Linear Transformation

Lecture 6 :

Consider, $T:\mathbb{R}^2 \to \mathbb{R}^2, T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ -5x+2y \end{pmatrix}$. The matrix of transformation is $A=\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$.

Now consider the image of $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under $T:\mathbb{R}^2 \to \mathbb{R}^2$. $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = AX = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Now with respect to Euclidean inner product,

$\|X\| = \sqrt{1+1} = \sqrt{2}$ and $\|TX\| = \|AX\| = \sqrt{4^2 + (-3)^2} = \sqrt{5}$.

Let $S:\mathbb{R}^2 \to \mathbb{R}^2, S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \end{pmatrix}$. The matrix of transformation is $B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. The image of $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under $S:\mathbb{R}^2 \to \mathbb{R}^2$, $S\begin{pmatrix} 1 \\ 1 \end{pmatrix} = BX = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$.

$\|SX\| = \|BX\| = \sqrt{0 + (\sqrt{2})^2} = \sqrt{2}$. Thus, from this example, we observe that

$\|TX\| = \|AX\| \neq \|X\|$, but $\|SX\| = \|BX\| = \|X\|$. Norm of X and its image under S are same, while norm of X and its image under T are not same.

Orthogonal Transformation : A linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is said to orthogonal if it is norm, i.e., distance preserving. If the transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ is represented by matrix A then $T:\mathbb{R}^n \to \mathbb{R}^n$ is orthogonal transformation if the matrix of transformation A is orthogonal matrix.

- **Orthogonal Matrix :** Matrix A is said to be orthogonal matrix if $AA^T = A^T A = I$, where I is a $n \times n$ identity matrix.

Results :

1. Every rotation matrix is an orthogonal matrix & vice versa.
2. Orthogonal transformation preserves the length.
   If A is an orthogonal matrix, then $\| AX \| = \| X \|$.
   $(\because \| AX \|^2 = \langle AX,\, AX \rangle = |A|^2 \langle X,\, X \rangle = \langle X,\, X \rangle = \| X \|^2)$
3. If the matrix is orthogonal, then its rows and columns are orthonormal.
4. If A is orthogonal then $|A| = \pm 1$.
5. If for certain matrix A, $|A| = \pm 1$ then A may or may not be orthogonal.
6. If for a matrix A, we observe that its rows (or columns) are orthogonal but Columns (rows) are not, then it can be converted into an orthogonal matrix by

simply normalizing each row (column) of the matrix.

Note that

$$\mathrm{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mi} & \cdots & a_{mn} \end{bmatrix}_{n\times n} \text{ and } \mathrm{A}^t=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

Each row of A is a vector in $\mathbb{R}^n$, $R_i=\left(a_{i1}, \quad a_{i2}, \quad \cdots, \quad a_{ij}, \quad \cdots, \quad a_{in}\right), i=1,2,\cdots,m$.

This is also a column vector of A^t. Also note that the diagonal entry of AA^t is $a_{i1}^2+a_{i1}^2+\cdots+a_{in}^2=\left\langle R_i, R_i\right\rangle=R_iR_i^{\,t}, i=1,2,\cdots,m$ and the other non diagonal entries are $a_{i1}a_{j1}+a_{i2}a_{j2}+\cdots+a_{in}a_{jn}=\left\langle R_i, R_j\right\rangle=R_iR_j^{\,t}, i=1,2,\cdots,m, j=1,2,\cdots,n$. Thus,

$$\mathrm{A}=\begin{bmatrix} R_1 \\ R_1 \\ \vdots \\ R_i \\ \vdots \\ R_m \end{bmatrix} \text{ while } \mathrm{A}^t=\begin{bmatrix} R_1 & R_2 & \cdots & R_i & \cdots & R_n \end{bmatrix}. \text{ Therefore}$$

$$\mathrm{AA}^t=\mathrm{A}^t\mathrm{A}=\begin{bmatrix} \langle R_1,R_1\rangle & \langle R_1,R_2\rangle & \cdots & \langle R_1,R_i\rangle & \cdots & \langle R_1,R_n\rangle \\ \langle R_2,R_1\rangle & \langle R_2,R_2\rangle & \cdots & \langle R_2,R_i\rangle & \cdots & \langle R_2,R_n\rangle \\ \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\ \langle R_i,R_1\rangle & \langle R_i,R_2\rangle & \cdots & \langle R_i,R_i\rangle & \cdots & \langle R_i,R_n\rangle \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ \langle R_n,R_1\rangle & \langle R_n,R_2\rangle & \cdots & \langle R_n,R_i\rangle & \cdots & \langle R_n,R_n\rangle \end{bmatrix}_{n\times n}=I_n$$

if and if $\left\langle R_i, R_i\right\rangle=1, i=1,2,\cdots,n$ and $\left\langle R_i, R_j\right\rangle=0, i\neq j, i, j=1,2,\cdots,n$.

7. Orthogonal matrix A is always invertible and $\mathrm{A}^{-1}=\mathrm{A}^T$.

Examples

1. Is $A=\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$ orthogonal?

   No, because rows are orthogonal but columns are not.

Note That: 1) If rows are orthogonal, normalize the rows to obtain an orthogonal matrix.

2) Similarly, if columns are orthogonal, then normalize columns to reduce it to orthogonal matrix.

2. Find l, m, n so that the matrix $A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$ is orthogonal**.**

A is an Orthogonal matrix, therefore every column and row must be orthonormal

$$\therefore \sqrt{o^2 + l^2 + l^2} = 1 \Rightarrow 2l^2 = 1 \Rightarrow l = \pm\frac{1}{\sqrt{2}}, \ \sqrt{(2m)^2 + m^2 + m^2} = 1 \Rightarrow 6m^2 = 1 \Rightarrow m = \pm\frac{1}{\sqrt{6}}$$

and $\sqrt{\mathrm{n}^2 + (-n)^2 + n^2} = 1 \Rightarrow 3n^2 = 1 \Rightarrow n = \pm\frac{1}{\sqrt{3}}$.

<u>Problem Session</u>

Q. 1 **Attempt the following**

1) Is the matrix given below an orthogonal? If not is it possible to reduce it to an orthogonal matrix. $A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \\ -2 & -4 & 5 \end{bmatrix}$.

2) Define an orthogonal Matrix. Is the matrix given below an orthogonal? if not is it possible to reduce it to an orthogonal Matrix and hence find its inverse.. $\begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

3) What conditions $a \text{ and } b$ must satisfy for the matrix $\begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$ to be orthogonal?

4) If A is orthogonal matrix then show that A^{-1} is also orthogonal. Further show that A' is also orthogonal.

5) Show that

$$a) \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{45}} & \frac{-4}{\sqrt{45}} & \frac{5}{\sqrt{45}} \end{bmatrix} \qquad b) \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$