Random Variables

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Study Material for Week 5

Lecture Three

Let X be a discrete random variable taking values $\{x_1, x_2, \cdots, x_n\}$ with probability mass function

$X = x_i$	x_1	x_2	x_3	$\cdots\cdots$	x_n
$p(X = x_i)$	p_1	p_2	p_3	$\cdots\cdots$	p_n

1. **Mathematical Expectation / Theoretical Mean** (analogous to Centre of Gravity)

   Theoretical mean or expectation of X denoted as $E(X)$ or μ is defined as

$$E(X) = \sum_{i=1}^{n} x_i p_i .$$

   Expectation value of X provides a central point of the distribution.

Note: Expected value of a random variable may not be actually taken by the variable.

2. **Variance**

   Variance of X denoted as $Var(X)$.

   $Var(X) = \sum_{i=1}^{n} \left(x_i - E(X)\right)^2 p_i$. This can be simplified as

$Var(X) = E(X^2) - \left(E(X)\right)^2$, where $E(X^2) = \sum_{i=1}^{n} x_i^2 p_i$

3. **Standard Deviation** $sd = +\sqrt{Var(X)}$

Results : Let X *and* Y be two random variables. Let a *and* b be any non zero constants.

i) $E(a) = a$

ii) $E(aX + b) = aE(X) + b$

iii) $E(X + Y) = E(X) + E(Y)$

iv) $Var(aX + b) = a^2 Var(X)$

v) $sd(aX + b) = |a| sd(X)$

Example

1. A box contains 8 items of which 2 are defective. A person draws 3 items from the box. Determine the expected number of defective items he has drawn.

Sol^n. Let X be the number of defective items drawn by a person. The pmf of X is

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X	0	1	2
$p(X=x)$	$\frac{20}{56}$	$\frac{30}{56}$	$\frac{6}{56}$

The expected number of defects $= E(X) = \sum x\, p(X=x) = \frac{3}{4}$.

2. A random variable has mean 2 and standard deviation $\frac{1}{2}$. Find

i) $E(2X-1)$

ii) $Var(X+2)$

iii) $sd\left(\frac{3X-1}{-4}\right)$

Sol^n. Given $E(X)=2$ and $sd(X)=\frac{1}{2} \Rightarrow Var(X)=\frac{1}{4}$

i) $E(2X-1)=2E(X)-1=2*2-1=3$.

ii) $Var(X+2)=Var(X)=\frac{1}{4}$.

iii) $sd\left(\frac{3X-1}{-4}\right)=\left|\frac{3}{-4}\right| sd(X)=\frac{3}{4}*\frac{1}{2}=\frac{3}{8}$.

3. A sample space of size 3 is selected at random from a box containing 12 items of which 3 are defective. Let X denote the number of defective items in the sample. Write the probability mass function and distribution function of X. Find the expected number of defective items.

Sol^n. X be the number of defective items in the sample.

X	0	1	2	3
$p(X=x)$	84/220	108/220	27/220	1/220
$F_X(x)$	84/220	192/220	219/220	220/220=1

The expected number of defective items in a sample is $E(X) = \frac{165}{220} = 0.75.$

4. A player tosses two fair coins. The player wins \$2 if two heads occur, and \$1 if one head occur. On the other hand, the player losses \$3 if no heads occur. Find the expected gain of the player. Is the game fair?

Sol^n. The sample space is $\{HH, HT, TH, TT\}$. Since coins are fair,

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$p(HH) = p(HT) = p(TH) = p(TT) = \frac{1}{4}$.

Let X be the player's gain. Then X takes values $-3, 1$ and 2 with

$p(-3) = \frac{1}{4}$, $p(1) = \frac{2}{4}$ and $p(2) = \frac{1}{4}$.

Expected gain $E(X) = -3\times\frac{1}{4} + 1\times\frac{2}{4} + 2\times\frac{1}{4} = \frac{1}{4} = 0.25$.

Thus the expected gain of the player is $\$0.25$. Further $E(X) > 0$, the game is

Favourable to the player.

Problem Session

Q. 1.		Attempt the following
	1)	A men's soccer team plays soccer zero, one, or two days a week. The probability that they play zero days is 0.2, the probability that they play one day is 0.5, and the probability that they play two days is 0.3. Find the expected value, μ, of the number of days per week the men's soccer team plays soccer.
	2)	The probability that a newborn baby does not cry after midnight is $\frac{2}{50}$ The probability that a newborn baby cries once after midnight is $\frac{11}{50}$. The probability that a newborn baby cries twice after midnight is $\frac{23}{50}$. The probability that a newborn baby cries thrice after midnight is $\frac{9}{50}$. The probability that a newborn baby cries for 4 times after midnight is $\frac{45}{50}$ The probability that a newborn baby cries for 5 times after midnight is $\frac{1}{50}$ Find the expected value of the number of times a newborn baby's crying wakes its mother after midnight. (The expected value is the expected number of times per week a newborn baby's crying wakes its mother after midnight.) . Also calculate the standard deviation of the variable as well.
	3)	Suppose you play a game of chance in which five numbers are chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. A computer randomly selects five numbers from zero to nine with replacement. You pay \$2 to play and could profit \$100,000 if you match all five numbers in order (you get your \$2 back plus \$100,000). Over the long term, what is your **expected** profit of playing the game?