Linear Transformation

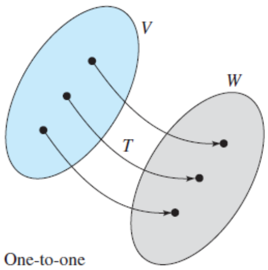
Lecture 3

One-to-One and Onto :

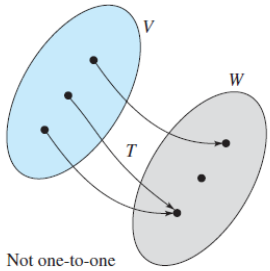
Let V and W be vector spaces and let $T:V\to W$ be linear transformation.

1. The mapping T is called **one-to-one** or **injective** if distinct elements of V must have distinct images in W.
   i.e. $T(u)=T(v)\Rightarrow u=v$ OR $u\neq v\Rightarrow T(u)\neq T(v)$.
2. The mapping T is called **onto** or **surjective** if the range of T is W.
   i.e. given any $w\in W$ there is $v\in V$ such that $w=T(v)$ OR $W=T(V)$.

A mapping is called **bijective** if it is both injective and surjective.



One-to-one



Not one-to-one

Example : Show that $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(v)=\mathrm{A}v,\mathrm{A}=\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$. Show that T is bijective.

Let $u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}, v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}\in\mathbb{R}^2$,

$$T\begin{pmatrix}u_1\\ u_2\end{pmatrix}=T\begin{pmatrix}v_1\\ v_2\end{pmatrix}\Rightarrow\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\Rightarrow\begin{pmatrix}u_1+u_2\\ -u_1\end{pmatrix}=\begin{pmatrix}v_1+v_2\\ -v_1\end{pmatrix}$$

This implies $u_1=v_1\,\&\,u_2=v_2, i.e., \begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}v_1\\ v_2\end{pmatrix}\Rightarrow u=v$. Therefore T is one-one.

To check T is onto. Let $w=\begin{pmatrix}a\\ b\end{pmatrix}\in\mathbb{R}^2$ be any general vector, to find $u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}$ such that $w=T(u)$.

$T(u)=\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=w\Rightarrow\begin{pmatrix}u_1+u_2\\ -u_1\end{pmatrix}=\begin{pmatrix}a\\ b\end{pmatrix}$. This solves to $u_1=-b\,\&\,u_2=a+b$. Thus given $w=\begin{pmatrix}a\\ b\end{pmatrix}\in\mathbb{R}^2$, there is $u=\begin{pmatrix}-b\\ a+b\end{pmatrix}$ such that $T\begin{pmatrix}-b\\ a+b\end{pmatrix}=\begin{pmatrix}a\\ b\end{pmatrix}$. Therefore T is onto.

As T is one-one as well as onto, T is bijective.

Results : 1) Let $T:V\to W$ be a linear transformation. Then T is one-to-one if and only if $Ker(T)=\{0\}$,i.e.,

$T:V\to W$ is One-one if and only if $Ker(T)$ only contains zero or null vector of W.

2) Let $T:V\to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if rank of T equals dimension of W, i. e, $\dim(R(T))=rank\ of\ T=\dim(W)$.

3) Let $T:V\to W$ be a linear transformation, where $\dim(V)=\dim(W)=n\ finite$. Then T is one-one if and only if T is onto.

Linear Transformation

Example :

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is represented by $T(X) = AX$. Find the nullity and rank of and determine whether is one-to-one, onto, or neither.

a) $T:\mathbb{R}^3 \to \mathbb{R}^3, \mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\,of\; T = 3$.

Further $\dim(V) = 3.$ By rank-nullity thm, nullity$= \dim(V) - rank = 0$. Hence $Ker(T) = \{0\}$.

Therefore T is one-one. By result 3 above, $\dim(V) = \dim(W) = 3$, T is onto also.

b) $T:\mathbb{R}^2 \to \mathbb{R}^3, \mathrm{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\,of\; T = 2$.

Further $\dim(V) = 2.$ By rank-nullity thm, nullity$= \dim(V) - rank = 0$. Hence $Ker(T) = \{0\}$.

Therefore T is one-one. By result 2 above, $\dim(W) = 3 \neq rank = 2$, T is not onto also.

c) $T:\mathbb{R}^3 \to \mathbb{R}^2, \mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\,of\; T = 2$.

Further $\dim(V) = 3.$ By rank-nullity thm, nullity$= \dim(V) - rank = 1$. Hence $Ker(T) \neq \{0\}$.

Therefore T is not one-one. By result 2 above, $\dim(W) = 2 = rank$, T is onto also.

d) $T:\mathbb{R}^3 \to \mathbb{R}^3, \mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\,of\; T = 2$.

Further $\dim(V) = 3.$ By rank-nullity thm, nullity$= \dim(V) - rank = 1$. Hence $Ker(T) \neq \{0\}$.

Therefore T is not one-one. By result 2 above, $\dim(W) = 3 \neq rank = 2$, T is not onto also.

Isomorphisms of Vector Spaces

A linear transformation is called an **isomorphism** if it is one-to-one and onto, i.e., bijective**.**

Two vector spaces V and W are said to be **isomorphic if there exist a map** $T: V \to W$ such that

i) T is one-one as well as onto, i.e., T is bijective

ii) T is linear, i.e., $T(ku+v) = kT(u) + T(v), u, v \in V, k \in \mathbb{R}$.

<u>**Result**</u> : Two finite dimensional vector spaces V and W are isomorphic if and only if they are of same dimension, i.e., $\dim(V) = \dim(W)$.

Isomorphic Vector Spaces :

The vector spaces $\mathbb{R}^4, M_{4\times 1}(\mathbb{R}), P_3, M_{2\times 2}(\mathbb{R}), M_{1\times 4}(\mathbb{R})$ are isomorphic to each other as dimension of each vector space is 4.

<u>**Result**</u> **:** Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A$=[T]$, then the Following conditions are equivalent.

1. T is invertible.
2. T is an isomorphism.
3. A is invertible.

And, if T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Linear Transformation

Problem Session

Q. 1 Attempt the following

1) Let $T, S : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ -x+3y \end{pmatrix}$ and $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix}$.

i) Find $-3S,\ 2T+S, T\circ S, S\circ T$

ii) Find rank and nullity of each of the above transformations.

iii) Which of the above transformations are one-one, onto? Justify your answer.

2) Which vector spaces are isomorphic to $\mathbb{R}^6$?

i) $M_{2\times 3}(\mathbb{R})$ ii) P_6 iii) $C[0,6]$ iv) $M_{6\times 1}(\mathbb{R})$ v) P_5

3) Verify that the matrix defines a linear function T that is one-to-one and onto. Justify your answer

i) $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$

4) Define $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $T(\mathrm{A}) = \mathrm{A} - \mathrm{A}^t$. Find the kernel and nullity of T . Hence determine whether the transformation is one-one or not. Is it onto also.

5) A linear transformation is represented by a matrix A . Determine which of the transformations are one- one and onto.

i) $\mathrm{A}=\begin{bmatrix} -1 & 3 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix}$ ii) $\mathrm{A} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

iii) $\mathrm{A} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -2 & -3 \\ 0 & -1 & 3 \end{bmatrix}$ iv) $\mathrm{A} = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix}$

6) Find the kernel and nullity of the linear transformation. Hence state which are one-one transformations.

i) $T : \mathbb{R}^3 \to \mathbb{R}^3, T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ii) $T : \mathbb{R}^3 \to \mathbb{R}^3, T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$

iii) $T : P_3 \to \mathbb{R}, T\left(a_0x^3 + a_1x^2 + a_2x + a_3\right) = a_3$

iv) $T : P_3 \to P_1, T\left(a_0x^2 + a_1x + a_2\right) = 2a_0x + a_1$

v) $T : \mathbb{R}^3 \to \mathbb{R}^2, T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-3y \\ 2y+4z \\ 4x+8z \end{pmatrix}$

7) Let A be a fixed $n\times n$ matrix. Show that $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(\mathrm{B}) = \mathrm{ABA}^{-1}$ is an isomorphism.