Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

Study Material for Week 3

In this section you will learn, what are eigen values and eigen vectors? What is a characteristic equation? How to find eigen values and eigen vectors?

Lecture One

First obvious question one may ask why to study eigen values and eigen vectors?

These are some applications where eigen values and eigen vectors are used. This shows why this concept is important in technology.

1. **Communication systems :** To determine a threshold for transmission of information through a communication medium
2. **Designing bridges :** The natural frequency of the bridge is the eigenvalue of smallest magnitude of a system that models the bridge.
3. **Electrical Engineering :** For decoupling three-phase systems through symmetrical component transformation
4. **Designing car stereo system :** Design of the car stereo systems, where it helps to reproduce the vibration of the car due to the music
5. **Mechanical Engineering :** Vectors in the principle directions are the eigenvectors and the percentage deformation in each principle direction is the corresponding eigenvalue
6. Oil companies frequently use eigenvalue analysis to explore land for oil

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **non-zero** vector X such that $\mathrm{A}X = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ .

Geometrically, eigen vectors are those non zero vectors which get mapped on to their scalar multiples by matrix A .

Now, By definition λ is eigen value then for non-zero vector X , $\mathrm{A}X = \lambda X$ which implies $(\mathrm{A} - \lambda I)X = 0$. This is a homogeneous system of linear equations in components of X and it will have a non trivial solution if and only if determinant of $\mathrm{A} - \lambda I$, $|A - \lambda I| = 0$.

This equation is known as **characteristic equation.** If A is a $n \times n$ matrix then this is n^{th} degree polynomial in λ and will have n roots which are called as **eigen values** of A .

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Eigen values are also called as **characteristic values / latent roots / proper values.**

The set of all eigen values of matrix A is known as **spectrum of** A.

- **Characteristic equation for** 2×2 **matrix.**

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ Then } |A-\lambda I|=0 \;\Rightarrow\; \lambda^2-S_1\lambda+|A|=0, \text{ where}$$

$S_1=$ sum of diagonal elements $=$ trace of $A=a_{11}+a_{22}$, $|A|=\det(A)=a_{11}a_{22}-a_{21}a_{12}$.

If λ_1,λ_2 are the roots of characteristic equation , then $\lambda_1+\lambda_2=S_1=$ trace of A,

$\lambda_1\cdot\lambda_2=|A|=$ determinant of A.

- **Characteristic equation for** 3×3 **equation**

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \text{ Then } |A-\lambda I|=0 \Rightarrow \lambda^3-S_1\lambda^2+S_2\lambda-|A|=0, \text{ where}$$

$S_1=$ sum of diagonal elements=trace of A$=a_{11}+a_{22}+a_{33}$.

$S_2=$ sum of minors of diagonal elements of A$=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

$\det(A)=|A|=a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{23})$.

If $\lambda_1,\lambda_2,\lambda_3$ are the roots of characteristic equation , then $S_1=\lambda_1+\lambda_2+\lambda_3$,

$S_2=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$ and $\det(A)=|A|=\lambda_1\lambda_2\lambda_3$.

<u>**Example :**</u>

1. Is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ a eigen vector of $A=\begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$? if so find the corrosponding eigen value.

Yes, because $\begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix}=\begin{bmatrix} 4+6 \\ -3-27 \end{bmatrix}=\begin{bmatrix} 10 \\ -30 \end{bmatrix}=10\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Therefore the corresponding eigen value is 10.

2. Are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ eigen vectors of $A=\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ corrosponding to same eigen value? If so find the corrosponding eigen value.

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -8+2 \\ 2+1 \\ -4+4 \end{bmatrix}=\begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}=3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} -12+3 \\ 3-3 \\ -6+9 \end{bmatrix}=\begin{bmatrix} -9 \\ 0 \\ 3 \end{bmatrix}=3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

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Therefore $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors of A corresponding to eigen value 3.

3. Is $\lambda = -2$ a eigen value of $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$? Why or why not?

No, because $\det(A + 2I) \neq 0$ or $\rho(A + 2I) = 3 = \text{order of matrix A}$.

- **Eigen space:** The set of all eigen vectors corresponding to eigen value λ, is a subspace of $\mathbb{R}^n$ called as eigen space of λ.
- **Algebraic Multiplicity:** Let *A* be a matrix of order *n*. The number of times the eigen value λ is repeated is known as algebraic multiplicity. e.g. If *A* has eigen values $\lambda_1 = \lambda_2 = \lambda_3, \underbrace{\lambda_4, \lambda_5, \dots \lambda_n}_{\text{distinct}}$ then algebraic multiplicity of λ_1 is 3.
- **Geometric Multiplicity: is the number of linearly independent eigen vectors corresponding** to Eigen value λ.
- **Relation betweenAlgebraic Multiplicity and Geometric Multiplicity :** $\boxed{\text{AM} \geq \text{GM}}$.

<u>**Example**</u> **:**

<u>**1.**</u> Find eigen values and eigen vectors of the matrix, $A = \begin{bmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.

Characteristic equation is $|A - \lambda I| = 0$ .i.e. $\lambda^{3.} - S_1\lambda^2 + S_2\lambda - |A| = 0$, where

$S_1 = \text{sum of diagonal elenents of } A = Trace(A) = 13,$

$S_2 = \text{sum of minors of diagonal elements of } A = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 3 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 0 \\ 2 & 2 \end{vmatrix} = 6 + 18 + 16 = 40,$

$|A| = \begin{vmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 36$. Therefore characteristic of matrix A is $\lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$

$\Rightarrow (\lambda - 2)^2(\lambda - 9) = 0 \Rightarrow \lambda = 2, 2, 9.$ Therefore Eigen values of A are 2, 2, 4.

- **Methods of finding Eigen vectors**

Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector of A for $\lambda = 9$. $[A - 9I]X_1 = 0 \Rightarrow$

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$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -7 & 1 \\ 2 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. We observed that 3^{rd} row of $A - 9I$ is -3 times 1^{st} . Therefore considering independent equations $-x_1 + 0x_2 + 3x_3 = 0$ and $2x_1 - 7x_2 + x_3 = 0$, we have

$x_1 = 3t,\ x_2 = t,\ x_3 = t$. Thus, solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. $\therefore\ X_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

Note that : Algebraic Multiplicity of $(\lambda = 9) = 1$ and Geometric multiplicity of $(\lambda = 9) = 1$.

The eigen space corresponding to eigen value $\lambda = 9$ is $E_{\lambda=9} = span\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$ and the basis is

$\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$. Hence dimension of $E_{\lambda=9}$ is **one**.

Eigen vector for $\lambda_2 = \lambda_3 = 2$, $[A - 2I]X_1 = 0$ i.e. $\begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$,

$\begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\therefore\ \rho[A-2I] = 1 < 3$. Therefore we have only one equation

$2x_1 + 0x_2 + 1x_3 = 0$. Here $\rho[A-2I] = 1$ and no. Of unknowns is 3, hence there are 2 free variables.

Let $x_2 = s$ & $x_3 = t$ $\therefore 2x_1 = -t$ *i.e.* $x_1 = \frac{-t}{2}$. Thus the solution is

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-t}{2} \\ s \\ t \end{bmatrix} = \frac{-t}{2}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore eigen vectors are $X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Algebraic Multiplicity of $(\lambda = 2) = 2$ and Geometric multiplicity of $(\lambda = 2) = 2$.

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The eigen space corresponding to eigen value $\lambda = 9$ is $E_{\lambda=2} = span\left\{\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ and the

basis is $\left\{\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. Hence dimension of $E_{\lambda=2}$ is **two**.

<u>Problem Session</u>

Q.1		**Attempt the following**
	1.	Is $\lambda = -2$ a eigen value of $A = \begin{bmatrix}1 & 0 & -1\\1 & -3 & 0\\4 & -13 & 1\end{bmatrix}$? Justify. If so find the corresponding eigen vectors.
	2.	Find eigen value of $\begin{bmatrix}3 & 2 & 4\\2 & 0 & 2\\4 & 2 & 3\end{bmatrix}$ whose eigen vector is $\begin{bmatrix}4\\2\\-5\end{bmatrix}$.
	3.	Find the eigen values and eigen vectors of $A = \begin{bmatrix}-14 & 1 & 0\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$. State algebraic and geometric multiplicities of each eigen value. Also find the eigen space of each eigen value and state the dimension of each.
	4.	Find a basis for the eigen space of $A = \begin{bmatrix}10 & -9\\4 & -2\end{bmatrix}$.
	5.	Find all eigen values and corresponding eigen vectors of $\begin{bmatrix}3 & -4\\2 & -6\end{bmatrix}$.

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