5

FUNCTIONS

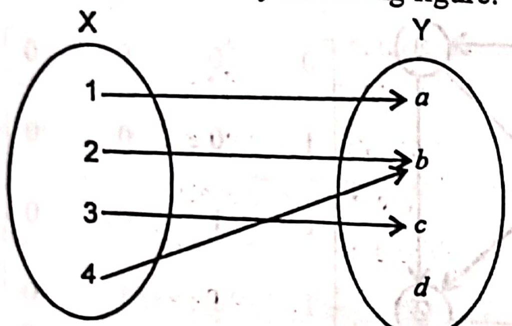
Art-1. Definition of Function

(P.T.U. B.C.A. I 2004)

Let X and Y be two non-empty sets. A subset f of $X \times Y$ is called a function from X to Y if for every $x \in X$, there exists a **unique** $y \in Y$ such that $(x, y) \in f$ or we say $y = f(x)$.

For example : $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d\}$ then subset $f = \{(1, a), (2, b), (3, c), (4, b)\}$ is a function from X to Y $\because$ for each $x \in X$, we have $y \in Y$ such that $y = f(x)$.

Above function is also represented by following figure.



Another Definition of Function

Let X and Y be two non-empty sets. Then a rule f which associates each element of X with a **unique** element of Y is called a function from X to Y.

The other terms used for functions are *mappings* or *transformations*. We denote this mapping by $f: X \to Y$ or $X \xrightarrow{f} Y$.

The set X is called the **domain** of f and is written as $D_f = X$. The set Y is called **co-domain** of f.

If an element $y \in Y$ is associated with an element x of X under the rule f, then y is called the **image** of x under the rule f, denoted by $f(x)$.

The set consisting of images of all the elements of X under f is called **Image** set or Range of f and is written as R_f or. ran f.

$\therefore$ $R_f = \{f(x) ; x \in X\} = f(X)$ or $R_f = \{y : y = f(x) \text{ where } x \in X\} = f(X)$

Clearly $f(X) \subset Y$

In above example $D_f = \{1, 2, 3, 4\}$

$\therefore R_f = \{a, b, c\}$ and co-domain is $\{a, b, c, d\}$.

Note. Difference between Relation and Function

Function is a special case of that of a relation. A relation may relate each element of the domain to more than one element of the range, but a function relates each element of the domain to one and only one element of the co-domain.

It should be noted that every function is a relation but every relation is not a function.

Consider $X = \{1, 2, 3, 4,\}$, $Y = \{5, 6, 7\}$

$X \times Y = \{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7)\}$

Let R be a subset of $X \times Y$ where $R = \{(1, 5), (2, 6), (2, 7), (3, 6), (4, 5)\}$

Here R is not a function from X to Y as $2 \in X$ is associated to two different elements 6, 7 of Y and for a function no two distinct ordered pairs have the same first element. But R is a relation as $R \subset X \times Y$.

Again take $R = \{(1, 5), (2, 7), (3, 6), (4, 5)\}$. In this case R is a function from X to Y as each element of X appears in the first element in one and only one ordered pair in R. R is also a relation from X to Y.

Remark : (*i*) To every $x \in X$, $\exists$ *a* unique $y \in Y$ such that $y = f(x)$. The unique element $y \in Y$ is also called the value of f at x and is denoted by $f(x)$.

(*ii*) Different element of X may be associated with the same element of Y.

(*iii*) There may be elements of Y which are not associated with any element of X.

(*iv*) We refer to a function as f and not as $f(x)$ which is the value of function f at x.

However, by *an abuse of language* it has become customary to call $f(x)$ as function instead of f.

(*v*) Since $f(x)$ or y depends upon the choice of x, x is called **independent** variable and y is called **dependent** variable.

e.g., $y = f(x) = x^2$, x is independent variable and y is dependent variable.

(*vi*) Functions are generally denoted by $f, g, h, \phi, \ldots\ldots$

Examples. The rule shown in the figure is not a function as each element of X is not associated. Here $5 \in X$ has no image in Y.

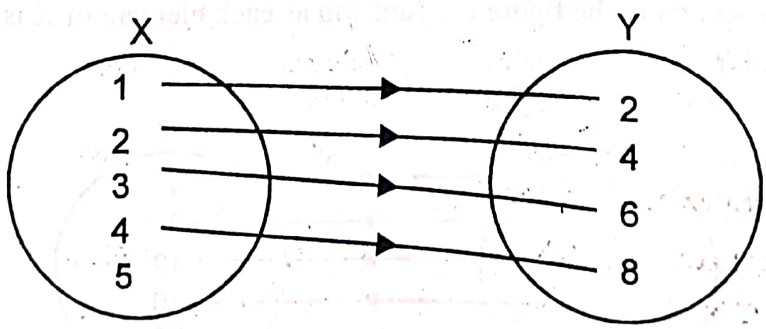


Fig. 1

(*ii*) The rule shown in the figure is not a function as $1 \in X$ is associated with more than one element namely a and b of Y.

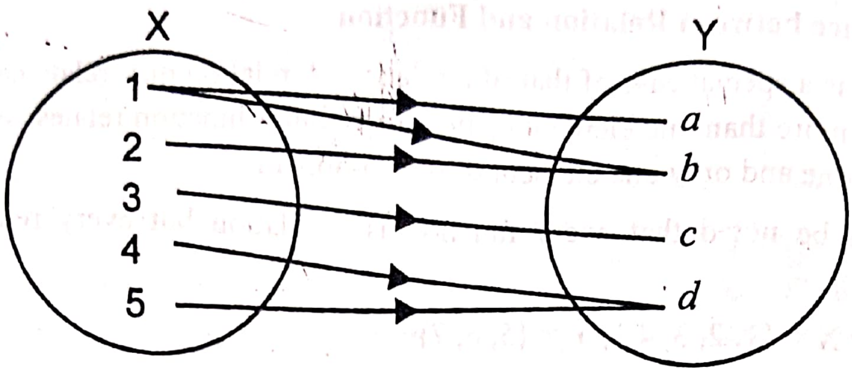


Fig. 2

(*iii*) The rule shown in the figure is a function as each element of X is associate with a unique element of Y.

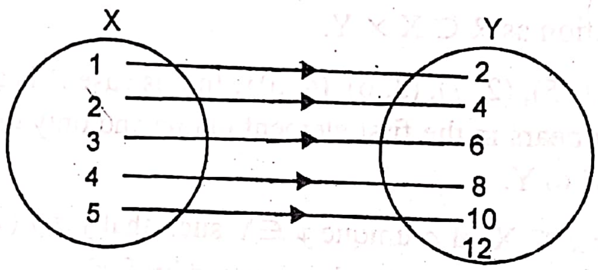


Fig. 3

(*iv*) The rule shown in the figure is a function as each element of X is associate with unique element of Y.

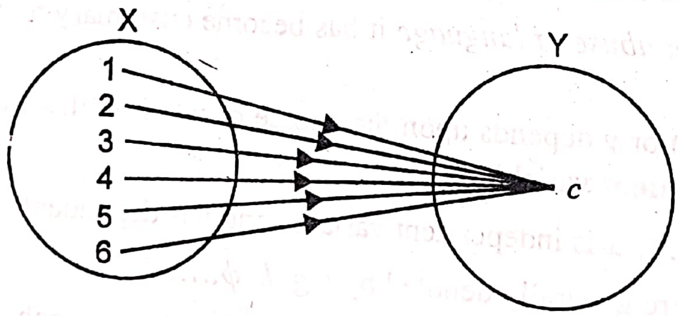


Fig. 4

(*v*) The rule shown in the figure is a function as each element of X is associated wit unique element of Y.

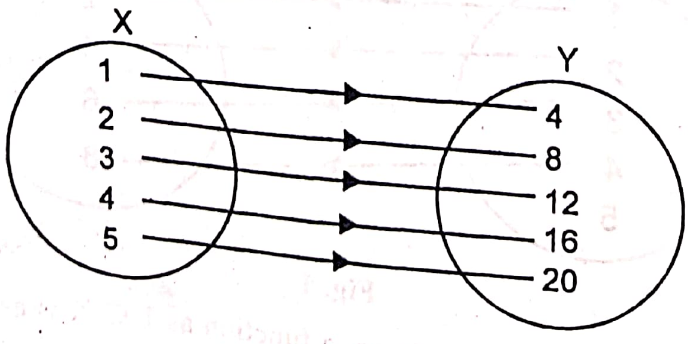


Fig. 5

TYPES OF FUNCTIONS :

ONE-ONE FUNCTION OR INJECTIVE FUNCTION

(P.T.U. B.C.A. I, 2004, 2005, 2007)

A function f from X to Y is said to be **one-one** (abbreviated 1-1) iff

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)\ \forall\ x_1, x_2 \in X, \text{ or equivalently}$$

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2\ \forall\ x_1, x_2 \in X.$$

In other words if different elements of X under the rule f have different images in Y, then f is called one-one function.

Function shown in fig. 3 is one-one function.

MANY-ONE FUNCTION

A function which is not 1-1 is called **many-one** function. Function in fig. 4 is many-one function.

ONTO FUNCTION OR SURJECTIVE FUNCTION (P.T.U. B.C.A. I 2004)

A function f from X to Y is called onto iff every element of Y is an image of at least one element of X. In other words we can say that for every $y \in Y$, there exist $x \in X$ such that $y = f(x)$. Function in fig. 5 is onto.

INTO FUNCTION : A function which is not onto is called into. Function in fig. 3 is into function.

Remark : In case of onto function $R_f = Y$ where as in case of into function R_f is proper subset of Y.

Examples. (*i*) Let X = {1, 2, 3, 4}, Y = {2, 4, 6, 8, 10}

Then the function f depicted by the diagram is 1-1 into

$\because$ $10 \in Y$ has no pre-image in X.

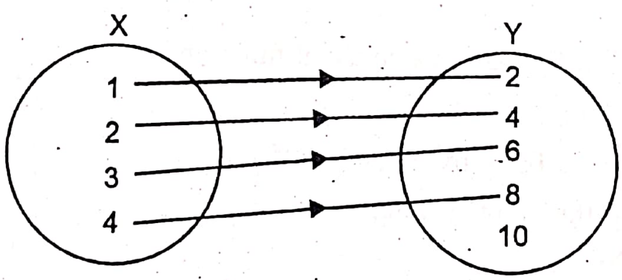


Fig. 6

(*ii*) Let X = {1, 2, 3, 4}, Y = {4, 8, 12, 16}

Then the function f depicted by the diagram is one-one onto.

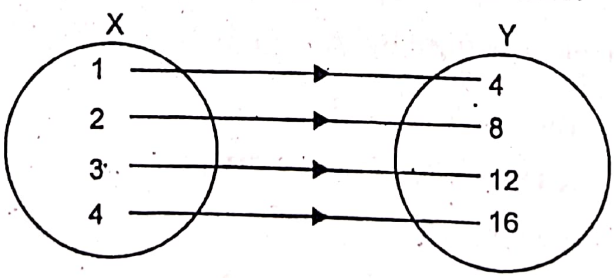
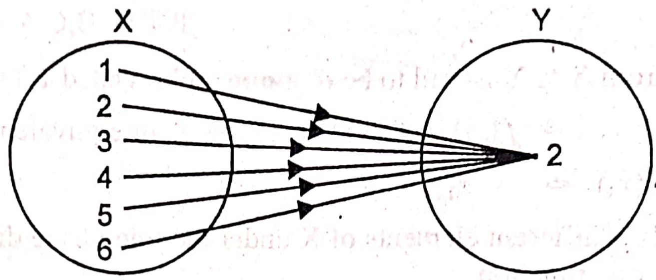


Fig. 7

(*iii*) Let $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{2\}$

Then the function f depicted by the diagram is many-one onto.



one one cross

Fig. 8

BIJECTIVE FUNCTION OR ONE-ONE ONTO FUNCTION

(P.T.U. B.C.A. I 2004)

A function which is one-one and onto is called bijective function. It is also called **one-one correspondence.**

Function shown in figure 5 is one-one onto.

REAL VALUED FUNCTION ON REAL VARIABLES

Let X, Y be two non-empty subsets of real numbers. Then every function f from X to Y is called a real valued function on real variables.

EQUAL FUNCTIONS

Two real valued functions f and g are said to be equal iff $D_f = D_g$ and

$f(x) = g(x)\ \forall\, x \in D_f$. We write it as $f = g$.

CONSTANT FUNCTION

A function $f : X \to Y$ is called a constant function if $f(x) = y$ for every $x \in X$ and for fixed $y \in Y$.

Function shown in figure 4 is a constant function.

IDENTITY MAPPING

Let $I_X : X \to X$ be defined by, $I_X(x) = x\ \forall\, x \in X$.

Then I_X is called the identity mapping on X.

INVERSE MAPPING

(P.T.U. B.C.A. I 2004)

Let $f : X \to Y$ be a one-one onto mapping. Then the mapping $f^{-1} : Y \to X$ which associates to each element $y \in Y$ the unique element $x \in X$ such that $f(x) = y$ is called the inverse map of f.

Let $X = \{1, 2, 3\}$ $Y = \{a, b, c\}$ then $f : \{(1, a), (2, b), (3, c)\}$ is one-one as well as onto. So f^{-1} exist and is defined by $f^{-1} : Y \to X$

$$f^{-1} = \{(a, 1), (b, 2), (c, 3)\}.$$

METHOD TO CHECK ONE-ONE (INJECTIVE)

Let $f : X \to Y$ be any function.

(*i*) Take two arbitrary elements x_1, x_2 in domain of f.

(ii) Solve $f(x_1) = f(x_2)$. If $f(x_1) = f(x_2)$ gives only $x_1 = x_2$ (i.e. only one solution) then we say function is one-one. Otherwise function is not one-one. Then it is called many one.

Note : Many one function (not 1-1) can also be proved by taking example. Take two numbers x_1 and x_2 from X such that $x_1 \neq x_2$. Show that $f(x_1) = f(x_2)$.

Example 1. Let $f: N \to N$ defined by $f(x) = 2x + 5$. Prove that f is one-one.

Sol. Let $x_1, x_2 \in N$

such that $f(x_1) = f(x_2)$

$$\Rightarrow \quad 2x_1 + 5 = 2x_2 + 5$$

$$\Rightarrow \quad 2x_1 + 5 - 2x_2 - 5 = 0$$

$$\Rightarrow \quad 2(x_1 - x_2) = 0$$

$$\Rightarrow \quad x_1 - x_2 = 0 \qquad \text{(as } 2 \neq 0\text{)}$$

$$\Rightarrow \quad x_1 = x_2.$$

So f is one-one.

Example 2. Let $f: Z \to Z$ defined by $f(x) = x^2$. Prove that f is not one-one.

(P.T.U. B.C.A. I 2007)

Sol. To show f is not one-one, we take one example.

Let us take two integers 2 and -2

then $f(2) = (2)^2 = 4$

$f(-2) = (-2)^2 = 4$

Since $2 \neq -2$ but $f(2) = f(-2)$

So f is not one-one.

METHOD TO CHECK ONTO (SURJECTIVE)

Let $f: X \to Y$ be any function.

(i) Take one arbitrary elements y in Y.

(ii) Take $y = f(x)$

(iii) Solve this equation and find x in terms of y.

(iv) If corresponding to every $y \in Y$, there exist $x \in X$ then f is called onto.

If for at least one $y \in Y$, there is no $x \in X$ then f is not onto (or into).

Example : Check whether $f: N \to N$ defined by $f(x) = 2x + 5$ is onto or not ?

Sol. We take one element from co-domain of f

Let $y \in N$

if possible, let $y = f(x)$

$\Rightarrow \quad y = 2x + 5$

$\Rightarrow \quad 2x = y - 5$ or $x = \dfrac{y-5}{2}$

We have to check wheater for every $y \in N$, we can find or not $x \in N$ (domain)

if $y = 6$, $x = \dfrac{6-5}{2} = \dfrac{1}{2} \notin N$

So $y = 6$ has no pre-image

$\therefore \quad f$ is not onto.

ILLUSTRATIVE EXAMPLES

Example 1. A function f is defined on the set of integers as follows,

$$f(x) = \begin{cases} 1+x & 1 \le x < 2 \\ 2x-1 & 2 \le x < 4 \\ 3x-10 & 4 \le x < 6 \end{cases}$$

(*i*) Find the domain of the function

(*ii*) Find the range of the function

(*iii*) Find the value of $f(4)$

(*iv*) State whether f is one-one or many one function.

Sol. $$f(x) = \begin{cases} 1+x & 1 \le x < 2 \\ 2x-1 & 2 \le x < 4 \\ 3x-10 & 4 \le x < 6 \end{cases}$$

(*i*) Domain of the function $= D_f = \{x : x \in I \text{ s.t. } f(x) \in I\}$, I = set of integers.

$= \{1, 2, 3, 4, 5\}$

(*ii*) Range of the function $= R_f = \{(f(x) : \text{for all } x \in D_f\}$

x	1	2	3	4	5
$f(x)$	2	3	5	2	5

Clearly $R_f = \{2, 3, 5\}$

(*iii*) Since $f(x) = 3x - 10$ for $4 \le x < 6$

$\therefore \quad f(4) = 3(4) - 10 = 12 - 10 = 2$

(*iv*) From (*ii*) we observe that

$f(1) = 2$ and $f(4) = 2$ and $1 \ne 4$.

$\therefore \quad f$ is many-one function.

Example 2. Let A = B = {1, 2, 3, 4, 5}. Define functions $f: A \to B$ (if possible) such that:

(i) f is one-to-one and onto

(ii) f is neither one-to-one nor onto

(iii) f is one-one but not onto.

(iv) f is onto but not one-to-one. (Pbi.U., B.C.A.-II, Sept. 2006)

Sol. (i) Let $f: A \to B$ defined by $f(x) = x, \forall x \in A$.

i.e. $f = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$ f is one-to-one and onto.

(ii) Let $f: A \to B$ defined by $f(x) = 1, \forall x \in A$.

i.e. $f = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$

Clearly f is neither one-to-one nor onto.

(iii) None exists. Reason is that as A and B contains equal number of elements. So if we define one-one function from A to B, then it will be onto also.

(iv) None exists. Again A and B contains equal number of elements. So if we define onto function from A to B, then it will be one-one also.

Example 3. Let A = {1, 2, 3, 4} and B = $\{x, y, z\}$ how many functions $f: A \to B$ satisfy $f(1) = f(2) = x$? Give reason. (Pbi.U., B.C.A.-II, April 2006)

Sol. A = {1, 2, 3, 4}, B = $\{x, y, z\}$

Since $f(1) = f(2) = x$

We know in a function $f: A \to B$

Every element of A is uniquely associated to an element of B.

It is given that image of 1 and 2 is x.

Image of 3 can be x or y or z.

Similarly image of 4 can be x or y or z.

i.e. 3 and 4 can have image in 3 ways each.

So, total number of functions $= 1 \times 1 \times 3 \times 3 = 9$.

Example 4. Consider $f: N \to Z_{10}$ defined by $f(a)$ = the remainder after dividing 10 into a. What is $f(23)$? Describe the set of elements of N whose image is zero.

(Pbi.U. B.C.A. April 2006)

Sol. $f: N \to Z_{10}$ defined by $f(a)$ = the remainder after dividing 10 into a.

Clearly $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$f(23) = 3$ [because when we divide 23 by 10, we get 3 as remainder]

The set of elements of N whose image is zero is

$\{10, 20, 30, ...\} = 10\,k, k \in N$.

Example 5. Under what condition a constant function can be (i) one-to-one (ii) an onto function. (Pbi.U. B.C.A. April 2005)

Sol. (*i*) For $f: A \to B$ to be one-one, different elements of A must have different images in B which is possible iff A contains only one element. So, f is one-one iff domain contains single element.

(*ii*) Also for $f: A \to B$ to be onto, every element of B must be an image of at least one element of A, which is true iff B contains only one element. So, condition for onto is that co-domain contains only one element.

Example 6. Prove that if A is a set then identity function I on A is one-one onto.

(P.T.U. B.C.A.-I 2006)

Sol. $I : A \to A$ is defined by $I(x) = x, \ \forall \ x \in A$

I is one-one :

Let $x_1, x_2 \in A$

such that $I(x_1) = I(x_2)$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ I is one-one.

I is onto :

Let $y \in A$

If possible, let $x \in A$

such that $\quad y = I(x)$

$\Rightarrow \quad y = x$

or $\quad x = y$

so $\forall \ y \in A, \ \exists \ x = y \in A$

such that $\quad y = I(x)$

$\therefore$ I is onto.

Hence I is one-one onto.

Example 7. Is function $f : R \to R$ defined by $f(x) = \dfrac{1}{x}$ is bijective in its domain.

(P.T.U., B.C.A.-I 2007)

Sol. $f : R \to R, \quad f(x) = \dfrac{1}{x}$

$D_f = R - \{0\} \quad R_f = R - \{0\}$

Let $x_1, x_2 \in R \ (x_1 \neq 0, \ x_2 \neq 0)$

s.t. $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad \dfrac{1}{x_1} = \dfrac{1}{x_2}$

$\Rightarrow \qquad x_1 = x_2$

so f is one-one.

Let $y \in R$

If possible, $\quad y = f(x)$

$$\Rightarrow \qquad y = \frac{1}{x}$$

$$\text{or} \qquad x = \frac{1}{y}$$

Now $\forall \;\; y \in R \;\; (y \neq 0)$

$\exists \;\; x \in R$ s.t. $\; y = f(x)$

so f is onto.

f is one-one and onto.

so f is bijective in its domain.

Example 8. Prove that the function $f : C \to \mathbf{R}$, defined by $f(z) = |z|$ is neither one-one nor onto. (Pbi.U. B.C.A. 2005, 2012)

Sol.
$$f : C \to \mathbf{R}$$
$$f(z) = |z|$$

Let $z_1 = 2 + 3i$, $z_2 = 2 - 3i$

$$\Rightarrow \qquad z_1 \neq z_2$$

$$f(z_1) = |z_1| = \sqrt{4+9} = \sqrt{13} \qquad [z = x + iy,\ |z| = \sqrt{x^2 + y^2}\ ]$$

$$f(z_2) = |z_2| = \sqrt{4+9} = \sqrt{13}$$

Here $\qquad f(z_1) = f(z_2).$

But $\qquad z_1 \neq z_2$

so, f is not one-one.

Onto : again let $-3 \in \mathbf{R}$.

But there does not exist any complex number such that $f(z) = -3$.

So, f is not onto.

Example 9. For a function $g : R \to R$, determine whether the following functions are one-to-one and onto. If the function is not onto, determine range $g(R)$.

(*i*) $g(x) = x + 7 \qquad$ (*ii*) $g(x) = x^2 + x$.

(Pbi.U., B.C.A.-II, April 2006)

Sol. $g : R \to R$

(*i*) $\qquad g(x) = x + 7$

One-one : Let $x_1, x_2 \in R$

such that $g(x_1) = g(x_2)$

$\Rightarrow \quad x_1 + 7 = x_2 + 7$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ g is one-one.

Onto : Let $y \in R$

such that $y = g(x)$

$\Rightarrow \quad y = x + 7$

$\Rightarrow \quad x = y - 7$

so $\forall\, y \in R\ \exists\, x \in R$ such that $y = g(x)$

Hence g is onto.

(*ii*) $g(x) = x^2 + x$

g is not one-one as $g(0) = 0^2 + 0 = 0$

$g(-1) = (-1)^2 + (-1) = 0$

But $0 \neq -1$

Again let $y \in R$

such that $y = g(x)$

$\Rightarrow \quad y = x^2 + x$

$\Rightarrow \quad x^2 + x - y = 0$

for $y = -2$ we have $x^2 + x + 2 = 0$.

Hence $D = b^2 - 4ac = 1 - 4 \times 1 \times 2 = -7 < 0$

So x will be imaginary so g is not onto.

Range of g : we have $x^2 + x - y = 0$

y should be such that $D \geq 0$

$1 - 4 \times 1 \times (-y) \geq 0$

$\Rightarrow \quad 1 + 4y \geq 0$ or $y \geq -\frac{1}{4} \qquad y \in R.$

$$\therefore \quad R_g = \left\{ y \in R,\ y \geq -\frac{1}{4} \right\}$$

Example 10. (*a*) Give an example of a map (*i*) which is one to one but not onto, (*ii*) which is not one to one but onto. (*iii*) which is neither one to one nor onto.

(*b*) Define the following functions on integers by

$$f(k) = k + 1,\ g(k) = 2k \text{ and } h(k) = \left[\frac{k}{2}\right].$$

(*i*) Which of these are one to one ?

(*ii*) Which of these are onto ? (P.T.U. B.C.A.-I 2005)

Sol. (*i*) Consider the function $f : \mathbf{N} \to \mathbf{N}$ defined by $f(x) = x^2$

then f is one-one but not onto function.

For one-one

Let $n_1, n_2 \in \mathbf{N}$ be such that

$$f(n_1) = f(n_2)$$

$\Rightarrow$ $$n_1^2 = n_2^2$$

$\Rightarrow$ $$n_1 = n_2$$ [$n_1 = \pm n_2$ but as $n_1 \in \mathbf{N}$ so reject negative]

$\therefore$ f is one-one

For onto Since $2 \in \mathbf{N}$ but $\nexists\ n \in \mathbf{N}$ such that

$$f(n) = 2$$

i.e., $$n^2 = 2$$ [There is no natural number whose square is 2]

$\therefore$ f is not onto

(*ii*) Consider the function $f : \mathbf{Z} \to \mathbf{N} \cup \{0\}$ defined by

$$f(n) = |n|$$

then f is not one-one function but onto.

For one-one

Let $n_1, n_2 \in \mathbf{Z}$ be such that

$$f(n_1) = f(n_2)$$

$\Rightarrow$ $$|n_1| = |n_2|$$

$\Rightarrow$ $$n_1 = \pm n_2$$

$\therefore$ f is not one-one function

For example $5, -5 \in \mathbf{Z}$

$$f(5) = 5,\ f(-5) = 5$$

So f is not one-one.

For onto Since $\mathbf{N} \cup \{0\} \subseteq \mathbf{Z}$

$\therefore$ for any $n \in \mathbf{N} \cup \{0\}, \exists\ n \in \mathbf{Z}$ such that

$$f(n) = |n| = n$$

$\therefore$ f is onto

(*iii*) Consider the function $f: \mathbf{Z} \rightarrow \mathbf{N} \cup \{0\}$ defined by

$$f(x) = |n^2|$$

For one-one : Let $n_1, n_2 \in \mathbf{Z}$ be such that

$$\therefore \quad f(n_1) = f(n_2)$$

$$|n_1^2| = |n_2^2|$$

$$\Rightarrow \quad n_1 = \pm n_2$$

$\therefore$ f is not one-one function

For onto : Since $2 \in \mathbf{N} \cup \{0\}$ but $\nexists\ n \in \mathbf{Z}$ such that

$$f(n) = 2$$

$\therefore$ f is not onto.

(*b*) Given $f(k) = k + 1$, $g(k) = 2k$ and $h(k) = \left[\frac{k}{2}\right]$

Let $k_1, k_2 \in I$ such that

$$f(k_1) = f(k_2)$$

$$\Rightarrow \quad k_1 + 1 = k_2 + 1$$

$$\Rightarrow \quad k_1 = k_2$$

$\therefore$ f is one-one function

Again, let $g(k_1) = g(k_2)$

$$\Rightarrow \quad 2k_1 = 2k_2$$

$$\Rightarrow \quad k_1 = k_2$$

$\therefore$ h is one-one function

Again, let $h(k_1) = h(k_2)$

$$\Rightarrow \quad \left[\frac{k_1}{2}\right] = \left[\frac{k_2}{2}\right]$$

$$\Rightarrow \quad \frac{k_1}{2} = \frac{k_2}{2} \text{ not necessary}$$

i.e. $k_1 = k_2$ not necessary

For example if $k_1 = 4, k_2 = 5$

$$h(k_1) = \left[\frac{4}{2}\right] = 2$$

$$h(k_2) = \left[\frac{5}{2}\right] = 2$$

But $k_1 \neq k_2$

so h is not one-one.

(*ii*) **onto** (*a*) Let $y \in Z$ (Integer)

such that $y = f(k)$

$\Rightarrow$ $y = k + 1$

$\Rightarrow$ $k = y - 1$

for all $y \in Z$, there exist $k \in Z$ such that $y = f(k)$ so f is onto.

(*b*) Again let $y \in Z$

such that $y = g(k)$

$\Rightarrow$ $y = 2k$

$\Rightarrow$ $k = \frac{y}{2}$

now for $y = 5$ $k = \frac{5}{2} \notin Z$ so g is not onto.

(*c*) Let $y \in Z$

such that $y = h(k)$

$\Rightarrow$ $y = \left[\frac{k}{2}\right]$

$\Rightarrow$ $y \le \frac{k}{2} < y + 1$

$\Rightarrow$ $2y \le k < 2y + 2$

so for every $y \in Z$ we have at two values of k, $2y$ and $2y + 1$ such that $y = h(k)$

So h is onto.

Example 11. Specify the types (one-to-one or onto or both or neither) of the following function :

(*i*) $f : N \to N$ and $f(j) = J(\text{mod } 4)$

(*ii*) $g : N \times N \to N$ such that $g(x, y) = x + y$

(*iii*) $X = R$, $Y = \{x : x \in R \text{ and } x > 0\}$ and $f(x) = |x|$.

(Pbi.U., B.C.A.-II, April 2007)

Sol. (*i*) $f : N \to N$

One-one : $f(j) = J\ (\text{mod } 4)$

[$f(j) = J(\text{mod } 4)$ means J is remainder when j is divided by 4]

f is not one-one

$\because$ $f(3) = 3\ (\text{mod } 4) = 3$

$f(7) = 7\ (\text{mod } 4) = 3$

Now $f(3) = f(7)$

But $3 \ne 7$

so, f is not one-one.

Onto : Again f is not onto

$\because$ $f(j)$ can be 0, 1, 2, 3 only

for $5 \in N$ there is no j such that $f(j) = 5$.

(*ii*) $g : N \times N \to N$

such that $g(x, y) = x + y$

One-one : Let $(x_1, y_1), (x_2, y_2) \in N \times N$ such that $g(x_1, y_1) = g(x_2, y_2)$

$\Rightarrow \quad x_1 + y_1 = x_2 + y_2$

which does not implies $x_1 = x_2,\ y_1 = y_2$

e.g. $g(3, 7) = 10$

$g(7, 3) = 10$

But $(3, 7) \neq (7, 3)$

so, g is not one-one.

Onto : Now for $1 \in N$

there does not exist $(x, y) \in N \times N$ such that $g(x, y) = 1$ so, g is not onto.

(*iii*) $f(x) = |x|$

One-one : Let $x_1, x_2 \in R$

such that $f(x_1) = f(x_2)$

$|x_1| = |x_2| \Rightarrow x_1 = x_2$ or $x_1 = -x_2$

so, f is not one-one. For example $f(3) = 3$ and $f(-3) = 3$. But $3 \neq -3$

Onto : Let $y \in \{x : x \in R \text{ and } x > 0\}$

i.e. $y > 0, y \in R$ then $\forall\ y \in R$

$\exists \quad x \in R$

such that $f(x) = y$

so, f is onto.

Example 12. Let $f : R \times R \to R \times R$ be defined by $f(x, y) = (x + y, x - y)$. Show that f is bijection.

Sol. $f : R \times R \to R \times R$ such that $f(x, y) = (x + y, x - y)$

Let $f(x, y) = f(u, v)$

$\therefore \quad (x + y, x - y) = (u + v, u - v)$

$\therefore \quad x + y = u + v$ ...(1)

and $x - y = u - v$ ...(2)

Adding (1) and (2), $2x = 2u \quad \Rightarrow \quad x = u$

Subtracting (2) from (1), $2y = 2v \Rightarrow y = v$

$\therefore$ $(x, y) = (u, v)$

$\therefore$ f is one-one *i.e.*, f is injective.

Let (s, t) be an element of the codomain R × R. We determine (x, y) such that

$$f(x, y) = (s, t)$$

$$\Rightarrow \quad (x + y,\ x - y) = (s\ ,\ t)$$

$$x + y = s \qquad \ldots(3)$$

and $x - y = t$ $\qquad \ldots(4)$

Adding (3) and (4), $2x = s + t \Rightarrow x = \dfrac{s+t}{2} \in \mathrm{R}$

Subtracting (4) from (3), $2y = s - t \Rightarrow y = \dfrac{s-t}{2} \in \mathrm{R}$

Since $\left(\dfrac{s+t}{2}, \dfrac{s-t}{2}\right)$ is in the domain of f

$\therefore$ f is onto

$\therefore$ f is one-one and onto

$\therefore$ f is a bijection.

Example 13. Prove that the set, 2P of even positive integers has the same cardinality as the set P of positive integers.

Sol. We know that two sets have the same cardinality if there exists a bijection between them.

$\therefore$ for proving the required result, we must find a map from P to 2P and show that this map is a bijection.

Define $f : \mathrm{P} \to 2\mathrm{P}$ such that $f(x) = 2x$

Let $x_1, x_2 \in \mathrm{P}$ and $f(x_1) = f(x_2)$

$\therefore$ $2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one-one map.

Let $y \in 2\mathrm{P}$. We will show that there exists an element $x \in \mathrm{P}$ such that $y = f(x)$.

Since $y \in 2\mathrm{P}$, $\therefore$ $y = 2p$ for some $p \in \mathrm{P}$

$\therefore$ $f(p) = 2p = y$

$\therefore$ each element of 2P comes from some element of P

$\therefore$ f is onto

$\therefore$ f is both 1-1 and onto

$\therefore$ f is bijection

Hence the result.

Example 14. Prove that P, set of positive integers, is countable.

Sol. A set is said to be countable if it has same cardinality as set of natural numbers N.

Let $f: N \to P$ such that $f(x) = x + 1$

Let $x_1, x_2 \in N$ such that

$f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 + 1 = x_2 + 1 \quad \Rightarrow \quad x_1 = x_2$

$\therefore$ f is one-one map

Let $y \in P$. Let $y = f(x)$

Then $x + 1 = y \Rightarrow x = y - 1 \in N$

Now $f(x) = f(y - 1) = y - 1 + 1 = y$

$\therefore$ for each $y \in P$, there exist $x \in N$ such that $f(x) = y$

$\therefore$ f is onto

$\therefore$ f is both 1 – 1 and onto

$\therefore$ f is a bijective map

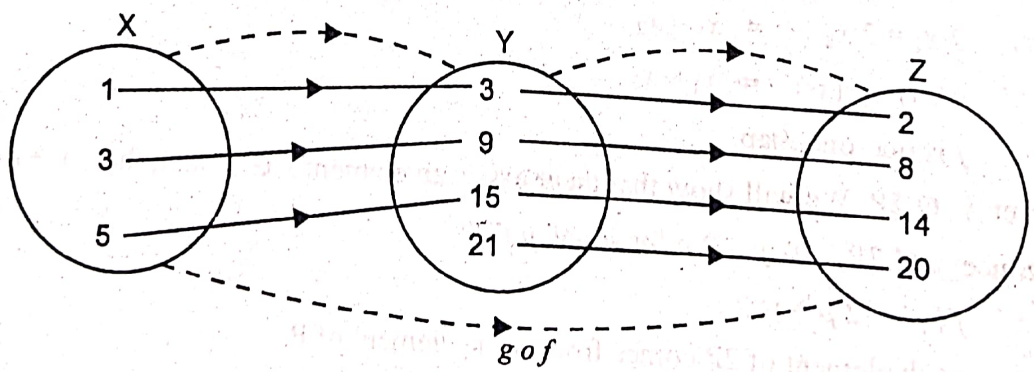
$\therefore$ N and P have same cardinality

$\Rightarrow$ P is countable.

Art-2. Composition of Functions

Let f be a function from X to Y and let g be a function from Y to Z. Let $x \in X$. Then the image of x under f *i.e.*, $f(x)$ is in Y. Now $g: Y \to Z$ and $f(x) \in Y$, therefore we can find the image of $f(x)$ under g *i.e.*, we can find $g[f(x)]$ which will be in Z. Also $f(x)$ is unique and consequently $g[f(x)]$ is unique. Thus we have a rule which assigns to element $x \in X$ a unique element $g[f(x)] \in Z$. In this way, we have a function from X to Z. This function is called the function of a function or composite function of g and f and is denoted by $g\,o\,f$.

Let X = {1, 3, 5}, Y = {3, 9, 15, 21}, Z = {2, 8, 14, 20}



Let f be a function from X to Y and g be a function from Y to Z such that

$f = \{(1, 3), (3, 9), (5, 15)\}$, $g = \{(3, 2), (9, 8), (15, 14), (21, 20)\}$

then $g\,o\,f = \{(1, 2), (3, 8), (5, 14)\}$

It must be noted that

(i) $g\,o\,f$ is defined only when $R_f \subset D_g$.

(ii) It is possible that one of fog may be defined while the other may not be defined.

(iii) $g\,o\,f$ and $f\,o\,g$ both may be defined but may not be equal.

COMPOSITE FUNCTION : Let f be a function with domain X and range in Y and let g be a function with domain Y and range in Z. The function with domain X and range in Z which maps an element $x \in X$, into $g\,(f(x))$, is called the composite of the functions f and g and is written as $g\,o\,f$.

PROPERTIES OF COMPOSITE FUNCTIONS (Pbi.U. B.C.A. 2009)

Let f, g, h be three functions and α be a real number, then

(i) $(fog)\,o\,h = fo\,(g\,o\,h)$ (Associative Law)

(ii) $fo\,(g+h) = fo\,g + fo\,h$ (Distributive Law)

(iii) $(\alpha f)\,o\,g = \alpha.\,(f\,o\,g)$ (Scalar multiplication)

(iv) $f\,o\,g \neq g\,o\,f$ (Non – Commutative)

Art-3. Equality of Maps

Two maps f and g are called equal maps if

(i) Domain of f = Domain of g

(ii) $f(x) = g\,(x)\ \forall\ x \in$ common domain of f and g

Art-4. Show that the composition of maps is associative.

Proof. Let $f : A \to B$, $g : B \to C$, $h : C \to D$ be mapping then ho $(g\,o\,f)$ and $(h\,o\,g)\,o\,f$ are both maps from A to D

$\therefore$ domain of $h\,o\,(g\,o\,f)$ = domain of $(h\,o\,g)\,o\,f$ [Each A] ...(1)

Let $x \in A$

then $((h\,o\,g)\,o\,f)\,(x) = (h\,o\,g)\,(f(x)) = h\,(g\,(f(x)))$

and $(h\,o\,(g\,o\,f))\,(x) = h\,((g\,o\,f)\,(x)) = h\,(g\,(f(x)))$

$\therefore$ $((h\,o\,g)\,o\,f)\,(x) = (h\,o\,(g\,o\,f))\,(x)\quad \forall\ x \in A$ ...(2)

$\therefore$ from (1) and (2) we get,

$$h\,o\,(g\,o\,f) = (h\,o\,g)\,o\,f$$

Hence composition of maps is associative.

Art-5. If $f : A \to B$ and $g : B \to C$ are both one - one maps, then $g\,o\,f$ is also one- one.

Proof. Since $f : A \to B$ and $g : B \to C$ are maps therefore $g\,o\,f$ is a map from A to C. Let $x_1, x_2 \in A$ such that

$(g\,o\,f)\,(x)_1 = (g\,o\,f)\,(x)_2$

$\Rightarrow$ $g\,(f(x_1)) = g\,(f(x_2))$

$\Rightarrow$ $f(x_1) = f(x_2)$, since g is one-one

$\Rightarrow$ $x_1 = x_2$ since f is one-one

$\therefore$ $g\,o\,f$ is a one-one map.

Art-6. If $f: A \to B$ and $g: B \to C$ are both onto maps, then $g \circ f$ is also onto.

Proof. Since f, g are onto

Let $c \in C$ be any element, then $\exists\ b \in B$ such that

$$g(b) = c \qquad (\because g \text{ being onto})$$

Again for this $b \in B$, $\exists$ some $a \in A$ such that

$$f(a) = b \qquad (\because f \text{ is onto})$$

$\therefore\ g \circ f(a) = g(f(a)) = g(b) = c$

Thus for $c \in C$, $\exists\ a \in A$ such that

$$g \circ f(a) = c$$

Hence $g \circ f: A \to C$ is onto.

Art-7. If $f: A \to B$ and $g: B \to C$ are both one-one and onto maps *i.e.*, bijective maps then $g \circ f$ is also both one-one and onto *i.e.*, bijective map.

(Pbi.U. M.Sc.I.T. 2010; Pbi. U. B.C.A. 2012)

Proof : Combining Art.5 and Art-6, we get required result.

Art-8. If $f: A \to B$ and $g: B \to C$ are two maps such that $g \circ f$ is one- one. Then f is one- one but g may not be one-one.

Proof. Since $f: A \to B$, $g: B \to C$ are maps

$\therefore\ g \circ f: A \to C$ is a map. Also $g \circ f$ is given to be one- one map.

If possible, suppose that f is not one-one

$\therefore\ \exists\ x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

But $\quad f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow \quad g \circ f(x_1) = g \circ f(x_2)$

$\therefore\ x_1, x_2 \in A$ such that $x_1 \neq x_2$, but $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\therefore\ g \circ f$ is not one-one, which is against the given hypothesis that $g \circ f$ is one-one.

Thus our supposition is wrong

$\therefore\ f$ is one-one

We, now give an example to illustrate that if $g \circ f$ is one-one, then g may not be one-one.

Let $A = \{1, 2\}$, $B = \{4, 5, 6\}$, $C = \{7, 8, 9, 10\}$

Let $f = \{(1, 4), (2, 6)\}$ and $g = \{(4, 7), (5, 8), (6, 8)\}$

then f and g are functions from A to B and from B to C respectively.

Have $\quad R_f = \{4, 6\} \subseteq D_g = \{4, 5, 6\}$

$\therefore \quad R_f \subseteq D_g$

$\Rightarrow\ g \circ f$ is defined and $D_{g \circ f} = D_f = A = \{1, 2\}$

$$g o f(1) = g\ (f(1)) = g\ (4) = 7$$

$$g o f(2) = g\ (f(2)) = g\ (6) = 8$$

$\therefore \quad g o f = \{(1, 7), (2, 8)\}$

Here, $g o f$ is one-one map since different elements of A have different image.

But g is not one-one since

$$g\ (5) = g\ (6) = 8. \text{ But } 5 \neq 6$$

Art-9. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two maps such that $g o f$ is onto then g is onto but f may not be onto.

Proof. Since $f : A \rightarrow B$ and $g : B \rightarrow C$ are maps, so $g o f$ is a map from A to C. We are given that $g o f : A \rightarrow C$ is onto. We now prove that g is onto

Let $z \in C$.

Since $g o f : A \rightarrow C$ is onto, so $\exists\ x \in A$ such that $g o f(x) = z$

$\Rightarrow \quad g\ (f(x)) = z$

$\Rightarrow \quad g\ (y) = z$ where $y = f(x)$

Since $x \in A$ and f is a map from A to B

Therefore $f(x) \in B$

$\Rightarrow \quad y \in B$

for given $z \in C$, we have determined $y \in B$ such that $g\ (y) = z$

$\therefore \quad g : B \rightarrow C$ is onto

Now, we show by an example that if $g o f$ is onto, then f may not be onto

Let $\quad A = \{1, 2\}, B = \{4, 5, 6\}, C = \{7\}$

Let $\quad f = \{(1, 4), (2, 6)\}$ and $g = \{(4, 7), (5, 7), (6, 7)\}$

Then f is a function from A to B and g is a function from B to C

$\therefore \quad g o f$ is a function from A to C such that $g o f = \{(1, 7), (2, 7)\}$

Here, g of is onto. But f is not onto since 5 belonging to B has no pre-image in A under the map f.

Art-10. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two maps such that $g o f : A \rightarrow C$ is both one-one and onto map then f is one-one and g is onto.

Proof. Combining Art-8, and Art-9, we get required result.

Art-11. Invertible Function

A function f defined from X to Y is said to be invertible if there exists a function g from Y to X such that $g o f = I_X$ and $f o g = I_Y$, where I_X is an identity mapping on X and I_Y is an identity mapping on Y.

Note : f and g are called inverse of each other.

Art-12. Let $f : X \to Y$. Then $f o I_X = f = I_Y o f$. (P.T.U. B.C.A. I 200

Proof. Let x be any element of X and let $f(x) = y, y \in Y$

Since $f : X \to Y$ and $I_Y : Y \to Y$

$\therefore \quad I_X o f : X \to Y$.

Now $(I_Y o f)(x) = I_Y(f(x)) = I_Y(y) = y = f(x) \ \forall \ x \in X$

$\therefore \quad I_Y o f = f$

Again $I_X : X \to X$ and $f : X \to Y$

$\therefore \quad f o I_X : X \to Y$

Now $(f o I_X)(x) = f(I_X(x)) = f(x) \ \forall \ x \in X$

$\therefore \quad f o I_X = f$.

Art-13. Let $f : X \to Y$ be one-one onto. Then the inverse map of f is unique.

Proof. Let $g : Y \to X$ and $h : Y \to X$ be two inverse maps of f.

Let y be an arbitrary element of Y.

Let $\quad g(y) = x_1$ and $h(y) = x_2$

Since g is an inverse map of f

$\therefore \quad g(y) = x_1$

$\Rightarrow \quad f(x_1) = y$ ...(1

Again h is an inverse map of f

$\therefore \quad h(y) = x_2$

$\Rightarrow \quad f(x_2) = y$ ...(2

From (1) and (2), we get, $f(x_1) = f(x_2)$

$\because \quad f$ is one-one

$\therefore \quad f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad g(y) = h(y) \ \forall \ y \in Y$

$\therefore \quad g = h$

$\therefore \quad$ inverse map of f is unique.

Art-14. A function $f : X \to Y$ is invertible iff f is one-one and onto.

Proof. (*i*) Assume that $f : X \to Y$ is invertible

$\therefore \quad \exists$ a function $g : Y \to X$ such that $f o g = I_Y$ and $g o f = I_X$

We will prove that f is one-one and onto.

To prove that f is one-one

Let $x_1 \in X, x_2 \in X$

Now $f(x_1) = f(x_2)$

$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$

$\Rightarrow \quad (g \, o \, f)(x_1) = (g \, o \, f)(x_2)$

$\Rightarrow \quad I_X(x_1) = I_X(x_2)$

$\Rightarrow \quad x_1 = x_2$

$\therefore \quad f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 = x_2 \; \forall \; x_1, x_2 \in X$

$\therefore \quad f$ is one-one.

'o prove f is onto

To each $y \in Y$, there exists $x \in X$ such that $g(y) = x$.

$\Rightarrow \quad f(g(y)) = f(x)$

$\Rightarrow \quad (f \, o \, g)(y) = f(x)$

$\Rightarrow \quad I_Y(y) = f(x)$

$\Rightarrow \quad y = f(x)$

$\therefore \quad f$ is onto.

(*ii*) Assume that $f : X \rightarrow Y$ is one-one and onto. We are to show that f is invertible.

Since f is one-one and onto

$\therefore$ to each $y \in Y$, there exists one and only one $x \in X$ such that $f(x) = y$.

$\therefore$ we can define a function $g : Y \rightarrow X$ such that $g(y) = x$ iff $f(x) = y$

Now $(g \, o \, f)(x) = g(f(x)) = g(y) = x, \; \forall \; x \in X$

$\therefore \quad g \, o \, f = I_X$

Again $(f \, o \, g)(y) = f(g(y)) = f(x) = y, \; \forall \; y \in Y$

$\therefore \quad f \, o \, g = I_Y$

$\therefore \quad f$ is invertible and g is inverse of f.

rt-15. If a function $f : X \rightarrow Y$ be one-one and onto then f^{-1} is also one-one and onto.

'oof. $\because \; f : X \rightarrow Y$ is one-one and onto

$\therefore \quad f^{-1} : Y \rightarrow X$ exists and $f^{-1} \, o \, f = I_X, f \, o \, f^{-1} = I_Y$

) prove f^{-1} is one-one

Let $y_1 \in Y, y_2 \in Y$.

Now $\quad f^{-1}(y_1) = f^{-1}(y_2)$

$\Rightarrow \quad f(f^{-1}(y_1)) = f(f^{-1}(y_2)) \quad \Rightarrow \quad (f \, o \, f^{-1})(y_1) = (f \, o \, f^{-1})(y_2)$

$\Rightarrow \quad I_Y(y_1) = I_Y(y_2) \quad \Rightarrow \quad y_1 = y_2$

$\therefore \quad f^{-1}(y_1) = f^{-1}(y_2) \quad \Rightarrow \quad y_1 = y_2 \; \forall \; y_1, y_2 \in Y$

$\therefore \quad f^{-1}$ is one-one.

To prove f^{-1} is onto

To each $x \in X$, there exists $y \in Y$ such that $y = f(x)$

$\Rightarrow \quad f^{-1}(y) = f^{-1}(f(x)) \quad \Rightarrow f^{-1}(y) = (f^{-1} o f)(x)$

$\Rightarrow \quad f^{-1}(y) = I_X(x) \quad \Rightarrow f^{-1}(y) = x$

$\therefore \quad f^{-1}$ is onto

Cor. $\because f^{-1}$ is invertible and its inverse is f.

$\therefore \quad (f^{-1})^{-1} = f.$

Art-16. Let $f : X \to Y$ and $g : Y \to Z$ and let f, g be one-one onto. Then $g o f : X \to Z$ is also one-one onto and $(g o f)^{-1} = f^{-1} o g^{-1}$.

(Pbi. U. M.Sc.-I.T. Dec. 2009)

Proof. (*i*) Here $f : X \to Y$ and $g : Y \to Z$ are 1-1 functions.

Now $g o f$ is a function from X to Z.

Let $x_1 \in X, x_2 \in X$.

Now $(g o f)(x_1) = (g o f)(x_2)$

$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$ [$\because$ g is one-one]

$\Rightarrow \quad f(x_1) = f(x_2)$ [$\because$ f is one-one]

$\Rightarrow \quad x_1 = x_2$

$\therefore \quad (g o f)(x_1) = (g o f)(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$

$\therefore \quad g o f$ is $1 - 1$.

(*ii*) Here $f : X \to Y$ and $g : Y \to Z$ are onto functions.

$\therefore \quad g o f$ is defined from X to Z

$\because \quad g$ is an onto mapping from $Y \to Z$

$\therefore \quad$ to each $z \in Z$, there exists $y \in Y$ such that $g(y) = z$.

Again as f is an onto mapping from X to Y.

$\therefore \quad$ to each $y \in Y$, there exists $x \in X$ such that $f(x) = y$

$\therefore \quad$ to each $z \in Z$, there exists $x \in X$ such that

$$z = g(y) = g(f(x)) = (g o f)(x)$$

$\therefore \quad g o f$ is onto.

(*iii*) Now $f : X \to Y$ and $g : Y \to Z$ are one-one and onto.

$\therefore \quad f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$ exist and the function $f^{-1} o g^{-1}$ is defined from Z to X.

Also $g o f : X \to Z$ is one-one and onto.

$\therefore\quad (g\,o\,f)^{-1}$ exists and is defined from Z to X.

Now $(f^{-1} o\, g^{-1})\, o\, (g\, o f) = f^{-1} o\, (g^{-1} o\, g)\, o f$

$= f^{-1} o\, I_Y\, o f = f^{-1} o\, (I_Y\, o f)$

$= f^{-1} o f = I_X$

Again $(g\, o f)\, o\, (f^{-1} o\, g^{-1}) = g\, o\, (f o f^{-1})\, o\, g^{-1}$

$= g\, o\, I_Y\, o\, g^{-1} = (g\, o\, I_Y)\, o\, g^{-1} = g\, o\, g^{-1} = I_Z$

$\therefore\quad (f^{-1} o\, g^{-1})\, o\, (g\, o f) = I_X$ and $(g\, o f)\, o\, (f^{-1} o\, g^{-1}) = I_Z$

$\therefore\quad (g\, o f)^{-1} = f^{-1} o\, g^{-1}$

xample 15. Let $A = \{1, 2, 3, 4, 5\}$. Let $f : A \to A$ and $g : A \to A$ be defined by

$f(1) = 3, f(2) = 5, f(3) = 3, f(4) = 1, f(5) = 2,$

$g(1) = 4, g(2) = 1, g(3) = 1, g(4) = 2, g(5) = 3.$

Find $(g\, o f)$ and $(f o\, g)$.

ol. $(f o\, g)(1) = f(g(1)) = f(4) = 1$

$(f o\, g)(2) = f(g(2)) = f(1) = 3$

$(f o\, g)(3) = f(g(3)) = f(1) = 3$

$(f o\, g)(4) = f(g(4)) = f(2) = 5$

$(f o\, g)(5) = f(g(5)) = f(3) = 3$

$(g\, o f)(1) = g(f(1)) = g(3) = 1$

$(g\, o f)(2) = g(f(2)) = g(5) = 3$

$(g\, o f)(3) = g(f(3)) = g(3) = 1$

$(g\, o f)(4) = g(f(4)) = g(1) = 4$

$(g\, o f)(5) = g(f(5)) = g(2) = 1$

$\therefore\quad f o\, g$ and $g\, o f$ are not equal in general.

xample 16. If $f, g : \mathbf{R} \to \mathbf{R}$ are defined respectively by

$f(x) = x^2 + 3x + 1, \quad g(x) = 2x - 3$

find formulae for (*i*) $f o\, g$ (*ii*) $g\, o f$ (*iii*) $f o f$ (*iv*) $g\, o\, g$.

ol. Here $f(x) = x^2 + 3x + 1, g(x) = 2x - 3$

(*i*) $(f o\, g)(x) = f(g(x)) = f(2x - 3)$

$= (2x - 3)^2 + 3(2x - 3) + 1$

$= 4x^2 - 12x + 9 + 6x - 9 + 1$

$= 4x^2 - 6x + 1$

(*ii*) $(g \circ f)(x) = g(f(x))$

$= g(x^2 + 3x + 1) = 2(x^2 + 3x + 1) - 3$

$= 2x^2 + 6x + 2 - 3 = 2x^2 + 6x - 1$

(*iii*) $(f \circ f)(x) = f(f(x))$

$= f(x^2 + 3x + 1)$

$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$

$= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1$

$= x^4 + 6x^3 + 14x^2 + 15x + 5$

(*iv*) $(g \circ g)(x) = g(g(x))$

$= g(2x - 3)$

$= 2(2x - 3) - 3 = 4x - 6 - 3 = 4x - 9$

Example 17. If $f(x) = \dfrac{1}{1-x}$, then what is $f[f\{f(x)\}]$? (Pbi. U. B.C.A. 2012)

Sol. Here $f(x) = \dfrac{1}{1-x}$ ...(1)

$\therefore \quad f\{f(x)\} = \dfrac{1}{1-f(x)} = \dfrac{1}{1-\dfrac{1}{1-x}}$ [$\because$ of (1)]

$= \dfrac{1-x}{1-x-1}$

$\therefore \quad f\{f(x)\} = \dfrac{1-x}{-x}$

$\therefore \quad f[f\{f(x)\}] = \dfrac{1-f(x)}{-f(x)} = \dfrac{1-\dfrac{1}{1-x}}{-\dfrac{1}{1-x}}$ [$\because$ of (1)]

$= \dfrac{1-x-1}{-1} = \dfrac{-x}{-1}$

$\therefore \quad f[f\{f(x)\}] = x.$

Example 18. Let f and g be functions from **R** to **R** defined by $f(x) = [x]$ and $g(x) = |x|$. Determine whether $f \circ g = g \circ f$. (Pbi.U., B.C.A.-II, April 2005)

Sol. Given $f(x) = [x]$ and $g(x) = |x|$

$f \circ g(x) = f[g(x)] = f(|x|) = [|x|]$

$g \circ f(x) = g[f(x)] = g([x]) = |[x]|$

Now $\quad fog \neq gof.$

As $\quad fog(-3.2) = f[g(-3.2)] = f(3.2) = 3$

$gof(-3.2) = g[f(-3.2)] = g(-4) = 4.$

Example 19. Let $f: R \to R$ and $g: R \to R$ be two functions on real number defined as $f(x) = 2x + 3$, $g(x) = x^2$ show that $fog \neq gof$ (Pbi.U., B.C.A. 2009)

Sol. $\quad f(x) = 2x + 3 \qquad g(x) = x^2$

$$fog(x) = f(g(x))$$
$$= f(x^2)$$
$$= 2x^2 + 3$$
$$gof(x) = g(f(x)) = g(2x + 3)$$
$$= (2x + 3)^2$$
$$fog \neq gof$$

Example 20. Let $A = \{1, 2, 3\}$. Define $f: A \to A$ by $f(1) = 2$, $f(2) = 1$, $f(3) = 3$.

Find f^2, f^4. (Pbi.U., B.C.A. 2009)

Sol. $\quad f^2(1) = f(f(1)) = f(2) = 1$

$f^2(2) = f(f(2)) = f(1) = 2$

$f^2(3) = f(f(3)) = f(3) = 3$

$f^2 : \{(1, 1), (2, 2), (3, 3)\}$

$f^4(1) = f^2(f^2(1)) = f^2(1) = 1$

$f^4(2) = f^2(f^2(2)) = f^2(2) = 2$

$f^4(3) = f^2(f^2(3)) = f^2(3) = 3$

$f^4 : \{(1, 1), (2, 2), (3, 3)\}$

Example 21. $f(x) = x^2 + 3x + 1$, $g(x) = 2x + 3$

$f, g : R \to R$. Find formulae for fog and gof. (Pbi. U. B.C.A., 2012)

Sol. Since $f: R \to R$ and $g: R \to R$

$\therefore \quad fog: R \to R$ and $gof: R \to R$

$$fog(x) = f(g(x))$$
$$= f(2x + 3)$$
$$= (2x+3)^2 + 3(2x + 3) + 1 = 4x^2 + 9 + 12x + 6x + 9 + 1$$
$$= 4x^2 + 18x + 19$$
$$gof(x) = g(f(x))$$
$$= g(x^2 + 3x + 1) = 2(x^2 + 3x + 1) + 3 = 2x^2 + 6x + 5$$

Example 22. If the mapping of f and g are given by

$$f = \{(1, 2), (3, 5), (4, 1)\}$$
$$g = \{(2, 3), (5, 1), (1, 3)\}$$

then write down the mapping $f \circ g$ and $g \circ f$. (Pbi. U. M.Sc., 2011)

Sol.
$$f \circ g\,(2) = f(g\,(2)) = f(3) = 5$$
$$f \circ g\,(5) = f(g\,(5)) = f(1) = 2$$
$$f \circ g\,(1) = f(g\,(1)) = f(3) = 5$$
$$\therefore \quad f \circ g = \{(2, 5), (5, 2), (1, 5)\}$$
$$g \circ f(1) = g\,(f(1)) = g\,(2) = 3$$
$$g \circ f(3) = g\,(f(3)) = g\,(5) = 1$$
$$g \circ f(4) = g\,(f(4)) = g\,(1) = 3$$
$$\therefore \quad g \circ f = \{(1, 3)\ (3, 1)\ (4, 3)\}$$

Example 23. Let $f : \mathrm{R} \to \mathrm{R}$ be real valued function defined by $f(x) = x^2$, $x \in \mathrm{R}$. Is f invertible ? Give reasons.

Sol. First we check weather f is a bijection or not.

For this let us check f to be one-one.

Let $x_1, x_2 \in \mathrm{R}$

such that $f(x_1) = f(x_2)$

$$\Rightarrow \quad x_1^2 = x_2^2$$
$$\Rightarrow \quad x_1 = \pm\, x_2$$
$$x_1 = x_2 \quad \text{or} \quad x_1 = -\,x_2.$$

So f is not 1-1.

Hence f is not a bijection $\therefore$ f is not invertible.

Example 24. Let $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = a\,x + b$, $a, b, x \in \mathbf{R}$ and $a \neq 0$. Show that f is invertible and find the inverse of f.

(Pbi.U., B.C.A.-II, April 2005)

Sol. $f(x) = ax + b,\ a, b \in \mathbf{R}, a \neq 0$

One-one : Let $x_1, x_2 \in \mathbf{R}$

such that $f(x_1) = f(x_2)$

$$\Rightarrow \quad ax_1 + b = ax_2 + b$$
$$\Rightarrow \quad ax_1 = ax_2$$
$$\Rightarrow \quad x_1 = x_2 \qquad [\because a \neq 0]$$

$\therefore$ f is one-one.

Onto : Let $y \in \mathbf{R}$

such that $y = f(x)$

$$\Rightarrow \qquad y = ax + b$$

$$\Rightarrow \qquad x = \frac{y-b}{a} \qquad ...(i)$$

as $a \neq 0 \therefore \forall\ y \in \mathbf{R}$

$\exists \qquad x \in \mathbf{R}$

such that $\qquad y = f(x)$

Hence f is onto.

Since f is one-one and onto so f is invertible.

Now we find inverse of f.

By definition $x = f^{-1}(y)$ iff $y = f(x)$

$$\Rightarrow \qquad \frac{y-b}{a} = f^{-1}(y) \qquad \text{[using } (i)\text{]}$$

or $\qquad f^{-1}(x) = \dfrac{x-b}{a}.$

Example 25. Define inverse function and find inverse of the function $y = -3x + 7$.

Sol. Definition : (Already given in Art-11.)

Let $f(x) = -3x + 7$

$f : \mathrm{R} \rightarrow \mathrm{R}$

Let $x_1, x_2 \in \mathrm{R}$

s.t. $\qquad f(x_1) = f(x_2)$

$$-3x_1 + 7 = -3x_2 + 7$$

$$-3x_1 + 7 - 3x_2 - 7 = 0$$

$$-3(x_1 - x_2) = 0$$

$$x_1 = x_2$$

so f is one-one

Let $y \in \mathrm{R}$

s.t. $\quad y = f(x)$

$\Rightarrow \quad y = -3x + 7$

$\Rightarrow \quad x = \dfrac{7-y}{3}$

$\forall\, y \in R, \exists\, x \in R$

s.t. $\quad y = f(x)$

so f is onto.

f is one-one and onto so $f^{-1} : R \to R$ exists.

By definition $x = f^{-1}(y)$ iff $y = f(x)$

$\Rightarrow \quad f^{-1}(y) = \dfrac{7-y}{3}$

or $\quad f^{-1}(x) = \dfrac{7-x}{3}$

Example 26. Is $f(x) = \dfrac{x-1}{x+1}$ invertible in its domain ? If so, find f^{-1}.

Further verify that $(f o f^{-1})(x) = x$.

Sol. Here $f(x) = \dfrac{x-1}{x+1}$

D_f = set of all reals except -1

R_f = set of all reals except 1

Let $x_1, x_2 \in D_f$ and $f(x_1) = f(x_2)$

$\Rightarrow \quad \dfrac{x_1 - 1}{x_1 + 1} = \dfrac{x_2 - 1}{x_2 + 1} \quad \Rightarrow x_1 x_2 - x_2 + x_1 - 1 = x_1 x_2 + x_2 - x_1 - 1$

$\Rightarrow \quad 2x_1 = 2x_2 \quad \Rightarrow \quad x_1 = x_2$

$\therefore \quad f(x_1) = f(x_2) \quad \Rightarrow x_1 = x_2$

$\therefore \quad f(x)$ is $1-1$ in D_f.

$\forall\, y \in R_f \;\; \exists \;\; x = \dfrac{1+y}{1-y} \in D_f$ (where $y \neq 1$)

s.t. $f(x) = f\left(\frac{1+y}{1-y}\right) = \frac{\frac{1+y}{1-y} - 1}{\frac{1+y}{1-y} + 1} = \frac{\frac{1+y-1+y}{1-y}}{\frac{1+y+1-y}{1-y}} = \frac{2y}{2} = y$

$\therefore$ the mapping f is onto

$\therefore$ f is both 1–1 and onto $\Rightarrow$ f^{-1} exists

Now to find f^{-1},

Let $y = f(x) = \frac{x-1}{x+1}$ (Cross Multiplying)

$\therefore$ $xy + y = x - 1 \Rightarrow x - xy = y + 1$

$\Rightarrow$ $x(1-y) = 1 + y \Rightarrow x = \frac{1+y}{1-y}$.

$\therefore$ $f^{-1}(y) = \frac{1+y}{1-y} \Rightarrow f^{-1}(x) = \frac{1+x}{1-x}$

and $D_{f^{-1}}$ = Set of all reals except 1

Verification :

$$(fof^{-1})(x) = f(f^{-1}(x)) = \frac{f^{-1}(x) - 1}{f^{-1}(x) + 1}$$

$$= \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{\frac{1+x-1+x}{1-x}}{\frac{1+x+1-x}{1-x}} = \frac{2x}{2} = x$$

$(fof^{-1})(x) = x$.

Example 27. Let $f: R \to R$ and $g: R \to R$ be real valued functions defined by

$f(x) = 2x^3 - 1$, $x \in R$ and $g(x) = \left[\frac{1}{2}(x+1)\right]^{1/3}$, $x \in R$. Show that f and g are bijective and each is inverse of other. (Pbi.U., B.C.A.-II, April 2007)

Sol. First we show f and g both are one-one

Let $x_1, x_2 \in R$ such that

$f(x_1) = f(x_2)$ and $g(x_1) = g(x_2)$

$\Rightarrow \quad 2x_1^3 - 1 = 2x_2^3 - 1$ and $\left[\frac{1}{2}(x_1+1)\right]^{1/3} = \left[\frac{1}{2}(x_2+1)\right]^{1/3}$

$\Rightarrow \quad 2x_1^3 - 1 - 2x_2^3 + 1 = 0$ and $\frac{1}{2}(x_1+1) = \frac{1}{2}(x_2+1)$ (By Cubing)

$\Rightarrow \quad 2(x_1^3 - x_2^3) = 0$ and $x_1 + 1 = x_2 + 1$

$\Rightarrow \quad x_1^3 - x_2^3 = 0$ and $x_1 + 1 - x_2 - 1 = 0$

$\Rightarrow \quad x_1^3 = x_2^3$ and $x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2$ and $x_1 = x_2$

$\therefore$ f and g both are one-one functions

Now we show f and g both are onto functions for this let $y \in R$

such that $y = f(x)$ and $y = g(x)$

$\Rightarrow \quad y = 2x^3 - 1$ and $y = \left[\frac{1}{2}(x+1)\right]^{1/3}$

$\Rightarrow \quad x^3 = \frac{y+1}{2}$ and $y^3 = \frac{1}{2}(x+1)$

$\Rightarrow \quad x = \left(\frac{y+1}{2}\right)^{1/3}$ ...(1) and $x = 2y^3 - 1$ ...(2)

so $\forall\, y \in R$ we have $x \in R$ such that $y = f(x)$

also $\forall\, y \in R$ we have, $x \in R$ such that $y = g(x)$

$\therefore$ f and g both are onto.

Hence f and g both are bijective and $\therefore$ Invertible.

From (1) $\quad x = \left(\frac{y+1}{2}\right)^{1/3}$

$\Rightarrow \quad f^{-1}(y) = \left(\frac{y+1}{2}\right)^{1/3}$ [Since $y = f(x)$ $\therefore x = f^{-1}(y)$]

or $\quad f^{-1}(x) = \left(\frac{x+1}{2}\right)^{1/3} = g(x)$

Similarly we can show $g^{-1}(x) = f(x)$.

Example 28. $f(a) = a + 1$

$$g(a) = \begin{cases} \dfrac{a}{2} & \text{if } a \text{ is even} \\ \dfrac{a-1}{2} & \text{if } a \text{ is odd} \end{cases}$$

Find $f o g$ and $g o f$. (P.T.U., B.C.A. I 2007)

Sol. $f o g(a) = f(g(a))$

$$= \begin{cases} \dfrac{a}{2}+1 & \text{if } a \text{ is even} \\ \dfrac{a-1}{2}+1 & \text{if } a \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{a+2}{2} & \text{if } a \text{ is even} \\ \dfrac{a+1}{2} & \text{if } a \text{ is odd} \end{cases}$$

$g o f(a) = g(f(a))$

$$= \begin{cases} \dfrac{f(a)}{2} & \text{if } f(a) \text{ is even} \\ \dfrac{f(a)-1}{2} & \text{if } f(a) \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{a+1}{2} & \text{if } a+1 \text{ is even} \\ \dfrac{a+1-1}{2} & \text{if } a+1 \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{a+1}{2} & \text{if } a \text{ is odd} \\ \dfrac{a}{2} & \text{if } a \text{ is even} \end{cases}$$

Example 29. f, g, h are functions from N to N ∪ {0} defined as

$$f(n) = n + 1,\ g(n) = 2n,\ h(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Find $f o h$, $(f o g) o h$.

(P.T.U. B.C.A.-I 2007)

Sol. $f(n) = n + 1$, $h(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$$\begin{aligned} f \circ h(n) &= f(h(n)) \\ &= h(n) + 1 \\ &= \begin{cases} 0+1 & \text{if } n \text{ is even} \\ 1+1 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

$$\begin{aligned} f \circ g(n) &= f(g(n)) = f(2\,n) \\ &= 2\,n + 1 \end{aligned}$$

$$\begin{aligned} (f \circ g) \circ h &= f \circ g\,(h(n)) \\ &= 2\,h(n) + 1 \\ &= \begin{cases} 2.0+1 & \text{if } n \text{ is even} \\ 2.1+1 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Example 30. Let $f : A \to B$ and $g : B \to C$ be two functions such that $g \circ f : A \to C$ is a bijective map. Then f is injective and g is surjective.

Sol. Since $f : A \to B$, $g : B \to C$ are maps

$\therefore$ $g \circ f : A \to C$ is a map. Also $g \circ f$ is given to be bijective *i.e.*, $g \circ f$ is one-one, onto map

If possible, let f be not injective *i.e.*, f is not one-one map

$\therefore$ $\exists$ $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

But $f(x_1) = f(x_2)$ $\Rightarrow$ $g(f(x_1)) = g(f(x_2))$

$\Rightarrow$ $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\therefore$ $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow$ $g \circ f$ is not one-one mapping which is against the given hyothesis that $g \circ f$ is one-one

$\therefore$ f is injective

Now, we show that g is surjective.

Let $z \in C$ be any element. Since $g o f : A \to C$ is onto

So, $\exists\ x \in A$ such that $(g o f)(x) = z$

$\Rightarrow\ g(f(x)) = z$

$\Rightarrow\ g(y) = z$, where $y = f(x)$

Since $x \in A$ and f is a map from A to B

$\therefore\ f(x) \in B$ *i.e.*, $y \in B$

$\therefore$ for given $z \in C$, we have determined $y \in B$ such that

$g(y) = z$

$\therefore\ g : B \Rightarrow C$ is onto *i.e.*, g is surjective.

Example 31. Let $X = Y = Z = R$ and let $f : X \to Y$ and $g : Y \to Z$ are such that $f(x) = 2x + 1$ and $g(y) = y/3$. Verify that $(g o f)^{-1} = f^{-1} o g^{-1}$.

(Pbi.U., B.C.A.-II, Sept. 2006)

Sol. $f : x \to y \quad f(x) = 2x + 1$

$$g(y) = \frac{y}{3}$$

L.H.S. : $\quad g o f(x) = g(f(x))$

$$= g(2x + 1)$$

$$= \frac{2x+1}{3}$$

Now we find $(g o f)^{-1}$

Let $\quad g o f(x) = y$

then $$y = \frac{2x+1}{3}$$

$$x = \frac{3y-1}{2}$$

$$\Rightarrow \quad (g o f)^{-1} y = \frac{3y-1}{2} \text{ or } (g o f)^{-1} x = \frac{3x-1}{2}$$

R.H.S. : $\quad f(x) = 2x + 1$

$$\Rightarrow \quad y = 2x + 1 \Rightarrow x = \frac{y-1}{2}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y-1}{2} \text{ or } f^{-1}(x) = \frac{x-1}{2}$$

Again $$g(y) = \frac{y}{3}$$

$$x = \frac{y}{3}$$

$$y = 3x$$

$$g^{-1}(x) = 3x \text{ or } g^{-1}(y) = 3y$$

Now $$f^{-1} o g^{-1}(x) = f^{-1}(g^{-1}(x))$$

$$= f^{-1}(3x)$$

$$= \frac{3x-1}{2}$$

so $$(g o f)^{-1} = f^{-1} o g^{-1}.$$

EXERCISE 5 (*a*)

1. Let a map $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Prove that f is neither one-one nor onto.
2. Let a function $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 3 \ \forall\ x \in \mathbf{R}$. Prove that f is one-one and onto.
3. Prove that a function $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x) = x^3$ is one-one onto.
4. Let A = {1, 2, 3}, B = {a, b, c, d} and define f : A → B by $f(1) = b$, $f(2) = c$, $f(3) = a$. Then show that f is injective.
5. Consider the function $f(x) = 2^x$ and $g(x) = x^2$. Determining which of the two function are one to one. Justify your answer.
6. f : A → B is a constant function. Can f be

   (*i*) 1 – 1 function (*ii*) Onto function ?
7. If $f(x) = \dfrac{x-4}{4x-1}$ for all $x \neq \dfrac{1}{4}$, find $f(f(x))$.
8. If $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.
9. If $y = f(x) =, \dfrac{x+2}{x-1}$, then show that $x = f(y)$.

10. If $f(x) = x^2-1$, $g(x) = 3x + 1$, then describe the following functions :

(i) $g o f$ (ii) $f o g$ (iii) $g o g$ (iv) $f o f$

11. (a) (i) Given $f(x + 1) = 3x + 5$, evaluate $f(2x)$.

(ii) $f(x) = x + 1$, $g(x) = x^2 + 1$, $h(x) = 3x - 2$ verify $(f o g) o h = f o (g o h)$

(iii) If a vertical line cuts a graph in two points then graph does not represent a function. Why ?

(b) Let $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$. For $x \in R$, where R is a set of real numbers, Find $g o f$, $f o f$, $h o g$.

12. Define composition of two functions. let f and g be two functions from $R \to R$ defined by $f(x) = x^2 + 3x + 2$ and $g(x) = 4x - 1$. Find $f og$ and $g o f$. Also calculate $(g o f)(-1)$ and $(f o g)(-1)$. Is composition commutative ?

13. Let $f : R \times R \to R \times R$ be defined by $f(x, y) = (x + 2y, y - x)$. Let $g : R \to R \times R$ be defined by $g(t) = (3t, t^2)$. Let $h = R \times R \to R$ be defined by $h(x, y) = x + 2y$. Find $f o g$, $h o f$, $h o (f o g)$.

14. If f: $\mathbf{R} \to \mathbf{R}$ is given by $f(x) = 3x - 1$, find f^{-1}.

15. The function $f(x) = 2^x$, $x > 0$ has an inverse. Why ?

16. Is the function $f(x) = \dfrac{x}{x+1}$ invertible in its domain ? If so, find the inverse function.

17. Let $A = \{-2, -1, 0, 1, 2, 3\}$, $B = \{0, 1, 2, \ldots\ldots, 10\}$ and $f : A \to B$ be a function defined by $f(x) = x^2$ for all $x \in A$, find $f^{-1}(C)$ where $C = \{0, 1, 2, 4\}$

18. Prove that each of the following function is a bijection :

(a) $f : R \times R \to R \times R$ defined by $f(x, y) = (x + y, 2x - y)$

(b) $f : Z \times Z \to Z \times Z$ defined by $f(m, n) = (2n + m, n + m)$

19. If S is a set containing finite number of elements and f is a function from S into S, then prove the following

(i) If f is one-one, then f is onto

(ii) If f is onto, then f is one-one.

20. Prove that if f is a bijection then $(f^{-1})^{-1} = f$.

21. Prove that $f(x) = 2^x$, $x > 0$ is invertible.

22. Let f and g be functions from R to R defined by $f(x) = [x]$ and $g(x) = |x|$. Determine whether $f o g = g o f$.

23. If #A and #B are both finite, how many different function are there from A into B.

ANSWERS

5. f is one-one, g is not one-one

6. (i) Not one-one (ii) onto

7. x

8. $x^4 - 6x^3 + 10x^2 - 3x$

10. (i) $3x^2 - 2$ (ii) $9x^2 + 6x$ (iii) $9x + 4$

(iv) $x^4 - 2x^2$

11. (a) (i) $6x + 2$ (ii) Two images correspond to same point

(b) x ; $x + 4$; $3x - 6$

12. $16x^2 + 4x$; $4x^2 + 12x + 7$; $-1, 12$; no

13. $(3t + 2t^2, t^2 - 3t)$, $4y - x$, $4t^2 - 3t$

14. $\dfrac{x+1}{3}$

15. Since f is one-one and onto

16. Yes ; $f^{-1}(x) = \dfrac{x}{1-x}$

17. $\{0, -1, 1, -2, 2\}$

23. $(\#B)^{\#A}$