Exercise 3.1 Let V be an n-dimensional complex vector space. If L is a Hermitian matrix then V has an orthonormal basis consisting of eigenvectors of L.

Proof. First we must show that L ***has*** an eigenvector. Then we will show how to extend it to a full basis. For this first step, L can be any matrix, not necessarily Hermitian.

$p_L(\lambda) = \mathbf{det}(L - \lambda I)$ is a polynomial in λ known as the characteristic polynomial of L. By the Fundamental Theorem of Algebra, there are complex roots λ_i of $p_L(\lambda)$ of multiplicity p_i such that $p_L(\lambda) = (\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$ where r is a positive integer and $p_1 + \cdots + p_r = n$. In particular, $p_L(\lambda)$ has at least one root, λ_1.

By the Lemma, below, there is an eigenvector $|\lambda_1\rangle$ having λ_1 as an eigenvalue. This proves that every matrix L has at least one non-zero eigenvector. WLOG we can assume $|\lambda_1\rangle$ is a unit vector because we can, if necessary, divide $|\lambda_1\rangle$ by its magnitude and it will still be an eigenvector having λ_1 as its eigenvalue. Define the null space of $|v_1\rangle$:

$$\mathrm{N}_1 = \left\{ |v\rangle : \langle v|\lambda_1\rangle = 0 \right\}.$$

It is easy to see that N_1 is a vector subspace of V. Claim dim $\mathrm{N}_1 = n - 1$:

Using the Gram-Schmidt Orthogonalization process, $|\lambda_1\rangle$ can be extended to an orthonormal basis $\left\{ |\lambda_1\rangle, |e_2\rangle, \cdots, |e_n\rangle \right\}$ of V, and the basis vectors $\left\{ |e_2\rangle, \cdots, |e_n\rangle \right\}$ belong to N_1 because $\langle e_i|\lambda_1\rangle = 0$ for all i. ✔

Claim $L\mathrm{N}_1 \subseteq \mathrm{N}_1$:

Let $|v\rangle \in \mathrm{N}_1$. Let $|w\rangle = L|v\rangle$. We need to show that $|w\rangle \in \mathrm{N}_1$. Since $\langle w| = (|w\rangle)^\dagger$ and L is Hermitian ($L = L^\dagger$),

$$\langle w|\lambda_1\rangle = (\langle w|)\,|\lambda_1\rangle = (|w\rangle)^\dagger\,|\lambda_1\rangle = (L|v\rangle)^\dagger\,|\lambda_1\rangle$$
$$= (\langle v|L^\dagger)\,|\lambda_1\rangle = \langle v|L^\dagger\,|\lambda_1\rangle = \langle v|\lambda_1|\lambda_1\rangle = \lambda_1\langle v|\lambda_1\rangle = 0 \quad ✔$$

In the Lemma, below, we showed that L is *associated* with a linear transformation T on V. Let T_2 be the linear transformation generated by restricting T to N_1, the $(n - 1)$ dimensional null space of $|v_1\rangle$, and let L_2 be the matrix associated with T_2.

Repeating our logic above, $p_{L_2}(\lambda)$ has a root λ_2 that is an eigenvalue of L_2 with corresponding unit eigenvector $|\lambda_2\rangle$. Since $|\lambda_2\rangle \in \mathrm{N}_1$, $\langle\lambda_1|\lambda_2\rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$.

Let T_3 be the linear transformation generated by restricting T_2 to N_2, the $(n-2)$ dimensional null space of $|v_2\rangle$, and let L_3 be the matrix associated with T_3. As above, we generate unit eigenvector $|\lambda_3\rangle$ such that $|\lambda_3\rangle \perp |\lambda_2\rangle$, and since $|\lambda_3\rangle \in \mathrm{N}_1$, $|\lambda_3\rangle \perp |\lambda_1\rangle$ also. Continuing this process, we eventually obtain the orthonormal basis $\left\{ |\lambda_i\rangle\text{:}\ i = 1,\cdots,n \right\}$. ■

Lemma Every real or complex matrix A has at least one (possibly complex) eigenvector corresponding to the root λ_1 of the characteristic polynomial.

Proof. We seek a non-zero vector $|x\rangle$ that satisfies $A|x\rangle = \lambda_1|x\rangle$. Let $\{|e_i\rangle\}$ be an orthonormal basis for V. Every matrix $A = (a_{ij})$ is **associated** with a linear transformation $T : V \to V$ defined by $T|e_i\rangle = \sum_j a_{ij}|e_j\rangle$ on the basis vectors and then extended linearly to all of V.

A linear transformation T is **singular** if dim $TV < n$, and T is singular iff det $A \neq 0$.

Set $B = A - \lambda I = (b_{ij})$. Since $\mathbf{det}\,B = 0$, B is singular, and T maps V into an m dimensional subspace, where m = dim $TV < n$. The set of n linear equations in n unknowns,

$$B|v\rangle = |0\rangle \Leftrightarrow \left\{ \begin{array}{c} b_{11}v_1 + \cdots + b_{1n}v_n = 0 \\ \vdots \\ b_{n1}v_1 + \cdots + b_{nn}v_n = 0 \end{array} \right. \text{, where } |v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

has $n - m$ redundant rows that can be eliminated by row reduction. There are then $n - m$ free variables that can be given any value. We set the free variables equal to 1 (i.e., non-zero) and then solve the remaining equations for the unique values of $v_1 - v_m$. The resulting solution $|v\rangle$ is a non-zero vector that satisfies

$$0 = B|v\rangle = (A - \lambda_1 I)|v\rangle = A|v\rangle - \lambda_1|v\rangle \quad \Rightarrow \quad A|v\rangle = \lambda_1|v\rangle.$$

That is, we have found an eigenvector having λ_1 as an eigenvalue. ■

Note: We have proven that the complex characteristic root λ_1 is in fact an eigenvalue. Since *L* is Hermitian, then λ_1 is in fact real. If *L*, in addition, has real elements, we just call it symmetric (rather than Hermitian) and the eigenvectors, also, are real because $L\left|v\right\rangle = \lambda_1\left|v\right\rangle$ constitutes a system of real linear equations that can be solved using just addition, subtraction, multiplication, and division.