Finite Difference

Full Notes

Topics Covered:

➢ Introduction to Finite Difference & its Real World Uses.

➢ Finite Difference Operators.

➢ Newton's Forward Interpolation.

➢ Newton's Backward Interpolation.

➢ Lagrange's Interpolation.

➢ Partial Fractions using Lagrange's Interpolation.

➢ Newton's Divide and Difference Interpolation.

➢ Numerical Differentiation in Newton's Forward and Backward Difference.

➢ Numerical Integration - Trapezoidal Rule Simpsons 1/3 & Simpsons 3/8 Rule.

➢ Difference Equation Introduction.

➢ Solution of Difference Equation.

Link for video Lecture: https://bit.ly/finitediff

For Doubts & Queries :

1. YouTube : www.youtube.com/ayushghurka
2. Instagram : https://www.instagram.com/ayushghurka/
3. Facebook : https://www.facebook.com/ayush.ghurka.1/

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CALCULUS OF FINITE DIFFERENCE

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The Calculus of finite difference deals with the change that takes place in the Value of function due to finite change in the independent variable.

Application of finite difference: The finite difference method (FDM) is an approximate method for Solving partial differential Equations. It has been used to Solve a wide range of problems. For Example:

Navier-Stokes differential Equations used to Simulate airflow around an obstruction. In Numerical analysis.

Finite difference Operator's

① Shift Operator: $Ef(x) = f(x+h)$

② Forward difference Operator: $\Delta f(x) = f(x+h) - f(x)$

③ Backward difference Operator: $\nabla f(x) = f(x) - f(x-h)$

④ Central difference Operator: $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

⑤ Average Operator: $\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$

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[$h \rightarrow$ length of interval]

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⑥ Derivative Operator : $Df(x) = \frac{d f(x)}{dx}$

★ Important property of Shift operator:

① $E^2 f(x) = f(x+2h)$ [Proff:- $EEf(x) = E[f(x+h)] = f(x+h+h)$
$= f(x+2h)$]

② $E^{-1} f(x) = f(x-h)$

③ $E^n f(x) = f(x+nh)$

④ $E^{1/2} f(x) = f(x+\frac{h}{2})$

Relation b/w Operator's:

① $\Delta = E - 1$

Proff: $\Delta f(x) = f(x+h) - f(x)$ [$Ef(x) = f(x+h)$]

$\Delta f(x) = Ef(x) - f(x)$

$\Delta f(x) = (E-1) f(x)$

$\boxed{\Delta = E - 1}$ H.P

② ~~Δ~~ $\nabla = 1 - E^{-1}$

Proff: $\nabla f(x) = f(x) - f(x-h)$ [$E^{-1} f(x) = f(x-h)$]

$\nabla f(x) = f(x) - E^{-1} f(x)$

$\nabla f(x) = (1 - E^{-1}) f(x)$

$\boxed{\nabla = 1 - E^{-1}}$

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③ $E = e^{hD}$ where $D = \frac{d}{dx}$

Proof:

$$Ef(x) = f(x+h)$$

using taylo's theorem

$$= f(x) + hf'(x) + h^2 \frac{f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} \cdots$$

$$= f(x) + h\frac{d}{dx}f(x) + \frac{h^2}{2!}\frac{d^2}{dx^2}f(x) + \frac{h^3}{3!}\frac{d^3}{dx^3}f(x) \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) \cdots$$

$$Ef(x) = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right] f(x)$$

$$E = 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots$$

$$\boxed{E = e^{hD}}$$

④ $(1+\Delta)(1-\nabla) = 1$

$$\left[\begin{array}{c} \Delta = E - 1 \\ \& \\ \nabla = 1 - E^{-1} \end{array}\right]$$

$E \times E^{-1}$

$\Rightarrow E^0$

$\Rightarrow 1.$

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(5) $\mu = \frac{1}{2}\left[\epsilon^{1/2} + \epsilon^{-1/2}\right]$ [$\mu \rightarrow$ average operator]

as we know μ (average operator)

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

$$\mu f(x) = \frac{\epsilon^{1/2} f(x) + \epsilon^{-1/2} f(x)}{2}$$

$$\mu f(x) = \frac{(\epsilon^{1/2} + \epsilon^{-1/2}) f(x)}{2}$$

$$\boxed{\mu = \frac{1}{2}(\epsilon^{1/2} + \epsilon^{-1/2})}$$

(6) $\delta = \epsilon^{1/2} - \epsilon^{-1/2}$ [$\delta \rightarrow$ Central difference operator]

$$\delta f(x) = f(x + \frac{h}{2}) - f(x - h/2)$$

$$\delta f(x) = \epsilon^{1/2} f(x) - \epsilon^{-1/2} f(x)$$

$$\delta f(x) = (\epsilon^{1/2} - \epsilon^{-1/2}) f(x)$$

$$\boxed{\delta = \epsilon^{1/2} - \epsilon^{-1/2}}$$

(7) $(\epsilon^{1/2} + \epsilon^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

$(\epsilon^{1/2} + \epsilon^{-1/2})(1+\Delta)^{1/2}$ [$1 + \Delta = \epsilon$]

$= (\epsilon^{1/2} + \epsilon^{-1/2})(\epsilon)^{1/2}$

$= \epsilon^1 + \epsilon^0$

$= \epsilon + 1$

$= 1 + \Delta + 1$

$= 2 + \Delta$ H.P //

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÷ EXAMPLES ÷

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① Evaluate : (i) $\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E\sin(x+h)}$

Solve: $\because \Delta = E - 1$

$$\therefore \left(\frac{(E-1)^2}{E}\right)\sin(x+h) + \frac{(E-1)^2\sin(x+h)}{E\sin(x+h)}$$

$$\Rightarrow \left(\frac{E^2-2E+1}{E}\right)\sin(x+h) + \frac{[E^2-2E+1]\sin(x+h)}{\sin(x+2h)}$$

$$\Rightarrow [E-2+E^{-1}]\sin(x+h) + \frac{\sin(x+h+2h) - 2\sin(x+h+h) + \sin(x+h)}{\sin(x+2h)}$$

$$\Rightarrow \sin(x+h+h) - 2\sin(x+h) + \sin(x+h-h) + \frac{\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)}{\sin(x+2h)}$$

$$\Rightarrow \sin(x+2h) - 2\sin(x+h) + \sin(x) + \frac{\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)}{\sin(x+2h)}$$

$$\left[\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$\Rightarrow 2\sin(x+h)\cos h - 2\sin(x+h) + \frac{2\sin(x+2h)\cos h - 2\sin(x+2h)}{\sin(x+2h)}$$

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$$\Rightarrow 2\sin(x+h)[\cosh-1] + \frac{2\sin(x+2h)[\cosh-1]}{\sin(x+2h)}$$

$$\Rightarrow 2\sin(x+h)[\cosh-1] + 2[\cosh-1]$$

$$\Rightarrow (\cosh-1)[2\sin(x+h)+2]$$

(ii) $\Delta\nabla = \nabla\Delta$

Solve:

$$\cancel{\Delta\nabla f(x) = \Delta(f(x)-f(x-h))}$$
$$\cancel{= \Delta f(x) - \Delta f(x-h)}$$
$$\cancel{= f(x) - f(x+h) - [f(x-h) - f(x-h+h)]}$$
$$\cancel{= f(x) - f(x+h) - (f(x-h) -}$$

$$\Delta\nabla f(x) = \Delta(f(x)-f(x-h))$$
$$= \Delta f(x) - \Delta f(x-h)$$
$$= f(x+h) - f(x) - [f(x-h+h) - f(x-h)]$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h) \quad \text{—(1)}$$

$$\nabla\Delta f(x) = \nabla(f(x+h)-f(x))$$
$$= \nabla f(x+h) - \nabla f(x)$$
$$= f(x+h) - f(x+h-h) - [f(x) - f(x-h)]$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h) \quad \text{—(2)}$$

by eq. (1) & (2) $\quad \Delta\nabla f(x) = \nabla\Delta f(x)$

$$\Delta\nabla = \nabla\Delta$$

Hence Proved.

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(iii) $hD = \log(1+\Delta) = -\log(1-\nabla)$

as we know $E = e^{hD}$ proved

before taking log both side

$$\log E = \log e^{hD}$$

$$\log E = hD$$

$\because$ $E = 1+\Delta$ also $(\nabla = 1 - E^{-1})$

$$\Rightarrow E^{-1} = 1-\nabla$$

$$\Rightarrow \frac{1}{E} = 1-\nabla$$

$$\Rightarrow E = \frac{1}{1-\nabla}$$

$$[E = (1-\nabla)^{-1}]$$

So, $\log(1+\Delta) = hD$ & $\log(1-\nabla)^{-1} = hD$

$\therefore$ $hD = \log(1+\Delta) = \log(1-\nabla)^{-1}$

Hence proved.

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★ Practice Questions : (Solution at :- → youtube.com/ayushghurka)

Prove that :-

① $\left(\frac{\Delta^2}{E}\right) e^x \frac{E e^x}{\Delta^2 e^x} = e^x$

② $\Delta - \nabla = \Delta \nabla$

③ $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

④ $\nabla = 1 - e^{-hD}$

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INTERPOLATION :-

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Interpolation is the process of estimating unknown values that fall between known values. For Example: A straight line passes through two points of known values. You can estimate the point of unknown value using interpolation because it appears to be midway between the other two points.

Newton's Forward Interpolation Formula :-

★ Forward difference table:

Let the function $y = f(x)$ be tabulated for equally spaced value of x.

x:	x_0	x_0+h	x_0+2h	x_0+3h	-------	x_0+nh
	x_0	x_1	x_2	x_3	------	x_n
y:	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	------	$f(x_n)$
	y_0	y_1	y_2	y_3	------	y_n

$h \rightarrow$ (Equal) interval

$\Delta \rightarrow$ forward difference Operator.

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x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

Newton's forward difference interpolation formula :-

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

where $p = \frac{x - x_0}{h}$ i.e $x = x_0 + ph$

Here $y_0, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ are called forward differences.

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Examples :- (FD)

Q) Find the Cubic Polynomial which takes following Values:

x :	0	1	2	3
$f(x)$:	1	2	1	10

find $f(4)$ & $f'(4)$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1	$\Delta Y_0 = 1$	$\Delta^2 Y_0 = -1-1$	
1	2	$\Delta Y_1 = -1$	$= -2$	$\Delta^3 Y_0 = 10-(-2)$
2	1	$\Delta Y_2 = 9$	$\Delta^2 Y_1 = 9-(-1)$	$= 12$
3	10		$= 10$	

$\therefore$ Here we have $x_0 = 0$ $y_0 = 1$ $h = 1$

$$\therefore p = \frac{x-0}{1} = x$$

$$\therefore y = f(x) = Y_0 + p\,\Delta Y_0 + \frac{p(p-1)}{2!}\Delta^2 Y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 Y_0$$

$$= 1 + x(1) + \frac{x(x-1)}{2} \times (-2) + \frac{x(x-1)(x-2)}{6} \times 12$$

$f(x) = 2x^3 - 7x^2 + 6x + 1$ [on solving ↑]

$f(4) = 41$

Now,

$f'(x) = 6x^2 - 14x + 6 \Rightarrow f'(4) = 46$

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Q2 From the table Estimate the Number of students who obtained marks between 40 & 45.

Marks:	30-40	40-50	50-60	60-70	70-80
No. of students:	31	42	51	35	31

Marks below (x)	No. of stud.	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 y$
40	31				
		$\Delta Y_0 = 42$			
50	73		$\Delta^2 Y_0 = 9$		
		$\Delta Y_1 = 51$		$\Delta^3 Y_0 = -15$	
60	124		$\Delta^2 Y_1 = -16$		$\Delta^4 y_0 = 37$
		$\Delta Y_2 = 35$		$\Delta^3 Y_1 = 12$	
70	159		$\Delta^2 Y_2 = -4$		
		$\Delta Y_3 = 31$			
80	190				

Here $x = 45$, $x_0 = 40$, $h = 10$ $\therefore$ $p = \frac{45-40}{10} = 0.5$

By Netwton's F.D Formula:-

$$f(x) = Y_0 + p\Delta Y_0 + p(p-1)\Delta^2 Y_0 + p(p-1)(p-2)\Delta^3 Y_0 + \cdots$$

$$f(45) = 31 + 0.5 \times 42 + \frac{0.5(-0.5)\,9}{2} + \frac{0.5(-0.5)(-1.5)(-25)}{6} + \frac{0.5(-0.5)(-1.5)(-2.5)(37)}{24}$$

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$f(45) = 47.87 \approx 48$ [No. of students having Marks below 45]

$\therefore$ No. of Students b/w 40 & 45

$= 48 - 31 = \underline{\underline{17}}$

Q3 Solve: $\Delta^2 ab^x$

$\Delta(\Delta ab^x) \Rightarrow \Delta(ab^{x+h} - ab^x)$

$\Rightarrow \Delta(ab^x(b^h - 1))$

$\Rightarrow a(b^h - 1)(b^{x+h} - b^x)$

$\Rightarrow ab^x(b^h - 1)^2$ Ans.

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PRACTICE QUESTIONS [Sol^n: youtube.com/ayushghurka]

① Fit a Cubic Polynomial which takes the following values. Hence find $y(4)$

x :	0	1	2	3
y :	1	0	1	10

[$x^2 - 2x^2 + 1$, $y(4) = 33$]

② Compute $y(1.5)$ & $y(8.5)$ from the following data:

x :	1	2	3	4	5	6	7	8
y :	1	8	27	64	125	216	343	512

[$y(1.5) = 3.375$ & $y(8.5) = 614.125$]

③ $\Delta^2 \left(\dfrac{5x+2}{x^2+5x+6} \right)$ for $h = 1$

④ $\Delta^n e^x$ for $h = 1$

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:- Newton's Backward Interpolation :- Formula.

★ Backward difference table:

let the function $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots$ corrosponding to $x_0, x_0+h, x_0+2h \ldots$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_0$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_0$		
x_3	y_3			

Newton's Backward difference interpolation formula :-

$$y = f(x) = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \ldots$$

where $p = \dfrac{x - x_n}{h}$ i.e $x = x_n + ph$

Here $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n \ldots$ are called forward differences.

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-: Examples :- (BD)

① Using Newton backward interpolation formula find $f(42)$ for following data.

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

x	$f(x)=y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	354					
		$\nabla y_1 = -22$				
25	332		$\nabla^2 y_2 = -19$			
		$\nabla y_2 = -41$		$\nabla^3 y_3 = -29$		
30	291		$\nabla^2 y_3 = 10$		$\nabla^4 y_4 = -37$	
		$\nabla y_3 = -31$		$\nabla^3 y_4 = -8$		$\nabla^5 y_5 = 25$
35	260		$\nabla^2 y_4 = 2$		$\nabla^4 y_5 = 6$	
		$\nabla y_4 = -29$		$\nabla^3 y_5 = 0$		
40	231		$\nabla^2 y_5 = 2$			
		$\nabla y_5 = -27$				
45	204					

Here $x = 42$, $h = 5$, $x_n = 45$ $\Rightarrow p = \frac{x - x_n}{h} = \frac{42-45}{5} = -0.6$

$$\therefore f(x) = y_n + p\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_n + \cdots$$

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$$f(42) = 204 + (-0.6)(-27) + \frac{(-0.6)(0.4)(2)}{2} + 0 + \frac{(-0.6)(0.4)(1.4)(2.4)(8)}{24}$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)(3.4)(4.5)}{120}$$

$$\therefore \boxed{f(42) = 219}$$

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Practice Questions [sol^n : youtube.com/ayushghurka]

① Using Newton's Backward Interpolation formula Compute $f(43)$, $f(84)$ from the following table. Fit a Polynomial.

x :	40	50	60	70	80	90
$f(x)$:	184	204	226	250	276	304

[$f(43) = 190$, $f(84) = 287$;
$f(x) = 84 + 2.9x - 0.01x^2$]

② Estimate the population in 1895 and 1925 from the following Statistics:

x :	1891	1901	1911	1921	1931
y :	46	66	81	93	101

[54.85 , 96.84]

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:- LAGRANGE'S INTERPOLATION :-

Lagrange's Interpolation formula is generally used for interpolation of Unequal intervals For any value of x using lagrange's Interpolation Formula $y(x)$ can be given as :-

X :	x_0	x_1	x_2	x_3	] → data points with unequal intervals
Y :	y_0	y_1	y_2	y_3	

$$Y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

So, if we have n-data points Generalised formula will be :-

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots}{(x_1-x_0)(x_1-x_2)\cdots} y_1$$

$$+ \cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})\cdots(x_i-x_n)} y_i$$

$$+ \cdots\cdots + \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots} \cdot y_n$$

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Note: In Inverse lagrange's interpolation ~~put~~ 'x' ki jagha 'y' & 'y' ki jagha 'x' krdena.

:- Example :-

① Using lagrange's interpolation formula find $f(9)$

x :	5	7	11	13	17
y :	150	392	1452	2366	5202

Sol^n: given $x_0 = 5,\ x_1 = 7,\ x_2 = 11,\ x_3 = 13,\ x_4 = 17$

using formula
i.e

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x-x_3)} y_4$$

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$\Rightarrow \boxed{f(9) = 810}$$

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-: PRACTICE QUESTIONS :-

[Sol^n at : youtube.com/ayushghurka]

① Find $f(27)$ using lagrange's interpolation

x :	14	17	31	35
$f(x)$:	68.7	64.0	44.0	39.1

$[f(27) = 49.3]$

② Express the function $\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions, using lagrange's formula.

$$\left[\frac{3}{35}\frac{1}{(x+1)} + \frac{1}{5}\frac{1}{(x-1)} - \frac{13}{10}\frac{1}{(x-4)} + \frac{71}{70}\frac{1}{(x-6)}\right]$$

③ Compute the value of x when $y = 15$

x :	5	6	9	11
y :	12	13	14	16

$[11.5]$

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Finding Partial Fraction Using Lagrange's Interpolation

Ex: we have:-

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Multiplying & dividing the terms by $(x-x_0)$, $(x-x_1)$, $(x-x_2)$

i.e

$$f(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x-x_0)(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x-x_1)(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x-x_2)(x_2-x_0)(x_2-x_1)} y_2$$

$$f(x) = (x-x_0)(x-x_1)(x-x_2)\left[\frac{y_0}{(x-x_0)(x_0-x_1)(x_0-x_2)} + \frac{y_1}{(x-x_1)(x_1-x_0)(x_1-x_2)}\right.$$

$$\left. + \frac{y_2}{(x-x_2)(x_2-x_0)(x_2-x_1)}\right]$$

$$\boxed{\frac{f(x)}{(x-x_0)(x-x_1)(x-x_2)} = \frac{y_0}{(x-x_0)(x_0-x_1)(x_0-x_2)} + \frac{y_1}{(x-x_1)(x_1-x_0)(x_1-x_2)} + \frac{y_2}{(x-x_2)(x_2-x_0)(x_2-x_1)}}$$

(required Partial fraction)

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Questions:

(1) $\dfrac{\omega^2 + 7\omega}{(\omega+2)(\omega-1)(\omega-4)}$ $\left[\dfrac{-5/9}{\omega+2} - \dfrac{8/9}{\omega-1} + \dfrac{22/9}{\omega-4}\right]$

(2) $\dfrac{3z^2+1}{(z+1)(z-5)^2}$ $\left[\dfrac{1/9}{z+1} + \dfrac{26/9}{z-5} + \dfrac{38/3}{(z-5)^2}\right]$

(3) $\dfrac{z^2+2z+3}{(z-6)(z^2+4)}$ $\left[\dfrac{51/40}{z-6} + \dfrac{-1/40\, z + 7/10}{z^2+4}\right]$

(4) $\dfrac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}$ $\left[\dfrac{3t}{3t^2+4} + \dfrac{-5t+2}{t^2+7}\right]$

(5) $\dfrac{6x^2-3x}{(x-2)(x+4)}$ $\left[\dfrac{3}{x-2} - \dfrac{18}{x+4}\right]$

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Ayush Ghurka

∴ NEWTON'S DIVIDE & DIFFERENCE FORMULA

When another interpolation point is added to the tabulated data, the Lagrangian Coefficients are to be recalculated, resulting the different lagrange's polynomial of higher degree. This difficulty is removed in Newton's divide & difference formula in which polynomial of higher degree is obtained simply by addition of new terms.

divide & difference of ~~any~~ x for any y is given by.

first order →

$$[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$[i = 0, 1, 2 \ldots]$$

second order →

$$[x_i, x_{i+1}, x_{i+2}] = \frac{[x_{i+1}, x_{i+2}] - [x_i, x_{i+1}]}{x_{i+2} - x_i}$$

for n^{th} order →

$$[x_i, x_{i+1}, \ldots, x_{i+n}] = \frac{[x_{i+1}, x_{i+2}, \ldots n] - [x_i, \ldots x_{i+n-1}]}{x_{(i+n)} - x_{(i)}}$$

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The divide & difference Operator is given by Δ

★ Divide & Difference Table

let the function $y = f(x)$ takes the values y_0, y_1, y_2, y_3, y_4 ---- for x_0, x_1, x_2, x_3, ----.

the divide & difference table can be given as :-

x	y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
x_0	y_0			
		$\Delta Y_0 = \frac{Y_1 - Y_0}{x_1 - x_0}$		
x_1	y_1		$\Delta^2 Y_0 = \frac{\Delta Y_1 - \Delta Y_0}{x_2 - x_0}$	
		$\Delta Y_1 = \frac{Y_2 - Y_1}{x_2 - x_1}$		$\Delta^3 Y_0 = \frac{\Delta^2 Y_1 - \Delta^2 Y_0}{x_3 - x_0}$
x_2	y_2		$\Delta^2 Y_2 = \frac{\Delta Y_2 - \Delta Y_1}{x_3 - x_1}$	
		$\Delta Y_3 = \frac{Y_3 - Y_2}{x_3 - x_2}$		
x_3	y_3			

so, divide & difference formula

$$y = f(x) = Y_0 + (x - x_0)\Delta Y_0 + (x - x_0)(x - x_1)\Delta^2 Y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 Y_0 \text{ ----}$$

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:- Example :-

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Q) find $f(4)$ & $f'(4)$ for the following

x :	0	2	3	6
$f(x)$:	648	704	729	792

Solⁿ: The divide & difference table for the given data :-

x	$f(x)$	ΔY	$\Delta^2 Y$	$\Delta^3 Y$
$x_0 = 0$	$Y_0 = 648$	$\Delta Y_0 = \frac{704-648}{2-0} = 28$	$\Delta^2 Y_0 = \frac{25-28}{3-0} = -1$	$\Delta^3 Y_0 = 0$
$x_1 = 2$	$Y_1 = 704$	$\Delta Y_1 = 25$	$\Delta^2 Y_1 = -1$	
$x_2 = 3$	$Y_2 = 729$	$\Delta Y_2 = 21$		
$x_3 = 6$	$Y_3 = 792$			

By Newton's divide & difference interpolation formula

$$f(x) = Y_0 + (x-x_0)\Delta Y_0 + (x-x_0)(x-x_1)\Delta^2 Y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 Y_0 + \dots$$

$$= 648 + (x-0)28 + (x-0)(x-2)(-1) + 0$$

$$= x^2 + 30x + 648$$

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So,

$f(4) = 752$

$f'(4) = 22$

★ **QUESTIONS FOR PRACTICE**

Q1 Fit a cubic Polynomial. Also, find $f(7)$ & $f'(7)$

x :	0	1	2	5
$f(x)$:	2	3	12	147

$[\, f(x) = x^3 + x^2 - x + 2 \,]$

$[\, f(7) = 387 \;,\; f'(7) = 160 \,]$

Q2 Find $f(1.5)$

x :	1	2	3	4	5
$f(x)$:	0	7	28	63	124

$[\, f(1.5) = 2.25 \,]$

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÷ NUMERICAL DIFFERENTIATION ÷

Ayush ghurka

Numerical differentiation or approximate differentiation is used when the function $y=f(x)$ is either given in tabular form or is highly Complex.

★ Numerical differentiation Using Newton's Forward formula

As we know Newton's forward interpolation Formula :-

$$y = f(x) = Y_0 + p\Delta Y_0 + \frac{p(p-1)}{2!}\Delta^2 Y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 Y_0 + \cdots$$

where $p = \frac{x - x_0}{h} \Rightarrow x = x_0 + ph$.

$$y(x_0+ph) = Y_0 + p\Delta Y_0 + \frac{(p^2-p)}{2!}\Delta^2 Y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 Y_0 + \cdots$$

Differentiating wrt x

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{dy}{dp} \times \frac{1}{h}$$

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$$y'(x) = \frac{1}{h}\left[\frac{d}{dp}\left(Y_0 + p\Delta Y_0 + \frac{p(p-1)}{2!}\Delta^2 Y_0 + \cdots\right)\right]$$

$$\boxed{y'(x) = \frac{1}{h}\left[\Delta Y_0 + \frac{2p-1}{2}\Delta^2 Y_0 + \frac{3p^2-6p+2}{6}\Delta^3 Y_0 + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 Y_0 + \cdots\right]}$$

Required formula

Similarly higher order derivatives can be computed.

Here if $p=0$ we get

** $$y'(x) = \frac{1}{h}\left[\Delta Y_0 - \frac{\Delta^2 Y_0}{2} + \frac{\Delta^3 Y_0}{3} - \frac{\Delta^4 Y_0}{4} + \frac{\Delta^5 Y_0}{5} - \cdots\right]$$

★ <u>Numerical Differentiation Using Newton's Backward formula</u>

For Newton's Backward formula as we did for Forward formula we have:

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}(2p+1)\nabla^2 y_n + \frac{3p^3+6p+2}{6}\nabla^3 y_n + \cdots\right]$$

where $p = \frac{x - x_n}{h}$

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Example :-

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Q) Find $\frac{dy}{dx}$ at $x = 1.5$ for the following

x :	1.5	2.0	2.5	3.0	3.5	4.0
y :	3.375	7.0	13.625	24	38.875	59

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		$\Delta y_0 = 3.625$				
2.0	7.0		$\Delta^2 y_0 = 3$			
		$\Delta y_1 = 6.625$		$\Delta^3 y_0 = 0.75$		
2.5	13.625		$\Delta^2 y_1 = 3.75$		$\Delta^4 y_0 = 0$	
		$\Delta y_2 = 10.375$		$\Delta^3 y_1 = 0.75$		$\Delta^5 y_0 = 0$
3.0	24		$\Delta^2 y_2 = 4.5$		$\Delta^4 y_1 = 0$	
		$\Delta y_3 = 14.875$		$\Delta^3 y_2 = 0.75$		
3.5	38.875		$\Delta^2 y_3 = 2.25$			
		$\Delta y_4 = 20.125$				
4.0	59					

$$p = \frac{x - x_0}{h} = \frac{1.5 - 1.5}{0.5} = 0$$

Now, $$\frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \dots\right]$$

$$= \frac{1}{0.5}\left[3.625 - \frac{1}{2} \times 3 + \frac{2}{6}(0.75)\right] = \underline{\underline{4.75}}$$

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PRACTICE QUESTIONS (Sol^n: youtube.com/ayush ghurka)

① Determine $y'(0)$ for

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

$[y'(0) = -27.9]$

② Find $f'(x)$ for $x = 15$

x:	15	17	19	21	23	25
$f(x) = \sqrt{x}$:	3.873	4.123	4.359	4.583	4.796	5.0

Compare with exact value

$[f'(15) = 0.128958]$

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-: NUMERICAL INTEGRATION :-

Ayush Ghurka

Numerical integration Constitutes a broad algorithms for Calculating the Numerical Value of a definite integral. Let the definite integral of $f(x)$ from a to b is given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

However Solving this definite integral becomes difficult or practically impossible when:

(i) The antiderivative $F(x)$ Can not be found by elementary means.

(ii) when the integrand $f(x)$ is specified in tabular form.

Hence to Solve these type of Problems Numerical integration Come into play.

* So, for $f(x)$ in the interval a to b

$$\int_a^b f(x)\,dx = \int_a^b y\,dx$$

choosing interval Size $h = \frac{b-a}{n}$, divide

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the interval $[a,b]$ into n intervals by means of $(n+1)$ equally spaced points

$x_0 = a$

$x_i = x_0 + ih \qquad (i = 1, 2, 3, \ldots n-1)$

$x_n = b$

for which $y_i = f(x_i)$ for $i = 0, 1, 2 \ldots, n$

We have Four major rules to Solve the integral :-

(Ayush Ghorka)

(i) <u>Trapezoidal Rule</u> :

$$\int_{x_0 = a}^{x_n = b} y \, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

For any value of 'n' we can use this Rule.

Here, $h = \frac{b-a}{n}$

<u>Example</u>: Using the following data evaluate approximate value of y by trapezoidal Rule.

x:	1	2	3	4	5	6	7
y:	2.105	2.808	3.614	4.604	5.857	7.451	9.467

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From the given data we have

$x_0 = 1$ $\quad x_n = 7$ $\quad$ & $n = 6$

$y_0 = 2.105$ $\quad \therefore h = \frac{7-1}{6} = 1$

$$\therefore \int_1^7 y\,dx = \frac{1}{2}\left[2.105 + 9.467 + 2(2.808 + 3.614 + 4.604 + 5.857 + 7.451)\right]$$

$$= 30.120 \text{ Ans.}$$

② <u>Simpson's $\frac{1}{3}$ Rule</u> :

$$\int_{x_0=a}^{x_n=b} y\,dx = \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + \dots + y_{n-1})\right]$$

For $n = 2$ or should be multiple of 2.

Here, $h = \frac{b-a}{n}$

<u>Example</u>: A river is 80 feet wide. The depth d in feet at a distance x feet from one bank is given by:

x :	0	10	20	30	40	50	60	70	80
d :	0	4	7	9	12	15	14	8	3

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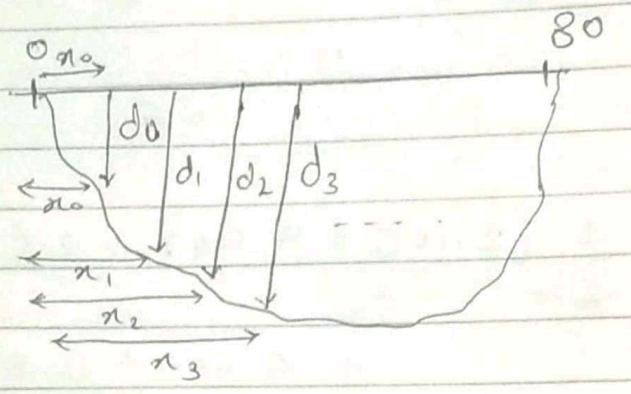
Find approximately the area of cross section of the river.



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$$\text{approximate area} = \int_0^{80} d\,dx$$

using Simpson's Rule

where $n = 8$, $h = \frac{80-0}{8} = 10$

$$\therefore \int_0^{80} d\,dx = \frac{1}{3}(10)\left[0 + 3 + 4(4+9+15+8) + 2(7+12+14)\right]$$

$$= 710 \text{ sq. feet}$$

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③ Simpson's $\frac{3}{8}$ Rule

$$\int_{x_0=a}^{x_n=b} y\,dx = \frac{3h}{8}\left[y_0 + y_n + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1} + y_{n-2})\right]$$

For $n = 3$ or Multiple of 3

Here, $h = \frac{b-a}{n}$

Example: Integrate $\int_0^{45} a(t)\,dt$ from the data given below:

t :	0	5	10	15	20	25	30	35	40	45
$a(t)$:	40	45.25	48.5	51.25	54.35	59.48	61.5	64.3	68.7	70

$x_0 = 0$ $\quad x_n = 45$ $\quad n = 9$

$y_0 = 40$ $\quad h = \frac{45-0}{9} = 5$.

Using Simpson's $\frac{3}{8}$ Rule

$$\int_0^{45} a(t)\,dt = 5\left(\frac{3}{8}\right)\left[40 + 70 + 2(51.25 + 61.5) + 3(45.25 + 48.5 + 54.35 + 59.48 + 64.3 + 68.7)\right]$$

$$= 5\left(\frac{3}{8}\right)(1357.24) \Rightarrow \boxed{2544.825}$$

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~~Example:~~

PRACTICE QUESTIONS [Soln: youtube.com/ayushghurka]

① Given that $y = \log x$ and

x :	4.0	4.2	4.4	4.6	4.8	5.0	5.2
y :	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

evaluate $I = \int_4^{5.2} \log x \, dx$ by

(a) Trapezoidal Rule. [1.8276551]

(b) Simpson's $\frac{1}{3}$ rule. [1.8278472]

(c) Simpson's $\frac{3}{8}$ rule. [1.8278470]

(e) Compare it with exact Value.

② Use trapezoidal Rule to Evaluate (a) $\int_1^{10} \frac{dx}{x}$ for $n=9$

[2.3788]

③ Use Simpson's $\frac{1}{3}$ rule to evaluate $\int_0^{\pi/2} \sqrt{\sin x} \, dx$

for $n=6$ & $h = \frac{\pi}{12}$. [1.18728]

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DIFFERENCE EQUATIONS

In Numerical Solution of ordinary differential Equation, the derivatives are discretized by replacing them by the finite (forward) differences. This give rise to difference Equation at higher order.

Or in other words a difference Equation is a relation between the differences of an unknown function at one or more general values of the argument.

Ex:

$$y_{n+2} + y_{n+1} - 4y_n = 16$$

$$y_{n+1} - y_n = 1$$

REAL WORLD APPLICATION OF DIFFERENCE EQUATIONS

it is a powerful tool for mathematician in Continuous processes. Generally difference Equation's are used in :

1. Theory of Probability.
2. Electrical Network.
3. Statistical Problems. etc.

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Order of difference Equation:

$\Rightarrow$ $\dfrac{\text{largest argument} - \text{Smallest argument}}{\text{unit of increment}}$.

Ex: Find the order of following difference equations:

(i) $y_{n+2} - y_{n+1} + y_n = 2$

According to formula $\Rightarrow$ $\dfrac{n+2-n}{1} \Rightarrow 2$

order = 2.

(ii) $y_{n+2} - 4y_{n+1} + y_n + y_{n-1} = 16$

According to formula $\Rightarrow$ $\dfrac{(n+2)-(n-1)}{1}$

$\Rightarrow n+2-n+1$

$\Rightarrow 3$

order = 3

* Remember while finding the order of difference equation it must always be expressed in a form free of Δs.

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FORMATION OF DIFFERENCE EQUATION

let's learn it directly by an example [For clear understanding watch video on: youtube.com/ayushghurka]

Ex 1: Form difference equation corresponding to the family of curves by eliminating the arbitrary constants.

(a) $y_n = A3^n + B5^n$

$$y_n = A3^n + B5^n$$

$$y_{n+1} = A3^{n+1} + B5^{n+1}$$

$$y_{n+2} = A3^{n+2} + B5^{n+2}$$

Eliminating A & B

$$\begin{vmatrix} y_n & 3^n & 5^n \\ y_{n+1} & 3^{n+1} & 5^{n+1} \\ y_{n+2} & 3^{n+2} & 5^{n+2} \end{vmatrix} = 0$$

$$\begin{vmatrix} y_n & 3^n & 5^n \\ y_{n+1} & 3^n \cdot 3 & 5^n \cdot 5 \\ y_{n+2} & 3^n \cdot 9 & 5^n \cdot 25 \end{vmatrix} = 0$$

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taking 3^n & 5^n common

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & 5 \\ y_{n+2} & 9 & 25 \end{vmatrix} = 0$$

Expanding determinant.

$$y_n(75-45) - 1(25y_{n+1} - 5y_{n+2}) + 1(9y_{n+1} - 3y_{n+2}) = 0$$

$$30y_n - 16y_{n+1} - 2y_{n+2} = 0$$

Required difference Equation.

(b) $$y = ax + bx^2 \quad \text{—(i)}$$

we have

$$\Delta y = a\Delta x + b\Delta x^2$$
$$= a[x+1-x] + b[(x+1)^2 - x^2]$$
$$= a + b[2x+1]$$

and $$\Delta^2 y = \Delta(\Delta y)$$
$$= \Delta[a + b(2x+1)]$$
$$= \Delta a + b\Delta(2x+1) \quad (\because \Delta a = 0)$$
$$= b[2(x+1-x)+0] \quad (\Delta 1 = 0)$$

$$b = \frac{\Delta^2 y}{2}$$

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$$a = \Delta y - \frac{\Delta^2 y}{2}(2x+1)$$

Substituting the values of a & b in (i)

$$y = \left[\Delta y - \frac{\Delta^2 y}{2}(2x+1)\right]x + \frac{\Delta^2 y}{2}x^2$$

$$2y = 2x\Delta y - x\Delta^2 y(2x+1) + \Delta^2 y x^2$$

$$2y = 2x\Delta y - \Delta^2 y\left(x(2x+1) - x^2\right)$$

$$2y = 2x\Delta y - \Delta^2 y(2x^2 + x - x^2)$$

$$2y = 2x\Delta y - \Delta^2 y(x^2 + x)$$

$$\Delta^2 y(x^2+x) - 2x\Delta y + 2y = 0$$

Required Eq.

it can be written as

$$(x^2+x)y_{x+2} - 2x y_{x+1} + 2y_x = 0$$

PRACTICE QUESTIONS : [Sol^n : youtube.com/ayushghurka]

(i) $y_n = A2^n + B(-3)^n$

$$[y_{n+2} + y_{n+1} - 6y_n = 0]$$

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SOLUTION OF DIFFERENCE EQUATION

it is an expression for y_n which satisfies the given difference Equation.

LINEAR DIFFERENCE EQUATION :-

an Equation of the form

$$(a_0 E^r + a_1 E^{r-1} + a_2 E^{r-2} + \dots + a_{n-1} E^{r-n-1} + a_n) y_n = f(n)$$

where
$a_0, a_1, a_2, a_3 \dots a_n$ are Constants is called a linear difference Equation of order r, with Constants Coefficients.

Symbolically it Can be written as :

$$\phi(E) y_n = f(n)$$

So, the function satisfies above eqn is called its Solution.

Solution of Difference Equation Contains two Parts : (i) Complementary function (C.F).
(ii) Particular Integral (P.I).

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let $\phi(E) y_n = f(n)$ be our difference Eq.

Note: (i) if difference equation involves Δ instead of E, use $\Delta = E - I$.

(ii) if it involves y_{n+r} instead of y_n, use

$$y_{n+r} = E^r y_n$$

(i) COMPLEMENTARY FUNCTION (CF)

To obtain the Complementary function we will first Solve $\phi(E) = 0$ by converting it into auxillary Equation.

By using the roots of auxillary Equation and the following Rules we will get Complementry function:-

let the roots of auxillary equation $\phi(E) = 0$ be $E = m_1, m_2, m_3 \ldots$

(i) if $m_1, m_2, m_3 \ldots$ are Real and distinct then

$$C.F = c_1 (m_1)^n + c_2 (m_2)^n + c_3 (m_3)^n + \ldots$$

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(ii) if two roots m_1 & m_2 are real & equal

let $m_1 = m_2 = m$

$$CF = (c_1 + c_2 n) m^n + c_3 (m_3)^n + \dots$$

Similarly if three roots are equal
i.e $m_1 = m_2 = m_3 = m$

$$CF = (c_1 + c_2 n + c_3 n^2) m^n + c_4 (m_4)^n + \dots$$

(iii) If two roots m_1 and m_2 are Conjugate Complex number say $\alpha \pm i\beta$ then

$$C.F = r^n (c_1 \cos n\theta + c_2 \sin n\theta) + c_3 (m_3)^n + \dots$$

where

$$r = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

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÷ EXAMPLES :-

① Solve the following difference Equations

(i) $y_{n+3} - 3y_{n+2} + 4y_n = 0$

firstly we will Convert it into 'E' by using

$$y_{n+r} = E^r y_n$$

$$\therefore \quad E^3 y_n - 3[E^2 y_n] + 4y_n = 0$$

$$[E^3 - 3E^2 + 4]y_n = 0$$

auxillary Equation $m^3 - 3m^2 + 4 = 0$

$$(m+1)(m^2 - 4m + 4) = 0$$

$$(m+1)(m-2)^2 = 0$$

$$m = -1, 2, 2$$

So, $CF = c_1(-1)^n + (c_2 + c_3 n)2^n$

$PI = 0$ (NO RHS)

$\therefore$ So, Complete $sol^n \Rightarrow c_1(-1)^n + (c_2 + c_3 n)2^n$

(ii) $\Delta^3 y_n - 5\Delta y_n + 4y_n = 0$

using $\Delta = E - 1$

$$(E-1)^3 y_n - 5(E-1)y_n + 4y_n = 0$$

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$$[(E-1)^3 - 5(E-1) + 4]\, y_n = 0$$

Req. auxillary equation

$$E^3 - 3E^2 - 2E + 8 = 0$$

Putting $[E = m]$

$$m^3 - 3m^2 - 2m + 8 = 0$$

$$(m-2)(m^2 - m - 4) = 0$$

$$m = 2,\ \frac{1 \pm \sqrt{17}}{2}$$

the Complete Solution is

$$y_n = c_1 2^n + c_2 \left(\frac{1+\sqrt{17}}{2}\right)^n + c_3 \left(\frac{1-\sqrt{17}}{2}\right)^n$$

(iii) 5

$$y_{n+2} - y_{n+1} + y_n = 0$$

given that $y_0 = 1$, $y_1 = \frac{1+\sqrt{3}}{2}$

Sol^n:

$$(E^2 - E - 1)\, y_n = 0$$

Auxillary Eq.

$$m^2 - m - 1 = 0$$

$$m = \frac{1 \pm j\sqrt{3}}{2} \quad \text{or} \quad m = \frac{1}{2} \pm \frac{j\sqrt{3}}{2}$$

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So, in this case complete soln is given by

$$y_n = r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

where $r = \sqrt{\alpha^2 + \beta^2} = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2}$

$= 1$

$$\theta = \tan^{-1} \beta/\alpha = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

$\therefore$ required soln

$$y_n = c_1 \cos \frac{n\pi}{3} + c_2 \sin \frac{n\pi}{3} \quad \text{--- (1)}$$

Now for $n=0$ $y_0 = c_1$

but given $y_0 = 1$ $\therefore$ $c_1 = 1$

putting $n = 1$ in (1) we get

$$y_1 = c_1 \cos \frac{\pi}{3} + c_2 \sin \frac{\pi}{3}$$

$$\frac{1+\sqrt{3}}{2} = \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = c_2 \frac{\sqrt{3}}{2}$$

$$\Rightarrow c_2 = 1$$

$\therefore$ Required particular soln

$$\boxed{y_n = \cos \frac{n\pi}{3} + \sin \frac{n\pi}{3}}$$

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PRACTICE QUESTION'S [sol^n at youtube.com/ayushghurka]

① $(\Delta^2 - 3\Delta + 2)\, y_n = 0$ $\quad [\, y_n = c_1 2^n + c_2 3^n \,]$

② $y_n = y_{n-1} + y_{n+2}$ $\quad \left[y_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$

③ $y_{n+2} - 6y_{n+1} + 9y_n = 0$ when $y_0 = 1$ and $y_1 = 0$

$[\, y_n = (1-n)3^n \,]$

④ $y_{x+2} - 4y_{x+1} + 13y_x = 0$

$[\, y_x = (13)^{x/2} [c_1 \cos x\theta + c_2 \sin x\theta]$

$\theta = \tan^{-1}\frac{3}{2} \,]$

(ii) PARTICULAR INTEGRAL

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★ Rules for finding Particular Integral (PI)

let $\phi(E)\, y_n = f(n)$ be the given difference Equation

then $\boxed{PI = \frac{1}{\phi(E)} f(n)}$

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Case I If $f(n) = a^n$, then

$$PI = \frac{1}{\phi(E)} a^n = \frac{1}{\phi(a)} a^n$$

Provided $\phi(a) \neq 0$ (i.e replace E by a)

if $\phi(a) = 0$ then $(E-a)$ must be a factor of $\phi(E)$ and we proceed as follow's :-

$$\frac{1}{(E-a)} \cdot a^n = n a^{n-1}$$

$$\frac{1}{(E-a)^2} a^n = \frac{n(n-1)}{2!} a^{n-2}$$

$$\frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)}{3!} a^{n-3}$$

and so on.

Ex: $y_{n+2} - 4y_n = 2^n$

Converting it into 'E' by using $y_{n+r} = E^r y_n$

$$(E^2 - 4) y_n = 2^n$$

AE is $E^2 - 4 = 0 \Rightarrow E = \pm 2$

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$$CF = c_1 2^n + c_2 (-2)^n$$

$$PI = \frac{1}{E^2-4} 2^n$$

$$= \frac{1}{(E-2)(E+2)} 2^n$$

$$= \frac{1}{E-2} \left[\frac{1}{(E+2)} 2^n \right]$$

$$= \frac{1}{E-2} \times \frac{1}{4} \times 2^n$$

$$= \frac{1}{4} \left[\frac{1}{(E-2)} 2^n \right]$$

↳ (failure of case 1)

$$= \frac{1}{4} \times n 2^{n-1}$$

$$PI = n 2^{n-3}$$

Ans.

Complete soln $\Rightarrow$ CF + PI

$\Rightarrow c_1 2^n + c_2 (-2)^n + n 2^{n-3}$

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Case II If $f(n) = n^p$ then

$$PI = \frac{1}{\phi(E)} n^p = \frac{1}{\phi(1+\Delta)} n^p$$

[i.e, replace E by $1+\Delta$ arrange in ascending powers of Δ after making the first term unity]

$$= [\phi(1+\Delta)]^{-1} n^p$$

Expand $[\phi(1+\Delta)]^{-1}$ in ascending order of powers of Δ upto the term containing Δ^p by Binomial theorem.

→ Express $f(n)$ in factorial & use $\Delta[x]^n = n[x]^{n-1}$

Ex: $y_{x+3} - 3y_{x+2} - 4y_{x+1} + 12y_x = 4-x$

In term's of E it will be

$$(E^3 - 3E^2 - 4E + 12) y_x = 4-x$$

A.E

$$m^3 - 3m^2 - 4m + 12 = 0$$
$$m^2(m-3) - 4(m-3) = 0$$
$$(m^2-4)(m-3) = 0$$
$$m = 3, \pm 2$$

$$C.F = c_1 3^x + c_2 2^x + c_3(-2)^x$$

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$$PI = \frac{1}{E^3 - 3E^2 - 4E + 12}(4 - x')$$

$$= \frac{1}{(1+\Delta)^3 - 3(1+\Delta)^2 - 4(1+\Delta) + 12}(4 - x')$$

$$= \frac{1}{6 - 7\Delta + \Delta^3}(4 - x')$$

$$= \frac{1}{6}\frac{1}{\left[1 - \frac{7}{6}\Delta + \frac{\Delta^3}{6}\right]}(4 - x')$$

$$= \frac{1}{6}\left[1 - \left(\frac{7\Delta}{6} + \frac{1}{6}\Delta^3\right)\right]^{-1}(4 - x')$$

$$= \frac{1}{6}\left[1 + \frac{7}{6}\Delta + \cdots\right](4 - x')$$

$$= \frac{1}{6}\left[4 - x' + \frac{7}{6}\Delta(4 - x') + \cdots\right]$$

$$= \frac{1}{6}\left[4 - x - \frac{7}{6}\right]$$

$$= \frac{17 - 6x}{36}$$

$\therefore$ Complete soln: $c_1 3^x + c_2 2^x + c_3(-2)^x + \frac{17 - 6x}{36}$

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Case III If $f(n) = \sin kn$ or $\cos kn$

Method-1 assume that PI is $y_n = A\cos kn + B\sin kn$ substitute in the given Equation and find A & B by comparing Coefficient of $\cos kn$ & $\sin kn$ both the sides

Method-2 Use $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2}$

and then apply Case I.

Ex: Find general soln of $(E^2+4)y_n = \cos\alpha n$.

A.E $m^2 + 4 = 0$

$m = \pm i2$

or

$0 \pm 2i$

$a = 0,\ b = 2$ $\qquad r = \sqrt{a^2+b^2} = 2$

$\tan\theta = \dfrac{2}{0} = \infty \therefore \theta = \dfrac{\pi}{2}$

So, $CF = r^n(c_1\cos n\theta + c_2\sin n\theta)$

$= 2^n\left(c_1\cos\dfrac{n\pi}{2} + c_2\sin\dfrac{n\pi}{2}\right)$

Now,

$PI = \dfrac{1}{E^2+4}(\cos\alpha n)$

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$$= \frac{1}{E^2+4}\left[\frac{1}{2}\left(e^{in\alpha}+e^{-in\alpha}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{E^2+4}e^{in\alpha}+\frac{1}{E^2+4}e^{-in\alpha}\right]$$

$$= \frac{1}{2}\left[\frac{1}{e^{2i\alpha}+4}e^{in\alpha}+\frac{1}{e^{-2i\alpha}+4}e^{-in\alpha}\right]$$

$$= \frac{1}{2}\left[\frac{(e^{-2i\alpha}+4)e^{i\alpha n}+(e^{2i\alpha}+4)e^{-i\alpha n}}{(e^{2i\alpha}+4)(e^{-2i\alpha}+4)}\right]$$

$$= \frac{4\cos\alpha n+\frac{1}{2}\{e^{i\alpha(n-2)}+e^{i\alpha(n-2)}\}}{1+4(e^{2i\alpha}+e^{-2i\alpha})+16}$$

$$= \frac{4\cos\alpha n+\cos\alpha(n-2)}{17+8\cos 2\alpha}$$

Required General soln

$$y_n = 2^n\left(c_1\cos\frac{n\pi}{2}+c_2\sin\frac{n\pi}{2}\right)+\frac{4\cos\alpha n+\cos\alpha(n-2)}{17+8\cos 2\alpha}$$

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Case IV : If $f(n) = a^n F(n)$ where $F(n)$ is a polynomial of degree p then

$$PI = \frac{1}{\phi(E)} a^n F(n)$$

$$= a^n \cdot \frac{1}{\phi(aE)} F(n)$$

Now Proceed as in Case II

Ex: Solve $(E^2 + E - 56) a_n = 2^n (n^2 - 3)$

A.E

$$m^2 + m - 56 = 0$$
$$(m+8)(m-7) = 0$$
$$m = -8, 7$$

$$CF = c_1 (-8)^n + c_2 7^n$$

Now **PI**:

$$PI = \frac{1}{E^2 + E - 56} \{2^n (n^2 - 3)\}$$

$$= \frac{1 \times 2^n}{(2E)^2 + 2E - 56} (n^2 - 3)$$

$$= \frac{2^n}{2(2E^2 + E - 28)} (n^2 - 3)$$

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$$= \frac{2^n}{2\left[2(1+\Delta)^2 + (1+\Delta) - 28\right]} \times (n^2-3)$$

where E is replaced by $(1+\Delta)$

$$PI = \frac{2^n}{2\left[2\Delta^2 + 5\Delta - 25\right]} (n^2-3)$$

$$= -\frac{2^n}{50}\left[1 - \left(\frac{5\Delta + 2\Delta^2}{25}\right)\right]^{-1} (n^2-3)$$

Expanding in binomial series we get

$$= -\frac{2^n}{50}\left[1 - \left(\frac{5\Delta + 2\Delta^2}{25}\right)\right]^{-1} (n^2-3)$$

Expanding in binomial series

$$PI = -\frac{2^n}{50}\left[1 + \frac{5\Delta + 2\Delta^2}{25} + \left(\frac{5\Delta + 2\Delta^2}{25}\right)^2 + \cdots\right] \times (n^2-3)$$

n^2-3 in factorial notation $\{[n]^2 + [n] - 3\}$

$$PI = -\frac{2^n}{50}\left[1 + \frac{5}{25}\Delta + \frac{3}{25}\Delta^2\right]\{[n]^2 + [n] - 3\}$$

$$P.I = -\frac{2^n}{50}\left[\{[n]^2 + [n] - 3\} + \frac{5}{25}\{2[n] + 1\} + \frac{3}{25}\{2\}\right]$$

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$$PI = -\frac{2^n}{50}\left[n^2 + \frac{2}{5}n - \frac{64}{25}\right]$$

general soln :

$$a_n = c_1(-8)^n + c_2(7)^n - \frac{2^{n-1}}{25}\left(n^2 + \frac{2}{5}n - \frac{64}{25}\right)$$

PRACTICE QUESTIONS: (soln: youtube.com/ayushghurka)

① $u_{n+2} - 7u_{n+1} + 10u_n = 12e^{3n} + 4n$

$$\left[u_n = c_1 2^n + c_2 5^n + \frac{12e^{3n}}{e^6 - 7e^3 + 10} - \frac{4n}{2}\right]$$

② $a_{n+2} - 5a_{n+1} + 6a_n = 2n^2 - 6n - 1.$

$$\left[a_n = c_1 2^n + c_2 3^n + n^2\right]$$

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[For clear understanding of this topic please watch the video]
[youtube.com/ayushghurka]

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★ HOW TO EXPRESS IN FACTORIAL NOTATION

Example 1: $2n^2 - 6n - 1$

let $y = 2n^2 - 6n - 1$

let $y = A[n^2] + B[n] + C$ in factorial notation

By using method of Synthetic division we divide by 0, 1

0	2	-6	-1	
×	+ #	$2\times0=0$	$-6\times0=0$	
1	2	-6	-1	⇒ C
	+ #	2		
	2	-4		
	⇓	⇓		
	A	B		

sum → -6; sum → -1

∴ $A = 2$ $B = -4$ & $C = -1$

So, in factorial Notation

$\Rightarrow 2[n^2] - 4[n] - 1$ Ans.

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Ex: $n^2 - 3$

let $y = A[n^2] + B[n] + C$

using Synthetic division we have: 0, 1 to divide

coefficient of n

$\therefore$

0	1	0	−3
	+ #	0	0
1	1	0	−3 → C
	+ #	1	
	1	1	
	↓ A	↓ B	

$\therefore$ $y = [n^2] + [n] - 3$ [required factorial notation]

Ex: $y = 2x^3 - 3x^2 + 3x - 10$

let $y = A[n^3] + B[n^2] + C[n] + D$

By Synthetic division we have to divide by 0, 1, 2

0	2	−3	3	−10
	+ #	0	0	0
1	2	−3	3	−10 → D
	+ #	2	−1	
2	2	−1	2 → C	
	+ #	4		
A ←	2	+3	→ B	

$\therefore$
A = 2
B = 3
C = 2
D = −10

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$\therefore$

$$y = 2[n^3] + 3[n^2] + 2[n] - 10$$

(Required factorial Notation)

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#3 Numerical Analysis || Newton's Backward Interpolation Formula || Backward Difference Table

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#4 Numerical Analysis | Lagranges Interpolation Formula-Lagranges Interpolation for unequal Interval

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#5 Numerical Analysis || Partial Fractions using Lagrange's Interpolation Formula

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#6 Numerical Analysis || Newton's Divide and Difference Interpolation Formula for Unequal Interval

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