

086761 - Homework 1

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1.1

$$f_x(x) = \frac{1}{\sqrt{|2\pi\Sigma_x|}} \cdot e^{-\frac{1}{2}(x-\mu_x)^T \Sigma_x^{-1} (x-\mu_x)} \quad (1.1)$$

1.2

μ_y, Σ_y can be calculated using expectation properties (and do not depend on y being Gaussian):

$$\mu_y = \mathbb{E}Ax + b = A\mathbb{E}x + b = \boxed{A\mu_x + b} \quad (1.2)$$

$$\Sigma_y = \mathbb{E} \left[(Ax + b - \mu_y)(Ax + b - \mu_y)^T \right] = \mathbb{E} \left[A(x - \mu_x)(x - \mu_x)^T A^T \right] = \quad (1.3)$$

$$A\mathbb{E} \left[(x - \mu_x)(x - \mu_x)^T \right] A^T = \boxed{A\Sigma_x A^T} \quad (1.4)$$

Next, we show that y is Gaussian. In the case where A is invertible this can be directly obtained from random vector transformation formula (1-to-1 transformation). Since $y = Ax + b$ the Jacobian matrix is A and since the mapping is one-to-one (A is invertible) according to the transformation theorem:

$$f_Y(y) = \frac{1}{|A|} f_X(A^{-1}(y - b)) \quad (1.5)$$

ignoring the constant normalization factors (since they don't depend on x and only guarantee the distributions' summing to 1) we calculate the exponent in the right hand side:

$$-\frac{1}{2} \left(A^{-1}(y - b) - \mu_x \right)^T \Sigma_x^{-1} \left(A^{-1}(y - b) - \mu_x \right) = \quad (1.6)$$

$$-\frac{1}{2} (y - b - A\mu_x)^T A^{-T} \Sigma_x^{-1} A^{-1} (y - b - A\mu_x) = \quad (1.7)$$

$$-\frac{1}{2} (y - (A\mu_x + b))^T \left(A\Sigma_x A^T \right)^{-1} (y - (A\mu_x + b)), \quad (1.8)$$

which exactly gives a Gaussian distribution (in particular, with the mean and variance we calculated beforehand).

We now deal with the case where $A \in \mathbb{R}^{m \times n}$ is not invertible - either rectangular or square with zero determinant. Consider the SVD decomposition of A :

$$A = U\Sigma V^T \quad (1.9)$$

Here, U and V are square invertible matrices of respective sizes $m \times m$ and $n \times n$. Note that $V^T = V^*$ as A is real. Σ is a rectangular matrix the same size as A with the main diagonal containing the singular values (and zero elsewhere). We can now write:

$$y = U\Sigma V^T x + b \quad (1.10)$$

Here, $V^T x$ is Gaussian, since V is invertible. Without loss of generality denote

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \quad V^T x = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (1.11)$$

where Σ_1 is (square) diagonal, with positive entries (and in particular, is invertible), and has the same number of columns as the length of vector v_1 . We have then that

$$\Sigma V^T x = \begin{pmatrix} \Sigma_1 v_1 \\ 0 \end{pmatrix}, \quad (1.12)$$

where $\Sigma_1 v_1$ is Gaussian since v_1 is and Σ_1 is invertible, and hence $\Sigma V^T x = U^T(y - b)$ is (possibly degenerate) Gaussian (and hence, so is y).

2

2.1

$$p(x | z) = \frac{p(z | x)p(x)}{p(z)} = \frac{p(z | x)p(x)}{\int_x p(z | x)p(x) \cdot dx} \propto p(z | x) \cdot p(x) \quad (2.1)$$

2.2

We are interested in the MAP estimate:

$$x_{MAP}^* \doteq \arg \max_x p(x | z) = \arg \max_x p(z | x) \cdot p(x) = \arg \max_x \log p(z | x) + \log p(x) = \quad (2.2)$$

$$\arg \min_x \|z - Hx\|_R^2 + \|x - x_0\|_{\Sigma_0}^2 \quad (2.3)$$

Develop the latter:

$$\|z - Hx\|_R^2 + \|x - x_0\|_{\Sigma_0}^2 = \left\| R^{-1/2}(Hx - z) \right\|^2 + \left\| \Sigma_0^{-1/2}(x - x_0) \right\|^2 = \quad (2.4)$$

$$\left\| \begin{pmatrix} R^{-1/2}(Hx - z) \\ \Sigma_0^{-1/2}(x - x_0) \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} R^{-1/2}H \\ \Sigma_0^{-1/2} \end{pmatrix} x - \begin{pmatrix} R^{-1/2}z \\ \Sigma_0^{-1/2}x_0 \end{pmatrix} \right\|^2, \quad (2.5)$$

from where x_{MAP}^* can be obtained as the least-squares solution using the pseudoinverse $A^\dagger b = (A^T A)^{-1} A^T b$, or equivalently proceed directly, noting that the function is convex and has a unique extremum which is the minimum. Develop the above into:

$$= x^T (H^T R^{-1} H + \Sigma_0^{-1}) x - 2 (z^T R^{-1} H + x_0^T \Sigma_0^{-1}) x + (z^T R^{-1} z + x_0^T \Sigma_0^{-1} x_0) \quad (2.6)$$

find the zero of the gradient:

$$\nabla(\cdot) = 2 (H^T R^{-1} H + \Sigma_0^{-1}) x^* - 2 (\Sigma_0^{-1} x_0 + H^T R^{-1} z) = 0 \quad (2.7)$$

$$\implies x_{MAP}^* = \left(H^T R^{-1} H + \Sigma_0^{-1} \right)^{-1} (\Sigma_0^{-1} x_0 + H^T R^{-1} z) \quad (2.8)$$

The associated covariance is

$$\Sigma = \left(H^T R^{-1} H + \Sigma_0^{-1} \right)^{-1}, \quad (2.9)$$

as can be seen from the quadratic term in Eq.(2.6).

3 Code

```
In [2]: import numpy as np
        from numpy.linalg import norm
        from math import sin, cos, pi, asin, atan2, degrees
        from tabulate import tabulate

        np.set_printoptions(suppress=True, precision=11)
```

3.1 Rotations

```
In [19]: def wedge(v):
        return np.matrix([[0, -v[2], v[1]],
                           [v[2], 0, -v[0]],
                           [-v[1], v[0], 0]]);

        def get_rotation_rodriguez(v, theta=None):
            if theta is None:
                theta = norm(v);
                n = v/theta;
            else:
                n = v;

            cM = wedge(n);
            return np.identity(3) + sin(theta)*cM + (1-cos(theta))*cM*cM
```

3.1.1

```
In [14]: def rotx(phi):
        v = phi*np.array([1, 0, 0]);
        return get_rotation_rodriguez(v)

        def roty(theta):
            v = theta*np.array([0, 1, 0]);
            return get_rotation_rodriguez(v)

        def rotz(psi):
            v = psi*np.array([0, 0, 1]);
            return get_rotation_rodriguez(v)

        def angles2rot(phi, theta, psi):
            return rotz(psi)*roty(theta)*rotx(phi)
```

3.1.2

```
In [16]: print('R= \n' + str(angles2rot(phi=pi/4, theta=pi/5, psi=pi/7)))
```

R=

```
[[ 0.72889912554  0.06766479742  0.68126906578]
 [ 0.35101931853  0.81741496597 -0.45674742629]
 [-0.58778525229  0.57206140282  0.57206140282]]
```

3.1.3

$$\begin{pmatrix} \cos(\psi) \cos(\theta) & \cos(\psi) \sin(\phi) \sin(\theta) - \cos(\phi) \sin(\psi) & \sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \sin(\theta) \\ \cos(\theta) \sin(\psi) & \cos(\phi) \cos(\psi) + \sin(\phi) \sin(\psi) \sin(\theta) & \cos(\phi) \sin(\psi) \sin(\theta) - \cos(\psi) \sin(\phi) \\ -\sin(\theta) & \cos(\theta) \sin(\phi) & \cos(\phi) \cos(\theta) \end{pmatrix} \quad (3.1)$$

For the case $\theta = \frac{\pi}{2}$:

$$\begin{pmatrix} 0 & \cos(\psi) \sin(\phi) - \cos(\phi) \sin(\psi) & \sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \\ 0 & \cos(\phi) \cos(\psi) + \sin(\phi) \sin(\psi) & \cos(\phi) \sin(\psi) - \cos(\psi) \sin(\phi) \\ -1 & 0 & 0 \end{pmatrix} \quad (3.2)$$

$$= \begin{pmatrix} 0 & \sin(\phi - \psi) & \cos(\phi - \psi) \\ 0 & \cos(\phi - \psi) & -\sin(\phi - \psi) \\ -1 & 0 & 0 \end{pmatrix} \quad (3.3)$$

For the case $\theta = -\frac{\pi}{2}$:

$$\begin{pmatrix} 0 & -\cos(\psi) \sin(\phi) - \cos(\phi) \sin(\psi) & \sin(\phi) \sin(\psi) - \cos(\phi) \cos(\psi) \\ 0 & \cos(\phi) \cos(\psi) - \sin(\phi) \sin(\psi) & -\cos(\phi) \sin(\psi) - \cos(\psi) \sin(\phi) \\ 1 & 0 & 0 \end{pmatrix} \quad (3.4)$$

$$= \begin{pmatrix} 0 & -\sin(\phi + \psi) & -\cos(\phi + \psi) \\ 0 & \cos(\phi + \psi) & -\sin(\phi + \psi) \\ 1 & 0 & 0 \end{pmatrix} \quad (3.5)$$

```
In [17]: def rot2angles(M):
    sin_theta = -M[2, 0];
    theta = asin(sin_theta);

    if sin_theta==1:
        psi = 0; # arbitrary
        phi = atan2(M[0,1],M[0,2]);
    elif sin_theta==-1:
        psi = 0; # arbitrary
        phi = atan2(-M[0,1],-M[0,2]);
    else:
        # another valid solution: pi-theta.
        cos_theta = cos(theta);
        phi = atan2(M[2,1]/cos_theta, M[2, 2]/cos_theta);
        psi = atan2(M[1,0]/cos_theta, M[0, 0]/cos_theta);

    return phi, theta, psi
```

3.1.4

```
In [22]: phi, theta, psi = (degrees(x) for x in rot2angles(
np.mat(' 0.813797681 -0.440969611 0.378522306; \
          0.46984631  0.882564119 0.0180283112; \
          -0.342020143 0.163175911 0.925416578'))))
print('phi=' + str(phi), 'theta=' + str(theta), 'psi=' + str(psi))

phi=9.999999994217536 theta=19.999999980143038 psi=29.99999998990158
```

3.2 3D Rigid Transformation

```
In [23]: trans = np.mat([-451.2459, 257.0322, 400])
rot = np.mat('0.5363 -0.8440 0; 0.8440 0.5363 0; 0 0 1');

pt = np.mat([450, 400, 50])

Tgl = rot.dot((pt + trans).T)
print(Tgl)

[[-555.20335297]
 [ 351.31482926]
 [ 450.         ]]
```

3.3 Pose Composition

Commanded:

$$x_{k+1}^c = x_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.6)$$

$$R_{k+1}^c = R_k^c \quad (3.7)$$

Actual:

$$x_{k+1}^a = x_k + \begin{pmatrix} 0 \\ 1.1 \end{pmatrix} \quad (3.8)$$

$$R_{k+1}^a = R(\theta = 1^\circ) \cdot R_k^c \quad (3.9)$$

```
In [24]: for ii in range(10):
```