Set 0

Problem 1

(a)

Let z be defined as follows:

$$z = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Solving the above equation for θ, we obtain:

$$e^{i\theta} - e^{-i\theta} = 2iz$$
$$e^{2i\theta} - 1 = 2ize^{i\theta}$$
$$e^{2i\theta} - 2ize^{i\theta} - 1 = 0$$
$$e^{i\theta} = iz \pm \sqrt{1 - z^2}$$
$$\theta = -i\log\left(iz \pm \sqrt{1 - z^2}\right)$$

To ensure continuity with the real-valued $\sin\theta$, we choose the $+$ branch:

$$\theta = \arcsin z = -i\log\left(iz + \sqrt{1 - z^2}\right)$$

Choosing the $+$ branch maps the domain $-1 \le z \le 1$ to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. This can be verified by the following calculation.

If $-1 \le z \le 1$, using $a,\ b \in \mathbb{R}$, we have:

$$z = \frac{b}{\sqrt{a^2 + b^2}}$$

Then, we can find θ as follows:

$$\arcsin z = -i\log\left(i\frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}}\right)$$
$$= -i\left\{\log\left|i\frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}}\right| + i\arg\left(i\frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}}\right)\right\}$$
$$= -i\left\{\log 1 + i\arctan\frac{b}{a}\right\}$$
$$= \arctan\frac{b}{a}$$

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Therefore, $\arcsin z = \arctan \frac{b}{a}$, which is in the range of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b)

Let z be defined as follows:

$$z = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

Solving the above equation for θ, we obtain:

$$e^{i\theta} - e^{-i\theta} = iz(e^{i\theta} + e^{-i\theta})$$
$$e^{2i\theta} - 1 = ize^{2i\theta} - iz$$
$$e^{2i\theta}(1 - iz) = 1 + iz$$
$$e^{2i\theta} = \frac{1 + iz}{1 - iz}$$
$$\theta = -\frac{i}{2} \log \frac{1 + iz}{1 - iz}$$
$$= \frac{i}{2}\{\log(1 - iz) - \log(1 + iz)\}$$

Therefore,

$$\theta = \arctan z = \frac{i}{2}\{\log(1 - iz) - \log(1 + iz)\}$$

(c)

$$i^{\frac{1}{3}} = e^{i\frac{\frac{\pi}{2} + 2k\pi}{3}}$$
$$= e^{i\frac{\pi}{6} + \frac{2k\pi}{3}} \quad (k = 0, 1, 2)$$

Therefore,

$$i^{\frac{1}{3}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \ -i, \ -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Problem 2

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

On the other hand,

$$e^{in\theta} = \left(e^{i\theta}\right)^n$$

$$= \left(\cos\theta + i\sin\theta\right)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}\left(\cos^{n-k}\theta\right)\left(i\sin^k\theta\right)$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k}\left(\cos^{n-2k}\theta\right)\left(\sin^{2k}\theta\right) + i\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1}\left(\cos^{n-2k-1}\theta\right)\left(\sin^{2k+1}\theta\right)$$

Therefore,

(a)

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k}\left(\cos^{n-2k}\theta\right)\left(\sin^{2k}\theta\right)$$

(b)

$$\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1}\left(\cos^{n-2k-1}\theta\right)\left(\sin^{2k+1}\theta\right)$$

Problem 3