## Question 4.

Prove that the following function is convex for :

We will use the composition statement from the book from Stephen Boyd. (page 86), which says:

If is concave and positive, then is convex.

Indeed: Let’s take concave positive function (). The composition function will be , which is convex (, and decreasing () for . Then:

Showing that its second derivative is positive:

In our case, proving that is convex is equal to proving that is concave.

From the previous statement, since is positive ( and concave, it is equal to proving that is convex. Rewriting. Need to prove:

Is convex.

Using exp and log trick to turn multiplication into sum:

Since is a concave function for positive values, thus is convex. So,

Is the sum of convex functions, thus is also convex.

Further, is also convex, since it is a composition of 2 convex functions, where is convex, and is convex and non-decreasing. Thus, we prove that is convex, thus is also convex.

Another way to prove it is to use the Hessian matrix of the original function, and prove that it’s PSD.

## Question 5.

Need to prove:

Let’s assume there’s a point . Extending:

Thus,

Since the function is convex, the following inequality holds:

Only if the conditions for is met (. And that those coefficients sum to 1.

And indeed:

And:

So we get:

Multiply by :

Which is what is required.

## Question 6.

Given function in . .

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*First direction.*

Expressing with the convex combination of :

We can then express and :

The function at point :

It is given that the function is convex, thus the following holds:

Inserting and back:

We multiply by . Since , it is a negative number, so we change the direction of inequality.

Which proves the statement.

*Second direction*

Given some 3 points , which satisfy and satisfy:

Prove that is convex.

Solution:

We will try to use the same characteristic of the convex function. Since , we can express as a combination of , .:

We will use the same derivations that we got from previous direction proof:

From this expression:

Divide both by :

Putting :

And since , it proves that the function is convex.

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If is convex and , then this holds:

We can use the expression from previous paragraph, reorganize it, and get:

Add and subtract :

Divide by (which is negative, changing sign):

Which proves the left part of the inequality.

To obtain the second part of the inequality, instead of adding and subtracting , we add and subtract . And follow same steps.

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Given:

Function is convex.

Prove:

We use the results from paragraph 3. From the definition of the gradient (approximation):

So we can mark the gradients:

Since: and using the inequation from paragraph 3:

Since and using the inequation from paragraph 3:

Summing all together, we obtain:

Which is what was required.

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Prove that if convex, if and only if for all is not decreasing.

Solution:

*Direction 1*

is convex. We will use the previous statement from paragraph 3 again.

*Case 1.*

We assume ; .

Then:

Thus:

Which means that the gradient is not decreasing.

*Case 2.*

We assume ; .

Then:

Thus,

Which means the gradient is not decreasing.

*Case 3.*

We assume ; .

Then:

Thus,

Which means the gradient is not decreasing.

*Direction 2*

is not decreasing.

Let’s take 3 points, s.t.

From the property of not decreasing gradient, we get:

Which means:

We multiply by which is negative. Change sign.

We can express as a convex combination of and :

Then:

Pasting:

We can divide both parts by . Since its negative, we change the sign:

And replace the with the declaration:

Which is the basic characteristic of the convex function.

Thus, is convex.