Exercise 3.1 Let V be an n-dimensional complex vector space. If L is a Hermitian matrix then V has an orthonormal basis consisting of eigenvectors of L.

Proof. First we must show that L ***has*** an eigenvector. Then we will show how to extend it to a full basis. For this first step, L can be any matrix, not necessarily Hermitian.

$p_L(\lambda) = \mathbf{det}(L - \lambda I)$ is a polynomial in λ known as the characteristic polynomial of L. By the Fundamental Theorem of Algebra, there are complex roots λ_i of $p_L(\lambda)$ of multiplicity p_i such that $p_L(\lambda) = (\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$ where r is a positive integer and $p_1 + \cdots + p_r = n$. In particular, $p_L(\lambda)$ has at least one root, λ_1.

By the Lemma, below, there is an eigenvector $|\lambda_1\rangle$ having λ_1 as an eigenvalue. This proves that every matrix L has at least one non-zero eigenvector. WLOG we can assume $|\lambda_1\rangle$ is a unit vector because we can, if necessary, divide $|\lambda_1\rangle$ by its magnitude and it will still be an eigenvector having λ_1 as its eigenvalue.

Define the null space $\mathrm{N} = \{|v\rangle : \langle v|\lambda_1\rangle = 0\}$. It is easy to see that N is a vector subspace of V.

Claim dim N = $n - 1$:

$|\lambda_1\rangle$ can be extended to a basis $\{|\lambda_1\rangle, |e_2\rangle, \cdots, |e_n\rangle\}$ of V, and the sub-basis vectors $\{|e_2\rangle, \cdots, |e_n\rangle\}$ belong to N because $\langle e_i|\lambda_1\rangle = 0$ for all i. ✔

Claim $L\mathrm{N} \subseteq$ N:

Let $|v\rangle \in \mathrm{N}$. We need to show that $L|v\rangle \in \mathrm{N}$. Since L is Hermitian, $L = L^\dagger$, the transpose of the complex conjugate of L. Thus, $L|v\rangle = \langle v|L^\dagger = \langle v|L$. So, we need to show that $\langle v|L|\lambda_1\rangle = 0$:

$$\langle v|L|\lambda_1\rangle = \langle v|\lambda_1|\lambda_1\rangle = \lambda_1\langle v|\lambda_1\rangle = 0 \quad ✔$$

Let $L_2 = L$ restricted to N. Repeating our logic above, $p_L(L_2)$ has a root λ_2 that is an eigenvalue of L_2 with corresponding unit eigenvector $|\lambda_2\rangle$. Since $|\lambda_2\rangle \in \mathrm{N}$, $\langle \lambda_1|\lambda_2\rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$.

Restricting L to the (n – 2)-dimensional null space of L_2, as above we generate unit eigenvector $|\lambda_3\rangle$ such that $|\lambda_3\rangle \perp |\lambda_2\rangle$, and since $|\lambda_3\rangle \in$ N, $|\lambda_3\rangle \perp |\lambda_1\rangle$ also.

Continuing this process, we eventually obtain the orthonormal basis $\{ |\lambda_i\rangle\text{:}\ i = 1, \cdots, n \}$. ■

Lemma Every real or complex matrix A has at least one (possibly complex) eigenvector corresponding to the root λ_1 of the characteristic polynomial.

Proof. We seek a non-zero vector $|x\rangle$ that satisfies $A|x\rangle = \lambda_1 |x\rangle$. Let $\{|e_i\rangle\}$ be an orthonormal basis for V. Every matrix A = (a_{ij}) represents a linear transformation $T: V \to V$ defined by $T|e_i\rangle = \sum_j a_{ij}|e_j\rangle$ on the basis vectors and then extended linearly to all of V.

A linear transformation T is **singular** if dim TV < n, and T is singular iff det A ≠ 0.

Set $B = A - \lambda I = (b_{ij})$. Since $\mathbf{det}\, B$ = 0, B is singular. That means that the n-equations in n-unknowns,

$$\left\{ \begin{array}{c} b_{11}v_1 + \cdots + b_{1n}v_n = 0 \\ \vdots \\ b_{n1}v_1 + \cdots + b_{nn}v_n = 0 \end{array} \right. \quad \Leftrightarrow \quad B|v\rangle = |0\rangle, \text{ where } |v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

has redundant rows. In theory, one or more rows can be eliminated by row reduction. Specifically, let m = dim TV. Assuming A ≠ 0, then 0 < m < n, and so n – m unknowns are redundant and thus are free variables that can be given any value. We set the free variables equal to 1 and then solve the remaining equations for the unique values of $v_1 - v_m$. Then $|v\rangle$ is a non-zero vector that satisfies

$$0 = B|v\rangle = (A - \lambda_1 I)|v\rangle = A|v\rangle - \lambda_1|v\rangle \quad \Rightarrow \quad A|v\rangle = \lambda_1|v\rangle .$$

That is, we have found an eigenvector having λ_1 as an eigenvalue. ■