Constrained multivariable optimization with inequality constraints.

optimize $z = f(x_1, x_2 - x_n)$
s.t $g(x_1, x_2 - x_n) \le c$ & $x_1, x_1 - x_n \geq 0$

Introducing the function $h(x_1 - - x_n) = g(x_1 - x_n) - c$

$z = f(x)$ s.t $h(x) \le 0$ & $x \geq 0$

Introduce a slack variable s s.t

$h(x) + s^2 = 0$ (we take s^2 to ensure its being non-negative)

Optimize $z = f(x)$
st ~~fc~~ $h(x) + s^2 = 0$ & $\lambda \geq 0$

Lagrangian function

$$L(x, s, \lambda) = f(x) - \lambda(h(x) + s^2)$$

where λ is the Lagrange multiplier. The necessary conditions for stationary points are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, \quad j = 1, -- n \qquad (1)$$

$$\frac{\partial L}{\partial \lambda} = -(h(x) + s^2) = 0 \qquad (2)$$

$$\frac{\partial L}{\partial s} = -2s\lambda = 0 \qquad (3)$$

Eqn (3) states that $\frac{\partial L}{\partial s} = 0$, which requires either $\lambda = 0$ or $s = 0$. If $s = 0$, (2) implies that $h(x) = 0$. Thus (2) & (3) together imply

$$\lambda h(x) = 0.$$

Necessary condition

$f_j - \lambda h_j = 0$
$\lambda h = 0$
$h \le 0$
$0 \geq 0$

Max f
s.t $h \le 0$

Minimize $z = f(x)$

s.t $g(x) \geq c, \quad x \geq 0$

$f_j - \lambda h_j = 0$

$\lambda h = 0$

$h \geq 0$

$\lambda \geq 0$

Ex Determine x_1 & x_2 so as to

Maximize $z = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

s.t. $x_2 \le 8$

$x_1 + x_2 \le 10$

$x_1, x_2 \geq 0$

Sol^n Here $f(x_1, x_2) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

$g_1(x_1, x_2) = x_2 - 8 \le 0$, $g_2(x_1, x_2) = x_1 + x_2 - 10 \le 0$

$L(x, s, \lambda) = f(x) - \lambda_1(g_1(x) + s_1^2) - \lambda_2(g_2(x) + s^2)$

The K-T conditions can be stated as

(i) $\frac{\partial f}{\partial x_j} - \lambda_1 \frac{\partial g_1}{\partial x_j} - \lambda_2 \frac{\partial g_2}{\partial x_j} = 0$

or $12 + 2x_2 - 4x_1 - \lambda_2 = 0$

$21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$

(ii) $\lambda_i h_i(x) = 0, \quad i = 1, 2$

$\Rightarrow$ $\lambda_1(x_1 - 8) = 0$

$\lambda_2(x_1 + x_2 - 10) = 0$

(iii) $h_i(x) \le 0$

$x_2 - 8 \le 0$

$x_1 + x_2 - 10 \le 0$

(iv) $\lambda_i \geq 0, \quad i = 1, 2$

There may arise 4 cases

Case 1 If $\lambda_1 = 0$, $\lambda_2 = 0$, then from condition (i),

we have $12 + 2x_2 - 4x_1 = 0$ & $21 + 2x_1 - 4x_2 = 0$

Solving these, $x_1 = 15/2$, $x_2 = 9$.

However, the solution violates condition (iii) &

therefore it may be ~~direr~~ discarded.

<u>Case 2</u> $\lambda_1 \neq 0, \lambda_2 \neq 0$, then from condition (ii),

$x_2 - 8 = 0 \Rightarrow x_2 = 8$

$x_1 + x_2 - 10 = 0 \Rightarrow x_1 = 2$

Substituting these values in condition (i), we get $\lambda_1 = -27, \quad \lambda_2 = 20$, this violates condition (iv), so discarded.

<u>Case 3</u> $\lambda_1 \neq 0, \quad \lambda_2 = 0$, then from (ii) & (i)

$x_1 + x_2 = 10$

$2x_2 - 4x_1 = -12$

$2x_1 - 4x_2 = -12 + \lambda_1$

Solving these eqn's, we get $x_1 = 2, \; x_2 = 8$ & $\lambda_1 = -16$. $\lambda_1 < 0$ violates condition (iv), so discarded.

<u>Case 4</u> $\lambda_1 = 0, \quad \lambda_2 \neq 0$, then from (i) & (ii)

$2x_2 - 4x_1 = -12 + \lambda_2$

$2x_1 - 4x_2 = -21 + \lambda_2$

$x_1 + x_2 = 10$

Solving these, $x_1 = 17/4, \; x_2 = 23/4$ & $\lambda_2 = 13/4$

This soln does not violate any of the K-T conditions & therefore, must be accepted.

Hence the optimum soln is $x_1 = 17/4, \; x_2 = 23/4$, $\lambda_1 = 0, \; \lambda_2 = 13/4$ & Max $Z = 1734/16$.