Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

Study Material for Week 4

Lecture Two

Recall

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **non-zero** vector X such that $AX = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ.

Let $Q(X) = X^T AX$ be the quadratic form associated with symmetric matrix A.

1) A quadratic form $Q(X) = X^T A X$ is said to be positive definite is $Q(X) > 0 \ \forall \ x \in \mathbb{R}^2 / x \in \mathbb{R}^3$.,i.e., if and only if all eigen values of A are strictly positive.
2) A Q. F. $Q(X) = X^T A X$ is positive semi definite if $Q(X) \geq 0 \ \forall \ X$, i.e. if and only if all eigen values of A are positive including zero.
3) $Q(X) = X^T A X$ is negative definite if $Q(X) < 0 \ \forall \ X$., i.e., if and only if all eigen values of A are strictly negative.
4) $Q(X) = X^T A X$ is negative semi definite if $Q(X) \leq 0$,i.e., if and only if all eigen values of A are negative including zero.
5) $Q(X) = X^T A X$ is indefinite if $Q(X)$ takes both positive and negative values, i.e., some eigen values of A are positive and negative including zero.

- **Index of a real symmetric matrix:** Let A be a real symmetric matrix, then number of positive eigen values is called as **Index** of quadratic form or matrix A.
- **Signature of real symmetric a Matrix:** The difference between the positive eigen values and negative eigen values is called **Signature** of quadratic form or matrix A.
- **Canonical Form :** The representation of a quadratic form free from product terms like $x_1x_2,\ x_2x_3$ etc. is also known as **'Sum of Squares Form' or 'Canonical form'**.

- **Result**

  Let $X \in \mathbb{R}^3$ and A be a 3×3 symmetric matrix. there is an orthogonal change of variable, $X = PY$, that transforms the quadratic form $Q(X) = X^T AX$, into a quadratic form $Q(Y) = Y^T DY$, with no cross product term, i.e., $Q(X) = X^T AX$ reduces to **'Canonical form'** Or **'Sum of Squares'** form $Q(Y) = Y^T DY$.

Eigen Values and Eigen Vectors

- **Method to reduce the quadratic form into a Canonical form by an Orthogonal Transformation**

  Consider, $Q(X) = ax_1^2 + 2hx_1x_2 + bx_2^2$ **OR**

  $$Q(X) = ax_1^2 + 2hx_1x_2 + bx_2^2 + cx_3^2 + 2gx_1x_3 + 2fx_2x_3$$

  This can be expressed in matrix form as $Q(X) = X^T \mathrm{A} X$, where A= $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ **OR**

  A= $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

Step 1 : Find eigen values of A= $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ **OR** A= $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

Step 2 : Find eigen vectors corresponding to each eigen value. If eigen values are distinct, eigen vectors will be mutually orthogonal. In case of repeated eigen values, find orthogonal eigen vectors.

Step 3 : Divide each eigen vector by its norm to normalize each eigen vector.

Step 4 : Construct modal matrix $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ consists of orthogonal eigen vectors of matrix A . P is a modal matrix which orthogonally diagonalizes A .

Step 5 : Consider the transformation $X = \mathrm{P}Y$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Now

$$Q(X) = (\mathrm{P}Y)^T \mathrm{A}(\mathrm{P}Y) = (Y^T \mathrm{P}^T)\mathrm{A}(\mathrm{P}Y) = Y^T(\mathrm{P}^T \mathrm{A}\mathrm{P})Y = Y^T \mathrm{D} Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,$$

where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ spectral matrix consisting of eigen values of matrix A.

Thus $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ is the canonical representation by the orthogonal change of variables $X = \mathrm{P}Y$.

<u>Example</u>

1. Reduce the Q. F. : $2x_1x_2 + 2x_1x_3 - 2x_2x_3 = X^T \mathrm{A} X$ to a canonical form by orthogonal transformation and discuss its nature.

Eigen Values and Eigen Vectors

Matrix of form is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. Characteristic equation is $|A - \lambda I| = 0$.

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$. $S_1 = 0, \quad S_2 = -3, \quad |A| = -1 - 1 = -2$

Thus char, eqn is $\quad \lambda^3 - 3\lambda + 2 = 0 \quad \lambda = 1, 1, -2$

Eigen vector for $\lambda = 1$, $|A - I|\ X = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. Thus there is only one equation as rank of $A - I$ is one. $-x + y + z = 0 \quad x = y + z$. Therefore solution is

$y = t, z = s, X = \begin{bmatrix} t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} s$. Therefore eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and

$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. For eigen vector for $\lambda = -2$, $[A + 2I]\ X = 0 \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Independent equations are $x - y + 2z = 0, y - z = 0$.

Solution is $y = z = t, x = -t, X = \begin{bmatrix} -t - \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t$. Therefore eigen vector is $X_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

As A is symmetric, eigen vectors are orthogonal.

$\langle X_1, X_3 \rangle = 0$ and $\langle X_2, X_3 \rangle = 0$. Let find the third vector which is orthogonal to X_1 and X_3. Let $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\langle X_1, V \rangle = 0$ and $\langle X_3, V \rangle = 0$. This gives, $x + y = 0$ and

$-x + y + z = 0$. Therefore the solution is $y = t, z = -2t, x = -t, \begin{bmatrix} -t \\ t \\ -2t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} t$. Thus, $V = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

Let normalized orthogonal eigen vectors.

Eigen Values and Eigen Vectors

$$W_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad W_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \text{ and } W_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}.$$

Let $P = [W_1 \quad W_2 \quad W_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$

Using the transformation $X = PY, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. $Q(X) = X^T \mathrm{A} X = Y^T D Y = y_1^2 - 2y_2^2 + y_3^2$,

where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. As eigen values of A are positive as well as negative, given form is indefinite. Further , number of positive eigen values is Two and there is only one negative eigen value. Therefore the difference between positive and negative eigen values is 2-1=1. Hence signature = 1.

Problem Session :

Q.1		**Attempt the following**
	1.	Find the orthogonal transformation which reduces the following quadratic form to canonical. Write the canonical representation. Also state nature, index and signature of the form $2x_1x_2 + 2x_1x_3 - 2x_2x_3 + 3x_1^2 + 3x_2^2 + 3x_3^2$
	2.	Find the orthogonal transformation which reduces the following quadratic form to canonical. Write the canonical representation. Also state nature, index and signature of the form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$