Various search methods permit us to approximate solutions to nonlinear optimization problems with a single independent variable. A unimodal function on an interval has exactly one point where a maximum or minimum occurs in the interval.

A function $f \in C[a; b]$ is called unimodal, if for some value x^*, it is monotonically increasing for $x <= x^*$ and monotonically decreasing for $x >= x^*$, that is the function satisfies the inequality
$x^* < x2 \Rightarrow f(x2) > f(x^*)$
$x1 < x^* \Rightarrow f(x1) > f(x^*)$
for all $x1\ x2 \in [a; b]$; $x1 < x2$.

Let $f \in C[a, b]$ be an unimodal function and $a < c < d < b$. Then

- If $f(c) < f(d)$, then $x^* \in [a, d]$.
- If $f(c) > f(d)$, then $x^* \in [c, b]$.
- If $f(c) = f(d)$, then $x^* \in [c, d]$.

The Dichotomous Search Method

The Dichotomous Search Method computes the midpoint $\frac{a+b}{2}$, and then moves slightly to either side of the midpoint to compute two test points: $\frac{a+b}{2} \pm \delta$, where δ is a very small number. The objective being to place the two test points as close together as possible. The procedure continues until it gets within some small interval containing the optimal solution Then:

$$\begin{aligned} c_k &= \frac{a_k + b_k}{2} - \delta = a_k + \frac{L_k}{2} - \delta, \\ d_k &= \frac{a_k + b_k}{2} + \delta = a_k + \frac{L_k}{2} + \delta. \qquad k = 1, 2, 3, \ldots \end{aligned}$$

We can determine the new interval in the following way:

- If $f(c_k) > f(d_k)$, then the new uncertainty interval is $[a_{k+1}, b_{k+1}] = [c_k, b_k]$.
- If $f(c_k) \leq f(d_k)$, then the new uncertainty interval is $[a_{k+1}, b_{k+1}] = [a_k, d_k]$.

It is easy to see, the length of the obtained interval is $L_{k+1} = \frac{L_k}{2} + \delta$.
We continue this procedure until get $L_k < 2\varepsilon$ for some k, where $\varepsilon > 0$ is a tolerance. Then if we choose the midpoint of the last interval $[a_k, b_k]$, then the error will be less then ε .

The algorithm of the Dichotomous Search Method

```
Input [a_1, b_1] and δ, ε > 0
WHILE b_k − a_k ≥ 2ε
        Let c_k = (a_k+b_k)/2 − δ = a_k + L_k/2 − δ,
        and d_k = (a_k+b_k)/2 + δ = a_k + L_k/2 + δ.
        IF f(c_k) > f(d_k)
                THEN a_{k+1} = c_k, b_{k+1} = b_k
                ELSE a_{k+1} = a_k, b_{k+1} = c_k
        k=k+1
```

Where: $c_k = \frac{a_k+b_k}{2} - \delta = a_k + \frac{L_k}{2} - \delta$, $d_k = \frac{a_k+b_k}{2} + \delta = a_k + \frac{L_k}{2} + \delta$.

The procedure continues until it gets within some small interval containing the optimal solution.

Determine the minimum point of the function $f(x) = x^2 - 7x + 12 \to \min!$ with Dichotomous search method, where $[a, b] = [2, 4]$, $\delta = 0.3$, $\varepsilon = 0.4$.

Solution

Step 1. The first interval is $[a_1, b_1] = [2, 4]$, which length is $L_1 = 2$. The two test points and values:

$$
\begin{aligned}
c_1 &= a_1 + \frac{L_1}{2} - \delta = 2.7, & f(c_1) = 0.39, \\
d_1 &= a_1 + \frac{L_1}{2} + \delta = 3.3, & f(d_1) = -0.21
\end{aligned}
$$

Step 2. Since $f(c_1) > f(d_1)$, then the new uncertainty interval is $[a_2, b_2] = [2.7, 4]$, which length is $L_2 = 1.3$. The two test points and values:

$$\begin{aligned} c_2 &= 3.05, & f(c_2) = -0.0475, \\ d_2 &= 3.65, & f(d_2) = -0.2275 \end{aligned}$$

Step 3. Since $f(c_2) > f(d_2)$, then the new uncertainty interval is $[a_3, b_3] = [2.7, 3.65]$, which length is $L_3 = 0.95$. The two test points and values:

$$\begin{aligned} c_3 &= 2.875, & f(c_3) = 0.140625, \\ d_3 &= 3.475, & f(d_3) = -0.24937500 \end{aligned}$$

Step 4. Since $f(c_3) > f(d_3)$, then the new uncertainty interval is $[a_4, b_4] = [2.875, 3.65]$, which length is $L_4 = 0.775$. We stop, because the length of obtained interval small enough $L_4 < 2\varepsilon$.
The approximation of the minimum point is $x_{\min} \approx 3.2625$.
We note the exact minimum point of the function f is $\bar{x} = 3.5$.

The Golden Section Search Method

Search Method chooses c_k and d_k such that the one of the two evaluations of the function in each step can be reused in the next step. At the end of the golden section method we get the local minimum point x^* is an interval $[a_k, b_k]$, which length is less then 2ε. We use the approximation $\widetilde{x} = \frac{a_k+b_k}{2}$, because the condition $|\widetilde{x} - x^*| < \varepsilon$ fulfils.

The algorithm of Golden Section Search Method

Let $\tau = \frac{\sqrt{5}-1}{2} \approx 0.618$.

INPUT $[a_1, b_1], \varepsilon > 0, k = 1$
$c_1 = a_1 + (1-\tau)(b_1 - a_1), \quad F_c = f(c_1)$
$d_1 = b_1 - (1-\tau)(b_1 - a_1), \quad F_d = f(d_1)$
WHILE $b_k - a_k \geq 2\varepsilon$
  IF $F_c < F_d$
    THEN $a_{k+1} = a_k,\ b_{k+1} = d_k,\ d_{k+1} = c_k$
      $c_{k+1} = a_{k+1} + (1-\tau)(b_{k+1} - a_{k+1})$
      $F_d = F_c, \quad F_c = f(c_{k+1})$
    ELSE $a_{k+1} = c_k,\ b_{k+1} = b_k,\ c_{k+1} = d_k$
      $d_{k+1} = b_{k+1} + (1-\tau)(b_{k+1} - a_{k+1})$
      $F_c = F_d, \quad F_d = f(d_{k+1})$
  $k = k + 1$

Example

Determine the minimum point of the function $f(x) = x^2 - 7x + 12$ with Golden Section search method, if the interval of uncertainty is $[a, b] = [2, 4]$ and $\varepsilon = 0.3$.

Solution

Step 1. In the first approximation the uncertainty interval $[a_1, b_1] = [2, 4]$, is of length $L_1 = 2$. The two intermediate points with their corresponding function values:

$$\begin{aligned} c_1 &= a_1 + 0.382L_1 = 2.764, & f(c_1) = 0.291696, \\ d_1 &= a_1 + 0.618L_1 = 3.236, & f(d_1) = -0.180304 \end{aligned}$$

$$
\begin{aligned}
c_3 &= d_2 = 3.527848, & f(c_3) = -0.24922449, \\
d_3 &= a_3 + 0.618L_3 = 3.70806, & f(d_3) = -0.20671104
\end{aligned}
$$

Step 4 Since $f(c_3) < f(d_3)$, the next uncertainty interval $[a_4, b_4] = [3.236, 3.70806]$, is of length $L_4 = 0.47206$. Since $L_4 < 2\varepsilon$ we stop the process. The minimum point is approximated by $x_{\text{min}} \approx 3.47203$.
Now we get closer to the exact minimum point (that is, the $\bar{x} = 3.5$) value as in the case of Dichotomous search.

Step 2 Since $f(c_1) > f(d_1)$, the uncertainty interval $[a_2, b_2] = [2.764, 4]$, is of length $L_2 = 1.236$. The two intermediate points and the associated function values are as follows. Do not forget that now the point c_2 and the point d_1 of the previous interval are identical.

$$
\begin{aligned}
c_2 &= d_1 = 3.236, & f(c_2) = -0.180304, \\
d_2 &= a_2 + 0.382L_2 = 3.527848, & f(d_2) = -0.24922449
\end{aligned}
$$

Step 3 Since $f(c_2) > f(d_2)$, the new uncertainty interval $[a_3, b_3] = [3.236, 4]$, is of length $L_3 = 0.764$. The two intermediate points and their associated function values are the following. In this case the point c_3 is taken to be d_2.

$$
\begin{aligned}
c_3 &= d_2 = 3.527848, & f(c_3) = -0.24922449, \\
d_3 &= a_3 + 0.618L_3 = 3.70806, & f(d_3) = -0.20671104
\end{aligned}
$$

Step 4 Since $f(c_3) < f(d_3)$, the next uncertainty interval $[a_4, b_4] = [3.236, 3.70806]$, is of length $L_4 = 0.47206$. Since $L_4 < 2\varepsilon$ we stop the process. The minimum point is approximated by $x_{\text{min}} \approx 3.47203$.
Now we get closer to the exact minimum point (that is, the $\bar{x} = 3.5$) value as in the case of Dichotomous search.