Problem of two variables with one constraint

Method of Langrange multipliers

Consider the problem Minimize $f(x_1, x_2)$

s.t $g(x_1, x_2) = 0$

The necessary condition can be generated by constructing a function L, known as Langrange function as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

By treating L as a function of the three variables x_1, x_2 & λ, the necessary conditions for its extremum are given by

$$\frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) = \frac{\partial f}{\partial x_1}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_1}(x_1, x_2) = 0$$

$$\frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) = \frac{\partial f}{\partial x_2}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_2}(x_1, x_2) = 0$$

$$\frac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) = g(x_1, x_2) = 0$$

Ex. Using the Langrange multiplier method, solve

Minimize $f(x, y) = k x^{-1} y^{-2}$

s.t $g(x, y) = x^2 + y^2 - a^2 = 0$

Soln The Langrange function is

$$L(x, y, \lambda) = k x^{-1} y^{-2} + \lambda(x^2 + y^2 - a^2)$$

The necessary conditions are

$$\frac{\partial L}{\partial x} = -k x^{-2} y^{-2} + 2x\lambda = 0 \quad - (1)$$

$$\frac{\partial L}{\partial y} = -2k x^{-1} y^{-3} + 2y\lambda = 0 \quad - (2)$$

$$\frac{\partial L}{\partial \lambda} = -(x^2 + y^2 - a^2) = 0 \quad - (3)$$

Solving ① & ②

$$2\lambda = -\frac{k}{x^3 y^2} = -\frac{2k}{x y^4}$$

$$\Rightarrow \quad x y^4 = 2x^3 y^2 \Rightarrow 2x^2 = y^2 \Rightarrow \quad x^* = \frac{y^*}{\sqrt{2}}$$

from ③, we have

$$\frac{y^{*2}}{2} + y^{*2} = a^2 \Rightarrow \quad 3y^{*2} = 2a^2$$

$$y^* = \sqrt{2}\frac{a}{\sqrt{3}}, \qquad x^* = \frac{a}{\sqrt{3}}$$

$$\text{Min } z = k\cdot(a/\sqrt{3})^{-1}\cdot(a\cdot\sqrt{2/3})^{-2} = 3\sqrt{3}\,k/2a^3$$

Ex Solve the following non-linear programming problem using the method of Langrge multiplier method

$$\text{Min } z = 6x_1^2 + 5x_2^2$$

$$\text{s.t} \quad x_1 + 5x_2 = 3, \quad x_1, x_2 \geq 0$$

Sol^n

$$L(x_1, x_2, \lambda) = 6x_1^2 + 5x_2^2 + \lambda(x_1 + 5x_2 - 3)$$

$$\frac{\partial L}{\partial x_1} = 12x_1 + \lambda = 0, \quad \frac{\partial L}{\partial x_2} = 10x_2 + 5\lambda$$

$$\frac{\partial L}{\partial \lambda} = x_1 + 5x_2 - 3 = 0$$

solving ① & ② $\quad \lambda = -12x_1 = -2x_2$

$$\therefore \quad x_2^* = 6x_1^*, \quad \text{using ③} \quad x_1 + 30x_1 = 3$$

$$x_1^* = 3/31, \quad x_2^* = 18/31$$

$$\text{Min } z = 54/31$$

Ex

$$\text{Min } f(x_1, x_2) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$$

$$\text{s.t} \quad 2x_1 - x_2 = 4, \quad x_1, x_2 \geq 0$$

Ans $x_1 = 1, \; x_2 = -2,$