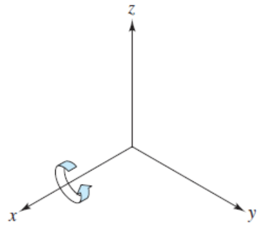
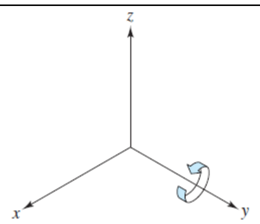
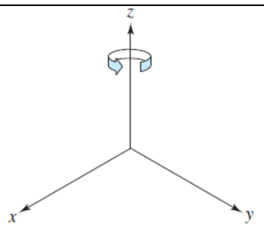
Linear Transformation

Study Material for Week 2

Lecture 4 : **Geometric Transformations in $\mathbb{R}^3$**

- Geometry of Linear Operators on $\mathbb{R}^3$

➢ **Rotation about standard axes**

1.	Rotation about X -axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$	
2.	Rotation about Y -axis	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	
3.	Rotation about Z -axis	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

➢ **Reflection**

4.	Reflection about XY -plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
5.	Reflection about XZ -plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
6.	Reflection about YZ -plane	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

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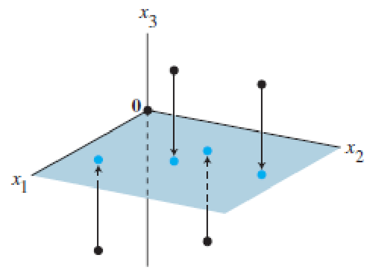
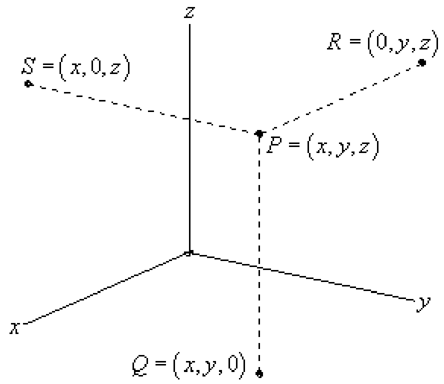
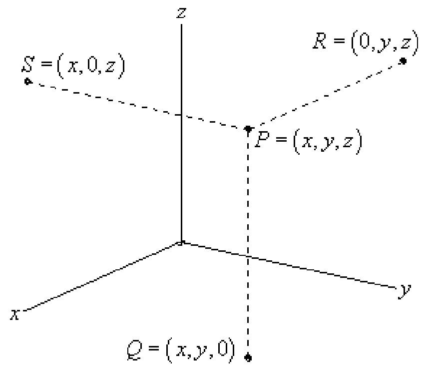
- **Projection**

7.	Projection on XY -plane	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$	
8.	Projection on YZ -plane	$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$	
9.	Projection on XZ -plane	$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$	

<u>**Examples**</u>

1. Find the standard matrix for $T:\mathbb{R}^3 \to \mathbb{R}^3$, that first reflects about xy-plane , then rotates the resulting vector in counterclockwise direction through an angle θ, about z - axis and then finally resultant vector is projected on xz-plane.

The standard matrix for reflection about xy-plane is $A_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$.

The standard matrix for rotation about z - axis is $A_2 = \begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$.

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The standard matrix for reflection about xy - plane is $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

The standard matrix for rotation about z - axis is $A_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore the required transformation is

$$A_3 A_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

2. The eight vertices of a rectangular box having sides of lengths 1, 2, and 3 are as follows.

$$\begin{array}{llll} V_1 = (0, 0, 0), & V_2 = (1, 0, 0), & V_3 = (1, 2, 0), & V_4 = (0, 2, 0), \\ V_5 = (0, 0, 3), & V_6 = (1, 0, 3), & V_7 = (1, 2, 3), & V_8 = (0, 2, 3) \end{array}$$

Find the coordinates of the box when it is rotated counter clockwise about the -axis through each angle. $60^0, 90^0, 120^0$.

The matrix that yields a rotation of 60^0 is

$$A = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by the eight vertices produces the rotated vertices listed below

Original Vertex	*Rotated Vertex*
$V_1 = (0, 0, 0)$	$(0, 0, 0)$
$V_2 = (1, 0, 0)$	$(0.5, 0.87, 0)$
$V_3 = (1, 2, 0)$	$(-1.23, 1.87, 0)$
$V_4 = (0, 2, 0)$	$(-1.73, 1, 0)$
$V_5 = (0, 0, 3)$	$(0, 0, 3)$
$V_6 = (1, 0, 3)$	$(0.5, 0.87, 3)$
$V_7 = (1, 2, 3)$	$(-1.23, 1.87, 3)$
$V_8 = (0, 2, 3)$	$(-1.73, 1, 3)$

The matrix that yields a rotation of is 90^0

$$A = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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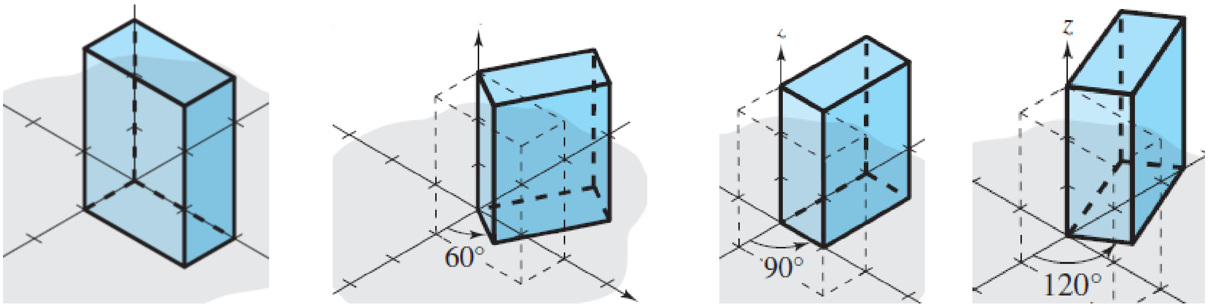
The matrix that yields a rotation of is 120^0

$$A = \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ & 0 \\ \sin 120^\circ & \cos 120^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$



<u>Problem Session</u>

Q. 1 **Attempt the following**

1) Find the matrix that will produce the indicated rotation and then find the image of the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. 30^0 about $Y-$ axis followed by 45^0 about $Z-$ axis.

2) Determine the matrix that will produce the indicated pair of rotations 60^0 about $X-$axis followed by 30^0 about Z-axis

3) Find the matrix that will produce the indicated rotation and find the image of the vector $(1, -1, 1)$, 45° about $z-$axis, 60° about the $x-$axis.

4) Find the matrix that will produce the indicated rotation and find the image of the vector $(1, -1, 1)$.

i) 45° about $z-$axis. ii) 60° about the $x-$axis.

5) Find the standard matrix for $T:\mathbb{R}^3 \to \mathbb{R}^3$, that first reflects about xy-plane, then rotates the resulting vector in counterclockwise direction through an angle $\pi/6$, about z- axis and then finally resultant vector is projected on xz-plane.

Q, 2 Find the matrix that will produce the indicated rotation.

1) 30° about the z-axis

2) 60° about the x-axis

3) 120° about the x-axis

4) projection onto XZ plane followed by reflection onto XY plane, followed by 30^0 about Z-axis