Exercise 3.1 . Let V be an n-dimensional complex vector space. If L is Hermitian then V has an orthonormal basis consisting of eigenvectors of L.

Proof. First we must show that L ***has*** an eigenvector. Then we will show how to extend it to a full basis. For this first step, L can be any matrix, not necessarily Hermitian.

$p_L(\lambda) = \det(L - \lambda \boldsymbol{I})$ is a polynomial in λ, known as the characteristic polynomial of L. By the Fundamental Theorem of Algebra, there are complex roots λ_i of $p_L(\lambda)$ of multiplicity p_i such that $p_L(\lambda) = (\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$ where r is a positive integer and $p_1 + \cdots + p_r = n$. In particular, $p_L(\lambda)$ has at least one root, λ_1.

By footnote (*), below, there is an eigenvector $|\lambda_1\rangle$ having λ_1 as an eigenvalue. This proves that every matrix L has at least one non-zero eigenvector. WLOG we can assume $|\lambda_1\rangle$ is a unit vector because we can, if necessary, divide $|\lambda_1\rangle$ by its magnitude and it will still be an eigenvector having λ_1 as its eigenvalue.

Define the null space $\mathrm{N} = \{|v\rangle : \langle v|\lambda_1\rangle = 0\}$. It is easy to see that N is a vector subspace of V.

Claim dim N = $n - 1$:

$|\lambda_1\rangle$ can be extended to a basis $\{|\lambda_1\rangle, |e_2\rangle, \cdots, |e_n\rangle\}$ of V, and the sub-basis vectors $\{|e_2\rangle, \cdots, |e_n\rangle\}$ belong to N because $\langle e_i|\lambda_1\rangle = 0$ for all i. ✔

Claim $L\mathrm{N} \subseteq$ N:

Let $|v\rangle \in \mathrm{N}$. We need to show that $L|v\rangle \in \mathrm{N}$. Since L is Hermitian, $L|v\rangle = \langle v|L^\dagger = \langle v|L$. So, we need to show that $\langle v|L|\lambda_1\rangle = 0$:

$$\langle v|L|\lambda_1\rangle = \langle v|\lambda_1|\lambda_1\rangle = \lambda_1\langle v|\lambda_1\rangle = 0 \quad ✔$$

Let $L_2 = L$ restricted to N. Repeating our logic above, $p_L(L_2)$ has a root λ_2 that is an eigenvalue of L_2 with corresponding unit eigenvector $|\lambda_2\rangle$. Since $|\lambda_2\rangle \in \mathrm{N}$, $\langle\lambda_1|\lambda_2\rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$.

Restricting L to the (n – 2)-dimensional null space of L_2, as above we generate unit eigenvector $|\lambda_3\rangle$ such that $|\lambda_3\rangle \perp |\lambda_2\rangle$, and since $|\lambda_3\rangle \in \mathrm{N}$, $|\lambda_3\rangle \perp |\lambda_1\rangle$ also.

Continuing this process, we eventually obtain the orthonormal basis $\left\{ |\lambda_i\rangle : \ i = 1, \cdots, n \right\}$. ■

Footnote (*)

First, suppose we have n equations in n unknowns:

$$\left\{ \begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{array} \right. \quad \Leftrightarrow \quad A|x\rangle = |0\rangle$$

By Cramer's Rule, det $A \neq 0 \Leftrightarrow$ there exists a unique vector $|x\rangle$ such that $A|x\rangle = 0$. So, if det A = 0, then while $|x\rangle = |0\rangle$ is a solution, it is not unique. That is, there is an $|x\rangle \neq 0$ such that $A|x\rangle = |0\rangle$.

Now we apply this fact to the characteristic polynomial of matrix L. Let $A = L - \lambda_1 \boldsymbol{I}$. Since λ_1 is a root of $p_L(\lambda)$,

$$\begin{aligned} p_L(\lambda_1) = 0 \quad &\Rightarrow \quad \det A = \det\left(L - \lambda_1 \boldsymbol{I}\right) = p_L(\lambda_1) = 0 \\ &\Rightarrow \quad \exists \text{ non-zero } |x\rangle \text{ such that } A|x\rangle = |0\rangle. \end{aligned}$$

So, $\left(L - \lambda_1 \boldsymbol{I}\right)|x\rangle = \mathrm{A}\,|x\rangle = 0$, or $L|x\rangle = \lambda_1|x\rangle$. That is, $|x\rangle$ is an eigenvector of L, and we denote it as $|\lambda_1\rangle$. Thus, we have found an eigenvector $|\lambda_1\rangle$ having λ_1 as an eigenvalue. ✔