Model Validation Report

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1 Executive Summary

Risk-management code for 5-day VaR and ES on a stock (and eventual stock+option) portfolio was reviewed. I implement four methods—parametric (5-year window), parametric (EWM), historical and Monte Carlo—and validate:

- **Accuracy** against closed-form GBM formulas
- **Stability** via flat-price and backtest exception-frequency tests
- **Consistency** Monte Carlo vs. parametric and across methods on a real portfolio

All tests passed within their acceptance criteria. The models reliably capture tail-risk for equity portfolios; extensions for options require adding volatility-surface dynamics and Greeks.

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2 Introduction

- **Review scope:** VaR/ES engines `parametric5yr`, `parametric_ewm`, `historical`, `montecarlo` in `/software`
- **Intended usage:** marking and risk-management of equity and option portfolios in a Python framework
- **Business unit:** Model Risk / Quantitative Risk Management

3 Product Description

A standalone risk-calculation library that:

- Takes time-series of asset prices (stocks, later options) and a portfolio definition
- Computes 5-day VaR and ES at user-specified confidence levels via:
    1. Parametric GBM (rolling window)
    2. Parametric GBM (exponential weighting)
    3. Historical empirical
    4. Monte Carlo GBM

Graphs and numeric outputs are produced for reporting and backtesting.

4 Model Description

Modeling theory / assumptions

- **Parametric VaR (5 yr window)**
  Assume the total portfolio value (P_t) follows a Geometric Brownian Motion:

  $$dP_t = \mu, P_t, dt + \sigma, P_t, dW_t.$$

  Estimate drift μ and volatility σ from the past five years of daily log-returns. Take the 1 % left tail of the resulting 5-day log-normal return distribution as VaR.

- **Parametric VaR (EWM "5 yr equivalent")**
  Same GBM assumption, but compute μ and σ by exponential weighting of all past daily log-returns with decay factor λ = 0.9992 (so that the effective memory ≈ 5 years). Define

  $$\alpha = 1 - \lambda = 0.0008.$$

- **Historical VaR (5 yr window)**
  No distributional assumptions. Collect the past five years of realized 5-day P&L, sort the losses, and take the 1 % empirical quantile.

- **Monte Carlo VaR (5 yr window)**
  Assume GBM with μ, σ estimated over a 5 year window. Simulate many 5-day paths of P_t, compute the loss distribution, and take the 1 % worst-case loss.

- **Expected Shortfall**

For each of the four VaR methods above, compute ES as the average of losses exceeding the VaR threshold (e.g. the worst 1 % tail for 99 % ES).

Options:

- **Underlying dynamics**
  We continue to assume each stock follows GBM under both the real and risk-neutral measures:

$$dS_t = \mu S_t, dt + \sigma S_t, dW_t \quad \text{and} \quad dS_t = r S_t, dt + \sigma S_t, dW_t^Q.$$

- **Volatility surface dynamics**
  Options depend on the entire implied-volatility surface (strike, maturity). We model its evolution via one of:

  1. **Simple "sticky delta" shocks**: treat ATM vol changes as a time-series, and shift the entire surface by historical Δσ_ATM.
  2. **Stochastic-volatility model** (e.g. Heston): jointly simulate S_t and instantaneous variance v_t under Q.
  3. **SABR or GARCH-based local-vol model**: calibrate to current surface and simulate vol-of-vol.

- **Portfolio P&L approximation**
  For parametric VaR/ES we use a **delta-vega-gamma approximation** of small changes:

$$\Delta P \approx \Delta_S, \Delta S + \tfrac{1}{2}\Gamma_S(\Delta S)^2 + \text{Vega}, \Delta\sigma$$

  where $(\Delta S, \Delta\sigma)$ are joint-normally distributed with parameters estimated from historical windows or EWM.

- **Historical and Monte Carlo**

  - **Historical**: collect five years of **total option P&L** at 5-day horizons, sorting and taking the 1% tail. Implied vols and underlying returns enter implicitly via realized option price moves.
  - **Monte Carlo**: simulate underlying paths (and vol paths if using Heston/SABR), reprice each option at $t + 5$ with the appropriate pricing model (Black-Scholes or chosen SV model), compute P&L.

Mathematical description

1. **Daily log-returns**

$$r_t = \ln\big(P_t/P_{t-1}\big).$$

2. **Parametric VaR**

   - Estimate

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} r_i, \quad \hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \hat{\mu})^2}$$

     either by rolling window or EWM.
   - The 5-day log-return $R_5 \sim N(5\hat{\mu}, ; 5\hat{\sigma}^2)$.
   - The 99 % VaR is

$$\mathrm{VaR}_{0.99} = -P_t\big(e^{q_{0.01}} - 1\big), \quad q_{0.01} = 5\,\hat{\mu} + z_{0.01}\,\sqrt{5}\,\hat{\sigma}, \quad z_{0.01} = \Phi^{-1}(0.01).$$

3. **Historical VaR**

   - Compute actual 5-day P&L: $\Delta P_i = P_{t_i+5} - P_{t_i}$.
   - Sort the losses $-\Delta P$ and take its 1 % quantile.

4. **Monte Carlo VaR**

   - Draw M samples $R_5^{(j)} \sim N(5\hat{\mu}, 5\hat{\sigma}^2)$.
   - Simulate $P_t^{(j)} = P_t, e^{R_5^{(j)}}$.
   - Losses $\ell^{(j)} = P_t - P_t^{(j)}$.
   - VaR = 1 % quantile of $\ell^{(j)}$.

5. **Expected Shortfall**

   - Compute the mean of the worst 1 % of losses for 99 % ES (or worst 2.5 % for 97.5 % ES).

Options:

1. **Greeks**

   - $\Delta = \partial P/\partial S, ; \Gamma = \partial^2 P/\partial S^2, ; \mathrm{Vega} = \partial P/\partial \sigma.$
   - Compute via closed-form formulas (Black-Scholes) or finite-difference on numerically-calibrated surfaces.

2. **Joint distribution**
    - Estimate daily covariance matrix of $(\Delta S/S, , \Delta\sigma)$ over your chosen window.
    - Scale to 5-day horizon:

$$\mathrm{Cov}\Big(\tfrac{\Delta S}{S},\ \Delta\sigma\Big)_{\text{5-day}} = 5 \times \mathrm{Cov}\Big(\tfrac{\Delta S}{S},\ \Delta\sigma\Big)_{\text{daily}}.$$

3. **Parametric VaR**
    - For each date t, approximate the 5-day P&L distribution as normal with mean μ_P and variance σ_P^2 given by the Greek-weighted covariance:

$$\mu_P = \Delta, \mu_S + \mathrm{Vega}, \mu_\sigma + \tfrac{1}{2}, \Gamma, \sigma_S^2,$$

$$\sigma_P^2 = \Delta^2, \sigma_S^2; +; \mathrm{Vega}^2, \sigma_\sigma^2; +; 2, \Delta, \mathrm{Vega}, \mathrm{Cov}\big(\Delta S, \Delta\sigma\big); +; \tfrac{1}{4}, \Gamma^2, \mathrm{Var}\big((\Delta S)^2\big).$$

    - $VaR_{0.99}$ is given by the 1% quantile of the normal distribution:

$$\mathrm{VaR}_{0.99} = \mu_P; +; \sigma_P, \Phi^{-1}(0.01),$$

      where Φ^{-1} is the inverse CDF of the standard normal.

4. **Historical VaR / ES**
    - Build the series of 5-day option P&L directly (price$_{t+5}$ – price$_t$) and take empirical quantiles or tail-averages.

5. **Monte Carlo VaR / ES**
    - Simulate scenarios (S_{t+5}, σ_{t+5}), reprice options via $P(S_{t+5}, \sigma_{t+5})$, compute P&L, and extract the 1% and ES statistics.

Model implementation & numerical methods

- **Parameter estimation**
    - Rolling: sample mean & std over N days.
    - EWM: use `.ewm(alpha=alpha, adjust=False)` for weighted mean & var.
- **VaR calculation**
    - Parametric: closed-form log-normal quantile.

- Historical: sort & index empirical distribution.
- Monte Carlo: vectorized simulation via `numpy.random.normal`.

- **Backtesting**

  - Compare realized 5-day P&L to previous-day VaR and count exceptions.

- **Greek calculation**

  - Use Black-Scholes formulas for $\Delta, \Gamma, \mathrm{Vega}$ with current implied vol.
  - If using a full SV model, compute via adjoint or finite-difference.

- **Volatility estimation**

  - For parametric VaR: estimate daily μ_σ, σ_σ, and $\mathrm{Cov}(\Delta S, \Delta\sigma)$ via rolling or EWM.

  - For Monte Carlo: calibrate SV parameters (e.g. Heston $\kappa, \theta, \xi, \rho$) to the implied surface.

- **Option repricing**

  - Historical: simply read back historical option prices from the data source.
  - Monte Carlo: vectorized pricing routines for large scenario sets (NumPy, Numba, or C extensions).

Calibration methodology

- **Window lengths**

  - 5 years ≈ 1 260 trading days.

- **Decay factor**

$$\lambda = 0.9992, \quad \alpha = 1 - \lambda = 0.0002.$$

- **Monte Carlo sample size**

  - $M = 10{,}000$ paths to stabilize tail estimates.

Options:

- **Window lengths**
  - Use the same 5-year windows for underlying and vol estimates.
- **SV calibration**
  - Fit model to current surface daily, update parameters with EWM.

Model usage

- The top-level **main.py** acts as a launcher, prompting the user to choose between:

    1. **historical_calibration.py** — loads CSV, calibrates μ/σ over historical window, then runs all VaR/ES methods
    2. **input_mu_sigma.py** — prompts user for manual μ/σ, then runs parametric VaR & ES only

- There are **four VaR engines**:

    - `parametric5yr` (rolling-window GBM)
    - `parametric_ewm` (exponentially weighted GBM, VaR only)
    - `historical` (empirical P&L)
    - `montecarlo` (GBM Monte Carlo)

- There are **three ES engines** (all except `parametric_ewm`).

Model exposure

- Captures market risk of the portfolio P.
- Supports both long- and short-position loss profiles.
- Future extensions: multi-asset portfolios, time-varying correlation, jumps.

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5 Validation Methodology and Scope

- (details in test plan)

We exercised a five-point test plan:

1. **Flat-Price Unit Test**
    - Constant prices → zero VaR/ES for all methods (ES skipped for EWM).
2. **Parametric Closed-Form Consistency**
    - Deterministic GBM (σ=0) → exact match to analytic VaR/ES.
3. **Backtest Exception Frequency**
    - 10-yr GBM path → realized 5-day exception frequency within ±0.005 of nominal 1%.
4. **Monte Carlo vs. Parametric Agreement**
    - σ=0 path → parametric and MC VaR series are identical.
5. **Portfolio Consistency Visualization**
    - Real multi-stock portfolio → overlaid VaR and ES time series showed no method drifting by

more than ±20% relative to the group median.

First 4 tests were automated with `pytest` and simple plotting scripts, covering both unit-level and end-to-end validation.

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6 Validation Results

- details in test result
- all tests passed

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7 Conclusions and Recommendations

The four VaR/ES engines:

- **Pass all unit and end-to-end tests**, demonstrating correct implementation of GBM assumptions.
- **Agree closely** on real-data portfolio risk metrics, ensuring robustness.

Limitations & next steps:

- Current scope: **stocks only**. Options require adding volatility-surface dynamics.
- Recommendation: implement Greeks-based parametric and full SV Monte Carlo for options, then re-run the validation suite with option P&L data.
- Investigate **non-GBM features** (jumps, fat tails) if backtests on extreme events show undercoverage.

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8 Bibliography

- [1] Stein, H. J. "Model Validation Report Template." Strategic Risk Research Group, November 18 2015.