Introduction to Soft Computing

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Outline

Course Overview

- Basics of Fuzzy Logic and problem-solving.
- Genetic Algorithm framework and optimization problems.
- Building and training Artificial Neural Networks for complex problems.

Class Organization

- **Semester:** Autumn, Session 2024-2025
- **Course:** Soft Computing
- **Code:** MC5145
- **Credit:** 3-0-0 = 3
- **Timing (Section A):**
  - Tuesday: 12:00 PM - 01:00 PM
  - Wednesday: 04:00 PM - 05:00 PM
  - Thursday: 12:00 PM - 01:00 PM
- **Timing (Section B):**
  - Tuesday: 09:30 AM - 10:30 AM
  - Wednesday: 03:00 PM - 04:00 PM
  - Thursday: 03:00 PM - 04:00 PM

Reference Books

1. S. Rajasekaran, G.A. VijayalakshmiPai, Neural Networks, Fuzzy Logic and Genetic Algorithms: Synthesis and Applications, 2nd edition, 2018.
2. S N Sivanandam Principles Of Soft Computing, 2nd Edition, John Wiley, 2011.
3. Davis E.Goldberg, Genetic Algorithms: Search, Optimization and Machine Learning, Addison Wesley, N.Y.,1989
4. J.S.R.Jang, C.T.Sun and E.Mizutani, Neuro-Fuzzy and Soft Computing, , PHI/Pearson Education, 2015

Evaluation Plan

- **Mid-Semester Test:** 20 Marks
- **End-Semester Test:** 50 Marks
  - Syllabus: 20% from the syllabus covered till Mid-semester.
  - 80% from the syllabus covered post-Mid-semester.
- **Other Assessment:** 30 Marks
  - Class Test 1: 10 Marks (Topic: Fuzzy Logic)
  - Class Test 2: 10 Marks (Topic: Artificial Neural Network)
  - Class Test 3: 10 Marks (Topic: Evolutionary Computing Techniques)
    (Note: Best two out of three tests will be considered.)
  - Practical problem solving: 5 Marks (Topic: Covering three major topics)
  - Attendance:
    - 5 Marks if more than 75% attendance
    - 4 Marks if more than 70% and less than 75
    - 3 Marks if more than 60% and less than 70
    - 2 Marks if more than 50% and less than 60
    - 1 Marks if more than 30% and less than 50
    - 0 Marks if less than 30%

Lesson Plan

<table>
<tr><th>Lecturer No.</th><th>Unit No.</th><th>Topic</th></tr>
<tr><td>Lecturer 1</td><td rowspan="7">Unit - I</td><td>Introduction to Soft Computing
Concept of computing systems.
"Soft" computing versus "Hard" computing
Characteristics of Soft computing
Some applications of Soft computing techniques</td></tr>
<tr><td>Lecturer 2</td><td rowspan="2">Fuzzy logic
Introduction to Fuzzy logic.
Crisp Logic
Fuzzy sets and membership functions.
Operations on Fuzzy sets.</td></tr>
<tr><td>Lecturer 3</td></tr>
<tr><td>Lecturer 4</td><td rowspan="4">Fuzzy relations, rules, propositions, implications and inferences.
Defuzzification techniques.
Some applications of Fuzzy logic.</td></tr>
<tr><td>Lecturer 5</td></tr>
<tr><td>Lecturer 6</td></tr>
<tr><td>Lecturer 7</td></tr>
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Lecturer 8	Unit - II	Neural Networks • Biological neurons and its working. • Simulation of biological neurons to problem solving.
Lecturer 9		• Different ANNs architectures.
Lecturer 10		
Lecturer 11		• Perceptron, Adaline
Lecturer 12		• Back propagation
Lecturer 13		• Multilayer Perceptron • Radial Basis Function Networks
Lecturer 14		• Unsupervised Learning Neural Networks
Lecturer 15		• Competitive Learning Networks, • Kohonen Self Organizing Networks
Lecturer 16		
Lecturer 17		• Hebbian Learning, • Hop-field networks
Lecturer 18		

Lecturer 19		Mid-Sem Revision
Lecturer 20		Mid-sem Paper discussion

Lecturer 21		Genetic Algorithms • Fundamentals of genetic algorithms: Encoding, Fitness functions, Reproduction.
Lecturer 22		
Lecturer 23		• Genetic Modeling: cross cover, inversion and deletion, Mutation operator, Bit-wise operators, Bitwise operators used in GA.
Lecturer 24		
Lecturer 25		
Lecturer 26		
Lecturer 27		
Lecturer 28	Unit - III	Optimization • Derivative-based Optimization, Descent Methods, The Method of Steepest Descent, Classical Newton's Method, Step Size Determination.
Lecturer 29		
Lecturer 30		• Derivative-free Optimization, Genetic Algorithms, Simulated Annealing, Random Search, Downhill Simplex Search
Lecturer 31		

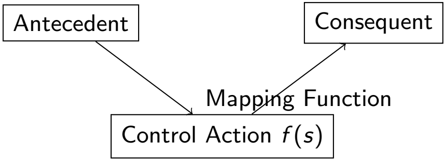
Lecturer 32	Unit-IV	Hybrid Systems • Hybrid system, neural Networks, fuzzy logic and Genetic algorithms hybrids.
Lecturer 33		
Lecturer 34		
Lecturer 35		
Lecturer 36		• Genetic Algorithm based Back propagation Networks
Lecturer 37		
Lecturer 38		• GA based weight determination applications: Fuzzy Back Propagation Networks.
Lecturer 39		
Lecturer 40		• End-Sem Revision

Introduction

- Concept of computation
- Hard computing
- Soft computing
- Comparison: Hard vs. Soft computing
- Hybrid computing

Concept of Computation

- Computation involves mapping functions and control actions.
- It is a formal method or algorithm to solve a problem.



Hard Computing

- Introduced by L. A. Zadeh in 1996.
- Guarantees precise results and unambiguous control actions.
- Requires formal mathematical models or algorithms.

Examples of Hard Computing

- Solving numerical problems (e.g., polynomial roots, integration).
- Searching and sorting techniques.
- Computational geometry problems (e.g., shortest path).

Soft Computing

- Exploits tolerance for imprecision and uncertainty.
- Achieves tractability, robustness, and low solution cost.
- Main components: fuzzy logic, neuro-computing, probabilistic reasoning.

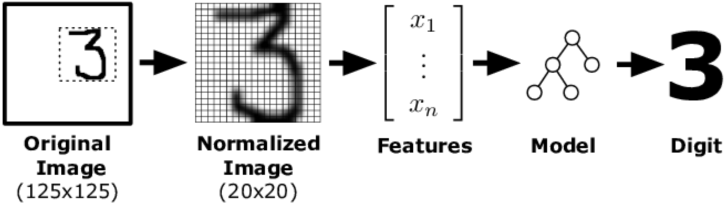
Characteristics of Soft Computing

- Does not require precise mathematical models.
- May not yield precise solutions.
- Algorithms are adaptive to dynamic environments.
- Inspired by biological methodologies (e.g., genetics, evolution).

Examples of Soft Computing

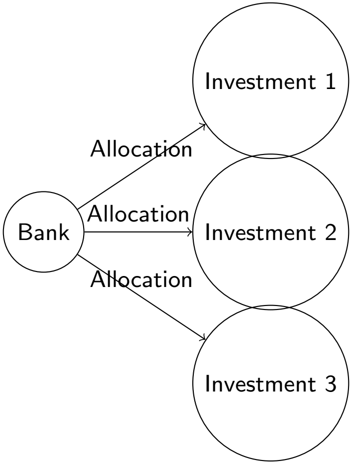
- Handwritten character recognition (Artificial Neural Networks).
- Money allocation problem (Evolutionary Computing).
- Robot movement (Fuzzy Logic).

Example: Handwritten Character Recognition



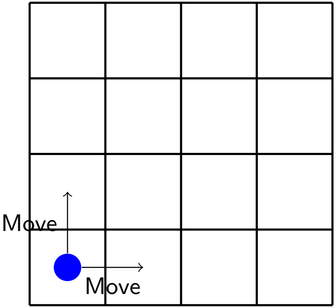
- Artificial Neural Networks are trained to recognize characters from handwriting samples.

Example: Money Allocation Problem



- Evolutionary Computing is used to optimize the allocation of money to different investments for maximum return.

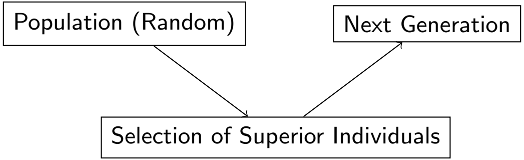
Example: Robot Movement



- Fuzzy Logic is applied to determine the movement of a robot in a grid environment.

Natural Selection Process

- Starts with a random population.
- Reproduces next generation.
- Selects superior individuals.
- Basis of Genetic Algorithms.



Medical Diagnosis Approach

- Doctor diagnoses based on symptoms and tests.
- Correlates with diseases despite uncertainties.
- Basis of Fuzzy Logic.

Hard Computing vs. Soft Computing

Hard Computing

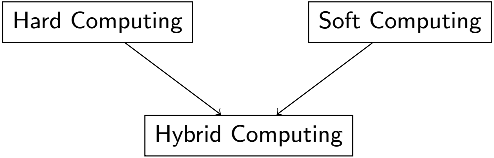
- Precise and unambiguous results.
- Requires exact input data.
- Deterministic.

Soft Computing

- Tolerant of imprecision and uncertainty.
- Can handle ambiguous and noisy data.
- Stochastic.

Hybrid Computing

- Combines hard and soft computing.
- Leverages strengths of both approaches.



Thank You!

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Crisp Set Theory

Crispness & Impreciseness

- **Crispness:** Clearly defined boundaries and characteristics.
  - Example: A set of even numbers: $A = \{2, 4, 6, 8, 10, \ldots\}$.
- **Impreciseness:** Lack of clarity in boundaries and characteristics.
  - Example: A set of "tall" people, where "tall" can have different interpretations.

Uncertainty & Vagueness

- **Uncertainty:** The state of being uncertain or not having exact information.
- **Vagueness:** Lack of precision or distinctness.

Example

Vague: I will come back soon.

Fuzzy: I will come back within 1 minute.

Crisp Set Theory

- Let A be a collection of well-defined objects x_i, and U is the universal set.
- $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$
- $\chi_A : x \to \{0, 1\}$

Example: Characteristic Function

Example

Let $A = \{1, 2, 3\}$ and $U = \{1, 2, 3, 4, 5\}$.

The characteristic function $\chi_A(x)$ is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

For $x = 2$, $\chi_A(2) = 1$.

For $x = 4$, $\chi_A(4) = 0$.

Operations on Sets

- $A^c = U - A$
- $(A^c)^c = A$
- $\phi^c = U$
- $U^c = \phi$
- $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- $A \cup U = U$
- $A \cap U = A$

Operations on Sets

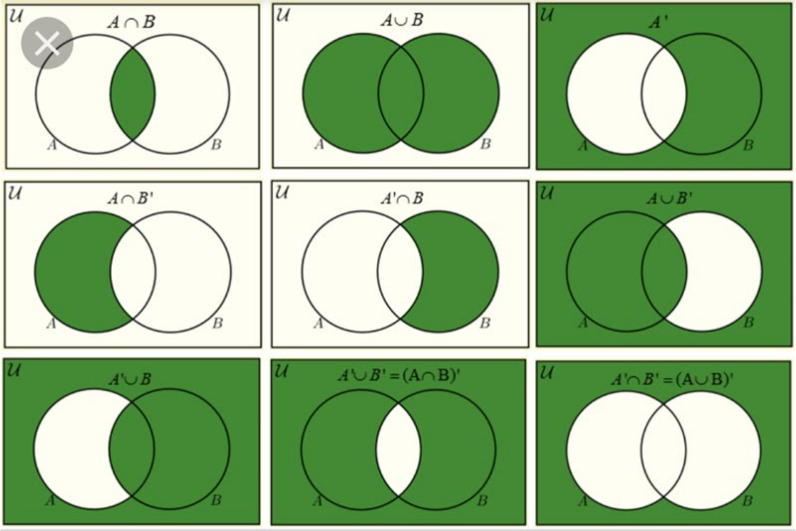


Figure: Operations on Sets

Example: Operations on Sets

Example

Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, and $U = \{1, 2, 3, 4, 5, 6\}$.

- **Union:** $A \cup B = \{1, 2, 3, 4, 5\}$
- **Intersection:** $A \cap B = \{3\}$
- **Complement:** $A^c = \{4, 5, 6\}$

Properties of Set Theory

- **Involution:** $(A^c)^c = A$
- **Commutativity:** $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Associativity:** $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Idempotency:** $A \cup A = A$, $A \cap A = A$
- **Absorption:** $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- **Identity:** $A \cup \phi = A$, $A \cap U = A$
- **De-Morgan's Law:** $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$
- **Law of Contradiction:** $A \cap A^c = \phi$
- **Law of Excluded Middle:** $A \cup A^c = U$

Example

Let $A = \{1,2\}$, $B = \{2,3\}$, and $C = \{1,3\}$.

- **Commutativity:** $A \cup B = B \cup A = \{1,2,3\}$
- **Associativity:** $A \cup (B \cup C) = (A \cup B) \cup C = \{1,2,3\}$
- **Distributive:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) = \{1,2\}$

Types of Sets

- **Convex Set:** A set in which the line segment between any two points in the set is also within the set.
- **Non-Convex Set:** A set that does not satisfy the convex set condition.

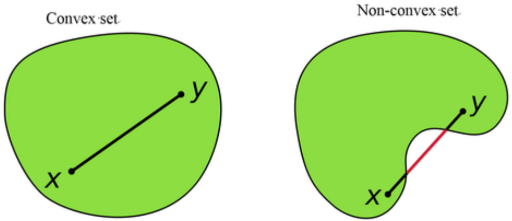


Figure: Convex and Non-Convex Set

Example: Types of Sets

Example

Convex Set: A circle or a square where any line segment between two points lies within the shape.

Non-Convex Set: A shape like a crescent moon where a line segment between two points can lie outside the shape.

Cardinality

- $|A|$ = Number of elements in set A.

Example

Let $A = \{a, b, c, d\}$.

The cardinality of A is $|A| = 4$.

Power Set

- $\mathcal{P}(A) =$ Collection of all subsets of A.

Example

Let $A = \{a, b, c\}$. Then, $|\mathcal{P}(A)| = 2^{|A|} = 8$.

$\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Crisp Set Theory(Continued...)

Crisp Relations

- Crisp relation is defined over the Cartesian product of two crisp sets.
- Suppose A and B are two crisp sets. The Cartesian product, denoted as $A \times B$, is a collection of ordered pairs such that:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Definition of Crisp Relation

- Crisp relation is a set of ordered pairs (a, b) from Cartesian product $A \times B$ such that $a \in A$ and $b \in B$.
- Relations represent the mapping of sets and define the interaction or association of variables.
- The strength of the relationship between ordered pairs of elements in each universe is measured by the characteristic function χ.

Properties of Cartesian Product

- $A \times B \neq B \times A$
- $|A \times B| = |A| \times |B|$
- $A \times B$ provides a mapping from $a \in A$ to $b \in B$

Applications of Crisp Relations

- Useful in logic, pattern recognition, control systems, classification, etc.

Characteristic Function

$$\chi_R(a,b) = \begin{cases} 1 & \text{if } (a,b) \in (A \times B) \\ 0 & \text{if } (a,b) \notin (A \times B) \end{cases}$$

Example: Crisp Relation

Example

Consider two crisp sets: $C = \{1, 2, 3\}$ and $D = \{4, 5, 6\}$.

- Find Cartesian product $C \times D$.
- Find relation R over this Cartesian product such that $R = \{(c, d) \mid d = c + 2, (c, d) \in C \times D\}$.

Solution: Cartesian Product

$$C \times D = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$$

$$R = \{(2,4),(3,5)\}$$

Representations of Crisp Relations

- Functional Form
- Sagittal (Pictorial) Representation
- Matrix Representation

Example: Representations

- Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $R = \{(2, 4), (3, 5)\}$.

Sagittal Representation

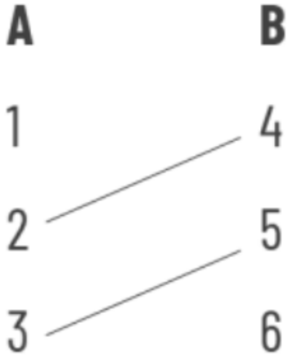


Figure: Sagittal Representation of Relation

Matrix Representation

$$
R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{ccc} 4 & 5 & 6 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

Figure: Matrix Representation of Relation

Special Types of Relations

- Null Relation: No mapping of elements from universe X to universe Y.
- Complete Relation: All elements of universe X are mapped to universe Y.
- Universal Relations: The universal relation on A is defined as $U_A = A \times A = A^2$.
- Identity Relations: The identity relation on A is defined as $I_A = \{(a, a) \mid \forall a \in A\}$.

Example: Special Relations

- Let $A = \{0, 1, 2\}$.
- Universal Relation:
  $U_A = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.
- Identity Relation: $I_A = \{(0, 0), (1, 1), (2, 2)\}$.

Operations on Crisp Relations

- Suppose $R(x, y)$ and $S(x, y)$ are two relations defined over two crisp sets, where $x \in A$ and $y \in B$.
- Operations include Union, Intersection, Complement, and Containment.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Relation R

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Relation S

Union of Crisp Relations

$$R \cup S = \chi_{R \cup S}(x, y) = \max(\chi_R(x, y), \chi_S(x, y))$$

$$R \cup S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure: Union of Crisp Relations

Intersection of Crisp Relations

$$R \cap S = \chi_{R \cap S}(x, y) = \min(\chi_R(x, y), \chi_S(x, y))$$

$$R \cap S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Figure: Intersection of Crisp Relations

Complement of Crisp Relation

$$R^c = \chi_{R^c}(x, y) = 1 - \chi_R(x, y)$$

$$R^c = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Figure: Complement of Crisp Relation

Containment of Crisp Relations

$$R \subseteq S = \chi_{R \subseteq S}(x, y) = \chi_R(x, y) \leq \chi_S(x, y)$$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure: Containment of Crisp Relations

- Relation R is not contained within relation S.
- Consider the relation T where $\chi_R(x, y) \leq \chi_T(x, y)$.
- Hence, R is contained within T.

Cardinality of Crisp Set

- Cardinality defines the number of elements in the given set.
- Let A and B be crisp sets with cardinality n and m, respectively.
- The cardinality of a crisp relation defined over Cartesian product $A \times B$ will be $n \times m$.

Example: Cardinality

Example

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$.

- $n = |A| = 2$
- $m = |B| = 3$
- So, $n \times m = 6$

The Cartesian product:

$$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$$

Cardinality of Cartesian product is $|A \times B| = 6 = n \times m$.

Composition of Crisp Relation

- The composition of relation R and S is denoted as $R \circ S$.
- $R \circ S = \{(x, z) \mid (x, y) \in R,\ \text{and } (y, z) \in S, \forall y \in Y\}$
- The composition of the relation is computed in two ways:
  - Max–min composition
  - Max–product composition
- For crisp relations, both methods yield identical results. For fuzzy relations, they give different results.

Max-Min Composition for Crisp Relations

- Max-Min composition is one way of computing the interaction between variables of different relations.
- The composition of relation R and S is denoted as $R \circ S$.
- Mathematically, it is defined as:

$$R \circ S = \{(x, z) \mid (x, y) \in R, \text{ and } (y, z) \in S, \forall y \in Y\}$$

Methods of Composition

- Max–min composition
- Max–product composition
- For crisp relations, both methods yield identical results.
- For fuzzy relations, the results of max-min composition and max-product composition would be different.

Example 1: Max-Min Composition

- Let $R = \{(x_1, y_1), (x_1, y_3), (x_2, y_4)\}$ and $S = \{(y_1, z_2), (y_3, z_2)\}$.
- Find the Max-Min composition of these relations.

$$R = \begin{array}{c|c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad S = \begin{array}{c|c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

Composition Matrix $T = R \circ S$

T	z_1	z_2
x_1	0	1
x_2	0	0
x_3	0	0

Calculation of T

$$\chi_T(x_1, z_1) = \max(\min(\chi_R(x_1, y_1), \chi_S(y_1, z_1)),$$
$$\min(\chi_R(x_1, y_3), \chi_S(y_3, z_1)), \min(\chi_R(x_1, y_4), \chi_S(y_4, z_1)))$$

$$\chi_T(x_1, z_1) = \max(\min(1, 0), \min(1, 0), \min(0, 0)) = \max(0, 0, 0) = 0$$

$$\chi_T(x_1, z_2) = \max(\min(1, 1), \min(1, 1), \min(0, 0)) = \max(1, 1, 0) = 1$$
$$\chi_T(x_2, z_1) = \max(\min(0, 0), \min(0, 0), \min(1, 0)) = \max(0, 0, 0) = 0$$
$$\chi_T(x_2, z_2) = \max(\min(0, 1), \min(0, 1), \min(1, 0)) = \max(0, 0, 0) = 0$$
$$\chi_T(x_3, z_1) = \max(\min(0, 0), \min(0, 0), \min(0, 0)) = \max(0, 0, 0) = 0$$
$$\chi_T(x_3, z_2) = \max(\min(0, 1), \min(0, 1), \min(0, 0)) = \max(0, 0, 0) = 0$$

Example 2: Max-Min Composition

Example

- Given $X = \{1, 3, 5\}$ and $Y = \{1, 3, 5\}$.
- $R = \{(x, y) \mid y = x + 2\} = \{(1, 3), (3, 5)\}$.
- $S = \{(x, y) \mid x < y\} = \{(1, 3), (1, 5), (3, 5)\}$.

Steps

$$X \times Y = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$$

R	1	3	5
1	0	1	0
3	0	0	1
5	0	0	0

S	1	3	5
1	0	1	1
3	0	0	1
5	0	0	0

T	1	3	5
1	0	0	1
3	0	0	0
5	0	0	0

Calculation of T

$$\chi_T(1,1) = \max(\min(0,0), \min(1,0), \min(0,0)) = \max(0,0,0) = 0$$
$$\chi_T(1,3) = \max(\min(0,1), \min(1,0), \min(0,0)) = \max(0,0,0) = 0$$
$$\chi_T(1,5) = \max(\min(0,1), \min(1,1), \min(0,0)) = \max(0,1,0) = 1$$
$$\chi_T(3,1) = \max(\min(0,0), \min(0,0), \min(1,0)) = \max(0,0,0) = 0$$
$$\chi_T(3,3) = \max(\min(0,1), \min(0,0), \min(1,0)) = \max(0,0,0) = 0$$
$$\chi_T(3,5) = \max(\min(0,1), \min(0,1), \min(1,0)) = \max(0,0,0) = 0$$
$$\chi_T(5,1) = \max(\min(0,0), \min(0,0), \min(0,0)) = \max(0,0,0) = 0$$
$$\chi_T(5,3) = \max(\min(0,1), \min(0,0), \min(0,0)) = \max(0,0,0) = 0$$
$$\chi_T(5,5) = \max(\min(0,1), \min(1,0), \min(0,0)) = \max(0,0,0) = 0$$

Fuzzy Set Theory

Outline

Introduction, or what is fuzzy thinking?

- Experts rely on common sense when they solve problems.
- How can we represent expert knowledge that uses vague and ambiguous terms in a computer?
- Fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness.
- Fuzzy logic is the theory of fuzzy sets, sets that calibrate vagueness.
- Fuzzy logic is based on the idea that all things admit of degrees.

Introduction, or what is fuzzy thinking? (cont.)

- Boolean logic uses sharp distinctions.
- Fuzzy logic reflects how people think.
- Fuzzy logic was introduced in the 1930s by Jan Lukasiewicz.
- Max Black published a paper on vagueness in 1937.
- Lotfi Zadeh published his paper "Fuzzy Sets" in 1965.

Why fuzzy?

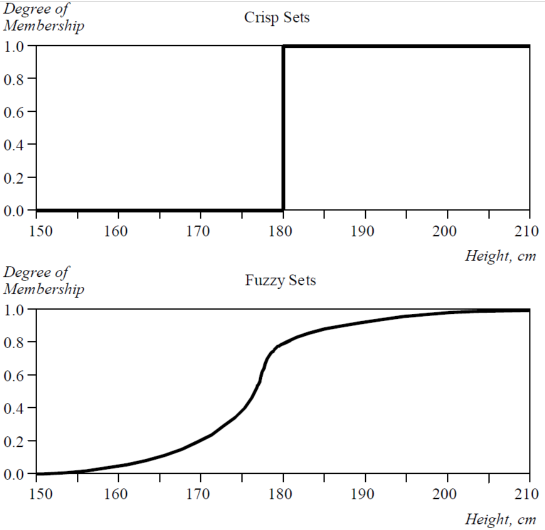
- The term is concrete, immediate, and descriptive.
- Fuzzy logic is a set of mathematical principles for knowledge representation based on degrees of membership.
- Unlike two-valued Boolean logic, fuzzy logic is multi-valued.
- Fuzzy logic uses the continuum of logical values between 0 and 1.

Fuzzy Sets

- The concept of a set is fundamental to mathematics and language.
- The classical example in fuzzy sets is tall men.
- Elements of the fuzzy set "tall men" have degrees of membership based on their height.

Fuzzy Sets (cont.)

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00



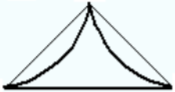
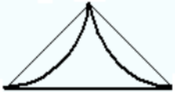
Fuzzy Sets (cont.)

- The x-axis represents the universe of discourse.
- The y-axis represents the membership value of the fuzzy set.
- A fuzzy set is a set with fuzzy boundaries.
- In classical set theory, a crisp set A of X is defined by the characteristic function $f_A(x)$.
- In fuzzy theory, a fuzzy set $\bar{A}$ of X is defined by the membership function $\mu_{\bar{A}}(x)$.
- This set allows a continuum of possible choices.

Linguistic Variables and Hedges

- At the root of fuzzy set theory lies the idea of linguistic variables.
- A linguistic variable is a fuzzy variable.
- In fuzzy expert systems, linguistic variables are used in fuzzy rules.
- The range of possible values of a linguistic variable represents the universe of discourse.
- A linguistic variable carries the concept of fuzzy set qualifiers, called hedges.
- Hedges are terms that modify the shape of fuzzy sets.

Fuzzy Sets with the Hedge *very*

Hedge	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

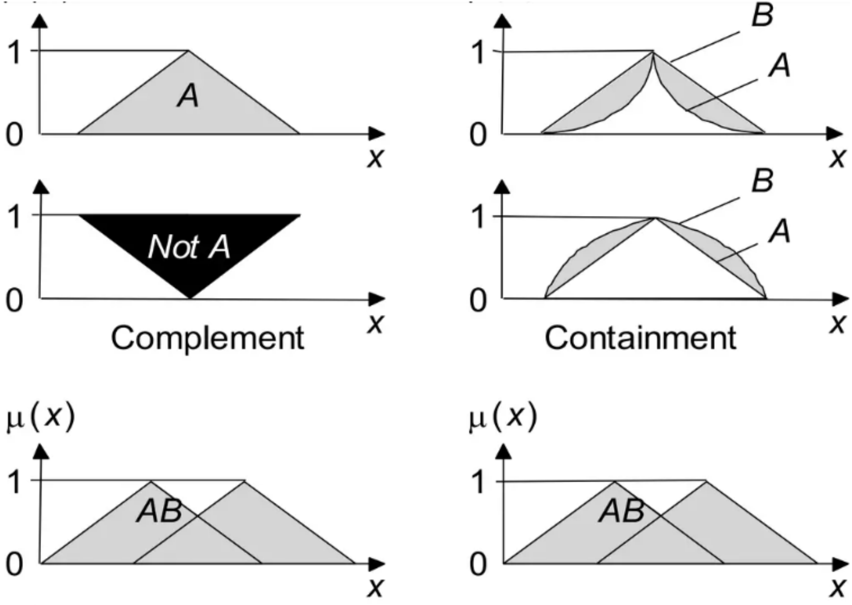
Operations of Fuzzy Sets

- The classical set theory describes interactions between crisp sets, called operations.
- Fuzzy sets include intersection, union, complement, and containment operations.
- **Complement** of a fuzzy set $\bar{A}$ is given by $\mu_{\bar{A}^c}(x) = 1 - \mu_{\bar{A}}(x)$.
- Example: If $\mu_{\bar{A}}(x) = 0.7$, then $\mu_{\bar{A}^c}(x) = 0.3$.
- **Containment**: Elements of a fuzzy subset have smaller memberships than in the larger set.
- Example: If $\mu_{\bar{B}}(x) = 0.4$ and $\mu_{\bar{A}}(x) = 0.7, \bar{B}$ is a subset of $\bar{A}$.

Operations of Fuzzy Sets (cont.)

- **Intersection**: The fuzzy intersection of two fuzzy sets $\bar{A}$ and $\bar{B}$ on X is given by $\mu_{A \bigcap B}(x) = min[_A(x), _B(x)]$.
- Example: If$_A(x) = 0.6$ and $_B(x) = 0.8$, then $\mu_{A \bigcap B}(x) = 0.6$.
- **Union**: The fuzzy operation for forming the union of two fuzzy sets A and B on X is given by $\mu_{A \bigcup B}(x) = max[_A(x), _B(x)]$.
- Example: If$_A(x) = 0.5$ and $_B(x) = 0.3$, then $\mu_{A \bigcup B}(x) = 0.5$.

Operations of Fuzzy Sets (cont.)



- $\bar{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$
- $\bar{B} = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$

Disjunctive Sum (Ex-OR) Simple

$$\boxed{\bar{A} \oplus \bar{B} = (\bar{A} \cap \bar{B}^c) \cup (\bar{A}^c \cap \bar{B})}$$

Example:

- $\bar{A} \oplus \bar{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.1)\}$

Disjoint Sum (Ex-OR)

$$\boxed{\bar{A} \Delta \bar{B}, \quad \mu_{A \Delta B}(x) = |\mu_A(x) - \mu_B(x)|}$$

Example:

- $\bar{A} \Delta \bar{B} = \{(x_1, 0.3), (x_2, 0.4), (x_3, 0), (x_4, 0.1)\}$

Simple Difference

$$\boxed{A - B = A \cap B^c = \min(\mu_A(x), 1 - \mu_B(x))}$$

Example:

- $\mu_{A-B}(x) = \{(x_1, 0.3), (x_2, 0.4), (x_3, 0), (x_4, 0.1)\}$

Simple Difference

$$\boxed{A - B = A \cap B^c = \min(\mu_A(x), 1 - \mu_B(x))}$$

Example:

- $\mu_{A-B}(x) = \{(x_1, 0.3), (x_2, 0.4), (x_3, 0), (x_4, 0.1)\}$

Bounded Difference

$$\boxed{\mu_{A \ominus B}(x) = \max[0, \mu_A(x) - \mu_B(x)]}$$

Example:

- $\mu_{A \ominus B}(x_1) = \max(0, 0.2 - 0.5) = 0$

m-th Power of a Fuzzy Set

$$\boxed{\mu_A^m(x) = [\mu_A(x)]^m}$$

Example:

- If $\mu_A(x) = 0.6$ and $m = 2$: $\mu_A^2(x) = 0.36$

Distances in Fuzzy Sets

Hamming Distance

$$\boxed{d(\bar{A}, \bar{B}) = \sum_{i=1}^{n} |\mu_{\bar{A}}(x_i) - \mu_{\bar{B}}(x_i)|}$$

Example:

- $d(\bar{A}, \bar{B}) = |0.2 - 0.5| + |0.7 - 0.3| + |1 - 1| + |0 - 0.1| = 0.8$

Relative Hamming Distance

$$\boxed{\delta(\bar{A}, \bar{B}) = \frac{d(\bar{A}, \bar{B})}{|X|}}$$

Example:

- If $|X| = 4$: $\delta(\bar{A}, \bar{B}) = \frac{0.8}{4} = 0.2$

Euclidean Distance

$$\boxed{e(\bar{A}, \bar{B}) = \sqrt{\sum_{i=1}^{n} (\mu_{\bar{A}}(x_i) - \mu_{\bar{B}}(x_i))^2}}$$

Example:

- $e(\bar{A}, \bar{B}) = \sqrt{(0.2 - 0.5)^2 + (0.7 - 0.3)^2 + (1 - 1)^2 + (0 - 0.1)^2} = \sqrt{0.09 + 0.16 + 0 + 0.01} = 0.5$

Minkowski Distance

$$\boxed{d_w(\bar{A}, \bar{B}) = \left(\sum_{i=1}^{n} |\mu_{\bar{A}}(x_i) - \mu_{\bar{B}}(x_i)|^w \right)^{1/w}}$$

Example:

- For $w = 3$: $d_3(\bar{A}, \bar{B}) = ((-0.3)^3 + 0.4^3 + 0^3 + (-0.1)^3)^{1/3} \approx 0.330$

Fuzzy Relations

Cartesian Product Let $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ be fuzzy sets in the universes $X_1, X_2, \ldots, X_n$. The **Cartesian product** $\bar{A}_1 \times \bar{A}_2 \times \ldots \times \bar{A}_n$ is a fuzzy set in the product space $X_1 \times X_2 \times \ldots \times X_n$ and its membership function is given by:

$$\boxed{\mu_{\bar{A}_1 \times \bar{A}_2 \times \ldots \times \bar{A}_n}(x_1, x_2, \ldots, x_n) = \min\{\mu_{\bar{A}_1}(x_1), \mu_{\bar{A}_2}(x_2), \ldots, \mu_{\bar{A}_n}(x_n)\}}$$

Example:

Let $\bar{A}_1 = \{(x_1, 0.5), (x_2, 0.7)\}$ and $\bar{A}_2 = \{(y_1, 0.8), (y_2, 0.6)\}$.
The Cartesian product $\bar{A}_1 \times \bar{A}_2$ will have the membership function values:

$$\boxed{\mu_{\bar{A}_1 \times \bar{A}_2}(x_i, y_j) = \min(\mu_{\bar{A}_1}(x_i), \mu_{\bar{A}_2}(y_j))}$$

Computed as:

$$\mathbf{M} = \begin{bmatrix} \mu_{\bar{A}_1\times\bar{A}_2}(x_1, y_1) & \mu_{\bar{A}_1\times\bar{A}_2}(x_1, y_2) \\ \mu_{\bar{A}_1\times\bar{A}_2}(x_2, y_1) & \mu_{\bar{A}_1\times\bar{A}_2}(x_2, y_2) \end{bmatrix} = \begin{bmatrix} \min(0.5, 0.8) & \min(0.5, 0.6) \\ \min(0.7, 0.8) & \min(0.7, 0.6) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.6 \end{bmatrix}$$

Thus, the Cartesian product $\bar{A}_1 \times \bar{A}_2$ is:

$$\mathbf{M} = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.6 \end{bmatrix}$$

Fuzzy Relation

$$\bar{R} : X \times Y \to [0, 1]$$

$$\boxed{\mu_R(x, y) = \min\{\mu_A(x), \mu_B(y)\}}$$

Example: Let:

$$\bar{A} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 1.0)\}$$
$$\bar{B} = \{(y_1, 0.7), (y_2, 0.4)\}$$

The fuzzy relation $\bar{R} = \bar{A} \times \bar{B}$ will have the membership values:

$$\mathbf{R} = \begin{bmatrix} \mu_{\bar{R}}(x_1, y_1) & \mu_{\bar{R}}(x_1, y_2) \\ \mu_{\bar{R}}(x_2, y_1) & \mu_{\bar{R}}(x_2, y_2) \\ \mu_{\bar{R}}(x_3, y_1) & \mu_{\bar{R}}(x_3, y_2) \end{bmatrix} = \begin{bmatrix} \min(0.2, 0.7) & \min(0.2, 0.4) \\ \min(0.5, 0.7) & \min(0.5, 0.4) \\ \min(1.0, 0.7) & \min(1.0, 0.4) \end{bmatrix}$$

Calculating the values:

$$\mathbf{R} = \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.4 \\ 0.7 & 0.4 \end{bmatrix}$$

Thus, the fuzzy relation $\bar{R} = \bar{A} \times \bar{B}$ is:

$$\mathbf{R} = \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.4 \\ 0.7 & 0.4 \end{bmatrix}$$

Operations on Fuzzy Relations

Intersection

$$\boxed{\mu_{R\cap S}(x,y) = \min\{\mu_R(x,y), \mu_S(x,y)\}}$$

Union

$$\boxed{\mu_{R\cup S}(x,y) = \max\{\mu_R(x,y), \mu_S(x,y)\}}$$

Complement

$$\boxed{\mu_{R^c}(x,y) = 1 - \mu_R(x,y)}$$

Projections:

- **First Projection:** $R^{(1)} = \{x, \max_y \mu_R(x,y)\}$
- **Second Projection:** $R^{(2)} = \{y, \max_x \mu_R(x,y)\}$
- **Total Projection:** $R^{(T)} = \max_{x,y} \mu_R(x,y)$

Example:

$$R = \begin{bmatrix} 0.1 & 0.2 & 0.4 \\ 0.2 & 0.8 & 0.6 \end{bmatrix}, \quad S = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.2 & 0.4 & 0.9 \end{bmatrix}$$

$$R \cap S = \begin{bmatrix} 0.1 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.6 \end{bmatrix}$$

Composition of Fuzzy Relations

Max-Min Composition

$$\boxed{R \circ S = \{(x,z), \max_y \min(\mu_R(x,y), \mu_S(y,z))\}}$$

Example:

- Given:

$$R = \begin{bmatrix} 0.2 & 0.6 \\ 0.4 & 0.8 \end{bmatrix}, \quad S = \begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.9 \end{bmatrix}$$

- $R \circ S = \begin{bmatrix} \max(\min(0.2, 0.3), \min(0.6, 0.5)) & \max(\min(0.2, 0.7), \min(0.6, 0.9)) \\ \max(\min(0.4, 0.3), \min(0.8, 0.5)) & \max(\min(0.4, 0.7), \min(0.8, 0.9)) \end{bmatrix}$

Max-Product Composition

$$\boxed{R \circ S = \{(x,z), \max_y(\mu_R(x,y) \cdot \mu_S(y,z))\}}$$

Example:

- Using the same matrices R and S:
- $R \circ S = \begin{bmatrix} \max(0.2 \cdot 0.3, 0.6 \cdot 0.5) & \max(0.2 \cdot 0.7, 0.6 \cdot 0.9) \\ \max(0.4 \cdot 0.3, 0.8 \cdot 0.5) & \max(0.4 \cdot 0.7, 0.8 \cdot 0.9) \end{bmatrix}$

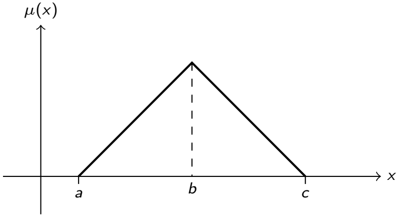
Membership Functions (MFs)

Triangular MF

$$\text{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a \le x \le b \\ \frac{c-x}{c-b}, & b \le x \le c \\ 0, & x \ge c \end{cases}$$

The triangular MF using the min-max formula:

$$\boxed{\text{triangle}(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)}$$

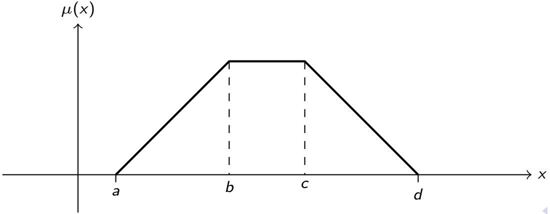


Trapezoidal MF

$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{d-x}{d-c}, & c \leq x \leq d \\ 0, & x \geq d \end{cases}$$

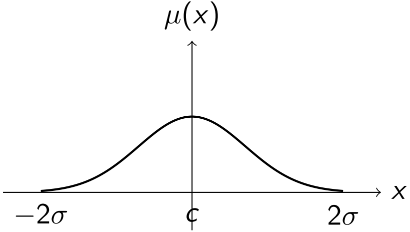
The trapezoidal MF using the min-max formula:

$$\boxed{\text{trapezoid}(x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)}$$



Gaussian MF

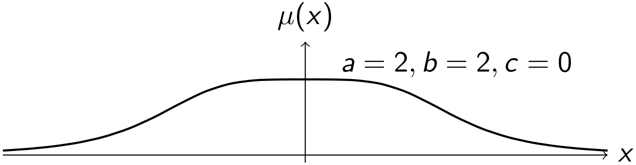
$$\boxed{\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}}$$



A **Generalized Bell MF**(also known as the Cauchy MF) is specified by three parameters (a, b, c):

$$\boxed{\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}}$$

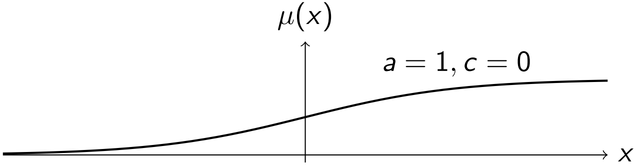
Visualization:



A sigmoidal MF is defined by:

$$\boxed{\text{sig}(x; a, c) = \frac{1}{1 + \exp[-a(x - c)]}}$$

Visualization:



Fuzzy Propositions

Fuzzy propositions involve logical connectives and are defined using truth values. Here are some basic fuzzy propositions:

S. No.	Symbol	Connective	Usage	Definition
1	$\neg$	NOT	$\neg P$	$1 - T(P)$
2	$\vee$	OR	$P \vee Q$	$\max(T(P), T(Q))$
3	$\wedge$	AND	$P \wedge Q$	$\min(T(P), T(Q))$
4	$\rightarrow$	Implication	$P \rightarrow Q$	$\max(1 - T(P), T(Q))$
5	$=$	Equality	$P = Q$	$1 - \|T(P) - T(Q)\|$

Examples:

Given fuzzy sets $A = \{0, 0, 0, 0.5, 0.5, 0.5, 1, 1, 1\}$ and $B = \{0, 0.5, 1.0, 0, 0.5, 1.0, 0, 0.5, 1.0\}$, compute:

- $\neg A$
- $A \wedge B$
- $A \vee B$
- $A \rightarrow B$

Given fuzzy propositions: - $P =$ "Max is fast" with truth value $T(P) = 0.8$ - $Q =$ "Daniel is fast" with truth value $T(Q) = 0.6$
Compute:

- (a) **Max is not fast:**

$$T(\neg P) = 1 - T(P) = 1 - 0.8 = 0.2$$

- (b) **Max is fast and so is Daniel:**

$$T(P \land Q) = \min(T(P), T(Q)) = \min(0.8, 0.6) = 0.6$$

- (c) **Either Max is fast or Daniel is:**

$$T(P \lor Q) = \max(T(P), T(Q)) = \max(0.8, 0.6) = 0.8$$

- (d) **If Max is fast, then so is Daniel:**

$$T(P \to Q) = \max(1 - T(P), T(Q)) = \max(1 - 0.8, 0.6) = 0.6$$

Fuzzy Implications

A fuzzy implication is a rule in the form: "If x is A, then y is B ($A \to B$)". Here, A and B are linguistic variables on universes X and Y, respectively.
Two main interpretations:

1. **A coupled with B:**

$$\boxed{R : \bar{A} \to \bar{B} = \bar{A} \times \bar{B} = \int_{X \times Y} [\mu_{\bar{A}}(x) \star \mu_{\bar{B}}(y)] \, (x, y)}$$

   T-norm operators like Algebraic Product: $T(ab) = a \cdot b$ and Minimum: $T(ab) = \min(a, b)$

2. **A entails B:**
   - Material implication: $R : \bar{A} \to \bar{B} = \bar{A}^c \cup \bar{B}$
   - Propositional calculus: $R : \bar{A} \to \bar{B} = \bar{A}^c \cup (\bar{A} \cap \bar{B})$
   - Extended propositional calculus: $R : \bar{A} \to \bar{B} = (\bar{A} \cap \bar{B}) \cup \bar{B}$

Implication Functions :

1. Zadeh's arithmetic rule:

$$\boxed{R_{za} = \bar{A}^c \cup \bar{B} = \int_{X\times Y} \left[1 \wedge (1 - \mu_{\bar{A}}(x) + \mu_{\bar{B}}(y))\right] (x, y)}$$

   Or, more compactly:

$$\boxed{f_{za}(a, b) = 1 \wedge (1 - a + b)}$$

2. Zadeh's max-min rule:

$$\boxed{R_{\mathrm{mm}} = \bar{A}^c \cup (\bar{A} \cap \bar{B}) = \int_{X\times Y} (1 - \mu_{\bar{A}}(x)) \vee \mu_{\bar{B}}(y)\, (x, y)}$$

   Or, more compactly:

$$\boxed{f_{zmm}(a, b) = (1 - a) \vee (a \wedge b)}$$

3. Boolean fuzzy rule:

$$\boxed{R_{\mathrm{bf}} = ((1 - a) \vee b)}$$

Example: Zadeh's Max-Min Rule

Given fuzzy sets:

$$X = \{a, b, c, d\}, Y = \{1, 2, 3, 4\}, \bar{A} = \{0, 0.8, 0.6, 0.1\}, \bar{B} = \{0.2, 1.0, 0.8, 0\}$$

$$\boxed{R_{mm} = (\bar{A} \times \bar{B}) \cup (\bar{A}^c \times I)}$$

Step-by-Step Solution:

1. **Compute the Complement of $\bar{A}$:**

$$\bar{A}^c = \{1 - 0, 1 - 0.8, 1 - 0.6, 1 - 0.1\} = \{1, 0.2, 0.4, 0.9\}$$

2. **Compute $(\bar{A} \times \bar{B})$ using min operation:**

$$\bar{A} \times \bar{B} = \begin{bmatrix} \min(0, 0.2) & \min(0, 1.0) & \min(0, 0.8) & \min(0, 0) \\ \min(0.8, 0.2) & \min(0.8, 1.0) & \min(0.8, 0.8) & \min(0.8, 0) \\ \min(0.6, 0.2) & \min(0.6, 1.0) & \min(0.6, 0.8) & \min(0.6, 0) \\ \min(0.1, 0.2) & \min(0.1, 1.0) & \min(0.1, 0.8) & \min(0.1, 0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.1 & 0.1 & 0.1 & 0 \end{bmatrix}$$

3. **Compute $(\bar{A}^c \times I)$ using min operation:**

$$\bar{A}^c \times I = \begin{bmatrix} \min(1,1) & \min(1,1) & \min(1,1) & \min(1,1) \\ \min(0.2,1) & \min(0.2,1) & \min(0.2,1) & \min(0.2,1) \\ \min(0.4,1) & \min(0.4,1) & \min(0.4,1) & \min(0.4,1) \\ \min(0.9,1) & \min(0.9,1) & \min(0.9,1) & \min(0.9,1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.9 & 0.9 & 0.9 & 0.9 \end{bmatrix}$$

4. **Combine Using Union ($\cup$):**

$$R_{mm} = \max\left(\bar{A} \times \bar{B}, \bar{A}^c \times I\right) = \begin{bmatrix} \max(0,1) & \max(0,1) & \max(0,1) & \max(0,1) \\ \max(0.2,0.2) & \max(0.8,0.2) & \max(0.8,0.2) & \max(0,0.2) \\ \max(0.2,0.4) & \max(0.6,0.4) & \max(0.6,0.4) & \max(0,0.4) \\ \max(0.1,0.9) & \max(0.1,0.9) & \max(0.1,0.9) & \max(0,0.9) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.9 & 0.9 & 0.9 & 0.9 \end{bmatrix}$$

Fuzzy Inferences / Fuzzy Reasoning

Fuzzy inferences or reasoning involves obtaining new knowledge through existing rules. Two main processes:

1. **Generalized Modus Ponens (GMP)**

Rule (p): If x is $\bar{A}_R$ then y is $\bar{B}_R$
Fact (q): If x is $\bar{A}_f$
Conclusion: Then y is $\bar{B}_c$

$$\bar{B}_c = \bar{A}_f \circ R_{\text{mm}}$$

$$\boxed{\mu_{\bar{B}_c}(y) = \max_{x \in X} \left(\min(\mu_{\bar{A}_f}(x), \mu_{R_{\text{mm}}}(x, y)) \right)}$$

Example:

- Let $\bar{A}_R = \{0.2, 0.5, 1.0\}$, $\bar{B}_R = \{0.7, 0.9, 0.4\}$, $\bar{A}_f = \{0.5, 0.8, 0.3\}$
- Compute $\bar{B}_c = \bar{A}_f \circ R_{\text{mm}}$

2. **Generalized Modus Tollens (GMT)**

Rule: If x is $\bar{A}_R$ then y is $\bar{B}_R$
Fact: If y is $\bar{B}_f$
Conclusion: Then x is $\bar{A}_c$

$$\bar{A}_c = \bar{B}_f \circ R_{\text{mm}}$$

$$\boxed{\mu_{\bar{A}_c}(x) = \max_{y \in Y} \left(\min(\mu_{\bar{B}_f}(y), \mu_{R_{\text{mm}}}(x, y))\right)}$$

Example:

- Let $\bar{B}_R = \{0.7, 0.9, 0.4\}$, $\bar{A}_R = \{0.2, 0.5, 1.0\}$, $\bar{B}_f = \{0.6, 0.8, 0.2\}$
- Compute $\bar{A}_c = \bar{B}_f \circ R_{\text{mm}}$

Defuzzification Techniques

1. Lambda-cut Method (Alpha-cut method)

The lambda-cut method, also known as the alpha-cut method, is defined by:

$$\bar{A}_\lambda = \{x \mid \mu_{\bar{A}}(x) \geq \lambda\}, \quad \forall \lambda \in (0 \leq \lambda \leq 1)$$

Example 1:

- For $\lambda = 0.6$:

$$\bar{A}_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\}$$

- For $\lambda = 0.2$:

$$\bar{A}_{0.2} = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.7), (x_4, 1)\}$$

2. Maxima Method

1. **Height Method:**

$$\mu_c(x^*) \geq \mu_c(x) \quad \forall x \in X$$

   - x^* is the height of the output fuzzy set $\bar{C}$. - This method is applicable when height is unique.

2. **First of Maxima (FoM):**

$$x^* = \min\{x|\bar{c}(x) = \max(\bar{c}(x))\}$$

3. **Last of Maxima (LoM):**

$$x^* = \max\{x|\bar{c}(x) = \max(\bar{c}(x))\}$$

4. **Mean of Maxima (MoM):**

$$x^* = \frac{\sum_{x \in M}(x_i)}{|M|}$$

   where $M = \{x|\mu_{\bar{c}}(x_i) = h(\bar{c})\}$

3. Centroid Method (COG)
The centroid method calculates the "center of gravity" of the fuzzy set:

$$x^* = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

where A_i denotes the area of region x_i and is the geometric center of the area A_i.
4. Weighted Average Method

$$x^* = \frac{\sum_i \mu_C(x_i) \cdot (x_i)}{\sum_i \mu_C(x_i)}$$

- Where x_i is the value where the middle of the fuzzy set C_i is observed.

Fuzzy Inference Process: Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic. The mapping then provides a basis from which decisions can be made, or patterns discerned. Major steps are as follows :

1. **Fuzzification of the input variables**: The first step is to take the inputs and determine the degree to which they belong to each of the appropriate fuzzy sets via membership functions (fuzzification).
2. **Application of fuzzy operators (AND, OR) in the antecedent**: After the inputs are fuzzified, it is known how each part of the antecedent is satisfied for each rule. If the antecedent of a rule has more than one part, the fuzzy operator is applied to obtain one number representing the result of the rule antecedent.
3. **Implication from the antecedent to the consequent** : A consequent is a fuzzy set represented by a membership function, which weights appropriately the linguistic characteristics attributed to it.
4. **Aggregation of the consequent across the rule**: Aggregation is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set. Aggregation only occurs once for each output variable, which is before the final defuzzification step.
5. **Defuzzification** : The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number. As much as fuzziness helps the rule evaluation during the intermediate steps, the final desired output for each variable is generally a single number.

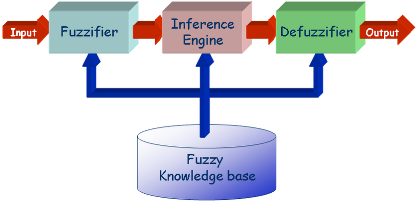
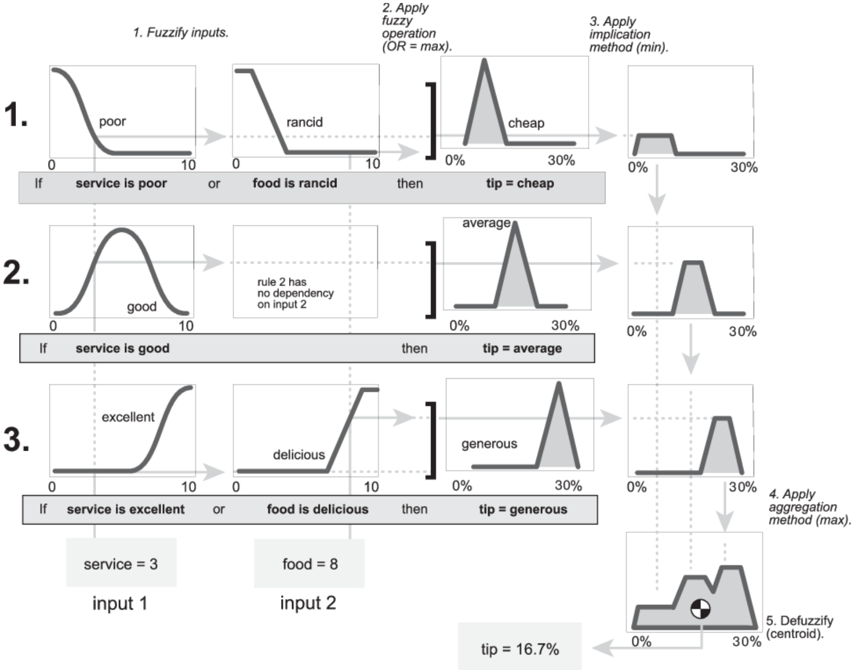


Figure: FIS System

1. **Mamdani Controllers**
   - Mamdani fuzzy inference was first introduced as a method to create a control system by synthesizing a set of linguistic control rules obtained from experienced human operators.
   - In a Mamdani system, the output of each rule is a fuzzy set.

Example: Two-input, one-output, three-rule tipping problem

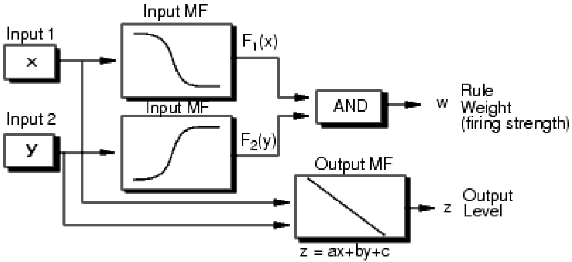
- **R1**: If the service is poor or the food is rancid, then the tip is cheap.
- **R2**: If the service is good, then the tip is average.
- **R3**: If the service is excellent or the food is delicious, then the tip is generous.



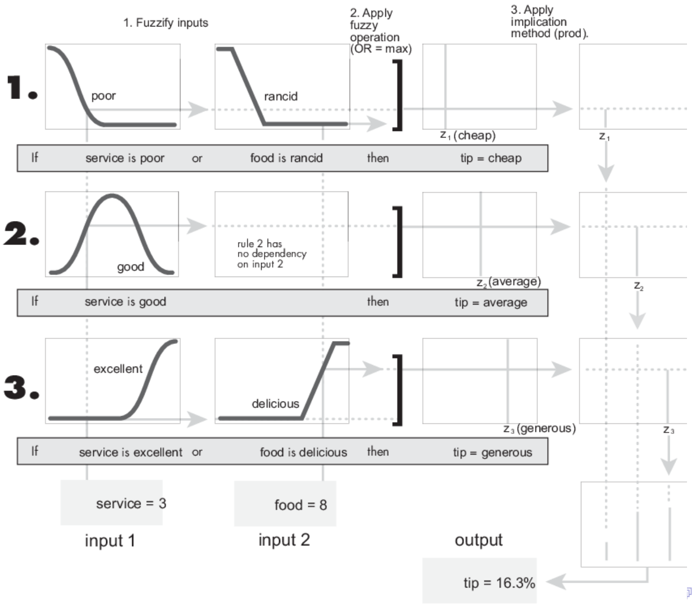
1. Fuzzify inputs.
2. Apply fuzzy operation (OR = max).
3. Apply implication method (min).
1.
poor
0
10
rancid
0
10
cheap
0%
30%
0%
30%
If service is poor or food is rancid then tip = cheap
2.
good
0
10
rule 2 has no dependency on input 2
average
0%
30%
0%
30%
If service is good then tip = average
3.
excellent
0
10
delicious
0
10
generous
0%
30%
0%
30%
If service is excellent or food is delicious then tip = generous
4. Apply aggregation method (max).
service = 3
input 1
food = 8
input 2
5. Defuzzify (centroid).
0%
30%
tip = 16.7%

2. Sugeno Fuzzy Inference Systems

- Sugeno fuzzy inference, also referred to as Takagi-Sugeno-Kang fuzzy inference, uses singleton output membership functions that are either constant or a linear function of the input values.
- The defuzzification process for a Sugeno system is more computationally efficient compared to that of a Mamdani system since it uses a weighted average or weighted sum of a few data points rather than compute a centroid of a two-dimensional area.
- Each rule generates two values w_i and z_i. The output of each rule is the weighted output level, which is the product of w_i and z_i.



Example


1. Fuzzify inputs
2. Apply fuzzy operation (OR = max)
3. Apply implication method (prod).
1.
poor
rancid
z_1 (cheap)
z_1
If service is poor or food is rancid then tip = cheap
2.
good
rule 2 has no dependency on input 2
z_2 (average)
z_2
If service is good then tip = average
3.
excellent
delicious
z_3 (generous)
z_3
If service is excellent or food is delicious then tip = generous
service = 3
input 1
food = 8
input 2
output
tip = 16.3%

Fuzzy Inference Systems (FIS)

FIS Types:

1. **Mamdani System:**
   - Intuitive.
   - Well-suited to human input.
   - More interpretable rule base.
   - Widespread acceptance.
2. **Sugeno System:**
   - Computationally efficient.
   - Works well with linear techniques (e.g., PID control).
   - Works well with optimization and adaptive techniques.
   - Guarantees output surface continuity.
   - Well-suited to mathematical analysis.