Sample Course – Example Sheet 1

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Problem 1

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + f(2x) = 0$.

Claim — The only such function is $f(x) = 0$.

Proof. With $x = 0$ we immediately have that $f(0) = 0$. Then rearranging the condition, we have $f(x) = -f(x/2)$, which can then be repeatedly applied to obtain

$$f(x) = -f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) = \cdots = (-1)^n f\left(\frac{x}{2^n}\right),$$

for any $n \in \mathbb{N}$. Then since f is continuous, we have

$$f(x) = \lim_{n\to\infty} (-1)^n f\left(\frac{x}{2^n}\right) = f(0) = 0.$$

Thus $f(x) = 0$ is the only possible function, which clearly works. □