Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

<u>Study Material for Week 3</u>

<u>Lecture Two</u>

<u>Recall</u>

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **<u>non-zero</u>** vector X such that $AX = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ.

<u>Note That</u> :

1) If matrix is singular, one of its eigen value is zero.
2) The eigen values of upper and lower triangular matrices are diagonal elements themselves.
3) Eigen vector is a null space of $A-\lambda I$ for every eigen value λ.
4) Eigen vector cannot be zero.
5) λ is an eigen value of A, if and only if the system of homogeneous equations $(A-\lambda I)X = 0$ has a non-trivial solution. This implies $\det(A-\lambda I)$ has to be zero, i.e., rank of $A-\lambda I$ must be less than n.
6) X is an eigen vector of A, if $AX = \lambda X,\ X \neq 0$.

Consider, A be a matrix of order 3×3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values.

1. If all eigen values are distinct, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then there are 3 linearly independent eigen vectors.
2. If one of the eigen values is repeated, say, $\lambda_1 \neq \lambda_2 = \lambda_3$, then

$$\begin{cases} 2 \text{ linearly independent eign vectors if } rank,\ \rho(A-\lambda_1 I) = 1 \\ 1 \text{ linearly independent eigen vectors if } rank,\ \rho(A-\lambda_1 I) = 2 \end{cases}.$$

Further eigen vectors corresponding to distinct eigen values λ_1 and λ_3 are linearly independent.

3. If all the eigen values are repeated, i. e., $\lambda_1 = \lambda_2 = \lambda_3$, then number of linearly independent eigen vectors $= 3 - r$, where $r = \rho[A-\lambda_1 I]$

<u>Note That</u> : Number of linearly independent eigen vectors corresponding to each eigen value is the dimension of the null space of $A-\lambda I$, i. e., dimension of null space of $A-\lambda I$, $\dim \text{Null}(A-\lambda I)$.

Eigen Values and Eigen Vectors

Properties of eigen values and eigen vectors

If X is an eigen vector of A ,corresponding to eigen value λ , then

1. λ^k is eigen value of A^k with same eigen vector X.
2. If all eigen values of A are non-zero the eigen values of A^{-1} are $\frac{1}{\lambda}$.
3. eigen values of kA is eigen value of $k\lambda, k \in \mathbb{R}$ with same eigen vector X.
4. eigen values of $A^3 + k_1A^2 + k_2A + k_3I$ is $\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3$, where k_1, k_2 and k_3 are real numbers.

Example

1. If 3 is eigen value of A then find the eigen value of $A^2 + 5A$.

By above property 1 and 3**, eigen value of** A^2 **is** $3^2 = 9$ **and eigen value of** $5A$ **is** $5 \times 3 = 15$**. Therefore** eigen value of $A^2 + 5A$ is $9 + 15 = 24$.

2. For what values of *a*, does the matrix $\begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix}$ have the characteristics listed below.

i) A has an eigen value of multiplicity 2.

ii) A has – 1 and – 2 as eigen values.

iii) A has – 1 and 2 as eigen values.

iv) A has real eigen values.

Characteristic equation of A is $\lambda^2 - S_1\lambda + |A| = 0$.

For given matrix $S_1 = 1$, $|A| = -a$. Therefore equation is $\lambda^2 - \lambda - a = 0$. This will have repeated roots if $b^2 - 4ac = 0$, i.e., $1 + 4a = 0$. This gives $\boxed{a = \frac{-1}{4}}$.

A has -1 and -2 as eigen values. Therefore roots of characteristic equation are -1 and -2. Now $Trace = \text{sum of eigen values} = \text{-1-2=-3}$, but for given matrix trace is one, so this is not possible. Hence there is no real value of a which satisfy given condition.

A has -1 and 2 as eigen values. With these eigen values $Trace = \text{sum of eigen values} = \text{-1+2=1}$

Now $\det = \text{product of eigen values} = -1 \times 2 = -2 = -a$ $\therefore a = 2$.

A has real eigen values, implies discriminant of $\lambda^2 - S_1\lambda + |A| = 0$ must be positive.

Thus $1 + 4a \geq 0 \Rightarrow \boxed{a \geq \frac{-1}{4}}$

Eigen Values and Eigen Vectors

3. Find the eigen values of $\mathrm{A}=\begin{bmatrix}3 & 2 & 4\\ 0 & 1 & 2\\ 0 & 0 & 3\end{bmatrix}$. State geometric and algebraic multiplicities of each eigen value. Is A inverible? If so, find eigen values of A^{-1}.

A is a upper triangular matrix, therefore diagonal elements are eigen values. Therefore eigen values of A are $3,1\&3$. Algebraic multiplicity of eigen value 1 is One as it appears only once, while

3 appears twice, so algebraic multiplicity of eigen value 3 is Two.

As AM of $\lambda=1$ is One, there will be only one eigen vector, hence geometric multiplicity is also One.

Now GM of $\lambda=3$ =dimension of kernel of $\mathrm{A\text{-}3I}$, so we simply check rank of $\mathrm{A\text{-}3I}$.

$\mathrm{A\text{-}3I}=\begin{bmatrix}0 & 2 & 4\\ 0 & -2 & 2\\ 0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 2 & 4\\ 0 & -2 & 2\\ 0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1 & 2\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$. $\rho\left(\mathrm{A\text{-}3I}\right)=2<3.$ Therefore there will be only one eigen vector. Therefore geometric multiplicity of $\lambda=3$ is also One.

As all eigen values of A are non-zero, A is invertible. Eigen values of A^{-1} are $\frac{1}{3},1\&\frac{1}{3}$.

In particular, if A is a **symmetric matrix,** of order *n* then it has *n* linearly independent eigen vectors. Further eigen vectors corresponding to distinct eigen values are always orthogonal. If eigen values are repeared, we can find orthogonal eigen vectors.

<u>Note That</u> :

1. Orthogonal set of vectors are always linearly independent.
2. Eigen values of symmetric matrices are real.
3. If A is a symmetric matrix, then eigen vectors from different eigen spaces are Orthogonal. (u and v are orthogonal if and only if $\langle u,v\rangle=u^Tv=0,\ u,\ v\neq 0$.

<u>Example</u>

1. Find the eigen values and eigen vectors of $\mathrm{A}=\begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}$.

Note that given matrix is symmetric.

Characteristic equation is $|\mathrm{A}-\lambda I|=0\Rightarrow\lambda^3-S_1\lambda^2+S_2\lambda-|\mathrm{A}|=0$

$S_1=6,\ S_2=-4-7-7=-15,\ |\mathrm{A}|=3(-4)-2(-2)+4(4)=-12+4+16=8$

Characteristic equation is $\lambda^3-6\lambda^2-15\lambda-8=0$. $\therefore\ \lambda=8,\ -1,\ -1$.

Eigen Values and Eigen Vectors

Consider eigen vector for $\lambda = 8,\ [\mathrm{A}-8I]X_1 = 0$ $\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - 4x_2 + x_3 = 0 \\ -2x_2 + x_3 = 0 \end{matrix}$. The solution is $x_3 = 2x_2, x_1 = 2x_2$, i.e.,

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 2t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} t, t \neq 0 \in \mathbb{R}$. Therefore eigen vector for $\lambda = 8$ is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Note : AM *and* GM of $\lambda = 8 = 1$.

Consider eigen vector for $\lambda = -1\, [\mathrm{A}+I]\ X = 0 \Rightarrow \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 2x_1 + x_2 + 2x_3 = 0$. $\rho[\mathrm{A+I}] = 1 < 3$. Therefore there are two

linearly independent vectors. The solution is $x_2 = -2x_1 - 2x_3, x_1 = t, x_3 = s, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t - 2s \\ s \end{bmatrix}$,

i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} s, t, s \neq 0 \in \mathbb{R}$. Thus the two linearly independent eigen vectors are

$X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Note AM and GM of λ=-1 are 2.

Note That: $X_1 \perp X_2,\ X_1 \perp X_3$, i.e. , X_1 is orthogonal to both X_2 as well as X_3 . But

$X_2 \not\perp X_3$, i.e., X_2 and X_3 are not orthogonal. Since A is symmetric $(A^T = A)$, we can find

$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for $\lambda = -1$ such that V is orthogonal to X_1 & X_2 OR X_1 & X_3.

$\langle X_1, V \rangle = 0$ & $\langle X_2, V \rangle = 0$ Implies $2x + y + 2z = 0 \,\&\, x - 2y = 0$. Therefore $V = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$.

Thus $\left\{ X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, V = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix} \right\}$ is a set of orthogonal eigen vectors.

If the choice is

$\langle X_1, V \rangle = 0$ & $\langle X_3, V \rangle = 0$ Implies $2x + y + 2z = 0 \& -2y + z = 0$. Therefore $V = \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$.

Thus $\left\{ X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, V = \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix} \right\}$ is a set of orthogonal eigen vectors.

OR apply **Gram-Schmidt** orthogonalization process to $\left\{ X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

to get a set of orthogonal eigen vectors.

2. Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

Charactristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$. $S_1 = 6,\ S_2 = 9,\ |A| = 0$.

Eigen values are $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 3$. Eigen vector for $\lambda_1 = 0$ is $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigen vectors for repeated eigen values $\lambda_2 = \lambda_3 = 3$ are $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

To obtained the orthgognal basis for $\mathbb{R}^3$, by Gram Schmidt process.

Observe that $\langle X_1, X_2 \rangle = 0$ and $\langle X_1, X_3 \rangle = 0$ but $\langle X_2, X_2 \rangle \neq 0$.

By **Gram-Schmidt** process

$$v_1 = X_1\ ,\ v_2 = X_2 - \frac{\langle v_1,\ X_2 \rangle}{\langle v_1,\ v_1 \rangle} v_1 = X_2\ \ as\ \langle v_1,\ X_2 \rangle = \langle X_1,\ X_2 \rangle = 0$$

$$v_3 = X_3 - \frac{\langle v_1,\ X_3 \rangle}{\langle v_1,\ v_1 \rangle} v_1 - \frac{\langle v_2,\ X_3 \rangle}{\langle v_2,\ v_2 \rangle} v_2,\ \langle v_1,\ X_3 \rangle = 0,\ \langle v_2,\ X_3 \rangle = 1,\ \langle v_2,\ v_2 \rangle = 2.$$

$$\therefore v_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\approx\begin{bmatrix}-1\\-1\\2\end{bmatrix}.\ \therefore \text{Orthoganal eigen vectors are }\begin{bmatrix}1\\1\\1\end{bmatrix},\ \begin{bmatrix}-1\\1\\0\end{bmatrix}\text{ and }\begin{bmatrix}-1\\-1\\2\end{bmatrix}.$$

Problem Session :

Q.1		Attempt the following
	1)	Is $\lambda=-2$ eigen value of $\begin{bmatrix}1&0&-1\\1&-3&0\\4&-13&1\end{bmatrix}$? If so find an eigen vector.
	2)	Find the values of a, b and c such that the chractistic polynomial of $A=\begin{bmatrix}0&1&0\\0&0&1\\a&b&c\end{bmatrix}$ is $-\lambda^3+4\lambda^2+5\lambda+6$.
	3)	Find the values of a and b if eigen values of $A=\begin{bmatrix}0&1\\a&b\end{bmatrix}$ are -4 and 7.
	4)	Find orthogonal eigen vectors of $A=\begin{bmatrix}-2&5&4\\5&7&5\\4&5&-2\end{bmatrix}$.
	5)	Find orthogonal eigen vectors of $A=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}$.

Eigen Values and Eigen Vectors