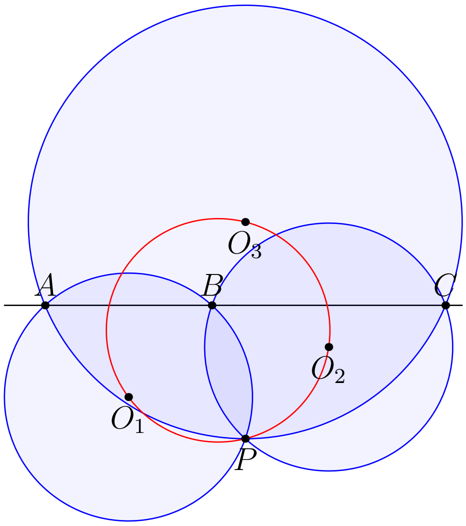
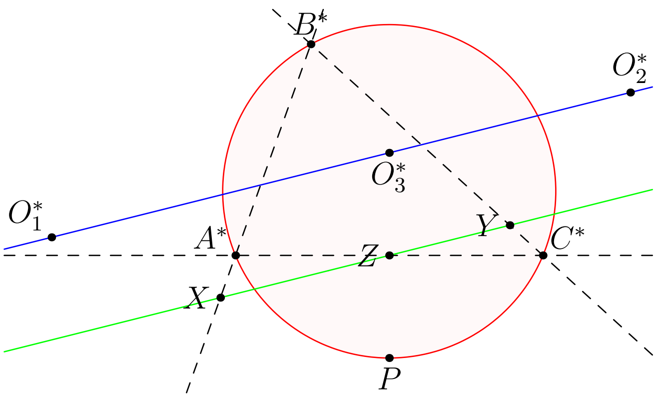
Random Problems

'Euclidean Geometry in Mathematical Olympiads' Chapter 8
August 21, 2020

Problem 1 (8.25)**.** Let A, B, C be three collinear points and P be a point not on this line. Prove that the circumcenters of $\triangle PAB, \triangle PBC$, and $\triangle PCA$ lie on a circle passing through P.



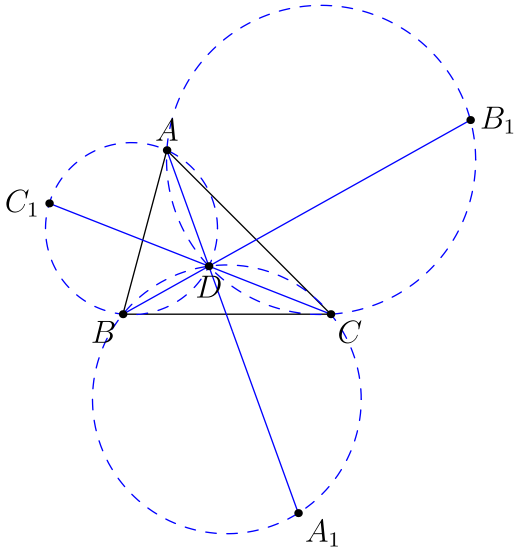
Solution. Inverting the diagram about P, we obtain the following diagram. It suffices to prove that O_1^*, O_2^* and O_3^* are collinear.



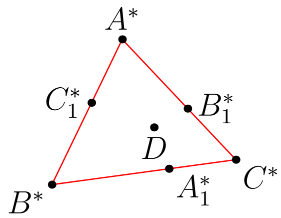
It is well known that O_1^* is the reflection of the point P across A^*B^* and so on for points O_2^* and O_3^*. Let the points X, Y and Z be the foot of the perpendiculars from P to A^*B^*, B^*C^* and A^*C^* respectively. As P lies on $(A^*B^*C^*)$, XYZ is a simson line. Then, the homothety at P with a scale factor of 2 sends X to O_1^*, Y to O_2^* and Z to O_3^*, thus O_1^*, O_2^* and O_3^* are collinear, as required. $\square$

Problem 2 (BAMO 2008/6)**.** A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1$$



Solution. We invert with respect to the the unit circle centered at D.



We note that A_1^*, B_1^* and C_1^* lie on the sides of the triangle $A^*B^*C^*$. Thus we have the well known fact

$$\frac{DA_1^*}{A^*A_1^*} + \frac{DB_1^*}{B^*B_1^*} + \frac{DC_1^*}{C^*C_1^*} = 1.$$

As we inverted about a unit circle, we have the following distance formulas.

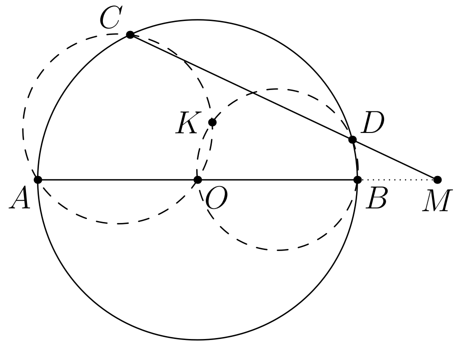
$$DX^* = \frac{1}{DX}, \qquad X^*Y^* = \frac{XY}{DX \cdot DY}.$$

Then, we note that by by the definition of inversion, $1 = AD \cdot A^*D \implies A^*D = \frac{1}{AD}$. Applying this, we have

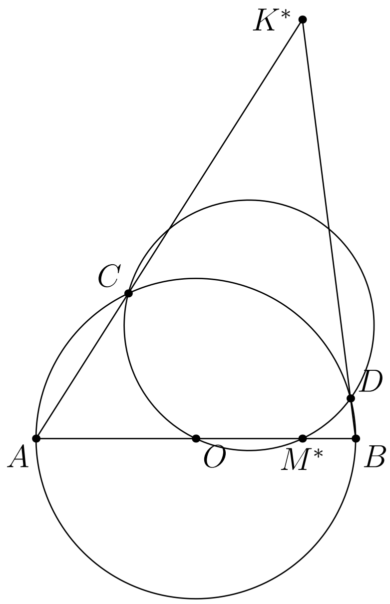
$$\begin{aligned} \frac{DA_1^*}{A^*A_1^*} + \frac{DB_1^*}{B^*B_1^*} + \frac{DC_1^*}{C^*C_1^*} &= \frac{1/DA_1}{AA_1/(DA \cdot DA_1)} + \frac{1/DB_1}{BB_1/(DB \cdot DB_1)} + \frac{1/DC_1}{CC_1/(DC \cdot DC_1)} \\ &= \frac{DA}{AA_1} + \frac{DB_1}{BB_1} + \frac{DC_1}{CC_1} = 1, \end{aligned}$$

as required. $\square$

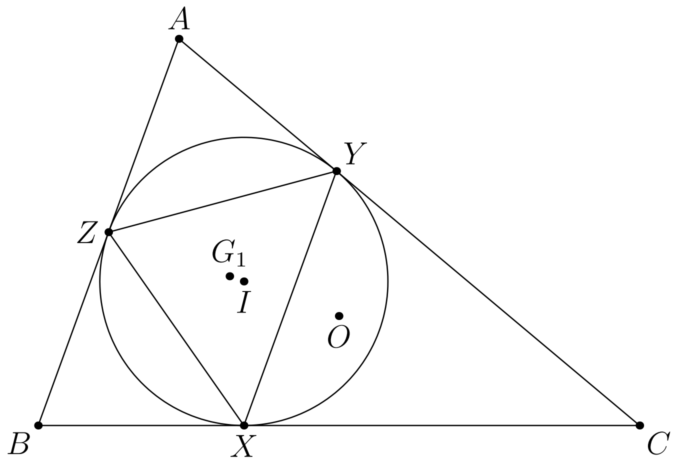
Problem 3 (Iran Olympiad 1996)**.** Consider a semicircle with center O and diameter $\overline{AB}$. A line intersects line AB at M and the semicircle at C and D such that $MC > MD$ and $MB < MA$. Suppose (AOC) and (BOD) meet at a point K other than O. Prove that $\angle MKO = 90°$.



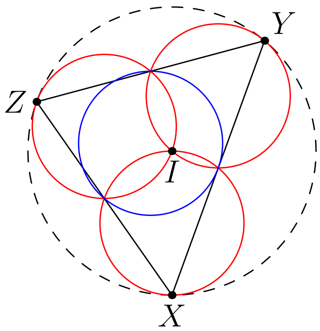
Invert about $(ABCD)$. It suffices to show that $AB \perp M^*K^*$. We have K^* is the intersection of lines AC and BD, and M^* is the intersection of AB with (COD). Then, as O is the midpoint of $\overline{AB}$, C is the foot of the perpendicular from B to AK^*, and D is the foot of the perpendicular from A to BK^*, thus (COD) is the nine-point circle of ABK^*, so $M^*K^* \perp AB$, as required.



Problem 4 (EGMO 8.29)**.** Let ABC be a triangle with incenter I and circumcenter O. Prove that line IO passes through the centroid G_1 of the contact triangle.



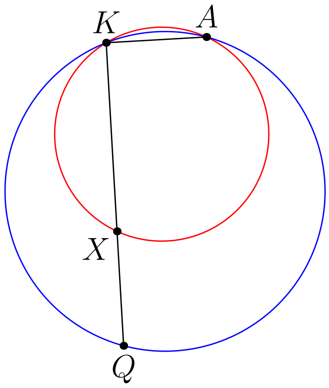
Invert about the incircle. It is well known that the circumcircle of $\triangle ABC$ inverts to the nine-point circle of the contact triangle. Thus the nine-point center lies on the line OI. As the circumcenter of the contact triangle is I, OI is the Euler line for the contact triangle. Thus implies G_1, the centroid of the contact triangle, lies on this line too.



Problem 5 (NIMO 2014)**.** Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

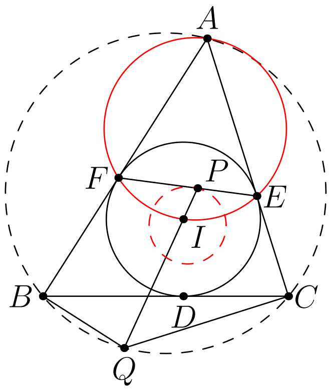
We begin with the following lemma.

Lemma. *Let Γ be a circle with diameter AQ, and let X be some point not on the circle. Then the intersection of the line QX and the circle with diameter AX intersect again on Γ.*



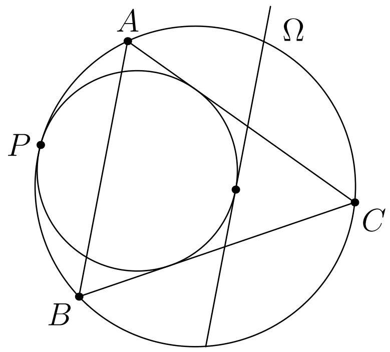
Proof. Let K be the point of intersection between QX and the circle with diameter AX. Then $\angle AKX = \angle AKQ = 90°$, so K lies on Γ. □

Now we return to the original problem.



Inverting about the incircle, we have that P^* is the intersection of QI and (FIE). As $\angle IFA = \angle IEA = 90°$, (FIE) is a circle with diameter AI. Similarly, as $\angle ABQ = \angle ACQ = 90°$, AQ is a diameter of (ABC). Thus we can apply our lemma, and P^* lies on (ABC). As (ABC) inverts to the nine-point circle of DEF, P is a point of intersection between EF and the nine-point circle of DEF, that is, it's the midpoint of EF or the foot of the altitude from D. P can only be the midpoint of EF if I, A and Q are collinear (in which case the foot of the perpendicular from D is the midpoint of EF), thus $DP \perp EF$, as required.

Problem 6 (EGMO 2013/5)**.** Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.



First, take a homothety from P sending Q to some point Q' on Γ.