

Set 0

Problem 1

(a)

Let z be defined as follows:

$$z = \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Solving the above equation for θ , we obtain:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2iz \\ e^{2i\theta} - 1 &= 2ize^{i\theta} \\ e^{2i\theta} - 2ize^{i\theta} - 1 &= 0 \\ e^{i\theta} &= iz \pm \sqrt{1 - z^2} \\ \theta &= -i \log(iz \pm \sqrt{1 - z^2}). \end{aligned}$$

To ensure continuity with the real-valued $\sin \theta$, we choose the $+$ branch:

$$\theta = \arcsin z = -i \log(iz + \sqrt{1 - z^2}).$$

Choosing the $+$ branch maps the domain $-1 \leq z \leq 1$ to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. This can be verified by the following calculation.

If $-1 \leq z \leq 1$, using $a, b \in \mathbb{R}$, we have:

$$z = \frac{b}{\sqrt{a^2 + b^2}}.$$

Then, we can find θ as follows:

$$\begin{aligned} \arcsin z &= -i \log \left(i \frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}} \right) \\ &= -i \left\{ \log \left| i \frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}} \right| + i \arg \left(i \frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}} \right) \right\} \\ &= -i \left\{ \log 1 + i \arctan \frac{b}{a} \right\} \\ &= \arctan \frac{b}{a}. \end{aligned}$$

Therefore, $\arcsin z = \arctan \frac{b}{a}$, which is in the range of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b)

Let z be defined as follows:

$$z = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

Solving the above equation for θ , we obtain:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= iz(e^{i\theta} + e^{-i\theta}) \\ e^{2i\theta} - 1 &= iz e^{2i\theta} - iz \\ e^{2i\theta}(1 - iz) &= 1 + iz \\ e^{2i\theta} &= \frac{1 + iz}{1 - iz} \\ \theta &= -\frac{i}{2} \log \frac{1 + iz}{1 - iz} \\ &= \frac{i}{2} \{\log(1 - iz) - \log(1 + iz)\}. \end{aligned}$$

Therefore,

$$\theta = \arctan z = \frac{i}{2} \{\log(1 - iz) - \log(1 + iz)\}.$$

(c)

$$\begin{aligned} i^{\frac{1}{3}} &= e^{i \frac{\frac{\pi}{2} + 2k\pi}{3}} \\ &= e^{i \frac{\pi}{6} + \frac{2k\pi}{3}} \quad (k = 0, 1, 2). \end{aligned}$$

Therefore,

$$i^{\frac{1}{3}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, -i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

Problem 2

$$e^{in\theta} = \cos n\theta + i \sin n\theta.$$

On the other hand,

$$\begin{aligned}
e^{in\theta} &= (e^{i\theta})^n \\
&= (\cos \theta + i \sin \theta)^n \\
&= \sum_{k=0}^n \binom{n}{k} (\cos^{n-k} \theta) (i \sin^k \theta) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\cos^{n-2k} \theta) (\sin^{2k} \theta) + i \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos^{n-2k-1} \theta) (\sin^{2k+1} \theta).
\end{aligned}$$

Therefore, we get the following:

(a)

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\cos^{n-2k} \theta) (\sin^{2k} \theta),$$

(b)

$$\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos^{n-2k-1} \theta) (\sin^{2k+1} \theta).$$

Problem 3

$$\begin{aligned}
\sin x/2 \sum_{n=0}^{N-1} \cos nx &= \frac{1}{2} \sum_{n=0}^{N-1} \left\{ \sin \left(\frac{2n+1}{2} x \right) - \sin \left(\frac{2n-1}{2} x \right) \right\} \\
&= \frac{1}{2} \left\{ \sin \left(\frac{2N-1}{2} x \right) - \sin \left(-\frac{x}{2} \right) \right\} \\
&= \frac{1}{2} \left\{ \sin \left(\frac{2N-1}{2} x \right) + \sin \left(\frac{x}{2} \right) \right\} \\
&= \frac{1}{2} \left\{ \sin \left(\frac{N}{2} x + \frac{N-1}{2} x \right) + \sin \left(\frac{N}{2} x - \frac{N-1}{2} x \right) \right\} \\
&= \sin \frac{Nx}{2} \cos \frac{(N-1)x}{2}.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin Nx/2 \cos (N-1)x/2}{\sin x/2}.$$

Problem 4

Let I be defined as follows:

$$I = \int_0^{\infty} e^{-at} \sin bt \, dt .$$

$$\begin{aligned} I &= \int_0^{\infty} e^{-at} \sin bt \, dt \\ &= \left[\frac{-e^{-at}}{a} \sin bt \right]_0^{\infty} - \int_0^{\infty} \frac{-e^{-at}}{a} b \cos bt \, dt \\ &= \frac{b}{a} \int_0^{\infty} e^{-at} \cos bt \, dt \\ &= \frac{b}{a} \left[\frac{e^{-at}}{a} \cos bt \right]_0^{\infty} - \frac{b}{a} \int_0^{\infty} \frac{e^{-at}}{a} (-b) \sin bt \, dt \\ &= \frac{b}{a^2} - \frac{b^2}{a^2} I . \end{aligned}$$

Therefore,

$$I = \frac{b}{a^2 + b^2} .$$

Problem 5

$$\begin{aligned} \int_0^{\infty} \frac{e^{-s}}{s} (1 - e^{-s}) \, ds &= \int_0^{\infty} \int_0^1 e^{-s(t+1)} \, dt \, ds \\ &= \int_0^1 \int_0^{\infty} e^{-s(t+1)} \, ds \, dt \\ &= \int_0^1 \frac{1}{t+1} \, dt \\ &= \ln 2 . \end{aligned}$$

The interchange of the order of integration was justified by Tonelli's theorem, as the integrand is non-negative within the integration domain.

Problem 6

e^{-x^2} is an even function, so

$$\int_0^{\infty} e^{-x^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \, dx .$$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\
&= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \\
&= 2\pi \int_0^{\infty} e^{-r^2} r dr \\
&= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} \\
&= \pi.
\end{aligned}$$

Therefore,

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Problem 7

$$\begin{aligned}
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{e^{-x^2-y^2-z^2}}{x^2+y^2+z^2} &= \int_0^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi r^2 \sin \theta \frac{e^{-r^2}}{r^2} \\
&= 2\pi \int_0^{\infty} dr \int_0^{\pi} d\theta e^{-r^2} \sin \theta \\
&= 4\pi \int_0^{\infty} dr e^{-r^2} \\
&= 2\pi^{\frac{3}{2}}.
\end{aligned}$$

Problem 8

(a)

First, we solve the following differential equation:

$$L \frac{dI(t)}{dt} + RI(t) = 0$$

The solution is:

$$I(t) = I(0)e^{-\frac{R}{L}t}.$$

Next, we solve the following differential equation:

$$L \frac{dI(t)}{dt} + RI(t) = V_0.$$

The solution is:

$$I(t) = \frac{V_0}{R} + \left(I(0) - \frac{V_0}{R} \right) e^{-\frac{R}{L}t}.$$

By the initial condition $I(0) = 0$, we get:

$$I(t) = \frac{V_0}{R} \left(1 - e^{-\frac{R}{L}t}\right).$$

(b)

We assume that particular solution be of the following form:

$$I(t) = A \cos \omega t + B \sin \omega t.$$

This results in:

$$\begin{aligned} L \frac{dI(t)}{dt} + RI(t) &= -LA\omega \sin \omega t + LB\omega \cos \omega t + RA \cos \omega t + RB \sin \omega t \\ &= (RA + LB\omega) \cos \omega t + (-LA\omega + RB) \sin \omega t. \end{aligned}$$

Therefore, we get the following equations:

$$\begin{aligned} RA - LB\omega &= V_0, \\ -LA\omega + RB &= 0. \end{aligned}$$

Solving the above equations, we get:

$$\begin{aligned} A &= \frac{V_0\omega L}{R^2 + \omega^2 L^2}, \\ B &= \frac{V_0 R}{R^2 + \omega^2 L^2}. \end{aligned}$$

Therefore, the particular solution is:

$$I(t) = \frac{V_0\omega L}{R^2 + \omega^2 L^2} \cos \omega t + \frac{V_0 R}{R^2 + \omega^2 L^2} \sin \omega t.$$

The general solution is the sum of the homogeneous solution and the particular solution:

$$I(t) = Ce^{-\frac{R}{L}t} + \frac{V_0\omega L}{R^2 + \omega^2 L^2} \cos \omega t + \frac{V_0 R}{R^2 + \omega^2 L^2} \sin \omega t.$$

C is determined by the initial condition $I(0) = 0$:

$$C = -\frac{V_0\omega L}{R^2 + \omega^2 L^2}.$$

Therefore, the solution is:

$$I(t) = \frac{V_0\omega L}{R^2 + \omega^2 L^2} \left(\cos \omega t - e^{-\frac{R}{L}t} \right) + \frac{V_0 R}{R^2 + \omega^2 L^2} \sin \omega t.$$

Problem 9

(a)

The solution of the homogeneous equation is:

$$y_h = Ae^{ix} + Be^{-ix}.$$

The particular solution is:

$$y_p = 2.$$

Therefore, the general solution is:

$$y = Ae^{ix} + Be^{-ix} + 2.$$

(b)

The solution of the homogeneous equation is:

$$y_h = Ae^{2ix} + Be^{-2ix}.$$

The particular solution is:

$$y_p = e^x.$$

Therefore, the general solution is:

$$y = Ae^{2ix} + Be^{-2ix} + e^x.$$

(c)

The solution of the homogeneous equation is:

$$y_h = Ae^{2x} + Be^x.$$

We assume that the particular solution is of the form:

$$y_p = C \cos x + D \sin x.$$

Then, we get the following equations:

$$\begin{aligned} C + 3D &= 0, \\ -3C + D &= 1. \end{aligned}$$

Solving the above equations, we get:

$$C = \frac{3}{10},$$

$$D = \frac{1}{10}.$$

Therefore, the general solution is:

$$y = Ae^{2x} + Be^x + \frac{3}{10} \cos x + \frac{3}{10} \sin x.$$

Problem 10

If $m > n$, z^{m-n-1} is analytic on \mathbb{C} , so

$$\frac{1}{2\pi i} \oint z^n dz = 0.$$

If $m < n$, z^n is a simple pole at $z = 0$.

The Laurent series expansion of z^{m-n-1} around $z = 0$ is simply z^{m-n-1} itself.

Thus, the residue is:

$$\text{Res}_{z=0} z^{m-n-1} = \begin{cases} 1 & (m = n), \\ 0 & (m \neq n). \end{cases}$$

By the residue theorem, we have:

$$\frac{1}{2\pi i} \oint z^{m-n-1} dz = \begin{cases} 1 & (m = n), \\ 0 & (m \neq n). \end{cases}$$

Therefore,

$$\frac{1}{2\pi i} \oint z^{m-n-1} dz = \delta_{m,n}.$$