HW1

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1.
By Law of Total Probability,
P($7\, out\, of\, 10\, heads$)=P($7\, out\, of\, 10\, heads|forged$)P(forged)+$P$ ($7\, out\, of\, 10\, heads\, |fair$)P(fair)
By Baye's Theorem,

$$P\left(forged|7\, out\, of\, 10\, heads\right)=\frac{P\left(\ forged\ \cap\ 7\, out\, of\, 10\, heads\right)}{P\left(\ 7\, out\, of\, 10\, heads\right)}$$

$$=\frac{\frac{1}{1000}\left(\begin{array}{c}10\\7\end{array}\right)\left(0.8\right)^{7}\left(0.2\right)^{3}}{\frac{1}{1000}\left(\begin{array}{c}10\\7\end{array}\right)\left(0.8\right)^{7}\left(0.2\right)^{3}+\frac{999}{1000}\left(\begin{array}{c}10\\7\end{array}\right)\left(0.5\right)^{7}\left(0.5\right)^{3}}$$

=0.00171675

2.Denote X-number of boys, Y-number of girls
P$(X=1)=P\left(boy\, is\, born\right)=1$ because families keep giving birth until boy is born
Then $EX=1P\left(X=1\right)=1$

$P\left(Y=0\right)=$ P(boy)=P(boy)=0.5because if boy is born then stop giving birth
P$(Y=n)=P\left(n\, girls\, then\, boy\right)$=P$(girl)^{n}\, P\left(boy\right)=0.5^{n+1}$
Then by the Complete Probability Formula:
EY=$\sum\limits_{n=0}^{\infty}\ P\left(Y=n\right)n=\sum\limits_{n=1}^{\infty}\ n\left(0.5\right)^{n+1}=1$

In conclusion there are the same number of boys and girls in the country.
3.a. $X_i\sim N\left(\mu_i,\sigma_i^2\right)$ iff probability density function of X_i is $f_{X_1}\left(x_1\right)=\frac{1}{\sqrt{2\pi}\sigma_i}e^{-\frac{\left(x_i-\mu_i\right)^2}{2\sigma_i}}$

$$f_{X_1}\left(x_1\right)=\int_{-\infty}^{\infty}f_{X_1,X_2}\left(x_1,x_2\right)dx_2=$$
$$=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}dx_2$$

Let $y_2=\frac{x_2-\mu_2}{\sigma_2}$. $dy_2=dx_2\frac{1}{\sigma_2}$
$...=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)y_2+y_2^2+\rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-\rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right]\right\}\sigma_2 dy_2=...$
$...=\left(\frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}}\exp\left\{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right\}\right)\left(\int_{-\infty}^{\infty}\exp\left\{-\frac{\left(y_2+\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\right)^2}{2(1-\rho^2)}\right\}dy_2\right)\underset{(1)}{=}\sqrt{1-\rho^2}\sqrt{2\pi}\frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}}\exp\left\{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right\}=...$
$=\frac{1}{\sigma_1\sqrt{2\pi}}\exp\left\{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right\}$
(1)$\int_{-\infty}^{\infty}\exp\left\{-\alpha\left(x+u\right)^2\right\}dx=\sqrt{\frac{\pi}{\alpha}}$
Analogously for $f_{X_2}\left(x_2\right)$.

3.b.By Baye's Law, (variation of Baye's Law applied to density functions)

$$f_{X_{1|}|X_2=x_2}(x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} =$$

$$=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}....$$

$$.../\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left\{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right\}=\frac{\sqrt{2\pi}\sigma_2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2-(1-\rho^2)\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}=...$$

$$...=\frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(p\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right]\right\}=...$$

$$=\frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)-p\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]^2\right\}=\frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\exp\left\{-\frac{\left(x_1-\left(\mu_1+\frac{\rho\sigma_1(x_2-\mu_2)}{\sigma_2}\right)\right)^2}{2\sigma_1^2(1-\rho^2)}\right\}$$

Therefore, $X_1|X_2=x_2 \sim N\left(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2),\sigma_1^2\left(1-\rho^2\right)\right)$

This shows that if X_1, X_2 are Gaussian and $Cov\left(X_{1,}X_2\right)=0$ then $X_1, X_2$are independent.

4. If Y=0 then $0 =$ Cov(X,0)$\leq 0\sigma_X^2 = 0$and $\rho = 0$

If X=0 then $0 =$ Cov(Y,0)$\leq 0\sigma_Y^2 = 0$and $\rho = 0$

Otherwise,

Let $Z = X - \frac{Cov(X,Y)}{\sigma_Y^2}Y$

$0 \leq_{(1)} \sigma_Z^2 \leq Cov\left(X-\frac{Cov(X,Y)}{\sigma_Y^2}Y, X-\frac{Cov(X,Y)}{\sigma_Y^2}Y\right) =_{(2)} \sigma_X^2 - 2Cov(\frac{Cov(X,Y)}{\sigma_Y^2}Y,X)+\frac{Cov(X,Y)}{\sigma_Y^4}\sigma_Y^2$

$= \sigma_X^2 - 2\frac{Cov(X,Y)^2}{\sigma_Y^2}+\frac{Cov(X,Y)^2}{\sigma_Y^2}$

(1)variance is non-negative

(2)lineaerity and homogenouity of covariance

Then $Cov\left(X,Y\right)^2 \leq \sigma_X^2\sigma_Y^2 \implies \left|Cov\left(X,Y\right)\right| \leq \sigma_X\sigma_Y \implies -1 \leq \frac{Cov(X,Y)}{\sigma_X\sigma_Y} \leq 1$

By definition, $-1 \leq \rho \leq 1$

5. a. σ_i~$Bin(p,\,10)$

5.b. Eσ_i=np=10p .

Let $\{X_j\}_{j=1}^{10}$ be independent bernoulli experiments (independent coin flips are bernolli experiments).

Then by definition $\sigma_i = X_1 + X_2 + ... + X_{10}$.

Then $E\sigma_i = EX_1 + X_2 + ... + X_{10}$=$EX_1 + ... + EX_n = p + ... + p = 10p$. (because $\{X_j\}$ i.i.d.)

5.c.

$$2\exp\left\{-\frac{2\left(1000\right)\epsilon^2}{\left(10-0\right)^2}\right\} = 2\exp\left\{-20\epsilon^2\right\} \leq 0.05$$

$$\implies 20\epsilon^2 \geq -\ln\left(0.025\right) \implies \epsilon \geq \sqrt{\frac{-\ln(0.025)}{20}}$$

Then $\epsilon_{min} = \sqrt{\frac{-\ln(0.025)}{20}} \approx 0.4295$

6. Let $\{A_i\}_{i=1}^n$ be events in a probability space and P a probability function.

Notice that $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n\left[A_i\backslash\bigcup_{j=1}^{i-1}A_j\right]$ from basic set theory and $\left\{A_i\backslash\bigcup_{j=1}^{i-1}A_j\right\}_{i=1}^n$ are disjoint events.

Claim If $\{B_i\}_{i=1}^n$ pairwise-disjoint events then $P\left(\bigcup_{i=1}^n B_i\right)$ =$\sum_{i=1}^n P\left(B_i\right)$

Proof Base case: n=1 trivial

n=2 :

By Inclusion-Exclusion Formula $P\left(A\cup B\right) = P\left(A\right)+P\left(B\right)-P\left(A\cap B\right)$ for events A, B.

Then if A, B are disjoint $P\left(A\cup B\right) = P\left(A\right)+P\left(B\right)$.

Assume induction hypothesis for n and induce for n+1.

$P\left(\bigcup_{i=1}^{n-1}B_i \cup B_n\right) = P\left(\bigcup_{i=1}^{n-1}B_i\right) + P\left(B_n\right)$ =$\sum_{i=1}^{n-1} P\left(B_i\right) + P\left(B\right)$ =$\sum_{i=1}^{n} P\left(B_i\right)$

QED

By claim, $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n}\left[A_i \backslash \bigcup_{j=1}^{i-1} A_j\right]\right) = \sum_{i=1}^{n} P\left(\left[A_i \backslash \bigcup_{j=1}^{i-1} A_j\right]\right) \leq \sum_{i=1}^{n} P\left(A_i\right)$
because if $A \subset B \implies P\left(A\right) \leq P\left(B\right)$.