4

RELATIONS

rt-1. Relation

(P.T.U. B.C.A. I 2004, 2006)

efinition : Let A and B are two sets. A **relation** from A to B is a subset of $A \times B$. In rms of symbols we can say $R \subset A \times B$.

If (x, y) is a member of relation then we say $(x, y) \in R$ or $x\,R\,y$, which means x is elated to y. Similarly if x is not related to y then we say $(x, y) \notin R$ or $x \not{R}\, y$.

xample : Let $A = \{1, 2, 3\}$, $B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$A \times B$ has 6 elements. As number of subsets of $A \times B$ is 2^6 so we have 2^6 *i.e.* 64 ifferent relations on A to B. Some of them are

$$R_1 = \{(1, a), (2, b), (3, b)\}$$

$$R_2 = \{(1, b), (2, b)\}$$

$$R_3 = \{(3, a), (3, b)\}$$

$$R_4 = \phi$$

ote : A relation from A to B is also called **binary relation** as elements of relation are in air form (x, y). In present text relation means binary relation.

Art-2. Domain and Range of a Relation

If R is a relation from a set A to a set B. Then the set of the first components of the lements of R is called the domain of R and the set of the second components of the lements of R is called the range of R.

Thus, domain of $R = \{a : (a, b) \in R\}$, and range of $R = \{b : (a, b) \in R\}$.

If R is a relation from a set A to the set A, then R is called **a relation on A**. Thus a relation on a set A is defined as any subset of $A \times A$.

Example : Let $A = \{1, 2, 3\}$

Then $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

Let $R = \{(1, 2), (2, 2), (3, 2), (3, 3)\}$.

Then $R \subseteq A \times A$. Therefore R is a relation on the set A.

Since $(1, 2) \in R$, therefore $1\, R\, 2$ *i.e.*, 1 is R related to 2.

Again, since $(1,1) \notin R$ so $1 \not{R}\, 1$ is not R related to 1.

Domain of $R = \{1, 2, 3\}$.

Range of $R = \{2, 3\}$.

Example. For any $a, b \in \mathbf{N}$, the set of natural numbers, define a relation R by $a\, R\, b$ if a divides b.

Then $R = \{(1, 1), (1, 2), (1, 3), \ldots, (2, 2), (2, 4), \ldots (3, 3), (3, 6), \ldots\}$

Then R is clearly a subset of $\mathbf{N} \times \mathbf{N}$ and hence a relation on **N**.

$(1, 2) \in R$ since 1 divides 2

$(2, 1) \notin R$ since 2 does not divide 1.

Example. Let $A = \{1, 2\}$ and $B = \{3, 4\}$

Then $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Let $R = \{(1, 3), (2, 4)\}$

Then $R \subseteq A \times B$ and hence R is a relation from A to B.

$1\, R\, 3$ since $(1,3) \in R$.

$1 \not{R}\, 4$ since $(1,4) \notin R$.

Domain of $R = \{1, 2\}$.

Range of $R = \{3, 4\}$.

Art-3. Reflexive Relation

(P.T.U. B.C.A.-I 2006)

A relation R on a set A is called a reflexive relation if $(x, x) \in R$ for all $x \in A$ *i.e.*, if $x\, R\, x$ for every $x \in R$.

Example. Let $A = \{1, 2\}$.

Then $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

Let $R = \{(1, 1), (2, 2), (1, 2)\}$.

Then $R \subseteq A \times A$ and so R is a relation on the set A.

Since $(x, x) \in R\ \forall\, x \in A$, so R is a reflexive relation on A.

Example. For $a, b \in \mathbf{N}$, the set of natural numbers, define $a\, R\, b$ if a divides b

Then $R = \{(1, 1), (1, 2), \ldots, (2, 2), (2, 4) \ldots\}$

Then $R \subseteq \mathbf{N} \times \mathbf{N}$ and so R is a relation on **N**. Since for any natural number x, x divides x, so $x\, R\, x\ \forall\, x \in \mathbf{N}$. Therefore R is a reflexive relation.

Example. We define a relation S on the set of real numbers R by $a\, S\, b$ if a is less than b where $a, b \in \mathbf{R}$. It is not a reflexive relation since for any $a \in \mathbf{R}$, a is not less then a and hence $(a, a) \notin S$.

Example : The relation R defined on set of lines by $l_1\, R\, l_2$ if l_2 is parallel to l_2 is reflexive, since every line is parallel to itself.

Example : The relation R defined on set of natural numbers a R b if $a > b$ is not reflexive

$\because$ $a > a$ is not true.

Art-4. Symmetric Relation

A relation R on a set A is called a symmetric relation if a R $b \Rightarrow b$ R a where $a, b \in A$. *i.e.*, if $(a, b) \in R \Rightarrow (b, a) \in R$ where $a, b \in A$.

Example. Let $A = \{1, 2, 3\}$

Then $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Let $R = \{(1, 1), (1, 3), (3, 1)\}$ and $R_1 = \{(1, 2), (1, 3), (3, 1)\}$

Then $R, R_1 \subseteq A \times A$

Therefore R and R_1 are both relations on A

Since $(x, y) \in R \Rightarrow (y, x) \in R$, therefore R is a symmetric relation on A.

Since $(1,2) \in R_1$ but $(2,1) \notin R_1$, therefore $(x, y) \in R_1 \Rightarrow (y, x) \in R_1$ does not hold in R_1 always.

$\therefore$ R_1 is not a symmetric relation.

Example. For $a, b \in$ **N**, the set of natural numbers define a relation R by a R b if $a < b$. Then $R = \{(1, 2), (1, 3),....,(2, 3), (2, 4)......\}$.

Since $R \subseteq$ **N** $\times$ **N**, so R is a relation on **N**. $(1, 2) \in R$ since 1 is less than 2.

But $(2, 1) \notin R$ since 2 is not less than 1.

Therefore R is not a symmetric relation on **N**.

Example : Relation R defined on set of lines by l_1 R l_2 if l_1 is perpendicular to l_2 is symmetric

$\because$ if $l_1 \perp l_2$ then $l_2 \perp l_1$.

Art-5. Transitive Relation

A relation R on a set A is called a transitive relation if

$$a \text{ R } b,\ b \text{ R } c \quad \Rightarrow \quad a \text{ R } c\ \forall\ a, b, c \in A$$

i.e., if $(a, b) \in R$

and $(b, c) \in R \quad \Rightarrow \quad (a, c) \in R$ where $a, b, c \in A$.

Example. Let $A = \{1, 2, 3\}$

Then $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Let $R = \{(1, 1), (2, 2), (1, 3), (2, 3), (2, 1)\}$

Let $R_1 = \{(1, 2), (2, 3), (2, 1)\}$

Then R and R_1 are both subsets of $A \times A$.

Therefore, R and R_1 are both relations on A.

Also $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. Thus R is a transitive relation.

Further, $(1, 2)$ and $(2, 3) \in R_1$, but $(1, 3) \notin R_1$.

Thus R_1 is not a transitive relation

Example. For $a, b \in N$, the set of natural numbers, define $a R b$ if $2a + b = 10$.

The natural numbers a and b satisfying the relation $2a + b = 10$ are given by :

$$a = 1, b = 8, a = 2, b = 6, a = 3, b = 4, a = 4, b = 2$$

$\therefore \quad R = \{(1, 8), (2, 6), (3, 4), (4, 2)\}$

Since $(3, 4) \in R$ and $(4, 2) \in R$ but $(3, 2) \notin R$. Therefore R is not a transitive relation.

Art-6. Anti-Symmetric Relation

(P.T.U. B.C.A.-I 2006)

A relation R on a set A is called an anti-symmetric relation if $a R b$ and $b R a$ implies that $a = b$.

i.e., if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

OR

A relation R on a set A is called anti-symmetric if $a, b \in A$ $(a \neq b)$

and $(a, b) \in R \Rightarrow (b, a) \notin R$.

Example. Let A be the set of all lines in a plane. Let $L_1, L_2 \in A$. We define a relation R on A by $L_1 R L_2$ if $L_1 \parallel L_2$ *i.e.,* if L_1 is parallel to L_2. Since in any plane there exist different lines L_1 and L_2 such that $L_1 \parallel L_2$ and $L_2 \parallel L_1$ but $L_1 \neq L_2$ *i.e.,* $L_1 R L_2$ and $L_2 R L_1$ but $L_1 \neq L_2$, therefore R is not an anti-symmetric relation.

Example. Let $A = \{1, 2, 3\}$

Then $R = \{(1,1), (1,2), (2,1)\}$ is a relation on the set A.

Since $(1,2) \in R$ and $(2,1) \in R$ but $1 \neq 2$, therefore R is not anti-symmetric relation.

But $R_1 = \{(3, 3)\}$ is an anti-symmetric relation on A.

Example. For $a, b \in N$ the set of natural numbers define $a R b$ if $a \leq b$.

Let $a, b \in N$ such that $a R b$ and $b R a$.

$\therefore \quad a \leq b$ and $b \leq a$.

$\Rightarrow \quad a = b$.

$\therefore$ R is an anti-symmetric relation.

Identity Relation : Relation R defined on A by

$$R = \{(x, x) : \forall x \in A\}$$

Note : Identity relation on a set is symmetric as well as anti-symmetric.

Example : $A = \{1, 2, 3\}$

then identity relation $(I_R) = \{(1, 1), (2, 2), (3, 3)\}$

which is symmetric as well as anti-symmetric.

Art-7. Equivalence Relation

(P.T.U. B.C.A.-I 2004)

A relation R on a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

Example. Let X be the set of all triangles in a plane.

For any two triangles Δ_1 and Δ_2 in X define $\Delta_1 \mathrm{R} \Delta_2$, if Δ_1 and Δ_2 are congruent triangles. Then

(*i*) **R is Reflexive.** Since each triangle is congruent to itself, so $\Delta \mathrm{R} \Delta$ for each Δ in X.

(*ii*) **R is Symmetric.** Let Δ_1 and $\Delta_2 \in \mathrm{X}$ such that $\Delta_1 \mathrm{R} \Delta_2$. Then Δ_2 and Δ_1 are congruent triangles. Hence $\Delta_2 \mathrm{R} \Delta_1$.

(*iii*) **R is Transitive.** Let $\Delta_1, \Delta_2, \Delta_3 \in \mathrm{X}$ such that $\Delta_1 \mathrm{R} \Delta_2$ and $\Delta_2 \mathrm{R} \Delta_3$. Then Δ_1, Δ_2 are congruent triangles and so are Δ_2 and Δ_3. This implies that the Δ_1 and Δ_3 are also congruent triangles. Hence $\Delta_1 \mathrm{R} \Delta_3$.

So, R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation on X.

Art-8. Inverse of A Relation

The inverse of a relation R, denoted by R^{-1}, is obtained from R by interchanging the first and second components of each ordered pair of R.

Therefore, $\mathrm{R}^{-1} = \{(a, b) : (b, a) \in \mathrm{R}\}$.

If R is a relation from a set A to set B, then R^{-1} is relation from the set B to the set A.

$\therefore$ Domain of R^{-1} = Range of R. And range of R^{-1} = domain of R.

Example. Let $A = \{1, 2, 3\}$.

Let $\mathrm{R} = \{(1, 2), (1, 3), (2, 3), (3, 2)\}$.

Then R is a relation on the set A, since $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{A}$.

$$\mathrm{R}^{-1} = \{(2, 1), (3, 1), (3, 2), (2, 3)\}$$

Art-9. Void Relation in a set

Since ϕ is a subset of $\mathrm{A} \times \mathrm{A}$, therefore the null set ϕ is also a relation in A, called the void relations in a set A. Void relation ϕ is symmetric and transitive but not reflexive.

Universal relation in a set : Let A be any set and R be the set $\mathrm{A} \times \mathrm{A}$. Then R is called the universal relation in A.

Compatible Relation : A relation R in A is said to be compatible relation if it is reflexive and symmetric.

Art-10. Partial Order Relation

A relation R on a set X is said to be a partial order relation if it satisfies the following three conditions :

(*i*) $x \mathrm{R} x$, for every $x \in \mathrm{X}$ (reflexivity)

(*ii*) $x \mathrm{R} y$ and $y \mathrm{R} x \Rightarrow x = y$ (anti-symmetry)

(*iii*) $x \mathrm{R} y$ and $y \mathrm{R} z \Rightarrow x \mathrm{R} z$ (transitivity), $x, y, z \in \mathrm{X}$

Remark : The only equivalence relation on a set X which is also a partial order relation on X is the identity relation I_X, that is, the relation defined by x R Y iff $x = y$.

Art-11. Composition of Relations

Let R be an relation from a set A to a set B and S be a relation from set B to a set C. Then the composition relation denoted by SoR is a relation from a set A to a set C defined as

$$\text{SoR} = \{(a, c) : \exists\, b \in \text{B for which } (a, b) \in \text{R}, (b, c) \in \text{S}\}$$

Also if A be any non-empty set and R, S be any two relations on A. Then composition of R and S denoted by SoR is defined as

$$\text{SoR} = \{(a, c) : \exists\, b \in \text{A for which } (a, b) \in \text{R}, (b, c) \in \text{S}\}$$

Example. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$

and $R = \{(1, 2), (2, 5), (3, 6), (7, 4)\}$

$S = \{(1, 4), (7, 5), (3, 7), (4, 3)\}$

be two relations on a set A

Then $\text{SoR} = \{(7, 3)\}$

$\text{RoS} = \{(3, 4), (4, 6)\}$

$R^{-1} = \{(2, 1), (5, 2), (6, 3), (4, 7)\}$

$S^{-1} = \{(4, 1), (5, 7), (7, 3), (3, 4)\}$

From above example it is clear that

$$\text{RoS} \neq \text{SoR}.$$

Now we discuss some theorems on composition of relations.

Art-12. Let A, B and C be sets, R is relation from A to B, and S is a relation from B to C. Then prove that

$$(\text{SoR})^{-1} = R^{-1} o S^{-1}.$$

Proof : Let $c \in C$ and $a \in A$

then $(c, a) \in (\text{SoR})^{-1}$ iff $(a, c) \in \text{SoR}$

Now $(a, c) \in \text{SoR}$ which means there exist $b \in B$

such that $(a, b) \in R$ and $(b, c) \in S$

$\Rightarrow$ $(b, a) \in R^{-1}$ and $(c, b) \in S^{-1}$

or $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$

$\Rightarrow$ $(c, a) \in R^{-1} o S^{-1}$

So $(\text{SoR})^{-1} = R^{-1} o S^{-1}$

Art-13. Let A, B, C, D be sets. Suppose R is a relation from A to B, S is relation from B to C and T is a relation from C to D. Then show that

$$(\text{RoS})o\text{T} = \text{Ro}(\text{SoT}).$$

(Pbi.U., B.C.A.-II, 2006)

Proof : Let $(a, d) \in (RoS)oT$

Then there exists some $c \in C$ such that

$$(a, c) \in RoS \text{ and } (c, d) \in T$$

Since $(a, c) \in RoS$, so there exist b in B such that

$$(a, b) \in R \text{ and } (b, c) \in S$$

Now $(b, c) \in S$ and $(c, d) \in T$

$\Rightarrow$ $(b, d) \in SoT$

again $(a, b) \in R$ and $(b, d) \in SoT$

$\Rightarrow$ $(a, b) \in Ro(SoT)$

$\therefore$ $(RoS)_o T \subset R_o(SoT)$ ...(1)

Similarly, $R_o(SoT) \subset (RoS)_o T$ ...(2)

From equations (1) and (2)

$$R_o(SoT) = (RoS)_o T.$$

Art-14. Let R be a relation from X to Y and X_1, X_2 be two subsets of X then

(*i*) $X_1 \subseteq X_2 \Rightarrow R(X_1) \subseteq R(X_2)$

(*ii*) $R(X_1 \cup X_2) = R(X_1) \cup R(X_2)$

(*iii*) $R(X_1 \cap X_2) \subseteq R(X_1) \cap R(X_2)$

Proof : (*i*) Let $b \in R(X_1)$

Since $b \in R(X_1)$ $\therefore$ there exist $a \in X_1$

such that $(a, b) \in R(X_1)$

But $X_1 \subseteq X_2$, $\therefore$ $a \in X_2$

as $a \in X_2 \Rightarrow b \in R(X_2)$

$\therefore$ $R(X_1) \subseteq R(X_2)$

(*ii*) Let $b \in R(X_1 \cup X_2)$

$\therefore$ there exist some $a \in X_1 \cup X_2$

s.t. $(a, b) \in R(X_1 \cup X_2)$

Now $a \in X_1 \cup X_2 \Rightarrow a \in X_1$ or $a \in X_2$

If $a \in X_1 \Rightarrow b \in R(X_1)$

Similarly if $a \in X_2 \Rightarrow b \in R(X_2)$

so $b \in R(X_1) \cup R(X_2)$

$\therefore$ $R(X_1 \cup X_2) \subseteq R(X_1) \cup R(X_2)$ ...(1)

Also we know $X_1 \subseteq X_1 \cup X_2$ and $X_2 \subseteq X_1 \cup X_2$

By part (*i*) $R(X_1) \subseteq R(X_1 \cup X_2)$

$R(X_2) \subseteq R(X_1 \cup X_2)$

$\Rightarrow$ $R(X_1) \cup R(X_2) \subseteq R(X_1 \cup X_2)$ ...(2)

From (1) and (2)

$$R(X_1 \cup X_2) = R(X_1) \cup R(X_2).$$

(*iii*) Let $b \in R(X_1 \cap X_2)$

$\therefore$ there exist some $a \in X_1 \cap X_2$

s.t. $(a, b) \in R(X_1 \cap X_2)$

Now $a \in X_1 \cap X_2 \Rightarrow a \in X_1$ and $a \in X_2$

$\Rightarrow$ $b \in R(X_1)$ and $b \in R(X_2)$

$\Rightarrow$ $b \in R(X_1) \cap R(X_2)$

$\therefore$ $R(X_1 \cap X_2) \subseteq R(X_1) \cap R(X_2)$

Art-15. Equivalence Class

Let R be an equivalence relation on a non-empty set X. let $a \in X$. Then the equivalence class of a, denoted by $[a]$, is defined as follows :

$$[a] = \{ b \in X : b \, R \, a \}.$$

Example. Let $A = \{1, 2, 3\}$.

$$R = \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 3)\}$$

$[1] = \{1, 2\}$ since only 1 and 2 are related to 1

Similarly, $[2] = \{2, 1\}$ and $[3] = \{3\}$

We observe that any two equivalence classes are either disjoint or identical. The distinct equivalence classes are $[1]$ and $[3]$.

Also $A = [1] \cup [3]$ and $[1] \cap [3] = \phi$

Then $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

is an equivalence relation on A.

Art-16. Suppose that R is an equivalence relation on a set X. Then

(*i*) $a \in [a] \; \forall \, a \in X$.

(*ii*) $a \in [b]$ if and only if $[a] = [b] \; \forall \, a, b \in X$.

(*iii*) $[a] = [b]$ or $[a] \cap [b] = \phi \; \forall \, a, b \in X$ *i.e.*, any two equivalence classes are disjoint or identical.

(Pbi.U., B.C.A. II 2007)

Proof : (*i*) Since R is an equivalence relation on X.

$\therefore$ R is reflexive.

$\therefore$ $a \, R \, a \; \forall \, a \in X$.

$\Rightarrow$ $a \in [a] \; \forall \, a \in X$.

(*ii*) Let $a, b \in X$ such that $a \in [b]$ ...(1)

$\therefore$ $a \, R \, b$

$\Rightarrow$ $b \, R \, a$, since R is equivalence relation ...(2)

Now we show that $[a] = [b]$.

Let $p \in [a]$.

$\therefore$ $p \, R \, a$ ...(3)

From, (1) and (3), $p \, R \, b$, since R is an equivalence relation

$\therefore \quad p \in [\, b \,]$.

So $p \in [\, a \,] \Rightarrow p \in [\, b \,]$. Therefore $[\, a \,] \subseteq [\, b \,]$.

Now let $q \in [\, b \,]$.

$\therefore \quad q \, R \, b$. Also $b \, R \, a$ (From (2).).

$\therefore \quad q \, R \, a$ and hence $q \in [\, a \,]$.

$\therefore \quad [\, b \,] \subseteq [\, a \,]$

Hence $[\, a \,] = [\, b \,]$

Conversely let $[\, a \,] = [\, b \,]$ for some $a, b \in X$.

From (*i*), $a \in [\, a \,]$.

$\therefore \quad a \in [\, b \,]$, since $[\, a \,] = [\, b \,]$.

(*iii*) Let $a, b \in X$.

If $[\, a \,] \cap [\, b \,] = \phi$, then we have nothing to prove.

If $[\, a \,] \cap [\, b \,] \neq \phi$, then there exists $p \in X$ such that $p \in [\, a \,] \cap [\, b \,]$.

$\therefore \quad p \in [\, a \,]$ and $p \in [\, b \,]$

$\Rightarrow \quad [p] = [a]$ and $[p] = [b]$, [From (*ii*)]

$\Rightarrow \quad [a] = [b]$

Art-17. The distinct equivalence classes of an equivalence relation on a set form a partition of that set. (Pbi.U., B.C.A. II, 2006)

Proof : Let R be an equivalence relation on a set X. Therefore R is also a reflexive relation.

$\therefore \quad a \, R \, a \; \forall \; a \in X$.

$\therefore \quad a \in [\, a \,] \; \forall \; a \in X$. ...(1)

where $[a]$ denotes the equivalence class of a.

We prove that $X = \bigcup_{a \in X} [\, a \,]$.

Let $\quad a \in X$.

Then $\quad a \in [\, a \,]$ [From (1)]

$\therefore \quad a \in \bigcup_{a \in X} [\, a \,]$

$\therefore \quad X \subset \bigcup_{a \in X} [\, a \,]$ ...(2)

Since $\quad [\, a \,] = \{\, b \in X : b \, R \, a \,\}$, therefore

$[\, a \,] \subseteq X \; \forall \; a \in X$

$\therefore \quad \bigcup_{a \in X} [\, a \,] \subseteq X$ ...(3)

From (2) and (3), we get $X = \bigcup_{a \in X} [a]$. If we delete the repetitions from this union, we get X as union of distinct equivalence of X under R.

Now we prove that any two distinct equivalence classes are disjoint.

Let $[a]$ and $[b]$ be any two distinct equivalence classes where $a, b \in X$

We want to prove that $[a] \cap [b] = \phi$.

If possible, let $[a] \cap [b] \neq \phi$.

$\therefore$ $\exists\, x \in X$ such that $x \in [a] \cap [b]$.

$\therefore$ $x \in [a]$ and $x \in [b]$.

$\therefore$ $x\,R\,a$ and $x\,R\,b$. ...(4)

$\Rightarrow$ $a\,R\,x$ and $b\,R\,x$. ...(5)

Now we prove that $[a] = [b]$.

Let $p \in [a]$.

$\therefore$ $p\,R\,a$. Also $a\,R\,x$ (From (5))

$\therefore$ $p\,R\,x$. Also $x\,R\,b$ (From (4))

$\therefore$ $p\,R\,b$ (Since R is an equivalence relation)

$\Rightarrow$ $p \in [b]$

$\therefore$ $[a] \subseteq [b]$

Now let $q \in [b]$

$\therefore$ $q\,R\,b$. Also $b\,R\,x$ (From (5))

$\therefore$ $q\,R\,x$. (Since R is equivalence relation)

Also $x\,R\,a$ (From (4))

$\therefore$ $q\,R\,a$

$\Rightarrow$ $q \in [a]$

$\therefore$ $[b] \subseteq [a]$

Hence $[a] = [b]$.

But this is against our supposition that $[a]$ and $[b]$ are distinct equivalence classes. So, our supposition that $[a] \cap [b] \neq \phi$ is wrong.

Thus $[a] \cap [b] = \phi$.

Therefore, X is union of distinct equivalence classes and any two distinct equivalence classes are disjoint.

Hence the set of distinct equivalence classes of R forms a partition of X.

Art-18. For any partition of X, there is an equivalence relation on X whose equivalence classes are the sets in the partition.

Proof. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a partition of X. Therefore we have

(i) $X = \underset{\lambda \in \Lambda}{\cup} A_\lambda$,

(ii) $A_\lambda \cap A_\mu = \phi$ if $\lambda \neq \mu$ where λ and $\mu \in \Lambda$.

For $a, b \in X$, define $a\,R\,b$ if and only if a, b are in the same A_λ.

Then for any $a, b, c, \in \Lambda$, we have :

(i) R is reflexive

Let $a \in X$.

$\therefore$ by (i) $a \in \underset{\lambda \in \Lambda}{\cup} A_\lambda$ so that $a \in A_\lambda$ for some $\lambda \in \Lambda$.

Therefore $a\,R\,a$.

(ii) R is symmetric

Let $a\,R\,b$.

$\therefore$ a and b belong to A_α for same $\alpha \in \Lambda$.

$\Rightarrow$ b and a belong to same A_α.

$\Rightarrow$ $b\,R\,a$.

(iii) Let $a\,R\,b$ and $b\,R\,c$.

$\therefore$ a and b belong to same A_α for some $\alpha \in \Lambda$ and b, c belong to same A_β for some $\beta \in \Lambda$.

$\therefore$ $b \in A_\alpha$ and A_β both *i.e.*, $b \in A_\alpha \cap A_\beta$.

$\Rightarrow$ $\alpha = \beta$, For if $\alpha \neq \beta$, then by (ii) $A_\alpha \cap A_\beta = \phi$.

$\therefore$ a and c belong to same A_α.

$\therefore$ $a\,R\,c$.

Therefore R is reflexive, symmetric as well as transitive relation on X. Hence R is n equivalence relation on X.

Now we prove that each equivalence class of X is equal to A_λ.

Let $[\,a\,]$ denote the equivalence of a for any $a \in X$.

Then $[\,a\,] = \{\,b \in X : b\,R\,a\,\}$.

$= [\,b \in X : b$ and a are in the same A_λ for some $\lambda \in \Lambda\}$.

$= A_\lambda$.

$\therefore$ Equivalence class of X is equal to A_λ.

Conversely we prove that each A_λ is equal to some equivalence of X.

Consider any A_λ

Take any $a \in A_\lambda$

Such an a exist, since A_λ is non-empty

We prove that $A_\lambda = [\,a\,]$

Let $b \in A_\lambda$. Then a and b belong to same A_λ

$\therefore$ $b \, R \, a$ and hence $b \in [a]$

$\therefore$ $b \in A_\lambda \Rightarrow x \in A_\lambda$

$\Rightarrow$ $A_\lambda \subseteq [a]$

Now, let $x \in [a]$

$\therefore$ $x \, R \, a$ and hence x, a belong to same A_a

But $a \in A_\lambda$ $\therefore$ $x \in A_\lambda$

$\therefore$ $x \in [a] \Rightarrow x \in A_\lambda$.

$\Rightarrow$ $[a] \subseteq A_\lambda$

Hence $A_\lambda = [a]$

$\therefore$ $\{A_\lambda\}_{\lambda \in \Lambda}$ is the set of all equivalence classes under the relation R.

ILLUSTRATIVE EXAMPLES

Example 1. How many relations are possible from a set A of m elements to another set B of n elements ? Why ? (P.T.U. B.C.A.-I 2006; Pbi. U. B.C.A. 2012

Sol. Number of elements in $A = m$

Number of elements in $B = n$

$\therefore$ number of elements in $A \times B = m \times n$

We know that number of subsets of any set with n elements is 2^n.

$\therefore$ number of subsets of $A \times B = 2^{mn}$

Since every subset of $A \times B$ is a relation from A to B, therefore 2^{mn} relations are possible from A to B.

Example 2. Let R be the relation defined on the set of natural numbers **N** as

$$R = \{ (x, y) : x \in \mathbf{N},\ y \in \mathbf{N},\ 2x + y = 41 \}.$$

Find the domain and range of this relation R. Also verify whether R is

(*i*) reflexive (*ii*) symmetric (*iii*) transitive (Pbi. U. B.C.A. 2012

Sol. $2x + y = 41 \Rightarrow y = 41 - 2x$

$x = 1 \Rightarrow y = 41 - 2(1) = 41 - 2 = 39$

$x = 2 \Rightarrow y = 41 - 2(2) = 41 - 4 = 37$

$x = 3 \Rightarrow y = 41 - 2(3) = 41 - 6 = 35$

$x = 4 \Rightarrow y = 41 - 2(4) = 41 - 8 = 33$

....

$x = 19 \Rightarrow y = 41 - 2(19) = 41 - 38 = 3$

$x = 20 \Rightarrow y = 41 - 2(20) = 41 - 40 = 1$

$x = 21 \Rightarrow y = 41 - 2(21) = 41 - 42 = -1 \notin \mathbf{N}$

$\therefore$ R = { (1, 39), (2, 37), (3, 35), (4, 33), ..., (20, 1) }

$\therefore$ domain of R = { 1, 2, 3, 4, ..., 20 }

and range of R = { 1, 3, 5, 7, ..., 39 }

(*i*) Now $1 \in N$ but $(1, 1) \notin R$

$\therefore$ R is not reflexive

(*ii*) $(1, 39) \in R$ but $(39, 1) \notin R$

$\therefore$ R is not symmetric

(*iii*) $(20, 1), (1, 39) \in R$ but $(20, 39) \notin R$

$\therefore$ R is not transitive.

Example 3. Give an example of a relation which is anti-symmetric and transitive but neither reflexive nor symmetric.

Sol. Let $A = \{1, 2, 3\}$.

Then $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Let $R = \{(1, 2), (2, 3), (1, 3)\}$.

R is a relation on the set A as $R \subseteq A \times A$.

R is not reflexive since $3 \in A$ and $(3, 3) \notin R$.

R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$.

R is transitive since $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$.

R is anti-symmetric since for no $(a, b) \in R$, We have $(b, a) \in R$.

Example 4. Give an example of relation which is transitive but neither reflexive nor symmetric nor anti-symmetric.

Sol. Let $A = \{1, 2, 3\}$

Then $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

Let $R = \{(1, 1), (2, 2), (1, 2), (2, 1), (1, 3), (2, 3)\}$.

Then R is transitive since $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$.

R is not reflexive since $3 \in A$ but $(3, 3) \notin R$.

R is not symmetric since $(1, 3) \in R$ but $(3, 1) \notin R$.

Also R is not anti-symmetric since $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

Example 5. Give example of relation R on A = {1, 2, 3} which is both symmetric and antisymmetric and R is neither symmetric nor antisymmetric. (B.C.A. II 2007)

Sol. $A = \{1, 2, 3\}$

(*i*) $R = \{(1, 1), (2, 2), (3, 3)\}$ is a relation which is both symmetric and antisymmetric

$\because$ if $(x, y) \in R \Rightarrow (y, x) \in R$ [$\because$ R is symmetric]

also $(x, y) \in R, (y, x) \in R \Rightarrow x = y$ [$\because$ R is antisymmetric]

(*ii*) $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3)\}$

R is not symmetric $\because$ $(2, 3) \in R$ but $(3, 2) \notin R$

R is not antisymmetric $\because$ $(1, 2) \in R, (2, 1) \in R$ but $1 \neq 2$.

Example 6. Give an example of relation which is both an equivalence relation and partial order relation. (P.T.U. B.C.A.-I 2006; Pbi.U., B.C.A.-II, 2008, 2009)

Sol. Let $A = \{1, 2, 3\}$

Define R on A by $R = \{(1, 1), (2, 2), (3, 3)\}$

i.e. $x R y$ iff $x = y$

Now R is reflexive as $1 R 1, 2 R 2, 3 R 3$

R is symmetric as $\forall$ $x R y$, $y R x$

R is antisymmetric as $x R y, y R x \Rightarrow x = y$

R is transitive.

So R is an equivalence as well as partial order relation.

Example 7. Give an example of relation R on $A = \{1, 2, 3\}$ having the stated property :

R is neither symmetric nor antisymmetric, and R is transitive but $R \cup R^{-1}$ is not transitive. (Pbi.U., B.C.A.-II 2007)

Sol. $A = \{1, 2, 3\}$

(*i*) $R = \{(1, 2), (2, 3), (2, 1)\}$

R is not symmetric

$\because$ $(2, 3) \in R$. But $(3, 2) \notin R$.

R is not antisymmetric.

$\because$ $(1, 2) \in R, (2, 1) \in R$ but $1 \neq 2$.

(*ii*) $R = \{(1, 2), (2, 3), (1, 3)\}$

R is transitive as $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

$$R^{-1} = \{(2, 1), (3, 2), (3, 1)\}$$

$$R \cup R^{-1} = \{(1, 2), (2, 3), (1, 3), (2, 1), (3, 2), (3, 1)\}$$

As $(1, 2)$ and $(2, 1) \in R \cup R^{-1}$

But $(1, 1) \notin R \cup R^{-1}$

$\therefore$ $R \cup R^{-1}$ is not transitive.

Example 8. Consider the following five relation on set $A = \{1, 2, 3\}$

$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$

$S = \{(1, 1,), (1, 2), (2, 1), (2, 2), (3, 3)\}$

$T = \{(1, 1) (2, 2) (1, 2) (2, 3)\}$

ϕ = Empty Relation

$A \times A$ = Universal Relation

Determine whether or not each of above relations on A is

(*i*) Reflexive

(*ii*) Symmetric

(*iii*) Transitive

(*iv*) Antisymmetric (Pbi. U., BCA, 2009)

Sol. R = {(1, 1), (1, 2), (1, 3), (3, 3)}

R is not reflexive $\because$ (2, 2) $\notin$ R

R is not Symmetric as (1, 2) $\in$R but (2, 1) $\notin$ R

R is antisymmetric as (a, b) $\in$R, (b, a) $\in$R $\Rightarrow$ a = b

R is Transitive as (a, b) $\in$ R, (b, c) $\in$R $\Rightarrow$ (a, c) $\in$R

S = {(1, 1) (1, 2) (2, 1) (2, 2) (3, 3)}

S is Reflexive, Symmetric and Transitive but not antisymmetric because (1, 2) $\in$ S, (2, 1) $\in$S but 1 $\neq$ 2.

T = {(1, 1), (2, 2), (1, 2), (2, 3)}

T is not Reflexive as (3, 3) $\notin$ T

T is not Symmetric as (1, 2) $\in$ T But (2, 1) $\notin$ T

T is not Transitive as (1, 2) $\in$T, (2, 3) $\in$T but (1, 3) $\notin$ T

T is Antisymmetric

ϕ = Empty Relation

Empty Relation is not reflexive but it is Symmetric, anti symmetric and Transitive.

A × A = Universal Relation

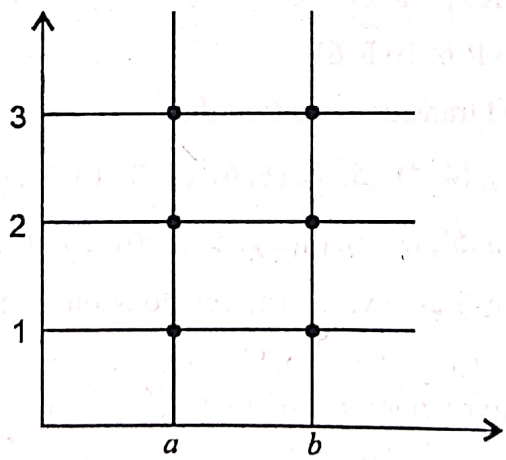
Universal Relation is reflexive, Symmetric, & Transitive but not Antisymmetric.

Example 9. Let A = $\{a, b\}$ and B = {1, 2, 3}. Represent A × B graphically.

What is | A × B | ?

Sol. A × B = $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Graphically A × B is shown below :



$|A \times B| = |A|.|B| = 2.3 = 6$

Example 10. Let X be set with 6 elements. How many relations can be there on A ? How many relations on X are reflexive ?

(Pbi.U., B.C.A. II 2007)

Sol. We know relation on A is subset of A × A

$$n(A \times A) = n(A) . n(A)$$
$$= 6 . 6 = 36$$

Number of subsets of A × A = 2^{36}

So different relations on A = 2^{36}

For relation to b reflexive we know $x R x \ \forall \ x \in A$ so relation must contain at least 6 elements.

The remaining 30 elements may or may not be taken so number of reflexive relations = 2^{30}.

Example 11. Let X = {1,2 ,3 4, 5, 6, 7, 8, 9, 10}. The family $\{\{1, 4, 8\}, \{3, 5, 9\}, \{2, 7\}, \{6, 10\}\}$ is a partition of X. Determine the equivalence relation corresponding to the above partition.

Sol. Here X = {1,2 ,3 4, 5, 6, 7, 8, 9, 10}

Let F = $\{\{1, 4, 8\}, \{3, 5, 9\}, \{2, 7\}, \{6, 10\}\}$ is a partition of X

Let R be the equivalence relation corresponding this partition F of X, then F should constitute the distinct equivalence classes of X under R.

∴ the distinct equivalence classes of X w.r.t. R will be

$\{1, 4, 8\}, \{3, 5, 9\}, \{2, 7\}, \{6, 10\}$

Since each element a belongs to equivalence class of a

$$\therefore \quad [1] = \{1, 4, 8\} = \{1 R 1, 4 R 1, 8 R 1\}$$
$$[3] = \{3, 5, 9\} = \{3 R 3, 5 R 3, 9 R 3\}$$
$$[2] = \{2, 7\} = \{2 R 2, 7 R 2\}$$
$$[6] = \{6, 10\} = \{6 R 6, 10 R 6\}$$

Also R is symmetric and transitive, we find that

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10), (1, 4), (1, 8), (2, 7), (3, 5), (3, 9), (4, 1), (5, 3), (6, 10), (7, 2), (8, 1), (9, 3) (10, 6)\}$$

Example 12. Suppose R and S are symmetric relations on a set A. Show that R ∩ S is also symmetric.

Sol. Since R and S are symmetric relation on a set A

$$\therefore \quad \text{if } (x, y) \in R \Rightarrow (y, x) \in R \quad \ldots(i)$$

$$\text{and if } (x, y) \in S \Rightarrow (y, x) \in S \quad \ldots(ii)$$

where $x, y \in A$.

Let $(x, y) \in R \cap S$ be any element

$\Rightarrow$ $(x, y) \in R$ and $(x, y) \in S$

$\Rightarrow$ $(y, x) \in R$ and $(y, x) \in S$ [$\because$ of (i) and (ii)]

$\Rightarrow$ $(y, x) \in R \cap S$

$\therefore$ $(x, y) \in R \cap S \Rightarrow (y, x) \in R \cap S$

$\therefore$ $R \cap S$ is symmetric.

Example 13. R is a relation on set of positive integers s.t.

$R = \{(a, b) : a - b \text{ is odd integer}\}$

Is R an equivalence relation ?

Sol. $R = \{(a, b) : a - b \text{ is odd integer}\}$

R is equivalence iff R is reflexive, symmetric and transitive.

(i) Reflexive : Let a be any positive integers then $a \, R \, a$

iff $a - a$ is odd integer

i.e. 0 is odd integer

which is not true.

So R is not equivalence as R is not reflexive.

Example 14. Given a set $S = \{1, 2, 3, 4, 5\}$, find the equivalence relation on S which generates partition $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$. Draw diagraph.

(P.T.U. B.C.A. I 2005, 2006)

Sol. Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Clearly R is reflexive, symmetric and transitive.

So R is equivalence.

Now we find equivalence classes generated by R

$$\bar{1} = \{1\}$$

$$\bar{2} = \{2\}$$

$$\bar{3} = \{3\}$$

$$\bar{4} = \{4\}$$

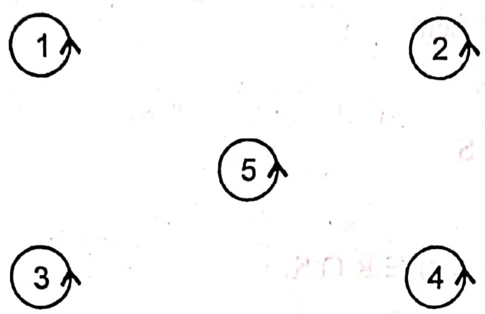
$$\bar{5} = \{5\}$$

$$\bar{1} \cup \bar{2} \cup \bar{3} \cup \bar{4} \cup \bar{5} = S$$

and all equivalence classes are pairwise disjoint.

So R generates partition $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

Diagraph of R



Example 15. Let R be a relation defined on the set of real numbers by a R b if $a \leq b$ where a, b are real number. Then R is a partial order relation.

Sol. (*i*) R is reflexive since $a \leq a$ for any real number a and hence a R a.

(*ii*) R is anti-symmetric. Let a and b be two real numbers such that a R b and b R a.

$\therefore$ $a \leq b$ and $b \leq a$.

$\Rightarrow$ $a = b$

$\Rightarrow$ R is anti-symmetric

(*iii*) R is transitive. Let a, b, c be any real numbers such that a R b and b R c.

$\therefore$ $a \leq b$ and $b \leq c$.

$\Rightarrow$ $a \leq c$ *i.e.*, a R c.

$\Rightarrow$ R is transitive.

$\therefore$ R is reflexive, anti-symmetric and transitive.

$\therefore$ R is a partial order relation on the set of real numbers.

Example 16. For any a, $b \in$ N, the set of natural numbers, define a R b if and only if a divides b. Then R is a partial order relation.

Or

Prove that relation of divisibility is partial order relation on N.

(Pbi.U. B.C.A. April 2011)

Sol. Let $a, b, c \in$ N. Then

(*i*) a R a since $a = 1.\, a$ implies that a divides a. Therefore R is reflexive

(*ii*) Let a R b and b R a

$\therefore$ a divides b and b divides a

$\therefore$ $a = b$

$\therefore$ R is anti-symmetric

(*iii*) Let a R b and b R c

$\therefore$ a divides b and b divides c

$\Rightarrow$ a divides c

$\Rightarrow$ a R c

$\Rightarrow$ R is transitive.

$\therefore$ R is reflexive, anti-symmetric and transitive,

$\therefore$ R is a partial order relation on **N**

Example 17. Prove that relation of divisibility is not a partial order relation on set of integers.

(Pbi. U. B.C.A.-II April 2011)

Sol. Let $a \in Z$ (integers)

then a divides a is true.

$\therefore$ relation is reflexive.

Let $a, b \in Z$

then a divides b and b divides a does not mean that $a = b$

e.g. $a = 5,\ b = -5$

Here a divides b and b divides a.

But $a \neq b$

So relation is not anti-symmetric and hence not partial order relation.

Example 18. If R is the relation in **N** $\times$ **N** defined by (a, b) R (c, d) if and only if $a + d = b + c$, show that R is equivalence relation.

Sol. Here (a, b) R (c, d) $\Leftrightarrow$ $a + d = b + c$.

(*i*) Now (a, b) R (a, b) if $a + b = b + a$, which is true.

$\therefore$ relation R is reflexive.

(*ii*) Now (a, b) R (c, d)

$\Rightarrow$ $a + d = b + c$ $\Rightarrow$ $d + a = c + b$

$\Rightarrow$ $c + b = d + a$ $\Rightarrow$ (c, d) R (a, b)

$\therefore$ relation R is symmetric.

(*iii*) Now (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ $a + d = b + c$ and $c + f = d + e$

$\Rightarrow$ $(a + d) + (c + f) = (b + c) + (d + e)$ $\Rightarrow$ $a + f = b + e$

$\Rightarrow$ (a, b) R (e, f)

$\therefore$ relation R is transitive.

Now R is reflexive, symmetric and transitive

$\therefore$ relation R is an equivalence relation.

Example 19. In **N** $\times$ **N**, show that the relation defined by (a, b) R (c, d) if $a\,d = b\,c$ is an equivalence relation.

Sol. Here (a, b) R (c, d) $\Leftrightarrow$ $a\,d = b\,c$ (*i*) Now (a, b) R (a, b) if $a\,b = b\,a$, which is true

$\therefore$ relation R is reflexive.

(*ii*) Now (a, b) R (c, d)

$\Rightarrow \quad a\,d = b\,c \quad \Rightarrow \quad d\,a = c\,b \quad \Rightarrow \quad c\,b = d\,a \quad \Rightarrow \quad (c, d)\text{ R }(a, b)$

$\therefore$ relation R is symmetric.

(*iii*) Now (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow \quad a\,d = b\,c$ and $c f = d\,e \Rightarrow (a\,d)\,(c f) = (b\,c)\,(d\,e)$

$\Rightarrow \quad a\,d\,c f = b\,c\,d\,e \Rightarrow (a f)\,(d\,c) = (b\,e)\,(d\,c)$

$\Rightarrow \quad a f = b\,e \Rightarrow (a, b)\text{ R }(e, f)$

$\therefore$ relation R is transitive

Now R is reflexive, symmetric and transitive

$\therefore$ relation R is an equivalence relation.

Example 20. For $\dfrac{a}{b}, \dfrac{c}{d} \in Q$ – the set of relational numbers, define $\dfrac{a}{b}\text{ R }\dfrac{c}{d}$ if and only if $a\,d = b\,c$. Show that R is an equivalence relation on Q.

Sol. Let $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f} \in Q$ arbitrarily, then

(*i*) Since $\dfrac{a}{b} \in Q$, therefore, a, b are integers.

$\therefore \quad a\,b = b\,a$, since multiplication is commutative in **Z**.

$\therefore \quad \dfrac{a}{b}\text{ R }\dfrac{a}{b}$

$\therefore$ R is reflexive.

(*ii*) Let $\dfrac{a}{b}\text{ R }\dfrac{c}{d}$

$\therefore \quad a\,d = b\,c$

$\Rightarrow \quad d\,a = c\,b$

$\Rightarrow \quad c\,b = d\,a$

$\Rightarrow \quad \dfrac{c}{d}\text{ R }\dfrac{a}{b}$

$\Rightarrow$ R is symmetric.

(*iii*) Let $\dfrac{a}{b}\text{ R }\dfrac{c}{d}$ and $\dfrac{c}{d}\text{ R }\dfrac{e}{f}$

$\therefore \quad a\,d = b\,c$ and $c f = de$

$\Rightarrow \quad (a\,d)\,(c f) = (b\,c)\,(d\,e)$

$\Rightarrow \quad (c\,d)\,(a f) = (c\,d)\,(b\,e)$,

by using commutative and associative laws of multiplication in **Z**.

$\Rightarrow \quad af = be$

$\Rightarrow \quad \frac{a}{b} \, R \, \frac{e}{f} \quad \Rightarrow$ R is transitive.

Thus R is an equivalence relation.

xample 21. Define the relation and an equivalence relation. The relation $R \subseteq N \times N$ is efined by $(a, b) \in R$ if and only if 5 divides $b - a$. Show that R is an equivalence lation. (Pbi. U. B.C.A. 2012)

ol. Definition of relation and equivalence relation have already been given

The relation $R \subseteq N \times N$ is defined by $(a, b) \in R$ if and only if 5 divides $b - a$.

This means that R is a relation on N defined by , if $a, b \in N$ then $(a, b) \in R$ if and nly if 5 divides $b - a$.

Let a, b, c belongs to N. Then

(*i*) $\quad a - a = 0 = 5 . 0.$

$\therefore \quad$ 5 divides $a - a$.

$\Rightarrow \quad (a, a) \in R. \quad \Rightarrow \quad$ R is reflexive.

(*ii*) $\quad$ Let $(a, b) \in R$.

$\therefore \quad$ 5 divides $a - b$.

$\Rightarrow \quad a - b = 5\,n$ for some $n \in N$.

$\Rightarrow \quad b - a = 5\,(-n)$.

$\Rightarrow \quad$ 5 divides $b - a \quad \Rightarrow \quad (b, a) \in R$.

$\therefore$ R is symmetric.

(*iii*) Let (a, b) and $(b, c) \in R$.

$\therefore \quad$ 5 divides $a - b$ and $b - c$ both

$\therefore \quad a - b = 5\,n_1$ and $b - c = 5\,n_2$ for some n_1 and $n_2 \in N$

$\therefore \quad (a - b) + (b - c) = 5\,n_1 + 5\,n_2$

$\Rightarrow \quad a - c = 5\,(n_1 + n_2)$

$\Rightarrow \quad$ 5 divides $a - c$

$\Rightarrow \quad (a, c) \in R$

$\therefore \quad$ R is transitive relation in N.

Example 22. If R is an equivalence relation on a set A, then so is R^{-1}

Sol. Let $a, b, c \in A$. Then

(*i*) $(a, a) \in R$, since R being equivalence relation is also a reflexive relation

$\Rightarrow \quad (a, a) \in R^{-1}$ [By definition of R^{-1}]

$\Rightarrow \quad R^{-1}$ is reflexive

(*ii*) Let $(a, b) \in R^{-1}$

$\therefore \quad (b, a) \in R$

$\Rightarrow \quad (a, b) \in R$; since R is symmetric

$\Rightarrow \quad (b, a) \in R^{-1}$.

$\therefore \quad (a, b) \in R^{-1} \Rightarrow (b, a) \in R^{-1}$.

So, R^{-1} is also symmetric

(*iii*) Let (a, b) and $(b, c) \in R^{-1}$

$\therefore \quad (b, a), (c, b) \in R$.

$\Rightarrow \quad (c, b), (b, a) \in R$.

$\Rightarrow \quad (c, a) \in R$, since R is transitive.

$\Rightarrow \quad (a, c) \in R^{-1}$

$\therefore \quad R^{-1}$ is also transitive.

$\therefore \quad R^{-1}$ is an equivalence relation.

Example 23. For any relation R in a set A, we can define the inverse relation R^{-1} by $a R^{-1} b$ iff $b R a$. Prove that R is symmetric iff $R^{-1} = R$.

Sol. Let $R^{-1} = \{(b, a) : a, b \in A \text{ s.t. } (a, b) \in R\}$

Firstly, let R be symmetric relation

Let $(a, b) \in R \Leftrightarrow (b, a) \in R$

$\Leftrightarrow (a, b) \in R^{-1}$

$\Rightarrow \quad R = R^{-1}$

Secondly let $R^{-1} = R$, To show that R is symmetric

Let $(a, b) \in R \Leftrightarrow (a, b) \in R^{-1}$ $\quad (\because R = R^{-1})$

$\Leftrightarrow (b, a) \in R$ $\quad$ (by def.)

$\therefore$ R is symmetric.

Example 24. For $a, b \in \mathbf{Z}$, the set of integers, define $a \equiv b \pmod 2$ iff 2 divides $a - b$. Then $\equiv$ is an equivalence relation on **Z**. Find the equivalence classes of 0 and 1. Further show that the equivalence class of any integer is equal to equivalence class of 0 or 1.

Sol. Let $a \in \mathbf{Z}$ arbitrarily, then 2 divides $a - a = 0$.

$\therefore \quad a \equiv a \pmod 2$.

$\therefore \quad \equiv$ is a reflexive relation

Let $a, b \in \mathbf{Z}$ such that $a \equiv b \pmod 2$.

Then $2 \mid a - b$ and hence $2 \mid b - a$.

Therefore, $\quad b \equiv a \pmod 2$.

$\therefore \quad \equiv$ is a symmetric relation

Let $a, b, c \in \mathbf{Z}$ such that $a \equiv b \pmod 2$ and $b \equiv c \pmod 2$.

$\therefore \quad 2 \mid a - b$ and $2 \mid b - c$.

This implies 2 divides $(a-b)+(b-c)=a-c$.

$\therefore \quad a \equiv c \pmod 2$.

$\therefore \quad \equiv$ is a transitive relation.

So $\equiv$ is reflexive, symmetric as well as transitive relation on **Z**.

Hence $\quad \equiv$ is an equivalence relation on **Z**.

Let $[n]$ denotes the equivalence class of integer n.

Now $[0] = \{n \in \mathbf{Z} : n \equiv 0 \pmod 2\}$

$= \{n \in \mathbf{Z} : 2 \mid n - 0 = n\}$

$= \{n \in \mathbf{Z} : n = 2k \text{ where } k \in \mathbf{Z}\}$

$= \{0, \pm 2, \pm 4, \pm 6, \ldots\ldots\}$

$[1] = \{n \in \mathbf{Z} : n \equiv 1 \pmod 2\}$

$= \{n \in \mathbf{Z} : 2 \mid n-1\}$

$= \{n \in \mathbf{Z} : n - 1 = 2k \text{ where } k \in \mathbf{Z}\}$

$= \{n \in \mathbf{Z} : n = 1 + 2k \text{ where } k \in \mathbf{Z}\}$

$= \{\pm 1, \pm 3, \pm 5, \ldots\ldots\}$

We observe that $[0] \cap [1] = \phi$ *i.e.*, $[0]$ and $[1]$ are disjoint

Also $\quad \mathbf{Z} = [0] \cup [1]$

Now let $n \in \mathbf{Z}$ arbitrarily Then $n = 2k$ or $n = 2k + 1$ for some integer n according as n is divisible by 2 or not

Now $[2k] = \{m \in \mathbf{Z} : m \equiv 2k \pmod 2\}$

$= \{m \in \mathbf{Z} : 2 \mid (m - 2k)\}$

$= \{m \in \mathbf{Z} : m - 2k = 2l \text{ where } l \in \mathbf{Z}\}$

$= \{m \in \mathbf{Z} : m = 2k + 2\ell \text{ where } k, \ell \in \mathbf{Z}\}$

$= \{m \in \mathbf{Z} : m \text{ is an even integer}\}$

$= \{0, \pm 2, \pm 4, \ldots\ldots\} = [0]$

$\therefore \quad [2k] = [0]$

Next $\quad [2k+1] = \{n \in \mathbf{Z} : n \equiv 2k + 1 \pmod 2\}$.

$= \{n \in \mathbf{Z} : 2 \mid n - (2k+1)\}$.

$= \{n \in \mathbf{Z} : n - (2k+1) = 2l \text{ for some } \ell \in \mathbf{Z}\}$.

$= \{n \in \mathbf{Z} : n = 2\ell + (2k+1)\}$.

$= \{n \in \mathbf{Z} : n = 2(\ell + k) + 1\}$.

$= \{n \in \mathbf{Z} : n \text{ is an odd integer}\}$.

$= \{\pm 1, \pm 3, \pm 5, \ldots\ldots\}$.

$= [1]$

$\therefore \quad [2k+1] = [1]$

Since $\quad n = 2k$ or $2k+1$ and $[2k] = [0]$, $[2k+1] = [1]$

$\therefore \quad [n] = [0]$ or $[1]$

Example 25. Find x and y if $(x+2, 4) = (5, 2x+y)$. (Pbi.U., B.C.A. II 2007)

Sol. $(x+2, 4) = (5, 2x+y)$

Comparing $x + 2 = 5$,

$\Rightarrow \quad x = 5 - 2, \quad \Rightarrow \quad x = 3$

$2x + y = 4$

$\Rightarrow \quad 2(3) + y = 4 \quad \Rightarrow \quad y = 4 - 6$

$\Rightarrow \quad y = -2$

Example 26. Let R be a relation on the set of all positive integers such that R = $\{(a, b) \mid a - b$ is an odd positive integer$\}$. Is R reflexive, symmetric, antisymmetric, transitive, an equivalence relation ? (Pbi. U. M.Sc. 2011)

Sol. R = $\{(a, b) \mid a - b$ is an odd positive integer$\}$

Let Z^+ denote set of all positive integers.

Reflexive : Let $a \in Z^+$

then $a - a = 0$, which is not odd positive integer.

So R is not reflexive.

Symmetric : Let $a, b \in Z^+$ such that $a \, R \, b$

$\Rightarrow \quad a - b$ is odd positive integer.

then $b - a$ is negative integer.

$\therefore \quad$ R is not symmetric.

Antisymmetric : Let $a, b \in Z^+$

Now if $a - b$ is odd positive integer

then $b - a$ is definitely not odd positive integer.

So R is antisymmetric.

Equivalence : Since R is not reflexive,

So it is not an equivalence relation.

EXERCISE 4 (a)

1. If R and R′ are reflexive relations on a set then so are R ∪ R′ and R ∩ R′.
2. Show that the union of two symmetric relations on a set is again a symmetric relation on that set.

3. (i) Give an example of a relation which is reflexive but neither symmetric nor transitive.

(ii) Give an example of a relation which is symmetric but neither reflexive nor transitive.

(iii) Given an example of a relation which is reflexive and symmetric but not transitive.

(iv) Give an example of a relation which is reflexive and transitive but not symmetric.

(v) Give an example of a relation which is reflexive and anti-symmetric but not transitive.

(vi) Give an example of a relation which is symmetric and transitive but not reflexive.

(vii) Give an example of a relation which is reflexive and anti-symmetric but neither symmetric nor transitive.

(viii) Give an example of a relation which is neither reflexive, nor symmetric, nor transitive nor anti-symmetric.

(ix) Give an example of a relation which is reflexive, symmetric, transitive and anti-symmetric.

4. Show that the relation '$\sim$' in the set of 2×2 invertible matrices with real entries given by $A \sim B$ iff $B = A^{-1}$ is symmetric but not reflexive. Is it transitive ?

5. Let R be the relation on N defined by ${}^xR_y$ if x and y share a common factor other than 1. Determine the reflexivity and transitivity of R.

6. For two lines ℓ_1, ℓ_2 in a plane σ, the relation R defined by $\ell_1 \, R \, \ell_2$ if and only if ℓ_1 is perpendicular to ℓ_2 is neither reflexive nor transitive but symmetric. Hence R is not an equivalence relation.

7. Let R be symmetric and transitive relation on a set A. If for each $x \in A$, $\exists \; y \in A$ such that $(x, y) \in R$. Then R is an equivalence relation.

8. Prove that the intersection of two equivalence relations on a non-empty set is again an equivalence relation on that set.

OR

If R and S are equivalence relations in the set X, prove that $R \cap S$ is an equivalence relation.

(Pbi.U., B.C.A.-II 2008)

9. Show that $R_1 \cup R_2$ may not be an equivalence relation on a set X if R_1, R_2 are equivalence relations on X.

Or

Through example show that union of two equivalence relations on a set is not necessarily an equivalence relation.

(Pbi. U. M.Sc. Dec. 2010, 2011).

10. If R and S are two equivalence relation then check R $\cup$ S for

(*i*) Reflexivity (*ii*) Transitivity and (*iii*) Symmetry

(Pbi.U., Sept. 2008)

11. Is inclusion of a subset in another, in the context of a universal set, an equivalence relation in the class of subsets of the sets ? Justify your answer.

(Pbi.U., B.C.A.-II 2007)

12. In the set N of all natural numbers. Let relation R be defined by

R = $\{(x, y)/x \in N, y \in N : x - y$ is divisible by $m\}$

Show that R is an equivalence relation.

13. For $a, b \in \mathbf{R}$ the set of real numbers, defined a S b if $|a| = |b|$ then S is an equivalence relation on **R.**

14. Prove that mod m relation is an equivalence relation.

(Pbi.U., B.C.A.-II 2007)

15. Prove that the following defines an equivalence relation on the $x\,y$-plane :

(x, y) R (s, t) if $x - s$ and $y - t$ are both integers.

16. For any relation R in a set A, we can define the inverse relation R^{-1} by a R^{-1} b if and only if b R a. Prove that

(*i*) As a subset of A $\times$ A, $R^{-1} = \{(b, a) : (a, b) \in R\}$

(*ii*) R is symmetric if and only if $R = R^{-1}$.

17. If R is an equivalence relation on a set A, and $x, y \in$ A, then prove that

(*i*) $x \in [x]$

(*ii*) x R y if and only if $[x] = [y]$ and

(*iii*) $[x] = [y]$ or $[x] \cap [y] = \phi$.

(Pbi.U., M.Sc.-I.T. Dec. 2009)

18. In power set P (X) of X, show that the relation '$\subseteq$' is a partial order. Is it an equivalence relation ?

19. Let A = $\{1, 2, 3, 4\}$ and P $\{\{1, 2, 3\}, \{4\}\}$ be a partition of A. Find equivalence relation determined by P.

(Pbi.U., B.C.A-II, 2006)

20. Given a set S = $\{1, 2, 3, 4, 5\}$, find the equivalence relation on S which generates the partition $\{\{1, 2\}, \{3\}, \{4, 5\}\}$.

(Pbi.U., B.C.A.-II, 2006)

21. Consider the following relation on $\{1, 2, 3, 4, 5, 6\}$ $r = \{(i, j) : |i - j| = 2\}$.

(*a*) Is r reflexive ? (*b*) Is r symmetric ?

(c) Is r transitive ? (*d*) Is r anti-symmetric ?

22. Let r_1, r_2, r_3 be relations on any set. Prove that if $r_1 \subseteq r_2$ then $r_1 \cdot r_3 \subseteq r_2 \cdot r_3$.

ANSWERS

4. No 5. Neither reflexive nor transitive

10. No

19. R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)}

20. R = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (4, 5), (5, 4)}

21. (*a*) No (*b*) Yes (*c*) No (*d*) No.

Art-19. Adjacency Matrix (Matrix of a Relation)

Let A and B be two finite sets and R be a relation from A to B. Then R can be represented by a matrix M_R called the adjacency matrix or matrix of relation R or Boolean matrix of R.

If $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots b_n)$ be two finite sets of order m and n and R be a relation from A to B. Then the adjacency matrix M_R is defined as

$$M_R = (r_{ij})_{m \times n}, \text{ where } \quad r_{ij} = \begin{cases} 1 \text{ if } a_i R b_j \\ 0 \text{ if } a_i \not R b_j \end{cases}$$

If B = A, then M_R is a square matrix called the matrix of relation R in the set A.

Art-20. Boolean Arithmetic

Boolean arithmetic is the arithmetic defined on the set {0, 1} using the rule of boolean algebra *i.e.*,

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

Multiplication of Boolean Matrices

If $A = (a_{ij})_{m \times n}$ and $B = (b_{jk})_{n \times p}$ be two Boolean matrices where each a_{ij}, b_{jk} are either 0 or 1. Then AB is the product of matrices A and B defined as

$$AB = (C_{ik})_{m \times p} \text{ where}$$

$$c_{ik} = \sum_{j=1}^{n} (a_{ij} . b_{jk}) \text{ for } 1 \le i \le m, 1 \le k \le p.$$

Art-21. Composition of Relation Using Matrices

Let R and S are two relations on set A and let associated adjacency matrices are M_r and M_S respectively. Then matrix of composition M_{SoR} is given by

$$M_{SOR} = M_R \cdot M_S$$

Note : To find $R^2, R^3, \ldots$, again we use adjacency matrix.

$$M_{R^2} = M_R \cdot M_R$$

$$M_{R^3} = M_{R^2} \cdot M_R \text{ and so on.}$$

Art-22. (Another proof of art-13)

Let P is a relation from A to B, Q is a relation from B to C and R is relation from to D then prove that Ro (QoP) = (RoQ)oP .

Proof : Let M_P, M_Q, M_R be the matrices associated to relations P, Q and R respective

then $\quad M_{Ro(QoP)} = M_{QoP} \cdot M_R$

$$= (M_P . M_Q) . M_R$$

$$= M_P \cdot (M_Q \cdot M_R) \text{ [Matrix multiplication is associative]}$$

$$= M_P \cdot M_{RoQ}$$

$$= M_{(RoQ)oP}$$

$\Rightarrow \quad$ Ro(QoP) = (RoQ)oP.

ILLUSTRATIVE EXAMPLES

Example 1. If R is relation "less than" from A = {1, 2, 3, 4, 5} to B = {1, 4, 5}. Wri down the adjacency matrix.

Sol. Here $R = \{(x, y) : x \in A, y \in B, \text{ s.t. } x < y\}$. Then

R = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

$\therefore$ the adjacency matrix, M_R is given by

$$M_R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \end{bmatrix}$$

$$\text{where } r_{ij} = \begin{cases} 1 \text{ if } (x_i, y_j) \in R \\ 0 \text{ if } (x_i, y_j) \notin R \end{cases}$$

We can also write matrix of relation by using the following table

R	1	4	5
1	0	1	1
2	0	1	1
3	0	1	1
4	0	0	1
5	0	0	0

$$\therefore \quad M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{5\times 3}$$

Example 2. Let A = {2, 3 5, 8}, B = {4, 6, 16}, C = {1, 4, 5, 7}. Let $R = \{(a, b) : a/b\}$ and $S = \{(b, c) : b \leq c\}$ be relations from A to B and B to C. Find the composite relation SoR. If M_{SoR}, M_R, M_S be the adjacency matrix of SoR, R and S respectively. Then show that $M_{SoR} = M_R M_S$.

Sol. Here $R = \{(a, b : a/b\} = \{(2, 4), (2, 6), (2, 16), (3, 6), (8, 16)\}$

and $S = \{(b, c) : b \leq c\} = \{(4, 4), (4, 5), (4, 7), (6, 7)\}$

$$\therefore \quad SoR = \{(a, c) : \exists\ b \in B \text{ s.t. } (a, b) \in R, (b, c) \in S\}$$
$$= \{(2, 4), (2, 5), (2, 7), (3, 7)\}$$

$$\text{Also } M_{SoR} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{4\times 3}$$

$$M_S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$

$$\text{Now } M_R M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_{SoR}$$

Example 3. Let A = {1, 2, 3}. Determine whether the relation R whose matrices M_R is given is an equivalence relation.

$$(a)\ M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (b)\ M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Sol. (*a*) Since $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\therefore$ R = {(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)}

(*i*) Since $(a, a) \in R \ \forall \ a \in A$.

$\therefore$ R is reflexive.

(*ii*) Since $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b \in A$

$\therefore$ R is Symmetric

(*iii*) Since $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

$\therefore$ R is transitive

Hence R is an equivalence relation.

(*b*) Since $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\therefore$ R = {(1, 1), (1, 3), (2, 2), (3, 1)}

Since $(3, 3) \notin R$ $\therefore$ R is not reflexive

Hence R is not an equivalence relation on A.

Example 4. Given A = {1, 2, 3, 4} and B = {x, y, z}. Let R be the relation from A to B R = {(1, y), (1, z), (3, y), (4, x), (4, z)}

(*a*) Determine matrix of relation.

(*b*) Determine domain and range of relation. (Pbi.U., B.C.A.-II 2007)

Sol. (*a*) Matrix of relation

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0
4	1	0	1

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(*b*) Domain of R = {1, 3, 4}

Range of R = {x, y, z}.

Example 5. Find the relation R if $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(Pbi.U., B.C.A.-II April 2006)

Sol. Let given set A = {a, b, c} then from M_R

$$R = \{(a, a), (b, b), (b, c), (c, a), (c, b)\}$$

Example 6. Determine whether the relation represented by zero-one matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$ is an Equivalence relation

(Pbi. U. B.C.A.-II April 2010)

Sol. Let A = {1, 2, 3, 4}

	1	2	3	4
1	1	0	1	0
2	0	1	0	1
3	1	0	1	0
4	0	1	0	1

Then given relation is

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (2, 4), (3, 1), (4, 2)\}$$

R is reflexive as $(x, x) \in R, \forall x \in A$

R is symmetric as $(x, y) \in R \Rightarrow (y, x) \in R$

R is transitive as $(x, y) \in R \Rightarrow (y, z) \in R \Rightarrow (x, z) \in R$

so R is an equivalence relation.

EXERCISE 4 (*b*)

1. Let A = Set of all positive divisors of 6 *i.e.*, A = {1, 2, 3, 6}. Define the relation R by $a R b$ iff a/b. Find adjacency matrix on A.
2. Let A = {2, 3, 5}, B = {4, 6, 16}, and $R_1 = \{(a, b) : a/b\}$, $R_2 = \{(a, b) : a < b\}$ be two relation from A to B. Find the product of the adjacency matrices M_{R_1} and M_{R_2}.
3. Let A = {a, b, c, d} and R be a relation defined on A by

$$R = \{(a, b), (b, d), (a, d), (d, a), (d, b), (b, a), (c, c)\}$$

Find the matrix that represent the above relation.

4. If M_R and M_S be adjacency matrices of the relation R and S defined on the A = {1, 2, 3, 4}, where

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find SoR.

5. If A = {1, 2,3} and R and S are relation defined on A whose matrices are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \text{ find } M_{SoR} \text{ and } SoR$$

6. If A = {2, 5, 6} and R = {(2, 2), (2, 5), (5, 6), (6, 6)} be a relation on A. Find and R^3.

ANSWERS

1. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ 4. {(1, 1), (1, 3), (1, 4), (3, 3)}

5. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$; {(1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)}

Art-23. Closures

Let R be a relation in a set A. R may not satisfy particular property like reflexivity symmetry or transitivity. The new relation, obtained after adding least number of new pairs so that R satisfies particular property, is called closure of R.

Reflexive Closure. Let R be a relation on A. A reflexive closure of R is the smallest reflexive relation that contains R.

Symmetric Closure. Let R be a relation on A which is not symmetric.

$\therefore$ there exists $(a, b) \in R$ but $(b, a) \notin R$

Now $(b, a) \in R^{-1}$

$\therefore$ to make R symmetric, we add all pairs of R^{-1}.

$\therefore$ $R \cup R^{-1}$ is symmetric closure of R.

Def. If R is a relation on A which is not symmetric. Then $R \cup R^{-1}$ is symmetric closure of R.

Transitive Closure : Let A be a set and R be a relation on A. The transitive closure of R, denoted by R^+, is the smallest relation which contains R as a subset and which is transitive.

Another Definition : Let A be a set and R be a relation on A.

The relation $R^+ = R \cup R^2 \cup R^3$ ......in A is called the transitive closure of R in A.

Methods to find Transitive Closure

Method I. Write ordered pairs (a, b) if vertex a joins b in directed graph of given relation R. Set of all such pairs will be transitive closure R^+ of R.

Method II. Let R be a given relation on a set A such that $O(A) = n$. Let M, M^+ be matrices of relation R, transitive closure R^+ respectively. Then

$$M^+ = M + M^2 + M^3 + \ldots\ldots + M^n.$$

ILLUSTRATIVE EXAMPLES

Example 1. Let R be a relation on a set A = {1, 2, 3} defined by R = {(1, 1), (1, 2), (2, 3)}. Find the reflexive closure of R and symmetric closure of R.

Sol. A = {1, 2, 3}

R = {(1, 1), (1, 2), (2, 3)}

R^{-1} = {(1, 1), (2, 1), (3, 2)}

$R \cup R^{-1}$ = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)}

Reflexive closure of R is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}

Symmetric closure of R is $R \cup R^{-1}$ = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)}

Example 2. Let R be a relation on set A = {1, 2, 3, 4} defined by

R = {(1, 2), (2, 3), (3, 4), (2, 1)}.

Find transitive closure of R.

Sol. A = {1, 2, 3, 4}

R = {(1, 2), (2, 3), (3, 4), (2, 1)}

$$\therefore \quad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{where M is matrix of R}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^3 = M^2 M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad M^+ = M + M^2 + M^3 + M^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $R^+ = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$
which transitive closure of R.

EXERCISE 4 (c)

1. Let R be relation on a set A = {1, 2, 3, 4} defined by
   R = {(1, 2), (2, 3), (1, 3), (3, 4)}.
   Find symmetric closure.
2. Let A = {1, 2, 3, 4} and relation on it R = $\{(a, b): |a - b| = 2\}$. Find transitive closure of R.

ANSWERS

1. {(1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1), (3, 4), (4, 3)}
2. {(1, 1), (1, 3), (2, 2),(2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}

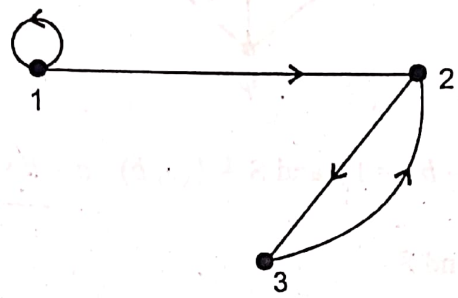
Art-24. Hasse Diagram or Ordering Diagram

The Hasse diagram of a relation R defined on a set X is a directed graph whose vertices are the elements of X and there is a undirected edge from a to b when ever $(a, b) \in R$ (Instead of drawing on arrow from a to b, we some times place b higher than a and draw a line between them). An arrow from a vertex to it self is not drawn when over $(a, a) \in R$.

Art-25. Diagraphs

Relation can be represented pictorially with the help of a graph. Let R be any relation on set A = $\{x_1, x_2, ..., x_n\}$. The elements of A are represented by points (or circles) which are called nodes or vertices. If $(x_i, x_j) \in R$ then we connect vertex x_i with vertex x_j with the help of a directed edge starting from x_i and ending at x_j. For $(x_i, x_i) \in R$ we create a self loop starting and ending at same vertex x_i.

Example : Let A = {1, 2, 3} and R = {(1, 1), (1, 2), (2, 3), (3, 2)}. The diagraph of R is shown below



Art-26. Paths in Relation and Digraph

In a diagraph of the relation R, a path is a succession of edges , when the directed direction of the edges are followed. The number of edges in the path is called the length of the path.

In other words, a path in relation R is of length n if there is a finite sequence

P : $x_i, x_1, x_2, \ldots\ldots, x_{n-1}, x_j$ beginning with x_i and ending with x_j such that

$(x_i, x_1) \in R, (x_1, x_2) \in R, \ldots\ldots (x_{n-1}, x_j) \in R$

When n is positive integer, the relation R^n on the set X is defined as

(*i*) $(x_1, x_j) \in R^n$ implies that there is a path in R from x_i to x_j of length n.

(*ii*) $(x_i, x_j) \in R^{\infty}$ implies that there is some path in R from x_i to x_j.

Cycle : A path of length n (≥ 1) from a vertex to itself is called a circle.

ILLUSTRATIVE EXAMPLES

Example 1. Draw the Hasse diagram of the relation $\subseteq$ on P(A), where A = $\{a, b, c\}$.

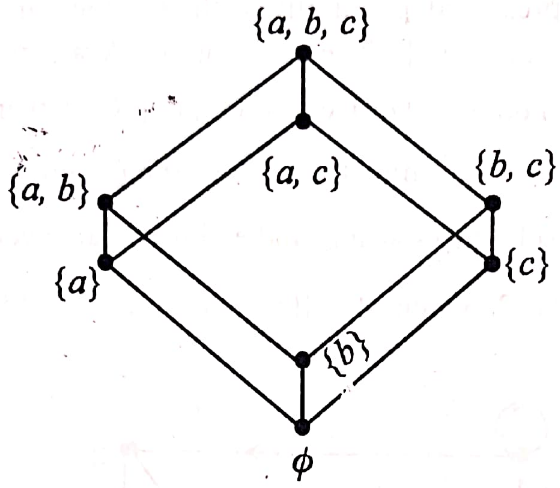
(P.T.U. B.C.A. I 2007)

Sol. Here A = $\{a, b, c\}$

$$\therefore \quad P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$R = \{(X, Y) : X, Y \in P(A) \text{ s.t. } X \subseteq Y\}$$

The Hasse diagram of R is as shown below :



Example 2. $R = \{(a, b) : |a - b| = 1\}$ and $S = \{(a, b) : a - b \text{ is even}\}$ are two relations on $A = \{1, 2, 3, 4\}$. Then

(*i*) Find matrices of R and S

(*ii*) Draw diagraphs of R and S

(*iii*) Using matrices of R and S, find the relation RS

(*iv*) Show that $R^2 = S^2$

Sol. $A = \{1, 2, 3, 4\}$

$R = \{(a, b) : |a - b| = 1\}$

$= \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3)\}$

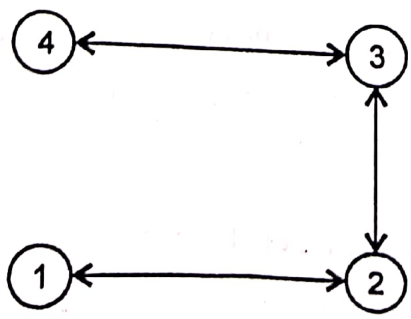
$$\therefore \quad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ is matrix of R}$$

$S = \{(a, b) : a - b \text{ is even}\}$

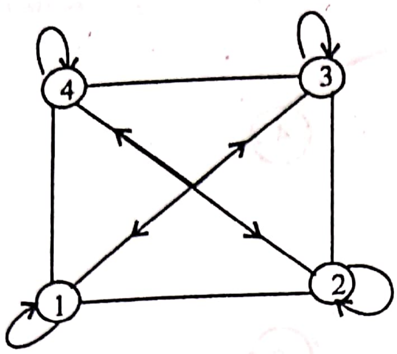
$= \{(1, 3), (3, 1), (2, 4), (4, 2), (1, 1), (2, 2), (3, 3), (4, 4)\}$

$$\therefore \quad N = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ is matrix of S}$$

(*ii*)



Diagraph of R



Diagraph of S

$$(iii)\quad MN = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

which is matrix of RS.

$\therefore \quad RS = \{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1), (4, 3)\}$

$$(iv)\quad M^2 = MM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$\therefore \quad R^2 = \{(1, 1), (1,3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 1), (4, 4)\}$

$$N^2 = NN = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$\therefore \quad S^2 = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$

$\therefore \quad R^2 = S^2.$

Example 3. Let $A = \{0, 1, 2, 3, 4\}$ find equivalence classes of equivalence relation $R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$.

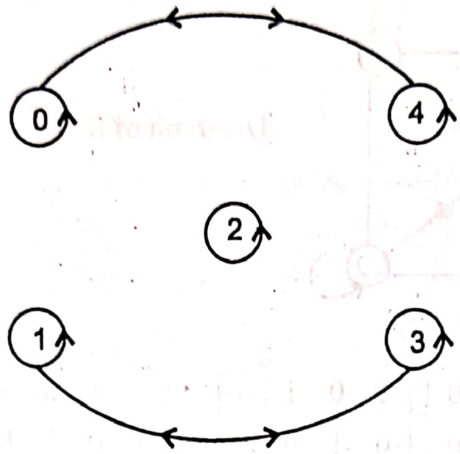
Draw diagraph of R and write down partition of A induced by R.

(Pbi.U., B.C.A.II, 2008)

Sol. Equivalence classes are given below

$$[0] = \{0, 4\}$$
$$[1] = \{1, 3\}$$
$$[2] = \{2\}$$
$$[3] = \{1, 3\}$$
$$[4] = \{0, 4\}$$

Diagraph of R is shown below :

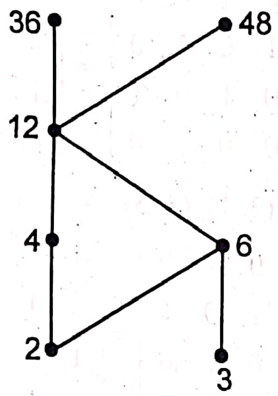


Partition of A induced by R is

$$P = \{\{0, 4\}, \{1, 3\}, \{2\}\}.$$

Example 4. Let B = {2, 3, 4, 6, 12, 36, 48} and S be the relation "divide" on B. Draw Hasse diagram of S. (PTU, B.C.A. I 2005)

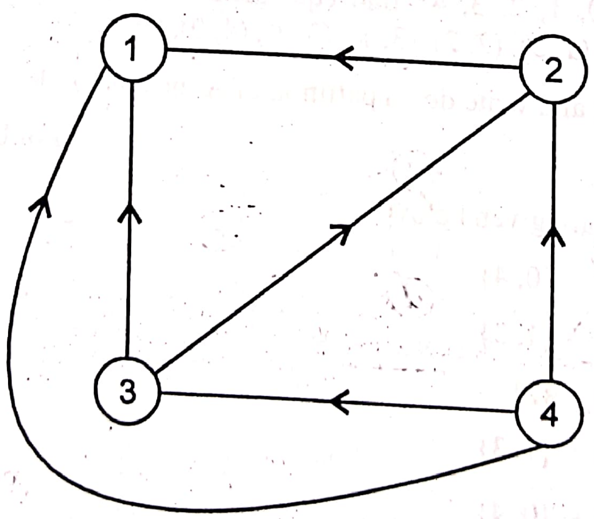
Sol. Hasse diagram is



Example 5. Let $x = \{1, 2, 3, 4\}$, R = $\{<x, y>> \mid x > y\}$. Draw the graph of R and also give its matrix.

Sol. R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}

Graph of R



Matrix of R

	1	2	3	4
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Example 6. Write all possible relations from A = {0} to B = {1, 2}.

(PTU, B.C.A. I 2003)

Sol. $A \times B = \{(0, 1), (0, 2)\}$ $\quad n(A \times B) = 2$

Total number of relations $= 2^2 = 4$

which are

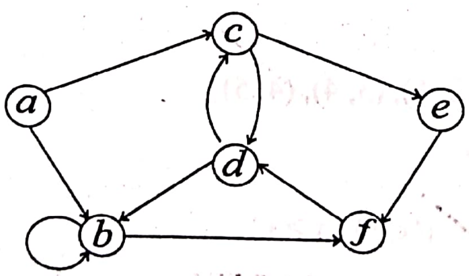
$R_1 = \phi$

$R_2 = \{(0, 1)\}$

$R_3 = \{(0, 2)\}$

$R_4 = \{(0, 1), (0, 2)\}$

Example 7. Let R be a Relation whose Graph is Given. Find all paths of length 3.



(Pbi.U., B.C.A., 2009)

Sol. Length of Path is the number of edges in the Path. The various Paths of length 3 are given below :

Starting Vertex	Path of Length 3
a	1. (a, c) (c, e) (e, f) 2. (a, c) (c, d) (d, b) 3. (a, b), (b, f), (f, d)
b	(b, f) (f, d) (d, c)
c	1. (c, e) (e, f) (f, d) 2. (c, d) (d, b) (b, f)
d	1. (d, b) (b, f) (f, d) 2. (d, c) (c, e) (e, f)
e	1. (e, f) (f, d) (d, c) 2. (e, f) (f, d) (d, b)
f	1. (f, d) (d, b) (b, f) 2. (f, d) (d, c) (c, e)

EXERCISE 4 (*d*)

1. Given relation is $R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$ in $A = \{1, 2, 3, 4\}$. Draw its graph.

2. Give an example of a non-empty set and a relation on the set that satisfies each of the following properties. Draw a digraph of the relation.

   (*i*) reflexive (*ii*) symmetric (*iii*) anti-symmetric (*iv*) transitive.

3. If $X = \{1, 2, 3, 4, 5\}$ and

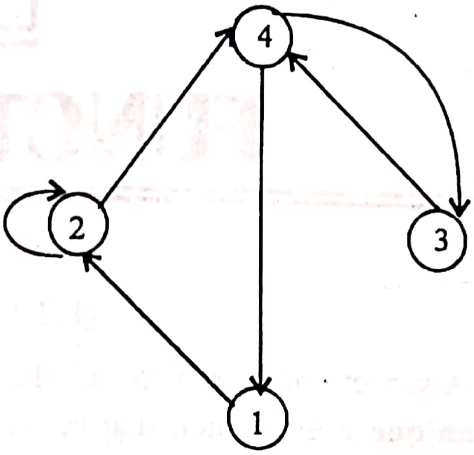
   $R = \{(1, 1,), (1, 2), (2, 3), (3, 5), (3, 4), (4, 5)\}$

   Determine (*i*) R^2 (*ii*) R^{∞}

4. Let $X = \{1, 2, 3, 4\}$ and $R = \{(x, y) : x > y\}$

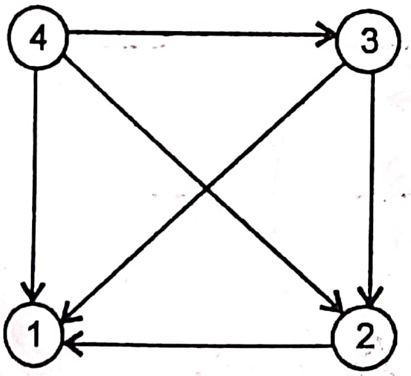
   Draw the diagraph and matrix of R.

ANSWERS

1. 

3. (*i*) $\{(1, 1), (1, 2), (1, 3), (2, 5), (2, 4), (3, 5)\}$

(*ii*) $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

4.  ;

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$