Set 0

Problem 1

(a)

Let z be defined as follows:

$$z = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Solving the above equation for θ, we obtain:

$$e^{i\theta} - e^{-i\theta} = 2iz$$
$$e^{2i\theta} - 1 = 2ize^{i\theta}$$
$$e^{2i\theta} - 2ize^{i\theta} - 1 = 0$$
$$e^{i\theta} = iz \pm \sqrt{1 - z^2}$$
$$\theta = -i\log\left(iz \pm \sqrt{1 - z^2}\right).$$

To ensure continuity with the real-valued $\sin\theta$, we choose the $+$ branch:

$$\theta = \arcsin z = -i\log\left(iz + \sqrt{1 - z^2}\right).$$

Choosing the $+$ branch maps the domain $-1 \le z \le 1$ to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. This can be verified by the following calculation.

If $-1 \le z \le 1$, using $a,\ b \in \mathbb{R}$, we have:

$$z = \frac{b}{\sqrt{a^2 + b^2}}.$$

Then, we can find θ as follows:

$$
\begin{aligned}
\arcsin z &= -i\log\left(i\frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}}\right)\\
&= -i\left\{\log\left|i\frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}}\right| + i\arg\left(i\frac{b}{\sqrt{a^2 + b^2}} + \sqrt{1 - \frac{b^2}{a^2 + b^2}}\right)\right\}\\
&= -i\left\{\log 1 + i\arctan\frac{b}{a}\right\}\\
&= \arctan\frac{b}{a}.
\end{aligned}
$$

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Therefore, $\arcsin z = \arctan \frac{b}{a}$, which is in the range of $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(b)

Let z be defined as follows:

$$z = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

Solving the above equation for θ, we obtain:

$$e^{i\theta} - e^{-i\theta} = iz(e^{i\theta} + e^{-i\theta})$$
$$e^{2i\theta} - 1 = ize^{2i\theta} - iz$$
$$e^{2i\theta}(1 - iz) = 1 + iz$$
$$e^{2i\theta} = \frac{1 + iz}{1 - iz}$$
$$\theta = -\frac{i}{2} \log \frac{1 + iz}{1 - iz}$$
$$= \frac{i}{2}\{\log(1 - iz) - \log(1 + iz)\}.$$

Therefore,

$$\theta = \arctan z = \frac{i}{2}\{\log(1 - iz) - \log(1 + iz)\}.$$

(c)

$$i^{\frac{1}{3}} = e^{i\frac{\frac{\pi}{2} + 2k\pi}{3}}$$
$$= e^{i\frac{\pi}{6} + \frac{2k\pi}{3}} \quad (k = 0, 1, 2).$$

Therefore,

$$i^{\frac{1}{3}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \ -i, \ -\frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

Problem 2

$$e^{in\theta} = \cos n\theta + i \sin n\theta.$$

On the other hand,

$$e^{in\theta} = \left(e^{i\theta}\right)^n$$
$$= (\cos\theta + i\sin\theta)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} (\cos^{n-k}\theta)\left(i\sin^k\theta\right)$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\cos^{n-2k}\theta)\left(\sin^{2k}\theta\right) + i\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos^{n-2k-1}\theta)\left(\sin^{2k+1}\theta\right).$$

Therefore, we get the following:

(a)

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\cos^{n-2k}\theta)\left(\sin^{2k}\theta\right),$$

(b)

$$\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos^{n-2k-1}\theta)\left(\sin^{2k+1}\theta\right).$$

Problem 3

$$\sin x/2 \sum_{n=0}^{N-1} \cos nx = \frac{1}{2}\sum_{n=0}^{N-1} \left\{ \sin\left(\frac{2n+1}{2}x\right) - \sin\left(\frac{2n-1}{2}x\right) \right\}$$
$$= \frac{1}{2}\left\{ \sin\left(\frac{2N-1}{2}x\right) - \sin\left(-\frac{x}{2}\right) \right\}$$
$$= \frac{1}{2}\left\{ \sin\left(\frac{2N-1}{2}x\right) + \sin\left(\frac{x}{2}\right) \right\}$$
$$= \frac{1}{2}\left\{ \sin\left(\frac{N}{2}x + \frac{N-1}{2}x\right) + \sin\left(\frac{N}{2}x - \frac{N-1}{2}x\right) \right\}$$
$$= \sin\frac{Nx}{2}\cos\frac{(N-1)x}{2}.$$

Therefore,

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin Nx/2 \cos(N-1)x/2}{\sin x/2}.$$

Problem 4

Let I be defined as follows:

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$$I = \int_0^\infty e^{-at} \sin bt \, dt \, .$$

$$
\begin{aligned}
I &= \int_0^\infty e^{-at} \sin bt \, dt \\
&= \left[\frac{-e^{-at}}{a} \sin bt \right]_0^\infty - \int_0^\infty \frac{-e^{-at}}{a} b \cos bt \, dt \\
&= \frac{b}{a} \int_0^\infty e^{-at} \cos bt \, dt \\
&= \frac{b}{a} \left[\frac{e^{-at}}{a} \cos bt \right]_0^\infty - \frac{b}{a} \int_0^\infty \frac{e^{-at}}{a} (-b) \sin bt \, dt \\
&= \frac{b}{a^2} - \frac{b^2}{a^2} I.
\end{aligned}
$$

Therefore,

$$I = \frac{b}{a^2 + b^2}.$$

Problem 5

$$
\begin{aligned}
\int_0^\infty \frac{e^{-s}}{s} (1 - e^{-s}) \, ds &= \int_0^\infty \int_0^1 e^{-s(t+1)} \, dt \, ds \\
&= \int_0^1 \int_0^\infty e^{-s(t+1)} \, ds \, dt \\
&= \int_0^1 \frac{1}{t+1} \, dt \\
&= \ln 2.
\end{aligned}
$$

The interchange of the order of integration was justified by Tonelli's theorem, as the integrand is non-negative within the integration domain.

Problem 6

e^{-x^2} is an even function, so

$$\int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} \, dx \, .$$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

$$= 2\pi \int_{0}^{\infty} e^{-r^2} r\, dr$$

$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_{0}^{\infty}$$

$$= \pi.$$

Therefore,

$$\int_{0}^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Problem 7

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \frac{e^{-x^2-y^2-z^2}}{x^2+y^2+z^2} = \int_{0}^{\infty} dr \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi\, r^2 \sin\theta \frac{e^{-r^2}}{r^2}$$

$$= 2\pi \int_{0}^{\infty} dr \int_{0}^{\pi} d\theta\, e^{-r^2} \sin\theta$$

$$= 4\pi \int_{0}^{\infty} dr\, e^{-r^2}$$

$$= 2\pi^{\frac{3}{2}}.$$

Problem 8

(a)

First, we solve the following differential equation:

$$L\frac{dI(t)}{dt} + RI(t) = 0$$

The solution is:

$$I(t) = I(0) e^{-\frac{R}{L} t}.$$

Next, we solve the following differential equation:

$$L\frac{dI(t)}{dt} + RI(t) = V_0.$$

The solution is:

$$I(t) = \frac{V_0}{R} + \left(I(0) - \frac{V_0}{R} \right) e^{-\frac{R}{L} t}.$$

By the initial condition $I(0) = 0$, we get:

$$I(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

(b)

We assume that particular solution be of the following form:

$$I(t) = A\cos\omega t + B\sin\omega t.$$

This results in:

$$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) = -LA\omega\sin\omega t + LB\omega\cos\omega t + RA\cos\omega t + RB\sin\omega t$$
$$= (RA + LB\omega)\cos\omega t + (-LA\omega + RB)\sin\omega t.$$

Therefore, we get the following equations:

$$RA - LB\omega = V_0,$$
$$-LA\omega + RB = 0.$$

Solving the above equations, we get:

$$A = \frac{V_0\omega L}{R^2 + \omega^2 L^2},$$
$$B = \frac{V_0 R}{R^2 + \omega^2 L^2}.$$

Therefore, the particular solution is:

$$I(t) = \frac{V_0\omega L}{R^2 + \omega^2 L^2}\cos\omega t + \frac{V_0 R}{R^2 + \omega^2 L^2}\sin\omega t.$$

The general solution is the sum of the homogeneous solution and the particular solution:

$$I(t) = Ce^{-\frac{R}{L}t} + \frac{V_0\omega L}{R^2 + \omega^2 L^2}\cos\omega t + \frac{V_0 R}{R^2 + \omega^2 L^2}\sin\omega t.$$

C is determined by the initial condition $I(0) = 0$:

$$C = -\frac{V_0\omega L}{R^2 + \omega^2 L^2}.$$

Therefore, the solution is:

$$I(t) = \frac{V_0\omega L}{R^2 + \omega^2 L^2}\left(\cos\omega t - e^{-\frac{R}{L}t}\right) + \frac{V_0 R}{R^2 + \omega^2 L^2}\sin\omega t.$$