Linear Transformation

Study Material for Week 2

In this section, you will learn about how Linear transformation provides a graphical view of matrix - vector multiplication and how it leads to the applications in computer graphics.

Lecture 4 : Geometric Transformations in $\mathbb{R}^2$

- Geometry of Linear Operators on $\mathbb{R}^2$

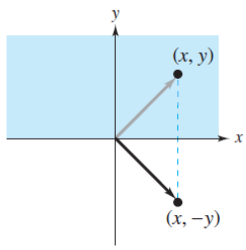
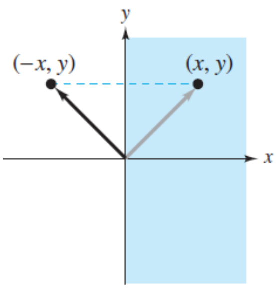
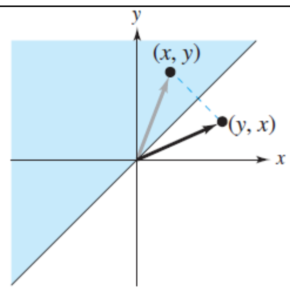
If $T:\mathbb{R}^2 \to \mathbb{R}^2$ is the matrix operator whose standard matrix is $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$Y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} ax_1+bx_2 \\ cx_1+dx_2 \end{bmatrix}.$$

It is natural question that Geometrically how can we view the obove transformation?
We may view entries in the matrices as components of vectors or as co-ordinates of points.
The important property of Linear Transformation useful in computer graphics is
Linear transformations map lines to lines, and hence polygons to polygons.

- **Reflections in the Plane**

The transformations defined by the matrices listed below are called **reflections.**

Sr. No.	Operator	Matrix Representation	Geometric Image
1.	Reflection about X - axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	
2.	Reflection about Y - axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	
3.	Reflection about the line $y=x$.	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	

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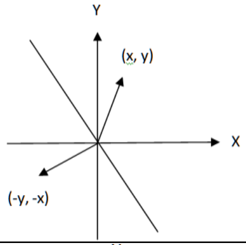
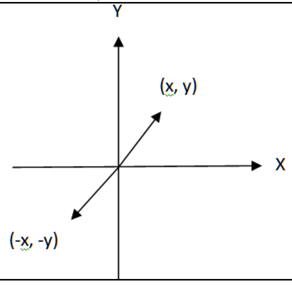
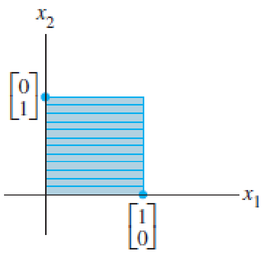
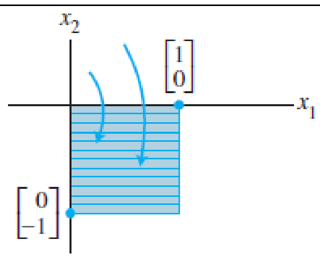
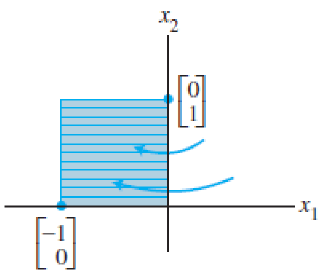
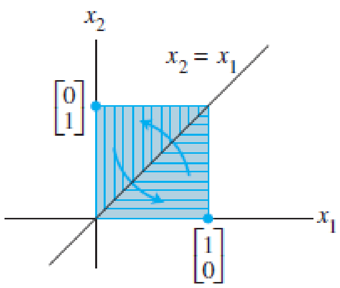
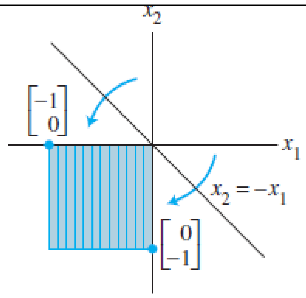
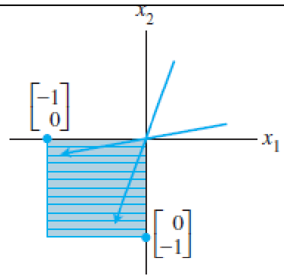
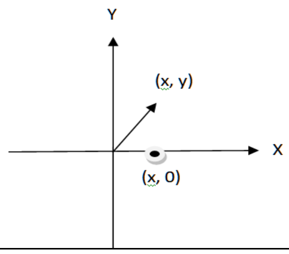
4.	Reflection about the line $y = -x$.	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	
5.	Reflection through origin	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	

Geometric Representation of Unit Square

Consider image of unit square

Original Image	Reflection about X -axis	Reflection about Y -axis
		
Reflection about the line $y = x$.	Reflection about the line $y = -x$.	Reflection through origin
		

- **Projection onto axes**

6.	Projection on x-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	

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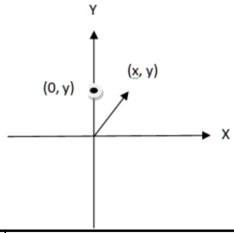
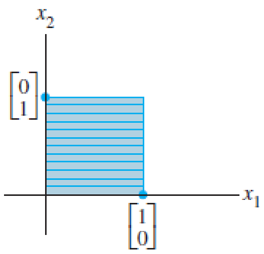
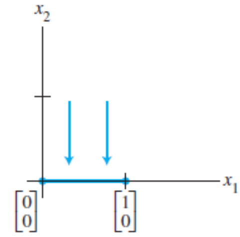
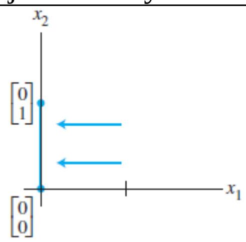
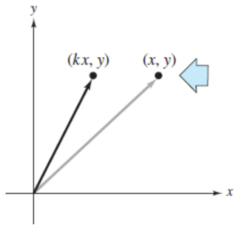
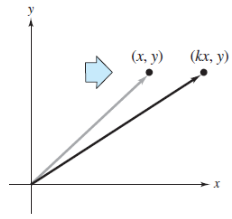
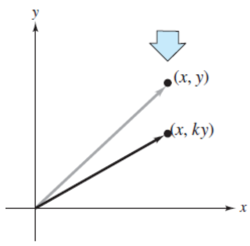
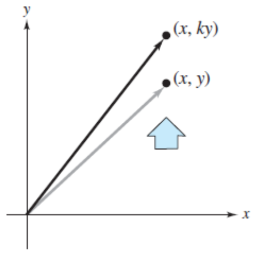
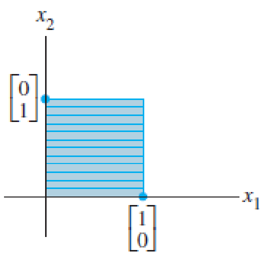
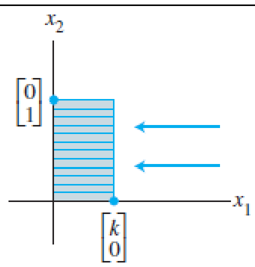
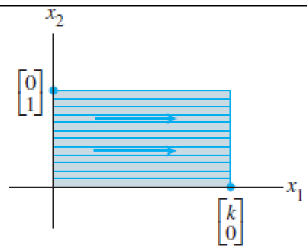
7.	Projection on y-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	

Original Image	Projection on x-axis	Projection on y-axis
		

- **Stretching or Expansion and Squeezing or Contraction in the Plane**

8.	Compression / Contraction $(0 < k < 1)$ or Expansion $(k > 1)$ in the x_1 - direction	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	 Contraction $(0 < k < 1)$  Expansion $(k > 1)$
9.	Compression / Contraction $(0 < k < 1)$ or Expansion $(k > 1)$ in the x_2 - direction	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	 Contraction $(0 < k < 1)$  Expansion $(k > 1)$
10.	Contraction /Contraction $(0 < k < 1)$ or Dilation / Expansion $(k > 1)$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	

Original Image	Compression $(0 < k < 1)$ in the X - direction	Expansion $(k > 1)$ in the X - direction
		 $k > 1$

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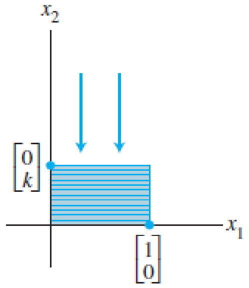
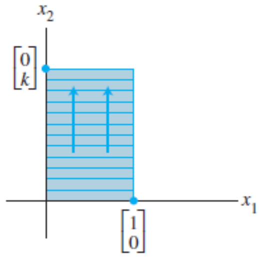
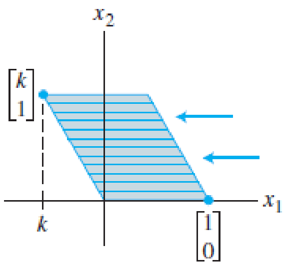
Compression $(0 < k < 1)$ in the Y - direction	Expansion $(k > 1)$ in the Y - direction	
		

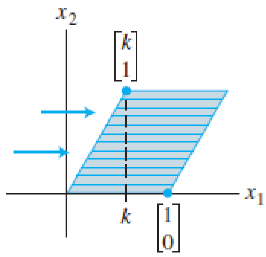
- **Shear in the Plane**

11.	Shear in the X -direction with factor k.	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
12.	Shear in the Y -direction with factor k.	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Horizontal Shear : Shear in the X -direction with factor k.

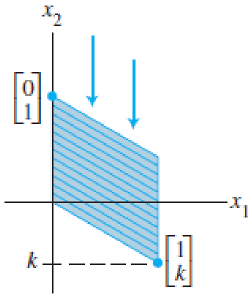


$k < 0$

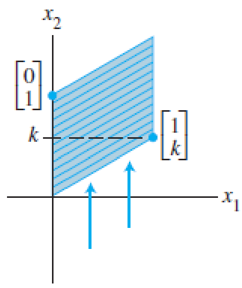


$k > 0$

Vertical Shear : Shear in the Y -direction with factor k.

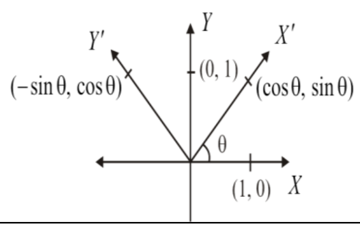


$k < 0$



$k > 0$

- **<u>Rotation</u>**

13.	Counterclock-wise/ Anticlockwise Rotation through an angle θ.	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	

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Examples

1. Find a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first shears in x_1 direction by a factor of 3 and

then reflects about $y = x$.

The standard shear matrix in x_1 direction by a factor of 3 is $A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

The standard matrix of reflection about $y = x$ is $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Hence the required matrix is $A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$.

2. Find a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first reflects about $y = x$ and then shears by a factor of 3 in x_1 direction.

The standard shear matrix in x_1 direction by a factor of 3 is $A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

The standard matrix of reflection about $y = x$ is $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Thus the required transformation is $A_1 A_2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$.

3. Find a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first reflects about $y = x$, followed by rotate in anticlockwise direction through an angle 45^0.

The standard matrix of reflection about $y = x$ is $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The standard matrix of rotation in anticlockwise direction through an angle 45^0 is $A_2 = \begin{bmatrix} \cos 45^0 & -\sin 45^0 \\ \sin 45^0 & \cos 45^0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Thus the required transformation is $A_2 A_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Result

Recall : 1. Elementary Matrix – A matrix obtained by a single row or column transformation on a identity matrix is known as a elementary matrix. For example

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 + 2R_2]{} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, obtained so is a elementary matrix.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 + 2R_2]{} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_{12}]{} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, the matrix in second operation is not a elementary matrix.

2. The matrix obtained by performing elementary transformation on a matrix, can be expressed as a product of elementary matrix and a matrix itself.

3. A elementary row operation is represented by left multiplication and a elementary column transformation by right multiplication.

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e.g. Consider A=$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Perform $R_1 + 2R_2$, A~$\begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix}$. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_2} E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Thus, $\begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix} = E_1$A=$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+8 \\ 0+3 & 0+4 \end{bmatrix}$.

Perform $C_1 + C_2$ on A , A~$\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{C_1 + C_2} E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

A$E_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 10 \end{bmatrix}$.

But note that E_2A=$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 4 & 6 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$.

Theorem : If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is multiplication by an invertible matrix A, then the geometric effect of T is the appropriate succession of shears, compressions, expansions and reflections.

Proof : Since A is invertible, it can be reduced to identity matrix by a finite sequence of elementary row transformation. An elementary row operations can be performed by multiplying on the left by elementary matrix and so there exist elementary matrices $E_1, E_2, ..., E_k$ such that $E_k \cdots E_2 E_1 A = I$. Therefore $A = E_1^{-1} E_2^{-1} \cdots E_n^{-1} I = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$.

1. Express the following matrix as a product of elementary matrices. Describe the effect of multiplication by the given matrix in terms of compression, expression, reflection and shear. A=$\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$.

A can be reduced to identity as follows : $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 4R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Two row operations can be performed on the left successively by

$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$. Therefore $E_2 E_1 A = I$. $A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

It follows that the effect of multiplication by A

i) Shearing by a factor 4 in the x_1 – direction.

ii) Followed by shearing by a factor 2 in the x_2 – direction.

2. Express the following matrix as a product of elementary matrices. Describe the effect of multiplication by the given matrix in terms of compression, expression, reflection and shear. A=$\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$.

A can be reduce to identity as follows :

$$\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Three row operations can be performed on the left successively by

$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Therefore $E_3 E_2 E_1 A = I$.

$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hence the effect of multiplication is

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i) Shearing by a factor 1 in the negative x_1 – direction.

ii) Shearing by a factor 3 in the negative x_2 – direction.

iii) Shearing by a factor 1 in the x_1 – direction.

<u>Problem Session</u>

Q. 1 Express the matrix as a product of the elementary matrices, and then describe the effect of multiplication by the given matrix A in terms of compressions, expansions, reflections and shears

1) $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ 2) $A=\begin{bmatrix}-1 & 2\\ 2 & -1\end{bmatrix}$ 3) $A=\begin{bmatrix}1 & 3\\ 5 & 2\end{bmatrix}$

Q. 2 Give a geometric description of the linear transformation defined by the matrix product.

1) $A=\begin{bmatrix}2 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$

2) $A=\begin{bmatrix}0 & 3\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$

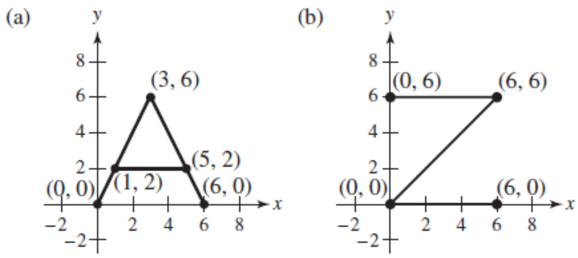
Q. 3 Sketch the image of the rectangle with vertices at $(0,0)$, $(0,2)$, $(1,2)$ and $(1,0)$ under the specified transformation.

1) reflection in the –axis 2) $T(x, y) = (x, y/2)$.

3) $T(x, y) = (2x, y)$. 4) $T(x, y) = (x + y, y)$

5) $T(x, y) = (x, y + 2x)$ 6) $T(x, y) = (x + 4y, y)$

Q. 4 Sketch each of the images with the given vertices under the specified transformations.



1) $T(x, y) = (x + y, y)$ 2) $T(x, y) = (x, x + y)$.

3) $T(x, y) = \left(2x, \frac{1}{2}y\right)$ 4) $T(x, y) = \left(\frac{1}{2}x, 2y\right)$

Q. 5 Identify the transformation and graphically represent the transformation for an arbitrary vector in the plane.

1) $T(x, y) = (x, y/2)$ 2) $T(x, y) = (x/4, y)$

3) $T(x, y) = (4x, y)$ 4) $T(x, y) = (x, 2y)$

5) $T(x, y) = (x + 3y, y)$ 6) $T(x, y) = (x, 2y)$

7) $T(x, y) = (x + 3y, y)$ 8) $T(x, y) = (x, 4x + y)$