Inverse of a matrix by elementary transformation :-

Invertible matrix :- If A is a square matrix of order 'n' and if $\exists$ another matrix B of the same order such that

$$AB = BA = I$$

then $B = A^{-1}$

$$A = IA$$
$$\Rightarrow I = A^{-1}A$$

Elementary transformation :- $R_i \leftrightarrow R_j$

$R_i \rightarrow kR_i$

$R_i \rightarrow R_i \pm kR_j$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = IA$$

$$[\quad] = [\quad] A$$

$$\vdots \qquad \vdots$$

$$I = BA$$

where $B = A^{-1}$

Eg:- 1) $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

$A = IA$

$$\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1/2 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + 4R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 3 \\ -1/2 & 1 \end{bmatrix}}_{B} A$$

$$\therefore B = A^{-1} = \begin{bmatrix} -1 & 3 \\ -1/2 & 1 \end{bmatrix}$$

Eg:- 2) $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

$A = IA$

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 8R_3$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 21 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -13 & 1 & 8 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 21 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -13 & 1 & 8 \\ -15 & 1 & 9 \end{bmatrix} A$$

Applying $R_3 \to \frac{1}{25} R_3$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 21 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -13 & 1 & 8 \\ -3/5 & 1/25 & 9/25 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 21R_3$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/5 & 4/25 & 11/25 \\ -3/5 & 1/25 & 9/25 \end{bmatrix} A.$$

Applying $R_1 \to R_1 + 2R_3$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/5 & 2/25 & 18/25 \\ -2/5 & 4/25 & 11/25 \\ -3/5 & 1/25 & 9/25 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2/5 & -3/5 \\ -2/5 & 4/25 & 11/25 \\ -3/25 & 1/25 & 9/25 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -2/5 & -3/5 \\ -2/5 & 4/25 & 11/25 \\ -3/5 & 1/25 & 9/25 \end{bmatrix}$$