

# Set 0

## Problem 1

(a)

Let  $z$  be defined as follows:

$$z = \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Solving the above equation for  $\theta$ , we obtain:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2iz \\ e^{2i\theta} - 1 &= 2ize^{i\theta} \\ e^{2i\theta} - 2ize^{i\theta} - 1 &= 0 \\ e^{i\theta} &= iz \pm \sqrt{1 - z^2} \\ \theta &= -i \log(iz \pm \sqrt{1 - z^2}). \end{aligned}$$

To ensure continuity with the real-valued  $\sin \theta$ , we choose the  $+$  branch:

$$\theta = \arcsin z = -i \log(iz + \sqrt{1 - z^2}).$$

Choosing the  $+$  branch maps the domain  $-1 \leq z \leq 1$  to  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . This can be verified by the following calculation.

If  $-1 \leq z \leq 1$ , using  $a, b \in \mathbb{R}$ , we have:

$$z = \frac{b}{\sqrt{a^2 + b^2}}.$$

Then, we can find  $\theta$  as follows:

$$\begin{aligned} \arcsin z &= -i \log \left( i \frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}} \right) \\ &= -i \left\{ \log \left| i \frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}} \right| + i \arg \left( i \frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}} \right) \right\} \\ &= -i \left\{ \log 1 + i \arctan \frac{b}{a} \right\} \\ &= \arctan \frac{b}{a}. \end{aligned}$$

Therefore,  $\arcsin z = \arctan \frac{b}{a}$ , which is in the range of  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

(b)

Let  $z$  be defined as follows:

$$z = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

Solving the above equation for  $\theta$ , we obtain:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= iz(e^{i\theta} + e^{-i\theta}) \\ e^{2i\theta} - 1 &= iz e^{2i\theta} - iz \\ e^{2i\theta}(1 - iz) &= 1 + iz \\ e^{2i\theta} &= \frac{1 + iz}{1 - iz} \\ \theta &= -\frac{i}{2} \log \frac{1 + iz}{1 - iz} \\ &= \frac{i}{2} \{\log(1 - iz) - \log(1 + iz)\}. \end{aligned}$$

Therefore,

$$\theta = \arctan z = \frac{i}{2} \{\log(1 - iz) - \log(1 + iz)\}.$$

(c)

$$\begin{aligned} i^{\frac{1}{3}} &= e^{i \frac{\frac{\pi}{2} + 2k\pi}{3}} \\ &= e^{i \frac{\pi}{6} + \frac{2k\pi}{3}} \quad (k = 0, 1, 2). \end{aligned}$$

Therefore,

$$i^{\frac{1}{3}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, -i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

## Problem 2

$$e^{in\theta} = \cos n\theta + i \sin n\theta.$$

On the other hand,

$$\begin{aligned}
e^{in\theta} &= (e^{i\theta})^n \\
&= (\cos \theta + i \sin \theta)^n \\
&= \sum_{k=0}^n \binom{n}{k} (\cos^{n-k} \theta) (i \sin^k \theta) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\cos^{n-2k} \theta) (\sin^{2k} \theta) + i \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos^{n-2k-1} \theta) (\sin^{2k+1} \theta).
\end{aligned}$$

Therefore, we get the following:

(a)

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\cos^{n-2k} \theta) (\sin^{2k} \theta),$$

(b)

$$\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos^{n-2k-1} \theta) (\sin^{2k+1} \theta).$$

### Problem 3

$$\begin{aligned}
\sin x/2 \sum_{n=0}^{N-1} \cos nx &= \frac{1}{2} \sum_{n=0}^{N-1} \left\{ \sin \left( \frac{2n+1}{2} x \right) - \sin \left( \frac{2n-1}{2} x \right) \right\} \\
&= \frac{1}{2} \left\{ \sin \left( \frac{2N-1}{2} x \right) - \sin \left( -\frac{x}{2} \right) \right\} \\
&= \frac{1}{2} \left\{ \sin \left( \frac{2N-1}{2} x \right) + \sin \left( \frac{x}{2} \right) \right\} \\
&= \frac{1}{2} \left\{ \sin \left( \frac{N}{2} x + \frac{N-1}{2} x \right) + \sin \left( \frac{N}{2} x - \frac{N-1}{2} x \right) \right\} \\
&= \sin \frac{Nx}{2} \cos \frac{(N-1)x}{2}.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin Nx/2 \cos (N-1)x/2}{\sin x/2}.$$

### Problem 4

Let  $I$  be defined as follows:

$$I = \int_0^{\infty} e^{-at} \sin bt \, dt .$$

Then,

$$\begin{aligned} I &= \int_0^{\infty} e^{-at} \sin bt \, dt \\ &= \left[ \frac{-e^{-at}}{a} \sin bt \right]_0^{\infty} - \int_0^{\infty} \frac{-e^{-at}}{a} b \cos bt \, dt \\ &= \frac{b}{a} \int_0^{\infty} e^{-at} \cos bt \, dt \\ &= \frac{b}{a} \left[ \frac{e^{-at}}{a} \cos bt \right]_0^{\infty} - \frac{b}{a} \int_0^{\infty} \frac{e^{-at}}{a} (-b) \sin bt \, dt \\ &= \frac{b}{a^2} - \frac{b^2}{a^2} I . \end{aligned}$$

Therefore,

$$I = \frac{b}{a^2 + b^2} .$$

## Problem 5

$$\begin{aligned} \int_0^{\infty} \frac{e^{-s}}{s} (1 - e^{-s}) \, ds &= \int_0^{\infty} \int_0^1 e^{-s(t+1)} \, dt \, ds \\ &= \int_0^1 \int_0^{\infty} e^{-s(t+1)} \, ds \, dt \\ &= \int_0^1 \frac{1}{t+1} \, dt \\ &= \ln 2 . \end{aligned}$$

The interchange of the order of integration was justified by Tonelli's theorem, as the integrand is non-negative within the integration domain.

## Problem 6

$e^{-x^2}$  is an even function, so

$$\int_0^{\infty} e^{-x^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \, dx .$$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy \\
&= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta \\
&= 2\pi \int_0^{\infty} e^{-r^2} r \, dr \\
&= 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_0^{\infty} \\
&= \pi.
\end{aligned}$$

Therefore,

$$\int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.$$

## Problem 7