Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

Study Material for Week 4

Lecture One

Recall

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **non-zero** vector X such that $AX = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ.

Consider $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad \therefore \; X^t = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$. Now

$$X^t A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2 .$$

Thus for $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and a fixed matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $X^t A X$ is a second degree homogeneous polynomial in two variables x_1 and x_2. In general, for $X \in \mathbb{R}^n$ and a fixed $n \times n$ matrix A then $X^t A X$ is a second degree homogeneous polynomial in n variables $x_1, x_2, \cdots,$ and x_n.

- **Quadratic Form :** A homogeneous polynomial of degree two in given number of a variable is called as a quadratic form.

  Mathematically, this can be defined as follows.

  Let $X \in \mathbb{R}^n$ and a fixed $n \times n$ matrix A. A function or a rule or an assignment which assigns every vector $X \in \mathbb{R}^n$ to a real number $X^t A X \in \mathbb{R}$, is defined as quadratic form.

Note That :

1. Quadratic Form in two variables is

$$Q(X) = ax_1^2 + 2h_1x_1x_2 + bx_2^2, \text{ where } a, h, \text{ and } b \in \mathbb{R}..$$

2. Quadratic Form in three variables is

$$Q(X) = ax_1^2 + bx_2^2 + cx_3^2 + 2fx_1x_2 + 2gx_1x_3 + 2hx_2x_3, \text{ where } a, b, c, f, g, h \in \mathbb{R}.$$

Matrix of the Quadratic Form

1) Given a 2×2 symmetric matrix $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, the associated quadratic form is

$$Q(X) = X^t A X = a\,x_1^2 + 2h\,x_1x_2 + b\,x_2^2, \text{where } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Eigen Values and Eigen Vectors

Conversely, if the quadratic form is $Q(X) = a\,x_1^2 + 2h\,x_1x_2 + b\,x_2^2$, the associated matrix is $\mathrm{A} = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$.

2) Given a 2×2 symmetric matrix $\mathrm{A} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, the associated quadratic form is

$$Q(X) = X^t\mathrm{A}X = a\,x_1^2 + b\,x_2^2 + c\,x_3^{\,2} + 2h\,x_1x_2 + 2g\,x_1x_3 + 2f\,x_2x_3.$$

Conversely, if the quadratic form is

$$Q(X) = a\,x_1^2 + b\,x_2^2 + c\,x_3^{\,2} + 2h\,x_1x_2 + 2g\,x_1x_3 + 2f\,x_2x_3,$$

the associated matrix is $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

Examples

Q 1) Write matrix associated with $2x_1^2 - 5x_2^2 + 6x_3^2 - 7x_1x_2 + 8x_1x_3 + 4x_2x_3$.

$$A = \begin{bmatrix} 2 & -\frac{7}{2} & 4 \\ -\frac{7}{2} & -5 & 2 \\ 4 & 2 & 6 \end{bmatrix}.$$

Q 2) Write the quadratic form corresponding to matrix $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 5 \\ 3 & 5 & 3 \end{bmatrix}$.

Form is $x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 + 6x_1x_3 + 10x_2x_3$.

Note that the quadratic form $Q(X) = X^t\mathrm{A}X$ is a real number , so it can be positive, negative or even zero. Based on the values taken by quadratic form, form is classified into following types.

1) A quadratic form $Q(X) = X^T\mathrm{A}X$ is said to be positive definite if it takes positive values only, i.e., $Q(X) > 0\ \forall\ X \in \mathbb{R}^2 / X \in \mathbb{R}^3$.

2) A quadratic form $Q(X) = X^T\mathrm{A}X$ is said to be positive definite if it takes positive values including zero, i.e., $Q(X) \ge 0\ \forall\ X \in \mathbb{R}^2 / X \in \mathbb{R}^3$.

3) A quadratic form $Q(X) = X^T\mathrm{A}X$ is said to be negative definite if it takes negative values only, i.e., $Q(X) < 0\ \forall\ X \in \mathbb{R}^2 / X \in \mathbb{R}^3$.

4) A quadratic form $Q(X) = X^T \mathrm{A} X$ is said to be negative semi-definite if it takes negative values including zero, i.e., $Q(X) \le 0 \ \forall \ X \in \mathbb{R}^2 / X \in \mathbb{R}^3$.

5) A quadratic form $Q(X) = X^T \mathrm{A} X$ is said to be indefinite if it takes all real values , i.e., positive as well as negative including zero.

To decide whether the quadratic form is positive definite, negative definite or indefinite, it is not feasible to find value of $Q(X)$ for every $X \in \mathbb{R}^2 / X \in \mathbb{R}^3$, so the nature is decided from the eigen values of matrix of quadratic form, A .

<u>Problem Session</u>

Q.1		Identify the matrix associated with the following quadratic forms		
	1)	$3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$	2)	$2x_1x_2 + 2x_1x_3 - 2x_2x_3$
	3)	$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3$	4)	$x_1^2 + 6x_1x_2 + 9x_2^2$
	5)	$2x_1x_2 + 2x_1x_3 - 2x_2x_3 + 3x_1^2 + 3x_2^2 + 3x_3^2$	6)	$2x_1x_2 + 2x_1x_3 - 2x_2x_3$
Q.2		Find the quadratic form associated with the following matrices		
	1)	$\mathrm{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$	2)	$\mathrm{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$
	3)	$\mathrm{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$	4)	$\mathrm{A} = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$
	5)	$\mathrm{A} = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$	6)	$\mathrm{A} = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$