Random Variables

Random Variables

Study Material for Week 5

Lecture Two

Random Variables

Probability Mass Function (p. m. f.)

The probability mass function of discrete random variable X with range set $\{x_1, x_2, \cdots, x_n\}$ defined on a sample space Ω is the assignment $p_i = p(x_i) = p(X = x_i)$ such that

i) $p(x_i) \geq 0 \ \forall \ i = 1, 2, 3, \cdots, n.$

ii) $\sum_{i=1}^{n} p(x_i) = 1.$

The table containing the value of X along with the probabilities given by probability mass function is called probability distribution of the random variable X .

Examples

1. Consider the experiment of tossing of 3 coins simultaneously.

   $\Omega = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}$.

   Let X be count of heads. Then c. d. f of X can be tabulated as follows.

$X = x_i$	0	1	2	3
$p(X = x_i)$	$1/8$	$3/8$	$3/8$	$1/8$

2. Consider the experiment of tossing of 2 fair dice simultaneously.

   $\Omega = \{(i, j) / 1 \leq i, j \leq 6\}$. Let X be count of sum of the numbers appear on the faces.

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(X = x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Distribution Function or Cumulative Distribution Function (c. d. f)

Let X be a discrete random variable with range set $\{x_0, x_1, .., x_n\}$. The distribution function of X denoted as F_X, is the probability of the event $\{X \leq a\}$, i. e.,

$$F_X(a) = p(X \leq a) = \sum_{x_i \leq a} p(X = x_i)$$

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Let X be a discrete random variable taking values $\{x_1, x_2, \cdots, x_n\}$ with the pmf

$X = x_i$	x_1	x_2	x_3	$\cdots\cdots$	x_n
$p(X = x_i)$	p_1	p_2	p_3	$\cdots\cdots$	p_n

Then the cumulative distribution function (cdf) of X is

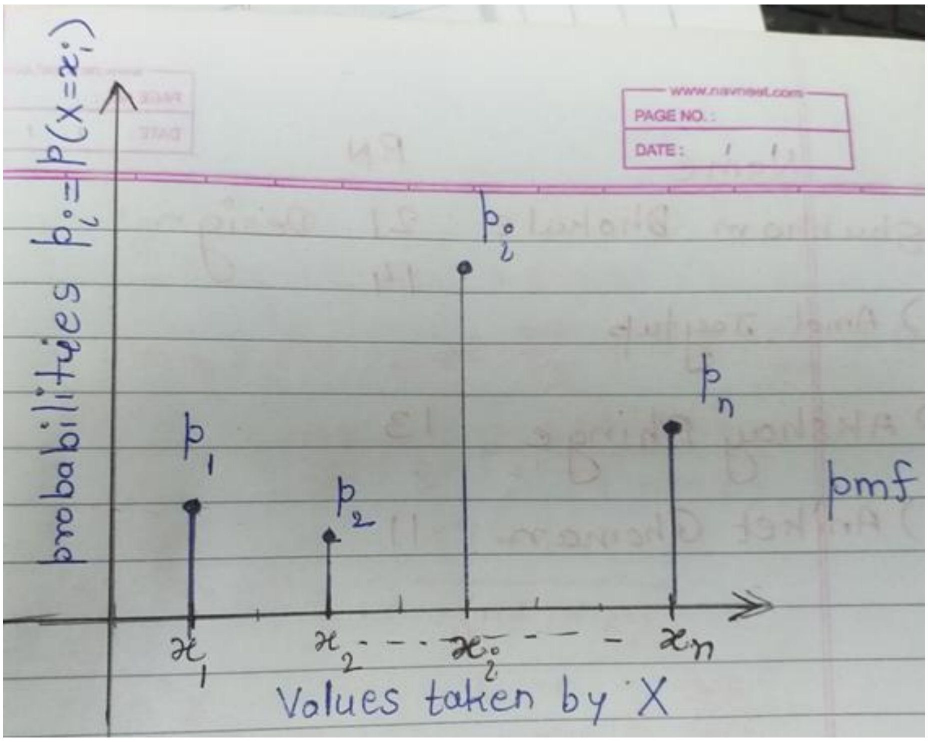
$X = x_i$	x_1	x_2	x_3	$\cdots\cdots$	x_n
$p(X = x_i)$	p_1	p_2	p_3	$\cdots\cdots$	p_n
$F_X(a)$	p_1	$p_1 + p_2$	$p_1 + p_2 + p_3$	$\cdots\cdots$	$\sum_{i=1}^{n} p_i = 1$

Note that:

i) Probability Mass Function (pmf) is a function of discrete variable. There are two ways for graphical representation of pmf. The bar chart and the histogram. The sum of the lengths of the bars in the bar chart is 1 whereas the sum of the areas of the rectangles in the histogram is 1.

ii) F_X is a function of real continuous variable a. Graph of this function is step or staircase.

Graphs of probability mass function:



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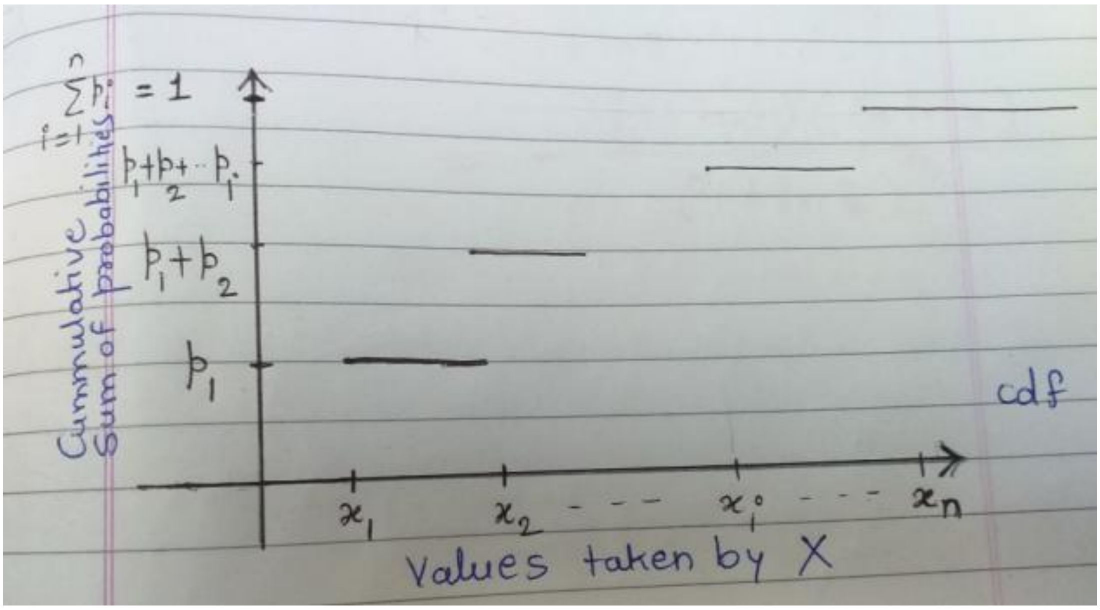
Graphs of cumulative distribution function :



The cumulative distribution function of random variable X has the following properties

1. F_X is a non decreasing function of real continuous variable a.
2. F_X ranges from 0 to 1. This makes sense since $F_X(a)$ is a probability.
3. If X is a discrete random variable whose minimum value is k, then $F_X(k) = p(X \leq k) = p(X = k) = p(X = k)$. If c is less than k, then $F_X(k) = 0$.
4. If the maximum value of X is m, then $F_X(m) = 1$.
5. This is also called the *distribution function*.
6. All probabilities concerning X can be stated in terms of F_X .

Note that : Probability mass function is a function of discrete variable while cumulative distribution function is a function of continuous variable.

Examples

1. If X is a random variable the difference between heads and tails obtained when a fair coin is tossed 3 times. What are the possible values of X and its probability mass function? Also write the distribution function of X .

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Sol^n. $\Omega = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}$. X can take values from -3 to 3, i.e., Range of $X = \{-3, -1, 1, 3\}$. The p.m.f and c.d.f are

$X = x_i$	-3	-1	1	3
$p(X = x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$F_X(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8} = 1$

2. A fair dice is rolled twice. Find the possible values of random variable X and its associated probability mass function, where X is the maximum of the values appearing in 2 rolls.

Sol^n. $\Omega = \{(i, j) / 1 \le i, j \le 6\}$. X can take values from 1 to 6, i.e.,

Range of $X = \{1, 2, 3, 4, 5, 6\}$. The p.m.f is

$X = x_i$	1	2	3	4	5	6
$p(X = x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

3. A random variable X takes values $-3, -1, 2, 5$ with respective probabilities $\frac{2k-3}{10}, \frac{k+1}{10}, \frac{k-1}{10}$ and $\frac{k-2}{10}$. Determine the distribution of X.

Sol^n. The assignments are probabilities, equating the to one

$$\frac{2k-3}{10} + \frac{k+1}{10} + \frac{k-1}{10} + \frac{k-2}{10} = 1 \Rightarrow k = 3 \qquad .$$

Hence the distribution of X is

X	-3	-1	2	5
$p(X = x)$	$3/10$	$4/10$	$2/10$	$1/10$

4. A random variable X has probability mass function (pmf) shown in the following tabular form. Find the value of unknown k. Hence write pmf and cdfof X. Draw graphs of pmf and cdf . Also find

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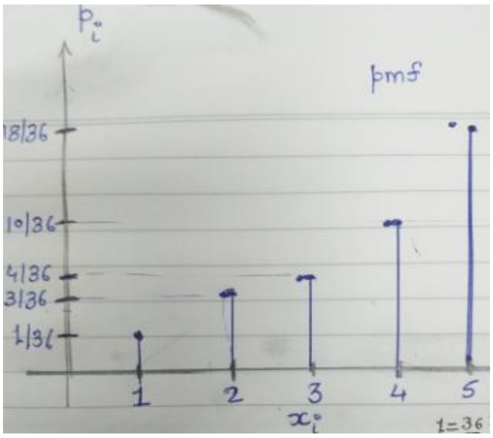
i) $p(1 \le X < 3)$ ii) $p(1 < X \le 3)$ iii) $p(X < 1)$ iv) $p(X > 5)$

X	1	2	3	4	5
$p(X = x)$	$k/36$	$3k/36$	$4k/36$	$10k/36$	$18k/36$

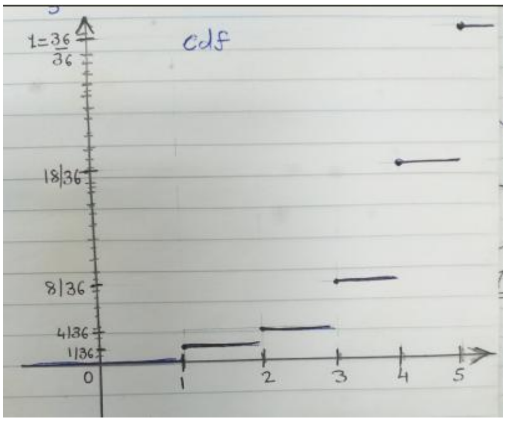
Sol$^{\mathbf{n}}$. Since the above assignment is probability distribution, sum of the probabilities is one implies $k = 1$. Thus the p.m.f and c.d.f of X are

X	1	2	3	4	5
$p(X = x)$	$1/36$	$3/36$	$4/36$	$10/36$	$18/36$
$F_X(x)$	$1/36$	$4/36$	$8/36$	$18/36$	$36/36=1$

Graphs of probability mass function



Graphs of cumulative distribution function



To find probabilities

i) $p(1 \le X < 3) = p(X = 1) + p(X = 2) = \frac{1}{36} + \frac{3}{36} = \frac{4}{36} = \frac{1}{9}$.

ii) $p(1 < X \le 3) = p(X = 2) + p(X = 3) = \frac{3}{36} + \frac{4}{36} = \frac{7}{36}$.

iii) $p(X < 1) = 0$, X is not taking values less than one

iv) $p(X > 5) = 0$, X is not taking values greater than five

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Problem Session

<table>
<tr><td>Q.1</td><td></td><td>Attempt the following</td></tr>
<tr><td></td><td>1)</td><td>A random variable X has the following p.m.f.
<table>
<tr><td>X</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>$p(X = x_i)$</td><td>k</td><td>$3k$</td><td>$5k$</td><td>$7k$</td><td>$9k$</td><td>$11k$</td><td>$13k$</td></tr>
</table>
Find (i) k (ii) $p\left(X \geq 2\right)$ (iii) $p\left(0 < X < 5\right)$
(iv) What is the minimum value of C for which $p\left(X \leq c\right) > 0.5$
(v) What is distribution function of X ?</td></tr>
<tr><td>Q.2</td><td></td><td>Determine k such that the following functions are p.m.f.s</td></tr>
<tr><td></td><td>1)</td><td>$P(x) = k\,x, \quad x = 1, 2, 3, \ldots, 10$</td></tr>
<tr><td></td><td>2)</td><td>$P(x) = k\,\frac{2^x}{x!}, \quad x = 0, 1, 2, 3$</td></tr>
<tr><td></td><td>3)</td><td>$P(x) = k(2x^2 + 3x + 1), x = 0, 1, 2, 3$</td></tr>
<tr><td>Q.3</td><td></td><td>Verify whether the assignment $p(X = n) = 2^{-n}, \; n = 1, 2, 3, \cdots$
is a probability mass function for random variable X .</td></tr>
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