 = 10^6 \Rightarrow k = 4.5 \Rightarrow [k] = 4$

Solution for Question No. 35 to 37

$$\begin{bmatrix}1&1&1\\1&0&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}1&1&0\\1&0&1\\0&1&0\end{bmatrix}\quad\begin{bmatrix}1&0&1\\0&0&1\\1&1&0\end{bmatrix}$$

(1) (2) (3)

$$\begin{bmatrix}0&1&1\\1&1&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&1&1\\1&1&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}$$

(4) (5) (6)

$$\begin{bmatrix}0&1&1\\1&0&0\\1&0&1\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&0&1\\0&1&1\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix}$$

(7) (8) (9)

$$\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$$

(10) (11) (12)

35. **Ans. (A)**

Total 12 matrix.

36. **Ans. (B)**

For following matrix equation has unique solution

(2), (3), (4), (5), (7), (8)

37. **Ans. (B)**

For the matrix (6), (9), (10), (11) the system of linear equation is inconsistent.

38. **Ans.** (A) – r, (B) – q, s, (C) – r, s, (D) – p, r

Sol. (A) $y = \dfrac{x^2+2x+4}{x+2}$

$\Rightarrow$ $x^2 + (2-y)x + 4 - 2y = 0$

x is real so

$y^2 + 4y - 12 \ge 0$

$y \le -6,\ y \ge 2$

so minimum value = 2

(A) $\longrightarrow$ (r)

(B) $(A+B)(A-B) = (A-B)(A+B)$

$\Rightarrow$ $AB = BA$

as A is symmetric & B is skew symmetric

$\Rightarrow$ $(AB)^t = -AB$

$\Rightarrow$ $k = 1, 3$

(C) $a = \log_3\log_3 2 \Rightarrow 3^{-a} = \log_3 2$

Now, $1 < 2^{(-K+3^{-a})} < 2$

$\Rightarrow$ $1 < 2^{(-K+\log_3 2)} < 2 \Rightarrow 1\ 3.2^{-k} < 2$

$\Rightarrow$ $\dfrac{1}{3} < 2^{-k} < \dfrac{2}{3} \Rightarrow \dfrac{3}{2} < 2^k < 3$

so, k = 1 which is less than 2 & 3

(C) $\longrightarrow$ (r, s)

(D) $\sin\theta = \cos\phi$

$\Rightarrow$ $\cos\left(\dfrac{\pi}{2}-\theta\right) = \cos\phi$

$\dfrac{\pi}{2} = 2n\pi \pm \phi$

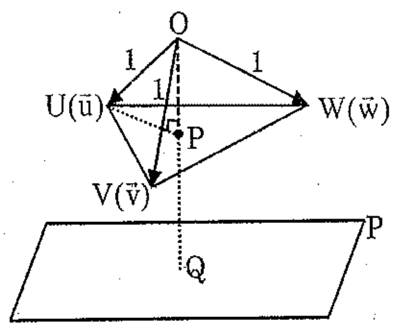
$\dfrac{1}{\pi}\left(\theta \pm \phi - \dfrac{\pi}{2}\right) = -2n$

which is an even number. So 0 and 2 are possible **(D) $\longrightarrow$ (p, r)**

VECTOR

1. **Ans. (45)**

Sol.



Given $|\vec{u}-\vec{v}| = |\vec{v}-\vec{w}| = |\vec{w}-\vec{u}|$

$\Rightarrow \Delta UVW$ is an equilateral Δ

Now distances of U, V, W from $P = \dfrac{7}{2}$

$\Rightarrow PQ = \dfrac{7}{2}$

Also, Distance of plane P from origin

$\Rightarrow OQ = 4$

$\therefore OP = OQ - PQ \Rightarrow OP = \dfrac{1}{2}$

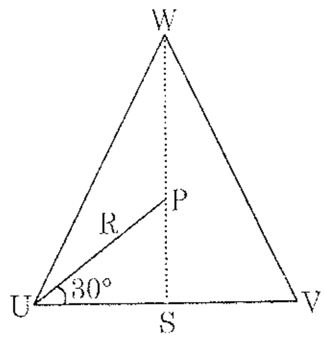
Hence, $PU = \sqrt{OU^2 - OP^2} \Rightarrow PU = \frac{\sqrt{3}}{2} = R$

Also, for ΔUVW, P is circumcenter

$\therefore$ for ΔUVW : $US = R\cos 30^\circ$

$\Rightarrow UV = 2R\cos 30^\circ$

$\Rightarrow UV = \frac{3}{2}$



$\therefore \quad Ar(\Delta UVW) = \frac{\sqrt{3}}{4}\left(\frac{3}{2}\right)^2 = \frac{9\sqrt{3}}{16}$

$\therefore$ Volume of tetrahedron with coterminous edges $\vec{u}$, $\vec{v}$, $\vec{w}$

$= \frac{1}{3}(Ar.\Delta UVW) \times OP = \frac{1}{3} \times \frac{9\sqrt{3}}{16} \times \frac{1}{2} = \frac{3\sqrt{3}}{32}$

$\therefore$ parallelopiped with coterminous edges

$\vec{u}, \vec{v}, \vec{w} = 6 \times \frac{3\sqrt{3}}{32} = \frac{9\sqrt{3}}{16} = V$

$\therefore \frac{80}{\sqrt{3}} V = 45$

2. Ans. (B)

Sol. $P\left(\hat{i} + 2\hat{j} - 5\hat{k}\right) = P\left(\vec{a}\right)$

$Q\left(3\hat{i} + 6\hat{j} + 3\hat{k}\right) = Q\left(\vec{b}\right)$

$R\left(\frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + 7\hat{k}\right) = R\left(\vec{c}\right)$

$S\left(2\hat{i} + \hat{j} + \hat{k}\right) = S\left(\vec{d}\right)$

$\frac{\vec{b} + 2\vec{d}}{3} = \frac{7\hat{i} + 8\hat{j} + 5\hat{k}}{3}$

$\frac{5\vec{c} + 4\vec{a}}{9} = \frac{21\hat{i} + 24\hat{j} + 15\hat{k}}{9}$

$\Rightarrow \frac{\vec{b} + 2\vec{d}}{3} = \frac{5\vec{c} + 4\vec{a}}{9}$

so [B] is correct.

option (D)

$\left|\vec{b} \times \vec{d}\right|^2 = \left|\vec{b}\right|\left|\vec{d}\right|^2 - \left(\vec{b}.\vec{d}\right)^2$

$= (9 + 36 + 9)(4 + 1 + 1) - (6 + 6 + 3)^2$

$= 54 \times 6 - (15)^2$

$= 324 - 225$

$= 99$

3. Ans. (B, C, D)

Sol. $\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$

$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3 - c_1 \\ 1 - c_2 \\ -1 - c_3 \end{pmatrix}$$

multiply & compare

$b_2c_3 - b_3c_2 = c_1 - 3 \quad \ldots(1)$

$c_3 - b_3c_1 = 1 - c_2 \quad \ldots(2)$

$c_2 - b_2c_1 = 1 + c_3 \quad \ldots(3)$

$(1)\hat{i} - (2)\hat{j} + (3)\hat{k}$

$\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(c_3 - b_3c_1) + \hat{k}(c_2 - b_2c_1)$

$= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} - 3\hat{i} - \hat{j} + \hat{k}$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Take dot product with $\vec{b}$

$0 = \vec{c}.\vec{b} - \vec{a}.\vec{b}$

$\vec{b}.\vec{c} = 0$

$\vec{b} \perp \vec{c}$

$\vec{b} \wedge \vec{c} = 90^\circ$

Take dot product with $\vec{c}$

$0 = |\vec{c}|^2 - \vec{a}.\vec{c}$

$\vec{a}.\vec{c} = |\vec{c}|^2$

$\vec{a}.\vec{c} \neq 0$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Squaring

$|\vec{b}|^2|\vec{c}|^2 = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a}$

$|\vec{b}|^2|\vec{c}|^2 = |\vec{c}|^2 + 11 - 2|\vec{c}|^2$

$|\vec{b}|^2|\vec{c}|^2 = 11 - |\vec{c}|^2$

$|\vec{c}|^2\left(|\vec{b}|^2 + 1\right) = 11$

$|\vec{c}|^2 = \dfrac{11}{|\vec{b}|^2 + 1}$

$|\vec{c}| \le \sqrt{11}$

given $\vec{a}.\vec{b} = 0$

$b_2 - b_3 = -3$ also

$b_2^2 + b_3^2 - 2b_2b_3 = 9$ $b_2b_3 > 0$

$b_2^2 + b_3^2 = 9 + 2b_2b_3$

$b_2^2 + b_3^2 = 9 + 2b_2b_3 > 9$

$b_2^2 + b_3^2 > 9$

$|\vec{b}| = \sqrt{1 + b_2^2 + b_3^2}$

$|\vec{b}| > \sqrt{10}$

4. Ans. (7)

Sol. Given, $|\vec{u}| = 1$; $|\vec{v}| = 1$; $\vec{u}.\vec{v} \ne 0$; $\vec{u}.\vec{w} = 1$;

$\vec{v}.\vec{w} = 1$;

$\vec{w}.\vec{w} = |\vec{w}|^2 = 4 \Rightarrow |\vec{w}| = 2$;

$[\vec{u}\ \ \vec{v}\ \ \vec{w}] = \sqrt{2}$

and $[\vec{u}\ \ \vec{v}\ \ \vec{w}]^2 = \begin{vmatrix} \vec{u}.\vec{u} & \vec{u}.\vec{v} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}.\vec{v} & \vec{v}.\vec{w} \\ \vec{w}.\vec{u} & \vec{w}.\vec{v} & \vec{w}.\vec{w} \end{vmatrix} = 2$

$\Rightarrow \begin{vmatrix} 1 & \vec{u}.\vec{v} & 1 \\ \vec{u}.\vec{v} & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2$

$\Rightarrow \vec{u}.\vec{v} = \dfrac{1}{2}$

So, $|3\vec{u} + 5\vec{v}| = \sqrt{9|\vec{u}|^2 + 25|\vec{v}|^2 + 2.3.5\vec{u}.\vec{v}}$

$= \sqrt{9 + 25 + 30\left(\dfrac{1}{2}\right)} = \sqrt{49} = 7$

5. Ans. (A, B, C)

Sol. $\overline{OB} \times \overline{OC} = \dfrac{1}{2}\overline{OB} \times \left(\overline{OB} - \lambda\overline{OA}\right)$

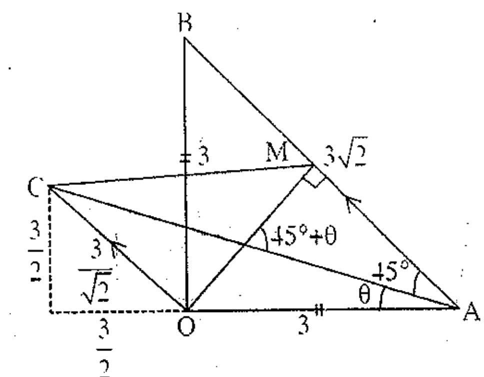
$= \dfrac{\lambda}{2}\left(\overline{OA} \times \overline{OB}\right)$

$|\overline{OB}| \times |\overline{OC}| = \dfrac{|\lambda|}{2}|\overline{OA}| \times |\overline{OB}|$

(Note $\overline{OA}$ & $\overline{OB}$ are perpendicular)

$\Rightarrow \dfrac{9\lambda}{2} = \dfrac{9}{2} \Rightarrow \lambda = 1$ (given $\lambda > 0$)

So $\overline{OC} = \dfrac{\overline{OB} - \overline{OA}}{2} = \dfrac{\overline{AB}}{2}$



M is mid point of AB

Note projection of $\overline{OC}$ on $\overline{OA} = -\dfrac{3}{2}$

$\tan\theta = \dfrac{1}{3}$

Area of $\Delta ABC = \dfrac{9}{2}$

Acute angle between diagonals is

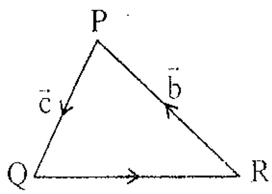
$\tan^{-1}\left(\dfrac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \tan^{-1} 2$

6. **Ans. (108.00)**

Sol. We have $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\Rightarrow \vec{c}=-\vec{a}-\vec{b}$$

Now, $\dfrac{\vec{a}.(-\vec{a}-2\vec{b})}{(-\vec{a}-\vec{b}).(\vec{a}-\vec{b})}=\dfrac{3}{7}$



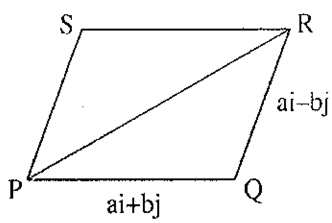
$$\Rightarrow \frac{9+2\vec{a}.\vec{b}}{9-16}=\frac{3}{7}$$

$$\Rightarrow \vec{a}.\vec{b}=-6$$

$$\Rightarrow |\vec{a}\times\vec{b}|^2=a^2b^2-(\vec{a}.\vec{b})^2=9\times 16-36=108$$

7. **Ans. (A, C)**

Sol.



$$\vec{u}=\left((i+j).\widehat{PQ}\right)\widehat{PQ}$$

$$\vec{u}=\left|(i+j).\widehat{PQ}\right|$$

$$|\vec{u}|=\left|(i+j).\frac{(ai+bj)}{\sqrt{a^2+b^2}}\right|=\frac{a+b}{a^2+b^2}$$

$$\vec{v}=(i+j).\widehat{PS}$$

$$|\vec{v}|=\left|\frac{(i+j).(ai-bj)}{\sqrt{a^2+b^2}}\right|=\frac{a-b}{\sqrt{a^2+b^2}}$$

$$|\vec{u}|+|\vec{v}|=|\vec{w}|$$

$$\frac{|(a+b)|+|(a-b)|}{\sqrt{a^2+b^2}}=\sqrt{2}$$

For $a\geq b$

$$2a=\sqrt{2}.\sqrt{a^2+b^2}$$

$$4a^2=2a^2+2b^2$$

$a^2=b^2 \quad \therefore a=b \qquad ...(1)$

$(a>0, b>0)$

similarly for $a\leq b$ we will get $a=b$

Now area of parallelogram

$$=\left|(ai+bj)\times(ai-bj)\right|$$

$=2ab$

$\therefore 2ab=8$

$ab=4 \qquad ...(2)$

from (1) and (2)

$a=2, b=2 \qquad \therefore a+b=4$ option (A)

length of diagonal is

$$\left|2a\hat{i}\right|=\left|4\hat{i}\right|=4$$

so option (C)

8. **Ans. (C, D)**

Sol. Let $P(\lambda, 0, 0)$, $Q(0, \mu, 1)$, $R(1, 1, \nu)$ be points. L_1, L_2 and L_3 respectively

Since P, Q, R are collinear, $\overline{PQ}$ is collinear with $\overline{QR}$

Hence $\dfrac{-\lambda}{1}=\dfrac{\mu}{1-\mu}=\dfrac{1}{\nu-1}$

For every $\mu\in R-\{0, 1\}$ there exist unique $\lambda, \nu\in R$

Hence Q cannot have coordinates (0, 1, 1) and (0, 0, 1).

9. **Ans. (18.00)**

Sol. $\vec{c}=(2\alpha+\beta)\hat{i}+\hat{j}(\alpha+2\beta)+\hat{k}(\beta-\alpha)$

$$\frac{\vec{c}.(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|}=3\sqrt{2}$$

$\Rightarrow \quad \alpha+\beta=2 \qquad ....(1)$

$$(\vec{c}-(\vec{a}\times\vec{b})).(\alpha\vec{a}+\beta\vec{b})$$

$$=|\vec{c}|^2=\alpha^2|\vec{a}|^2+\beta^2|b|^2+2\alpha\beta(\vec{a}.\vec{b})$$

$$=6(\alpha^2+\beta^2+\alpha\beta)$$

$$=6(\alpha^2+(2-\alpha)^2+\alpha(2-\alpha))$$

$$=6((\alpha-1)^2+3)$$

$\Rightarrow \quad$ Min. value $=18$

10. Ans. (3)

Sol. $\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b}$

$\vec{c}.\vec{a} = x$ and x = 2cosα

$\vec{c}.\vec{b} = y$ and y = 2cosα

Also, $|\vec{a}\times\vec{b}| = 1$

$\therefore \quad \vec{c} = 2\cos\alpha(\vec{a}+\vec{b})+\vec{a}\times\vec{b}$

$\vec{c}^2 = 4\cos^2\alpha(\vec{a}+\vec{b})^2 + (\vec{a}\times\vec{b})^2 +$

$2\cos\alpha\ (\vec{a}+\vec{b}).(\vec{a}\times\vec{b})$

$4 = 8\cos^2\alpha + 1$

$8\cos^2\alpha = 3$

11. Ans. (B)

Sol. Let position vector of $P(\vec{p}), Q(\vec{q}), R(\vec{r})$ & $S(\vec{s})$ with respect to $O(\vec{o})$

Now, $\overline{OP}.\overline{OQ} + \overline{OR}.\overline{OS} = \overline{OR}.\overline{OP} + \overline{OQ}.\overline{OS}$

$\Rightarrow \quad \vec{p}.\vec{q} + \vec{r}.\vec{s} = \vec{r}.\vec{p} + \vec{q}.\vec{s}$

$\Rightarrow \quad (\vec{p}-\vec{s}).(\vec{q}-\vec{r}) = 0 \qquad ....(1)$

Also, $\overline{OR}.\overline{OP} + \overline{OQ}.\overline{OS} = \overline{OQ}.\overline{OR} + \overline{OP}.\overline{OS}$

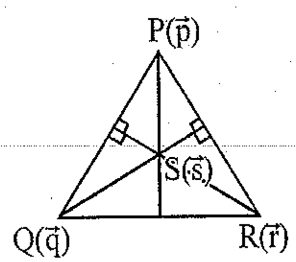
$\Rightarrow \quad \vec{r}.\vec{p} + \vec{q}.\vec{s} = \vec{q}.\vec{r} + \vec{p}.\vec{s}$

$\Rightarrow \quad (\vec{r}-\vec{s}).(\vec{p}-\vec{q}) = 0 \qquad ....(2)$

Also, $\overline{OP}.\overline{OQ} + \overline{OR}.\overline{OS} = \overline{OQ}.\overline{OR} + \overline{OP}.\overline{OS}$

$\Rightarrow \quad \vec{p}.\vec{q} + \vec{r}.\vec{s} = \vec{q}.\vec{r} + \vec{p}.\vec{s}$

$\Rightarrow \quad (\vec{q}-\vec{s}).(\vec{p}-\vec{r}) = 0 \qquad ....(3)$



$\Rightarrow$ Triangle PQR has S as its orthocentre

$\therefore$ option (B) is correct.

12. Ans. (D)

Sol. $\overline{OX} = \dfrac{\overline{QR}}{QR}$

$OY = \dfrac{\overline{RP}}{RP}$

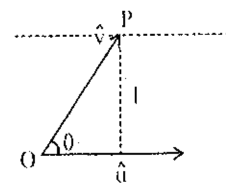
$|OX \times OY| = \sin R = \sin(P+Q)$

13. Ans. (B)

Sol. $-(\cos P + \cos Q + \cos R) \geq -\dfrac{3}{2}$ as we know cosP + cosQ + cosR will take its maximum value when $P = Q = R = \dfrac{\pi}{3}$

14. Ans. (B, C)

Sol. $|\hat{w}||\hat{u}\times\hat{v}|\cos\phi = 1 \Rightarrow \phi = 0$



$\Rightarrow \quad \hat{u}\times\vec{v} = \hat{w}$ also $|\vec{v}|\sin\theta = 1$

$\Rightarrow$ there may be infinite vectors $\vec{v} = \overline{OP}$ such that P is always 1 unit dist. from $\hat{u}$

For option (C) : $\hat{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$\hat{w} = (u_2v_3)\hat{i} - (u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$

$u_2v_3 = \dfrac{1}{\sqrt{6}}, -u_1v_3 = \dfrac{1}{\sqrt{6}}$

$\Rightarrow \quad |u_1| = |u_2|$

for option (D) : $\hat{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & 0 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

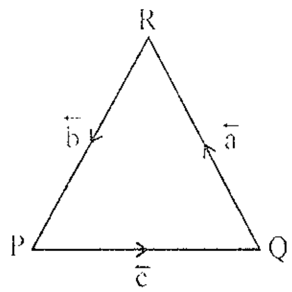
$\hat{w} = (-v_2u_3)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2)\hat{k}$

$-v_2u_3 = \dfrac{1}{\sqrt{6}}, u_1v_2 = \dfrac{2}{\sqrt{6}}$

$\Rightarrow 2|u_3| = |u_1|$ So, (D) is wrong

15. Ans. (A, C, D)

Sol.



$|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}$

$\vec{a}+\vec{b}+\vec{c} = 0$ ....(1)

$a^2 = b^2 + c^2 + 2\vec{b}.\vec{c}$

$144 = 48 + c^2 + 48$

$c^2 = 48 \Rightarrow c = 4\sqrt{3}$

Also $c^2 = a^2 + b^2 + 2\vec{a}.\vec{b}$

$48 = 144 + 48 + 2\vec{a}.\vec{b}$

$\Rightarrow \vec{a}.\vec{b} = -72$

Also by (1)

$\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

$\Rightarrow |\vec{a}\times\vec{b}+\vec{c}\times\vec{a}| = 2|\vec{a}\times\vec{b}|$

$= 2\sqrt{a^2b^2 - (\vec{a}.\vec{b})^2}$

$= 2\sqrt{12^2.48 - (72)^2}$

$= 2.12\sqrt{48-36} = 48\sqrt{3}$

$\therefore$ A, C, D are correct & B incorrect.

16. Ans. (A) $\to$ (P,R,S); (B) $\to$ (P); (C) $\to$ (P,Q); (D) $\to$ (S,T)

Sol. (A) $2\left(\sin^2 x - \sin^2 y\right) = \sin^2 z$

$2\sin(x-y)\sin(x+y) = \sin^2 z$

$\because x + y + z = \pi$

$\frac{\sin(x-y)}{\sin z} = \frac{1}{2} \Rightarrow \lambda = \frac{1}{2}$

$\Rightarrow \cos\left(\frac{n\pi}{2}\right) = 0 \Rightarrow n = 1,3,5$

(B) $1 + 1 - 2\sin^2 x - 2(1 - 2\sin^2 y)$

$= 2\sin x \sin y$

$\Rightarrow -2a^2 + 4b^2 = 2ab$

$\Rightarrow a^2 + ab - 2b^2 = 0$

$\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 2 = 0 \Rightarrow \frac{a}{b} = -2, 1$

$\Rightarrow \frac{a}{b} = 1$ as -2 rejected

(C) Angle bisector of $\overline{OX}$ & $\overline{OY}$ is along the line $y = x$ and its distance from $(\beta, 1-\beta)$ is

$\left|\frac{\beta-(1-\beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}} \Rightarrow 2\beta - 1 = \pm 3$

$\Rightarrow \beta = 2, -1$

$\Rightarrow |\beta| = 1, 2$

(D) 7

$6 - \int_0^2 2\sqrt{x}dx$ $\quad$ $5 - \int_0^2 2\sqrt{x}\,dx$

$6 - \frac{8}{3}\sqrt{2}$ ...(1) $\quad 5 - \frac{8}{3}\sqrt{2}$ ...(2)

By (1) & (2) $F(\alpha) + \frac{8}{3}\sqrt{2}$

can be 5 or 6.

17. Ans. (Bonus)

Sol. Although the language of the question is not appropriate (incomplete information) and it must be declare as bonus but as per the theme of problem it must be as follows

$\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

$\vec{s} = x(-\vec{p}+\vec{q}+\vec{r}) + y(\vec{p}-\vec{q}+\vec{r}) + z(-\vec{p}-\vec{q}+\vec{r})$

$\vec{s} = \vec{p}(-x+y-z) + \vec{q}(x-y-z) + \vec{r}(x+y+z)$

$-x + y - z = 4$

$x - y - z = 3$

$x + y + z = 5$

$\Rightarrow x = 4,\ y = \frac{9}{2}, z = -\frac{7}{2}$

$\Rightarrow 2x + y + z = 8 - \frac{7}{2} + \frac{9}{2} = 9$

18. Ans. (A, B, C)

Sol. Given that $|\vec{x}|=|\vec{y}|=|\vec{z}|=\sqrt{2}$

and angle between each pair is $\frac{\pi}{3}$

$\therefore \quad \vec{x}.\vec{y}=\vec{y}.\vec{z}=\vec{z}.\vec{x}=\sqrt{2}.\sqrt{2}.\frac{1}{2}=1$

Now $\vec{a}$ is $\perp$ to $\vec{x}$ & $(\vec{y}\times\vec{z})$

Let $\vec{a}=\lambda(\vec{x}\times(\vec{y}\times\vec{z}))$

$=\lambda((\vec{x}.\vec{z})\vec{y}-(\vec{x}.\vec{y})\vec{z})=\lambda(\vec{y}-\vec{z})$

$\vec{a}.\vec{y}=\lambda(\vec{y}.\vec{y}-\vec{y}.\vec{z})=\lambda(2-1)=\lambda$

$\Rightarrow \vec{a}=(\vec{a}.\vec{y})(\vec{y}-\vec{z})$

Now let $\vec{b}=\mu(\vec{y}\times(\vec{z}\times\vec{x}))=\mu(\vec{z}-\vec{x})$

$\vec{b}.\vec{z}=\mu(2-1)=\mu$

$\Rightarrow \quad \vec{b}=(\vec{b}.\vec{z})(\vec{z}-\vec{x})$

Now $\vec{a}.\vec{b}=(\vec{a}.\vec{y})(\vec{y}-\vec{z}).(\vec{b}.\vec{z})(\vec{z}-\vec{x})$

$=(\vec{a}.\vec{y})(\vec{b}.\vec{z})(\vec{y}.\vec{z}-\vec{y}.\vec{x}-\vec{z}.\vec{z}+\vec{z}.\vec{x})$

$=(\vec{a}.\vec{y})(\vec{b}.\vec{z})(1-1-2+1)$

$=-(\vec{a}.\vec{y})(\vec{b}.\vec{z})$

19. Ans. (4)

Sol. We know $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c}\\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c}\\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c}\end{vmatrix}$

$$=\begin{vmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 1\end{vmatrix}=\frac{5}{4}-\frac{3}{4}=\frac{1}{2}$$

$\therefore \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=\frac{1}{\sqrt{2}} \quad ...(1)$

as given $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$

take dot product with $\vec{a}$

$\Rightarrow \vec{a}.\left(\vec{a}\times\vec{b}\right)+\vec{a}.\left(\vec{b}\times\vec{c}\right)=p\vec{a}^2+q\vec{b}.\vec{a}+r\vec{c}.\vec{a}$

$\Rightarrow 0+\frac{1}{\sqrt{2}}=p+\frac{q}{2}+\frac{r}{2} \quad ...(2)$

Now, take dot product with $\vec{b}$ & $\vec{c}$

$0=\frac{p}{2}+q+\frac{r}{2} \quad ...(3)$

& $\frac{1}{\sqrt{2}}=\frac{p}{2}+\frac{q}{2}+r \quad ...(4)$

equation (2) – equation (4)

$\Rightarrow \frac{p}{2}-\frac{r}{2}=0\Rightarrow p=r\Rightarrow p+q=0$ by equation (3)

$\therefore \frac{p^2+2q^2+r^2}{q^2}=\frac{p^2+2p^2+p^2}{p^2}=4$

20. Ans. (A)

Sol. (P) $y=\cos(3\cos^{-1}x)=(4x^3-3x)$

$\frac{dy}{dx}=12x^2-3,\frac{d^2y}{dx^2}-24x$

then $\frac{1}{y}\left[\left(x^2-1\right)\frac{d^2y}{dx^2}+\frac{xdy}{dx}\right]$

$\frac{1}{4x^3-3x}\left[\left(x^2-1\right).24x+x\left(12x^2-3\right)\right]$

$=9$

(Q) let $\vec{a}_1=\hat{i}$,

then $\vec{a}_2=\cos\frac{2\pi}{n}\hat{i}+\sin\frac{2\pi}{n}\hat{j}$

$\vec{a}_3=\cos\frac{4\pi}{n}\hat{i}+\sin\frac{4\pi}{4}\hat{j}...$

now

$|\vec{a}_1\times\vec{a}_2+\vec{a}_2\times\vec{a}_3+.....+\vec{a}_{n-1}\times\vec{a}_n|$

$=|\vec{a}_1.\vec{a}_2+\vec{a}_2.\vec{a}_3+......+\vec{a}_{n-1}.\vec{a}_n|$

$=\left|(n-1)\sin\frac{2\pi}{n}\hat{k}\right|=\left|(n-1)\cos\frac{2\pi}{n}\right|$

$\Rightarrow \tan\frac{2\pi}{n}=1$

$\Rightarrow$ for minimum n $\frac{2\pi}{n}=\frac{\pi}{4}\Rightarrow n=8$

(R) $\frac{x^2}{6}+\frac{y^2}{3}=1 \Rightarrow \frac{dy}{dx}=-\frac{x}{2y} \Rightarrow \frac{2y}{x}=1$

$\Rightarrow \frac{4y^2}{6}+\frac{y^2}{3}=1 \Rightarrow y=\pm 1 \ \& \ x=\pm 2$

as normal passes through $(-2,-1)$ and $(h,1)$ slope of normal

$=\frac{2}{h+2}=1 \Rightarrow h=0$

OR

if normal passes through $(2,1)$ then

$h=2$

(S) $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

$$\Rightarrow \tan^{-1}\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{2x+1}\cdot\frac{1}{4x+1}}=\tan^{-1}\frac{2}{x^2}$$

$\Rightarrow x=0,-\frac{2}{3},3$

but only +ve integral $x=3$

21. **Ans. (B, D)**

Sol $\ell_1: \vec{r}=(3,-1,4)+(1,2,2)t$

$\ell_2: \vec{r}=(3,3,2)+(2,2,1)s$

vector perpendicular to ℓ_1 and ℓ_2 :

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 2\\ 2 & 2 & 1\end{vmatrix}=-2\hat{i}+3\hat{j}-2\hat{k}$$

$\therefore$ Equation of line ℓ :

$\vec{r}=0+(-2,3,-2)\lambda$

Point of intersection of ℓ_1 and ℓ :

$3+t=-2\lambda$

$-1+2t=3\lambda.$

$4+2t=-2\lambda.$

On solving we get $\lambda=-1, t=-1$

$\therefore$ Point of intersection of ℓ_1 & ℓ : P(2, –3, 2)

A point on ℓ_2 at distance of $\sqrt{17}$ from P :

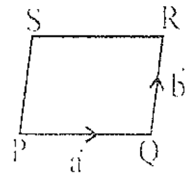
$\Rightarrow \quad (1+2s)^2+(6+2s)^2+s^2=17$

$\Rightarrow \quad s=-\frac{10}{9}; s=-2$

for above s, point will be (B), (D)

22. **Ans. (C)**

Sol.



$\vec{a}+\vec{b}=\overline{PR}$ & $\vec{a}-\vec{b}=\overline{QS}$

$\vec{a}=\frac{\overline{PR}+\overline{QS}}{2}$ & $\vec{b}=\frac{\overline{PR}-\overline{QS}}{2}$

$\vec{a}=2\hat{i}-\hat{j}-3\hat{k}$ & $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$

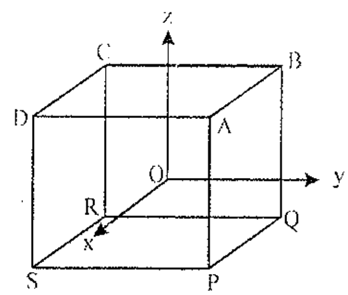
$$\text{Volume}=\begin{vmatrix}2 & -1 & -3\\ 1 & 2 & 1\\ 1 & 2 & 3\end{vmatrix}$$

$2(4)+(3-1)-3(2-2)$

$8+2=10$

23. **Ans. (5)**

Sol



The 8 vectors will represent

$\overline{OA}, \overline{OB}......\overline{OD}, \overline{OP}, ...\overline{OS}$

O is at the centre of cube

ABCDPQRS

any three out of these 8 will be coplanar when two of them are collinear. There are 4 pairs of collinear vectors

$\overline{OA}$ & $\overline{OR}$, $\overline{OB}$ & $\overline{OS}$, $\overline{OC}$ & $\overline{OP}$, $\overline{OD}$ & $\overline{OQ}$

(it will generate $4\times 6=24$

set of coplanar vectors) rest of the combination of 3 vectors will form three edges of a tetrahedron so they will be not coplanar.

So number of non-coplanar vectors

${}^8C_3-4.6=32$

24. Ans. (C)

Sol. (P) Given $[\vec{a}\ \vec{b}\ \vec{c}] = 2$

$[2(\vec{a}\times\vec{b})\ \ 3(\vec{b}\times\vec{c})\ \ (\vec{c}\times\vec{a})]$

$= 6[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]$

$= 6[\vec{a}\ \vec{b}\ \vec{c}]^2 = 24$

(Q) Given $[\vec{a}\ \vec{b}\ \vec{c}] = 5$

$[3(\vec{a}+\vec{b})\ \ (\vec{b}+\vec{c})\ \ 2(\vec{c}+\vec{a})]$

$= 12[\vec{a}\ \vec{b}\ \vec{c}] = 60$

(R) Given $\frac{1}{2}|\vec{a}\times\vec{b}| = 20 \Rightarrow |\vec{a}\times\vec{b}| = 40$

$\left|\frac{1}{2}(2\vec{a}+3\vec{b})\times(\vec{a}-\vec{b})\right|$

$= \frac{1}{2}\left|0+3\vec{b}\times\vec{a}-2\vec{a}\times\vec{b}\right|$

$= \frac{1}{2}|-5\vec{a}\times\vec{b}| = \frac{5}{2}|\vec{a}\times\vec{b}| = \frac{5}{2}.40 = 100$

(S) Given $|\vec{a}\times\vec{b}| = 30$

$|(\vec{a}+\vec{b})\times\vec{a}| = |0+\vec{b}\times\vec{a}| = 30$

25. Ans. (3)

Sol. $\left|\vec{a}-\vec{b}\right|^2 + \left|\vec{b}-\vec{c}\right|^2 + \left|\vec{c}-\vec{a}\right|^2 = 9$

$\Rightarrow \quad 6 - 2\Sigma\vec{a}.\vec{b} = 9$

$\Rightarrow \quad \Sigma\vec{a}.\vec{b} = -\frac{3}{2} \qquad ...(1)$

$\left|\vec{a}+\vec{b}+\vec{c}\right|^2 \geq 0$

$\Sigma\vec{a}^2 + 2\Sigma\vec{a}.\vec{b} \geq 0$

$\Sigma\vec{a}.\vec{b} \geq -\frac{3}{2}$

for equality $|\vec{a}+\vec{b}+\vec{c}| = 0$

$\Rightarrow \vec{a}+\vec{b}+\vec{c} = 0$

$5\vec{b}+5\vec{c} = -5\vec{a}$

$2\vec{a}+5\vec{b}+5\vec{c} = -3\vec{a}$

$\left|2\vec{a}+5\vec{b}+5\vec{c}\right| = 3\left|\vec{a}\right| = 3$

26. Ans. (C)

Sol. $(\vec{a}+\vec{b})\times(2\hat{i}+3\hat{j}+4\hat{k}) = 0$

$\Rightarrow \vec{a}+\vec{b} = \lambda(2\hat{i}+3\hat{j}+4\hat{k})$

$|\vec{a}+\vec{b}| = \sqrt{29} \qquad \Rightarrow |\lambda| = 1$

$\vec{a}+\vec{b} = (2\hat{i}+3\hat{j}+4\hat{k})$

$(\vec{a}+\vec{b}).(-7\hat{i}+2\hat{j}+3\hat{k}) = -14+6+12 = 4$

27. Ans. (C)

Sol. $\vec{v} = x\vec{a}+y\vec{b}$

$= \hat{i}(x+y)+\hat{j}(x-y)+\hat{k}(x+y) \qquad ....(i)$

Given, $\vec{v}.\hat{c} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \frac{x+y-x+y-x-y}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$y - x = 1$

$\Rightarrow \quad x - y = -1 \qquad ....(ii)$

using (ii) in (i) we get

$\vec{v} = (x+y)\hat{i}-\hat{j}+(x+y)\hat{k}$

28. Ans. (A, D)

Sol. $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$

$\vec{b} = \hat{i}+2\hat{j}+\hat{k}$

$\vec{c} = \hat{i}+\hat{j}+\hat{k}$

$\vec{v} = \lambda((\vec{a}\times\vec{b})\times\vec{c}) = \lambda((\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$

$\vec{v} = \lambda[4(\hat{i}+2\hat{j}+\hat{k})-4(\hat{i}+\hat{j}+2\hat{k})]$

$\vec{v} = 4\lambda(\hat{j}-\hat{k})$

29. Ans. (9)

Sol. $\vec{a} = -\hat{i}-\hat{k}$

$\vec{b} = -\hat{i}+\hat{j}$

$\vec{c} = \hat{i}+2\hat{j}+3\hat{k}$

$\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$

Taking cross product by

$(\vec{r}\times\vec{b})\times\vec{a} = (\vec{c}\times\vec{b})\times\vec{a}$

$\Rightarrow (\vec{r}.\vec{a})\vec{b}-(\vec{b}.\vec{a})\vec{r} = (\vec{c}.\vec{a})\vec{b})-(\vec{b}.\vec{a})\vec{c}$

$\Rightarrow 0 - \vec{r} = (-1-3)(-\hat{i}+\hat{j})-(1)(\hat{i}+2\hat{j}+3\hat{k})$

$\vec{r} = -3\hat{i}+6\hat{j}+3\hat{k}$

$\vec{r}.\vec{b} = 3+6 = 9$

30. Ans. (5)

Sol. $(2\vec a + \vec b).[(\vec a \times \vec b) \times \vec a - 2(\vec a \times \vec b) \times \vec b]$

$= (2\vec a + \vec b).[a^2\vec b - (\vec a . \vec b)\vec a - 2\{(\vec a . \vec b)\vec b - b^2\vec a\}]$

$= (2\vec a + \vec b).[a^2\vec b + 2b^2\vec a]$

$= (2\vec a + \vec b).[2\vec a + \vec b]$ as $[a^2 = b^2 = 1]$

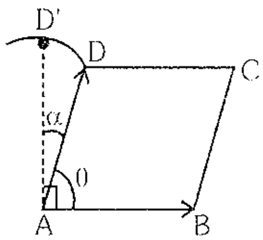
$= 4a^2 + b^2 = 5$

31. Ans. (B)

Sol. Let θ be the angle between $\vec{AB}$ and $\vec{AD}$

$\Rightarrow \theta + \alpha = 90^\circ$

$\Rightarrow \alpha = 90^\circ - \theta$



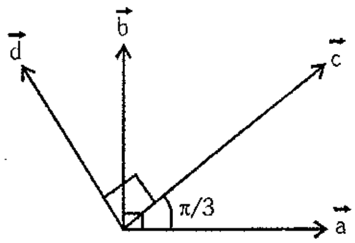
$\Rightarrow \cos\alpha = \sin\theta$ ....(i)

Now, $= \frac{8}{9}$

$\Rightarrow \cos\theta = \frac{\sqrt{17}}{9}$ from (i)

32. Ans. (C)

Sol.



$(\vec a \times \vec b).(\vec c \times \vec d) = 1$ ...(1)

Let $\vec a \wedge \vec b = \alpha$

$\vec c \wedge \vec d = \beta$

angle between plane of $(\vec a, \vec b)$ & $(\vec c, \vec d)$ be θ

equation (1) becomes

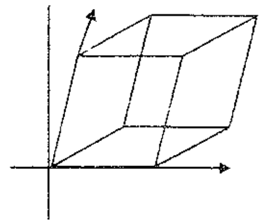
$\sin\alpha . \sin\beta\cos\theta = 1$

$\Rightarrow \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}, \theta = 0$

$\Rightarrow \vec b$ & $\vec d$ are non-parallel.

33. Ans. (A)

Sol.



Let $\hat c = \hat i = x\hat i + y\hat j + z\hat k$

$\vec a = \hat i$ then $\vec b = \frac{1}{2} + \hat i\frac{\sqrt3}{2}$

so $x = \frac{1}{2}$

& $\frac{x}{2} + \frac{y\sqrt3}{2} = \frac{1}{2}$

$= |\vec a(\vec b \times \vec c)|$

$\Rightarrow y\sqrt3 = \frac{1}{2}$ $\therefore$ $y = \frac{1}{2\sqrt3}$

Alternative

volume

$$\sqrt{\begin{vmatrix} \vec a\cdot\vec a & \vec a\cdot\vec b & \vec a\cdot\vec c \\ \vec b\cdot\vec a & \vec b\cdot\vec b & \vec b\cdot\vec c \\ \vec c\cdot\vec a & \vec c\cdot\vec b & \vec c\cdot\vec c \end{vmatrix}} = \sqrt{\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}} = \frac{1}{\sqrt2}$$

also $x^2 + y^2 + z^2 = 1$

$\Rightarrow z^2 = 2/3 \Rightarrow z = \sqrt{2/3}$

$$\text{so volume} = \begin{vmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt3/2 & 0 \\ 1/2 & 1/2\sqrt3 & \sqrt{2/3} \end{vmatrix} = 1/\sqrt2$$

34. Ans. (A)

Sol. $|\vec{OP}| = |\hat a\cos t + \hat b\sin t|$

$= (\cos^2 t + \sin^2 t + 2\sin t\cos t\ \hat a . \hat b)^{1/2}$

$= (1 + \sin 2t\,\hat a.\hat b)^{1/2}$

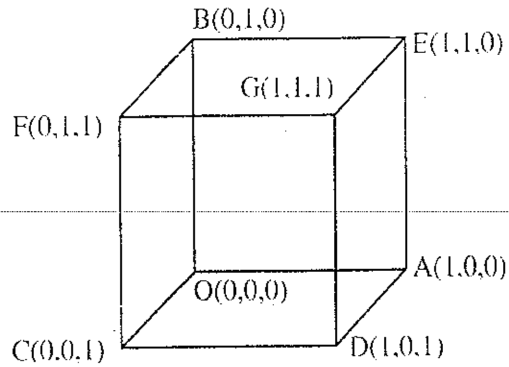
$\therefore$ $|\vec{OP}|_{max} = (1 + \hat a.\hat b)^{1/2}$, when $t = \frac{\pi}{4}$

Now $\hat u = \frac{\hat a + \hat b}{\sqrt2\frac{|\hat a + \hat b|}{\sqrt2}} = \frac{\hat a + \hat b}{|\hat a + \hat b|}$

3D GEOMETRY

1. Ans. (A)

Sol.



DR'S of OG = 1, 1, 1

DR'S of AF = –1, 1, 1

DR'S of CE = 1, 1, –1

DR'S of BD = 1, –1, 1

Equation of OG $\Rightarrow \frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

Equation of AB $\Rightarrow \frac{x-1}{1}=\frac{y}{-1}=\frac{z}{0}$

Normal to both the line's

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & -1 & 0\end{vmatrix}=\hat{i}+\hat{j}-2\hat{k}$$

$\overline{OA}=\hat{i}$

$$\text{S.D.}=\frac{\left|\hat{i}.(\hat{i}+\hat{j}-2\hat{k})\right|}{\left|\hat{i}+\hat{j}-2\hat{k}\right|}=\frac{1}{\sqrt{6}}$$

2. **Ans. (B)**

Sol. $L_1:\vec{r}_1=\lambda\left(\hat{i}+\hat{j}+\hat{k}\right)$

$L_2:\vec{r}_2=\hat{j}-\hat{k}+\mu\left(\hat{i}+\hat{k}\right)$

Let system of planes are

$ax+by+cz=0$ ....(1)

$\because$ It contain L_1

$\therefore a+b+c=0$ ....(2)

For largest possible distance between plane (1) and L_2 the line L_2 must be parallel to plane (1)

$\therefore a+c=0$ .... (3)

$\Rightarrow \boxed{b=0}$

$\therefore$ Plane H_0 : $\boxed{x-z=0}$

Now $d(H_0)=\perp$ distance from point (0, 1, –1) on L_2 to plane.

$$\Rightarrow d\left(H_0\right)=\left|\frac{0+1}{\sqrt{2}}\right|=\frac{1}{\sqrt{2}}$$

$\therefore P\to 5$

for 'Q' distance $=\left|\frac{2}{\sqrt{2}}\right|=\sqrt{2}$

$\therefore Q\to 4$

$\therefore$ (0, 0, 0) lies on plane

$\therefore R\to 3$

for 'S' $x=z$; $y=z$; $x=1$

$\therefore$ point of intersection p(1, 1, 1).

$\therefore\ OP=\sqrt{1+1+1}=\sqrt{3}$

$\therefore S\to 2$

$\therefore$ option (B) is correct

3. **Ans. (A, B, D)**

Sol. line of intersection is $\frac{x}{0}=\frac{y-4}{-4}=\frac{z}{5}$

(1) Any skew line with the line of intersection of given planes can be edge of tetrahedron.

(2) any intersecting line with line of intersection of given planes must lie either in plane P_1 or P_2 can be edge of tetrahedron.

4. **Ans. (A, B, C)**

Sol. $\vec{r}=\hat{k}+t\left(-\hat{i}+\hat{j}\right)+p\left(-\hat{i}+\hat{k}\right)$

$\vec{n}=\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow x+y+z=1$

$Q(10,15,20)$ and $S(\alpha,\beta,\gamma)$

$$\frac{\alpha-10}{1}=\frac{\beta-15}{1}=\frac{\gamma-20}{1}=-2\left(\frac{10+15+20-1}{1+1+1}\right)$$

$$=-\frac{88}{3}$$

$$\Rightarrow\left(\alpha,\beta,\gamma\right)\equiv\left(-\frac{58}{3},-\frac{43}{3},-\frac{28}{3}\right)$$

$\Rightarrow$ A, B, C are correct options

5. Ans. (1.00)

6. Ans. (1.50)

Solutions for 5 & 6

Sol. $7x + 8y + 9z - (\gamma - 1) = A(4x + 5y + 6z - \beta) + B(x + 2y + 3z - \alpha)$

$x : 7 = 4A + B$

$y : 8 = 5A + 2B$

$A = 2, B = -1$

const. term : $-(\gamma - 1) = -A\beta - \alpha B$

$\Rightarrow -(\gamma - 1) = 2\beta + \alpha$

$\alpha - 2\beta + \gamma = 1$

$$M = \begin{pmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \Rightarrow |M| = \alpha - 2\beta + \gamma = 1$$

Plane $P : x - 2y + z = 1$

Perpendicular distance

$$= \left|\frac{3}{\sqrt{6}}\right| = P \Rightarrow D = P^2 = \frac{9}{6} = 1.5$$

7. **Ans. (A, B)**

Sol. Point of intersection of L_1 & L_2 is (1, 0, 1)

Line L passes through (1, 0, 1)

$$\frac{1-\alpha}{\ell} = -\frac{1}{m} = \frac{1-\gamma}{-2} \quad ...(1)$$

acute angle bisector of L_1 & L_2

$$\vec{r} = \hat{i} + \hat{k} + \lambda\left(\frac{\hat{i} - \hat{j} + 3\hat{k} - 3\hat{i} - \hat{j} + \hat{k}}{\sqrt{11}}\right)$$

$$\vec{r} = \hat{i} + \hat{k} + t\left(\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$\Rightarrow \frac{\ell}{1} = \frac{m}{1} = \frac{-2}{-2} \quad \Rightarrow \quad \ell = m = 1$$

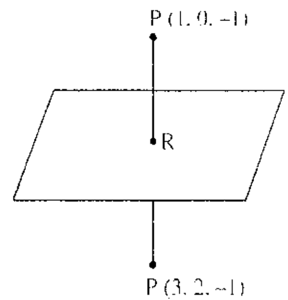
From (1)

$$\frac{1-\alpha}{1} = -1 \quad \Rightarrow \quad \alpha = 2$$

$$\& \ \frac{1-\gamma}{-2} = -1 \quad \Rightarrow \quad \gamma = -1$$

8. Ans. (A, B, C)

Sol.



R is mid point of PQ

$\therefore$ R(2,1,–1) and it lies on plane

equation of plane is $\alpha a + \beta y + \gamma z = \delta$

$\therefore 2\alpha + \beta - \gamma = \delta \quad ...(1)$

Normal vector to plane is

$\vec{n} = 2i + 2j$

$\therefore \frac{\alpha}{2} = \frac{\beta}{2} = \frac{\gamma}{0} = k$

$\therefore \alpha = 2k, \beta = 2k, \gamma = 0 \quad ...(2)$

and $\alpha + \gamma = 1$ (given) $\quad ...(3)$

from (2) and (3)

$\therefore \alpha = 1, \beta = 1, \gamma = 0$

and from (1)

$2(1) + 1 - 0 = \delta$

$\delta = 3$

Now :

$\alpha + \beta = 2$

$\delta - \gamma = 3$

$\delta + \beta = 4$

so, A,B,C are correct.

9. **Ans. (A, B, D)**

Sol. Points on L_1 and L_2 are respectively

$A(1 - \lambda, 2\lambda, 2\lambda)$ and $B(2\mu, -\mu, 2\mu)$

So, $\overline{AB} = (2\mu + \lambda - 1)\hat{i} + (-\mu - 2\lambda)\hat{j} + (2\mu - 2\lambda)\hat{k}$

and vector along their shortest distance

$= 2\hat{i} + 2\hat{j} - \hat{k}$.

Hence, $\frac{2\mu + \lambda - 1}{2} = \frac{-\mu - 2\lambda}{2} = \frac{2\mu - 2\lambda}{-1}$

$\Rightarrow \quad \lambda = \frac{1}{9} \ \& \ \mu = \frac{2}{9}$

Hence, $A \equiv \left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right)$ and $B \equiv \left(\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right)$

$\Rightarrow$ Mid point of AB $\equiv \left(\frac{2}{3}, 0, \frac{1}{3}\right)$

10. **Ans. (0.75)**

Sol. $A(1,0,0)$, $B\left(\frac{1}{2},\frac{1}{2},0\right)$ & $C\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Hence.

$$\overline{AB}=-\frac{1}{2}\hat{i}+\frac{1}{2}\hat{j} \ \& \ \overline{AC}=-\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{3}\hat{k}$$

$$\text{So, } \Delta=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|=\frac{1}{2}\sqrt{\frac{1}{2}\times\frac{2}{3}-\frac{1}{4}}$$

$$=\frac{1}{2\times2\sqrt{3}}$$

$$\Rightarrow \quad (6\Delta)^2=\frac{3}{4}=0.75$$

11. **Ans. (C, D)**

Sol. D.C. of line of intersection (a, b, c)

$$\Rightarrow \quad 2a+b-c=0$$

$$a+2b+c=0$$

$$\frac{a}{1+2}=\frac{b}{-1-2}=\frac{c}{4-1}$$

$\therefore$ D.C. is $(1,-1,1)$

(B) $\frac{3x-4}{9}=\frac{1-3y}{9}=\frac{z}{3}$

$$\Rightarrow \quad \frac{x-4/3}{3}=\frac{y-1/3}{-3}=\frac{z}{3}$$

$\Rightarrow$ lines are parallel.

(C) Acute angle between P_1 and P_2

$$=\cos^{-1}\left(\frac{2\times1+1\times2-1\times1}{\sqrt{6}\sqrt{6}}\right)$$

$$=\cos^{-1}\left(\frac{3}{6}\right)=\cos^{-1}\left(\frac{1}{2}\right)=60^\circ$$

(D) Plane is given by

$$(x-4)-(y-2)+(z+2)=0$$

$$\Rightarrow \quad x-y+z=0$$

Distance of (2, 1, 1) from plane

$$=\frac{2-1+1}{\sqrt{3}}=\frac{2}{\sqrt{3}}$$

12. **Ans. (8)**

Sol. Let $P(\alpha,\beta,\gamma)$

$Q(0,0,\gamma)$ &

$R(\alpha,\beta,-\gamma)$

Now, $\overline{PQ}\parallel \hat{i}+\hat{j} \Rightarrow \left(\alpha\hat{i}+\beta\hat{j}\right)\parallel\left(\hat{i}+\hat{j}\right)$

$\Rightarrow \alpha=\beta$

Also, mid point of PQ lies on the plane

$$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}=3 \Rightarrow \alpha+\beta=6\Rightarrow\alpha=3$$

Now, distance of point P from X-axis is

$$\sqrt{\beta^2+\gamma^2}=5$$

$$\Rightarrow \beta^2+\gamma^2=25 \ \Rightarrow \ \gamma^2=16$$

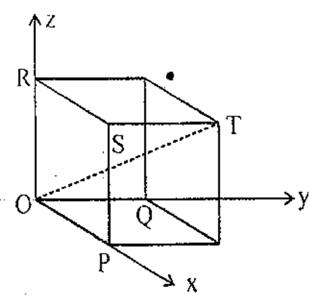
as $\beta=\alpha=3$

as $\gamma=4$

Hence, $PR=2\gamma=8$

13. **Ans. (0.5)**

Sol.



$$\vec{p}=\overline{SP}=\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{2}\left(\hat{i}-\hat{j}-\hat{k}\right)$$

$$\vec{q}=\overline{SQ}=\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{2}\left(-\hat{i}+\hat{j}-\hat{k}\right)$$

$$\vec{r}=\overline{SR}=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\left(-\hat{i}-\hat{j}+\hat{k}\right)$$

$$\vec{t}=\overline{ST}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$\left|(\vec{p}\times\vec{q})\times\left(\vec{r}\times\vec{t}\right)\right|=\frac{1}{4}\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&-1\\-1&1&-1\end{vmatrix}\times\frac{1}{4}\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-1&-1&1\\1&1&1\end{vmatrix}$$

$$=\frac{1}{16}\left|\left(2\hat{i}+2\hat{j}\right)\times\left(-2\hat{i}+2\hat{j}\right)\right|=\left|\frac{\hat{k}}{2}\right|=\frac{1}{2}$$

14. **Ans. (A)**

Sol. The normal vector of required plane is parallel to vector

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = -14\hat{i} - 2\hat{j} - 15\hat{k}$$

$\therefore$ The equation of required plane passing through (1, 1, 1) will be

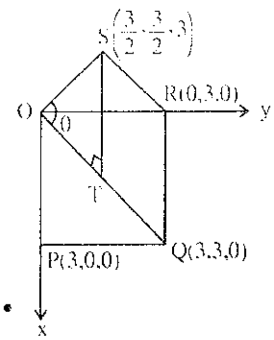
$-14(x-1) - 2(y-1) - 15(z-1) = 0$

$\Rightarrow \boxed{14x + 2y + 15z = 31}$

$\therefore$ Option (A) is correct

15. **Ans. (B, C, D)**

Sol.



Given OP = OR = 3 and OPQR is a square

$\Rightarrow \quad OQ = 3\sqrt{2} \Rightarrow OT = \dfrac{3}{\sqrt{2}}$ and $ST = 3$

using ΔSOT, $\tan\theta = \dfrac{ST}{OT} = \sqrt{2}$

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$

clearly, equation of plane containing triangle OQS is $Y - X = 0$

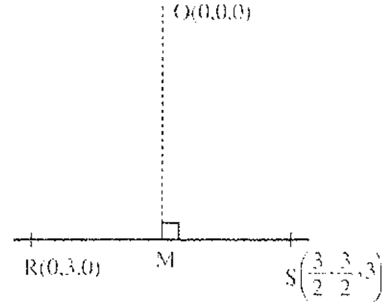
Also, length of perpendicular from P to the plane containing the triangle OQS is $PT = \dfrac{3}{\sqrt{2}}$

Also equation of RS is

$$\bar{r} = 3\hat{j} + t\left(\frac{3}{2}\hat{i} - \frac{3}{2}\hat{j} + 3\hat{k}\right)$$

$$= \left(\frac{3t}{2}, 3 - \frac{3t}{2}, 3t\right)$$

Let co-ordinates of $M = \left(\dfrac{3t}{2}, 3 - \dfrac{3t}{2}, 3t\right)$



$\because \quad \overline{OM}.\overline{RS} = 0$

$\Rightarrow \quad \dfrac{9}{4}t - \dfrac{3}{2}\left(3 - \dfrac{3t}{2}\right) + 9t = 0$

$\Rightarrow \quad \dfrac{9t}{2} + 9t = \dfrac{9}{2} \quad \Rightarrow \quad t = \dfrac{1}{3}$

$\therefore \quad M = \left(\dfrac{1}{2}, \dfrac{5}{2}, 1\right)$

$\Rightarrow \quad OM = \sqrt{\dfrac{1}{4} + \dfrac{25}{4} + 1} = \sqrt{\dfrac{30}{4}} = \sqrt{\dfrac{15}{2}}$

16. **Ans. (C)**

Sol. Line AP : $\dfrac{x-3}{1} = \dfrac{y-1}{-1} = \dfrac{z-7}{1} = \lambda$

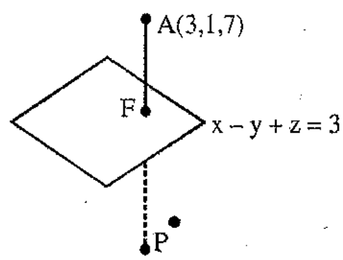
$\Rightarrow F(3 + \lambda, 1 - \lambda, \lambda + 7)$ lies in the plane

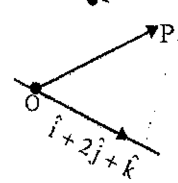
$\therefore 3 + \lambda - (1 - \lambda) + \lambda + 7 = 3$

$3\lambda = -6 \Rightarrow \lambda = -2$

$\Rightarrow F(1,3,5)$

$\Rightarrow P(-1,5,3)$





so required plane is $\begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 2 & 1 \\ -1 & 5 & 3 \end{vmatrix} = 0$

$\therefore x - 4y + 7z = 0$

17. Ans. (B, D)

Sol. Let $P_3 : (x + z - 1) + \lambda y = 0$

$x + \lambda y + z - 1 = 0$ ...(i)

distance of (0,1,0) from P_3 is 1

$\Rightarrow \dfrac{|\lambda - 1|}{\sqrt{2+\lambda^2}} = 1$

$\Rightarrow (\lambda - 1)^2 = 2 + \lambda^2$

$\Rightarrow \lambda = -\dfrac{1}{2}$

$\therefore$ P_3 is $2x - y + 2z - 2 = 0$

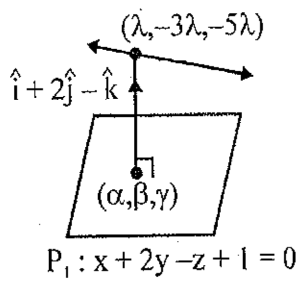
distance from (α,β,γ) is $\left|\dfrac{2\alpha - \beta + 2\gamma - 2}{\sqrt{9}}\right| = 2$

$\therefore 2\alpha - \beta + 2\gamma - 2 = 6$ or $2\alpha - \beta + 2\gamma - 2 = -6$

$2\alpha - \beta + 2\gamma - 8 = 0$ or $2\alpha - \beta + 2\gamma + 4 = 0$

18. Ans. (A, B)

Sol. Straight line 'L' is parallel to line of intersection of plane P_1 & plane P_2.



$\therefore$ Equation of line 'L' is

$\dfrac{x}{1} = \dfrac{y}{-3} = \dfrac{z}{-5} = \lambda$

$\dfrac{\alpha - \lambda}{1} = \dfrac{\beta + 3\lambda}{2} = \dfrac{\gamma + 5\lambda}{-1} = k$

$$\left.\begin{aligned} \alpha &= k + \lambda \\ \beta &= 2k - 3\lambda \\ \gamma &= -k - 5\lambda \end{aligned}\right\} \text{...(1)}$$

satisfying in plane P_1

$k + \lambda + 4k - 6\lambda + k + 5\lambda + 1 = 0$

$6k = -1$

putting in (1) required locus is

$x = -\dfrac{1}{6} + \lambda$

$y = -\dfrac{1}{3} - 3\lambda$

$z = \dfrac{1}{6} - 5\lambda$

Now check the options.

19. Ans. (C)

Sol. Line L_1 given by $y = x$; $z = 1$ can be expressed as

L_1: $\dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z-1}{0}$

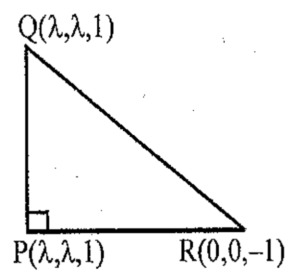
Similarly $L_2(y = -x;\ z = -1)$ can be expressed as

$L_2 : \dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z+1}{0}$

Let any point Q $(\alpha,\alpha,1)$ on L_1 and $R(\beta,-\beta,-1)$ on L_2

Given that PQ is perpendicular to L_1

$\Rightarrow (\lambda - \alpha).1 + (\lambda - \alpha).1 + (\lambda - 1).0 = 0 \Rightarrow \lambda = \alpha$



$\therefore Q(\lambda, \lambda, 1)$

Similarly PR is perpendicular to L_2

$(\lambda - \beta).1 + (\lambda + \beta)(-1) + (\lambda + 1).0 = 0 \Rightarrow \beta = 0$

$\therefore R(0,0,-1)$

Now as given

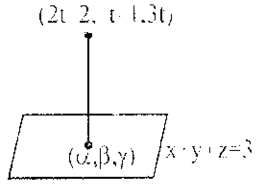
$\Rightarrow \overline{PR}.\overline{PQ} = 0$

$0.\lambda + 0.\lambda + (\lambda - 1)(\lambda + 1) = 0$

$\lambda \neq 1$ as P & Q are different points $\Rightarrow \lambda = -1$

20. Ans. (D)

Sol.



$$\frac{\alpha-2t+2}{1}=\frac{\beta+t+1}{1}=\frac{\gamma-3t}{1}=k$$

$\alpha = k + 2t - 2$

$\beta = k - t - 1$

$\gamma = k + 3t$

$\alpha+\beta+\gamma=3$

$$k=\frac{6-4t}{3}$$

$$\alpha=\frac{6-4t}{3}+2t-2=\frac{2t}{3}$$

$$\beta=\frac{6-4t}{3}-t-1=\frac{3-7t}{3}$$

$$\gamma=\frac{6-4t}{3}+3t=\frac{5t+6}{3}$$

$$\Rightarrow \frac{3\alpha}{2}=\frac{3\beta-3}{-7}=\frac{3\gamma-6}{5}$$

$$\Rightarrow \frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$$

21. **Ans. (A, D)**

Sol. $L_1:\frac{x-5}{0}=\frac{y}{3-\alpha}=\frac{z}{-2}$

$L_2:\frac{x-\alpha}{0}=\frac{y}{-1}=\frac{z}{2-\alpha}$

for lines to be coplanar

$$\begin{vmatrix}5-\alpha & 0 & 0\\ 0 & 3-\alpha & -2\\ 0 & -1 & 2-\alpha\end{vmatrix}=0$$

$\Rightarrow\quad (5-\alpha)\,((3-\alpha)\,(2-\alpha)-2)=0$

$\Rightarrow\quad (5-\alpha)(\alpha^2-5\alpha+4)=0$

$\Rightarrow\quad \alpha = 1, 4, 5$

22. Ans. (A)

Sol. For point of intersection of L_1 and L_2

$$\begin{cases}2\lambda+1=\mu+4\\ -\lambda=\mu-3\\ \lambda-3=2\mu-3\end{cases}\quad\Rightarrow\quad \mu=1$$

$\Rightarrow$ point of intersection is $(5, -2, -1)$

Now, vector normal to the plane is

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & 1 & 2\\ 3 & 5 & -6\end{vmatrix}$$

$= -16(\hat{i}-3\hat{j}-2\hat{k})$

Let equation of required plane be

$x - 3y - 2z = \alpha$

$\because$ it passes through $(5, -2, -1)$

$\therefore$ $\alpha = 13$ $\Rightarrow$ equation of plane is

$x - 3y - 2z = 13$

23. Ans. (A)

Sol. Line QR :

$$\frac{x-2}{1}=\frac{y-3}{4}=\frac{z-5}{1}=\lambda$$

Any point on line QR :

$(\lambda+2, 4\lambda+3, \lambda+5)$

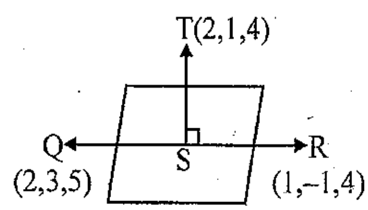
$\therefore$ Point of intersection with plane :

$5\lambda + 10 - 16\lambda - 12 - \lambda - 5 = 1$

$$\Rightarrow \lambda=-\frac{2}{3}$$

$$\therefore\ P\left(\frac{4}{3},\frac{1}{3},\frac{13}{3}\right)$$

Also



$\because TQ = TR = \sqrt{5}$

$\Rightarrow$ S is the mid-point of QR

$$\Rightarrow S\left(\frac{3}{2},1,\frac{9}{2}\right)$$

$$\Rightarrow PS=\frac{1}{\sqrt{2}}\text{ units}$$

24. Ans. (A)

Sol. Let required plane be

$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$\because$ plane is at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.

$$\Rightarrow \left|\frac{(3+2-3-2)+\lambda(3-1-1-3)}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \lambda^2 = \frac{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}{3}$$

$$\Rightarrow 3\lambda^2 = 3\lambda^2 + 2\lambda - 4\lambda + 6\lambda + 14$$

$$\Rightarrow \lambda = -\frac{7}{2}$$

$\therefore$ required plane is

$$(x + 2y + 3z - 2) + \left(-\frac{7}{2}\right)(x-y+z-3) = 0$$

$$\Rightarrow 5x - 11y + z = 17$$

25. Ans. (B, C)

Sol. $(1, -1, 0); (-1, -1, 0)$

For coplanarity of lines

$$\begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0 \Rightarrow 2(k^2 - 4) = 0$$

$\Rightarrow \quad k = \pm 2$

for $k = 2$

Normal vector $\vec{n} = \hat{j} - \hat{k}$

$\therefore$ Required plane : $y - z = \lambda$

$\because$ Passes through $(1, -1, 0) \Rightarrow \lambda = -1$

$\therefore$ $y - z = -1$

for $k = -2$

$\vec{n} = \hat{j} + \hat{k}$

$\therefore$ Required plane : $y + z = \lambda$

$\because$ Passes through $(1, -1, 0) \Rightarrow \lambda = -1$

$\therefore$ $y + z = -1$

26. Ans. (C)

Sol. Normal vector to the plane containing the

Lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

$$\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 8\hat{i} - \hat{j} - 10\hat{k}$$

Let direction ratios of required plane be a. b. c.

Now $8a - b - 10c = 0$

and $2a + 3b + 4c = 0$

($\because$ plane contains the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$)

$$\Rightarrow \frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$$

$\Rightarrow$ equation of plane is $x - 2y + z = d$

$\because$ plane contains the line, which passes through origin, hence origin lies on a plane.

$\Rightarrow$ equation of required plane is $x - 2y + z = 0$.

27. Ans. (6)

Sol. Plane containing the line

Direction ratio's of normal to the plane :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$$

Hence equation of plane

$1(x - 1) - 2(y - 2) + 1(z - 3) = 0$

i.e. $x - 2y + z = 0$

As given plane must be parallel $\Rightarrow A = 1$

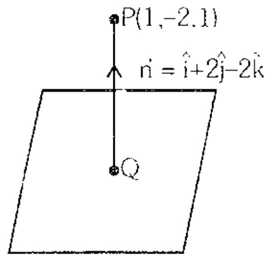
& distance between the planes

$$\left|\frac{d-0}{\sqrt{1^2+2^2+1^2}}\right| = \sqrt{6}$$

$|d| = 6$

28. **Ans. (A)**

Sol.



$\because \left|\dfrac{1-4-2-\alpha}{3}\right| = 5$

$\Rightarrow \alpha = 10 \because \alpha > 0$

Now, let Q (α, β, γ) be the foot of perpendicular from P to the plane $x + 2y - 2z = 10$

Equation of line PQ is

$\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-1}{-2} = r$ (Let)

$\Rightarrow \alpha = r + 1, \beta = 2r - 2$ and $\gamma = -2r + 1$

$\because$ Q lies in the plane

$\therefore (r + 1) + 2(2r - 2) - 2(-2r+1) = 10$

$\Rightarrow r = \dfrac{5}{3}$

foot of the perpendicular is $\left(\dfrac{8}{3}, \dfrac{4}{3}, -\dfrac{7}{3}\right)$

29. (A) $\to$ (t), (B) $\to$ (p,r), (C) $\to$ (q,s), (D) $\to$ (r)

Sol. (A) $P(\lambda+2, -2\lambda+1, \lambda-1)$

$Q\left(2k+\dfrac{8}{3}, -k-3, k+1\right)$

$3\lambda+6 = a(6k+8)$ ....(i)

$-2\lambda + 1 = a(-k - 3)$ ....(ii)

$2\lambda - 2 = 2a(k + 1)$ ....(iii)

(ii) + (iii) $\Rightarrow -1 = ak - a$

$k = \dfrac{a-1}{a}$ ....(iv)

Put the value of k in equation (iii)

$\Rightarrow \lambda = 2a$ ....(v)

Put the values of l & k in equation (i)

$6a+6 = a\left(\dfrac{6a-6}{a}+8\right)$

$\Rightarrow 6 = 6a - 6 + 8a$

$\Rightarrow a = \dfrac{3}{2}$

Put the value of a in equation (iv) & (v)

$k = \dfrac{\frac{3}{2}-1}{\frac{3}{2}} = \dfrac{1}{3}$ & $l = 3$

$P(5, -5, 2)$ & $Q\left(\dfrac{10}{3}, \dfrac{-10}{3}, \dfrac{4}{3}\right)$

$d = \sqrt{\left(5-\dfrac{10}{3}\right)^2 + \left(5-\dfrac{10}{3}\right)^2 + \left(\dfrac{2}{3}\right)^2}$

$= \sqrt{\dfrac{25}{9}+\dfrac{25}{9}+\dfrac{4}{9}}$

$\Rightarrow d = \sqrt{6} \Rightarrow d^2 = 6$

(B) $\tan^{-1}(x + 3) - \tan^{-1}(x - 3) = \tan^{-1}\left(\dfrac{3}{4}\right)$

$\tan^{-1}\left(\dfrac{(x+3)-(x-3)}{1+(x^2-9)}\right) = \tan^{-1}\left(\dfrac{3}{4}\right)$

$\Rightarrow 1 + x^2 - 9 = 8 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

(C) $\mu b^2 + 4\,\vec{b}.\vec{c} = 0$

$b^2 - \vec{a}.\vec{c} + \vec{b}.\vec{c} = 0$

$b^2 - \left(\mu\vec{b}+4\vec{c}\right).\vec{c} + \vec{b}.\vec{c}$

$= b^2 + \vec{b}.\vec{c}(1-\mu) - 4c^2 = 0$

$b^2 - \dfrac{\mu}{4}b^2(1-\mu) = 4c^2$

$b^2(4-\mu+\mu^2) = 16c^2$ ....(i)

$4b^2 + 8\vec{b}.\vec{c} + 4c^2 = b^2 + a^2$

$3b^2 - 2\mu b^2 + 4c^2 = \left(\mu\vec{b}+4\vec{c}\right)^2$

$3b^2 - 2\mu b^2 + 4c^2$

$= \mu^2 b^2 + 8\mu\vec{b}.\vec{c} + 16c^2$

$b^2(3-2\mu-\mu^2) = 12c^2 - 2\mu^2 \times b^2$

$b^2(3 - 2\mu + \mu^2) = 12c^2$ ....(ii)

$\dfrac{4-\mu+\mu^2}{3-2\mu+\mu^2} = \dfrac{4}{3}$

$12 - 3\mu + 3\mu^2 = 12 - 8\mu + 4\mu^2$

$\mu^2 - 5\mu = 0$

$\mu = 0,5$

(D) $$I=\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{\sin\frac{9x}{2}}{\sin\frac{x}{2}}dx$$

$$I=\frac{4}{\pi}\int_{0}^{\pi}\frac{\sin\frac{9x}{2}}{\sin\frac{x}{2}}dx \qquad ....(i)$$

$$I=\frac{4}{\pi}\int_{0}^{\pi}\frac{\cos\frac{9x}{2}}{\cos\frac{x}{2}}dx \qquad ....(ii)$$

(i) + (ii)

$$I=\frac{2}{\pi}\int_{0}^{\pi}\frac{\sin 5x}{\sin\frac{x}{2}\cos\frac{x}{2}}dx \quad -\frac{4}{\pi}\int_{0}^{\pi}\frac{\sin 5x}{\sin x}dx$$

$f(x) = f(p - x)$

$$I=\frac{8}{\pi}\int_{0}^{\pi/2}\frac{\sin 5x}{\sin x}dx \qquad ....(i)$$

$$I=\frac{8}{\pi}\int_{0}^{\pi/2}\frac{\cos 5x}{\cos x}dx \qquad ....(ii)$$

(i) + (ii)

$$I=\frac{4}{\pi}\int_{0}^{\pi/2}\frac{\sin 6x}{\sin x\cos x}dx$$

$$I=\frac{8}{\pi}\int_{0}^{\pi/2}\frac{\sin 6x}{\sin 2x}dx$$

$$=\frac{8}{\pi}\int_{0}^{\pi/2}(3-4\sin^2 2x)dx$$

$$=\frac{8}{\pi}\int_{0}^{\pi/2}3-2(1-\cos 2x)dx$$

$$=\frac{8}{\pi}\int_{0}^{\pi/2}(1+2\cos 2x)dx \quad =\frac{8}{\pi}\times\frac{\pi}{2}=4$$

30. **Ans. (A)**

Sol. Let Q be $(1-3\mu, \mu-1, 5\mu+2)$

$\Rightarrow \overline{PQ} = (-3\mu-2)\hat{i} + (\mu-3)\hat{j} + (5\mu-4)\hat{k}$

$\Rightarrow \overline{PQ}\,.\,\hat{n} = 0$ (where $\hat{n}$ is $\perp^{er}$ to plane)

$\Rightarrow (-3\mu-2)\,1 + (\mu-3)\,.\,(-4) + (5\mu-4)\,3 = 0$

$\Rightarrow \mu = \frac{1}{4}.$

31. **Ans. (C)**

Sol. Let DC's be $(\cos\alpha, \cos\alpha, \cos\alpha)$

$3\cos^2\alpha = 1$

$\cos\alpha = \frac{1}{\sqrt{3}}$

Line PQ is $\frac{x-2}{1/\sqrt{3}} = \frac{y+1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}} = \lambda$

$Q\left(\frac{\lambda}{\sqrt{3}}+2, \frac{\lambda}{\sqrt{3}}-1, \frac{\lambda}{\sqrt{3}}+2\right)$

Putting in plane

$\frac{2\lambda}{\sqrt{3}} + 4 + \frac{\lambda}{\sqrt{3}} - 1 + \frac{\lambda}{\sqrt{3}} + 2 = 9$

$\frac{4\lambda}{\sqrt{3}} = 4$

$\lambda = \sqrt{3}$

$Q = (3, 0, 3)$

$(PQ)^2 = 1+1+1$

$PQ = \sqrt{3}$

32. **Ans. (D)**

Sol. d.r.s of $L_1 = 0, -4, -4$

$L_2 = 0, -2, -2$

$L_3 = 0, 2, 2$

So 3 lines are parallel

Hence S(I) is false

Now $\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{vmatrix} = 0$

so there will be no solution.

Hence S(II) is true.

33. **Ans. (B)**

Sol. $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$

$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

a vector perpendicular to L_1 & L_2 will be

$= \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -i - 7j + 5k$

Hence unit vector $= \frac{-i-7j+5k}{5\sqrt{3}}$

34. **Ans. (D)**

Sol. Shortest distance

$= (3\hat{i} - 4\hat{k}) \cdot \frac{(-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}} = \frac{17}{5\sqrt{3}}$

35. **Ans. (C)**

Sol. Eq. of plane $-(x + 1) - 7(y + 2) + 5(z + 1) = 0$

$x + 7y - 5z + 10 = 0$

distance from $(1, 1, 1) = \frac{1 + 7 - 5 + 10}{5\sqrt{3}} = \frac{13}{5\sqrt{3}}$

COMPLEX NUMBER

1. **Ans. (281)**

Sol. $A = \frac{1967 + 1686 i \sin\theta}{7 - 3i\cos\theta}$

$= \frac{281(7 + 6i\sin\theta)}{7 - 3i\cos\theta} \times \frac{7 + 3i\cos\theta}{7 + 3i\cos\theta}$

$= \frac{281(49 - 18\sin\theta\cos\theta + i(21\cos\theta + 42\sin\theta))}{49 + 9\cos^2\theta}$

for positive integer

$\text{Im}(A) = 0$

$21\cos\theta + 42\sin\theta = 0$

$\tan\theta = \frac{-1}{2}$; $\sin 2\theta = \frac{-4}{5}$, $\cos^2\theta = \frac{4}{5}$

$\text{Re}(A) = \frac{281(49 - 9\sin 2\theta)}{49 + 9\cos^2\theta}$

$= \frac{281\left(49 - 9 \times \frac{-4}{5}\right)}{49 + 9 \times \frac{4}{5}} = 281$ (+ve integer)

2. **Ans. (B)**

Sol. $\because |z|^3 + 2z^2 + 4\bar{z} - 8 = 0$ .... (1)

Take conjugate both sides

$\Rightarrow |z|^3 + 2\bar{z}^2 + 4z - 8 = 0$ .... (2)

By (1) – (2)

$\Rightarrow 2(z^2 - \bar{z}^2) + 4(\bar{z} - z) = 0$

$\Rightarrow \boxed{z + \bar{z} = 2}$ .... (3)

$\Rightarrow |z + \bar{z}| = 2$ .... (4)

Let $z = x + iy$

$\therefore \boxed{x = 1}$ $\therefore z = 1 + iy$

Put in (1)

$\Rightarrow (1 + y^2)^{3/2} + 2(1 - y^2 + 2iy) + 4(1 - iy) - 8 = 0$

$\Rightarrow (1 + y^2)^{3/2} = 2(1 + y^2)$

$\Rightarrow \sqrt{1 + y^2} = 2 = |z|$

Also $\boxed{y = \pm\sqrt{3}}$

$\therefore z = 1 \pm i\sqrt{3}$

$\Rightarrow z - \bar{z} = \pm 2i\sqrt{3}$

$\Rightarrow |z - \bar{z}| = 2\sqrt{3}$

$\Rightarrow |z - \bar{z}|^2 = 12$

Now $z + 1 = 2 + i\sqrt{3}$

$|z + 1|^2 = 4 + 3 = 7$

$\therefore P \to 2 ; Q \to 1 ; R \to 3 ; S \to 5$

$\therefore$ Option (B) is correct.

3. **Ans. (512)**

Sol. $z^8 - 2^8 = (z - 2)(z - \alpha)(z - \alpha^2)\ldots(z - \alpha^7)$

Put $z = 2e^{i\theta}$

$2^8(e^{i8\theta} - 1) = (2e^{i\theta} - 2)(2e^{i\theta} - \alpha)\ldots\ldots(2e^{i\theta} - \alpha^7)$

Take mod

$2^8|e^{i8\theta} - 1| = PA_1\, PA_2 \ldots PA_8$

$2^8|2\sin 4\theta| = PA_1\, PA_2 \ldots PA_8$

$(PA_1 \cdot PA_2 \ldots PA_8)_{max} = 512$

4. **Ans. (0.50)**

Sol. Given that

$z \neq \bar{z}$

Let $\alpha = \frac{2 + 3z + 4z^2}{2 - 3z + 4z^2} = \frac{(2 - 3z + 4z^2) + 6z}{2 - 3z + 4z^2}$

$\therefore \alpha = 1 + \frac{6z}{2 - 3z + 4z^2}$

If α is a real number, then

$\alpha = \bar{\alpha}$

$\Rightarrow \frac{z}{2 - 3z + 4z^2} = \frac{\bar{z}}{2 - 3\bar{z} + 4\bar{z}^2}$

$\therefore 2(z - \bar{z}) = 4z\bar{z}(z - \bar{z})$

$\Rightarrow (z - \bar{z})(2 - 4z\bar{z}) = 0$

As $z \neq \bar{z}$ (Given)

$\Rightarrow z\bar{z} = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow |z|^2 = 0.50$

5. Ans. (4.00)

Sol. Given .

$$\overline{z}-z^2=i\left(\overline{z}+z^2\right)$$

$$\Rightarrow (1-i)\overline{z}=(1+i)z^2$$

$$\Rightarrow \frac{(1-i)}{(1+i)}\overline{z}=z^2$$

$$\Rightarrow \left(-\frac{2i}{2}\right)\overline{z}=z^2$$

$$\therefore z^2=-i\overline{z}$$

Let $z = x + iy$,

$\therefore (x^2-y^2)+i(2xy)=-i(x-iy)$

so, $x^2-y^2+y=0$ ...(1)

and $(2y+1)x=0$ ...(2)

$$\Rightarrow x=0 \text{ or } y=-\frac{1}{2}$$

Case I : When $x = 0$

$\therefore (1) \Rightarrow y(1-y)=0 \Rightarrow y=0,1$

$\therefore (0,0), (0,1)$

Case II : When $y=-\frac{1}{2}$

$$\therefore \quad (1) \quad \Rightarrow \quad x^2-\frac{1}{4}-\frac{1}{2}=0 \Rightarrow x^2=\frac{3}{4}$$

$$\Rightarrow x=\pm\frac{\sqrt{3}}{2}$$

$$\therefore \left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right),\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

$\Rightarrow$ Number of distinct 'z' is equal to 4.

6. Ans. (A)

Sol. Let $(\overline{z})^2+\frac{1}{z^2}=m+in$, $m,n\in\mathbb{Z}$

$$(\overline{z})^2+\frac{\overline{z}^2}{|z|^4}=m+in$$

$$\Rightarrow (x^2-y^2)\left(1+\frac{1}{|z|^4}\right)=m \quad ...(i)$$

$$\& \ -2xy\left(1+\frac{1}{|z|^4}\right)=n \quad ...(2)$$

Equation $(1)^2+(2)^2$

$$\left(1+\frac{1}{|z|^4}\right)^2\left[\left(x^2+y^2\right)^2\right]=m^2+n^2$$

$$\left(1+\frac{1}{|z|^4}\right)^2(|z|)^4=m^2+n^2$$

$$\Rightarrow |z|^4+\frac{1}{|z|^4}+2=m^2+n^2$$

Now for option (A)

$$|z|^4=\frac{43+3\sqrt{205}}{2}$$

$\Rightarrow m^2+n^2=45$

$\Rightarrow m=\pm6,\ n=\pm3$

Option (B)

$$|z|^4+\frac{1}{|z|^4}+2=\frac{7+\sqrt{33}}{4}+\frac{7-\sqrt{33}}{4}+2$$

$$=\frac{7}{2}+2=\frac{11}{2}$$

Option (C)

$$|z|^4+\frac{1}{|z|^4}+2=\frac{9+\sqrt{65}}{4}+\frac{9-\sqrt{65}}{4}+2$$

$$=\frac{9}{2}+2=\frac{13}{2}$$

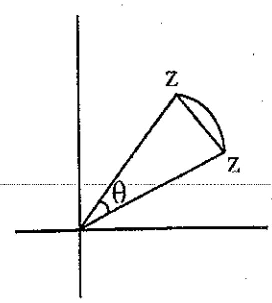
Option (D)

$$|z|^4+\frac{1}{|z|^4}+2=\frac{7+\sqrt{13}}{6}+\frac{7-\sqrt{13}}{6}+2$$

$$=\frac{7}{3}+2=\frac{13}{2}$$

7. Ans. (C)

Sol.



$|z_1|=|z_2|=\ldots|z_{10}|=1$

$$\text{angle}=\frac{\text{arc}}{\text{rad}}$$

$\theta_2 = \text{arc}(z_1z_2) > |z_2 - z_1|$

$P : |z_2 - z_1| + \ldots + |z_1 - z_{10}| \le \theta_1 + \theta_2 + \ldots + \theta_{10}$

$\Rightarrow |z_2 - z_1| + \ldots + |z_1 - z_{10}| \le 2\pi$ P is true

$z_1^2 = e^{i2\theta_1}$, $z_k^2 = z_{k-1}^2 . e^{i2\theta_k}$

Let $2\theta_k = \alpha k$

$z_1^2 = e^{i\alpha_1}$, $z_k^2 = z_{k-1}^2 . e^{i\alpha_k}$

$\alpha_1 + \alpha_2 + \ldots + \alpha_k = 4\pi$

one similar sense

$|z_1^2 - z_2^2| + \ldots |z_1^2 - z_{10}^2| \le 4\pi$

Q is also true

8. Ans. (B, D)

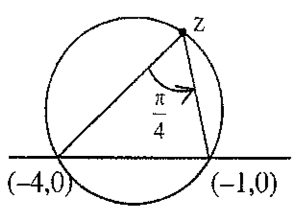
Sol. $\arg\left(\dfrac{z+\alpha}{z+\beta}\right) = \dfrac{\pi}{4}$ implies z is

on arc and $(-\alpha, 0)$ & $(-\beta, 0)$ subtend $\dfrac{\pi}{4}$ on z.

And z lies on $x^2 + y^2 + 5x - 3y + 4 = 0$

So put $y = 0$;

$x^2 + 5x + 4 = 0 \Rightarrow x = -1$; $x = -4$



Now, $\arg\left(\dfrac{z+\alpha}{z+\beta}\right) = \dfrac{\pi}{4}$

$\Rightarrow z + \alpha = (z + \beta) . r. e^{i\frac{\pi}{4}}$

So, $z + \beta = z + 4 \Rightarrow \beta = 4$ & $z + \alpha = z + 1$

$\Rightarrow \alpha = 1$

9. Ans. (B, C)

Sol. $|z^2 + z + 1| = 1$

$$\Rightarrow \left|\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}\right| = 1$$

$$\Rightarrow \left|\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}\right| \le \left|z + \frac{1}{2}\right|^2 + \frac{3}{4}$$

$$\Rightarrow 1 \le \left|z + \frac{1}{2}\right|^2 + \frac{3}{4} \quad \Rightarrow \left|\left(z + \frac{1}{2}\right)\right|^2 \ge \frac{1}{4}$$

$$\Rightarrow \left|z + \frac{1}{2}\right| \ge \frac{1}{2}$$

also $|(z^2 + z) + 1| = 1 \ge ||z^2 + z| - 1|$

$\Rightarrow |z^2 + z| - 1 \le 1$

$\Rightarrow |z^2 + z| \le 2$

$\Rightarrow ||z^2| - |z|| \le |z^2 + z| \le 2$

$\Rightarrow |r^2 - r| \le 2$

$\Rightarrow \quad r = |z| \le 2$; $\forall z \in S$

Also we can always find root of the equation

$z^2 + z + 1 = e^{i\theta}$; $\forall \theta \in R$

Hence set 'S' is infinite

10. Ans. (8)

Sol. Let $z = x + iy$

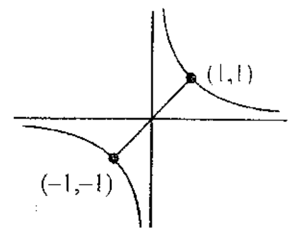
$z^4 - |z|^4 = 4iz^2$

$\Rightarrow \quad z^4 - (z\bar{z})^2 = 4iz^2$

$\Rightarrow \quad z = 0$ or $z^2 - (\bar{z})^2 = 4i$

$\Rightarrow \quad 4ixy = 4i$

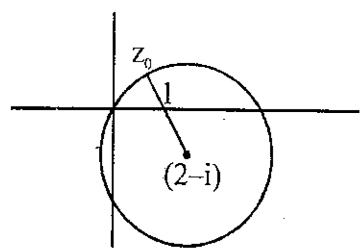
$\Rightarrow \quad xy = 1$



$|z_1 - z_2|^2_{\min} = 8$

11. Ans. (B)

Sol.



$$\arg\left(\frac{4 - (z_0 + \bar{z}_0)}{(z_0 - \bar{z}_0) + 2i}\right)$$

$$= \arg\left(\frac{4 - 2\text{Re}\,z_0}{2i\,\text{Im}\,z_0 + 2i}\right) = \arg\left(\frac{2 - \text{Re}\,z_0}{(1 + \text{Im}\,z_0)i}\right)$$

$$= \arg\left(-\left(\frac{2 - \text{Re}\,z_0}{1 + \text{Im}\,z_0}\right)i\right)$$

$= \arg(-ki)$; $k > 0$ (as $\text{Re}z_0 < 2$ & $\text{Im}z_0 > 0$)

$= -\dfrac{\pi}{2}$

12. Ans. (3.00)

Sol. $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)\left(\overline{a + b\omega + c\omega^2}\right)$

$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$= \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

$\geq \frac{1+1+4}{2} = 3$ (when $a = 1, b = 2, c = 3$)

13. Ans. (A, B, D)

Sol. (A) $\arg(-1-i) = -\frac{3\pi}{4}$.

(B) $f(t) = \arg(-1 + it)$

$= \begin{cases} \pi - \tan^{-1}(t), & t \geq 0 \\ -\pi + \tan^{-1}(-t), & t < 0 \end{cases}$

Discontinuous at $t = 0$.

(C) $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$

$= \arg z_1 - \arg(z_2) + 2n\pi - \arg(z_1) + \arg(z_2)$

$= 2n\pi$.

(D) $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$

$\Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ is real.

$\Rightarrow$ z, z_1, z_2, z_3 are concyclic.

14. Ans. (A, C, D)

Sol. Given

$sz + t\bar{z} + r = 0$ ....(1)

on taking conjugate $\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0$ ....(2)

from (1) and (2) eliminating $\bar{z}$

$z\left(|s|^2 - |t|^2\right) = \bar{r}t - r\bar{s}$

(A) If $|s| \neq |t|$ then z has unique value

(B) If $|s| = |t|$ then $\bar{r}t - r\bar{s}$ may or may not be zero so L may be empty set

(C) locus of z is null set or singleton set or a line in all cases it will intersect given circle at most two points.

(D) In this case locus of z is a line so L has infinite elements

15. Ans. (A, D)

Sol. $\text{Im}\left(\frac{az+b}{z+1}\right) = y$ and $z = x + iy$

$\therefore \quad \text{Im}\left(\frac{a(x+iy)+b}{x+iy+1}\right) = y$

$\Rightarrow \quad \text{Im}\left(\frac{(ax+b+iay)(x+1-iy)}{(x+1)^2+y^2}\right) = y$

$\Rightarrow \quad -y(ax + b) + ay(x + 1)$

$= y((x + 1)^2 + y^2)$

$\Rightarrow \quad (a - b)y = y((x + 1)^2 + y^2)$

$\because \quad y \neq 0$ and $a - b = 1$

$\Rightarrow \quad (x + 1)^2 + y^2 = 1$

$\Rightarrow \quad x = -1 \pm \sqrt{1-y^2}$

16. Ans. (1)

Sol. $z = \omega$

$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}, P^2 = -I$

$\Rightarrow P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}((-1)^r + 1) \\ \omega^{r+2s}((-1)^r + 1) & \omega^{4s} + \omega^{2r} \end{bmatrix} = -I$

$\Rightarrow (-1)^r + 1 = 0 \Rightarrow r$ is odd $\Rightarrow r = 1,3$

also $\omega^{2r} + \omega^{4s} = -1 \therefore \quad r \neq 3$

by $r = 1 \Rightarrow \omega^2 + \omega^{4s} = -1 \Rightarrow s = 1$

$(r, s) = (1, 1)$

only 1 pair

17. Ans. (A, C, D)

Sol.. $x + iy = \frac{1}{a + ibt}$

$x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

Let $a \neq 0$ & $b \neq 0$

$x = \frac{a}{a^2 + b^2t^2}$ ....(1)

$y = \frac{-bt}{a^2 + b^2t^2}$ ....(2)

$\frac{y}{x} = \frac{-bt}{a} \Rightarrow t = -\frac{ay}{bx}$

put in (1)

$$x\left\{a^2+b^2.\frac{a^2y^2}{b^2x^2}\right\}=a$$

$a^2(x^2+y^2)=ax$

$$x^2+y^2-\frac{1}{a}x=0$$

$$\left(x-\frac{1}{2a}\right)^2+y^2=\frac{1}{4a^2}$$

$\Rightarrow$ option (A) is correct

for $a\neq 0$, $b=0$

$$x+iy=\frac{1}{a}$$

$x=\frac{1}{a}$, $y=0$ $\Rightarrow$ z lies on x-axis

$\Rightarrow$ option (C) is correct

for $a=0$, $b\neq 0$

$$x+iy=\frac{1}{ibt}$$

$$y=-\frac{1}{bt}i,\ x=0$$

$\Rightarrow$ z lies on y-axis.

$\Rightarrow$ option (D) is correct

18. Ans. (A) $\to$ (P,Q); (B) $\to$ (P,Q);
(C) $\to$ (P,Q,S,T); (D) $\to$ (Q,T)

Sol. (A) $\left|\frac{\alpha\sqrt{3}+\beta}{2}\right|=\sqrt{3}\Rightarrow\alpha\sqrt{3}+\frac{\alpha-2}{\sqrt{3}}=\pm 2\sqrt{3}$

$$\Rightarrow\alpha\left(\frac{4}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}}\pm 2\sqrt{3}$$

$$\alpha=2,-1\Rightarrow|\alpha|=1,2$$

(B) By continuity $-3a-2=b+a^2$
By differentiability $-6a=b$
$a^2-3a+2=0\Rightarrow a=1,2$

(C) $\left(\left(-3+2\omega+3\omega^2\right)\omega\right)^{4n+3}+$

$$\left(\left(-3+2\omega+3\omega^2\right)\omega^2\right)^{4n+3}+$$

$$\left(\left(-3+2\omega+3\omega^2\right)^{4n+3}\right)=0$$

$$\Rightarrow\left(-3+2\omega+3\omega^2\right)^{4n+3}\left[\omega^{4n+3}+\omega^{8n-6}+1\right]=0$$

$$\Rightarrow\ \omega^n+\omega^{2n}+1=0$$

$\Rightarrow$ n is not a multiple of 3.

(D) $\frac{2ab}{a+b}=4$, $2(5-a)=b-5$

$b=15-2a$

$2a(15-2a)=4(15-a)\Rightarrow 15a-2a^2$

$=30-2a$

$2a^2-17a+30=0$

$\Rightarrow 2a^2-12a-5a+30=0$

$2a(a-6)-5(a-6)=0$

$$a=\frac{5}{2},6$$

$$\Rightarrow|q-a|=|10-2a|=5\text{ or }2$$

19. Ans. (4)

Sol. α_k are vertices of 14 sided regular polygon

$|\alpha_{k+1}-\alpha_k|$ length of a side of the regular polygon

$|\alpha_{4k-1}-\alpha_{4k-2}|$ length of a side of the regular polygon

$$\Rightarrow\frac{12(S)}{3(S)}=4$$

20. Ans. (D)

Sol. Let $p(x)=ax^2+b+c$

$$p(x)=0\Rightarrow x=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

so $b=0$ as roots are purely imaginary

so equation will be $ax^2+c=0$

Now $p(p(x))=0$

$$\Rightarrow\quad ap^2(x)+c=0\Rightarrow p(x)=\pm\sqrt{-\frac{c}{a}}$$

$$ax^2+c=\pm\sqrt{-\frac{c}{a}}\Rightarrow x\notin R$$

if $x=i\beta$ then

$$-a\beta^2+c=\pm\sqrt{-\frac{c}{a}}\ \text{not possible}$$

(real) (imaginary)

So, neither real nor purely imaginary roots.

21. Ans. (C)

Sol. (P) $e^{\frac{i2k\pi}{10}}.e^{\frac{i2j\pi}{10}}=1$

$e^{i\frac{\pi}{10}\times 2(k+j)}=1$

$\frac{\pi}{10}\left(2(k+j)\right)=2n\pi$

$(k+j)=10$

Possible

(Q) $e^{\frac{i2\pi}{10}}.z=e^{\frac{i2\pi k}{10}}$

$z=\frac{e^{\frac{i2\pi k}{10}}}{e^{\frac{i2\pi}{10}}}$ is possible

(R) $z^{10}-1=(z-1)(z-z_1)(z-z_2)\ldots(z-z_9)$

put $z=1$

$\lim_{z\to 1}\frac{z^{10}-1}{(z-1)}=(1-z_1)(1-z_2)\ldots(1-z_9)$

$\lim_{z\to 1}\frac{10z^9}{1}=(1-z_1)(1-z_2)\ldots(1-z_9)$

$=|(1-z_1)(1-z_2)\ldots(1-z_9)|=10$

(S) $1+\cos\frac{2\pi}{10}+\cos\frac{4\pi}{10}+\ldots\ldots+\cos\frac{18\pi}{10}=0$

since they are sum of ten, tenth roots of unity

$\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)=-1$

$1+1=2$

22. **Ans. (C)**

Sol. Given : α satisfies $|z-z_0|=r \Rightarrow |\alpha-z_0|=r$ ...(1)

& $\frac{1}{\bar\alpha}$ satisfies $|z-z_0|=2r \Rightarrow \left|\frac{1}{\bar\alpha}-z_0\right|=2r$ ...(2)

squaring (1) and (2) we get

$(\alpha-z_0)(\bar\alpha-\bar z_0)=r^2$

$\Rightarrow \alpha\bar\alpha-z_0\bar\alpha-\alpha\bar z_0+z_0\bar z_0=r^2=2|z_0|^2-2$ ...(3)

& $\left(\frac{1}{\bar\alpha}-z_0\right)\left(\frac{1}{\alpha}-\bar z_0\right)=4r^2$

$\Rightarrow \frac{1}{\alpha\bar\alpha}-\frac{z_0}{\alpha}-\frac{\bar z_0}{\bar\alpha}+z_0\bar z_0=4r^2$

$\Rightarrow 1-z_0\bar\alpha-\bar z_0\alpha+|z_0|^2|\alpha|^2=4\left(2|z_0|^2-2\right)|\alpha|^2$

$\Rightarrow 1+2|z_0|^2-2-|\alpha|^2-|z_0|^2+|z_0|^2|\alpha|^2$

$=8|z_0|^2|\alpha|^2-8|\alpha|^2$

$\Rightarrow -1+|z_0|^2-7|z_0|^2|\alpha|^2+7|\alpha|^2=0$

$\Rightarrow (|z_0|^2-1)(7|\alpha^2|-1)=0$

$\Rightarrow |z_0|=1$ (rejected as $r\neq 0$)

& $|\alpha|=\frac{1}{\sqrt7}$

23. **Ans. (B, C, D)**

Sol. $P^2=[\alpha_{ij}]_{n\times n}$

$\alpha_{ij}=\sum_{k=1}^{n}p_{ik}.p_{kj}$

$=\sum_{k=1}^{n}\omega^{i+k}.\omega^{k+j}=\omega^{i+j}\sum_{k=1}^{n}\omega^{2k}$

$=\omega^{i+j}(\omega^2+\omega^4+\omega^6+\ldots..+\omega^{2n})$

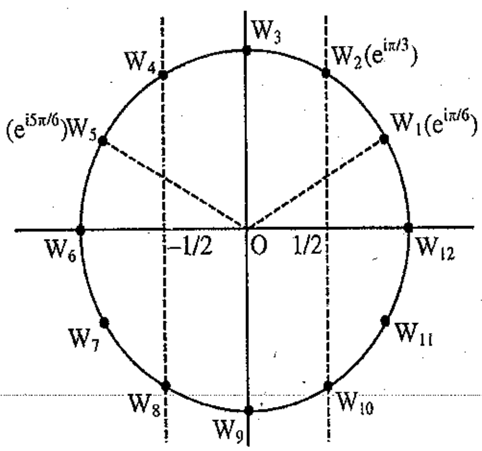
If n is a multiple of 3 then $P^2=\mathbf{0}$

$\Rightarrow$ n is not a multiple of 3

$\Rightarrow$ n can be 55, 58, 56

24. **Ans. (D)**

Sol.



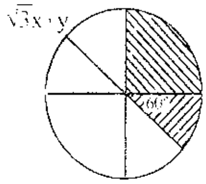
$z_1=\{w_1, w_{11}, w_{12}\}$

$z_2=\{w_5, w_6, w_7\}$

$\angle w_1Ow_5=\frac{2\pi}{3}$ & $\angle w_1Ow_6=\frac{5\pi}{6}$

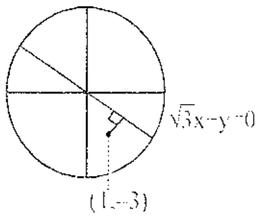
25. Ans. (C)

Sol.



S_1 is interior of circle centred at (0,1) & radius = 4.

Re(z) > 0 is in I^{st} & IV^{th} quadrant.



$$\frac{\left(z-\left(1-i\sqrt{3}\right)\right)}{\left(1-i\sqrt{3}\right)}=\frac{\left(\left(x-1\right)+i\left(y-\sqrt{3}\right)\right)}{\left(1-i\sqrt{3}\right)}$$

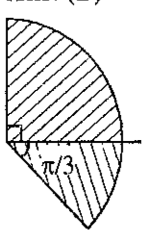
$$=\frac{\left(\left(x-1\right)+i\left(y-\sqrt{3}\right)\right)\left(1+i\sqrt{3}\right)}{2}$$

$$I_m\left(S_2\right)=\sqrt{3}x+y>0$$

perpendicular distance from (1,–3) to the line is

$$P=\left|\frac{\sqrt{3}-3}{2}\right|=\left(\frac{3-\sqrt{3}}{2}\right)$$

26. Ans. (B)



Sol.

Area of $S=\frac{\pi(4)^2}{4}+\frac{1}{2}(4)^2\frac{\pi}{3}$

$\frac{8\pi}{3}+4\pi=\frac{20\pi}{3}$

27. Ans. (D)

Sol. $z^2+z+1-a=0$

$\because$ z is imaginary $\Rightarrow D<0$

$1-4(1-a)<0$

$4a<3$

$a<\frac{3}{4}.$

Aliter : $a=z^2+z+1$

$\because \quad a=\bar{a}$ (given a is real)

$\therefore \quad z^2+z=\bar{z}^2+\bar{z}$

$\Rightarrow \quad z^2-\bar{z}^2=\bar{z}-z$

$\Rightarrow \quad z+\bar{z}=-1$ ($\because$ Im(z) is non zero)

$\Rightarrow \quad Re(z)=-\frac{1}{2}$

$\therefore \quad$ z can be taken as $-\frac{1}{2}+iy$

where $y\in R$

$\therefore \quad a=\left(-\frac{1}{2}+iy\right)^2+\left(\frac{-1}{2}+iy\right)+1$

$\Rightarrow \quad a=\frac{1}{4}-\frac{1}{2}+1-iy+iy-y^2$

$\Rightarrow \quad a=\frac{3}{4}-y^2\Rightarrow a<\frac{3}{4}$

$\therefore \quad a\neq\frac{3}{4}$

28. Ans. (D)

Sol. (K, 6K, –7K)

$2x+y+z=1$

$2K+6K-7K=1$

($\because$ point lies on the plane)

$\Rightarrow \quad K=1$

$\Rightarrow \quad 7a+b+c=7K+6K-7K=6$

29. Ans. (A)

Sol. $x^3-1=0$

$\Rightarrow x=1,\omega,\omega^2$

$\omega=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ since $Im(\omega)>0$

If $a=2=K \Rightarrow \quad b=12$ & $c=-14$

Hence $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}=\frac{3}{\omega^2}+\frac{1}{\omega^{12}}+\frac{3}{\omega^{-14}}$

$=3\omega+1+3\omega^2$

$=-3+1=-2$

30. **Ans. (B)**

Sol. $\because \quad b = 6 \Rightarrow 6K = 6 \Rightarrow K = 1$

$\Rightarrow \quad a = 1, \quad b = 6 \quad \& \quad c = -7$

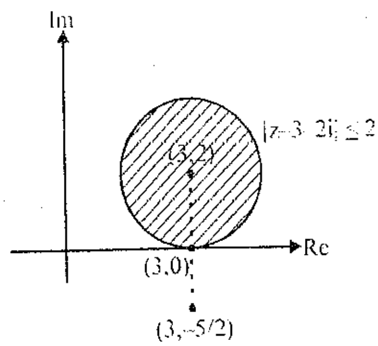
$x^2 + 6x - 7 = 0$

$\Rightarrow \quad \alpha + \beta = -6, \ \alpha\beta = -7$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\left(\frac{\alpha+\beta}{\alpha\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = \frac{1}{1-\frac{6}{7}} = 7$$

31. **Ans. (5)**

Sol.



We have to find minimum value of

$$2\left|z-\left(3-\frac{5}{2}i\right)\right|$$

$= 2 \times$ (minimum distance between z and point $\left(3,-\frac{5}{2}\right)$)

$= 2 \times$ (distance between (3,0) and $\left(3,-\frac{5}{2}\right)$)

$= 2 \times \frac{5}{2} = 5$ units.

32. **Ans. (3) (Bonus)**

Sol. (Comment : If $\omega = e^{i\pi/3}$ then

$\dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is not always an integer.

For example if a = b = c = 1 then the value of

$\dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is $\dfrac{17}{3}$.

Now if we consider $\omega = e^{i2\pi/3}$ then the solution is)

$|x|^2 = (a+b+c)\left(\bar{a}+\bar{b}+\bar{c}\right)$

$= |a|^2 + |b|^2 + |c|^2 +$

$a\bar{b} + a\bar{c} + b\bar{a} + b\bar{c} + c\bar{a} + c\bar{b}$

$|y|^2 = (a + b\omega + c\omega^2)\left(\bar{a}+\bar{b}\omega^2+\bar{c}\omega\right)$

$= |a|^2 + |b|^2 + |c|^2 +$

$a\bar{b}\omega^2 + a\bar{c}\omega + b\bar{a}\omega + b\bar{c}\omega^2 + c\bar{a}\omega^2 + c\bar{b}\omega$

$|z|^2 = (a + b\omega^2 + c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)$

$= |a|^2 + |b|^2 + |c^2| +$

$a\bar{b}\omega + a\bar{c}\omega^2 + b\bar{a}\omega^2 + b\bar{c}\omega + c\bar{a}\omega + c\bar{b}\omega^2$

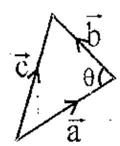
$\therefore \quad |x|^2 + |y|^2 + |z|^2 = 3\,(|a|^2 + |b|^2 + |c|^2)$

$$\Rightarrow \quad \frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2} = 3$$

33. **Ans.** (A) → (q); (B) → (p) or (p,q,r,s,t);

(C) → (s); (D) → (t)

Sol. (A)



$|\vec{a}| = 2$
$|\vec{b}| = 2$
$|\vec{c}| = 2\sqrt{3}$

$$\cos\theta = \frac{|\vec{a}|^2+|\vec{b}|^2-|\vec{c}|^2}{2|\vec{a}||\vec{b}|}$$

$$= \frac{4+4-12}{2.2.2} = \frac{-1}{2}$$

$$\Rightarrow \quad \theta = \frac{2\pi}{3}$$

(B) $\displaystyle\int_a^b (f(x)-3x)dx = a^2 - b^2 = \int_a^b (-2x)dx$

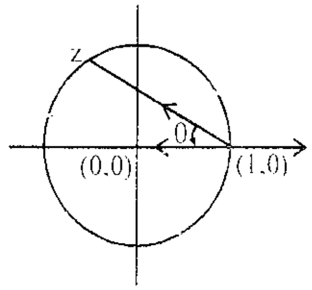
$\Rightarrow \quad \displaystyle\int_a^b (f(x)-x)dx = 0$

$\Rightarrow$ one of the possible solution of this equation is

$f(x) = x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

(C) $\frac{\pi^2}{\ell n\,3}\int_{7/6}^{5/6}(\sec \pi x)dx$

$=\frac{\pi^2}{\ell n\,3}\,\frac{1}{\pi}\left[\ell n\,|\sec \pi x+\tan \pi x|\right]_{7/6}^{5/6}$



$=\frac{\pi}{\ell n\,3}\ell n\left|\frac{\sec\frac{5\pi}{6}+\tan\frac{5\pi}{6}}{\sec\frac{7\pi}{6}+\tan\frac{7\pi}{6}}\right|=\ell n\,3=\pi$

(D) Let $\theta=\text{Arg}\left(\frac{1}{1-z}\right)$

$\Rightarrow\quad \theta=\text{Arg}\left(\frac{0-1}{z-1}\right)$

which is shown in adjacent diagram.

$\Rightarrow$ Maximum value of θ is approaching to $\frac{\pi}{2}$ but θ will never obtained the value equal to $\frac{\pi}{2}$.

Hence there is an error in aksing the problem.

34. Ans. (A) $\rightarrow$ (s); (B) $\rightarrow$ (t); (C) $\rightarrow$ (r); (D) $\rightarrow$ (r)

Sol. (A) Let $z=\cos\theta+i\sin\theta$

$\text{Re}\left(\frac{2i(\cos\theta+i\sin\theta)}{1-(\cos\theta+i\sin\theta)^2}\right)$

$=\text{Re}\left(\frac{\cos\theta i-\sin\theta}{\sin^2\theta-i\cos\theta\sin\theta}\right)$

$=\text{Re}\left(-\frac{1}{\sin\theta}\right)=\frac{-1}{\sin\theta}$

$\therefore$ Set will be $(-\infty,-1]\cup[1,\infty)$

(B) $-1\le\frac{8.3^{(x-2)}}{1-3^{2(x-1)}}\le 1\quad x\ne 1$

$\Rightarrow\quad -1\le\frac{8.3^x}{(3-3^x)(3+3^x)}\le 1$

$3^x=t\quad\therefore\quad t>0$

$\frac{8t}{(3-t)(t+3)}\ge -1$

$\Rightarrow\quad t\in(0,3)\cup[9,\infty)$

$\Rightarrow\quad x\in(-\infty,1)\cup[2,\infty)$

$\frac{8t}{(3-t)(t+3)}\le 1$

$\Rightarrow\quad t\in(0,1]\cup(3,\infty)$

$\Rightarrow\quad x\in(-\infty,0]\cup(1,\infty)$

Taking intersection.

$x\in(-\infty,0]\cup[2,\infty)$

(C) $f(\theta)=\begin{vmatrix}1 & \tan\theta & 1\\ -\tan\theta & 1 & \tan\theta\\ -1 & -\tan\theta & 1\end{vmatrix}$

$C_1\rightarrow C_1+C_3$

$\Rightarrow\quad f(\theta)=\begin{vmatrix}2 & \tan\theta & 1\\ 0 & 1 & \tan\theta\\ 0 & -\tan\theta & 1\end{vmatrix}$

$\Rightarrow\quad f(\theta)=2\sec^2\theta$

$\Rightarrow\quad f(\theta)\in[2,\infty)$

(D) $f(x)=3x^{5/2}-10x^{3/2}$

$f'(x)=\frac{15}{2}x^{3/2}-\frac{30}{2}x^{1/2}>0$

$\Rightarrow\quad \frac{15}{2}\sqrt{x}(x-2)\ge 0$

$\Rightarrow\quad x\ge 2$

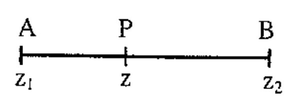
35. Ans. (A, C, D)

Sol. $z=z_1+t(z_2-z_1)$

$\frac{z-z_1}{z_2-z_1}=t,\ t\in(0,1)\Rightarrow z=\frac{z_1(1-t)+tz_2}{(1-t)+t}$

point $P(z)$ divides point $A(z_1)$ & $B(z_2)$ internally in ratio $(1-t):t$

Hence locus is a line segment such that $P(z)$ lies between $A(z_1)$ & $B(z_2)$ as shown in figure.



Hence options A, C & D are correct.

36. Ans. (1)

Sol. $R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z = 0 \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ \omega & z+(\omega^2-\omega) & 1-\omega \\ \omega^2 & (1-\omega^2) & z-(\omega^2-\omega) \end{vmatrix} = 0$$

$$z^2 - (\omega^2-\omega)^2 - (1-\omega)(1-\omega^2) = 0$$

$$z^2 - \omega - \omega^2 + 2 - 1 + \omega + \omega^2 - 1 = 0$$

$$z^2 = 0 \Rightarrow z = 0$$

37. Ans. (A) → (q, r), (B) → (p),
(C) → (p, s, t), (D) → (q, r, s, t)

Sol. (A) $|z - i|z||^2 = |z + i|z||^2$

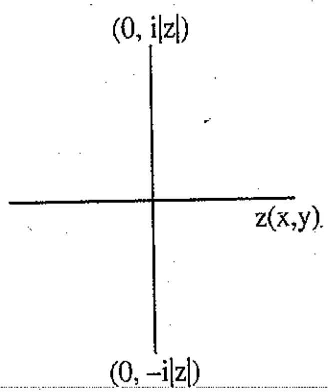
$$\Rightarrow (\bar{z} + i|z|) = (z + i|z|)(\bar{z} - i|z|)$$

$$\Rightarrow 2i|z|z = 2i|z|\bar{z}$$

$\Rightarrow z = \bar{z}$ ∴ z is purely real.

∴ z lies on real axis.

Alternate :



By geometry z lies on x-axis.

(B) Locus is ellipse having focii (–4, 0) & (4, 0)

$2ae = 8$ & $2a = 10$

$\Rightarrow a = 5$ & $e = 4/5$

It is ellipse having eccentricity 4/5.

(C) $w = 2(\cos\theta + i\sin\theta)$

$$z = 2(\cos\theta + i\sin\theta) - \frac{1}{2(\cos\theta + i\sin\theta)}$$

$$x + iy = \frac{3}{2}\cos\theta + \frac{i5}{2}\sin\theta$$

$$\Rightarrow x = \frac{3}{2}\cos\theta \quad \& \quad y = \frac{5}{2}\sin\theta$$

It is a locus $\dfrac{x^2}{9/4} + \dfrac{y^2}{25/4} = 1$

$$\frac{9}{4} = \frac{25}{4}(1 - e^2) \Rightarrow e = \frac{4}{5}$$

Since $x = \dfrac{3}{2}\cos\theta \Rightarrow |\text{Re}(z)| \le \dfrac{3}{2}$

$$|\text{Re}(z)| \le \frac{3}{2} \qquad \Rightarrow |\text{Re}(z)| \le 2$$

Consider the circle $x^2 + y^2 - 9 = 0$

By putting $x = \dfrac{3}{2}\cos\theta$

& $y = \dfrac{5}{2}\sin\theta$ into $x^2 + y^2 - 9$

$$\frac{9\cos^2\theta}{4} + \frac{25}{4}\sin^2\theta - 9 < 0$$

(D) $z = (\cos\theta + i\sin\theta) + \dfrac{1}{(\cos\theta + i\sin\theta)}$

$z = 2\cos\theta$

where z is real value & $z \in [-2, 2]$

38. Ans. (D)

Sol. $z = \cos\theta + i\sin\theta$

$z^{2m-1} = \cos(2m-1)\theta + i\sin(2m-1)\theta$

$\Rightarrow \text{Im}(z^{2m-1}) = \sin(2m-1)\theta$

$$\Rightarrow \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \sin\theta + \sin 3\theta + \ldots + \sin 29\theta$$

$$= \frac{\sin 15\theta}{\sin\theta}\sin(\theta + 14\theta) = \frac{\sin^2 15\theta}{\sin\theta}$$

$$\because \theta = 2° \Rightarrow \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \frac{\sin^2 30°}{\sin 2°}$$

$$= \frac{1}{4\sin 2°}$$

39. Ans. (A)

Sol. $|z|^2\left\{\bar{z}^2+z^2\right\}=350$

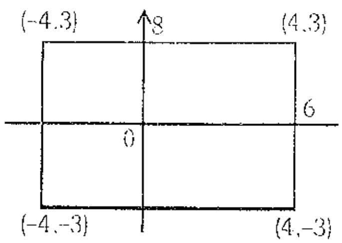
$\Rightarrow r^2\left\{(\bar{z}+z)^2-2z\bar{z}\right\}=350$

$\Rightarrow r^2\{4x^2-2r^2\}=350 \Rightarrow r^2\{2x^2-r^2\}=175$

$\Rightarrow (x^2+y^2)(x^2-y^2)=5^2.7$

$\Rightarrow x^2+y^2=25\ \&\ x^2-y^2=7$

$\Rightarrow x=\pm 4\ \&\ y=\pm 3$



Area = 48.

40. Ans. $A \to (P)$; $B \to (S, T)$;

$C \to (R)$; $D \to (Q, S)$

Sol. (P) $\left|\dfrac{1}{\sqrt{h^2+k^2}}\right|=2$

$\therefore$ Locus of (h, k) is

$x^2+y^2=1/4$

(Q) $|z+2|-|z-2|=\pm 3\ \because PS-PS'=2a,$

$SS'>2a \Rightarrow$ Locus of z is hyperbola.

(R) $x=\sqrt{3}\cos 2\theta,\ y=\sin 2\theta$

$\dfrac{x^2}{3}+y^2=1$ which is an ellipse.

(S) $1 \le x < \infty$ As x is eccentricity

$\Rightarrow$ Parabola, Hyperbola.

(T) $(z+1)^2=(x+iy+1)^2$

$=(x+1)^2-y^2+2iy(x+1)$

$\therefore \text{Re}(z+1)^2=(x+1)^2-y^2$

$\text{Re}(z+1)^2=|z|^2+1$

$\Rightarrow (x+1)^2-y^2=x^2+y^2+1$

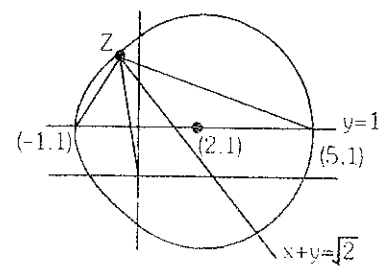
$\Rightarrow x=y^2$ which is parabola.

41. Ans. (B)

Sol. $A=\{z : \text{Im}\, z \ge 1\}$ $\quad y \ge 1$

$B=\{z : |z-2-i|=3\}$ $\quad (x-2)^2+(y-1)^2=9$

$C=\{z : \text{Re}((1-i)z)=\sqrt{2}\}$ $\quad x+y=\sqrt{2}$



As we can see 3 curves intersects at only one point

So $A \cap B \cap C$ contains exactly one element

42. Ans. (C)

Sol. $|z+1-i|^2+|z-5-i|^2$

$=(-1-5)^2+(1-1)^2=36$

so exactly 36

43. Ans. (B, C, D)

Sol. As $\quad 3-\sqrt{5} \le |z| \le 3+\sqrt{5}$

As $\quad -3+\sqrt{5} \le |\omega| \le 3+\sqrt{5}$

$-3-\sqrt{5} \le -|\omega| \le 3-\sqrt{5}$

$-\sqrt{5} \le -|\omega|+3 \le 6-\sqrt{5}$

$-3 \le |z|-|\omega|+3 \le 9$

44. Ans. (D)

Sol. $Z_0=1+2i$

Now Z_1 will be $6+5i$

Now the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i}+\hat{j}$.

So it will reach $7+6i$.

Now for Z_2

$Z_2=i(7+6i)=-6+7i$

PROBABILITY

1. **Ans. (B)**

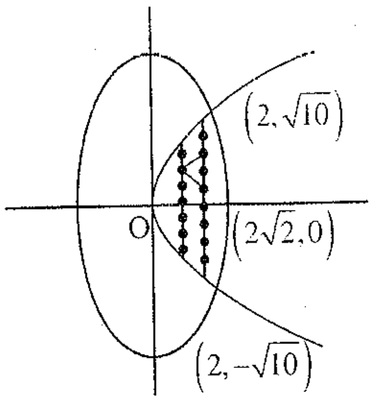
Sol. $\dfrac{x^2}{8}+\dfrac{y^2}{20}<1$ & $y^2<5x$

Solving corresponding equations

$\dfrac{x^2}{8}+\dfrac{y^2}{20}=1$ & $y^2=5x$

$$\Rightarrow\left\{\begin{matrix} x=2 \\ y=\pm\sqrt{10}\end{matrix}\right\}$$

X = {(1,1), (1,0), (1,–1), (1,2), (1,–2), (2, 3), (2,2), (2,1), (2,0), (2,–1), (2,–2), (2,–3)}



Let S be the sample space & E be the event n(S) $={}^{12}C_3$

<u>For E</u>

Selecting 3 points in which 2 points are either or x = 1 & x = 2 but distance b/w then is even

Triangles with base 2 :

$=3\times7+5\times5=46$

Triangles with base 4 :

$=1\times7+3\times5=22$

Triangles with base 6 :

$=1\times5=5$

$$P(E)=\frac{46+22+5}{{}^{12}C_3}=\frac{73}{220}$$

2. **Ans. (B)**

Sol. $P(H)=\dfrac{1}{3};P(T)=\dfrac{2}{3}$

Req. prob = P(HH or HTHH or HTHTHH or ....) + P(THH or THTHH or THTHTHH or ....)

$$=\frac{\frac{1}{3}\cdot\frac{1}{3}}{1-\frac{2}{3}\cdot\frac{1}{3}}+\frac{\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}}{1-\frac{2}{3}\cdot\frac{1}{3}}=\frac{5}{21}$$

3. **Ans. (31)**

Sol. No. of elements in X which are multiple of 5

$$\left.\begin{array}{l}
\underbrace{_\,_\,_\,_}_{1,2,2,2}\ \underbrace{0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 4}{\lfloor 3}=4 \\
\underbrace{_\,_\,_\,_}_{1,4,2,2}\ \underbrace{0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 4}{\lfloor 2}=12 \\
\underbrace{_\,_\,_\,_}_{4,2,2,2}\ \underbrace{0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 4}{\lfloor 3}=4 \\
\underbrace{_\,_\,_\,_}_{2,2,4,4}\ \underbrace{0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 4}{\lfloor 2\lfloor 2}=6 \\
\underbrace{_\,_\,_\,_}_{1,2,4,4}\ \underbrace{0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 4}{\lfloor 2}=12
\end{array}\right\}\text{Total}=38$$

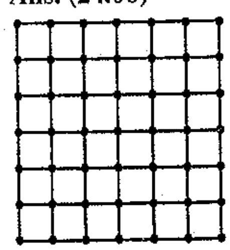
Among these 38 elements, let us calculate when element is not divisible by 20

$$\left.\begin{array}{l}
\underbrace{_\,_\,_}_{2,2,2}\ \underbrace{1\ 0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 3}{\lfloor 3}=1 \\
\underbrace{_\,_\,_}_{2,2,4}\ \underbrace{1\ 0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 3}{\lfloor 2}=3 \\
\underbrace{_\,_\,_}_{2,4,4}\ \underbrace{1\ 0}_{\text{fixed}}\rightarrow\dfrac{\lfloor 3}{\lfloor 2}=3
\end{array}\right\}\text{Total}=7$$

$\therefore\ p=\dfrac{38-7}{38}\quad\therefore\ 38p=31$

4. **Ans. (24.00)**

Sol.



P_i = Probability that randomly selected points has friends

$P_0 = 0$ (0 friends)

$P_1 = 0$ (exactly 1 friends)

$P_2=\dfrac{{}^4C_1}{{}^{49}C_1}=\dfrac{4}{9}$ (exactly 2 friends)

$P_3=\dfrac{{}^{20}C_1}{{}^{49}C_1}=\dfrac{20}{49}$ (exactly 3 friends)

$P_4=\dfrac{{}^{25}C_1}{{}^{49}C_1}=\dfrac{25}{49}$ (exactly 4 friends)

x	0	1	2	3	4
P(x)	0	0	$\frac{4}{49}$	$\frac{20}{49}$	$\frac{25}{49}$

Mean = E(x) =

$$\sum x_i P_i = 0+0+\frac{8}{49}+\frac{60}{49}+\frac{100}{49}=\frac{168}{49}$$

$$7\left(E(x)\right)=\frac{168}{49}\times 7=24$$

5. **Ans. (0.50)**

Sol. Total number of ways of selecting 2 persons $= {}^{49}C_2$

Number of ways in which 2 friends are selected $= 6\times 7\times 2 = 84$

$$7P=\frac{84\times 2}{49\times 48}\times 7=\frac{1}{2}$$

6. **Ans. (0.80)**

Sol. $n(U) = 900$

Let $A \equiv$ Fever, $B \equiv$ Cough

$C \equiv$ Breathing problem

$\therefore n(A) = 190, n(B) = 220, n(C) = 220$

$n(A\cup B) = 330, n(B\cup C) = 350,$

$n(A\cup C) = 340, n(A\cap B\cap C) = 30$

Now $n(A\cup B) = n(A) + n(B) - n(A\cap B)$

$\Rightarrow 330 = 190 + 220 - n(A\cap B)$

$\Rightarrow n(A\cap B) = 80$

Similarly,

$350 = 220 + 220 - n(B\cap C)$

$\Rightarrow n(B\cap C) = 90$

and $340 = 190 + 220 - n(A\cap C)$

$\Rightarrow n(A\cap C) = 70$

$\therefore n(A\cup B\cup C) = (190 + 220 + 220) - (80 + 90 + 70) + 30$

$= 660 - 240 = 420$

$\Rightarrow$ Number of person without any symptom

$= n(\cup) - n(A\cup B\cup C)$

$= 900 - 420 = 480$

Now, number of person suffering from exactly one symptom

$= (n(A) + n(B) + n(C)) - 2(n(A\cap B) + n(B\cap C) + n(C\cap A)) + 3n(A\cap B\cap C)$

$= (190 + 220 + 220) - 2(80 + 90 + 70) + 3(30)$

$= 630 - 480 + 90 = 240$

$\therefore$ Number of person suffering from atmost one symptom

$= 480 + 240 = 720$

$$\Rightarrow \text{Probability}=\frac{720}{900}=\frac{8}{10}=\frac{4}{5}=0.80$$

7. **Ans. (A)**

Sol. $P(\text{draw in 1 round})=\frac{6}{36}=\frac{1}{6}$

$$P(\text{win in 1 round})=\frac{1}{2}\left(1-\frac{1}{6}\right)=\frac{5}{12}$$

$$P(\text{loss in 1 round})=\frac{5}{12}$$

$P(X_2 > Y_2) = P(10,0) + P(7,2)$

$$=\frac{5}{12}\times\frac{5}{12}+\frac{5}{12}\times\frac{1}{6}\times 2=\frac{45}{144}=\frac{5}{16}$$

$P(X_2 = Y_2) = P(5,5) + P(4,4)$

$$=\frac{5}{12}\times\frac{5}{12}\times 2+\frac{1}{6}\times\frac{1}{6}=\frac{25+2}{72}=\frac{3}{8}$$

$P(X_3 = Y_3) = P(6,6) + P(7,7)$

$$=\frac{1}{6\times6\times6}+\frac{5}{12}\times\frac{1}{6}\times\frac{5}{12}\times 6=\frac{2}{432}+\frac{75}{432}$$

$$=\frac{77}{432}$$

$$P(X_3 > Y_3)=\frac{1}{2}\left(1-\frac{77}{432}\right)=\frac{355}{864}$$

8. **Ans. (C)**

Sol. Box I 8(R) 3(B) 5(G)

Box II 24(R) 9(B) 15(G)

Box III 1(B) 12(G) 3(y)

Box IV 10(G) 16(o) 6(w)

A (one of the chosen balls is white)

B (at least one of the chosen ball is green)

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$

$A\cap B \rightarrow (wG)$

$$=\frac{\frac{5}{16}\times\frac{6}{32}}{\frac{5}{16}\times 1+\frac{8}{16}\times\frac{15}{48}+\frac{3}{16}\times\frac{12}{16}}$$

$$=\frac{15}{156}=\frac{5}{52}$$

9. Ans. (A)

Sol. $P = \dfrac{P(S_1 \cap (E_1 = E_3))}{P(E_1 = E_3)} = \dfrac{P(B_{1,2})}{P(B)}$

$$P(B) = \underset{\substack{\uparrow \\ \text{If 1,2} \\ \text{chosen} \\ \text{at start}}}{P(B_{1,2})} + \underset{\substack{\uparrow \\ \text{If 1,3} \\ \text{chosen} \\ \text{at start}}}{P(B_{1,3})} + \underset{\substack{\uparrow \\ \text{If 2,3} \\ \text{chosen} \\ \text{at start}}}{P(B_{2,3})}$$

$$P(B_{1,2}) = \frac{1}{3} \times \underbrace{\frac{1 \times {}^3C_1}{{}^4C_2}}_{\substack{\text{1 is definitely} \\ \text{chosen from } F_2}} \times \underbrace{\frac{1}{{}^5C_2}}_{\substack{\text{1,2 chosen} \\ \text{from } G_2}}$$

$$P(B_{1,3}) = \frac{1}{3} \times \underbrace{\frac{1 \times {}^2C_1}{{}^3C_2}}_{\substack{\text{1 is definitely} \\ \text{chosen from } F_2}} \times \underbrace{\frac{1}{{}^5C_2}}_{\substack{\text{1,2 chosen} \\ \text{from } G_2}}$$

$$P(B_{2,3}) = \frac{1}{3} \times \left[\underbrace{\frac{{}^3C_2 \times 1}{{}^4C_2}}_{\substack{\text{If 1 is not chosen} \\ \text{from } F_2}} \times \frac{1}{{}^4C_2} + \underbrace{\frac{1 \times {}^3C_1}{{}^4C_2}}_{\substack{\text{If 1 is chosen} \\ \text{from } F_2}} \times \frac{1}{{}^5C_2} \right]$$

$$\frac{P(B_{1,2})}{P(B)} = \frac{1}{5}$$

10. Ans. (76.25)

Sol. p_1 = probability that maximum of chosen numbers is at least 81

p_1 = 1 – probability that maximum of chosen number is at most 80

$$p_1 = 1 - \frac{80 \times 80 \times 80}{100 \times 100 \times 100} = 1 - \frac{64}{125}$$

$$p_1 = \frac{61}{125}$$

$$\frac{625p_1}{4} = \frac{625}{4} \times \frac{61}{125} = \frac{305}{4} = 76.25$$

the value of $\dfrac{625p_1}{4}$ is 76.25

11. Ans. (24.50)

Sol. p_2 = probability that minimum of chosen numbers is at most 40

= 1 – probability that minimum of chosen numbers is at least 41

$$= 1 - \left(\frac{60}{100}\right)^3$$

$$= 1 - \frac{27}{125} = \frac{98}{125}$$

$$\therefore \frac{125}{4} p_2 = \frac{125}{4} \times \frac{98}{125} = 24.50$$

12. Ans. (A, B, C)

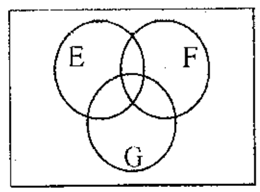
Sol. $P(E) = \dfrac{1}{8}$; $P(F) = \dfrac{1}{6}$; $P(G) = \dfrac{1}{4}$; $P(E \cap F \cap G) = \dfrac{1}{10}$

(C) $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(G \cap E) + P(E \cap F \cap G)$

$$= \frac{1}{8} + \frac{1}{6} + \frac{1}{4} - \sum P(E \cap F) + \frac{1}{10}$$

$$= \frac{3+4+6}{24} + \frac{1}{10} - \sum P(E \cap F)$$

$$= \frac{13}{24} + \frac{1}{10} - \sum P(E \cap F)$$



$\Rightarrow P(E \cup F \cup G) \le \dfrac{13}{24}$ [(C) is Correct]

(D) $P(E^C \cap F^C \cap G^C)$

$$= 1 - P(E \cup F \cup G) \ge 1 - \frac{13}{24}$$

$\Rightarrow P(E^C \cap F^C \cap G^C) \ge \dfrac{11}{24}$ [(D) is Incorrect]

(A) $P(E) = \dfrac{1}{8} \ge P(E \cap F \cap G^C) + P(E \cap F \cap G)$

$$\Rightarrow \frac{1}{8} \ge P(E \cap F \cap G^c) + \frac{1}{10}$$

$$\Rightarrow \frac{1}{8} - \frac{1}{10} \ge P(E \cap F \cap G^c)$$

$\Rightarrow \dfrac{1}{40} \ge P(E \cap F \cap G^c)$ [(A) is Correct]

(B) $P(F) = \dfrac{1}{6} \ge P(E^C \cap F \cap G) + P(E \cap F \cap G)$

$$\Rightarrow \frac{1}{6} - \frac{1}{10} \ge P(E^C \cap F \cap G)$$

$$\Rightarrow \frac{4}{60} \ge P(E^C \cap F \cap G)$$

$\Rightarrow \dfrac{1}{15} \ge P(E^C \cap F \cap G)$ [(B) is Correct]

13. Ans. (214)

Sol. A = set of numbers divisible by 3

$A = \{3, 6, 9, 12, \ldots\ldots\ldots\ldots 1998\}$

$\therefore n(A) = 666$

B = set of numbers divisible by 7

$B = \{7, 14, 21, \ldots\ldots 1995\}$

$\therefore n(B) = 285$

$A \cap B = \{21, 42, \ldots\ldots\ldots 1995\}$

$\therefore n(A \cup B) = 606 + 285 - 95 = 856$

required probability $= \frac{856}{2000} = P$

so, $500\,P = \frac{856}{2000} \times 500 = 214$

14. Ans. (B)

Sol. $P(H) = \frac{2}{3}$ for C_1

$P(H) = \frac{1}{3}$ for C_2

for C_1

No. of Heads (α)	0	1	2
Probability	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

for C_2

No. of Heads (β)	0	1	2
Probability	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

for real and equal roots

$\alpha^2 = 4\beta$

$(\alpha, \beta) = (0, 0), (2, 1)$

So, probability $= \frac{1}{9}\times\frac{4}{9}+\frac{4}{9}\times\frac{4}{9}=\frac{20}{81}$

15. Ans. (6)

Sol. Let P(r) = probability of r successes

$= {}^nC_r\left(\frac{3}{4}\right)^r\left(\frac{1}{4}\right)^{n-r}$

$1 - (P(0) + P(1) + P(2)) \ge 0.95$

$\Rightarrow \quad 1 - {}^nC_0\left(\frac{1}{4}\right)^n - {}^nC_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n-1} - {}^nC_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2} > 0.95$

$\Rightarrow \quad 1 - \left(\frac{1+3n+\frac{9n(n-1)}{2}}{4^n}\right) \ge 0.95$

$\Rightarrow \quad 9n^2 - 3n + 2 \le 0.05 \times 4^n \times 2 \le \frac{4^n}{10}$

for $n = 5$ $\quad 212 \le 102.4$ (Not true)

for $n = 6$ $\quad 308 \le 409.6$ true

$\therefore$ least value of $n = 6$

16. Ans. (8.00)

Sol. Prime : 2, 3, 5, 7, 11

1 2 4 6 2

$P(\text{Prime}) = \frac{15}{36}$

Perfect square = 4, 9 •

$P(\text{perfect square}) = \frac{7}{36}$

required probability

$$= \frac{\frac{4}{36}+\frac{14}{36}\times\frac{4}{36}+\left(\frac{14}{36}\right)^2\frac{4}{36}+\ldots}{\frac{7}{36}+\frac{14}{36}\times\frac{7}{36}+\left(\frac{14}{36}\right)^2\frac{7}{36}+\ldots}$$

$P = \frac{4}{7}$

$14P = 14.\frac{4}{7} = 8$

17. Ans. (B, C)

Sol.

Ball	Balls composition	$P(B_i)$
B_1	$5R + 5G$	$\frac{3}{10}$
B_2	$3R + 5G$	$\frac{3}{10}$
B_3	$5R + 3G$	$\frac{4}{10}$

(A) $P(B_3 \cap G) = P\left(\frac{G_1}{B_3}\right)P(B_3)$

$= \frac{3}{8} \times \frac{4}{10} = \frac{3}{20}$

(B) $P(G) = P\left(\frac{G_1}{B_1}\right)P(B_1) + P\left(\frac{G}{B_2}\right)P(B_2) + P\left(\frac{G}{B_3}\right)P(B_3)$

$= \frac{3}{20} + \frac{3}{16} + \frac{3}{20} = \frac{39}{80}$

(C) $P\left(\frac{G}{B_3}\right) = \frac{3}{8}$

(D) $P\left(\frac{B_3}{G}\right) = \frac{P(G \cap B_3)}{P(G)} = \frac{3/20}{39/80} = \frac{4}{13}$

18. Ans. (0.50)

Sol. $n(E_2) = {}^9C_2$ (as exactly two cyphers are there)

Now, det A = 0, when two cyphers are in the same column or same row

$\Rightarrow$ $n(E_1 \cap E_2) = 6 \times {}^3C_2$.

Hence, $P\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{36} = \frac{1}{2}$

$= 0.50$

19. Ans. (422.00)

Sol. $P\left(\frac{B}{A}\right) = P(B)$

$\Rightarrow$ $n\frac{(A \cap B)}{n(A)} = \frac{n(B)}{n(s)}$ ....(1)

$\Rightarrow$ n(A) should have 2 or 3 as prime factors

$\Rightarrow$ n(A) can be 2, 3, 4 or 6 as $n(A) > 1$

n(A)= 2 does not satisfy the constraint (1).

for $n(A) = 3$. $n(B) = 2$ and $n(A \cap B) = 1$

$\Rightarrow$ No. of ordered pair $= {}^6C_4 \times \frac{4!}{2!} = 180$

for $n(A) = 4$. $n(B) = 3$ and $n(A \cap B) = 2$

$\Rightarrow$ No. of ordered pairs $= {}^6C_5 \times \frac{5!}{2!2!} = 180$

for $n(A) = 6$. n(B) can be 1, 2, 3, 4, 5.

$\Rightarrow$ No. of ordered pairs $= 2^6 - 2 = 62$

Total ordered pair = 180 + 180 + 62 = 422.

20. Ans. (A)

Sol. Required probability $= \frac{4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)}{5!}$

$= \frac{9}{120} = \frac{3}{40}$

21. Ans. (C)

Sol. $n(T_1 \cap T_2 \cap T_3 \cap T_4)$

$= \text{Total} - n(\bar{T}_1 \cup \bar{T}_2 \cup \bar{T}_3 \cup \bar{T}_4)$

$= 5! - \left({}^4C_1 4!2! - \left({}^3C_1.3!2! + {}^3C_1 3!2!2!\right) + \left({}^2C_1 2!2! + {}^4C_1.2.2!\right) - 2\right)$

$= 14$

Probability $= \frac{14}{5!} = \frac{7}{60}$

22. Ans. (A, D)

Sol. $P(x) = \frac{1}{3}$; $\frac{P(X \cap Y)}{P(Y)} = \frac{1}{2}$; $\frac{P(Y \cap X)}{P(X)} = \frac{2}{5}$

from this information, we get

$P(X \cap Y) = \frac{2}{15}$; $P(Y) = \frac{4}{15}$

$\therefore$ $P(X \cup Y) = \frac{1}{3} + \frac{4}{15} - \frac{2}{15} = \frac{7}{15}$

$P(\bar{X}/Y) = \frac{P(\bar{X} \cap Y)}{P(Y)}$

$= \frac{P(Y) - P(X \cap Y)}{P(Y)}$

$\Rightarrow$ $P(\bar{X}/Y) = 1 - \frac{2/15}{4/15} = \frac{1}{2}$

23. Ans. (B)

Sol. Let z = 2k, where k = 0, 1, 2, 3, 4, 5

$\therefore$ $x + y = 10 - 2k$

Number of non negative integral solutions

$$\sum_{k=0}^{5} {}^{11-2k}C_1 = \sum 11 - 2k = 36$$

Total cases $= {}^{10+3-1}C_{3-1} = 66$

Reqd. prob. $= \frac{36}{66} = \frac{6}{11}$

24. **Ans. (C)**

Sol. $P(T_1)=\dfrac{20}{100}\qquad P(T_2)=\dfrac{80}{100}$

Let $P\left(\dfrac{D}{T_2}\right)=x$

$P\left(\dfrac{D}{T_1}\right)=10x$

$P(D)=\dfrac{7}{100}$ (given)

$P(T_1)P\left(\dfrac{D}{T_1}\right)+P(T_2)P\left(\dfrac{D}{T_2}\right)=\dfrac{7}{100}$

$\dfrac{20}{100}\times 10x+\dfrac{80}{100}\times x=\dfrac{7}{100}$

$x=\dfrac{1}{40}$

$P\left(\dfrac{D}{T_2}\right)=\dfrac{1}{40}\Rightarrow P\left(\dfrac{\bar{D}}{T_2}\right)=\dfrac{39}{40}$

$P\left(\dfrac{D}{T_1}\right)=\dfrac{10}{40}\Rightarrow P\left(\dfrac{\bar{D}}{T_1}\right)=\dfrac{30}{40}$

$$P\left(\frac{T_2}{\bar{D}}\right)=\frac{\frac{80}{100}\times\frac{39}{40}}{\frac{20}{100}\times\frac{30}{40}+\frac{80}{100}\times\frac{39}{40}}=\frac{78}{93}$$

25. **Ans. (B)**

Sol. $P(X>Y)=P(T_1\text{ win})\,P(T_1\text{ win})+P(T_1\text{ win})\,P(\text{match draw})+P(\text{match draw}).P(T_1\text{ win})$

$=\dfrac{1}{2}\times\dfrac{1}{2}+\dfrac{1}{2}\times\dfrac{1}{6}+\dfrac{1}{6}\times\dfrac{1}{2}=\dfrac{5}{12}$

26. **Ans. (C)**

Sol. $P(X=Y)=P(\text{match draw})\,P(\text{match Draw})+P(T_1\text{ win})\,P(T_2\text{ win})+P(T_2\text{ win})\,P(T_1\text{ win})$

$=\dfrac{1}{6}\times\dfrac{1}{6}+\dfrac{1}{2}\times\dfrac{1}{3}+\dfrac{1}{3}\times\dfrac{1}{2}=\dfrac{13}{36}$

27. **Ans. (8)**

Sol. Let the number of tosses be n

$\therefore$ Probability of getting at least two heads

$=1-\left(\dfrac{1}{2}\right)^n-{}^nC_1\cdot\left(\dfrac{1}{2}\right)^{n-1}\cdot\left(\dfrac{1}{2}\right)$

$\therefore\quad 1-\dfrac{(n+1)}{2^n}\ge\dfrac{24}{25}\quad\Rightarrow\quad\dfrac{n+1}{2^n}\le\dfrac{1}{25}$

$\therefore\quad n=8$

28. **Ans. (A, B)**

Sol. Required probability $=\dfrac{\left(\dfrac{n_3}{n_3+n_4}\right)}{\dfrac{n_1}{n_1+n_2}+\dfrac{n_3}{n_3+n_4}}=\dfrac{1}{3}$

now check options.

29. **Ans. (C, D)**

Sol. Required probability =

$\dfrac{n_1}{(n_1+n_2)}\dfrac{(n_1-1)}{(n_1+n_2-1)}+\dfrac{n_2}{(n_1+n_2)}\dfrac{n_1}{(n_1+n_2-1)}=\dfrac{1}{3}$

$\Rightarrow\dfrac{n_1^2+n_1n_2-n_1}{(n_1+n_2)(n_1+n_2-1)}=\dfrac{1}{3}$

now check options.

30. **Ans. (A)**

Sol. Total ways of arranging all boys & girls = 5!
= 120

unfavourable case will be

$\left.\begin{array}{l}\text{I}\ \_\ \_\ \_\ \_\ \_\ g\\ \_\ \_\ \_\ \_\ g\ g\end{array}\right\}\ 2.4!=48$

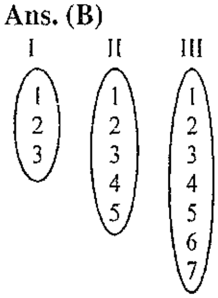
$\text{II}\ \_\ \_\ \_\ g\ g\ B\ \}\ 2!.3!=12$

Favourable ways are $120-48-12=60$

$P=\dfrac{60}{120}=\dfrac{1}{2}$

31. **Ans. (B)**

Sol.



x_1 = number on the card drawn from I

x_2 = number on the card drawn from II

x_3 = number on the card drawn from III

$\because\quad x_1+x_2+x_3=\text{odd}$

odd + odd + odd $\Rightarrow\dfrac{2}{3}.\dfrac{3}{5}.\dfrac{4}{7}=\dfrac{24}{105}$

odd + even + even $\Rightarrow\dfrac{2}{3}.\dfrac{2}{5}.\dfrac{3}{7}=\dfrac{12}{105}$

even + odd + even $\Rightarrow\dfrac{1}{3}.\dfrac{3}{5}.\dfrac{3}{7}=\dfrac{9}{105}$

even + even + odd $\Rightarrow\dfrac{1}{3}.\dfrac{2}{5}.\dfrac{4}{7}=\dfrac{8}{105}$

$\Rightarrow$ Probability that $x_1+x_2+x_3$ is odd is

$\dfrac{24+12+9+8}{105}=\dfrac{53}{105}$

32. Ans. (C)

Sol. $2x_2 = x_1 + x_3$

$\Rightarrow$ $x_1 + x_3 =$ even for every x_2

even + even $\Rightarrow \left(\frac{1}{3}\cdot\frac{3}{7}\right)\frac{1}{5} = \frac{3}{105}$

odd + odd $\Rightarrow \left(\frac{2}{3}\cdot\frac{4}{7}\right)\frac{1}{5} = \frac{8}{105}$

$\Rightarrow$ probability that x_1, x_2, x_3 are in AP is

$$\frac{3}{105} + \frac{8}{105} = \frac{11}{105}$$

33. Ans. (A)

Sol. P(Problem is solved by at least one of them)

$= 1 -$ P(solved by none)

$$= 1 - \left(\frac{1}{2} \times \frac{1}{4} \times \frac{3}{4} \times \frac{7}{8}\right)$$

$$= 1 - \frac{21}{256} = \frac{235}{256}$$

34. Ans. (6)

Sol. Let $P(E_1) = p_1$, $P(E_2) = p_2$, $P(E_3) = p_3$

given that

$p_1(1 - p_2)(1 - p_3) = \alpha$ ......(i)

$p_2(1 - p_1)(1 - p_3) = \beta$ ......(ii)

$p_3(1 - p_1)(1 - p_2) = \gamma$ ......(iii)

and

$(1 - p_1)(1 - p_2)(1 - p_3) = p$ ....(iv)

$$\Rightarrow \frac{p_1}{1 - p_1} = \frac{\alpha}{p}, \ \frac{p_2}{1 - p_2} = \frac{\beta}{p} \ \& \ \frac{p_3}{1 - p_3} = \frac{\gamma}{p}$$

Also $\beta = \dfrac{\alpha p}{\alpha + 2p} = \dfrac{3\gamma p}{p - 2\gamma}$

$\Rightarrow \alpha p - 2\alpha\gamma = 3\alpha\gamma + 6p\gamma$

$\Rightarrow \alpha p - 6p\gamma = 5\alpha\gamma$

$$\Rightarrow \frac{p_1}{1 - p_1} - \frac{6p_3}{1 - p_3} = \frac{5p_1p_3}{(1 - p_1)(1 - p_3)}$$

$\Rightarrow p_1 - 6p_3 = 0$

$$\Rightarrow \frac{p_1}{p_3} = 6$$

35. Ans. (D)

B_1: 1W, 3R, 2B $\quad B_2$: 2W, 3R, 4B $\quad B_3$: 3W, 4R, 5B

Sol.

A = Total drawn balls are drawn & one is white, another is Red

$P(B_2|A)$ is to be determined

$P(B_2|A) =$

$$\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}$$

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$P\left(\frac{A}{B_1}\right) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2}$$

$$P(A|B_2) = \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2}$$

$$P(A|B_3) = \frac{{}^3C_1 \times {}^4C_1}{{}^9C_2}$$

By putting the values

$$P(B_2|A) = \frac{55}{181}$$

36. Ans. (A)

Sol. $B_1\begin{cases}1W\\3R\\2B\end{cases} \quad B_2\begin{cases}2W\\3R\\4B\end{cases} \quad B_3\begin{cases}3W\\4R\\5B\end{cases}$

Probability of 3 drawn balls of same colour

$$= \frac{1}{6} \times \frac{2}{9} \times \frac{3}{12} + \frac{3}{6} \times \frac{3}{9} \times \frac{4}{12} + \frac{2}{6} \times \frac{4}{9} \times \frac{5}{12}$$

$$= \frac{82}{648}$$

37. Ans. (B, D)

Sol. $P(X) = E_1E_2E_3 + E_1E_2\bar{E}_3 + E_1\bar{E}_2E_3 + \bar{E}_1E_2E_3$

$$= \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times \frac{3}{4} + \frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4}$$

$$\Rightarrow P(X) = \frac{1}{4}$$

$$P\left(\frac{X_1^c}{X}\right) = \frac{P(X_1^c \cap X)}{P(X)} = \frac{1/32}{1/4} = \frac{1}{8}$$

P(Exactly two engines are functioning |x)

$$= \frac{7/32}{1/4} = \frac{7}{8}$$

$$P\left(\frac{X}{X_2}\right) = \frac{P(X \cap X_2)}{P(X_2)} = \frac{5/32}{1/4} = \frac{5}{8}$$

$$P\left(\frac{X}{X_1}\right) = \frac{P(X \cap X_1)}{P(X_1)} = \frac{7/32}{1/2} = \frac{7}{16}$$

38. Ans. (A)

Sol. $1 - \frac{{}^6C_1 . 5^3}{6^4} = \frac{91}{216}$

39. **Ans. (A, B)**

Sol. $P(X \cap Y) = P(X).\,P(Y/X)$

$\Rightarrow \quad P(X) = \frac{1}{2}$

Also $P(X \cap Y) = P(Y).P(X/Y)$

$\Rightarrow \quad P(Y) = \frac{1}{3}$

$\Rightarrow \quad P(X \cap Y) = P(X).P(Y)$

$\Rightarrow \quad X, Y$ are independent

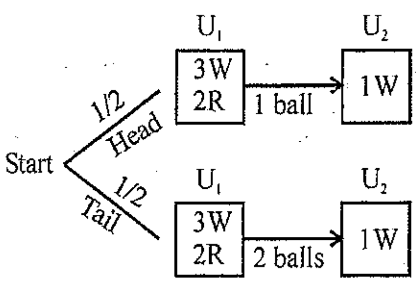
$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}$

$P(X^C \cap Y) = P(Y) - P(X \cap Y) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

$\Rightarrow$ (A, B) are correct.

Solution for Question No. 40 and 41



40. **Ans. (B)**

Sol. Required probability =

$$\frac{1}{2}\left(\frac{3}{5}.1 + \frac{2}{5}.\frac{1}{2}\right) + \frac{1}{2}\left(\frac{{}^3C_2}{{}^5C_2}.1 + \frac{{}^2C_2}{{}^5C_2}.\frac{1}{3} + \frac{{}^3C_1\,{}^2C_1}{{}^5C_2}.\frac{2}{3}\right)$$

$$= \frac{1}{2}\left(\frac{4}{5}\right) + \frac{1}{2}\left(\frac{3}{10} + \frac{1}{30} + \frac{2}{5}\right) = \frac{2}{5} + \frac{11}{30} = \frac{23}{30}$$

41. **Ans. (D)**

Sol. Required probability

$$= \frac{2/5}{2/5 + 11/30} \text{ (using Baye's theorem)}$$

$$= \frac{12}{23}$$

42. **Ans. (A, D)**

Sol. Let $P(E) = x$ & $P(F) = y$

According to given condition

$$x(1-y) + y(1-x) = \frac{11}{25}$$

$$\Rightarrow \quad x + y - 2xy = \frac{11}{25} \qquad ....(i)$$

$$\text{Also, } (1-x)(1-y) = \frac{2}{5}$$

$$\Rightarrow \quad x + y - xy = \frac{23}{25} \qquad ....(ii)$$

from (i) & (ii)

$$xy = \frac{12}{25},\ x + y = \frac{7}{5}$$

Solving this $x = \frac{4}{5}, y = \frac{3}{5}$ or $x = \frac{3}{5}, y = \frac{4}{5}$

43. **Ans. (C)**

Sol. r_1, r_2, r_3 can be from the set (3, 6), (1, 4) or (2,5) which can be done in $2 \times 2 \times 2 = 8$ ways and these can be arranged in 3! ways

$\therefore$ Probability $= \frac{3! \times 8}{216} = \frac{2}{9}$

44. **Ans. (C)**

Sol. C : Correct signal is transmitted

$\bar{C}$: false signal is transmitted

G : Original signal is green

R : Original signal is red

K : Signal recieved at station B is green.

$$P(G/K) = \frac{P(G).P(K/G)}{P(K)}$$

$$= \frac{P(GCC) + P(G\bar{C}\bar{C})}{P(GCC) + P(G\bar{C}\bar{C}) + P(RC\bar{C}) + P(R\bar{C}C)}$$

$$= \frac{\frac{4}{5}\times\frac{3}{4}\times\frac{3}{4} + \frac{4}{5}\times\frac{1}{4}\times\frac{1}{4}}{\frac{4}{5}\times\frac{3}{4}\times\frac{3}{4} + \frac{4}{5}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{5}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{5} + \frac{1}{4}\times\frac{3}{4}}$$

$$= \frac{40}{46} = \frac{20}{23}$$

45. Ans. (A)

Sol. $P(X=3) = P(F\,F\,S) = \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} = \frac{25}{216}$

46. Ans. (B)

Sol. $P(X \le 3)$

$= 1 - P(X \le 2) = 1 - P(X=1) - P(X=2)$

$= 1 - \frac{1}{6} - \frac{5}{6}\cdot\frac{1}{6} = \frac{25}{36}.$

47. Ans. (D)

Sol. $P(x \ge 6 / x > 3) = \frac{P(x \ge 6)}{P(x > 3)}$

$= \frac{(1 - P(x \le 5))}{(1 - P(x \le 3))}$

$$= \frac{1 - \frac{1}{6} - \frac{5}{6}\cdot\frac{1}{6} - \ldots\ldots - \left(\frac{5}{6}\right)^4 \frac{1}{6}}{1 - \frac{1}{6} - \frac{5}{6}\cdot\frac{1}{6} \ldots\ldots \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}} = \frac{25}{36}$$

48. Ans. (B)

Sol. $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Total possible no. of determinants = 16

For unique solution $\Delta \neq 0$

$ad - bc \neq 0$

No. of such determinants = 6

Hence probability $= \frac{3}{8}$

For infinite solution $\Delta = 0 \Rightarrow ad - bc = 0$

No. of such determinants = 10

Hence probability $= \frac{5}{8}$

Hence S(I) is true & S(II) is true but S(II) is not the correct explaination of S(I)

49. Ans. (D)

Sol. Let B have n number of outcomes.

so $P(B) = \frac{n}{10}$, $P(A) = \frac{4}{10}$

$P(A \cap B) = \frac{4}{10} \times \frac{n}{10} = \frac{2n/5}{10} \Rightarrow \frac{2n}{5}$ is an integer $\Rightarrow n = 5$ or 10

STATISTICS

1. Ans. (A)

Sol.

x_i	3	4	5	8	10	11
f_i	5	4	4	2	2	3

(P) Mean

(Q) Median

(R) Mean deviation about mean

(S) Mean deviation about median

x_i	f_i	x_if_i	C.F.	$\lvert x_i - \text{Mean}\rvert$
3	5	15	5	3
4	4	16	9	2
5	4	20	13	1
8	2	16	15	2
10	2	20	17	4
11	3	33	20	5
	$\Sigma f_i = 20$	$\Sigma x_if_i = 120$		

$f_i\lvert x_i - \text{Mean}\rvert$	$\lvert x_i - \text{Median}\rvert$	$f_i\lvert x_i - \text{Median}\rvert$
15	2	10
8	1	4
4	0	0
4	3	6
8	5	10
15	6	18
$\Sigma f_i\lvert x_i - \text{Mean}\rvert = 54$		$\Sigma f_i\lvert x_i - \text{Median}\rvert = 48$

(P) Mean $= \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{120}{20} = 6$

(Q) Median $= \left(\frac{20}{2}\right)^{th}$

observation $= 10^{th}$ observation = 5

(R) Mean deviation about mean

$= \frac{\Sigma f_i \lvert x_i - \text{Mean}\rvert}{\Sigma f_i} = \frac{54}{20} = 2,70$

(S) Mean deviation about median

$= \frac{\Sigma f_i \lvert x_i - \text{Median}\rvert}{\Sigma f_i} = \frac{48}{20} = 2.40$