Maxima or minima

Sufficient condition

Let $f'(x^*) = f''(x^*) = \cdots = f^{(n-1)}(x^*) = 0$ but $f^{(n)}(x^*) \neq 0$.
Then $f(x^*)$ is (i) a minimum value of $f(x)$ if $f^{(n)}(x^*) > 0$ & n is even (ii) a maximum value of $f(x)$ if $f^{(n)}(x^*) < 0$ & n is even (iii) neither a maximum nor a minimum if n is odd. In this case the point x^* is called a point of inflection.

How do we find them

1) Given $f(x)$, we differentiate once to find $f'(x)$
2) Set $f'(x) = 0$ & solve for x. The values of x are called critical points
3) Calculate $f''(x)$

   If $f''(a) < 0$ — a local maximum

   $f''(a) > 0$ — " " minimum

   $f''(a) = 0$, then the test fails.

Q. $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$$f'(x) = 12x^3 + 12x^2 - 24x$$

$$f'(x) = 0 \Rightarrow x^3 + x^2 - 2x = 0 \Rightarrow x(x+2)(x-1) = 0$$

$$\Rightarrow x = 0, 1, -2$$

$$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

At $x = -2$, $f''(-2) = 72 > 0$

$x = 1$, $f''(1) = 36 > 0$

$x = 0$, $f''(0) = -24 < 0$

Thus ~~at~~ $x = 0$ is the point of maxima, while $x = 1$ & -2 are the points of local minima.

Q Determine the maximum & minimum values of the function $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$

Sol^n $f'(x) = 60(x^4 - 3x^3 + 2x^2) = 60x^2(x-1)(x-2)$

$f'(x) = 0$ at $x = 0, 1$ & 2.

$f''(x) = 60(4x^3 - 9x^2 + 4x)$

At $x = 1$ $f''(1) = -60$, relative ~~minimum~~ maximum at $x = 1$

$f''(2) = 240 > 0$ " minimum at $x = 2$

$\cancel{f''(0) = 240 > 0}$ $f''(0) = 0$ & hence we investigate the next derivative

$f''(x) = 60(12x^2 - 18x + 4) = 240$ at $x = 0$

Since $f'''(x) \neq 0$ at $x = 0$, $x = 0$ is neither a max nor a minimum and it is an inflection point.

Q Find the maxima & minima, if any, of the function

$f(x) = 4x^3 - 18x^2 + 27x - 7$

An $f'(x) = 12x^2 - 36x + 27$

$f'(x) = 0 \Rightarrow$ $\cancel{2x}$ $4x^2 - 12x + 9 = 0 \Rightarrow (2x-3)^2 = 0$

$\Rightarrow 2x = 3 \Rightarrow x = 3/2$

$f''(x) = 24x - 36$

$f''(3/2) = 0$, hence we investigate the next der

$f'''(x) = 24$ at $x = 0$

Since $f'''(x) \neq 0$ at ~~$x = 0$~~,

$\therefore$ $x^* = 1.5$ is an inflection point