Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

<u>Study Material for Week 3</u>

<u>Lecture Three</u>

<u>Recall</u>

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **<u>non-zero</u>** vector X such that $AX = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ.

Diagonalization of a Matrix

In this section, we are going to find the condition on square matrices so that they are similar to diagonal matrices.

- **Similarity of Matrices**

  A matrix A is said to be similar to B if there exists a non-singular matrix P such that A is expressible as $A = P^{-1}BP$. **Notation is** $A \approx B$

NoteThat : If 2 matrices are similar, then their Eigen values are same.

- A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix.

Diagonalization is the process of finding the non singular matrix P and the diagonal matrix D such that $D = P^{-1}AP$.

- **Method of diagonalization :**

  Let A be a matrix of order 3×3. Let $X_1,\ X_2,\ X_3$ be linearly independent eigen vectors corresponding to eigen values $\lambda_1,\ \lambda_2,\ \lambda_3$, i.e., $AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, AX_3 = \lambda_3 X_3$. Let $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$, consist of columns as linearly independent eigen vectors $X_1,\ X_2,\ X_3$. P is invertible as eigen vectors are linearly independent. Find P^{-1}. Then

$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Matrix P which diagonalizes A is called as **Modal** matrix and D the diagonal matrix containing eigen values of A as diagonal elements, is called as **spectral** matrix.

<u>Results</u> :

1. If a matrix A has distinct eigen values, then it is always diagonalizable.
2. A matrix A of order n if and only if it has n linearly independent eigen vectors.
3. A matrix is diagonalizable if and only if algebraic multiplicity (AM) of each eigen value equals its geometric multiplicity(GM).

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4. A matrix is diagonalizable if its eigen vectors form basis of $\mathbb{R}^n$.

5. If A is symmetric matrix then eigen vectors corresponding to distinct eigen values are always orthogonal. Further if eigen values are repeated then we can find orthogonal eigen vectors. Hence every symmetric matrix is orthogonally diagonalizable.

6. Note that the sequence of eigen vectors selected to construct P, the eigen values also appear with same sequence as diagonal elements in D. This means, X_1, X_2, X_3 are eigen vectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively.

If $P=\begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ then $D=\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. If the choice of P is $P=\begin{bmatrix} X_3 & X_1 & X_2 \end{bmatrix}$

then $D=\begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$.

<u>Example</u>

1. Consider $A=\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$. Is A diagonalizable? If yes, find the model and spectral matrices.

Characteristic equation of A is $\lambda^2-3\lambda-28=0$. Eigen values of A are $\lambda=7,-4$.

Eigen vector corresponding to $\lambda=7$ is $X_1=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\lambda=-4$ is $X_2=\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

As algebraic multiplicity equal to the geometric multiplicity of both the eigen values, A is diagonalizable.

The model matrix $P=\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}$. A has distinct eigen values, hence eigen vectors are linearly independent. $|P|\neq 0$, therefore P is invertible and $P^{-1}=\frac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$.

$$P^{-1}AP=\frac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}=\frac{1}{11}\begin{bmatrix} 77 & 0 \\ 0 & -44 \end{bmatrix}=\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}.$$

Thus the resulting matrix is a diagonal matrix with diagonal entries as eigen values of A. Thus spectral matrix $D=\begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$.

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2. Is $\begin{bmatrix} -14 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ diagonalizable?

Characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |\mathrm{A}| = 0$. $S_1 = -10$, $S_2 = 4 + (-28) + (-28) = -52$,

$|\mathrm{A}| = -14(4) - 1(0) = -56 \quad \therefore \lambda^3 + 10\lambda^2 - 52\lambda + 56 = 0 \Rightarrow \lambda = -14, 2, 2$

Algebraic multiplicity of $\lambda = -14$ is **One**, hence geometric multiplicity of $\lambda = -14$ is also **One**. Algebraic multiplicity of $\lambda = 2$ is **TWO**. We now check geometric multiplicity of $\lambda = 2$. Consider the matrix $\mathrm{A} - \lambda I$.

$$\mathrm{A} - 2I = \begin{bmatrix} -16 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore \rho\left(\mathrm{A} - 2I\right) = 2$$

. Therefore there is only one eigen vector for $\lambda = 2$. Hence geometric multiplicity of $\lambda = 2$ is **ONE**. Thus, AM of $\lambda = 2 \neq$ GM of $\lambda = 2$. Therefore given matrix is not diagonalizable.

Note That: In case of a symmetric matrix, the matrix P is formed by considering orthonormal Eigen vectors as its columns.

Procedure to diagonalize symmetric matrix

- Step 1 Find eigen values of A.
- Step 2 Find eigen vectors of A.
- Step 3 If eigen values are distinct, eigen vectors will be orthogonal.
- If eigen values are repeated, find orthogonal eigen vectors as discussed in Lecture 8
- Step 4 Consider orthogonal eigen vectors. Divide each vector by its norm.
- Step 5 Construct P by taking these orthonormal eigen vectors as columns, i.e.,

$$P = \begin{bmatrix} \dfrac{X_1}{\|X_1\|} & \dfrac{X_2}{\|X_2\|} & \dfrac{X_3}{\|X_3\|} \end{bmatrix}$$

- Step 6 Find inverse of P. As P is orthogonal, $\mathrm{P}^{-1} = \mathrm{P}^t$.
- Step 7 $\mathrm{P}^{-1}\mathrm{AP} = \mathrm{P}^t\mathrm{AP} = \mathrm{D}$ diagonal matrix with diagonal entries as eigen values of A.

Example :

1. Is $\mathrm{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ orthogonally diagonalizable? If so, find the orthogonal modal matrix and the spectral matrix.

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Note that A is a symmetric matri$\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\\2\end{bmatrix}$.x, hence is orthogonally diagonalizable.

To find the orthogonal modal matrix and the spectral matrix.

Characteristic equation $\lambda^3 - S_1\lambda^2 + S_2\lambda - |\mathrm{A}| = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda = 0$. Eigen values are $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 3$. To find eigen vectors

$$\left[\mathrm{A} - 0I\right]X_1 = 0 \Rightarrow \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 0\,.\ \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 1 & -2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}.$$

The reduce system is $x_1 + x_2 - 2x_3 = 0, x_2 - x_3 = 0$. Let $x_3 = t \Rightarrow x_1 = x_2 = t$. Therefore

Eigen vector is $X_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Consider $\mathrm{A} - 3I = \begin{bmatrix}-1 & -1 & -1\\-1 & -1 & -1\\-1 & -1 & -1\end{bmatrix}$. $\begin{bmatrix}-1 & -1 & -1\\-1 & -1 & -1\\-1 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$.

Reduce system $\left(\mathrm{A} - 3I\right)X = 0$ is $x_1 + x_2 + x_3 = 0$. There are two free variables.

Let $y = s,\ z = t \therefore x = -s - t$. The solution is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-s-t\\s\\t\end{bmatrix} = s\begin{bmatrix}-1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

Eigen vectors are $X_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $X_3 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$. We apply Gram-Schmidt process on

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ to find orthogonal set of eigen vectors. Observe that $\langle X_1, X_2\rangle = 0$,

$\langle X_1, X_3\rangle = 0$ but $\langle X_2, X_3\rangle \neq 0$. Let $w_1 = X_1$, $w_2 = X_2 - \dfrac{\langle w_1,\ X_2\rangle}{\langle w_1,\ w_1\rangle}w_1 = X_2$ as

$\langle w_1, X_2\rangle = \langle X_1,\ X_2\rangle = 0$ and $w_3 = X_3 - \dfrac{\langle w_1, X_3\rangle}{\langle w_1, w_1\rangle}w_1 - \dfrac{\langle w_2, X_3\rangle}{\langle w_2, w_2\rangle}w_2, \langle w_1,\ X_3\rangle = 0$,

$\langle v_2,\ X_3\rangle = 1$, $\langle w_2,\ w_2\rangle = 2$. $w_3 = \begin{bmatrix}-1\\0\\1\end{bmatrix} - \dfrac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} \approx \begin{bmatrix}-1\\-1\\2\end{bmatrix}$. Therefore orthogonal

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Eigen vectors are $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\\2\end{bmatrix}$. The orthogonal modal matrix is

$P=\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}$. As P is orthogonal $P^{-1}=P^t=\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\ -1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6}\end{bmatrix}$.

The spectral matrix

$$P^{-1}AP=P^tAP=\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\ -1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6}\end{bmatrix}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}.$$

Application of Diagonalization (powers of matrices)

To find higher powers of given matrix.

If A is diagonalizable with P as modal matrix and D as spectral matrix then

$$A^n=(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})=(PD)(P^{-1}P)(DP^{-1})\cdots(P^{-1}P)(DP^{-1})$$

$$=PD^nP^{-1} \quad \boxed{\therefore A^n=PD^nP^{-1}}$$

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Problem Session

Q. 1		**Attempt the following**
	1)	Is $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ Diagonalizable, if yes diagonalize A.
	2)	Orthogonally diagonalize the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$.
	3)	Diagaonalize $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
	4)	Let $A_{2\times 2}$ be a symmetric matrix having eigen values 1, 1 and one of its eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find A.
	5)	Which of the matrices cannot be diagonalized? a) $\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$ b) $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 3 & -3 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$
	6)	For the matrix $A = \begin{bmatrix} 3 & a & 0 & 0 \\ 0 & 3 & b & 0 \\ 0 & 0 & 3 & c \\ 0 & 0 & 0 & 3 \end{bmatrix}$ i) Find eigen values of A. ii) Find the condition on a, b, and c such that A is diagonalizable. iii) Under what condition does the eigen space of $\lambda = 3$ have dimensions 1? 2? 3?