STRUCTURE FROM MOTION

by

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ABSTRACT

AKASH CHANDRA SHEKAR. Structure From Motion. (Under the direction of DR. ANDREW WILLIS)

The Structure from motion involves two aspects, one is to track the camera trajectory by solving nonlinear equation and other is to estimate the depth by solving Stereo correspondence of sub-set of pixels. The traditional Structure From Motion functions by finding the correspondence between pair of images, by tracking the features like corner points, lines etc. However, these features do not necessarily provide all information regarding the image. The more accurate method is to use the pixel intensity itself to minimize the Photometric and Geometric errors, by defining the robust error functions.

ACKNOWLEDGEMENTS

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DEDICATION

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The Dedication page, if used, pays a special tribute to a person(s) who has given extraordinary encouragement or support to one's academic career.

INTRODUCTION

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Depending on the discipline or the requirements of the student's advisory committee, an Introduction may be included as a preliminary page.

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ECE An acronym for Electrical and Computer Engineering.

CHAPTER 1: INTRODUCTION

Structure from Motion(SfM) is a photogrammetric process of estimating the three-dimensional structure of a scene from a set of two-dimensional images, this is achieved by tracking the motion of the cameras corresponding to these images. SfM has many application in the field of robotics, augmented reality and geoscience etc. In Robotics, SfM is mainly applied to implement the visual odometry, the process where the ego-motion of an robot is estimated using only the inputs of cameras attached to it. In the field of augmented reality, SfM used to estimate the depth maps of the scene, which are later used to implement basic physical interaction with the environment. The core of the SfM involves solving the trigonometry on set of images from camera with unknown calibration to extract the depth of an object. The main requirement to do this is to find a correspondence between pair of images, traditionally this was done by feature extraction and matching[1]. The feature extraction is the process of selecting the key points in the image which are unique and distinguishable and represent them in a efficient descriptor,which are later used for matching in the other image. There are many possible choices for features and descriptors, like SIFT, SURF, ORB etc. However in more recent implementations[2], instead of features, the image intensities are used directly for matching. Once the correspondence are established, the Pose of the cameras are estimated by imposing the Epipolar constraint, now with the estimated camera pose the three-dimensional structure (depth) is computed.

There mainly two variants of SfM, Incremental and Global SfM.Incremental SfM[3] begins by first estimating the 3D structure and camera poses of just two cameras based on their relative pose. Then additional cameras are added on incrementally and 3D structure is refined as new parts of the scene are observed.On the other hand Global

SfM[4] consider the entire problem at once,it tries to estimate the global camera poses and 3D structure by removing outliers and by applying averaging scheme.

Over the years many approaches have be suggested to tackle this problem, one of them is the Parallel Tracking and Mapping (PTAM)[1], it is a feature based system, here the tracking and mapping are split into two separate tasks, processed in parallel threads on dual core computer. The map is represented by a collection of point features located in a world coordinate frame W. These points feature represents a locally planar textured patch in the world, each point has coordinates in world frame, an unit patch normal and a reference to the patch source pixels. The map also contains N key frames, each key frame has an associated camera-center coordinate frame and the transformation between the frames and also stores a four level pyramid of gray-scale 8bpp images. The tracking is a two-stage process done from coarse-to-fine,when the new image is acquired, an initial coarse search searches only for 50 map points which appear at the highest levels of the current frame's image pyramid, and this search is performed (with sub-pixel refinement) over a large search radius. A new pose is then calculated from these measurements. After this, up to 1000 of the remaining potentially visible image patches are re-projected into the image, and now the patch search is performed over a far tighter search region. Sub-pixel refinement is performed only on a high-level subset of patches. The final frame pose is calculated from both coarse and fine sets of image measurements together.The pose update is computed iteratively by minimizing a robust objective function of the re-projection error:

$$\mu' = \underset{\mu}{argmin} \sum_{j \in s} Obj\left(\frac{\|e_j\|}{\sigma_j}, \sigma_T\right)$$

Where $e_j$is the re-projection error vector:

$$e_j = \begin{pmatrix} \hat{u_j} \\ \hat{v_j} \end{pmatrix} - CamProj(exp(\mu)E_{CWp_j})$$

$Obj(., \sigma_T)$ is the Tukey bi-weight objective function and σ_T a robust (median-based) estimate of the distribution's standard deviation derived from all the residuals.

Associated with the the key-frame in the map is a set S_i of image measurements.. For example, the jth map point measured in key-frame i would have been found at $(\hat{u}_{ji}, \hat{v}_{ji})^T$ with standard deviation of σ_{ji} pixels. Writing the current state of the map as $\{E_{k_1W}....E_{k_Nw}\}$ and $\{p_1...p_M\}$, each image measurement also has an associated re-projection error e_{ji}. Bundle adjustment is applied to iteratively adjust the map so as to minimize the robust objective function:

$$\{\{\mu_{2..}\mu_N\}, \{p'_1..p'_M\}\} = \underset{\{\{\mu\},\{p\}\}}{argmin} \sum_{i=1}^{N} \sum_{j \in s} Obj\left(\frac{\|e_{ji}\|}{\sigma_{ji}}, \sigma_T\right)$$

LSD SLAM [2] on the other hand is the direct method, this circumvent the drawback of PTAM, which is a feature base method and only the information that conforms to the feature type can be used. LSD LSD SLAM on the other hand optimizes the geometry directly on the image intensities, which enables using all information in the image.In addition to higher accuracy and robustness in particular in environments with little key-points, this provides substantially more information about the geometry of the environment.Images are aligned by Gauss -Newton minimization of the photometric error.

$$E(\xi) = \sum_i (I_{ref}(P_i) - I(\omega(p_i, D_{ref}(p_i), \xi)))^2$$

Where D_{ref} is the estimated depth of reference frame,$\xi \in se(3)$ is Lie-algebra representation of rigid body motion and ω is the affine wrap function. The above error function gives the maximum-likelihood estimator for ξ assuming i.i.d. Gaussian

residuals. $\delta\xi^{(n)}$ is computed for each iteration by solving for the minimum of Gauss-Newton second-order approximation of E:

$$\delta\xi^{(n)} = -(J^T J)^{-1} J^T r(\xi^{(n)})$$

with

$$J = \frac{\partial r(\epsilon \circ \delta\xi^{(n)})}{\partial \epsilon}$$

The new estimate is then obtained by multiplication with the computed update

$$\xi^{(n+1)} = \delta\xi^{(n)} \circ \xi^{(n)}$$

The overall system is composed of three major components, tracking, depth map estimation and map optimization.

The tracking component continuously estimates the rigid body pose with respect to the current keyframe, using the pose of the previous frame as initialization.LSD slam tracks new frame by minimizing the variance-normalized photometric error

$$E_p(\xi_{ji}) = \sum_{p \in \Omega_{D_i}} \left\| \frac{r_p^2(p, \xi_{ji})}{\sigma^2_{r_p(p,\xi_{ji})}} \right\|_\delta$$

with

$$r_p(p, \xi_{ji}) \coloneqq I_i(p) - I_j(\omega(p, D_i(p), \xi_{ji}))$$

$$\sigma^2_{r_p(p,\xi_{ji})} \coloneqq 2\sigma_I^2 + \left(\frac{\partial r_p(p, \xi_{ji})}{\partial D_i(p)} \right)^2 V_i(p)$$

where $\|.\|$ is the Huber norm

$$\left\| r^2 \right\|_\delta \coloneqq \begin{cases} \frac{r^2}{2\delta} if \ \|r\| \leq \delta \\ , \|r\| - \frac{\delta}{2} \ otherwise \end{cases}$$

applied to the normalized residual. The depth map estimation component uses tracked frames to either refine or replace the current key-frame. Depth is refined by filtering over many per-pixel, small-baseline stereo comparisons coupled with interleaved spatial regularization. If the camera has moved too far, a new key-frame is initialized by projecting points from existing, close-by key-frames into it. Each key-frame is scaled such that its mean inverse depth is one, which enables more small-baseline stereo comparisons. For every new key-frame added, the possibility of loop closure is checked by performing the reciprocal tracking check. The map optimization component is responsible for the updating depth map into global map, it detect loop closure and scale drift by estimating similarity transform (sim(3)) to close by existing key-frames.The global map is represented as a pose graph consisting of key-frames as vertices's with 3D similarity transforms as edges, elegantly incorporating changing scale of the environment and allowing to detect and correct accumulated drift. Each key-frame consists of a camera image, an inverse depth map and variance of the inverse depth.Edges between key-frames contain their relative alignment as similarity transform, as well as corresponding covariance matrix. The map, consisting of a set of key-frames and tracked sim(3)-constraints, is continuously optimized in the background using pose graph optimization. The error function that is minimized is defined by (W defining the world frame)

$$E(\xi_{W1}....\xi_{Wn}) \coloneqq \sum_{(\xi_{ji},\Sigma_{ji})\in\varepsilon} (\xi_{ji} \circ \xi_{Wi}^{-1} \circ \xi_{Wj})\Sigma_{ji}^{-1}(\xi_{ji} \circ \xi_{Wi}^{-1} \circ \xi_{Wj})$$

CHAPTER 2: Methodology

Structure from motions (SfM) is the process of triangulating the three-dimensional structure from two-dimensional images, along with estimating the motion of camera (visual odometer), hence it is some time called as Visual SLAM.

2.1 Rigid Body Motion

One of the goals of SfM it to estimate the Camera trajectory, which is a rigid body motion. We need an efficient model to represent and compute this rigid body motion. The camera position is represented by a 3D Vector in an Euclidean Space, this camera position is chosen to represent the world frame and specify the translation and rotation of the scene relative to that frame. The rigid body motion itself is composed of a rotation and translation.

Traditionally, rotation is represented by a 3×3 special orthogonal matrix called rotational matrix. Special Orthogonal matrix SO(3) are the matrix which satisfy $R^T R = RR^T = I$ and have a determinant of $+1$.

$$SO(3) = \{R \in \mathbb{R}^{3\times3} \mid R^T R = I, det(R) = +1\}$$

The rotation transformation of the coordinates $X_c$of a point p relative to frame C to its coordinates X_w relative to frame W is

$$X_w = R_{wc} X_c$$

Also, because the rotational matrix are orthogonal, we have$R^{-1} = R^T$, on this line the inverse transformation of coordinates are

$$X_c = R_{wc}^{-1} X_w = R_{wc}^T X_w$$

The continuous rotation of a camera is described as a trajectory $R(t) : t \to SO(3)$ in the space $SO(3)$.When the starting time is not t = 0, the relative motion between time t_2 and time t_1 will be denoted as $R(t_2, t_1)$. The composition law of the rotation group implies

$$R(t_2,t_0) = R(t_2,t_1) \times R(t_1,t_0), \vee t_0 < t_1 < t_2 \in R$$

On the other hand the translation is represented by a $T \in R^3$,1×3 vector which adds the translation values in each dimension.With this, the complete rigid body motion is represented by

$$X_w = R_{wc} X_c + T_{wc} \tag{2.1}$$

However, the above equation is not linear but affine. We may convert this to linear by using homogeneous coordinates, where we append 1 for 1×3 vector and make it a 1×4 vector,

$$\bar{X} = \begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

With this new notation for point, we can rewrite the transformation from equation 6 as following

$$\bar{X}_w = \begin{bmatrix} X_w \\ 1 \end{bmatrix} = \begin{bmatrix} R_{wc} & T_{wc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_c \\ 1 \end{bmatrix} = \bar{g}_{wc} \bar{X}_c \tag{2.2}$$

where the 4×4 matrix $\bar{g}_{wc} \in R^{4\times4}$is called the homogeneous representation of the rigid-body motion.

The set of all possible configurations of a rigid body can then be described by the space of rigid-body motions or special Euclidean transformations called SE(3)

$$SE(3) = \{\bar{g} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \mid R \in SO(3), T \in R^3\} \subset \mathbb{R}^{4\times4}$$

2.2 Exponential Map

The special orthogonal group in three dimensions can be represented by a 3×3 rotation matrix $R \in SO(3)$, which must satisfy the constraint $R^T R = I$, this implies that the $SO(3)$ transformations leaves the quantity $x^2 + y^2 + z^2$ invariant. The group $SO(3)$ has 9 parameters, but the invariance of the length produces six independent conditions, leaving three free parameters, Hence, the dimension of the space of rotation matrices $SO(3)$ should be only three, and six parameters out of the nine are in fact redundant.We can use this to have better representation of Rigid body motion.

We know that the continuous rotational motion represented by $R(t) : R \in SO(3)$, must satisfy the following constraint

$$R(t)R^T(t) = I$$

Differentiating the above equation with respect to time t gives

$$\dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0$$

$$\dot{R}(t)R^T(t) = -(\dot{R}(t)R^T(t))^T$$

This shows that the matrix $\dot{R}(t)R^T(t) \in \mathbb{R}^{3\times3}$is a skew-symmetric matrix.This implies that there must exist a vector, say $\omega(t) \in \mathbb{R}^3$,such that

$$\dot{R}(t)R^T(t) = \hat{\omega}(t) \tag{2.3}$$

Multiplying both sides by $R(t)$ on the right yields

$$\dot{R}(t) = \hat{\omega}(t)R(t) \tag{2.4}$$

In above equation, if $R(t_0) = I$ for for $t = t_0$, we have $\dot{R}(t_0) = \hat{\omega}(t_0)$. Hence, around the identity matrix I, a skew-symmetric matrix gives a first- order approximation to a rotation matrix:

$$R(t_0 + dt) \approx I + \hat{\omega}(t_0)dt$$

This space of all skew-symmetric matrix represents the tangent space of the rotation group $SO(3)$ and it is the lie algebra $so(3)$ of the corresponding lie group.

$$so(3) \doteq \left\{\hat{\omega} \in \mathbb{R}^{3\times3} \mid \omega \in \mathbb{R}^3\right\}$$

Once we have so(3), we need a way to map SO(3) to so(3),It is obvious that the solution for 2.4 is the matrix exponential $e^{\hat{\omega}t}$, where

$$e^{\hat{\omega}t} = I + \hat{\omega}t + \frac{(\hat{\omega}t)^2}{2!} + \cdots + \frac{(\hat{\omega}t)^n}{n!} + \cdots$$

Hence, we have

$$R(t) = e^{\hat{\omega}t} \tag{2.5}$$

The above equation represents a rotation around the axis $\omega \in \mathbb{R}^3$ by an angle of t

radians.This map from the space $so(3)$ to $SO(3)$ is called the exponential map.

$$exp : so(3) \to SO(3); \hat{\omega} \mapsto e^{\hat{\omega}}$$

And the inverse mapping is obtained by logarithm of SO(3)

$$log : SO(3) \to so(3); log(R) \mapsto \hat{\omega}$$

We can extend this to full rigid body motion which also involves the translation along with the rotation.From 2.2, the continuous rigid body trajectory on SE(3) is given by

$$g(t) = \begin{bmatrix} R(t) & T(t) \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

With above representation, we have

$$\dot{g}(t)g^{-1}(t) = \begin{bmatrix} \dot{R}(t)R^T(t) & \dot{T}(t) - \dot{R}(t)R^T(t)T(t) \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

with $\upsilon(t) = \dot{T}(t) - \hat{\omega}(t)T(t) \in \mathbb{R}^3$ and 2.3

$$\hat{\xi(t)} = \begin{bmatrix} \hat{\omega}(t) & \upsilon(t) \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{2.6}$$

then we have

$$\dot{g}(t) = (\dot{g}(t)g^{-1}(t))g(t) = \hat{\xi(t)}g(t)$$

where $\hat{\xi}$ is called the twist and can be used to approximate g(t) locally

$$g(t + dt) \approx g(t) + \hat{\xi(t)}g(t)d(t) = (I + \hat{\xi(t)}dt)g(t)$$

Also the twist represent the tangent space (or Lie algebra) of the matrix group SE(3).

$$se(3) \doteq \left\{ \hat{\xi} = \begin{bmatrix} \hat{\omega} & \upsilon \\ 0 & 0 \end{bmatrix} \mid \hat{\omega} \in so(3), \upsilon \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}$$

Similar 2.5. with the initial condition of $g(0) = 1$,we have

$$g(t) = e^{\hat{\xi}t}$$

where

$$e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \cdots + \frac{(\hat{\xi}t)^n}{n!} + \cdots$$

This defines the the exponential map from the space $se(3)$ to $SE(3)$

$$exp : se(3) \to SE(3); \hat{\xi} \mapsto e^{\hat{\xi}}$$

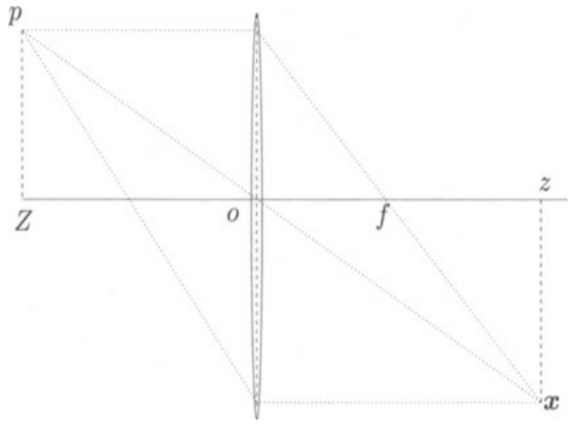
and as before inverse to the exponential map is defined by logarithm

$$log : SE(3) \to se(3); log(g) \mapsto \hat{\xi}$$

2.3 Camera model

The 2D image is formed by capturing the light energy (irradiance) for every pixel, this process can be mathematically represented by thin lens camera model, which describes the relationship between the three-dimensional coordinators to its projection onto the image plane. The thin lens model is represented by a optical axis and a perpendicular plane called the focal plane.The thin lens itself is characterized by its focal length and diameter, the focal length is the distance from optic center to where all the ray intersect the optic axis, while the point of intersection itself is called

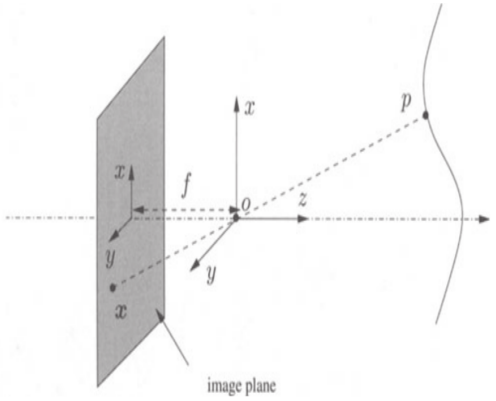
the focus of the lens. One of the important properties to consider is that the rays entering the lens through optic center are undeflected while the rest of the rays are. With this model the irradiance on the image plane is obtained by the integration of all the energy emitted from region of space contained in the cone determined by the geometry of the lens.



Using similar triangles, from above figure, we obtain the following fundamental equation of the thin lens

$$\frac{1}{Z} + \frac{1}{z} = \frac{1}{f}$$

For the simplification of calculation, we consider a Ideal camera model called the pinhole camera model, here the aperture of a thin lens is assumed to decreased to zero, all rays are forced to go through the optical center o, and therefore they remain undeflected.Consequently,as the aperture of the cone decreases to zero, the only points that contribute to the irradiance at the image point $x = [x, y]$ are on a line through the center 0 of the lens. If a point p has coordinates $X = [X, Y, Z]$ relative to a reference frame centered at the optical center 0, with its z-axis being the optical axis (of the lens), then it is immediate to see from similar triangles in Figure that the coordinates of p and its image x are related by the so-called ideal perspective projection.



$$x = -f\frac{X}{Z} \tag{2.7}$$

$$y = -f\frac{Y}{Z} \tag{2.8}$$

This mapping of 3D point to 2D is called projection and is represented by π

$$\pi : R^3 \rightarrow R^2$$

This is also written as $x = \pi(X)$.

The negative sign in the 2.7 and 2.8 makes the object appear upside down on the image plan, we can handle this by moving the image plane to front of optic center to $\{z = -f\}$ this will make $(x, y) \rightarrow (-x, -y)$.There for the equation 2 and 3 changes to

$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

This can be represented in matrix form as

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix}$$

In homogeneous coordinates, this relationship can be modified as

$$Z \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

The above equation can be decomposed into

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

with

$$K_f = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in R^{3\times3}, \Pi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times4}$$

The matrix $\Pi_0$is a standard projection matrix.

With the rigid body transformation from 2.2, we can represent the overall geometric model for an ideal camera as below

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} \tag{2.9}$$

Where λ is the unknown scale factor.

However, the above equation represents the ideal model where the retinal frame centered at the optical center with one axis aligned with the optical axis.But in practice, this does not true and the origin of the image coordinate frame typically in the upper-left corner of the image.we need to address this relationship between the retinal plane coordinate frame and the pixel array in our camera model. This can be represented by a special matrix as following

$$K_s = \begin{bmatrix} s_x & s_\theta & o_x \\ 0 & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

with these new parameters the we can represent the interstice parameters of camera as following

$$K = K_s K_f = \begin{bmatrix} s_x & s_\theta & o_x \\ 0 & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} fs_x & s_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

where

- o_x: x-coordinate of the principal point in pixels,
- o_y: y-coordinate of the principal point in pixels,
- $fs_x = \alpha_x$: size of unit length in horizontal pixels,
- $fs_y = \alpha_y$: size of unit length in vertical pixels,
- $\frac{\alpha_x}{\alpha_y}$: aspect ratio σ,
- fs_θ: skew of the pixel, often close to zero.

Now, the ideal camera model 2.9can be updated as

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & s_\theta & o_x \\ 0 & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} \tag{2.10}$$

In the matrix notation,

$$\lambda x = K \Pi_0 g X_0 \tag{2.11}$$

To summarize, 2.11 represents the projection of three-dimensional coordinates X_0 by camera with orientation (extrinsic parameters)g and intrinsic parameters K, onto two-dimensional coordinate x with known scale λ

In addition to above linear distortion, if a camera has a wide field of view, there will be a significant distortion along radial directions called radial distortion. Such a distortion can be models by

$$x = x_d(1 + a_1 r^2 + a_2 r^4)$$

$$y = y_d(1 + a_1 r^2 + a_2 r^4)$$

where (x_d, y_d) are the coordinates of the distorted points, $a_1, a_2$are the coefficients of radial distortion and r is the radius of the radial distortion.

2.4 Epipolar geometry

Consider two images of the same point p from two camera position with relative pose (R, T), where $R \in SO(3)$ is the relative orientation and $T \in \mathbb{R}^3$is the relative position,then if $X_1, X_2 \in \mathbb{R}^3$ are the 3-D coordinates of a point p relative to the two camera frames, by the rigid-body transformation we have

$$X_2 = RX_1 + T$$

Now,let $x_1, x_2 \in \mathbb{R}^3$be the homogeneous coordinates of the projection of the same point p in the two image planes with respective unknown scales of $\lambda_1$and λ_2.

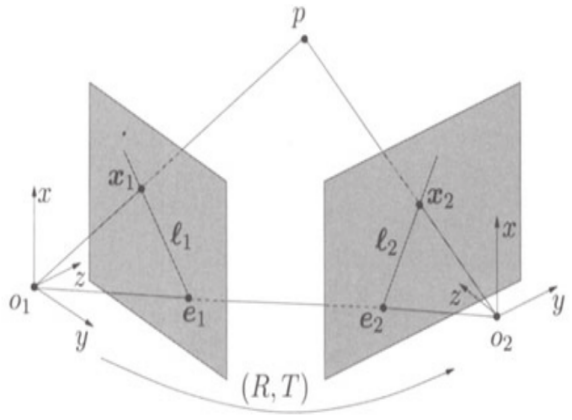
$$\lambda_2 x_2 = R\lambda_1 x_1 + T$$

By multiplying both the side by $\hat{T}$

$$\lambda_2 \hat{T} x_2 = \hat{T} R \lambda_1 x_1$$

By multiplying both the side by x_2

$$0 = x_2^T \hat{T} R \lambda_1 x_1 \tag{2.12}$$



This is the epipolar constraint and the matrix $E = \hat{T}R$ is called the essential matrix.It encodes the relative pose between the two cameras. Geometrically, it impose that the The vector connecting the first camera center o_1 and the point p, the vector connecting o_2 and p, and the vector connecting the two optical centers o_1 and o_2 clearly form a triangle. Therefore, the three vectors lie on the same plane.Hence the their triple product which measures the volume of the parallelepiped is zero.

2.4.1 The eight-point linear algorithm

With epipolar constrain between two images, we should be able to retrieve the relative pose of the cameras. The eight-point linear algorithm is a simple closed-form algorithm, it consists of two steps: First a matrix E is recovered from a number of epipolar constraints; then relative translation and orientation are extracted from E.

The entries of E are denoted by

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

The matrix E is reshaped into vector $E \in \mathbb{R}^9$

$$E = [e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}, e_{13}, e_{23}, e_{33}]^T$$

and for $x_1 = [x_{1,}y_1, z_1]^T \in \mathbb{R}^3$and $x_2 = [x_{2,}y_2, z_2]^T \in \mathbb{R}^3$ define

$$a = [x_1x_2, x_1y_2, x_1z_2, y_1x_2, y_1y_2, y_1z_2, z_1x_2, z_1y_2, z_1z_2] \in \mathbb{R}^9$$

With these new notations, we can rewrite the 2.12 as below

$$a^T E = 0$$

this representation emphasizes the linear dependence of the epipolar constraint on the elements of the essential matrix.

Now,with a set of corresponding image points $(x_1^j, x_2^j), j = 1, 2, ..., n$ we can define a matrix $\chi \in \mathbb{R}^{n\times 9}$ associated with these measurements to be

$$\chi = \left[a^1, a^2, ..., a^n\right]^T$$

In the absence of noise, the vector E satisfies

$$\chi E = 0$$

In order to obtain the unique solution, the rank of the matrix $\chi \in \mathbb{R}^{n\times 9}$ needs to be exactly eight. This should be the case when we have $n \geq 8$ "ideal" corresponding points, hence the name The eight-point linear algorithm.

Because of errors in correspondences, we try to find the E that minimizes the least-squares error function $\|\chi E\|^2$.We do this by choosing eigenvector of $\chi^T\chi$ that corresponds to its smallest eigenvalue.

Once we have E, we need to extract the pose ($R \in SO(3)$ and $T \in \mathbb{R}^3$) from E, we know that [5]a nonzero matrix $E \in \mathbb{R}^3$ is an essential matrix if and only if E has a singular value decomposition $(SVD)E = U\Sigma V^T$with

$$\Sigma = diag\,\{\sigma, \sigma, 0\}$$

for some $\sigma > 0$ and $U, V, \in SO(3)$

With this we can obtain two relative pose

$$(\hat{T}_1, R_1) = (UR_Z(+\frac{\pi}{2})\Sigma U^T, UR_Z^T(+\frac{\pi}{2})V^T)$$

$$(\hat{T}_2, R_2) = (UR_Z(-\frac{\pi}{2})\Sigma U^T, UR_Z^T(-\frac{\pi}{2})V^T)$$

Among these one with that gives the meaningful (positive) depth are selected as the valid pose.

With $R_Z(\pm\frac{\pi}{2}) = \begin{bmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.4.2 Structure Reconstruction

One remaining thing to find is the position of points in three-dimension by recovering their depths relative to each camera frame.With the estimated pose (Translation is T is defined up to the scale γ) and point correspondence, we have

$$\lambda_2^j x_2^j = \lambda_1^j R x_1^j + \gamma T$$

for $j = 1, 2, 3..., n$

where, $\lambda_1$and $\lambda_2$are depths with respect to the first and second camera frames, respectively. One of the depths is redundant, if λ_1 is known, we can estimate λ_2 as a function of (R, T).Hence we can eliminate, say, λ_2 from the above equation by multiplying both sides by $\hat{x_2}$

$$0 = \lambda_1^j x_2^j R x_1^j + \gamma x_2^j T$$

This is represented as linear equations

$$M^j \bar{\lambda^j} = \left[\hat{x_2^j} R x_1^j, \hat{x_2^j} T\right] \begin{bmatrix} \lambda_1^j \\ \gamma \end{bmatrix} = 0$$

where $M^j = \left[\hat{x_2^j} R x_1^j, \hat{x_2^j} T\right] \in \mathbb{R}^{3\times 2}$and $\bar{\lambda^j} = \begin{bmatrix} \lambda_1^j \\ \gamma \end{bmatrix} \in \mathbb{R}^2$ for $j = 1, 2, 3..., n$

since all n equations above share the same γ; we define a vector $\vec{\lambda} = [\lambda_1^1, \lambda_1^2, ..., \lambda_1^2, \gamma]$ and a matrix $M \in \mathbb{R}^{3n\times(n+1)}$as

$$M = \begin{bmatrix} \hat{x}_2^1 R x_1^1 & 0 & 0 & 0 & 0 & \hat{x}_2^1 T \\ 0 & \hat{x}_2^2 R x_1^2 & 0 & 0 & 0 & \hat{x}_2^2 T \\ 0 & 0 & \ddots & 0 & 0 & \vdots \\ 0 & 0 & 0 & \hat{x}_2^{n-1} R x_1^{n-1} & 0 & \hat{x}_2^{n-1} T \\ 0 & 0 & 0 & 0 & \hat{x}_2^n R x_1^n & \hat{x}_2^n T \end{bmatrix}$$

Then the equation

$$M \overrightarrow{\lambda} = 0$$

determines all the unknown depths up to a single universal scale. The linear least-squares estimate of $\overrightarrow{\lambda}$ is simply the eigenvector of $M^T M$s that corresponds to its smallest eigenvalue.

2.5 Non linear Optimization

In practice, because of the noise in image correspondence and other errors we cannot measure the actual coordinates but only their noisy versions, say

$$\widetilde{x}_1^j = x_1^j + \omega_1^j$$

$$\widetilde{x}_2^j = x_2^j + \omega_2^j$$

where $x_1^j$and $x_2^j$are the ideal image coordinates and $\omega_1^j = \left[\omega_{11}^j, \omega_{12}^j, 0\right]^T$and $\omega_2^j = \left[\omega_{21}^j, \omega_{22}^j, 0\right]^T$are localization errors (called residuals) in the correspondence.Therefore, we need a way to optimize the parameters (x, R, T) that minimize this errors.

One of the minimalistic approach to optimality is to minimize the squared 2-norm of residuals, if we choose the first camera frame as the reference

$$\phi(x, R, T, \lambda) = \sum_{j=1}^{n} \left\|\omega_1^j\right\| + \left\|\omega_2^j\right\|^2 = \sum_{j=1}^{n} \left\|\widetilde{x}_1^j - x_1^j\right\|^2 + \left\|\widetilde{x}_2^j - \pi(R\lambda_1^j x_1^j + T)\right\|^2$$

The above error is often called the "re-projection error", since x_1^j and $x_2^j$are the recovered 3-D points projected back onto the image planes. This process of minimizing the above expression for the unknowns (R, T, x_1, λ) is known as bundle adjustment[6].

One of the many ways to minimize this squared error is the Gauss-Newton method. Gauss-Newton is a iterative method for finding the value of the variables which minimizes the sum of squares,it starts with the initial guess and this method does not need the second derivatives (Hessian matrix) of the of function, which is often expensive and sometimes not possible to compute, instead the Hessian is approximated with the Jacobian matrix of the function.

For the least-square function of form

$$\min_x \sum_i r_i(x)^2$$

Gauss-Newton method iteratively solves

$$x_{t+1} = x_t - H^{-1} g$$

with gradient g

$$g_j = 2\sum_i r_i \frac{\partial r_i}{\partial x_i}$$

and the Hessian H

$$H_{jk} = 2\sum_i r_i \left(\frac{\partial r_i}{\partial x_j}\frac{\partial r_i}{\partial x_k} + r_i \frac{\partial^2 r_i}{\partial x_i \partial x_k}\right)$$

by dropping the second order term we can approximate the Hessian matrix with Jacobian matrix

$$H_{jk} = 2\sum_{i} J_{ij} J_{ik}$$

with

$$J_{ij} = \frac{\partial r_i}{\partial x_j}$$

The modification to Gauss-Newton method is The Levenberg-Marquardt algorithm,

$$x_{t+1} = x_t - (H + \lambda I)^{-1} g$$

this modification is a hybrid between the Newton method ($\lambda = 0$) and a gradient descent step size $1/\lambda$ for $\lambda \longrightarrow \infty$

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