Olympaid Graph Theory
Irish Mathematical Olympiad Training

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Basic Definitions

Definition (Graph)

A **graph** $G = (V, E)$ is a set V of **vertices** and a set E of unordered pairs of distinct vertices known as **edges**.

We have a natural way of drawing a graph:

- For each vertex we have a point in the plane,
- For each edge we draw a line between the corresponding pair of vertices.

Example (Path)

The ordered pair (V, E) where $V = \{1, 2, \ldots, 6\}$ and $E = \{(1, 2), (2, 3), \ldots, (5, 6)\}$ is a graph.

This graph is known as P_6, a path on 6 vertices.

Common Graphs

There are some graphs that will appear repeatedly when doing graph theory problems, and we will define them now.

> ### Definition (Path)
>
> We define P_n to be the graph $V = \{1, \ldots, n\}$, and $E = \{(1,2), (2,3), \ldots, (n-1, n)\}$ as shown.
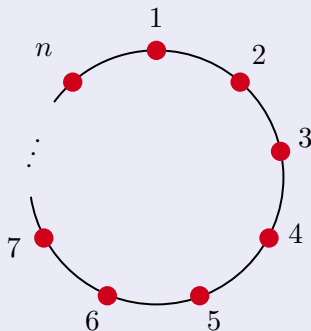>
>
>
> We call this a **path** on n vertices, and say it has **length** $n - 1$.

Common Graphs

Definition (Cycle)

We define C_n (for $n \geq 3$) to be the graph $V = \{1, \ldots, n\}$, and $E = \{(1, 2), \ldots, (n - 1, n), (n, 1)\}$ as shown.
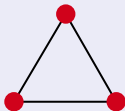
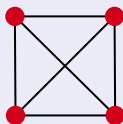We call this the **cycle** on n vertices.

Common Graphs

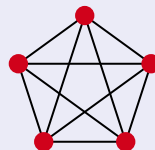## Definition (Complete Graph)

The **complete graph** on n vertices K_n is the graph with vertices $\{1, \ldots, n\}$ and edges $E = \{(i, j) \mid i \neq j \in V\}$.

$n = 3$ $n = 4$ $n = 5$

Note that there is an edge between every pair of vertices.

A Remark on Conventions

Remark. In our definition of a graph, we *don't allow* loops, and there *cannot* be multiple edges between the same set of vertices.

Loops

Multiple edges

You can define graphs where such things are allowed, but for now we will outlaw them. We also note that edges are *unordered pairs*, so for now edges have no direction.

Notation

Notation. If $G = (V, E)$ is a graph, and we have some edge $\{x, y\} \in E$, we will denote it by xy. We will also define $|G| = |V|$, and $e(G) = |E|$.

Example

How many vertices and edges does the graph K_n have?

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Example

How many vertices and edges does the graph K_n have?

Solution. We have $|K_n| = n$ and $e(K_n) = \binom{k}{2}$, as there is an edge between any pair of vertices.