

**Exercise 3.1.** Let  $V$  be an  $n$ -dimensional complex vector space and  $L$  a matrix.  $L$  has at least one eigenvalue, and each distinct eigenvalue has a corresponding (non-zero) eigenvector. In particular,  $L$  has at least one eigenvector. If  $L$  is Hermitian then  $V$  has an orthonormal basis consisting of eigenvectors of  $L$ .

**Proof.** Let  $p_L(\lambda) = \det(L - \lambda I)$ .  $p_L(\lambda)$  is clearly a polynomial in  $\lambda$ , and it is known as the characteristic polynomial of  $L$ . By the Fundamental Theorem of Algebra, there are complex roots  $\lambda_i$  of  $p_L(\lambda)$  of multiplicity  $p_i$  such that

$p_L(\lambda) = (\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$  where  $r$  is a positive integer and  $p_1 + \cdots + p_r = n$ . In particular, there is at least one  $\lambda_i$ , and for all  $i$ ,  $p_L(\lambda_i) = 0$ . For each  $i$ , by footnote (\*), there is a non-zero vector  $|\lambda_i\rangle$  such that  $L|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ . That is,  $\lambda_i$  is an eigenvalue of  $L$  with eigenvector  $|\lambda_i\rangle$ . This proves that every matrix  $L$  has at least one non-zero eigenvector and that each distinct eigenvalue has a corresponding (non-zero) eigenvector.

Now suppose that  $L$  is Hermitian. Then all eigenvalues  $\lambda$  are real. WLOG we can assume  $|\lambda_1\rangle$  is a unit vector,  $\langle \lambda_1 | \lambda_1 \rangle = 1$ . Define the null space

$N = \{ |v\rangle : \langle v | \lambda_1 \rangle = 0 \}$ . It is easy to see that  $N$  is a vector subspace of  $L$ . Since  $\dim \{ \alpha |\lambda_1\rangle : \alpha \in \mathbb{C} \}$  is a 1-dimensional subspace, the orthogonal subspace  $N$  has dimension  $n - 1$ . Claim  $LN \subseteq N$ :

Let  $|v\rangle \in N$ . We need to show that  $L|v\rangle \in N$ . Since  $L$  is Hermitian,

$L|v\rangle \leftrightarrow \langle v | L$ . So, we need to show that  $\langle v | L | \lambda_1 \rangle = 0$ :

$$\langle v | L | \lambda_1 \rangle = \langle v | \lambda_1 | \lambda_1 \rangle = \lambda_1 \langle v | \lambda_1 \rangle = 0 \quad \checkmark$$

Let  $L_2 = L$  restricted to  $N$ . Repeating our logic above,  $p_{L_2}(\lambda)$  has a real root  $\lambda_2$  that is an eigenvalue of  $L_2$  with corresponding unit eigenvector  $|\lambda_2\rangle$ . Since  $|\lambda_2\rangle \in N$ ,  $\langle \lambda_1 | \lambda_2 \rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$ .

Using the  $(n - 2)$ -dimensional null space of  $L_2$  as above we generate  $|\lambda_3\rangle \perp |\lambda_2\rangle$ , and since  $|\lambda_3\rangle \in N$ ,  $|\lambda_3\rangle \perp |\lambda_1\rangle$  also.

Continuing this process, we eventually obtain the orthonormal basis

$$\{ |\lambda_i\rangle : i = 1, \dots, n \}. \quad \blacksquare$$

Footnote (\*)

First, suppose we have  $n$  equations in  $n$  unknowns:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{cases} \Leftrightarrow A|x\rangle = |0\rangle$$

By Cramer's Rule,  $\det A \neq 0 \Leftrightarrow$  there exists a unique vector  $|x\rangle$  such that  $A|x\rangle = 0$ . Since  $A|0\rangle = 0$ ,  $|x\rangle = |0\rangle$  is the unique solution. That is,  $\det A \neq 0$  if and only if  $|x\rangle = |0\rangle$  is the unique solution to  $A|x\rangle = |0\rangle$ .

So, if  $\det A = 0$ , then while  $|x\rangle = |0\rangle$  is a solution it is not unique. That is, there is an  $|x\rangle \neq 0$  such that  $A|x\rangle = |0\rangle$ .

Now we apply this fact to the characteristic polynomial. Fix  $i$  and let  $A = L - \lambda_i I$ . Then

$$\begin{aligned} p_L(\lambda_i) = 0 &\Rightarrow \det A = \det (L - \lambda_i I) = p_L(\lambda_i) = 0. \\ &\Rightarrow \exists \text{ non-zero } |x\rangle \text{ such that } A|x\rangle = |0\rangle. \end{aligned}$$

Define the vector  $|\lambda_i\rangle = |x\rangle \neq |0\rangle$ . Then

$$(L - \lambda_i I)|\lambda_i\rangle = A|x\rangle = 0.$$

That is, for each  $\lambda_i$  there is a vector  $|\lambda_i\rangle \neq 0$  such that  $L|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ .