Set 0

Problem 1

(a)

Let z be defined as follows:

$$z = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Solving the above equation for θ, we obtain:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2iz \\ e^{2i\theta} - 1 &= 2ize^{i\theta} \\ e^{2i\theta} - 2ize^{i\theta} - 1 &= 0 \\ e^{i\theta} &= iz \pm \sqrt{1 - z^2} \\ \theta &= -i\log\Big(iz \pm \sqrt{1 - z^2}\Big). \end{aligned}$$

To ensure continuity with the real-valued $\sin\theta$, we choose the $+$ branch:

$$\theta = \arcsin z = -i\log\Big(iz + \sqrt{1 - z^2}\Big).$$

Choosing the $+$ branch maps the domain $-1 \leq z \leq 1$ to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. This can be verified by the following calculation.

If $-1 \leq z \leq 1$, using $a,\ b \in \mathbb{R}$, we have:

$$z = \frac{b}{\sqrt{a^2 + b^2}}.$$

Then, we can find θ as follows:

$$\begin{aligned} \arcsin z &= -i\log\left(i\frac{b}{\sqrt{a^2+b^2}} + \sqrt{1 - \frac{b^2}{a^2+b^2}}\right) \\ &= -i\left\{\log\left|i\frac{b}{\sqrt{a^2+b^2}} + \sqrt{1 - \frac{b^2}{a^2+b^2}}\right| + i\arg\left(i\frac{b}{\sqrt{a^2+b^2}} + \sqrt{1 - \frac{b^2}{a^2+b^2}}\right)\right\} \\ &= -i\left\{\log 1 + i\arctan\frac{b}{a}\right\} \\ &= \arctan\frac{b}{a}. \end{aligned}$$

Therefore, $\arcsin z = \arctan\frac{b}{a}$, which is in the range of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b)

Let z be defined as follows:

$$z = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

Solving the above equation for θ, we obtain:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= iz(e^{i\theta} + e^{-i\theta}) \\ e^{2i\theta} - 1 &= ize^{2i\theta} - iz \\ e^{2i\theta}(1 - iz) &= 1 + iz \\ e^{2i\theta} &= \frac{1+iz}{1-iz} \\ \theta &= -\frac{i}{2}\log\frac{1+iz}{1-iz} \\ &= \frac{i}{2}\{\log(1-iz) - \log(1+iz)\}. \end{aligned}$$

Therefore,

$$\theta = \arctan z = \frac{i}{2}\{\log(1-iz) - \log(1+iz)\}.$$

(c)

$$\begin{aligned} i^{\frac{1}{3}} &= e^{i\frac{\frac{\pi}{2}+2k\pi}{3}} \\ &= e^{i\frac{\pi}{6}+\frac{2k\pi}{3}} \quad (k = 0, 1, 2). \end{aligned}$$

Therefore,

$$i^{\frac{1}{3}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

Problem 2

$$e^{in\theta} = \cos n\theta + i\sin n\theta.$$

On the other hand,

$$
\begin{aligned}
e^{in\theta} &= \left(e^{i\theta}\right)^n \\
&= (\cos\theta + i\sin\theta)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\cos^{n-k}\theta\right)\left(i\sin^k\theta\right) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \left(\cos^{n-2k}\theta\right)\left(\sin^{2k}\theta\right) + i \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} \left(\cos^{n-2k-1}\theta\right)\left(\sin^{2k+1}\theta\right).
\end{aligned}
$$

Therefore, we get the following:

(a)

$$
\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \left(\cos^{n-2k}\theta\right)\left(\sin^{2k}\theta\right),
$$

(b)

$$
\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} \left(\cos^{n-2k-1}\theta\right)\left(\sin^{2k+1}\theta\right).
$$

Problem 3

$$
\begin{aligned}
\sin x/2 \sum_{n=0}^{N-1} \cos nx &= \frac{1}{2} \sum_{n=0}^{N-1} \left\{ \sin\left(\frac{2n+1}{2}x\right) - \sin\left(\frac{2n-1}{2}x\right) \right\} \\
&= \frac{1}{2} \left\{ \sin\left(\frac{2N-1}{2}x\right) - \sin\left(-\frac{x}{2}\right) \right\} \\
&= \frac{1}{2} \left\{ \sin\left(\frac{2N-1}{2}x\right) + \sin\left(\frac{x}{2}\right) \right\} \\
&= \frac{1}{2} \left\{ \sin\left(\frac{N}{2}x + \frac{N-1}{2}x\right) + \sin\left(\frac{N}{2}x - \frac{N-1}{2}x\right) \right\} \\
&= \sin\frac{Nx}{2} \cos\frac{(N-1)x}{2}.
\end{aligned}
$$

Therefore,

$$
\sum_{n=0}^{N-1} \cos nx = \frac{\sin Nx/2 \cos(N-1)x/2}{\sin x/2}.
$$

Problem 4

Let I be defined as follows:

$$I = \int_0^\infty e^{-at} \sin bt \, \mathrm{d}t \, .$$

Then,

$$\begin{aligned}
I &= \int_0^\infty e^{-at} \sin bt \, \mathrm{d}t \\
&= \left[\frac{-e^{-at}}{a} \sin bt \right]_0^\infty - \int_0^\infty \frac{-e^{-at}}{a} b \cos bt \, \mathrm{d}t \\
&= \frac{b}{a} \int_0^\infty e^{-at} \cos bt \, \mathrm{d}t \\
&= \frac{b}{a} \left[\frac{e^{-at}}{a} \cos bt \right]_0^\infty - \frac{b}{a} \int_0^\infty \frac{e^{-at}}{a} (-b) \sin bt \, \mathrm{d}t \\
&= \frac{b}{a^2} - \frac{b^2}{a^2} I.
\end{aligned}$$

Therefore,

$$I = \frac{b}{a^2 + b^2}.$$

Problem 5

$$\begin{aligned}
\int_0^\infty \frac{e^{-s}}{s} (1 - e^{-s}) \, \mathrm{d}s &= \int_0^\infty \int_0^1 e^{-s(t+1)} \, \mathrm{d}t \, \mathrm{d}s \\
&= \int_0^1 \int_0^\infty e^{-s(t+1)} \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_0^1 \frac{1}{t+1} \, \mathrm{d}t \\
&= \ln 2.
\end{aligned}$$

The interchange of the order of integration was justified by Tonelli's theorem, as the integrand is non-negative within the integration domain.

Problem 6

e^{-x^2} is an even function, so

$$\int_0^\infty e^{-x^2} \, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} \, \mathrm{d}x \, .$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi \int_0^{\infty} e^{-r^2} r\,\mathrm{d}r \\
&= 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \\
&= \pi.
\end{aligned}
$$

Therefore,

$$
\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.
$$

Problem 7

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \frac{e^{-x^2-y^2-z^2}}{x^2+y^2+z^2} &= \int_0^{\infty} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, r^2 \sin\theta \frac{e^{-r^2}}{r^2} \\
&= 2\pi \int_0^{\infty} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta\, e^{-r^2} \sin\theta \\
&= 4\pi \int_0^{\infty} \mathrm{d}r\, e^{-r^2} \\
&= 2\pi^{\frac{3}{2}}.
\end{aligned}
$$