Linear Transformation

Linear Transformation
<u>Study Material for Week 1</u>

In this section you will learn about a function that maps a vector space V into a vector space W. This type of functions are denoted by $T: V \to W$. The matrix representation of such map. Isomorphism between two vector spaces.

<u>Lecture One</u>

Mapping / Transformation : Let V and W be two vector spaces. A mapping T from V to W is a function that assigns to each vector $v \in V$ a unique vector $w \in W$. In this case we say that T maps V into W and is written as $T: V \to W$. For each $v \in V$ the vector $w = T(v) \in W$ is the image of v under T.

- **Linear Mapping/Linear Transformation :** Let V and W be two vector spaces. A mapping $T: V \to W$ is called linear transformation or linear mapping if
  i) $T(u+v) = T(u) + T(v),\ u, v \in V$ (Additivity)
  ii) $T(ku) = kT(u),\ k \in \mathbb{R},\ u \in V$ (Homogeneity)
  OR $T(ku+v) = kT(u) + T(v),\ k \in \mathbb{R},\ u, v \in V$

When $V = W$, T is called as linear operator.

Note That :1. Substituting , in condition (ii), we get $T(0) = 0$, thus every linear mapping maps zero vector into zero vector.
2. If for transformation $T: U \to V,\ T(0) = 0,$ then T may or may not be linear.

Some Examples of Linear Transformation

1) Any matrix transformation is a linear transformation.
If A is a matrix of order $m \times n$ then the map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(X) = \mathrm{A}X$, is a linear transformation.
2) Derivative operator, integration operator is linear operators.
3) $T : \mathbb{R}^3 \to \mathbb{R}^2$ is projection of mapping is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$A_{2\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

4) Multiplication by a fixed polynomial is a linear transformation.
5) $T : M_{m\times n}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$ defined by $T(A) = \mathrm{A}^t$ is a linear transformation.

Illustrative Examples

Q 1) Let A be $n \times n$ matrix. Define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(x) = Ax,\ x \in \mathbb{R}^n$.
i) Show that T is a linear transformation.
ii) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$. Find the image of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$ under the mapping $T(x) = Ax.$

Sol$^{\mathbf{n}}$. i) Let $u, v \in \mathbb{R}^n$ & $\alpha \in \mathbb{R}$. Then $T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$ and $T(\alpha u) = A(\alpha u) = \alpha Au = \alpha T(u)$. Hence T is linear.

ii) $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note That : Every matrix of order $m \times n$ determines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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Q 2) Define $T:\mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y+z \\ -x+5y+z \end{bmatrix}$ i) Show that T is a linear transformation. ii) Find all vectors that are mapped to 0.

Sol^n. i) Let $u, v \in \mathbb{R}^3$ & $k \in \mathbb{R}$. Then

a) $T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1+v_1)+2(u_2+v_2)+(u_3+v_3) \\ -(u_1+v_1)+5(u_2+v_2)+(u_3+v_3) \end{bmatrix}$

$= \begin{bmatrix} (u_1+2u_2+u_3)+(v_1+2v_2+v_3) \\ (-u_1+5u_2+u_3)+(-v_1+5v_2+v_3) \end{bmatrix} = \begin{bmatrix} (u_1+2u_2+u_3) \\ (-u_1+5u_2+u_3) \end{bmatrix} + \begin{bmatrix} (v_1+2v_2+v_3) \\ (-v_1+5v_2+v_3) \end{bmatrix} = T(u)+T(v)$

$= \begin{bmatrix} (u_1+2u_2+u_3) \\ (-u_1+5u_2+u_3) \end{bmatrix} + \begin{bmatrix} (v_1+2v_2+v_3) \\ (-v_1+5v_2+v_3) \end{bmatrix} = T(u)+T(v)$.

b) $T\left(k\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \end{bmatrix}\right) = \begin{bmatrix} (ku_1+2ku_2+ku_3) \\ (-ku_1+5ku_2+ku_3) \end{bmatrix} = \begin{bmatrix} k(u_1+2u_2+u_3) \\ k(-u_1+5u_2+u_3) \end{bmatrix}$

$= k\begin{bmatrix} (u_1+2u_2+u_3) \\ (-u_1+5u_2+u_3) \end{bmatrix} = kT\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = kT(u)$.

ii) To find u such that $T(u)=0$, *i.e.*, $\begin{bmatrix} (u_1+2u_2+u_3) \\ (-u_1+5u_2+u_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *i.e.*, $u_1+2u_2+u_3=0$ & $-u_1+5u_2+u_3=0$.

Solving the above homogeneous system of 2 equations in 3 unknowns, the possible set of vectors is $\left\{\begin{bmatrix} 11 \\ -2 \\ 7 \end{bmatrix} t \;/\; t \in \mathbb{R}\right\}$.

Q 3) Determine whether the function $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (2x, x+y)$ is a linear transformation.

Sol^n. $T(x_1, y_1, z_1) = (2x_1, x_1+y_1) = T(u)$ $\quad T(x_2, y_2, z_2) = (2x_2, x_2+y_2) = T(v)$

$T(u+v) = (x_1+x_2, y_1+y_2, z_1+z_2)$

$= \{2(x_1+x_2), [(x_1+x_2)+(y_1+y_2)]\} = (2x_1, x_1+y_1)+(2x_2, x_2+y_2) = T(u)+T(v)$

$T(ku) = T(kx_1, ky_1, kz_1) = (2kx_1, kx_1+ky_1) = k(2x_1, x_1+y_1) = kT(u)$

$\Rightarrow$ It is linear transformation.

Q 4) Show that $T:\mathbb{R} \to \mathbb{R}$ defined by $T(x) = x+1$ is not linear, i. e., translation is not a linear transformation.

Sol^n. Let $x_1, x_2 \in \mathbb{R}$ & $\alpha \in \mathbb{R}$ then $T(x_1+x_2) = (x_1+x_2)+1 = x_1+x_2+1$ while $T(x_1)+T(x_2) = x_1+1+x_2+1 = (x_1+x_2)+2$. Thus, $T(x_1+x_2) \neq T(x_1)+T(x_2)$.

Also, $T(k\,x) = (k\,x)+1$ and $k\,T(x) = k\,(x+1)$ Thus, $T(k\,x) \neq k\,T(x)$.

$\therefore$ T is not a linear operation.

Or equivalently $T(0) = 1 \neq 0$. Therefore T is not a linear transformation.

Q 5) Show that $T:\mathbb{R} \to \mathbb{R}$ defined by $T(x) = x^2$ is not linear.

Sol^n. Let $x_1, x_2 \in \mathbb{R}$. $T(x_1+x_2) = (x_1+x_2)^2$ $\quad T(x_1)+T(x_2) = x_1^2+x_2^2$ $\quad$ Thus, $T(x_1+x_2) \neq T(x_1)+T(x_2)$. $\therefore$ T is not additive.

$T(k\,x) = (k\,x)^2 = k^2x^2$ while $kT(x) = k\,x^2 \neq T(k\,x)$

$\therefore$ T is not satisfying homogeneity also. $\therefore$ T is not linear.

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Q 6) Let $T: M_{n\times n} \to \mathbb{R}$ be the transformation that maps an $n\times n$ matrix to a number set by $T(A)=\det(A)$. Show that the transformation is not linear.

Sol$^{\mathbf{n}}$. $T(A+B)=\det(A+B)\neq\det(A)+\det(B)$ and $\det(kA)=k^n\det(A)\neq k\det(A)$**.**

Therefore T is not linear transformation.

- **Method of Finding Standard Matrix for Linear Transformation**

Consider the linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\mathrm{T}(\mathrm{e}_1)=\begin{bmatrix} a_{11}\\ a_{21}\\ \cdot\\ \cdot\\ \cdot\\ a_{m1}\end{bmatrix},\ \ \mathrm{T}(\mathrm{e}_2)=\begin{bmatrix} a_{12}\\ a_{22}\\ \cdot\\ \cdot\\ \cdot\\ a_{m2}\end{bmatrix},\ \ldots,\ \mathrm{T}(\mathrm{e}_n)=\begin{bmatrix} a_{1n}\\ a_{2n}\\ \cdot\\ \cdot\\ \cdot\\ a_{mn}\end{bmatrix},\quad \text{where}$$

$\{e_1, e_2,\cdots,e_n\}$ is a standard basis for $\mathbb{R}^n$. Then the $m\times n$ matrix whose n correspond to images of $e_1, e_2,\cdots,e_n$ under T, i.e., $T(e_1),\mathrm{T}(\mathrm{e}_2),\ \ldots\ \mathrm{T}(\mathrm{e}_n)$ is called the standard matrix of T. Thus

$$A=\begin{bmatrix}\mathrm{T}(\mathrm{e}_1) & \mathrm{T}(\mathrm{e}_2) & \ldots & \mathrm{T}(\mathrm{e}_n)\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix}.$$

Hence for every $v\in\mathbb{R}^n$, $T(v)=Av.$

Example :

Consider the transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by

$$T(x,\ y,\ z)=\left(x-2y+3z,\ -2x+3y-2z,\ x-y-z\right).$$

This can be expressed as $T(x,\ y,\ z)=\begin{bmatrix}1 & -2 & 3\\ -2 & 3 & -2\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=AX$. Because the standard basis of $\mathbb{R}^3$ is $\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$. $T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\3\\-1\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\-1\end{pmatrix}$.

- **Method of Finding Standard Matrix for general vector spaces**

Consider the linear transformation $T:V\to W$ from a n-dimensional vector space V to a m-dimensional vector space W such that $\mathrm{T}(\mathrm{e}_1)=w_1$, $\mathrm{T}(\mathrm{e}_2)=w_2$, ..., $\mathrm{T}(\mathrm{e}_n)=w_n$, where $\{e_1, e_2,\cdots,e_n\}$ is a standard basis for V. Let $\{f_1,\ f_2,\cdots,\ f_m\}$ be standard basis of W.

Express each image $T(e_i)=w_i$ as a linear combination of $\{f_1,\ f_2,\cdots,\ f_m\}$, basis of W.

Thus $w_1=T(e_1)=a_{11}f_1+a_{21}f_2+\cdots+a_{m1}f_1$, $w_2=T(e_2)=a_{12}f_1+a_{22}f_2+\cdots+a_{m2}f_m,\cdots,$

$w_j=T(e_j)=a_{1j}f_1+a_{2j}f_2+\cdots+a_{mj}f_m,\ \cdots\cdots,\ w_n=T(e_n)=a_{1n}f_1+a_{2n}f_2+\cdots+a_{mn}f_m$.

To obtain the matrix representation of T, arrange the coefficients in linear combination of each $w_j=T(e_j)$ as the j^{th} column of a $m\times n$ matrix, denoted as $[T]$. Thus,

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$$[T]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Example

Consider the transformation $T: M_2(\mathbb{R}) \to P_3$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=ax^3+bx^2+cx+d$.

T is a linear map. $T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}+\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)=T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}+\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)=T\left(\begin{bmatrix} ka_1+a_2 & kb_1+b_2 \\ kc_1+c_2 & kd_1+d_2 \end{bmatrix}\right)$

$$=(ka_1+a_2)x^3+(kb_1+b_2)x^2+(kc_1+c_2)x+(kd_1+d_2)$$

$$=\left(ka_1x^3+kb_1x^2+kc_1x+kd_1\right)+\left(a_2x^3+b_2x^2+c_2x+d_2\right)$$

$$=k\left(a_1x^3+b_1x^2+c_1x+d_1\right)+\left(a_2x^3+b_2x^2+c_2x+d_2\right)=kT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)+T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

Standard basis of $M_2(\mathbb{R})$ is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and standard basis of P_3 is $\left\{x^3, x^2, x, 1\right\}$.

Now $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)=1x^3+0x^2+0x+0, T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)=0x^3+1x^2+0x+0, T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)=0x^3+0x^2+1x+0$ and $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)=0x^3+0x^2+0x+1$. Hence the matrix of T, $[T]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Note That : If T is a linear transformation from a $n-$dimensional vector space V to a $m-$dimensional vector space W, then the matrix of T is of order $m\times n$.

- **Operations with Linear Transformations**

1. Let V and W be vector spaces and let $S,\ T: V \to W$ be linear transformations. The function $S+T$ defined by $(S+T)(v)=S(v)+T(v)$ is a linear transformation from V into W.

   Let $u, v \in V$ and $k \in \mathbb{R}$.

$$(S+T)(ku+v)=S(ku+v)+T(ku+v)$$
$$=\left\{kS(u)+S(v)\right\}+\left\{kT(u)+T(v)\right\}$$
$$=k\left\{S(u)+T(u)\right\}+\left\{S(v)+T(v)\right\}$$
$$=k\left\{\left(S+T\right)(u)\right\}+\left\{\left(S+T\right)(v)\right\}$$

2. If c is any scalar, the function cS defined by $\left(cS\right)(v)=cS(v)$ is a linear transformation from V into W.

   Let $u, v \in V$ and $k \in \mathbb{R}$.

$$cS(ku+v)=cS(ku+v)=c\left\{kS(u)+S(v)\right\}$$
$$=ckS(u)+cS(v)=k\left(cS\right)(u)+\left(cS\right)(v)$$

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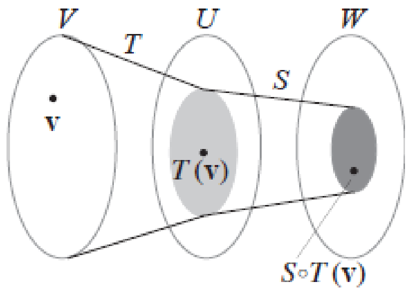
3. Composite Linear Transformation

   Let U, V, and W be vector spaces. If $T:V\to U$ and $S:U\to W$ are linear transformations, then the composition map $S\circ T:V\to W$, defined by $(S\circ T)(\mathbf{v})=S(T(\mathbf{v}))$ is a linear transformation.

   Let $u,v\in V$ and $k\in\mathbb{R}$.

$$(S\circ T)(ku+v)=S\left(T(ku+v)\right)=S\left\{kT(u)+T(v)\right\}$$
$$=kS\left(T(u)\right)+S\left(T(v)\right)=k\left(S\circ T\right)(u)+\left(S\circ T\right)(v)$$



Note That : If $T:V\to U$ is a linear transformation from a n dimensional vector space V to a p dimensional vector space U and $S:U\to W$ is a linear transformation from a p dimensional vector space V to a m dimensional vector space W, then $S\circ T:V\to W$ is a linear transformation from a n dimensional vector space V to a m dimensional vector space W. Hence matrix of $S\circ T$ is of order $m\times n$, $[S\circ T]=[S][T]$.

Example

1. $S,\ T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $S\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x\end{pmatrix}$ and $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x-y\\x-3y\end{pmatrix}$ then find $(S+T)$ and cS.

   $(S+T)\begin{pmatrix}x\\y\end{pmatrix}=S\begin{pmatrix}x\\y\end{pmatrix}+T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x\end{pmatrix}+\begin{pmatrix}2x-y\\x-3y\end{pmatrix}=\begin{pmatrix}3x\\2x-3y\end{pmatrix}$ and

   $(cS)\begin{pmatrix}x\\y\end{pmatrix}=cS\begin{pmatrix}x\\y\end{pmatrix}=c\begin{pmatrix}x+y\\x\end{pmatrix}=\begin{pmatrix}cx+cy\\cx\end{pmatrix}$.

2. Define $T:\mathbb{R}^3\to\mathbb{R}^2$ by $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+2y+z\\-x+5y+z\end{pmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ by $S\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2y\\x+y\\2x\\x-y\end{pmatrix}$.

   Find $S\circ T$ explicitly. Hence find matrix of $S\circ T$. Also verify $[S\circ T]=[S][T]$.

   $T:\mathbb{R}^3\to\mathbb{R}^2$ and $S:\mathbb{R}^2\to\mathbb{R}^4$, therefore $S\circ T:\mathbb{R}^3\to\mathbb{R}^4$.

$$(S\circ T)\begin{pmatrix}x\\y\\z\end{pmatrix}=S\left(T\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=S\begin{pmatrix}x+2y+z\\-x+5y+z\end{pmatrix}=\begin{pmatrix}2(-x+5y+z)\\(x+2y+z)+(-x+5y+z)\\2(x+2y+z)\\(x+2y+z)-(-x+5y+z)\end{pmatrix}=\begin{pmatrix}-2x+10y+2z\\7y+2z\\2x+4y+2z\\2x-3y\end{pmatrix}.$$

To find matrix of $S\circ T$

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Standard basis of $\mathbb{R}^3$ is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and $(S\circ T)\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\0\\2\\2\end{pmatrix}$, $(S\circ T)\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}10\\7\\4\\-3\end{pmatrix}$,

$(S\circ T)\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\2\\2\\0\end{pmatrix}$. Hence $[S\circ T]=\begin{bmatrix}-2&10&2\\0&7&2\\2&4&2\\2&-3&0\end{bmatrix}_{4\times3}$.

To verify $[S\circ T]=[S][T]$

Standard basis of $\mathbb{R}^3$ is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. $T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$. Therefore

$[T]=\begin{bmatrix}1&2&1\\-1&5&1\end{bmatrix}_{2\times3}$. Standard basis of $\mathbb{R}^2$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$. $S\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\2\\1\end{pmatrix}$, $S\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2\\1\\0\\-1\end{pmatrix}$.

Therefore $[S]=\begin{bmatrix}0&2\\1&1\\2&0\\1&-1\end{bmatrix}_{4\times2}$.

Now $[S][T]=\begin{bmatrix}0&2\\1&1\\2&0\\1&-1\end{bmatrix}_{4\times2}\begin{bmatrix}1&2&1\\-1&5&1\end{bmatrix}_{2\times3}=\begin{bmatrix}-2&10&2\\1-1&2+5&1+1\\2&4&2\\1+1&2-5&1-1\end{bmatrix}_{4\times3}\begin{bmatrix}-2&10&2\\0&7&2\\2&4&2\\2&-3&0\end{bmatrix}_{4\times3}$.

<u>Problem Session</u>

Q. 1 Check whether the following maps are linear transformations or not.

1) $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(x,\ y,\ z)=(2x-y+3z,\ x-y+5z)$
2) $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(x,\ y)=(ax+by,\ cx+dy)$, where a, b, c and d are reals.
3) $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T(x,\ y,\ z)=(x+y,\ z+2)$
4) $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T(x,\ y)=(x,\ xy)$
5) $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T(x,\ y,\ z)=(|x|,\ y+z)$
6) $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(x,\ y)=(x^2,\ y)$
7) $T:V\to\mathbb{R}$, where V is an inner product space and $T(v)=\|v\|$.
8) $\mathrm{T}:\mathrm{M}_{m\times m}\to\mathbb{R}$ defined as $T(A)=tr(A)$.
9) $T:P_2\to P_2$ defined as $T(a_0+a_1x+a_2x^2)=a_0+a_1(x+1)+a_2(x+1)^2$.
10) $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $\mathrm{T(x,\ y)=(cosx,\ y)}$.

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Q.2 **Attempt the following**

1) Find a 2×2 that maps $(1,\ 2)^T$ and $(2,\ -3)^T$ into $(-2,\ 5)^T$ and $(3,\ 2)^T$ respectively.

2) Is there exists a 2×2 singular matrix that maps $(1,\ 2)^T$ into $(2,\ -3)^T$? If so, find the linear map represented by the matrix.

3) Is there exists a 2×2 singular matrix that maps $(1,\ 2)^T$ into $(2,\ 4)^T$? If so, find the linear map represented by the matrix.

4) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices. Define $T: M_{2\times 2}\to M_{2\times 2}$ by $T(A)=A+A^T$, where $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   a) Show that T is a linear transformation.

   b) Let $B\in M_{2\times 2}$ be such that $B^T=B$. Find $A\in M_{2\times 2}$ such that $T(A)=B$.

5) Let $B=\{v_1,\ v_2\}$ be the basis of $\mathbb{R}^2$, where $v_1=(-2,\ 1)$ and $v_2=(1,\ 2)$. Let $\mathrm{T}:\mathbb{R}^2\to\mathbb{R}^3$ such that $T(v_1)=(1,\ 2,\ -1)$ and $T(\mathrm{v}_2)=(-1,\ -2,\ 4)$

   Find $T(4,\ -3)$.

6) Let $B=\{v_1,\ v_2,\ v_3\}$ be the basis of $\mathbb{R}^3$, where $v_1=(-2,\ 1,\ 0)$, $v_2=(1,\ 2,\ 1)$ and $v_3=(1,\ 1,\ 1)$. Let $\mathrm{T}:\mathbb{R}^3\to\mathbb{R}^3$ such that

   $T(v_1)=(2,1,-1)$, $T(\mathrm{v}_2)=(-1,\ 1,\ 1)$ and $\mathrm{T}(\mathrm{v}_3)=(1,\ 0,\ 0)$. Find $T(2,\ 4,\ -1)$.

7) Which of the following satisfy $T(v+w)=T(v)+T(w)$ and $T(\alpha u)=\alpha T(u)$.

   a) $T(u)=\dfrac{u}{\|u\|}$ b) $T(u)=u_1+u_2$

   c) $T(u)=(u_1,\ 3u_2,\ -5u_3)$ d) $T(u)=$ largest component of u

8) Find the standard matrix for the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x,\ y,\ z)=(x-2y,\ 2x+y)$.

9) Define $T: C[0,1]\to\mathbb{R}$ by $T(f)=\int_0^1 f(x)dx$. Show that T is a linear operator.

   Also find $T(2x^2-x+3)$

10) Let $T: P_2\to P_2$ such that $T(1)=x, T(x)=1+x$, $T(x^2)=1+x+x^2$.

   Find $T(2-6x+x^2)$.

11) Define $T:\mathbb{R}^2\to\mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x-2y \\ 3x+y \\ 2y \end{pmatrix}$. Find the standard matrix of T.

   Hence find $T\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

12) Let T_1 and T_2 be two linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined as $T_1(x,y,z)=(2x+y,0,x+z)$ and $T_2(x,y,z)=(x-y,z,y)$. Find the matrix of the composite map $T=T_2\circ T_1$ relative to the standard basis.

13) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be a linear operator and $B=\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$ be a basis of

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$\mathbb{R}^3$. If $T\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $T\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-1\\-2\\-3\end{pmatrix}$ and $T\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}2\\2\\4\end{pmatrix}$. Find the explicit formula of T for $\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3$. Also find the image of $\begin{pmatrix}2\\3\\6\end{pmatrix}\in\mathbb{R}^3$ under T.

14) If $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map defined by $T(x,y,z)=(-x+2z,3x-2y)$ and $S:\mathbb{R}^2\to\mathbb{R}^3$ is a map defined by $S(x,y)=(x,-5y,-x+y)$.

a) Compute the matrix of transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ with respect to the standard basis of $\mathbb{R}^2$.

b) Compute the matrix of transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ with respect to the standard basis of $\mathbb{R}^3$.

15) Find the standard matrix A of the linear transformation T. Use A to find image of v. Sketch the graph of v and $T(v)$.

$T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\-x\end{pmatrix}$ and $v=\begin{pmatrix}3\\4\end{pmatrix}$.