[1.1] Given

(0) $\langle A|B\rangle \in \mathbb{C}$ (since it is defined to be an inner product)

(1) $\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle = \langle C|B\rangle$

(2) $\langle B|A\rangle = \langle A|B\rangle^*$

Prove

(a) $\{\langle A| + \langle B|\}|C\rangle = \langle A|C\rangle + \langle B|C\rangle$

(b) $\langle A|A\rangle \in \mathbb{R}$

Solution.

(a) From (0), let $\langle C|A\rangle = a + bi$ and $\langle C|B\rangle = c + di$. Then

(3) Claim $\left(\langle C|A\rangle + \langle C|B\rangle\right)^* = \langle C|A\rangle^* + \langle C|B\rangle^*$:

$$\begin{aligned}\left[(a+bi)+(c+di)\right]^* &= \left[(a+c)+(b+d)i\right]^* = (a+c)-(b+d)i \\ &= (a-bi)+(c-di) = (a+bi)^* + (c+di)^*\end{aligned}$$

So,

$$\begin{aligned}\{\langle A| + \langle B|\}|C\rangle &\overset{(2)}{=} \langle C|\{|A\rangle + |B\rangle\}\rangle^* \overset{(1)}{=} \left(\langle C|A\rangle + \langle C|B\rangle\right)^* \\ &\overset{(3)}{=} \langle C|A\rangle^* + \langle C|B\rangle^* \overset{(2)}{=} \langle A|C\rangle + \langle B|C\rangle\end{aligned}$$

(b) From (0), let $\langle A|A\rangle = x + yi$. By (2),

$$x - yi = \langle A|A\rangle^* \overset{(2)}{=} \langle A|A\rangle = x + yi \quad \Rightarrow \quad y = 0$$

$$\Rightarrow \quad \langle A|A\rangle = x \in \mathbb{R}$$