TAYLOR SERIES

Taylor Series. Suppose that the function f is infinitely differentiable (smooth) at $x = a$.

Then the Taylor series for $f(x)$ centered at $x = a$ is $\sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n$, i.e.,

$$f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n + \cdots \qquad (1)$$

The above series is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***).

$\frac{f^{n}(a)}{n!}(x-a)^n$ is the nth term of the series.

In particular, if $a = 0$ then the series becomes,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots}$$

This series is known as Maclaurin Series.

Being a power series, Taylor series converges for $|x-a| < R$ and diverges for $|x-a| > R$ for some real number R. This number R is known as radius of convergence.

To find the radius of convergence

1) Ratio Test

Let a_n be the nth term and a_{n+1} be the (n+1)th term of the series.

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $L < 1$ series converges and $L > 1$ series diverges.

$R = \frac{1}{L}$ is the radius of convergence.

Different forms of Taylor Series

1. $f(a+h) \approx f(a) + hf^{'}(a) + \frac{h^2}{2!} f^{''}(a) + \cdots + \frac{h^n}{n!} f^{n}(a) + \cdots$

2. $f(a+x) \approx f(a) + xf^{'}(a) + \frac{x^2}{2!} f^{''}(a) + \cdots + \frac{x^n}{n!} f^{n}(a) + \cdots$

Standard Expansions

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

2) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{(2k+1)!}$

4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

5) $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \frac{17}{315} x^7 \cdots\cdots$

6) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \cdots\cdots = \sum_{k=0}^{\infty} \frac{x^{(2k+1)}}{(2k+1)!}$ 7) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots\cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

8) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\cdots$ 9) $\sin^{-1}(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots$

10) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$ 11) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots$

12) $\frac{1}{1+x} = 1 - x^2 + x^3 - x^4 + x^5 \cdots$ 13) $\frac{1}{1-x} = 1 + x^2 + x^3 + x^4 + x^5 + \cdots$

14) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-3)}{3!} x^3 + \cdots$

Note that : Not every infinitely differentiable function equal to its Taylor series.

Applications of Taylor Series

1. Evaluation of Integrals
2. Evaluation of limits in indeterminate form.

Methods to obtain Taylor series

1. Use of Taylor series formula
2. Use of standard expansions
3. Multiplication and division of infinite expansions