Set 1

Problem 1

(a)

$$\begin{aligned} e^{-2x} &= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} - \frac{32x^5}{5!} + \cdots \\ &= \sum_{n=0}^{\infty} (-2)^n \frac{x^n}{n!}. \end{aligned}$$

(b)

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 = \sum_{n=0}^{3} \binom{3}{n} x^n.$$

(c)

If $0 < r < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Therefore,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Problem 2

(a)

$$\begin{aligned} e^{-2} &= e^{-2} - 2e^{-2}(x-1) + \frac{4}{2!}e^{-2}(x-1)^2 - \frac{8}{3!}e^{-2}(x-1)^3 + \cdots \\ &= e^{-2} \sum_{n=0}^{\infty} (-2)^n \frac{(x-1)^n}{n!}. \end{aligned}$$

(b)

$$
\begin{aligned}
(1+x)^3 &= 1+(t+1)^3 \quad \text{where } t = x-1 \\
&= t^3+6t^2+12t+8 \\
&= (x-1)^3+6(x-1)^2+12(x-1)+8.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{1}{1-2x} &= \frac{1}{1-2(t+1)} \quad \text{where } t = x-1 \\
&= \frac{1}{1-2t-2} \\
&= -\frac{1}{1+2t} \\
&= -\sum_{n=0}^{\infty}(-2t)^n \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}2^n(x-1)^n.
\end{aligned}
$$

Problem 3

(a)

$$(3)_2 = 3 \times 4 = 12.$$

(b)

$$\left(\frac{1}{2}\right)_3 = \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} = \frac{15}{8}.$$

(c)

$$(-3)_2 = (-3) \times (-2) = 6.$$

(d)

$$(-3)_4 = (-3) \times (-2) \times (-1) \times 0 = 0.$$

Problem 4

(a)

$$(1-x)^{1/2} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n}{n!} x^n.$$

(b)

$$(2+3x)^{-2} = \left(\frac{1}{2}\right)^{-2} \left(1 + \frac{3}{2}x\right)^{-2} = 4 \sum_{n=0}^{\infty} \frac{(2)_n}{n!} \left(\frac{3}{2}x\right)^n.$$

Problem 5

$$\begin{aligned} E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} &= mc^2(1-x)^{-1/2} \quad \text{where } x = \frac{v^2}{c^2} \\ &= mc^2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} x^n \\ &= mc^2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{v^2}{c^2}\right)^n. \end{aligned}$$

Problem 6

(a)

$$\begin{aligned} (x)_{2n} &= x(x+1)(x+2)\cdots(x+2n-1) \\ &= x(x+2)(x+4)\cdots(x+2n-2)(x+1)(x+3)\cdots(x+2n-1) \\ &= 2^{2n} \frac{x}{2}\left(\frac{x}{2}+1\right)\left(\frac{x}{2}+2\right)\cdots\left(\frac{x}{2}+n-1\right)\frac{x+1}{2}\left(\frac{x+1}{2}+1\right)\cdots\left(\frac{x+1}{2}+n-1\right) \\ &= 2^{2n}\left(\frac{x}{2}\right)_n \left(\frac{x+1}{2}\right)_n. \end{aligned}$$

(b)

$$
\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^2)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{\left(\frac{1}{2}\right)_n (1)_n} \left(\frac{-x^2}{4}\right)^n \qquad \because \text{Problem 6 (a)} \\
&= \sum_{n=0}^{\infty} \frac{1}{\left(\frac{1}{2}\right)_n n!} \left(\frac{-x^2}{4}\right)^n .
\end{aligned}
$$

Problem 7

$$
\begin{aligned}
\cos(x+y) + i\sin(x+y) = e^{i(x+y)} &= e^{ix}e^{iy} \\
&= (\cos x + i\sin x)(\cos y + i\sin y) \\
&= \cos x\cos y - \sin x\sin y + i(\cos x\sin y + \sin x\cos y).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\cos(x+y) &= \cos x\cos y - \sin x\sin y, \\
\sin(x+y) &= \cos x\sin y + \sin x\cos y.
\end{aligned}
$$

Problem 8

First, we compute the following integral:

$$
\int_0^x \frac{1}{\sqrt{1-t^2}}dt.
$$

Using the substitution $t = \sin u$, we have $dt = \cos u du$ and the limits of integration transform as follows: $t = 0 \to u = 0$,and $t = x \to u = \arcsin x$. Substituting, we get:

$$
\int_0^x \frac{1}{\sqrt{1-t^2}}dt = \int_0^{\arcsin x} \frac{1}{\cos u}\cos u du = \int_0^{\arcsin x} du = \arcsin x.
$$

Next, we expand $\arcsin x$ as a power series:

$$
\begin{aligned}
\arcsin x &= \int_0^x \frac{1}{\sqrt{1-t^2}} dt \\
&= \int_0^x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} (t^2)^n dt \\
&= \sum_{n=0}^{\infty} \int_0^x \frac{\left(\frac{1}{2}\right)_n}{n!} t^{2n} dt \\
&= \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{\left(\frac{1}{2}\right)_n}{n!} x^{2n+1} \\
&= x \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n} \frac{(1/2)_n}{n!} x^n.
\end{aligned}
$$

Problem 9

(a)

$$
\begin{aligned}
\left(x \frac{\mathrm{d}}{\mathrm{d}x} - a\right) y(x) = 0 \\
x \frac{\mathrm{d}}{\mathrm{d}x} y(x) &= a y(x) \\
\int \frac{1}{y}\,\mathrm{d}y &= a \int \frac{1}{x}\,\mathrm{d}x \\
\ln y &= a \ln x + C' \quad \text{where } C' \text{ is const} \\
y &= C x^a \quad \text{where } C = e^{C'}.
\end{aligned}
$$

(b)

$$
x \frac{\mathrm{d}}{\mathrm{d}x} y(x) - a y(x) = c x^b.
$$

The homogeneous solution is given by $y_h = Cx^a$. For the particular solution, assume $y_p = Kx^b$. Substituting y_p into the differential equation yields:

$$
\begin{aligned}
bKx^b - aKx^b &= cx^b \\
bK - aK &= c \\
K &= \frac{c}{b-a}.
\end{aligned}
$$

Thus, the general solution is:

$$
y = Cx^a + \frac{c}{b-a} x^b.
$$

(c)

Applying $(\theta - a)$ to x^b, we get:

$$(\theta - a)x^b = (b - a)x^b.$$

If $b - a \neq 0$, the operator $(\theta - a)$ is invertible when acting on x^b. Its inverse operates as:

$$\frac{1}{\theta - a}x^b = \frac{1}{b - a}x^b.$$

Therefore, we can express the particular solution $y_p(x)$ formally as:

$$y_p(x) = c\frac{1}{\theta - a}x^b = \frac{c}{b - a}x^b.$$

This result aligns with the particular solution found in Problem 9 (b). Hence, the special solution of the differential equation is:

$$y(x) = c\frac{1}{\theta - a}x^b.$$

(d)

First, consider a particular solution of the form:

$$y_p(x) = Cx^{a+\varepsilon} \quad \text{where } C \text{ is const.}$$

Applying $(\theta - a)$ to $y_p(x)$:

$$\begin{aligned}
(\theta - a)y_p(x) &= \left(x\frac{d}{dx} - a\right)Cx^{a+\varepsilon} \\
&= \left(x\frac{d}{dx}Cx^{a+\varepsilon} - aCx^{a+\varepsilon}\right) \\
&= \{(a+\varepsilon)Cx^{a+\varepsilon} - aCx^{a+\varepsilon}\} \\
&= \{(a+\varepsilon - a)Cx^{a+\varepsilon}\} \\
&= \varepsilon Cx^{a+\varepsilon}.
\end{aligned}$$

Thus, the particular solution is:

$$y_p(x) = Cx^{a+\varepsilon}.$$

In the limit $\varepsilon \to 0$, the particular solution becomes:

$$\lim_{\varepsilon \to 0} y_p(x) = Cx^a.$$

Because of Problem 9 (a), the homogeneous solution is given by $y_h = Cx^a$.
Therefore, in the limit $\varepsilon \to 0$, the specialsolution of the differential equation

$$(\theta - a)y(x) = \varepsilon x^{a+\varepsilon}$$

reduces to the solution of the homogeneous equation.

Problem 10

The left-hand side of the differential equation can be written as:

$$\text{LHS} = \theta f(x) = x\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

The right-hand side of the differential equation can be written as:

$$\text{RHS} = x(\theta + a)f(x) = x^2\frac{\mathrm{d}}{\mathrm{d}x}f(x) + axf(x).$$

Therefore, the differential equation can be written as:

$$x\frac{\mathrm{d}}{\mathrm{d}x}f(x) - x^2\frac{\mathrm{d}}{\mathrm{d}x}f(x) - axf(x) = 0.$$

If $x \neq 0$,

$$\begin{aligned}
(1-x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - af(x) &= 0 \\
\frac{1}{f(x)}\frac{\mathrm{d}}{\mathrm{d}x}f(x) - af(x) &= 0 \\
\int \frac{1}{f(x)}\,\mathrm{d}f(x) &= a\int \frac{1}{1-x}\,\mathrm{d}x \\
\ln f(x) &= a\ln(1-x) + C' \quad \text{where } C' \text{ is const} \\
f(x) &= C(1-x)^{-a} \quad \text{where } C = e^{C'} \\
&= C\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n.
\end{aligned}$$

Problem 11

The homogeneous solution is

$$f_h = C(1-x)^{-a} = C\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n \quad \because \text{Problem 10.}$$

For the general solution, assume

$$f = C(x)(1-x)^{-a} = C(x)\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n$$

Substituting it into the differential equation yields:

$$\begin{aligned}
\theta f - x(\theta + a)f &= cx^b \\
(1-x)\theta f - axf &= cx^b \\
(1-x)x\left[C'(x)(1-x)^{-a} + aC(x)(1-x)^{-a-1}\right] - axC(x)(1-x)^{-a} &= cx^b \\
x(1-x)^{-a+1}C'(x) &= cx^b
\end{aligned}$$

where $C'(x) = \frac{\mathrm{d}C(x)}{\mathrm{d}x}$. Therefore,

$$
\begin{aligned}
C(x) &= c \int x^{b-1}(1-x)^{a-1} \,\mathrm{d}x \\
&= c \int x^{b-1} \sum_{n=0}^{\infty} \frac{(1-a)_n}{n!} x^n \,\mathrm{d}x \\
&= c \sum_{n=0}^{\infty} \frac{(1-a)_n}{n!} \int x^{b+n-1} \,\mathrm{d}x \\
&= c \sum_{n=0}^{\infty} \frac{(1-a)_n}{n!} \frac{x^{b+n}}{b+n} + C'' \quad \text{where } C'' \text{ is const} \\
f &= c \left(\sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \right) \left(\sum_{n=0}^{\infty} \frac{(1-a)_n}{n!} \frac{x^{b+n}}{b+n} + C'' \right)
\end{aligned}
$$