# Linear Transformation

**Lecture 2 :**

- **Invertible Linear Transformation**

A linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ defined by $T(X) = \text{A}X = Y$ is said to be **invertible** or **non singular** or **regular** if the matrix of transformation A is non singular matrix, i.e., invertible. The corresponding inverse transformation $S:\mathbb{R}^m \to \mathbb{R}^n$ is given by $S(Y) = \text{A}^{-1}Y = X$.

**Note That :** If $T:\mathbb{R}^n \to \mathbb{R}^m$ and its inverse is $S:\mathbb{R}^m \to \mathbb{R}^n$ then $S \circ T:\mathbb{R}^n \to \mathbb{R}^n$ is such that $S \circ T(X) = S(T(X)) = S(\text{A}X) = \text{A}^{-1}\text{A}X = X$. Also $T \circ S:\mathbb{R}^m \to \mathbb{R}^m$ is such that $T \circ S(Y) = T(S(Y)) = T(\text{A}^{-1}Y) = \text{A}\text{A}^{-1}Y = Y$. This implies $S \circ T$ is identity map on $\mathbb{R}^n$ and $T \circ S$ is identity map on $\mathbb{R}^m$.

Example :

Is $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{pmatrix}$ regular? If regular, find the inverse transformation.

$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Thus $T(X) = \text{A}X$, where $\text{A} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

Further $\det(\text{A}) \neq 0$. Therefore $T$ is a regular transformation. The inverse transformation $S:\mathbb{R}^2 \to \mathbb{R}^2$ is given by $S\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (y_1 + y_2)/3 \\ (-2y_1 + y_2)/3 \end{pmatrix}$.

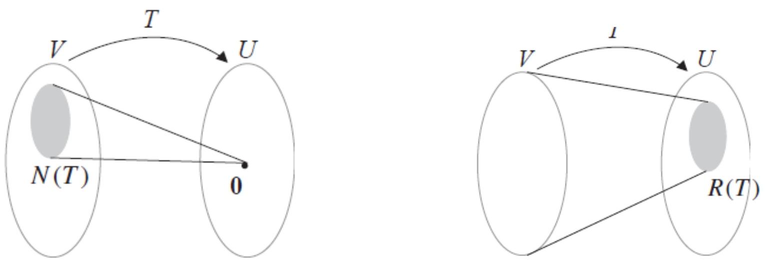
- **Kernel and Range of a Linear transformation**

Let $V$ and $W$ be vector spaces. For a linear transformation $T:V \to W$ the **Kernel** or **null space** of $T$, denoted by $Ker(T)$ *or* $N(T)$, is the collection of all vectors in $v \in V$ which are map to zero vector of $W$. Thus $Ker(T) = N(T) = \{v \in V : T(v) = 0\}$.

The **range** of $T$, denoted by $R(T)$ is the collection of all vectors $w \in W$ which are images of vectors $v \in V$ under the map $T$. Thus $R(T) = \{w = T(v) : v \in V\}$.

Note That : The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is with the matrix representation $[T] = \text{A}$ then

- Range of $T = \{Y \in \mathbb{R}^m :$ such that $\text{A}X = Y\} = col(\text{A}) = column\ space\ of\ \text{A}$.
- Kernel of $T = \{X \in \mathbb{R}^n :\ \text{A}X = 0\} = Null(\text{A}) = Null\ space\ of\ \text{A}$.




**Result :** 1) $Ker(T)$ *or* $N(T)$ is subspace of $V$.

Let $u, v \in Ker(T) \subseteq V \Rightarrow T(u) = 0, T(v) = 0$, $k \in \mathbb{R}$. Now $T(u+v) = T(u) + T(v) = 0$ and $T(ku) = kT(u) = 0$. Therefore $Ker(T)$ is closed under addition and scalar multiplication. Therefore $Ker(T)$ is subspace of $V$.

2) $R(T)$ is subspace of $W$.

Let $w_1, w_2 \in R(T) \subseteq W \Rightarrow \exists v_1, v_2 \in V$ such that $T(v_1) = w_1, T(v_2) = w_2$.

Now $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$. Thus $w_1 + w_2 \in W$ is image of $v_1 + v_2 \in V$. Therefore

$w_1 + w_2 \in R(T)$. For $k \in \mathbb{R}$, $kw_1 = kT(v_1) = T(kv_1)$. Thus $kw_1$ is image of $kv_1$. Therefore $kw_1 \in R(T)$. Therefore $R(T)$ is closed under addition and scalar multiplication.Therefore $R(T)$ is subspace of $W$.

**Examples :**

**1)** Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3, x_4) = \left(x_1 - x_2 + x_3 + x_4,\ 2x_1 - 2x_2 + 3x_3 + 4x_4,\ 3x_1 - 3x_2 + 4x_3 + 5x_4\right).$$ Find

a) Basis and dimension of the range of $T$.
b) Basis and dimension of the kernel of $T$.

$T(1, 0, 0, 0) = (1, 2, 3)$, $T(0, 1, 0, 0) = (-1, -2, -3)$, $T(0, 0, 1, 0) = (1, 3, 4)$ and $T(0, 0, 0, 1) = (1, 4, 5)$. Therefore the matrix of the transformation is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix}.$$

a) To find the basis of image of $T$ which is nothing but column space of $A$, we reduce $A$ to row Echelon form

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the basis of range of $T$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$ and the dimension of $R(T)$ the space is $2$.

b) To find the basis for Kernel of $T$ which is the null space of $A$, consider $AX = 0$. Solving this homogeneous system of linear equations the solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} l+k \\ l \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} l + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} k$$. Therefore the basis for kernel is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ and the dimension of the kernel is $2$.

2) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator and $B = \{v_1, v_2, v_3\}$ a standard basis for $\mathbb{R}^3$. Suppose that $T(v_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, T(v_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, T(v_3) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. a) Is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in R(T)$? b) Find basis and dimension of $R(T)$. c**)** Find basis and dimension of null space $N(T) = Ker(T)$.

$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in R(T)$ if there exist $k_1, k_2, k_3 \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = k_1 T(v_1) + k_2 T(v_2) + k_3 T(v_3)$.

**i.e.,** $$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{array}{r} k_1 + k_2 + 2k_3 = 1 \\ k_1 + k_3 = 2 \\ -k_2 - k_3 = 1 \end{array}.$$

Augmented matrix $\left(\text{A:B}\right)=\begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 1 & 0 & 1 & \vdots & 2\\ 0 & -1 & -1 & \vdots & 1\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 2 & \vdots & 1\\ 0 & -1 & -1 & \vdots & 1\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix}\therefore \rho\left(\text{A:B}\right)=\rho\left(\text{A}\right)$.

Thus the system is consistent. Therefore $\begin{pmatrix}1\\2\\1\end{pmatrix}\in R(T)$.

b) To find basis and dimension of $R(T)$

As images of basis vectors are given, matrix of $T:\mathbb{R}^3\to\mathbb{R}^3$ is A=$\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 0 & -1 & -1\end{bmatrix}$.

As $R(T)=Col(\text{A})$, Reduce A to Echelon form. A $\sim\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & -1\\ 0 & 0 & 0\end{bmatrix}$. Pivot columns of reduce matrix are $1^{st}$ and $2^{nd}$. Therefore $R(T)=span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$. Further vectors $\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}$ are not scalar multiples of each other, hence are linearly independent. Thus basis of $R(T)$ is $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$ and dimension of $R(T)$ is $2$.

c) $Ker(T)=\left\{v\in\mathbb{R}^3:\ T(v)=0\right\}$. Now every $v\in\mathbb{R}^3$ can be expressed as $v=k_1v_1+k_2v_2+k_3v_3$, as $B=\{v_1,v_2,v_3\}$ is basis. Therefore $0=T(v)=k_1T(v_1)+k_2T(v_2)+k_3T(v_3)$

**i.e.,** $\begin{pmatrix}0\\0\\0\end{pmatrix}=k_1\begin{pmatrix}1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}1\\0\\-1\end{pmatrix}+k_3\begin{pmatrix}2\\1\\-1\end{pmatrix}\Rightarrow\begin{matrix}k_1+k_2+2k_3=0\\ k_1+k_3=0\\ -k_2-k_3=0\end{matrix}$. This homogeneous system has reduce form $\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & -1\\ 0 & 0 & 0\end{bmatrix}$. Hence the system possesses 1-parametric solution, $\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}-1\\-1\\1\end{pmatrix}t, t\in\mathbb{R}$.

Therefore $Ker(T)=span\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$. Basis of $Ker(T)$ is $\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$. Dimension of $Ker(T)$ is $1$.

**<u>Also note that</u> :** $\dim\left(Ker(T)\right)+\dim\left(R(T)\right)=1+3=\dim(\mathbb{R}^3)$

**<u>Results</u> :** 1) Dimension of $Ker(T)$ is known as nullity.

2) Dimension of $R(T)$ is known as rank.

3) Rank-Nullity Theorem : Let $T:V\to W$ be a linear then

$$\dim\left(Ker(T)\right)+\dim\left(R(T)\right)=\dim(V).$$

**dim(range)+ dim(kernel)= dim(domain)**

# Linear Transformation

## Problem Session

**Attempt the following**

1) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by
$T(x_1,\ x_2,\ x_3,\ x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4)$.
Find range and kernel of $T$. Find the basis and dimension for range and kernel of $T$.

2) Find range and kernel of $T$, where $T(X) = AX$. Find the basis and dimension for range and kernel of $T$. Verify rank-nullity theorem. $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$

3) Let $T: M_{3\times 2}(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$ be the linear transformation $T(\mathrm{A}) = \mathrm{A}^t$ Find the kernel of *T*.

4) Let $T:\mathbb{R}^5 \to \mathbb{R}^4$ defined by $T(X) = AX$, $A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$
Find range and kernel of $T$. Find the basis and dimension for range and kernel of $T$. Verify rank-nullity theorem.

5) Let $T:\mathbb{R}^5 \to \mathbb{R}^7$ be the linear transformation
a) Find the dimension of the kernel of $T$ if the dimension of the range is 2.
b) Find the rank of $T$ if the nullity of $T$ is 4.
c) Find the rank of $T$ if $Ker(T) = \{0\}$.

6) Show that $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 + x_3 \\ 3x_1 + 3x_2 + x_3 \\ 2x_1 + 4x_2 + x_3 \end{pmatrix}$ is invertible. Hence find it's inverse.

7) Given the transformation $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Find the coordinates $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of $X$ corresponding to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ in $Y$.

8) Find the kernel of $T: P_3 \to P_2$ defined by
$T\left(a_0x^3 + a_1x^2 + a_2x + a_3\right) = 3a_0x^2 + 2a_1x + a_2$. Find kernel of $T$.

9) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Use the given information to find the nullity of $T$ and give a geometric description of the kernel and range of $T$.
a) $rank(T) = 2$ b) $rank(T) = 1$ c) $rank(T) = 0$ d) $rank(T) = 3$

10) Let $T: P_2 \to \mathbb{R}$ be a linear transformation defined as $T(p) = \int_0^1 p(x)dx$. Find kernel of $T$.

11) Find the nullity of $T$
a) $T:\mathbb{R}^4 \to \mathbb{R}^3,\ rank(T) = 2$ b) $T:\mathbb{R}^5 \to \mathbb{R}^2,\ rank(T) = 2$
c) $T:\mathbb{R}^4 \to \mathbb{R}^4,\ rank(T) = 0$ d) $T: P_3 \to P_1,\ rank(T) = 2$