Olympiad Graph Theory

Irish Mathematical Olympiad Training

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Basic Definitions

Definition (Graph)

A **graph** $G = (V, E)$ is a set V of **vertices** and a set E of unordered pairs of distinct vertices known as **edges**.

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Example (Path)

The ordered pair (V, E) where $V = \{1, 2, \ldots, 6\}$ and $E = \{(1, 2), (2, 3), \ldots, (5, 6)\}$ is a graph.

This graph is known as P_6, a path on 6 vertices.

Common Graphs

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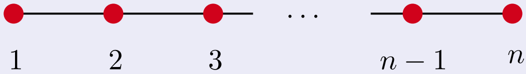
Definition (Path)

We define P_n to be the graph $V = \{1, \ldots, n\}$, and $E = \{(1,2), (2,3), \ldots, (n-1, n)\}$ as shown.

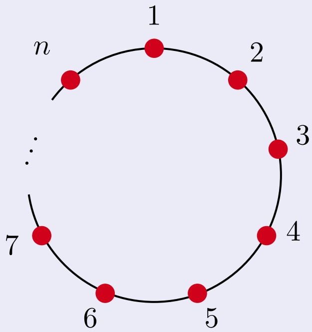


We call this a **path** on n vertices, and say it has **length** $n-1$.

Common Graphs

Definition (Cycle)

We define C_n (for $n \geq 3$) to be the graph $V = \{1, \ldots, n\}$, and $E = \{(1,2), \ldots, (n-1, n), (n, 1)\}$ as shown.

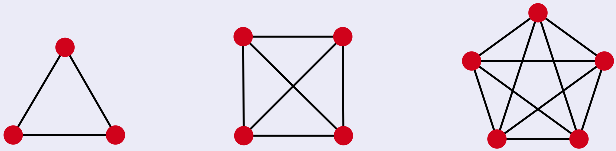


We call this the **cycle** on n vertices.

Common Graphs

Definition (Complete Graph)

The **complete graph** on n vertices K_n is the graph with vertices $\{1, \ldots, n\}$ and edges $E = \{(i, j) \mid i \neq j \in V\}$.

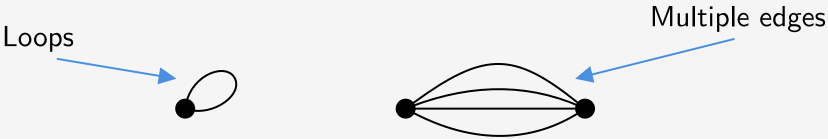


$n = 3$ $\quad$ $n = 4$ $\quad$ $n = 5$

Note that there is an edge between every pair of vertices.

A Remark on Conventions

Remark. In our definition of a graph, we *don't allow* loops, and there *cannot* be multiple edges between the same set of vertices.



You can define graphs where such things are allowed, but for now we will outlaw them. We also note that edges are *unordered pairs*, so for now edges have no direction.

Notation

Notation. If $G = (V, E)$ is a graph, and we have some edge $\{x, y\} \in E$, we will denote it by xy. We will also define $|G| = |V|$, and $e(G) = |E|$.

Example

How many vertices and edges does the graph K_n have?

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Example

How many vertices and edges does the graph K_n have?

Solution. We have $|K_n| = n$ and $e(K_n) = \binom{k}{2}$, as there is an edge between any pair of vertices.

Subgraphs

Now we will define the notion of a *subgraph*, in the natural way.

Definition (Subgraph)

We say that $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Informally, H is a subgraph of G if we can remove vertices and edges from G to get H. Let's look at some examples.

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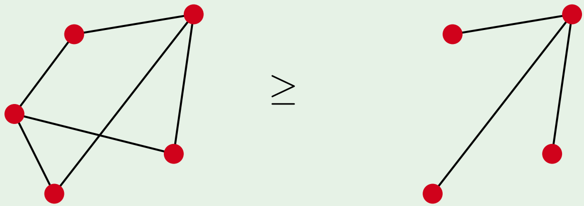
Example

The graph on the right is a subgraph of the graph on the left.



Induced Subgraph

An easy way to get a subgraph is by taking a subset of the vertices and seeing what edges you get from the original graph.

Definition (Inducted Subgraph)

If $G = (V, E)$ is a graph and $X \subseteq V$, the **subgraph inducted by X** is defined to be $G[X] = (X, \{xy \in E : x, y \in X\})$.

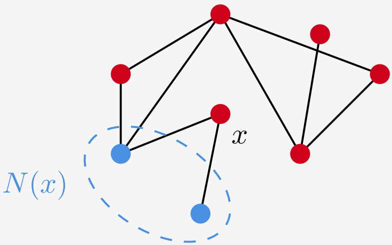
Neighbors and Degree

Now for the following discussion, fix some graph $G = (V, E)$, and let $x \in V$.

Definition (Neighborhood)

If $xy \in E$, then we say that x and y are **adjacent**. We define the **neighborhood** of x to be the set $N(x) = \{y \in V \mid xy \in E\}$ of all vertices adjacent to x.

Note that as in the diagram below, x is not in its own neighborhood.



We can define the neighborhood of a subset of vertices in a similar way.

Neighbors and Degree

Definition (Degree)

We define the **degree** of a vertex x to be $d(x) = |N(x)|$. This is equal to the number of edges that are incident to x.

Definition (Regularity)

A graph G is said to be **regular** if all of the degrees are the same. We say G is k-regular if $d(x) = k$ for all $x \in V$.

Example (Regular and Non-Regular Graphs)

The graphs K_n is $n-1$ regular, and C_n is 2-regular. The graph P_n is not regular.

Neighbors and Degree

Definition (Minimum/Maximum Degree)

Let G be a graph. The **maximum degree** of G, $\Delta(G)$ is defined to be

$$\Delta(G) = \max_{x \in V} d(x).$$

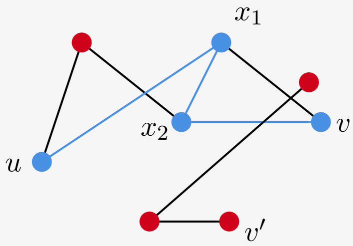
Similarly, we define the **minimum degree** of G, $\delta(G)$ to be

$$\delta(G) = \min_{x \in V} d(x).$$

In a k-regular graph as mentioned above, we have $\Delta(G) = \delta(G) = k$.

Connectivity

We now want to define some notion of *connectivity*.



For example, in the graph above we want to say somehow that u and v are connected, but u and v' are not.

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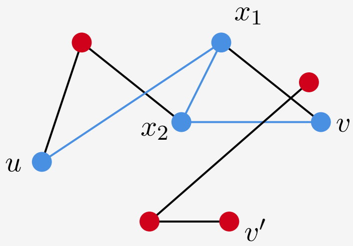

For example, in the graph above we want to say somehow that u and v are connected, but u and v' are not.

Definition (Connectivity)

In a graph $G = (V, E)$, we say that two vertices $u, v \in V$ are **connected** if there is some path between u and b using only edges in E.

Connectivity

Definition (Connected Graph)

If there is a path between any two vertices in G then we say that G is **connected**.

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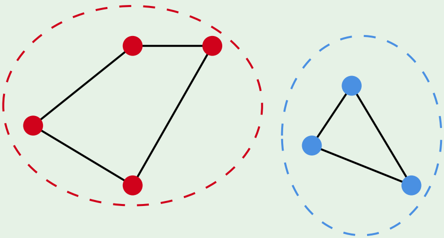
Definition (Connected Components)

A **component** of a graph G is a subgraph in which any two vertices in the subgraph are connected, but no vertex in the subgraph is connected to a vertex outside of the subgraph.

Example

In the graph below, the vertices that are the same colour are in the same component.



Trees

We will now discuss a special class of graph called *trees*. This class is quite restrictive (yet is quite useful), and they have some nice properties.

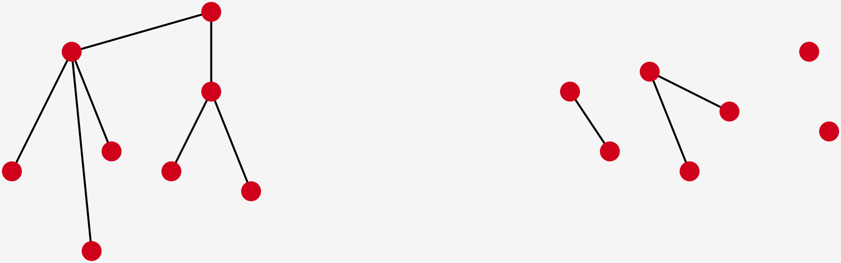
Trees

To define what a tree is, we first need a notion of when a graph is acyclic.

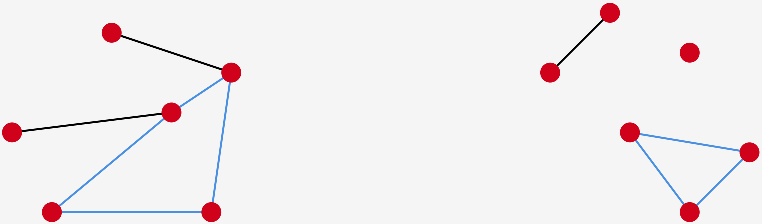
Definition (Acyclic)

A graph G is said to be **acyclic** if it does not contain any subgraph isomorphic to a cycle, C_n.

In the example below, the two graphs are both *acyclic*.



Two *non-acyclic* graphs are shown below. The subgraphs isomorphic to C_4 and C_3 are highlighted.



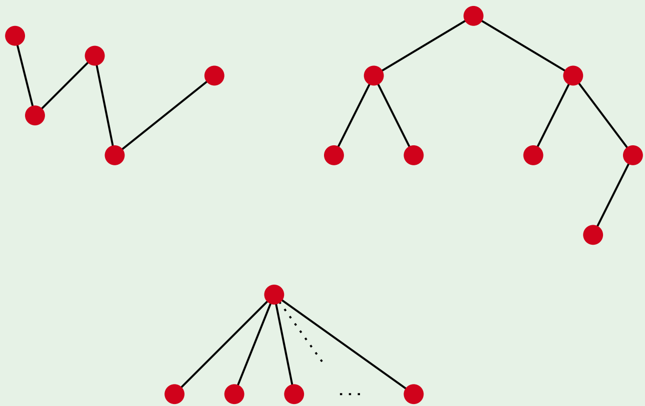
Trees

Definition (Tree)

A **tree** is a connected, acyclic graph.

Example

The following three graphs are trees.



Trees

Proposition (Characterising Trees)

The following are equivalent.

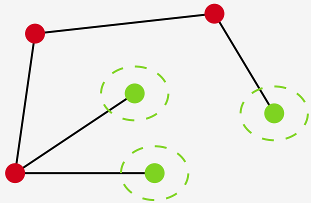
1. G *is a tree.*
2. G *is a maximal acyclic graph (adding any edge creates a cycle).*
3. G *is a minimal connected graph (removing any edge disconnects the graph).*

Trees

Definition (Leaf)

Let G be a graph. A vertex $v \in V(G)$ is a **leaf** if $d(v) = 1$.

For example, the tree below has three leaves.



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In general, trees have leaves.

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Every tree T with $|T| \geq 2$ has a leaf.

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Every tree T with $|T| \geq 2$ has a leaf.

Proof. Let P be a path of maximum length in T, with $P = x_1 x_2 \ldots x_k$. We claim $\deg(x_k) = 1$. Suppose x_k was adjacent to some vertex y other than x_{k-1}. Then if y already occurred in the path the graph would contain a cycle which is a contradiction, and otherwise if y did not occur in the path, it could be added to it, increasing the length and contradicting maximality. Thus there can be no other vertex adjacent to x_k.

Remark. This proof gives us two leaves in T, which is the best we can hope for considering P_n is a tree with exactly two leaves.

Trees

Proposition (Edges of a Tree)

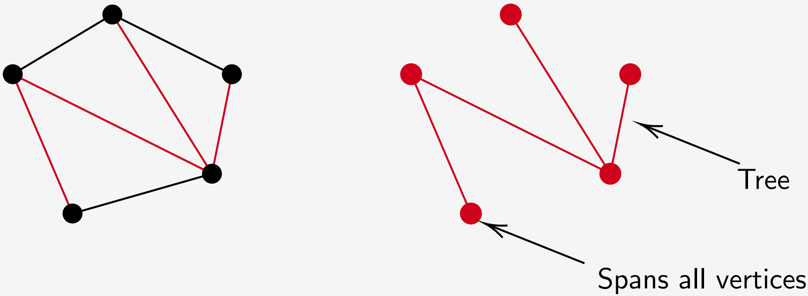
Let T be a tree. Then $e(T) = |T| - 1$.

Proof. Left as an exercise.

Now lets think about trees as subgraphs of other graphs.

Definition (Spanning Tree)

Let $G = (V, E)$ be a graph. We say T is a **spanning tree** of G if T is a tree on V and is a subgraph of G.



Trees

Spanning trees are useful in lots of contexts, one of which is giving a sensible ordering to the vertices of a graph. They are particularly useful because of the following result.

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Every connected graph contains a spanning tree.

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Proposition (Connected Graphs have Spanning Trees)

Every connected graph contains a spanning tree.

Proof. A tree is a minimal connected graph. So take the connected graph and remove edges until it becomes a minimal connected graph. Then this will be a subgraph of the original graph, and will thus be a spanning tree.

Problem Solving

So far we haven't really proved many theorems about graphs, but we will soon change that as we look at some problems. Before we do that though, it's worth talking about some common strategies that are used in graph theory problems.

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One strategy you should always have to hand is using *induction*.

Problem Solving

So far we haven't really proved many theorems about graphs, but we will soon change that as we look at some problems. Before we do that though, it's worth talking about some common strategies that are used in graph theory problems.

One strategy you should always have to hand is using *induction*. Usually inductive proofs in graph theory look like:

1. Suppose that the result you care about hold for $n - 1$.
2. Take a graph with n. Remove things so that it has $n - 1$, then apply the inductive hypothesis.
3. Add back on what you removed, and show that either everything still works or that you can do something to make it work.

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Prove that the edges of a graph can be partitioned into cycles if and only if each vertex has even degree.

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2. Prove that a cycle exists.

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Walkthrough.

1. Suppose the graph can be partitioned into cycles, then show that every vertex has even degree.
2. Prove that a cycle exists.
3. Remove it from the graph and apply the inductive hypothesis.

Example: Euler's Formula

Example (Euler's Formula)

Let G be a connected planar[a] graph with V vertices, E edges and F faces. Then

$$V - E + F = 2.$$

[a] We say that a graph is **planar** if it can be drawn on the plane $\mathbb{R}^2$ that no edges overlap. We will also call the connected components of $\mathbb{R}^2 - G$ the **faces** of the graph.

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Walkthrough.

1. Prove it for when G is a tree.
2. Suppose the graph contained a cycle. Remove it, apply the inductive hypothesis.

Problems!

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