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PRINCIPLE OF MATHEMATICAL INDUCTION

Art-1. Introduction

Induction is the process of establishing a valid general result from particular results. The principle of mathematical Induction is used to establish the validity of general results involving natural number. In other words, we can say, Mathematical Induction is one of the technique which can be used to prove a variety of mathematical statements which are formulated in terms of n, where n is a positive integer. Induction is a very important tool in computer science for several reasons, one of which is the fact that a characteristic of most programs is repetition of a sequence of statements.

Art-2. Principle of Mathematical Induction

Let $P(n)$ be the given statement. Then to prove the validity of $P(n)$ we have to perform following three steps :

(*i*) **Basis :** First we prove the given statement is true for $n = 1$ *i.e.* $P(1)$ is true.

(*ii*) **Assumption :** We assume result is true for $n = k$.

(*iii*) **Induction :** We prove that the given statement is true for $n = k + 1$ *i.e.* $P(k+1)$ is true.

Then we conclude by principle of mathematical induction that statement is true for all $n \in N$.

ILLUSTRATIVE EXAMPLES

Example 1. Show by Mathematical Induction that for all $n \geq 1$

$$1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

Sol. Let $P(n)$ be the statement

$$1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

Basis : We must first show that $P(1)$ is true so put $n = 1$

$$1 = \frac{1(1+1)}{2}, \text{ which is clearly true.}$$

Assumption : Now we suppose that P(k) is true. So taking $n = k$

$$1 + 2 + 3 + \ldots\ldots + k = \frac{k(k+1)}{2} \qquad \ldots(\text{I})$$

Induction : At last, we prove P(k +1) is true. Taking $n = k + 1$

$$1 + 2 + 3 + \ldots\ldots + (k+1) = \frac{(k+1)(k+1+1)}{2}$$

The left hand side can be written as

$$1 + 2 + 3 + \ldots\ldots + k + (k+1) = [1 + 2 + 3 + \ldots\ldots + k] + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \ldots[\text{using I}]$$

$$= (k+1)\left[\frac{k}{2}+1\right] = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+1+1)}{2} \text{ which is right hand side of P}(k+1)$$

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So by Mathematical Induction, P(n) is true for $n \geq 1$.

Example 2. By mathematical induction, prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall \ n \in \text{N}.$$

(Pbi.U. B.C.A. 2002, 2006; M.Sc. I.T., 2006, 2009)

Sol. We are to prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \ldots(1)$$

For $n = 1$,

L.H.S. $= 1^2 = 1$

R.H.S. $= \dfrac{1(1+1)(2+1)}{6} = \dfrac{1 \times 2 \times 3}{6} = \dfrac{6}{6} = 1$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ result (1) is true for $n = 1$

Assume that result (1) is true for $n = k$

$$\therefore \quad 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \ldots(2)$$

Adding $(k+1)^2$ to both sides of (2), we get,

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right]$$

$$= \frac{(k+1)}{6}\left[k(2k+1)+6(k+1)\right] = \frac{(k+1)}{6}[2k^2+k+6k+6]$$

$$= \frac{(k+1)}{6}[2k^2+7k+6] = \frac{(k+1)}{6}[2k^2+3k+4k+6]$$

$$= \frac{(k+1)}{6}\left[k(2k+3)+2(2k+3)\right] = \frac{(k+1)}{6}[(k+2)(2k+3)]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Comparison with result (1) shows that result is true for $n = k+1$.

$\therefore$ by method of induction, result (1) is true for all $n \in \mathbf{N}$.

Note. We have added $(k+1)^2$ to both sides. This is obtained by changing n to $k+1$ in the last term of L.H.S. of (1).

Example 3. Use the principle of mathematical induction to prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \quad \forall \; n \in \mathbf{N}.$$

Sol. We have to prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \qquad \ldots(1)$$

Basic : For $n = 1$

L.H.S. $= 1^3 = 1$

$$\text{R.H.S.} = \frac{1^2(1+1)^2}{4} = \frac{1 \times 4}{4} = 1$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ result (1) is true for $n = 1$.

Assumption : Assume that result (1) is true for $n = k$

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$$

Induction: Now $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \dfrac{k^2(k+1)^2}{4} + (k+1)^3$

$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right] = (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$\therefore \quad 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2\,(k+2)^2}{4}$$

which by comparison with result (1) shows that result is true for $n = k + 1$.

$\therefore$ by method of induction, result (1) is true for all $n \in \mathbf{N}$.

Example 4. Use the principle of mathematical induction to prove that

$$1.3 + 2.4 + 3.5 + \ldots + n\,(n+2) = \frac{n\,(n+1)\,(2n+7)}{6} \quad \forall \; n \in \mathbf{N}.$$

Sol. We have to prove that

$$1.3 + 2.4 + 3.5 + \ldots + n\,(n+2) = \frac{n\,(n+1)\,(2n+7)}{6} \qquad \ldots(1)$$

For $n = 1$,

L.H.S. $= 1.3 = 3$

R.H.S. $= \dfrac{1\,(1+1)\,(2+7)}{6} = \dfrac{1 \times 2 \times 9}{6} = 3$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ result (1) is true for $n = 1$.

Assume that result (1) is true for $n = k$.

$$\therefore \quad 1.3 + 2.4 + 3.5 + \ldots + k\,(k+2) = \frac{k\,(k+1)\,(2k+7)}{6}$$

Adding $(k+1)\,(k+3)$ to both sides, we get,

$$1.3 + 2.4 + 3.5 + \ldots + k\,(k+2) + (k+1)\,(k+3) = \frac{k\,(k+1)\,(2k+7)}{6} + (k+1)\,(k+3)$$

$$= (k+1)\left[\frac{k\,(2k+7)}{6} + k + 3\right] = (k+1)\left[\frac{k\,(2k+7) + 6k + 18}{6}\right]$$

$$= (k+1)\left[\frac{2k^2 + 13k + 18}{6}\right] = (k+1)\left[\frac{(k+2)\,(2k+9)}{6}\right]$$

$$\therefore \quad 1.3 + 2.4 + 3.5 + \ldots + k\,(k+2) + (k+1)\,(k+3) = \frac{(k+1)\,(k+2)\,(2k+9)}{6}$$

which by comparison with result (1), shows that result is true for $n = k + 1$.

$\therefore$ by method of induction, result (1) is true for all $n \in \mathbf{N}$.

Example 5. Use the principle of mathematical induction to prove that

$$1.2 + 2.3 + 3.4 + \ldots + n\,(n+1) = \frac{1}{3}\,n\,(n+1)\,(n+2) \;\; \forall \;\; n \in \mathbf{N}.$$

(Pbi.U. B.C.A. Sept. 2008; M.Sc.I.T. 2010)

Sol. We have to prove that

$$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{1}{3} n(n+1)(n+2) \quad \ldots(1)$$

For $n = 1$,

L.H.S. $= 1.2 = 2$

R.H.S. $= \frac{1}{3} \times 1 \times (1+1)(1+2) = \frac{1}{3} \times 1 \times 2 \times 3 = 2$

L.H.S. = R.H.S.

$\therefore$ result (1) is true for $n = 1$

Assume that result (1) is true for $n = k$.

$$\therefore \quad 1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \frac{1}{3} k(k+1)(k+2)$$

Adding $(k+1)(k+2)$ to both sides, we get,

$1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)(k+2)$

$$= \frac{1}{3} k(k+1)(k+2) + (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3} + 1\right)$$

$$\therefore \quad 1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$$

which by comparison with result (1) shows that result is true for $n = k + 1$.

$\therefore$ by method of induction, result (1) is true for all $n \in \mathbf{N}$.

Example 6. Use the principle of mathematical induction to prove that

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \forall \quad n \in \mathbf{N}.$$

(Pbi.U. M.Sc.I.T. 2011)

Sol. We have to prove that

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \ldots(1)$$

For $n = 1$,

L.H.S. $= \frac{1}{1.3} = \frac{1}{3}$

R.H.S. $= \frac{1}{2+1} = \frac{1}{3}$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ result (1) is true for $n = 1$

Assume that result (1) is true for $n = k$.

$$\therefore \quad \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Adding $\dfrac{1}{[2(k+1)-1][2(k+1)+1]}$ *i.e.*, $\dfrac{1}{(2k+1)(2k+3)}$ to both sides, we

et

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{1}{2k+1}\left[\frac{k}{1}+\frac{1}{2k+3}\right]=\frac{1}{2k+1}\left[\frac{2k^2+3k+1}{2k+3}\right]$$

$$=\frac{1}{2k+1}\left[\frac{(k+1)(2k+1)}{2k+3}\right]$$

$$\therefore \quad \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Comparing this result with result (1), we see that result (1) is true for $n = k + 1$.

$\therefore$ by method of induction, the result is true for all natural numbers n.

xample 7. Prove by Mathematical Induction that for all $n \geq 1$

$$1+a+a^2+\ldots..+a^{n-1}=\frac{a^n-1}{a-1}$$

ol. Let P(n) be the statement

$$1+a+a^2+\ldots..+a^{n-1}=\frac{a^n-1}{a-1}$$

asis : First we show that P(1) is true, so put $n = 1$

$$a^{1-1}=\frac{a^1-1}{a-1}$$

$$1 = 1 \text{ which is true.}$$

ssumption : Now we suppose that P(k) is true, so taking $n = k$

$$1+a+a^2+\ldots..+a^{k-1}=\frac{a^k-1}{a-1} \qquad \ldots(\text{I})$$

nduction : At last, we prove P(k +1) is true, taking $n = k + 1$

$$1+a+a^2+\ldots..+a^{k+1-1}=\frac{a^{k+1}-1}{a-1}$$

The left hand side can be written as

$$1 + a + a^2 + \ldots\ldots + a^k = (1 + a + a^2 + \ldots\ldots + a^{k-1}) + a^k$$

$$= \frac{a^k - 1}{a - 1} + a^k \qquad \text{[usin}$$

$$= \frac{a^k - 1 + a^k (a-1)}{a-1} = \frac{a^k - 1 + a^k \cdot a - a^k}{a - 1}$$

$$= \frac{a^{k+1} - 1}{a - 1}$$

which is equal to right hand side of P $(k + 1)$

So by mathematical induction, $P(n)$ is true for $n \geq 1$.

Example 8. Use Mathematical Induction to show that (P.U.P, M.C.A., 2004, 20

$$1 + 2 + 4 + \ldots\ldots 2^n = 2^{n+1} - 1$$

Sol. Let $P(n)$ be the statement

$$1 + 2 + 4 \ldots\ldots 2^n = 2^{n+1} - 1$$

or $$2^0 + 2^1 + 2^2 \ldots\ldots 2^n = 2^{n+1} - 1$$

Basis : First we show that P(1) is true, so put $n = 1$

$$2^0 + 2^1 = 2^{1+1} - 1$$

$$1 + 2 = 4 - 1$$

$$3 = 3, \text{ which is true.}$$

Assumption : Suppose that $P(k)$ is true, so taking $n = k$

$$2^0 + 2^1 + 2^2 \ldots\ldots 2^k = 2^{k+1} - 1$$

Induction : Now we prove $P(k + 1)$ is true, taking $n = k + 1$

$$2^0 + 2^1 + 2^2 \ldots\ldots 2^k + 2^{k+1} = 2^{k+1+1} - 1$$

L.H.S.

$$2^0 + 2^1 + 2^2 \ldots\ldots 2^k + 2^{k+1} = (2^0 + 2^1 + 2^2 \ldots\ldots 2^k) + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1} \qquad \text{...[usin}$$

$$= 2^{k+1} + 2^{k+1} - 1$$

$$= 2 \cdot 2^{k+1} - 1 = 2^{k+1+1} - 1$$

$$= \text{R.H.S.}$$

$\therefore$ P $(k + 1)$ is true.

Hence $P(n)$ is true by Induction.

Example 9. Prove by induction

$$a + ar + ar^2 \ldots\ldots\ldots ar^{n-1} = \frac{a(1 - r^n)}{1 - r}, r \neq 0$$

(Pbi.U. B.C.A., 200

Sol. Basis : for $n = 1$

$$a \, . \, r^{1-1} = \frac{a(1-r^1)}{1-r}$$

$$a = a$$

Result is True for $n = 1$

Assumption : Let result is true for $n = k$

$$a + ar + ar^2 \, \ldots\ldots\ldots + ar^k = a\,\frac{(1-r^k)}{1-r}, \; r \neq 0 \qquad \ldots(1)$$

Induction : Put $n = k + 1$

$$a + ar + ar^2 \, \ldots\ldots\ldots + ar^k = a\,\frac{(1-r^{k+1})}{1-r}$$

$$\text{L.H.S.} = a + ar + ar^2 \, \ldots\ldots\ldots + ar^k$$

$$= a + ar + ar^2 \, \ldots\ldots\ldots \, ar^{k-1} + ar^k$$

$$= a\,\frac{(1-r^k)}{1-r} + ar^k \qquad \text{[Using (1)]}$$

$$= a\left[\frac{1-r^k + r^k - r^{k+1}}{1-r}\right]$$

$$= a\left[\frac{1-r^{k+1}}{1-r}\right]$$

$$= \text{R.H.S.}$$

So result is True for $n = k + 1$

Example 10. Use Mathematical Induction to show that

$$1 \, . \, 2 \, . \, 3 + 2 \, . \, 3 \, . \, 4 + \ldots\ldots\ldots + n\,(n+1)\,(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

(Pbi.U. M.C.A., 2008)

Sol. Basis : for $n = 1$

$$1\,(1+1)\,(1+2) = \frac{1(1+1)(1+2)(1+3)}{4}$$

$$6 = \frac{2.3.4}{4}$$

$$6 = 6$$

so result is true for $n = 1$

Assumption : Let result is true for $n = k$

$$1 \, . \, 2 \, . \, 3 + 2 \, . \, 3 \, . \, 4 + \ldots\ldots\ldots + k\,(k+1)\,(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

Induction : Put $n = k+1$

$$1\,.\,2\,.\,3+2\,.\,3\,.\,4\,\ldots\ldots\ldots\,(k+1)\,(k+2)\,(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$\begin{aligned}
\text{L.H.S.} &= 1\,.\,2\,.\,3+2\,.\,3\,.\,4\,\ldots\ldots\ldots + (k+1)\,(k+2)\,(k+3)\\
&= 1\,.\,2\,.\,3+2\,.\,3\,.\,4\,\ldots\ldots\ldots + k\,(k+1)\,(k+2) + (k+1)\,(k+2)\,(k+3)\\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)\\
&= (k+1)(k+2)(k+3)\left[\frac{k}{4}+1\right] = \frac{k(k+1)(k+2)(k+3)(k+4)}{4}\\
&= \text{R.H.S.}
\end{aligned}$$

so result in True for $n = k+1$

Example 11. Show that $1^3+2^3+3^3\ldots\ldots+n^3 = (1+2+3+\ldots\ldots+n)^2$

(Pbi. U. M.Sc. I.T., 2011)

Sol. Basis : Put $n = 1$

$$1^3 = (1)^2$$

$$1 = 1$$

Result is true for $n = 1$

Assumption : Let result is true for $n = k$

$$1^3+2^3+3^3\ldots\ldots\ldots+k^3 = (1+2+3+\ldots\ldots\ldots+k)^2 \qquad \ldots(1)$$

Induction : Put $n = k+1$

$$1^3+2^3+3^3\ldots\ldots\ldots+(k+1)^3 = (1+2+3+\ldots\ldots\ldots+k+1)^2$$

$$\begin{aligned}
\text{L.H.S.} &= 1^3+2^3+3^3\ldots\ldots\ldots+(k^3)+(k+1)^3\\
&= (1+2+3+\ldots\ldots\ldots+k)^2+(k+1)^3 \qquad \text{[Using (1)]}\\
&= \left[\frac{k\,(k+1)}{2}\right]^2+(k+1)^3 \qquad \left[1+2+3\ldots\ldots.k = \frac{k\,(k+1)}{2}\right]\\
&= \frac{k^2\,(k+1)^2}{4}+(k+1)^3 = (k+1)^2\left[\frac{k^2}{4}+(k+1)\right] = (k+1)^2\left[\frac{k^2+4k+4}{4}\right]\\
&= \frac{(k+1)^2\,(k+2)^2}{4} = \left[\frac{(k+1)\,(k+2)}{2}\right]^2 = [1+2+3+\ldots\ldots.(k+1)]^2\\
&= \text{R.H.S.}
\end{aligned}$$

Example 12. Prove that if $n \geq 1$, then $1\underline{|1}+2\underline{|2}+3\underline{|3}+\ldots\ldots+n\underline{|n}=\underline{|n+1}-1$.

Sol. We have to prove that $1\underline{|1}+2\underline{|2}+3\underline{|3}+\ldots\ldots+n\underline{|n}=\underline{|n+1}-1$

For $n = 1$,

$$\text{L.H.S.} = 1\underline{|1} = 1\times 1 = 1$$

$$\text{R.H.S.} = \underline{|1+1}-1 = \underline{|2}-1 = 1\times 2-1 = 2-1 = 1$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ result is true for $n = 1$

Assume that result is true for $n = k$

$\therefore$ $1\underline{|1}+2\underline{|2}+3\underline{|3}+\ldots+k\underline{|k}=\underline{|k+1}-1$

Adding $(k+1)\underline{|k+1}$ to both sides, we get,

$$1\underline{|1}+2\underline{|2}+3\underline{|3}+\ldots\ldots+k\underline{|k}+(k+1)\underline{|k+1}=\underline{|k+1}-1+(k+1)\underline{|k+1}$$

$$= (1+k+1)\underline{|k+1}-1 = (k+2)\underline{|k+1}-1$$

$$= \underline{|k+2}-1$$

$\therefore$ $1\underline{|1}+2\underline{|2}+\ldots+k\underline{|k}+(k+1)\underline{|k+1}=\underline{|k+2}-1$

$\therefore$ result is true for $n = k+1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k+1$.

But the result is true for $n = 1$

$\therefore$ by method of induction, result follows.

Example 13. Prove by induction that for all natural number n

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta) = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Sol. Consider P (n) : $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)},$$ for all natural number n.

Basis : Put $n = 1$

$$P(1) : \sin\alpha = \frac{\sin(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}, \text{ which is true}$$

Assumption : Assume that P(n) is true for some natural numbers k, *i.e.*,

$$P(k) : \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(k-1)\beta)$$

$$= \frac{\sin\left(\alpha+\frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Induction : Now, to prove that P (k + 1) is true, we have

$$P(k+1) : \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(k-1)\beta) + \sin(\alpha+k\beta)$$

$$= \frac{\sin\left(\alpha+\frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha+k\beta)$$

$$= \frac{\sin\left(\alpha+\frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha+k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+k\beta-\frac{\beta}{2}\right) + \cos\left(\alpha+k\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+k\beta+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+k\beta+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}} = \frac{\sin\left(\alpha+\frac{k\beta}{2}\right)\sin\left(\frac{k\beta+\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha+\frac{k\beta}{2}\right)\sin(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thus P (k + 1) is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for al natural number n.

:xample 14. Prove that for any positive integer n, then number $2^{3n}-1$ is divisible by 7.

(P.T.U., B.C.A.-I, 2003; Pbi.U. B.C.A.-II 2011).

ol. Let P(n) be the statement : $2^{3n}-1$ is divisible by 7.

:asis : First we prove that P(1) is true. Taking $n = 1$

$2^{3\cdot 1}-1$ is divisible by 7

i.e. 7 is divisible by 7, which is true.

ıssumption : Suppose P(k) is true. Taking $n = k$ we get $2^{3k}-1$ is divisible by 7

$\therefore$ $2^{3k}-1 = 7m$, where m is a positive integer.

$\Rightarrow$ $2^{3k} = 7m + 1$ ...(I)

nduction : Now we prove that P($k+1$) is true. Taking $n = k+1$

$2^{3(k+1)}-1$ is divisible by 7

Now $2^{3(k+1)}-1 = 2^{3k+3}-1 = 2^{3k}\cdot 2^3 - 1 = 2^{3k}\cdot 8 - 1$

$= (7m+1)\,8 - 1$ [Using I]

$= 56m + 8 - 1 = 56m + 7 = 7\,(8m+1)$

which is divisible by 7.

$\therefore$ P($k+1$) is true.

Hence by Mathematical Induction $2^{3n}-1$ is divisible by 7.

:xample 15. Prove that $x-y$ divides $x^n - y^n$ for $n \geq 1$.

;ol. Let P(n) be the statement $x-y$ divides x^n-y^n.

3asis : First we show that P(1) is true. Put $n = 1$

$x-y$ divides x^1-y^1, which is true.

Assumption : Suppose P(k) is true. Taking $n = k$

we get $x-y$ divides x^k-y^k

i.e. $x^k - y^k = m\,(x-y)$ where m is an integer.

$x^k = m\,(x-y) + y^k$ ...(I)

Induction : Now we prove that P($k+1$) is true

Taking $n = k+1$

$x-y$ divides $x^{k+1}-y^{k+1}$

Now $x^{k+1}-y^{k+1} = x^k\cdot x - y^{k+1} = [m\,(x-y)+y^k]\cdot x - y^{k+1}$ [using I]

$= m\,(x-y)\cdot x + y^k\cdot x - y^{k+1}$

$= m\,(x-y)\,x + y^k\cdot x - y^k\cdot y$

$= m\,(x-y)\,x + y^k\,(x-y)$

$= (x-y)\,[mx + y^k]$

which is divisible by $x-y$.

$\therefore$ P ($k+1$) is true.

Hence by mathematical induction $x-y$ divides x^n-y^n for $n \geq 1$.

Example 16. Prove that the sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for any positive integer n.

Sol. Here $S_n = n^3 + 3n^2 + 5n + 3$

Basis : for $n = 1$, $S_1 = (1)^3 + 3(1)^2 + 5(1) + 3 = 1 + 3 + 5 + 3 = 12$

which is divisible by 3

$\therefore$ result is true for $n = 1$.

Assumption : Assume that the result is true for $n = k$.

$\therefore$ $S_k = k^3 + 3k^2 + 5k + 3$ is divisible by 3

Let $k^3 + 3k^2 + 5k + 3 = 3l$ ...(1)

where l is an integer

Induction :

$$S_{k+1} = (k+1)^3 + 3(k+1)^2 + 5(k+1) + 3$$
$$= (k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + (5k + 5) + 3$$
$$= (k^3 + 3k^2 + 5k + 3) + (3k^2 + 6k + 3 + 5k + 5 + 3 - 2k - 2)$$
$$= (k^3 + 3k^2 + 5k + 3) + (3k^2 + 9k + 9) = 3l + 3(k^2 + 3k + 3) \quad [\because \text{ of (1)}]$$
$$= 3[l + (k^2 + 3k + 3)]$$

which is divisible by 3.

$\therefore$ result is true for $n = k + 1$.

$\because$ by the method of induction, result is true for all positive integers n.

Example 17. Use the method of induction to prove that $n(n+1)(n+2)$ is a multiple of $6 \ \forall \ n \in \mathbb{N}$.

Sol. Let $P(n) = n(n+1)(n+2)$

$\therefore$ $P(1) = 1(1+1)(1+2) = 6$, which is a multiple of 6

$\therefore$ result is true for $n = 1$.

Assume that result is true for $n = k$

$\therefore$ $P(k) = k(k+1)(k+2)$ is a multiple of 6

Let $k(k+1)(k+2) = 6l$

where l is integer ...(1)

$$P(k+1) = (k+1)(k+2)(k+3) = (k+1)(k+2)(k) + (k+1)(k+2)(3)$$
$$= k(k+1)(k+2) + 3(k+1)(k+2)$$

Now $k + 1$ and $k + 2$ are two consecutive integers and, therefore, their product $(k+1)(k+2)$ is even.

Let $(k+1)(k+2) = 2m$ ...(2)

$\therefore$ $P(k+1) = 6l + 3(2m)$ $[\because$ of (1) and (2)$]$

$\therefore$ $P(k+1) = 6(l+m)$, which is a multiple of 6

$\therefore$ result is true for $n = k+1$.

$\therefore$ by method of induction, result is true for all $n \in \mathbf{N}$.

Example 18. Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

(Pbi. U. M.Sc-I.T. April 2010)

Sol. $P(n) = n^3 - n$

Basis : $n = 1$ $P(1) = 1^3 - 1 = 0$, which is divisible by 3

$\therefore$ results is true for $n = 1$

Assumption : Let result is true for $n = k$

$k^3 - k$ is divisible by 3

$k^3 - k = 3m$, m is an integer.

$$k^3 = 3m + k \quad \text{...(1)}$$

Induction : $n = k+1$

$$(k+1)^3 - (k+1) = k^3 + 1 + 3k(k+1) - k - 1$$

$$= k^3 + 1 + 3k^2 + 3k - k - 1$$

$$= 3m + k + 1 + 3k^2 + 2k - 1 \quad \text{[using (1)]}$$

$$= 3m + 3k^2 + 3k$$

$$= 3(m + k^2 + k)$$

which is divisible by 3.

Example 19. Prove by mathematical induction that $3^{4n+1} + 2^{2n+2}$ is divisible by 7 for every natural number n.

Sol. Let $P(n) = 3^{4n+1} + 2^{2n+2}$

$P(1) = 3^{4+1} + 2^{2+2} = 3^5 + 2^4 = 243 + 16 = 259$

$= 7 \times 37$, which is divisible by 7

$\therefore$ $P(n) = 3^{4n+1} + 2^{2n+2}$ is divisible by 7, is true for $n = 1$

Assume that result is true for $n = k$

$\therefore$ $P(k) = 3^{4k+1} + 2^{2k+2}$ is divisible by 7.

Let $3^{4k+1} + 2^{2k+2} = 7l$ where l is an integer.

$$\therefore \quad 3^{4k+1} = 7l - 2^{2k+2} \quad \text{...(1)}$$

$$P(k+1) = 3^{4(k+1)+1} + 2^{2(k+1)+2}$$

$$= 3^{4k+1} . 3^4 + 2^{2k+2} . 2^2 = 81.3^{4k+1} + 4.2^{2k+2}$$

$$= 81(7l - 2^{2k+2}) + 4.2^{2k+2} \qquad [\because \text{ of } (1)]$$

$$= 7.81\,l - 81.2^{2k+2} + 4.2^{2k+2} = 7.81\,l - 77.2^{2k+2}$$

$$= 7(81\,l - 11.2^{2k+2}) \text{, which is a multiple of 7.}$$

$\therefore$ result is true for $n = k + 1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$.

$\therefore$ by method of induction, the result is true for all $n \in \mathbf{N}$.

Example 20. Prove that $10^{2n-1} + 1$ is divisible by 11. (P.T.U. B.C.A.-I 2003)

Sol. Basis : Put $n = 1$

$$10^{2.1-1} + 1 = 10^{2-1} + 1$$

$$= 10 + 1$$

$$= 11 \text{ which is divisible by 11.}$$

Assumption : Let result is true for $n = k$.

$10^{2k-1} + 1$ is divisible by 11

$$\therefore \quad 10^{2k-1} + 1 = 11\,m$$

$$10^{2k-1} = 11\,m - 1 \qquad \ldots(1)$$

Induction : Put $n = k + 1$

$$10^{2(k+1)-1} + 1 = 10^{2k+2-1} + 1$$

$$= 10^{2k-1} . 10^2 + 1$$

$$= (11\,m - 1)\,100 + 1 \qquad [\text{using } (1)]$$

$$= 1100\,m - 100 + 1$$

$$= 1100\,m - 99$$

$$= 11(100\,m - 9) \text{ which is divisible by 11.}$$

Example 21. Prove by induction that 21 divides $4^{n+1} + 5^{2n-1}$.

(P.T.U. B.C.A. I 2004)

Sol. Basis : For $n = 1$

$$4^{1+1} + 5^{2.1-1} = 4^2 + 5^1$$

$$= 16 + 5$$

$$= 21 \text{ which is true.}$$

Assumption : Let result is true for $n = k$.

21 divides $4^{k+1} + 5^{2k-1}$

$\Rightarrow \quad 4^{k+1} + 5^{2k-1} = 21\,m$

$\Rightarrow \quad 5^{2k-1} = 21\,m - 4^{k+1}$ ...(1)

Induction : Put $n = k + 1$

$$4^{k+1+1} + 5^{2(k+1)-1} = 4^{k+1}.4^1 + 5^{2k-1}.5^2$$

$$= 4^{k+1}.4 + (21\,m - 4^{k+1})\,25 \quad \text{[Using (1)]}$$

$$= 4^{k+1}.4 + 21m.25 - 4^{k+1}.25$$

$$= 4^{k+1}(4 - 25) + 21\,m\,.\,25$$

$$= 4^{k+1}(-21) + 21\,m\,.\,25$$

$= 21\,(-4^{k+1} + 25\,m)$ which is divisible by 21.

Example 22. Prove by mathematical induction that $5^{2n+2} - 24\,n - 25$ is divisible by 576.

Sol. Let $p(n) = 5^{2n+2} - 24\,n - 25$

$\therefore \quad p(1) = 5^4 - 24 - 25 = 625 - 24 - 25 = 576$

which is divisible by 576

$\therefore$ result is true for $n = 1$

Assume that result is true for $n = k$

$\therefore \quad p(k) = 5^{2k+2} - 24\,k - 25$ is divisible by 576

Let $5^{2k+2} - 24\,k - 25 = 576\,m$ where m is an integer

$\therefore \quad 5^{2k+2} = 24\,k + 25 + 576\,m$ ...(1)

$$p(k+1) = 5^{2(k+1)+2} - 24(k+1) - 25 = 5^{(2k+2)+2} - 24\,k - 24 - 25$$

$$= 5^{2k+2}.5^2 - 24\,k - 24 - 25$$

$$= (24\,k + 25 + 576\,m)\,25 - 24\,k - 24 - 25 \quad [\because \text{ of (1)}]$$

$$= 25 \times 24\,k + 625 + 25 \times 576\,m - 24\,k - 24 - 25$$

$$= 24 \times 24\,k + 576 + 25 \times 576\,m$$

$$= 576\,k + 576 + 576 \times 25\,m = 576\,(k + 1 + 25\,m)$$

which is divisible by 576

$\therefore$ result is true for $n = k + 1$

$\therefore$ if result is true for $n = k$, then it is also true for $n = k + 1$

But the result is true for $n = 1$

$\therefore$ result follows by method of induction.

Example 23. Prove that $7^{2n} + 2^{3n-3} . 3^{n-1}$ is divisible by 25, $n \in N$.

Sol. Let $P(n) = 7^{2n} + 2^{3n-3} . 3^{n-1}$

$\therefore$ $P(1) = 7^2 + 2^{3-3} . 3^{1-1} = 49 + 1 = 50$, which is divisible by 25

$\therefore$ result is true for $n = 1$

Assume that result is time for $n = k$

$\therefore$ $P(k) = 7^{2k} + 2^{3k-3} . 3^{k-1}$ is divisible by 25

Let $P(k) = 7^{2k} + 2^{3k-3} . 3^{k-1} = 25\,m$

$\therefore$ $7^{2k} = 25\,m - 2^{3k-3} . 3^{k-1}$ ...(1)

Now $P(k+1) = 7^{2k+2} + 2^{3k} . 3^k = 7^{2k} . 7^2 + 2^{3k-3} . 2^3 . 3^{k-1} . . 3$

$= 49 . 7^{2k} + 24 . 2^{3k-3} . 3^{k-1}$

$= 49\,(25\,m - 2^{3k-3} . 3^{k-1}) + 24 . 2^{3k-3} . 3^{k-1}$ [$\because$ of (1)]

$= 49 \times 25\,m - 49 . 2^{3k-3} . 3^{k-1} + 24 . 2^{3k-3} . 3^{k-1}$

$= 49 \times 25\,m - 25 . 2^{3k-3} . 3^{k-1} = 25\,(49\,m - 2^{3k-3} . 3^{k-1})$

which is divisible by 25

$\therefore$ result is true for $n = k + 1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$

But the result is true for $n = 1$

$\therefore$ by method of induction, result is true for all $n \in N$.

Example 24. Prove using induction

$$2^n \geq 2n \ \forall\ n \geq 1$$

Sol. Let $P(n)$ be the statement

$$2^n \geq 2n$$

Basis : Put

$$n = 1$$

$$2^1 \geq 2 \cdot 1$$

$$2 \geq 2, \text{ which is true.}$$

Assumption : Suppose that $P(k)$ is true, so taking $n = k$

$$2^k \geq 2\,k \quad ...(I)$$

Induction : Now we prove $P(k+1)$ is true, put $n = k + 1$

$$2^{k+1} \geq 2\,(k+1)$$

$$\text{L.H.S.} = 2^{k+1}$$

$$= 2^k \cdot 2^1 \geq 2\,k \cdot 2 \quad ...[\text{using I}]$$

$$= 4k$$

$$= 2k + 2k$$

$$\geq 2k + 2 \qquad [\because k \geq 1, \ \therefore 2k \geq 2]$$

$$= 2(k+1)$$

$$\therefore \qquad 2^{k+1} \geq 2(k+1)$$

Hence P $(k + 1)$ is true.

$\therefore$ P(n) is true by Induction.

Example 25. Show that $2^n \geq n^3$ for $n \geq 10$

Sol. Basis : for $n = 10$

$$2^{10} \geq 10^3$$

$$1024 \geq 1000$$

so result is true for $n = 10$

Assumption : Let result is True for $n = k$, $k \geq 10$

$$2^k \geq k^3 \qquad ...(1)$$

Induction : Now, we prove result for $n = k + 1$

$$2^{k+1} \geq (k+1)^3$$

Now $\quad 2^{k+1} = 2^k . 2^1$

$$\geq k^3 . 2 \qquad \text{[Using (1)]}$$

$$= k^3 + k^3$$

$$\geq k^3 + 3k^2 + 3k + 1 - [k^3 \geq 3k^2 + 3k + 1 \text{ for } k \geq 10]$$

$$= (k+1)^3$$

$$\therefore \qquad 2^{k+1} \geq (k+1)^3$$

Hence result is true for $n = k + 1$

Example 26. Prove the following by PMI :

$$1 + 2 + 3 + ... + n < \frac{(2n+1)^2}{8}. \qquad \text{(Pbi.U., B.C.A. 2006)}$$

Sol. Let P(n) : $1 + 2 + 3 + ...n < \dfrac{(2n+1)^2}{8}$

Basis : P(1) mean $1 < \dfrac{(2 \cdot 1 + 1)^2}{8}$

i.e. $\quad 1 < \dfrac{9}{8}$, which is true.

$\therefore$ P(1) is true.

Basis : Let us assume that P(m) is true $\forall\, m \in$ N and $m \geq 1$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots\ldots m < \frac{(2m+1)^2}{8} \quad \ldots(i)$$

Induction : Adding (m + 1) to both sides ; we get

$$1 + 2 + 3 + \ldots. m + (m + 1) < \frac{(2m+1)^2}{8} + (m + 1)$$

$$= \frac{1}{8}(4m^2 + 4m + 1 + 8m + 8)$$

$$= \frac{1}{8}(2m+3)^2 = \frac{1}{8}(2(m+1)+1)^2$$

$\Rightarrow$ P (m + 1) is true

Thus, we see that P(m) $\Rightarrow$ P(m + 1). Also P(1) is true.

Hence by induction, P(n) is true for all $n \in$ N.

Example 27. Use mathematical induction to show that (Pbi.U. B.C.A. 2007)

$$n\,! \geq 2^n,\ n = 4, 5, 6, \ldots.$$

Sol. Basis : Let P (n) : "$2^n < n\,!$"

Now P(4) means $2^4 < 4!$

i.e. $16 < 24$, which is true.

Assumption : Let us assume that P(m) is true $\forall\, m \in$ N and $m \geq 4$

$$\Rightarrow \quad 2^m < m! \quad \ldots(i)$$

Induction : Multiply both sides of (i) by 2 ; we get

$2^{m+1} < 2 \times m\,! \leq (m + 1)\, m!$ $\quad [\because m \geq 1 \ \therefore m + 1 \geq 2]$

$\Rightarrow$ $2^{m+1} < (m + 1)!$

$\Rightarrow$ P(m + 1) is true

Thus, we see that P(m) $\Rightarrow$ P (m + 1). Also P(4) is true.

Hence by induction, P(n) is true for all $n \geq 4$.

Example 28. Prove by induction that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is a natural number for every natural number n.

(Pbi.U. M.Sc.I.T. 2008)

Sol. Let $P(n) = \frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$

$$P(1) = \frac{1}{5}.1^5 + \frac{1}{3}.1^3 + \frac{7}{15}.1$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1$$ which is a natural number.

$\therefore$ result is true for $n = 1$

Assume that result is true for $n = k$ where k is a natural number.

$\therefore$ $P(k) = \frac{1}{5}k^5 + \frac{1}{3}k^3 + \frac{7}{15}k$ is a natural number.

$$P(k+1) = \frac{1}{5}(k+1)^5 + \frac{1}{3}(k+1)^3 + \frac{7}{15}(k+1)$$

$$= \frac{1}{5}(k^5+5k^4+10k^3+10k^2+5k+1)+\frac{1}{3}(k^3+3k^2+3k+1)+\frac{7}{15}(k+1)$$

$$= \left(\frac{1}{5}k^5 + \frac{1}{3}k^3 + \frac{7}{15}k\right) + (k^4+2k^3+3k^2+2k+1)$$

$= P(k) + (k^4+2k^3+3k^2+2k+1)$, which is a natural number for all k.

$\therefore$ result is true for $n = k+1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k+1$

But the result is true for $n = 1$

$\therefore$ by method of induction, the result is true for all $n \in N$.

Example 29. Prove by mathematical induction if $A_1, A_2, \ldots\ldots, A_n$ are any sets then

$$\overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \overline{A}_i \text{ for } n \geq 1$$ (P.T.U., B.C.A. I, 2007)

Sol. Let $P(n)$ be the statement $\overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \overline{A}_i$

Basis : First we prove that P(2) is true. Taking $n = 2$

$$\overline{\left(\bigcap_{i=1}^{2} A_i\right)} = \bigcup_{i=1}^{2} \overline{A}_i$$

which can be written as $\overline{A_1 \cap A_2} = \overline{A}_1 \cup \overline{A}_2$

$\overline{A_1 \cap A_2} = \{x : x \in \overline{A_1 \cap A_2}\}$

$\Leftrightarrow$ $\{x : x \notin (A_1 \cap A_2)\}$

$\Leftrightarrow$ $\{x : x \notin A_1 \text{ or } x \notin A_2\}$

$\Leftrightarrow$ $\{x : x \in \overline{A}_1 \text{ or } x \in \overline{A}_2\}$

$\Leftrightarrow$ $\{x : x \in \overline{A}_1 \cup \overline{A}_2\} = \overline{A}_1 \cup \overline{A}_2$

$\therefore$ $\overline{A_1 \cap A_2} = \overline{A}_1 \cup \overline{A}_2$ ...(I)

Hence P(2) is true.

Assumption : Suppose P(k) is true taking $n = k$

$$\overline{\left(\bigcap_{i=1}^{k} A_i\right)} = \bigcup_{i=1}^{k} \overline{A}_i \quad \text{...(II}$$

Induction : Now we prove that P($k + 1$) is true, taking $n = k + 1$

$$\overline{\bigcap_{i=1}^{k+1} A_i} = \bigcup_{i=1}^{k+1} \overline{A}_i$$

L.H.S. can be written as

$$\overline{\left(\bigcap_{i=1}^{k} A_i \cap A_{k+1}\right)} = \overline{\left(\bigcap_{i=1}^{k} A_i\right)} \cup \overline{A_{k+1}} \quad \text{...[using I]}$$

$$= \bigcup_{i=1}^{k} \overline{A}_i \cup \overline{A_{k+1}} \quad \text{...[using II]}$$

$$= \bigcup_{i=1}^{k+1} \overline{A}_i \text{ which is equal to R.H.S. of P}(k + 1)$$

$\therefore$ P($k + 1$) is true.

EXERCISE 3 (*a*)

1. By mathematical induction, prove that

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2 \quad \forall n \in N$$

Or

Sum of first n odd natural numbers is n^2.

(P.T.U., B.C.A. I. 2005)

2. Use the principle of mathematical induction to prove that

$$1 + 4 + 7 + \ldots\ldots + (3n - 2) = \frac{n(3n - 1)}{2} \quad \forall n \in N.$$

3. Use the principle of mathematical induction to prove that

$$4 + 8 + 12 + \ldots + 4n = 2n(n + 1) \quad \forall n \in N.$$

4. Use the principle of mathematical induction to prove that

$$x + 4x + 7x + \ldots + (3n - 2)x = \frac{1}{2}n(3n - 1)x \quad \forall n \in N.$$

5. Use the principle of mathematical induction to prove that

$$a+(a+d)+(a+2d)+\ldots+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d] \ \forall\ n\in \mathbf{N}.$$

6. By mathematical induction, prove that

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3} \ \forall n\in \mathbf{N}.$$

7. Prove by principle of mathematical induction that

$$1.3+3.5+5.7+\ldots+(2n-1)(2n+1)=\frac{n}{3}(4n^2+6n-1) \ \forall n\in \mathbf{N}$$

8. Apply the principle of mathematical induction to prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \forall\ n\in \mathbf{N}.$$

(Pbi.U., B.C.A., 1999; M.C.A., 2006)

9. Use principle of mathematical induction to prove that

$$\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)} \quad \forall\ n\in \mathbf{N}.$$

10. Prove by induction that

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1} \quad \forall\ n\in \mathbf{N}.$$

11. In the arithmetic progression a, $a+d$, $a+2d$, ..., prove that the nth term is $a+(n-1)d$.

12. If 3^{2n}, where n is a natural number, is divided by 8, the remainder is always 1.

13. Prove by the method of induction that every even power of every odd number greater than 1 when divided by 8 leaves 1 for a remainder.

14. Prove that n^3+2n is divisible by 3, where n is a natural number.

(Pbi. U., M.Sc.-I.T. 2006, 2008)

15. Prove that n^2+n is even, where n is a natural number.

16. Use mathematical induction to prove : If n is any odd positive integer then $n(n^2-1)$ is divisible by 24.

17. Prove that $4^n+15n-1$ is divisible by 9 for all natural number n.

18. Show that $11^{n+2}+12^{2n+1}$ is divisible by 133, $n\in \mathbf{N}$.

19. Show that $2.7^n+3.5^n-5$ is divisible by 24, $n\in \mathbf{N}$.

20. Prove that $\frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$ is an integer for every positive integer n.

21. Let $p(n)$ be "$8^n - 3^n$ is a multiple of 5." Prove that $p(n)$ is a tautology over N.

(Pbi. U., M.C.A., 2003)

22. Prove that every positive integer greater than or equal to 2, has a prime decomposition.

23. Suppose that there are n persons in a room, $n \geq 1$, and that they all shake hands with one another. Prove that $\frac{(n-1)n}{2}$ hand shakes will have occurred.

24. Prove that if $n \geq 2$, then the generalized De Morgan's Law is true :

$$\sim(p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots \wedge p_n) \Leftrightarrow (\sim p_1) \vee (\sim p_2) \vee \ldots \vee (\sim p_n)$$