Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

Study Material for Week 4

Lecture Three

Recall

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **non-zero** vector X such that $AX = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ.

- **Principal Axes Theorem: -**

  For a conic whose equation is of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by $X = PY$ eliminates the xy term if P is an orthogonal matrix, such that $|P| = 1$, that diagonalizes $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and equation of the rotated conic is given by $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \begin{bmatrix} d & e \end{bmatrix} PY + f = 0$ where eigen vectors are the principle axes.

  Principle axes – These are those directions along which conic is in standard form.

  Example

  1. Find out what type of conic sections / pair of straight lines is represented by given Q. F. Transform it into the principle axis and plot. $Q(x) = x_1^2 + 24x_1x_2 - 6x_2^2 = 5$.

     $Q(x) = x_1^2 + 24x_1x_2 - 6x_2^2 = 5$ which in matrix form is $X^T AX = 5$ ---------- (i)

     where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$.

     Characteristic equation is $|A - \lambda I| = 0$. $\lambda^2 - S_1\lambda^2 + S_2\lambda + |A| = 0$. $S_1 = -5$, $|A| = -6 - 144 = -150$. $\therefore \lambda^2 + 5\lambda - 150 = 0 \Rightarrow \lambda = 10, -15$.

  Eigen vector for $\lambda = -15$, $[A + 15I]\ X = 0 \Rightarrow \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Independent equation is $4x_1 + 3x_2 = 0$. Solution is $x_2 = t, x_1 = \frac{-3}{4}t$. $\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(3/4)t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} t$.

  Therefore eigen vector is $X_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

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Eigen vector for $\lambda = 10$, $[A - 10I]\ X = 0 \Rightarrow \begin{bmatrix} -9 & 12 \\ 12 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Independent equation is $-3x_1 + 4x_2 = 0$. Solution is $x_2 = t, x_1 = \frac{4}{3}t$. $\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (4/3)t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} t$.

Therefore eigen vector is $X_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

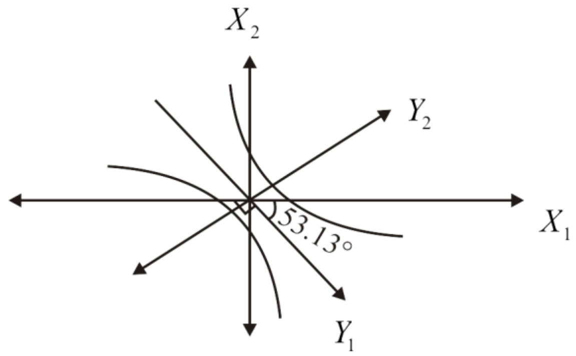
To construct orthogonal modal matrix, divide each eigen vector by its norm.

$W_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ and $W_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$. Consider the transformation $X = PY$

where, $P = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$. Thus $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Putting $X = PY$ in (i),

we have $Y^T DY = 5$, i.e., $\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} -15 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} y_1 & y_2 \end{bmatrix} = 5$, i.e., $-15y_1^2 + 10y_2^2 = 5$.

This can be rearranged as $\frac{y_1^2}{-\frac{1}{3}} + \frac{y_2^2}{\frac{1}{2}} = 1$.



The Curve is Hyperbola.

It intersect Y_2 axis at $\left(0, \frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{-1}{\sqrt{2}}\right)$ respectively.

2. Find out an orthogonal matrix P such that $P^T AP$ is diagonal. Sketch the graph in each of the equations. $3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0$.

Consider, $3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0$. In matrix form this can be written as

$$X^T AX + \begin{bmatrix} 16\sqrt{2} & 0 \end{bmatrix} X = 32 \text{ ---------- (i)}, \text{ where } A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Characteristic equation is $[A - \lambda I] = 0 \Rightarrow \lambda^2 - S_1\lambda + |A| = 0$. $S_1 = 6, |A| = 9 - 25 = -16$.

$\therefore \lambda^2 - 6\lambda - 16 = 0 \quad \lambda = 8, -2$.

Eigen vector for $\lambda = 8$. $[A - 8I]\ X = 0 \Rightarrow \begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Independent equation is

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$x_1 + x_2 = 0$. Therefore the solution is $x_1 = t, x_2 = -t$. $\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t$. Therefore eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Eigen vector for $\lambda = -2$, $[A + 2I]\ X = 0 \Rightarrow \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Independent equation is

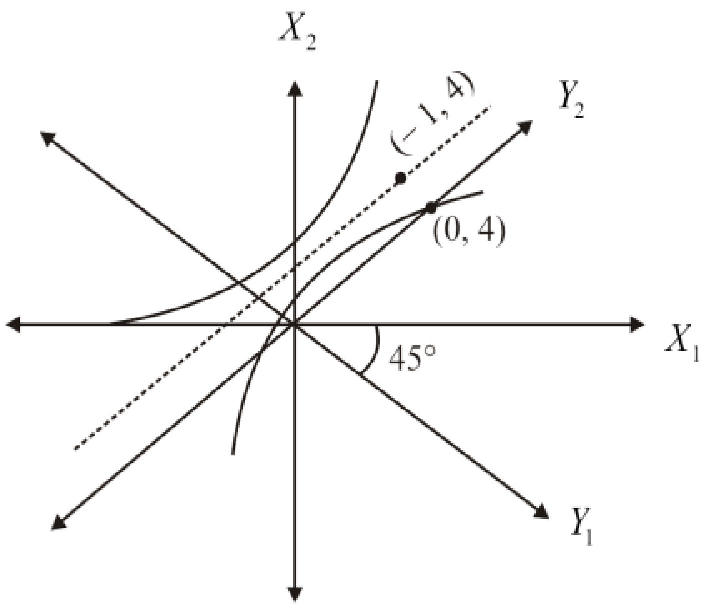
$x_1 - x_2 = 0$. Therefore the solution is $x_1 = t, x_2 = t$. $\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$. Therefore eigen vector is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To construct orthogonal modal matrix, divide each eigen vector by its norm. $W_1 = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $W_2 = \dfrac{X_2}{\|X_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Consider the transformation $X = PY$ where, $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Thus $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Putting $X = PY$ in (i), we have $Y^T DY + \begin{bmatrix} 16\sqrt{2} & 0 \end{bmatrix} PY = 32$, i.e.,

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 16\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 32.$$ This gives

$8y_1^2 - 2y_2^2 + 16y_1 + 16y_2 = 32 \Rightarrow 8(y_1^2 + 2y_1) - 2(y_2^2 - 8y_2) = 32$. Competing square

$8[(y_1+1)^2 - 1] - 2[(y_2-4)^2 - 16] = 32 \Rightarrow 8(y_1+1)^2 - 2(y_2-4)^2 = 8$, i.e.,

$(y_1+1)^2 - (y_2-4)^2 = 1$.



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<u>Problem Session</u>

Q.1		Find out what type of conic section (or pair of straight line) is represented by the following equation. Transform it to Principal axis. Express $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in terms of new the new coordinate vector $Y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Sketch the conic with respect to old and new axes.		
	1)	$x_1^2+24x_1x_2-6x_2^2=5$	2)	$x_1^2+4\sqrt{3}x_1x_2+7x_2^2=9$
	3)	$x_1^2+6x_1x_2+9x_2^2=10$	4)	$-30x_1x_2+17x_2^2=128$
	5)	$53x^2-72xy+32y^2=80.$	6)	$16x^2-24xy+9y^2-60x-80y+100=0$