## Question 4.

Prove that the following function is convex for $x > 0\,; x \in R$:

$$g(x) = -\prod_{i=1}^{n} x_i^{p_i}, \quad p_i \geq 0\,; \sum_i p_i = 1$$

We will use the composition statement from the book from Stephen Boyd. (page 86), which says:

If $f(x)$ is concave and positive, then $1/f(x)$ is convex.

Indeed: Let's take $f(x)$ concave positive function ($f(x) > 0$). The composition function will be $h(x) = \frac{1}{x}$, which is convex ($h''^{(x)} > 0$), and decreasing ($h'^{(x)} < 0$) for $x \in R^+$. Then:

$$g(x) = h\big(f(x)\big) = \frac{1}{f(x)}$$

Showing that its second derivative is positive:

$$g''(x) = \underbrace{h''\big(f(x)\big)}_{>0}\underbrace{f'(x)^2}_{>0} + \underbrace{h'\big(f(x)\big)}_{<0}\underbrace{g''(x)}_{<0} > 0$$

In our case, proving that $-\prod_{i=1}^{n} x_i^{p_i}$ is convex is equal to proving that $\prod_{i=1}^{n} x_i^{p_i}$ is concave.

From the previous statement, since $\prod_{i=1}^{n} x_i^{p_i}$ is positive ($x > 0\,;\ p_i \geq 0\ \forall\ i$ ) and concave, it is equal to proving that $\frac{1}{\prod_{i=1}^{n} x_i^{p_i}}$ is convex. Rewriting. Need to prove:

$$g_2(x) = \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{-1} = \prod_{i=1}^{n} x_i^{-p_i}$$

Is convex.

Using exp and log trick to turn multiplication into sum:

$$g_2(x) = e^{\log\left(\prod_{i=1}^{n} x_i^{-p_i}\right)} = e^{\sum_{i=1}^{n}\log\left(x_i^{-p_i}\right)} = e^{\sum_{i=1}^{n}-\log\left(x_i^{p_i}\right)}$$

Since $log$ is a concave function for positive values, thus $-\log$ is convex. So,

$$y = \sum_{i=1}^{n} -\log\left(x_i^{p_i}\right)$$

Is the sum of convex functions, thus is also convex.

Further, $e^y$ is also convex, since it is a composition of 2 convex functions, where $y$ is convex, and $e$ is convex and non-decreasing. Thus, we prove that $g_2(x)$ is convex, thus $g(x)$ is also convex.