7

GRAPH THEORY

Art-1. Introduction

Graph theory was born in 1736 with Euler's paper in which he solved the Konigsberg bridge problem. In 1947, G.R. Kirchoff developed the theory of trees for their application in electrical network. A Caylay also discovered trees while he was trying to enumerate the isomers of saturated hydrocarbon $C_n H_{2n+2}$. They also lay down four color conjective, which states that four colour are sufficient for colouring any atlas such that the countries with common boundaries have different colour.

Now a day Graph Theory is employed in many areas, such as Communications, Engineering, Physical Sciences, Social Sciences etc. On account of diversity of its application, it is useful to develop and study the subject in abstract form and then import its results. In general areas of computer science such as switching theory and logical design artificial intelligence, formal languages, computer graphics, operating system, graph theory is very useful.

In this chapter, we shall define the various components of the graph theory along with suitable examples. An attempt has been made to show that graphs can be useful to represent any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with the relationships along them.

Art-2.Terminology

Graph :

(P.T.U. B.C.A.-I, 2006)

A graph (or undirected graph) is a diagram consisting of a collection of vertices together with edges joining certain pair of these vertices. Mathematically, we can write

A graph G = [V(G), E(G)]

where V(G) and E (G) are sets defined as

V(G) = Vertex set (points set or nodes set) of the graph G,

E(G) $\subseteq$ V(G) $\times$ V (G), a relation on V(G), called edge set of G

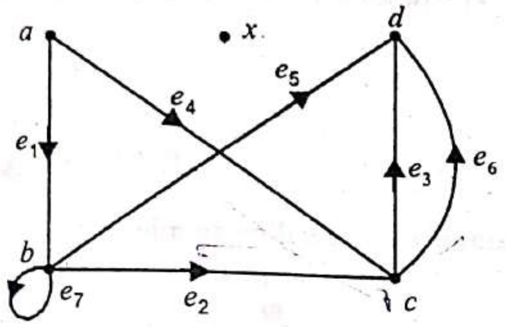
Each element e of E(G) is assigned on **unordered pair** of vertices (a, b) called the end vertices of e.

DIRECTED GRAPH

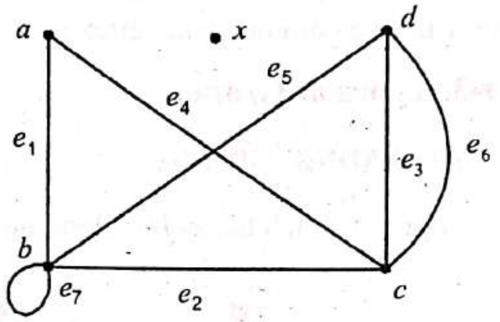
(P.T.U. B.C.A.-I, 2006)

A directed graph is a graph in which each element e of E (G) is assigned an **ordered pair** of vertices (a, b) along with arrow starting from a to b, where a is called the **initial vertex** and b is called the **terminal vertex** of the edge e.

The graphs directed and undirected are shown in the following figures :



(DIRECTED GRAPH)



(UNDIRECTED GRAPH)

REMARK : (*i*) A graph is represented by means of a diagram in which the vertices are denoted by points and edges are represented by line segments joining its end vertices.

(*ii*) It does not matter whether the joining of the two vertices in a graph is a straight line or a curve, longer or shorter.

ADJACENT VERTICES

Two vertices u and v of a graph G = (V, E) are said to be adjacent if there is an edge $e = (u, v)$ connecting u and v. Also the edge e is said to be **incident** on each of its end points u and v.

For example : In the above diagram a and b are adjacent vertices. Since there is an edge $e_1 = (a, b)$ joining a and b. Also the vertices a and d are not adjacent, as there is no edge joining the vertices a and d.

LOOP (OR SELF LOOP)

An edge that is incident from and into itself starts and ends at same vertex is called self loop or sting.

For Example : In the above diagram the edge e_7 is a loop. Since the edge $e_7 = (b, b)$ starts and ends at b.

ISOLATED VERTEX

A vertex of a graph G = (V, E), which is not joined to any vertex by an edge in G, is called an isolated vertex.

For example : In the above diagram the vertex x is an isolated vertex.

PARALLEL EDGES

If two (or more) edges of a graph G have the same end vertices, then these edges are called parallel edges.

For example : In the above diagram the edges $e_3 = (c, d)$ and $e_6 = (c, d)$ are parallel edges.

INCIDENCE

An edge e of a graph G = (V, E) is said to be **incident** with the vertex v if v is an end vertex of e (or u incident with e)

ADJACENT EDGES

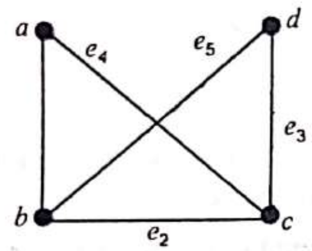
Two non-parallel edges of the graph are called adjacent if they have one common vertex.

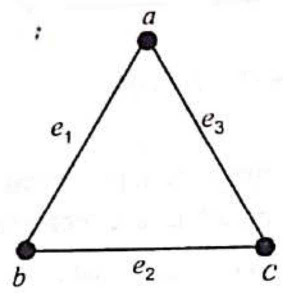
For example : In the above diagram the edges $e_1 = (a, b)$ and $e_4 = (a, c)$ are adjacent vertices, as they have a common end vertex a.

Art-3. Types of Graphs

(i) SIMPLE GRAPH (P.T.U. B.C.A.-I, 2004)

A graph which has neither loop nor parallel edge is called simple graph.

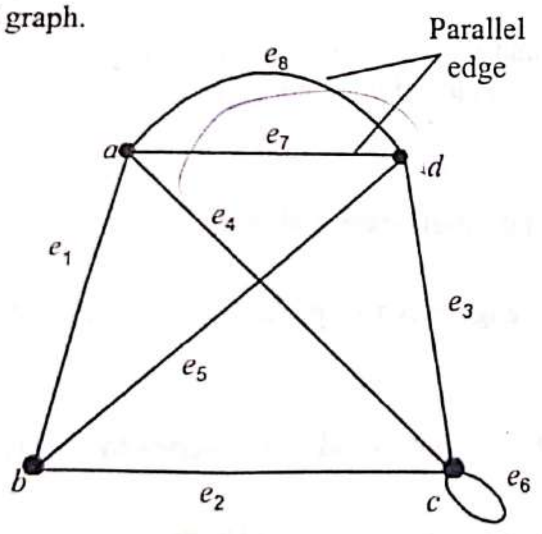


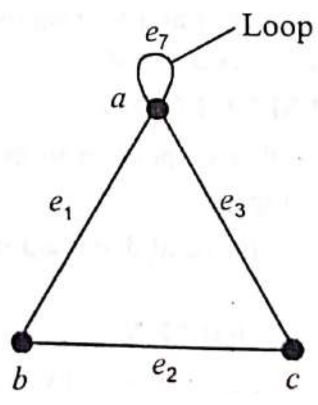


The above graphs are simple graphs.

(ii) GENERAL GRAPH (or Multi graph) (P.T.U. B.C.A.-I, 2004)

A graph which have either loop or parallel edge or both, is called a general graph or a multi graph.



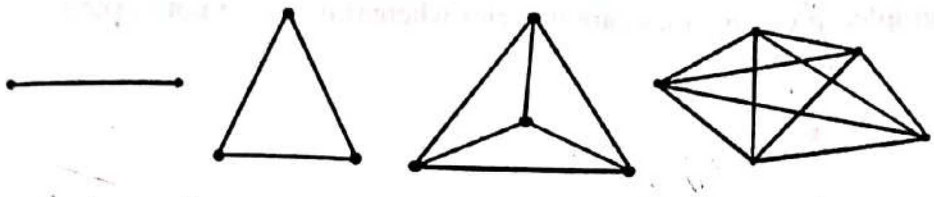


The above graphs are general graphs.

(iii) COMPLETE GRAPH

A simple graph in which there exists an edge between every pair of vertices is called a complete graph. It is also known as universal graph.

For example : Following graphs are complete graph.

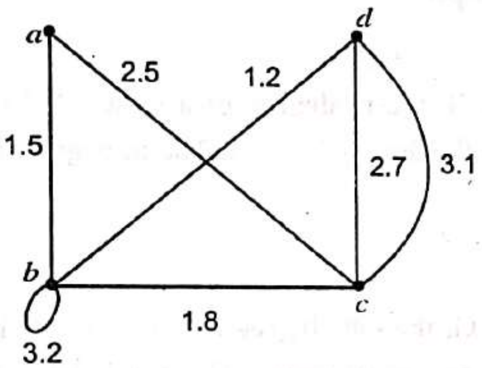


Note I : A complete graph with n vertices is usually denoted by K_n.

II : A complete graph has $C(n, 2)$ edges.

(iv) WEIGHTED GRAPH :

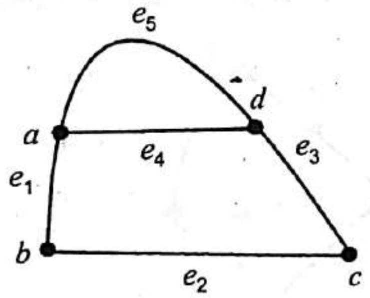
Let $G = (V, E)$ be any graph and $\omega : E \to R$ be a function from edge set E to set real numbers R. Then the graph $G = (V, E, \omega)$ in which each edge is assigned a number called the weight of the edge, is known as weighted graph.



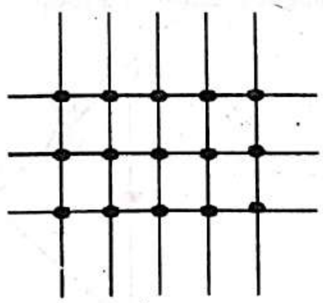
The above graph is a weighted graph, as each edge is assigned with a number.

(v) FINITE GRAPH : A graph $G = (V, E)$ is called a finite graph if the vertex set V is a finite set.

(vi) INFINITE GRAPH : A graph $G = (V, E)$ is called an infinite graph if the vertex set V is an infinite set.



(Finite graph)



(Infinite graph)

(vii) ORDER OF A GRAPH : Let G be a graph then the number of vertices denoted by $|V(G)|$ is called order of G.

(viii) TRIVIAL GRAPH, NULL OR EMPTY GRAPH

The trivial graph is the graph with one vertex and no edges.

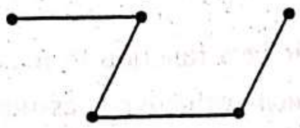
The empty graph is the graph with No vertices and no edges.

(ix) EDGES IN SERIES : When two edges in a graph have exactly one vertex in common and this vertex is of degree two, then two edges are said to be in series.

Example. 

e_1, e_2 are in series whereas e_2, e_3 are not in series.

(x) Acyclic : An Acyclic is a simple graph which does not have any cycles. *i.e.* No loop exists in such graphs.

Example : 

Art-4. Degree in a Graph

IN-DEGREE

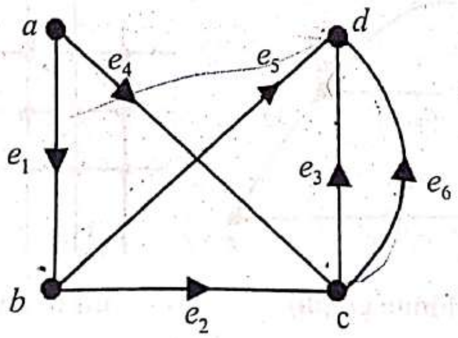
In a directed graph G, the **in-degree** of a vertex "a" is defined as the number of edges which have "a" as the terminal vertex. The in degree of the vertex a is denoted by deg $G^+(a)$ or $d^+(a)$.

OUT-DEGREE

In a directed graph G, the **out-degree** of a vertex "a" is defined as the number of edges which have "a" as the initial vertex. The out-degree of the vertex a is denoted by deg $G^-(a)$ or $d^-(a)$

Remark. (*i*) A vertex in a directed graph with in-degree zero is called a **source** and out-degree zero is called a **Sink**.

(*ii*) The direction of a loop in a directed graph has no significance.



In the above figure

deg $G^-(a) = 2$, deg $G^+(a) = 0$

deg $G^-(d) = 0$, deg $G^+(d) = 3$

deg $G^-(c) = 2$, deg $G^+(c) = 2$

In above graph, vertex "a" is called a **source** and the vertex "d" is called a **Sink**.

Even or odd (Parity) of a Vertex : The vertex V is said to be even or odd according as deg.(V) is even or odd.

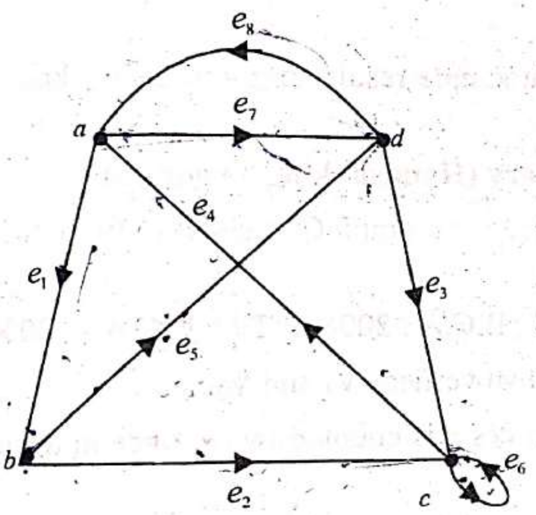
DEGREE

The degree of a vertex "a" in a directed or undirected graph is defined as the total number of edges incident with a. The degree of a vertex a is denoted by deg G (a) or $d(a)$.

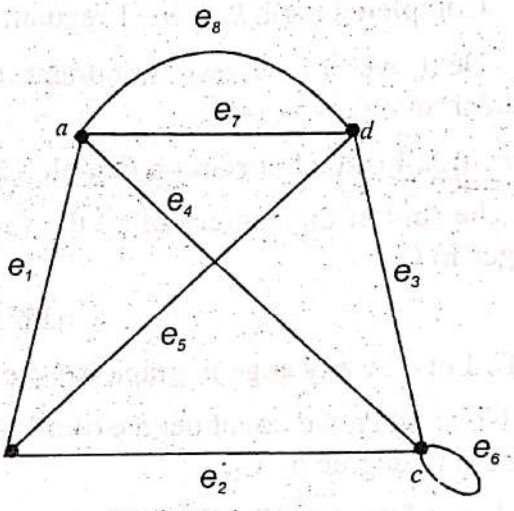
Thus in a direct graph deg G (a) = deg G^+ (a) + deg G^- (a).

Remark. For calculating degree of a a vertex in a general graph, a loop is counted twice.

For example : In the following graph directed or undirected we have



(Directed



(Undirected

In directed graph

$$\deg G(x) = \deg G^+(x) + \deg G^-(x)$$

$$\therefore \quad \deg G(a) = 2 + 2 = 4$$

$$\deg G(b) = 1 + 2 = 3$$

$$\deg G(c) = 2 + 3 = 5$$

$$\deg G(d) = 3 + 1 = 4$$

In undirected graph

$$\deg G(a) = 4$$

$$\deg G(b) = 3$$

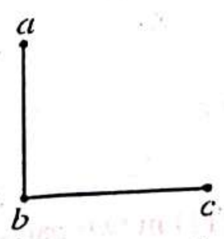
$$\deg G(c) = 5$$

$$\deg G(d) = 4$$

Pendent vertex (End vertex)

A vertex whose degree in a graph is one is called pendent vertex.

For example :- In the following graph



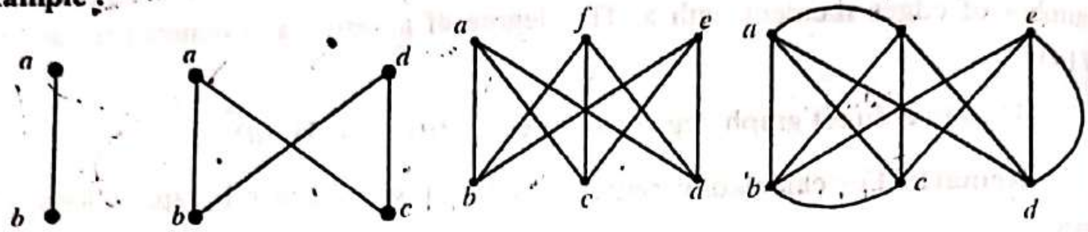
The vertices a and c are pendent vertices.

since deg G (a) = 1 and deg G (c) = 1

REGULAR GRAPH : A graph in which all the vertices are of same degree is called a regular graph.

k-REGULAR GRAPH : A graph in which all the vertices have the same degree equal to k, is called a k-Regular graph.

For example :



(1-Regular graph) (2-Regular graph) (3-Regular graph) (4-Regular graph)

Note : Complete Graph k_n is $n-1$ regular.

Next, we have a very important, but simple result on graph theory known as the first theorem on graph theory.

Theorem 1. First Theorem on Graph Theory (Handshaking Theorem)

The sum of the degrees of all the vertices in a graph G is equal to twice the number of edges in G.

(Pbi.U., B.C.A., 2006; P.T.U. B.C.A.-I, 2004, 2007)

Proof : Let e be any edge in graph between two vertices V_1 and V_2.

Now, when we count degree of all vertices e is counted twice, once in degree of V_1 and again in degree of V_2.

Also, if V_1 and V_2 are identical, again e will be counted twice.

($\because$ e is self-loop)

Hence every edge is counted twice.

So total degree is twice number of edges.

or $$\sum_{i=1}^{n} \deg(v_i) = 2e$$

mainly

Theorem 2. Prove that in a graph the number of vertices of odd degree is even.

(P.T.U. B.C.A.-I, 2007)

Proof. Let $v_1, v_2, \ldots, v_n$ be n-vertices and $e_1, e_2, \ldots, e_e$ be e-edges in the graph G. Then by first theorem on graph theory

$$\sum_{i=1}^{n} d(v_i) = 2e \qquad \ldots(1)$$

Now, divide the sum on the L.H.S of (1) in two parts

(*i*) One part contains the sum of degree of vertices which have even degree.

(*ii*) Second part contains the sum of the vertices which have odd degree.

Then equation (1) can be written as

$$\sum_{\text{even}} d(v_i) + \sum_{\text{odd}} d(v_k) = 2e \quad \ldots(2)$$

Since the R.H.S of (2) is an even number. Also $\sum_{\text{even}} d(v_i)$ is also even. This implies that $\sum_{\text{odd}} d(v_k)$ is also even.

i.e. the sum of the degree of vertices having odd degrees is even

$\therefore$ The number of vertices having odd degree must be even.

Theorem 3. Prove that the maximum degree of any vertex in a simple graph having n vertices is $n-1$.

Proof. Since, in a simple graph, there are no parallel edges and no loops. Therefore a vertex can be connected to the remaining $n-1$ vertices at the most by $(n-1)$ edges.

Hence, the maximum degree of any vertex in a simple graph having n vertices is $n-1$.

Theorem 4. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Proof. Let $v_1, v_2, \ldots\ldots, v_n$ be n-vertices and $e_1, e_2, \ldots\ldots, e_e$ be e-edges in a simple graph G. Then

By First theorem on graph theory

$$\sum_{i=1}^{n} d(v_i) = 2e \quad \ldots(1)$$

Also, we know that

In a simple graph, the maximum degree of any vertex with n vertices is $n-1$.

Sum of maximum degrees of n vertices $= \underbrace{(n-1) + (n-1) + \ldots\ldots + (n-1)}_{n \text{ terns}}$

$$= n(n-1)$$

$\therefore$ from (1), we have

$$2e = n(n-1)$$

$$e = \frac{n(n-1)}{2}$$

$\therefore$ The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Theorem 5. Prove that the number of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$.

Proof. Since every vertex in a complete graph is joined with every other vertex through one edge

$\therefore$ The degree of every vertex in a complete graph of n vertices is $n-1$.

$\therefore$ If e be the total number of edges in G. Then by first theorem on graph theory, we have

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$n(n-1) = 2e \qquad [\because d(v_i) = n-1 \text{ for } 1 \le i \le n]$$

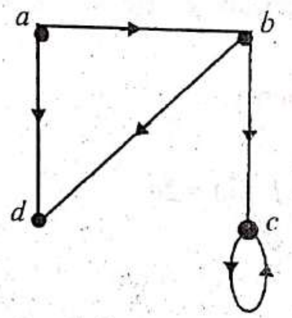
$$\Rightarrow \qquad e = \frac{n(n-1)}{2}$$

$\therefore$ Total number of edges in G $= \dfrac{n(n-1)}{2}$.

ILLUSTRATIVE EXAMPLES

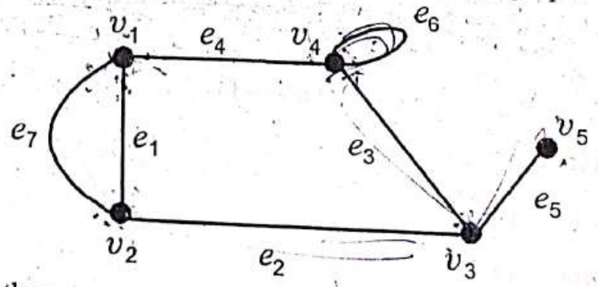
Example 1. If $V = \{a, b, c, d\}$ and $E = \{(a, b), (a, d), (b, c), (b, d), (c, c)\}$ be the vertex set and edge set of a graph G. Draw the directed graph $G = (V, E)$. Is it a simple graph?

Sol. The directed graph $G = (V, E)$ is as shown below :



Since it contain a loop. Therefore it is not a simple graph.

Example 2. Find the degree of each vertices of the following graph.



Also verify first theorem on graph theory.

Sol. Here $d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 3$, $d(v_4) = 4$, $d(v_5) = 1$

By first theorem of graph theory

$$\sum_{i=1}^{n} d(v_i) = 2e$$

where e is the number of edges and n is the number of vertices in the graph.

Here $n = 5$ and $e = 7$

Also $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 3 + 3 + 3 + 4 + 1 = 14 = 2(7) = 2e$

Thus first theorem on graph theory is verified.

Example 3. A graph G has 21 edges, 3 vertices of degree 4 and all other vertices are of degree 3. Find the number of vertices in G.

Sol. Let n be the number of vertices in G.

According of first theorem on graph theory.

$$\sum_{i=1}^{n} d(v_i) = 2e, \text{ where } e \text{ is the no. of edges.}$$

Let v_1, v_2, v_3 be the vertices of degree 4 and $v_4, v_5, \ldots\ldots, v_n$ be the remaining vertices of degree 3

$$\therefore \quad \sum_{i=1}^{3} d(v_i) + \sum_{k=4}^{n} d(v_k) = 2(21)$$

$$3\times 4 + (n-3)\times 3 = 42$$

$$12 + 3n - 9 = 42$$

$$3n = 39$$

$$n = 13$$

$\therefore$ Number of vertices in G be 13.

Example 4. Prove that there does not exist a graph with 5 vertices with degree equal to 1, 3, 4, 2, 3 respectively.

Sol. Here $n = 5$, Let e be the number of edges in the graph

By first Theorem on graph theory

$$\sum_{i=1}^{5} d(v_i) = 2e$$

$$\Rightarrow \quad 1+3+4+2+3 = 2e$$

$$\Rightarrow \quad 13 = 2e$$

$$\Rightarrow \quad e = \frac{13}{2}, \text{ which is not possible}$$

Hence there does not exist a graph with 5 vertices of given degrees.

Example 5. Is there a simple graph G with six vertices of degree 1, 1, 3, 4, 6, 7 ?

Sol. Here number of vertices in the graph, $n = 6$

we know that

Maximum degree of any vertices in a simple graph $= n - 1 = 6 - 1 = 5$

But G has a vertices of degree 7, which is not possible in a simple graph.

Hence there is no simple graph G of six vertices having the given degrees.

Example 6. Find k, if a k-regular graph with 8 vertices has 12 edges. Also draw k-regular graph.

Sol. We know that a graph G will be a k-regular graph if the degree of all the vertices in G are same and equal to k.

Also, number of vertices, $n = 8$

number of edges, $e = 12$

∴ By the first theorem on graph theory, we have

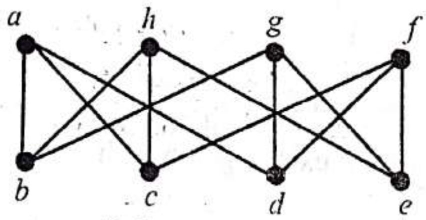
$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\sum_{i=1}^{8} k = 2(12)$$

$\Rightarrow$ $8k = 24$

$\Rightarrow$ $k = 3$ so graph is 3-regular graph

and the 3-regular graph is



(3-Regular graph)

Art-5. Isomorphic Graphs

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Then G is isomorphic to G′ written as $G \cong G'$ if there exists a bijection f, from V onto V′ such that $(v_i, v_j) \in E$, if and only if $(f(v_i), f(v_j)) \in E'$.

In other words, two graphs are isomorphic if there exists a one-one correspondence between their vertices and edges such that incidence relationship is preserved.

Remark. (*a*) Two graphs which are isomorphic will have

(*i*) same number of vertices

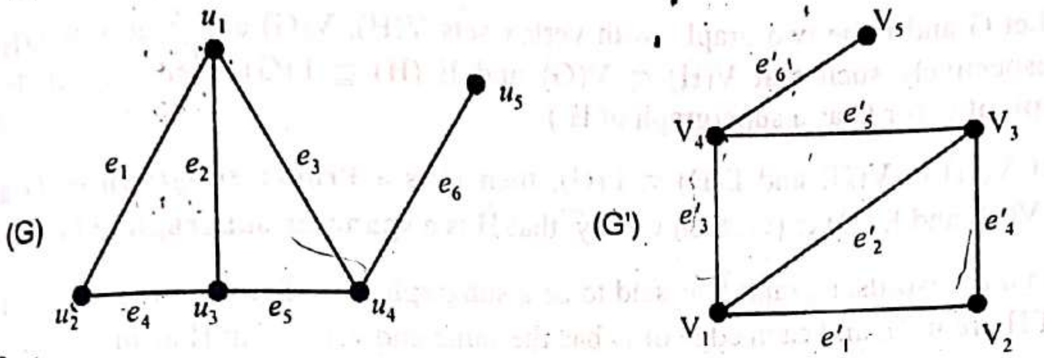
(*ii*) same number of edges

(*iii*) an equal number vertices with given degrees

(*b*) The converse of (*a*) need not be true.

or example.

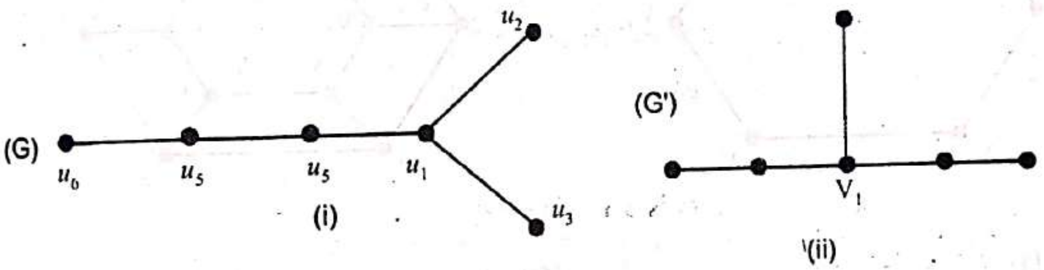
(*a*) The following two graphs are isomorphic to each other



Because $\exists$ a mapping $u_i \xrightarrow{f} v_i$ for $i = 1; 2, 3, 4, 5$

and $e_i \xrightarrow{f} e`_i$ for $i = 1, 2, 3, 4, 5$

(*b*) The two graphs may be non-isomorphic even though they have the same umber of vertices and edges and an equal number of vertices of given degrees.



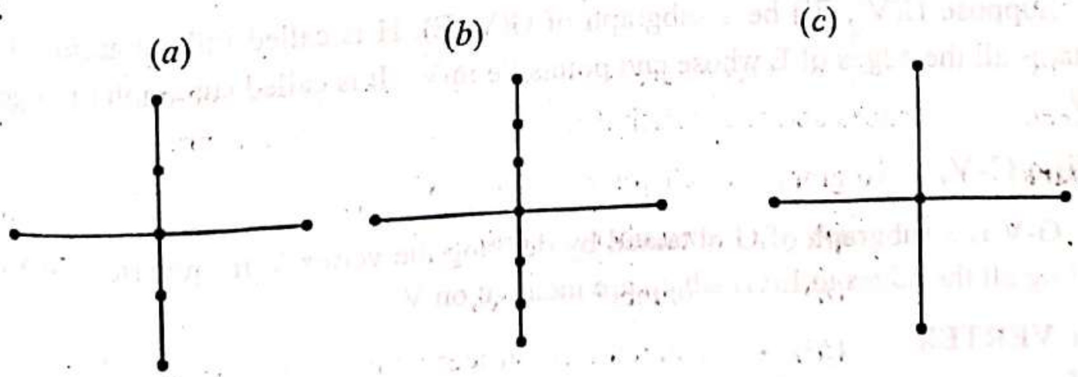
If the graph G are the to be isomorphic to graph G′, then the vertex u_1 must orresponds to v_1, because there is no other vertex of degree 3 in G′. Also in G′ there is ily one pendent vertex v_2 adjacent to v_1 , while in G there are two pendent vertices u_2 id u_3 adjacent to u_1.

Hence $G \not\equiv G'$.

OMEOMORPHIC GRAPHS

Given any graph G, obtain a new graph by dividing an edge of G with additional ertices.

e.g.



(*a*) and (*b*) Homeomorphic obtained from (*c*).

SUB-GRAPHS

(P.T.U. B.C.A.-I, 2005)

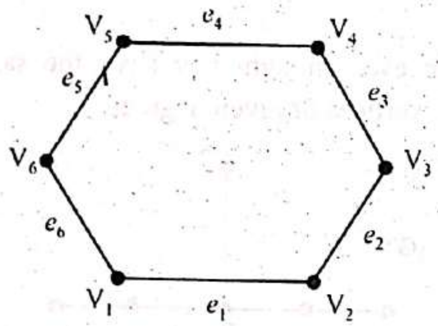
Let G and H be two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we call H as a **Subgraph** of G (or G as a supergraph of H).

If $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then H is a **Proper subgraph** of G and if $V(H) = V(G)$ and $E(H) \subset (G)$ then we say that H is a **spanning subgraph** of G.

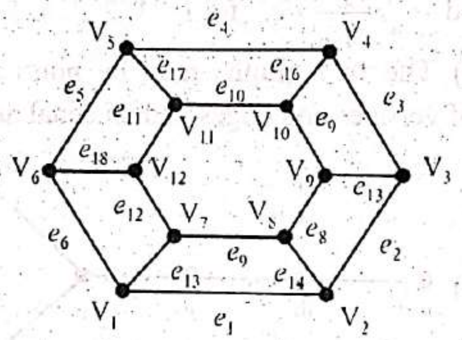
In other words, a graph H is said to be a subgraph of G if all the vertices and all the edges of H are in G, and each edge of H has the same end vertices in H as in G.

For example. In the following examples, H is a subgraph of G.

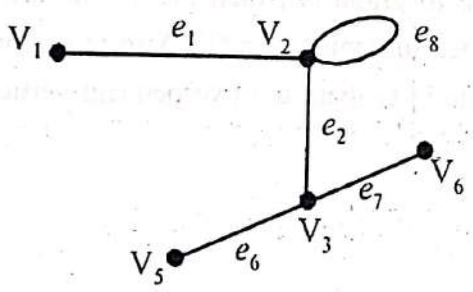
(*i*) (H)



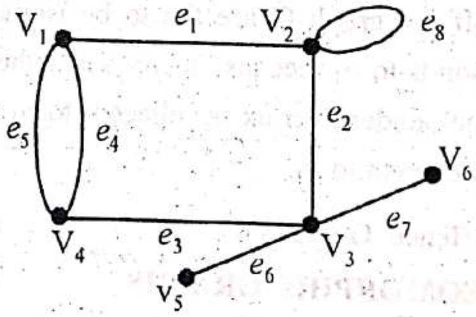
(G)



(*ii*) (H)



(G)



FULL SUBGRAPH

Suppose H(V′, E′) be a subgraph of G(V, E). H is called full subgraph of G if E′ contains all the edges of E whose end points lie in V′. It is called subgraph of G generated by V′.

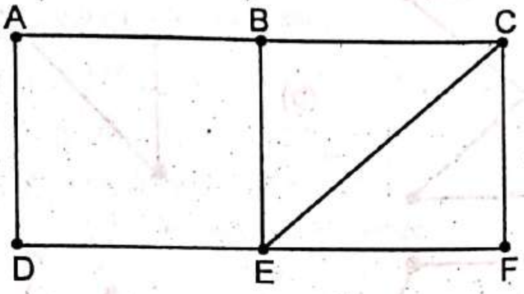
Define G-V

G-V is a subgraph of G obtained by deleting the vertex V from vertex set V(G) and deleting all the edges in E(G) which are incident on V.

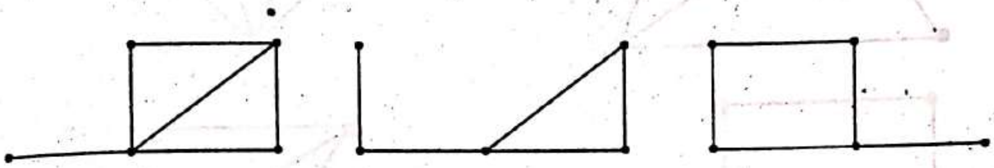
CUT VERTEX

A vertex V is called a cut vertex for G if G-V is disconnected.

Example. Let G be the graph find G-A, G-B, G-C



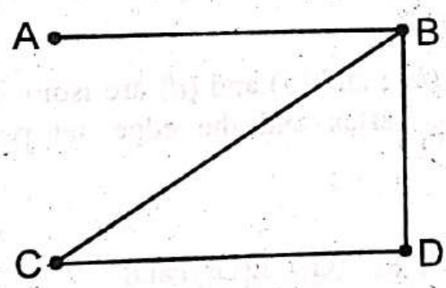
Sol.



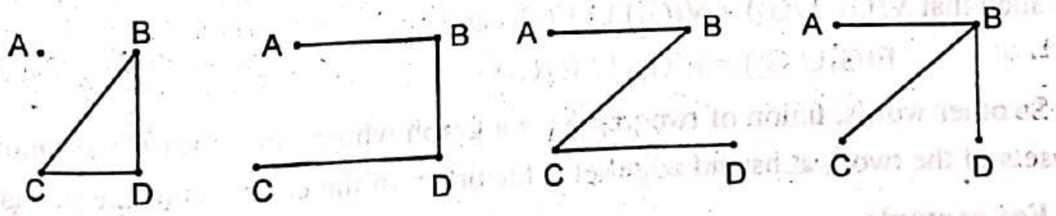
(*a*) (*b*) (*c*)

Define G – *e* : *e* is an edge in G. G– *e* is the graph obtained by simply deleting *e* from the edge set of G.

Example. Let G be graph.



Find (*a*) G – {A, B} (*b*) G – {B, C} (*c*) G – {B, D} (*d*) G – {C, D}

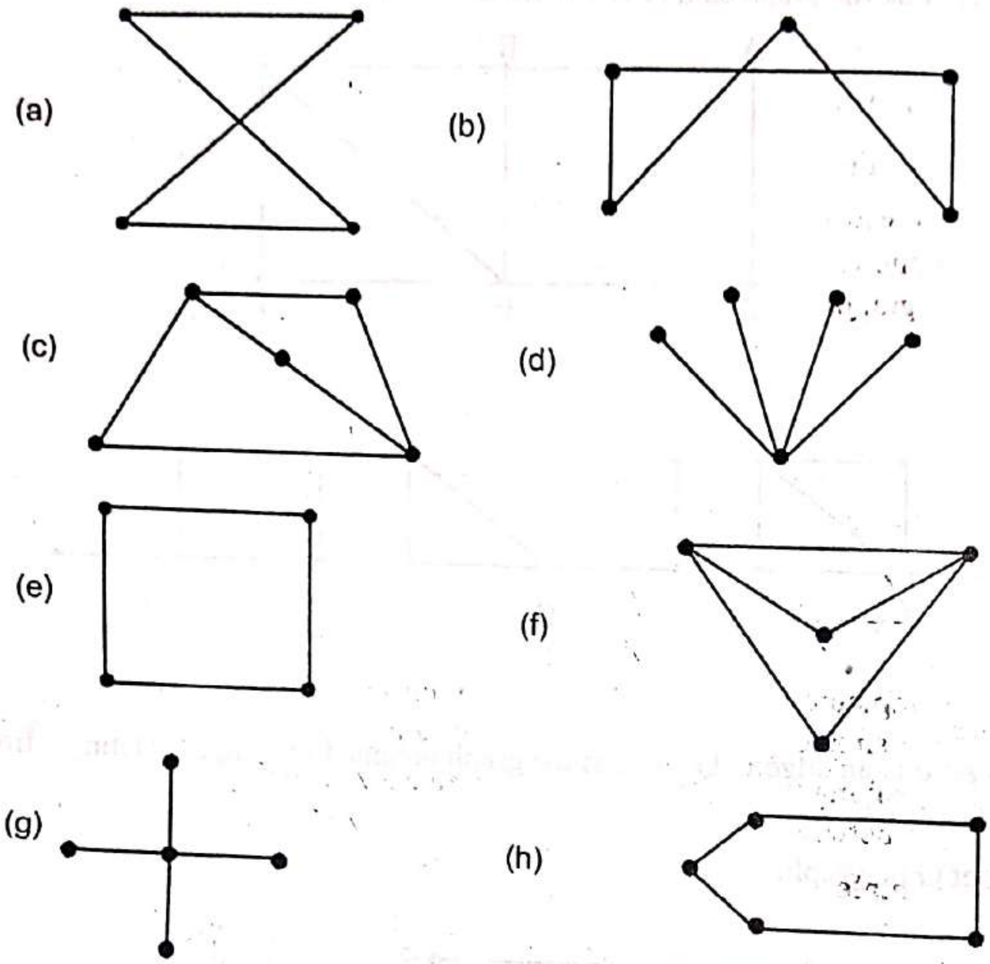


G – {A, B} G – {B, C} G – {B, D} G – {C, D}

Remark. (*i*) Every graph is its own subgraph

(*ii*) The null graph obtained from G by deleting all the edges of G is a subgraph of G.

Example 7. Which of the following pair of graphs are isomorphic ?



Sol. The graphs (a) and (e) ; (b) and (h) ; and (g) and (d) are isomorphic. Since there a one-one correspondence between the vertex and the edge set preserving incidence relation.

OPERATIONS OF GRAPH

(i) Union of two graphs : Let $G_1 = (V(G_1), E(G_1))$ and

$G_2 = (V(G_2), E(G_2))$ be two graphs.

Then their union is denoted by $G_1 \cup G_2$, is a graph

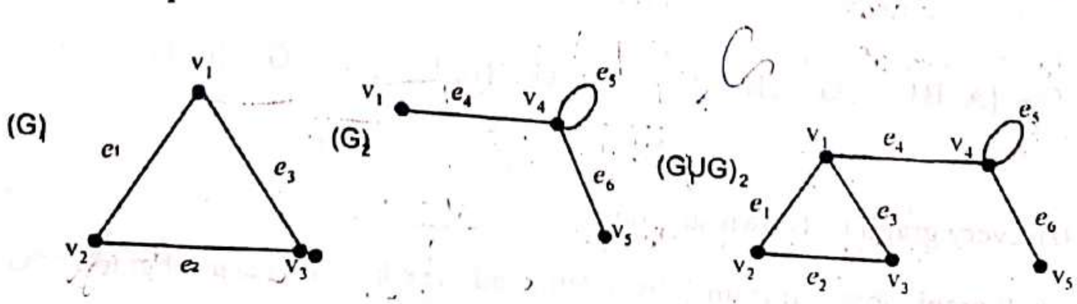
$$G_1 \cup G_2 = (V(G_1 \cup G_2), E(G_1 \cup G_2)).$$

such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$

In other words, union of two graphs is a graph whose vertex set is the union of the vertex sets of the two graphs and edge set is the union of the edge sets of the two graphs.

For example.



***(ii)* Intersection of two graphs :** Let $G_1 = (V(G_1), E(G_2))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. Then their intersection is denoted by $G \cap G_2$, is a graph

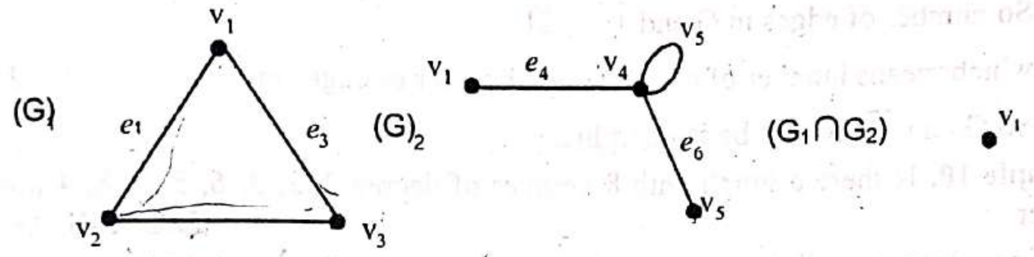
$$G_1 \cap G_2 = (V(G_1 \cap G_2), E(G_1 \cap G_2))$$

such that $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$

$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2).$$

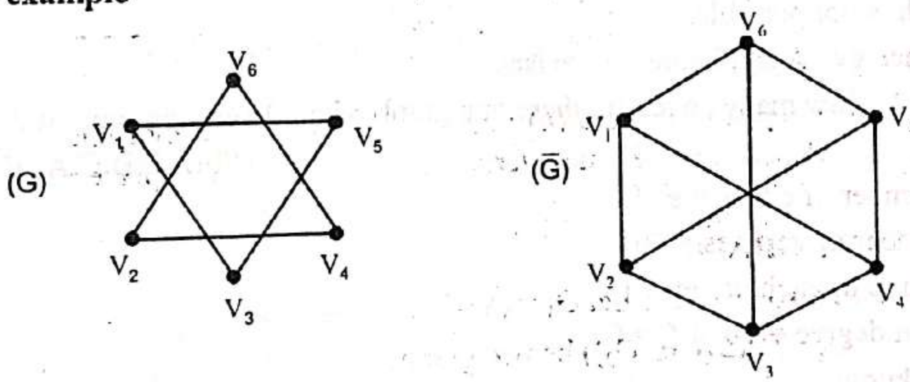
In other words, intersection of two graphs is a graph whose vertex set is the intersection of the vertex sets of the two graphs and edge set is the intersection of the edge sets of the two graphs.

For example.



***(iii)* Complement of a graph :** The compliment of a graph G is denoted by $\overline{G}$ and is defined as the simple graph with the vertex set same as the vertex set of G together with the edge set satisfying the property that there is an edge between two vertices in $\overline{G}$, when there is no edge between these vertices in G.

For example



Note : If the degree of a vertex v in a simple graph G having n vertices is k. Then degree of v in $\overline{G}$ is n-k-1.

Example 8. What is total number of edges in k_n, the complete graph on n vertices ? Justify your answer ?

Sol. We know number of vertices in $k_n = n$

also in complete graph there is an edge between every two vertices.

So we have to make pairs of n vertices

for this number of ways $= c\,(n, 2)$

$$= \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

$\therefore$ number of edges in complete graph $= \dfrac{n(n-1)}{2}$

Example 9. Can a graph with seven vertices be isomorphic to its complement ? Justify.

(B.C.A. II Sept. 2005)

Sol. Let G be the given graph and $\overline{G}$ is complement of G. We know, if an edge $e \in G$ then $e \notin \overline{G}$.

So total number of edges in G and $\overline{G}$ = Maximum number of possible edges in complete graph.

Here we have number of vertices = 7

Using 7 vertices max. number of edge $= \dfrac{7(7-1)}{2} = 21$

So number of edges in G and $\overline{G} = 21$

which means number of edges in G $\neq$ number of edges in $\overline{G}$ [$\because$ 21 is odd]

So G and $\overline{G}$ cannot be isomorphic.

Example 10. Is there a graph with 8 vertices of degree 2, 2, 3, 6, 5, 7, 8, 4 justify your answer.

(B.C.A.-II, April 2007)

Sol. Total degree of all the vertices = 2 + 2 + 3 + 6 + 5 + 7 + 8 + 4 = 37

We know total degree = 2 | E | where | E | = number of edges

$\therefore \quad 37 = 2\,|E|$

$\Rightarrow \quad |E| = \dfrac{37}{2}$

which is not possible.

Hence given graph does not exists.

Example 11. How many edges are there in a graph with 10 vertices each of degree 6 ?

(Pbi. U. B.C.A.-II April 2011)

Sol. Let number of edges = e

Number of vertices = 10

Degree of each vertex = 6

Total degree = 10 × 6 = 60

We know,

$$\text{Total degree} = 2 \times \text{number of edges}$$

$$60 = 2 \times e$$

$$e = 30.$$

EXERCISE 7 (*a*)

1. Draw all simple graphs of one, two and three vertices.
2. Draw graphs representing problems of
   (*a*) Two houses and three utilities
   (*b*) Three houses and three utilities
   (*c*) Four houses and four utilities, say water, gas, electricity, and telephone.

3. Draw graphs of the following chemical compounds

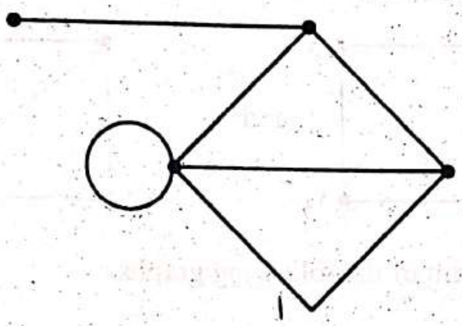
(a) CH_4 (b) $C_2 H_6$ (c) $C_6 H_6$

4. Differentiate between directed graph and undirected graphs.

5. How many nodes are necessary to construct a 2-regular graph with exactly 6 edges ?

6. Is it possible to construct a graph with 12 edges such that two of its vertices have degree 3 and remaining vertices have degree 4 ?

7. Find the degree of each vertices in the following graph :



8. Does there exist a graph with 6 vertices with degree equal to 3, 2, 4, 1, 3, 2 respectively.

9. Find k, if a k-regular graph with 7 vertices has 14 edges. Also draw the k-regular graph.

10. Find n, if a complete graph having n vertices has 15 edges.

11. Draw 3-regular graph with eight vertices.

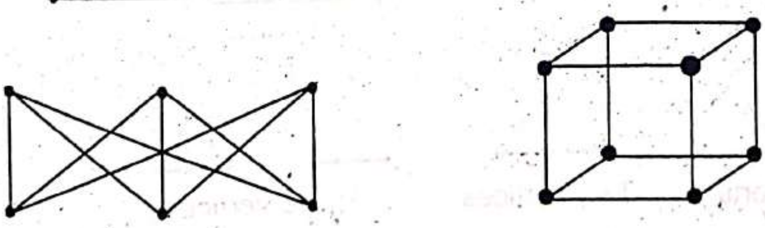
12. Draw 3-regular graphs with nine vertices.

13. Which of the following pair of graphs are isomorphic ?

(a)

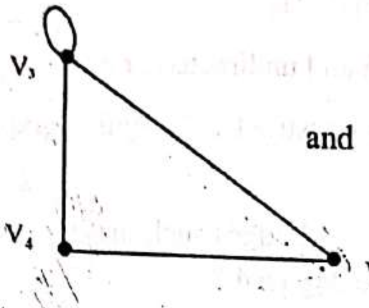
(b)

(c)

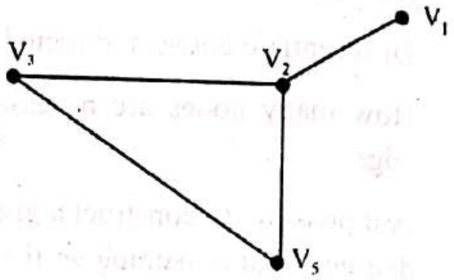


14. Find the union and intersection of the following graphs.

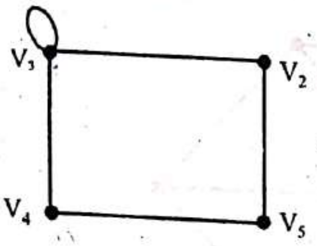
(*i*)



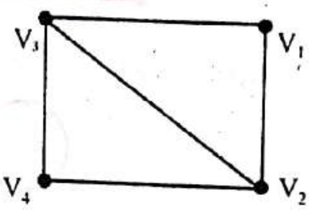
and



(*ii*)



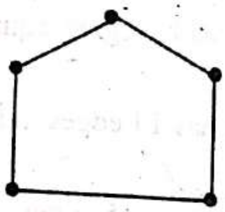
and



15. Find the compliment of the following graphs.

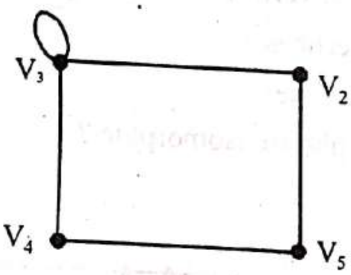
(*i*)

(G)



(*ii*)

(G)



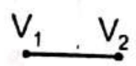
16. Let G be a complete graph of n vertices. Find the compliment of G.

17. Let G = (V, E) be a loop free undirected graph, where $V = \{v_1, v_2, v_3, \ldots\ldots, v_{10}\}$. If $\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 3$, $\deg(v_4) = 5$, $\deg(v_5) = 1$, $\deg(\overline{v_6}) = 2$, $\deg(v_7) = 5$, $\deg(v_8) = 2$, $\deg(v_9) = 3$, $\deg(v_{10}) = 2$. Determine the $\deg(v_i)$ in the compliment $\overline{G}$, for all $1 \le i \le 10$.

ANSWERS

1.

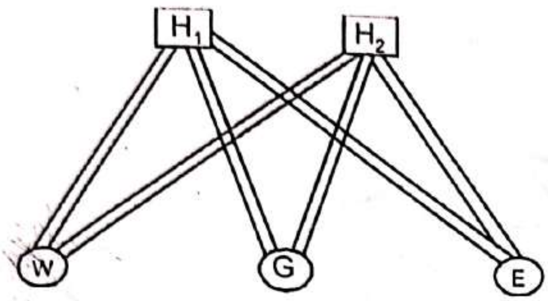
One Vertix



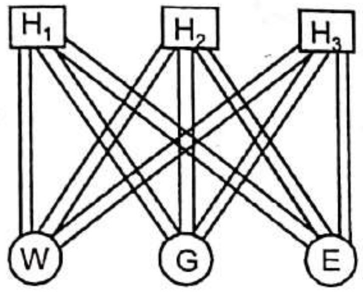
Two Vertices

Three Vertices

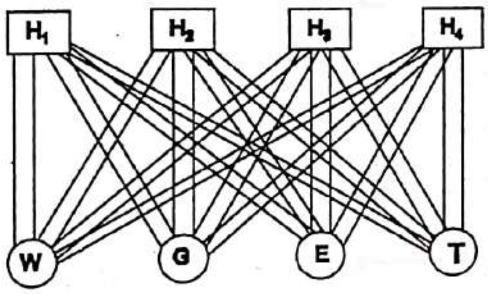
2. (*a*)



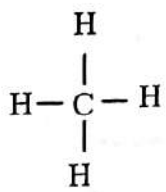
(*b*)



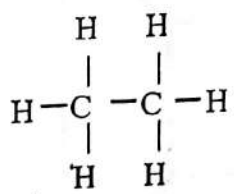
(*c*)



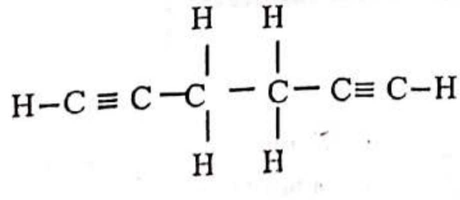
3. (*a*) CH_4



(*b*) C_2H_6



(*c*) C_6H_6



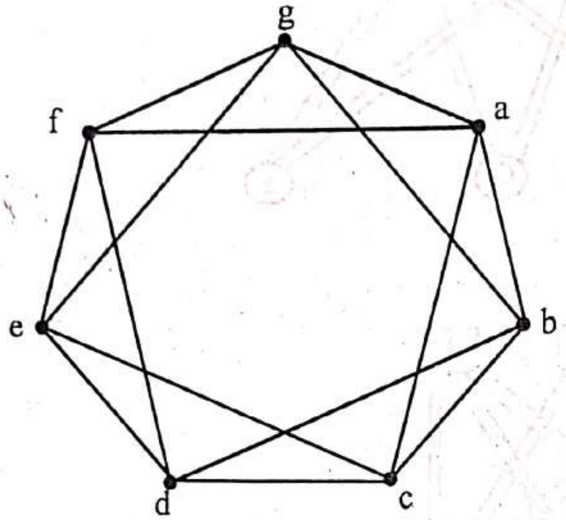
5. 6

6. No

7. 1, 3, 5, 2, 3

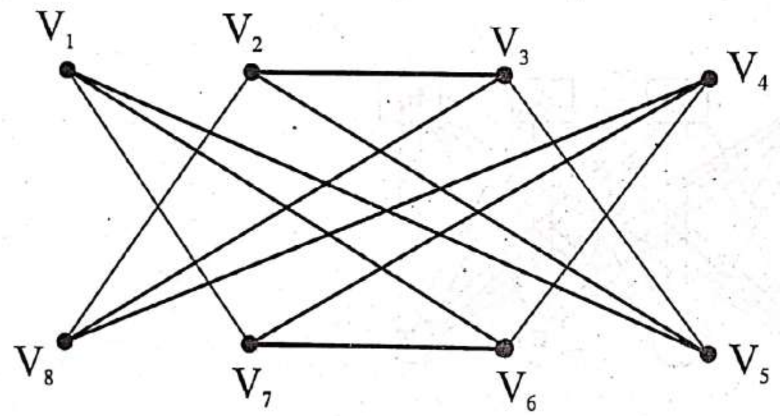
8. No

9. $k = 4$

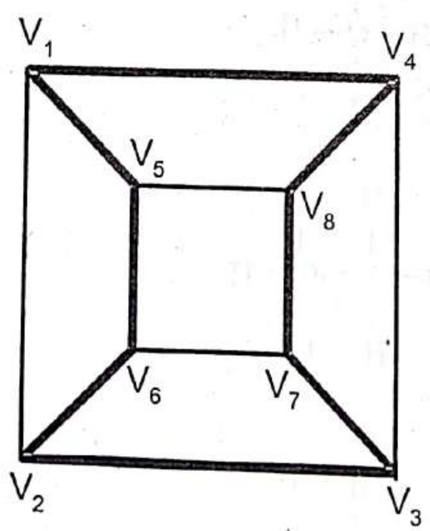


10. $n = 6$

11.

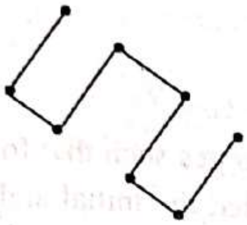


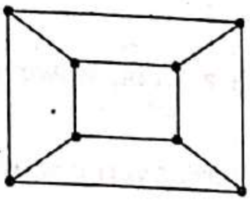
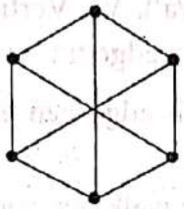
OR

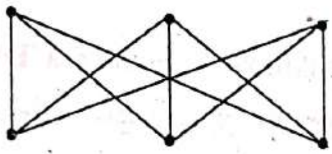
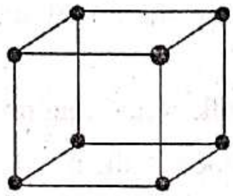


12. There is no 3-regular graph with nine vertices because the sum of degree of all the vertices is $3 \times 9 = 27$ which is not even.

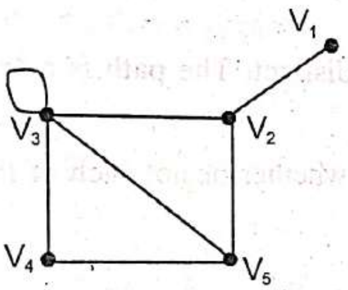
13.

(a) Yes  ≅ 

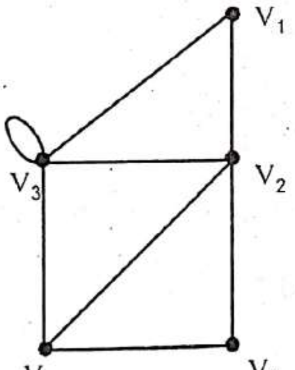
(b) No  ≇ 

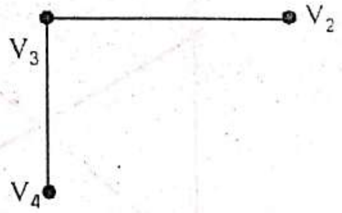
(c) No  ≇ 

14. (*i*)

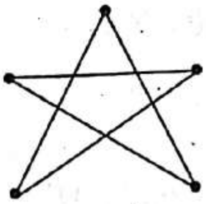
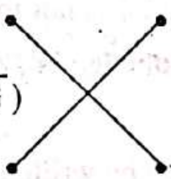


(*ii*)





15. (*i*) (*ii*)

($\overline{G}$)  ($\overline{G}$) 

16. Graph with n vertices and no edge

17. 7, 6, 6, 4, 8, 7, 4, 7, 6, 7

Art-6. Walks, Paths and Circuits

(P.T.U. B.C.A.-I 2007)

Walk : A walk in a graph G is finite sequence

$$W = V_0, e_1, V_1, e_2, \ldots\ldots, V_{k-1}, e_{k-1}, V_k$$

whose terms are alternatively vertices and edges such that for $1 \le i \le k-1$, the edge e_i has end vertices v_{i-1} and v_i. The vertex V_0 is called the initial and the vertex V_k is called terminal of the walk W. Vertices $V_1, V_2, \ldots, V_{k-1}$ are called internal vertices. A walk is also refereed as an edge train or chain.

Remark. (*i*) Each edge can appear only once in a walk, however vertices may appear more than once.

Open Walk : If a walk begin and end with the different vertices, it is called an open walk.

Closed Walk : If the initial and terminal vertices of a walk are same, it is called a closed walk.

Remark. A walk containing no edge and has length zero is called a **Trivial walk.**

PATH : An open walk in which no vertex appear more than once is called a path or simple path.

Note : A path do not intersect it self.

Simple Path and Trail

A path is **simple** if all vertices are distinct. The path is a **trail** if all the edges are distinct.

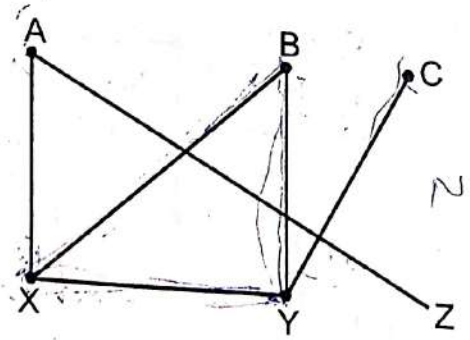
Example. Let G be the graph. Determine whether or not each of the following sequences of edges forms a path

(*a*) {(A, X), (X, B), (C, Y), (Y, X)}

(*b*) {(A, X), (X, Y), (Y, Z), (Z, A)}

(*c*) {(X, B), (B, Y), (Y, C)}

(*d*) {(B, Y), (X, Y), (A, X)}



Sol. (*a*) No, the edge {X, B} is not followed by edge (C, Y).

(*b*) No, Graph has not edge (Y, Z)

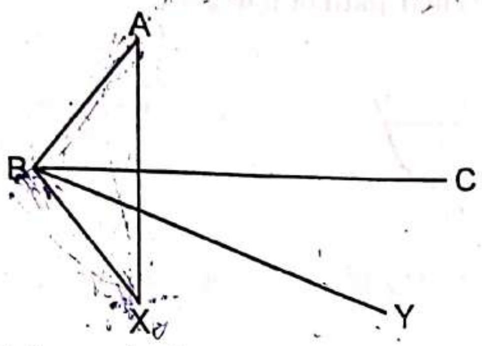
(*c*) Yes

(*d*) Yes, sequence can be written as {(B, Y), (Y, X), (X, A)}

[In undirected graph (Y, X) and (X, Y) are same]

Example. Let G be the graph. Determine whether each of the following is a closed path, trail, simple path or cycle.

(*a*) (B, A, X, B) (*b*) (X, A, B, Y) (*c*) (B, X, Y, B)



(*a*) This path is a cycle since it is closed and has distinct vertices

(*b*) This path is simple since it vertices are distinct. It is not cycle since it is not closed.

(*c*) This not even a path since {X, Y} is not an edge.

Length of path : The number of edges appearing in the sequence of the path is called the length of the path.

For example. Consider the following graph

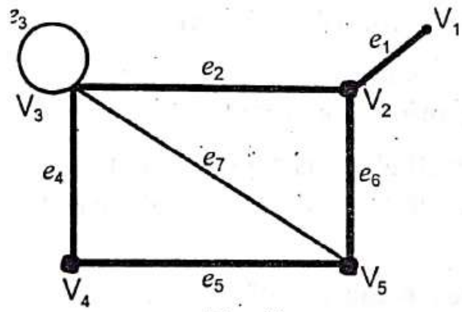


Fig. (i)

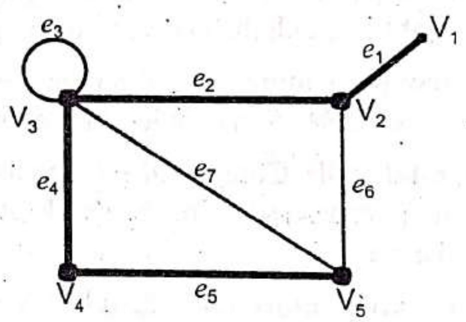


Fig. (ii)

$$W = V_1, e_1, V_2, e_2, V_3, e_3, V_3, e_4, V_4, e_5, V_5, e_6, V_2.$$

Then W is a walk of length 6 as shown by the bold line in fig. (*i*). The above walk is not a path as the vertices V_3 and V_2 appear twice in the walk W. However the walk

$W' = V_1, e_1, V_2, e_2, V_3, e_4, V_4, e_5, V_5$

is a path of length 4 as shown by the bold line in fig. (*ii*). Moreover, the above walk W and W′ are open walk as their terminus vertices are different.

But the walk

$W'' = V_1, e_1, V_2, e_3, V_3, e_3, V_3, e_4, V_4, e_5, V_5, e_6, V_2, e_1, V_1$

is a closed walk as the terminus vertices are same.

Remark. (*i*) An edge which is not a self loop is a path of length 1.

(*ii*) A self loop can be included in a walk but not in a path.

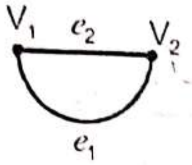
(*iii*) The terminus vertices of a path are of degree 1 and the internal vertices of the walk are of degree 2.

CIRCUIT : A circuit is a closed walk in which no vertex (except the initial and terminal vertex) appears more than once.

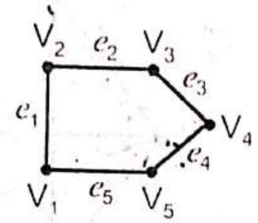
In other words, a circuit is a closed, non-intersecting walk. A circuit is also called the **cycle** or **elementary cycle** or **circular path** or **polygon.**

For example.

$W = V_1 e V_1$



$W = V_1 e_1 V_2 e_2 V_1$



$W = V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5 e_5 V_1$

are all circuits.

***k*-cycle :** A cycle with k-edges is called a k-cycle and it is denoted by C_k.

Remark : (*i*) A self loop is also a circuit, but converse is not true.

(*ii*) The degree of every vertex in a circuit is two.

(*iii*) 1 cycle is loop, 2 cycle is a pair of parallel edges, 3 cycle is a triangle,......., n cyclic is a polygon of n sides.

CONNECTED GRAPHS, DISCONNECTED GRAPHS, AND COMPONENTS

Connectivity : An undirected graph is said to be connected, if for any pair of vertices of the graph the two vertices are reachable from one another.

Strongly Connected : If any pair of vertices of the digraph both the vertices of the pair are reachable from another, then graph is strongly connected.

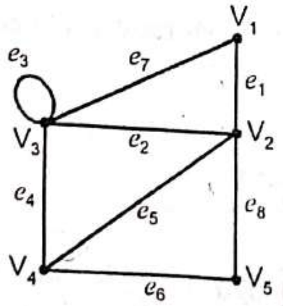
Unilaterally Connected : A simple directed graph is said to unilaterally connected if for any pair of vertices of the graph, at least one of the vertices of the pair is reachable from other vertex.

Weakly Connected Digraph : A directed graph is called weakly connected if its undirected graph is connected.

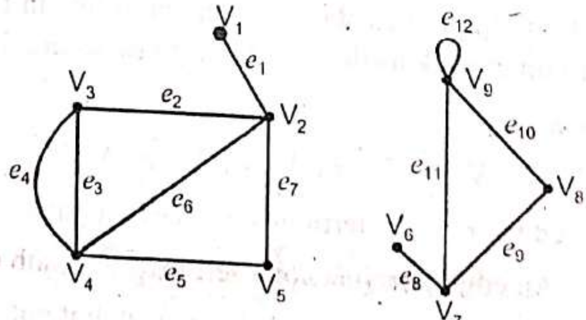
Connected graph : A graph G is said to be connected graph if there is atleast one path between every pair of vertices in G.

Disconnected graph : A graph which is not a connected graph is called disconnected graph.

For example.



Connected Graph

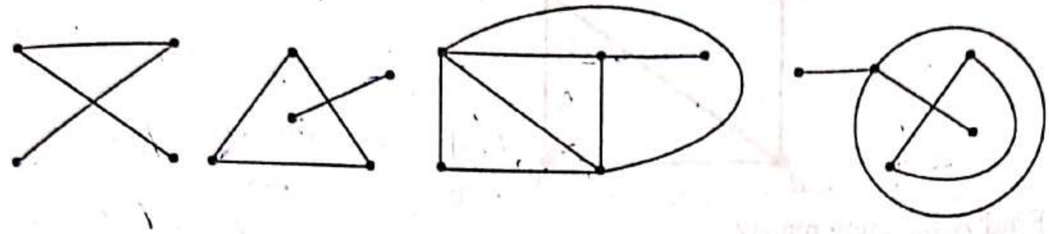


Disconnected Graph
(with two components)

Component : Each connected subgraph of a disconnected graph are called component.

Example. Consider the multigraph which of them are

(*a*) connected (*b*) loop-free (*c*) simple graphs



I II III IV

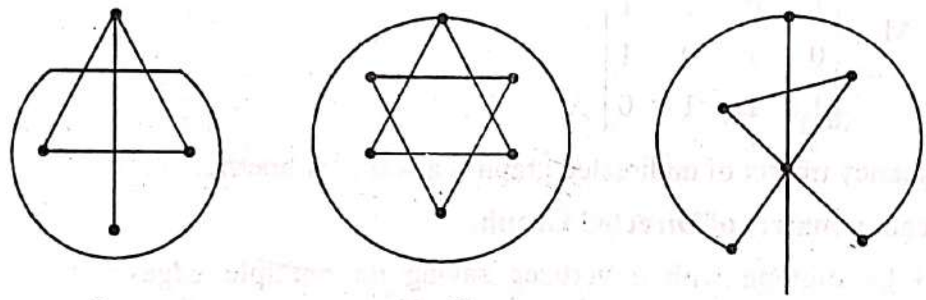
(*a*) I and III are connected.

(*b*) only IV has a loop.

(*c*) only (I) and (II) are simple graphs

(III) has multiple edges and IV has multiple edges and a loop.

Example. Which of following are connected, graphs.



I II III

(*a*) Only III is connected.

(*b*) All are graphs.

Art-7. Matrix Representation of Graphs

A graph can be represented by a matrix in two ways :

(*i*) Adjacency matrix

(*ii*) Incidence matrix.

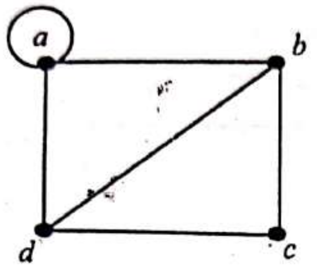
Adjacency Matrix (for undirected graph) :

Let G be an undirected graph with n vertices. Further suppose G has no multiple edges. Then G is represented by $n \times n$ matrix defined as $M = [a_{ij}]_{n \times n}$

$$a_{ij} = \begin{cases} 1 & \text{if } a_i \text{ and } a_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

i.e. an entry is 1 if there is an edge between a_i and a_j.

Example : Consider the graph



Find Adjacency matrix.

Sol.

	a	b	c	d
a	1	1	0	1
b	1	0	1	1
c	0	1	0	1
d	1	1	1	0

So $$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

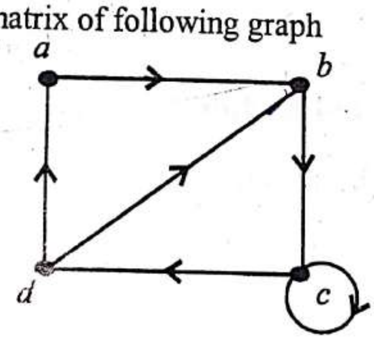
Note : Adjacency matrix of undirected graph is always symmetric.

Adjacency matrix of Directed Graph.

Let G be digraph with n vertices having no multiple edges. Then G can be represented by $n \times n$ adjacency matrix m defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is edge from } a_i \text{ to } a_j \\ 0 & \text{otherwise} \end{cases}$$

Example : Write adjacency matrix of following graph



Sol.

	a	b	c	d
a	0	1	0	0
b	0	0	1	0
c	0	0	1	1
d	1	1	0	0

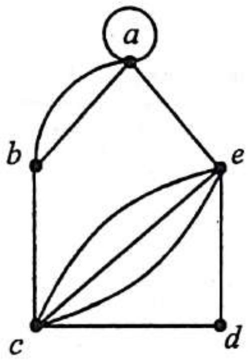
So $$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Adjacency matrix of multi-graph (undirected)

Let G be undirected graph of n vertices that may contain parallel edges. Then acency matrix M is $n \times n$ matrix defined by $M = [a_{ij}]_{n \times n}$

where $a_{ij} = \begin{cases} k, & k \text{ is number of edges between } a_i \text{ and } a_j \\ 0 & \text{otherwise} \end{cases}$

ample : Find adjacency matrix of Multi-graph.



	a	b	c	d	e
a	1	2	0	0	1
b	2	0	1	0	0
c	0	1	0	1	3
d	0	0	1	0	1
e	1	0	3	1	0

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & 0 \end{bmatrix}$$

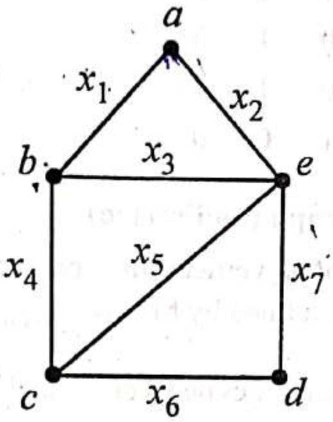
Note : In similar way we can find adjacency matrix of directed multi-graph.

Incidence matrix :

Let G be a graph have m vertices and n edges. Then incidence matrix of graph is $\times n$ matrix written as $A(G) = [a_{ij}]_{m \times n}$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Example : Write incident matrix of graph.



Sol. Number of vertices = 5

Number of edges = 7

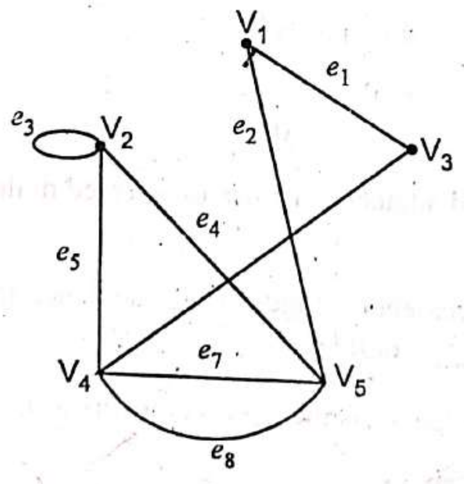
So **incidence** matrix is 5 × 7 matrix.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	1	1	0	0	0	0	0
b	1	0	1	1	0	0	0
c	0	0	0	1	1	1	0
d	0	0	0	0	0	1	1
e	0	1	1	0	1	0	1

so $$A(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

ILLUSTRATIVE EXAMPLES

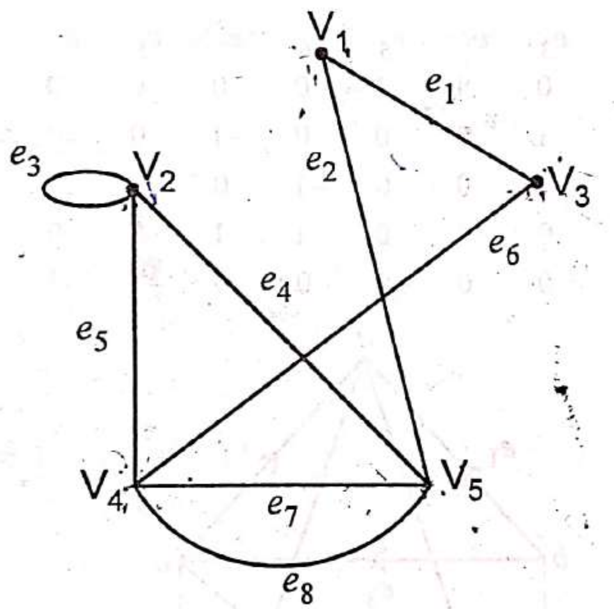
Example 1. Find the adjacency matrix A of the multigraph.



Set $a_{ij} = n$, where n is the number of edges between V_i and V_j and set $a_{ij} = 0$ otherwise.

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

Example 2. Find the incidence matrix M of the multigraph.

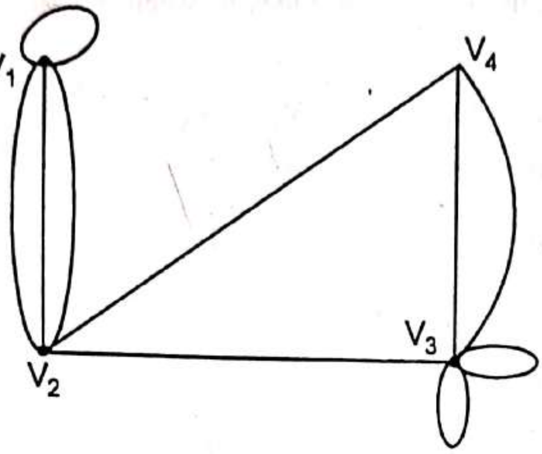


$$M = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 3. Draw the multigraph G whose adjacency matrix A = follows :

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Sol.

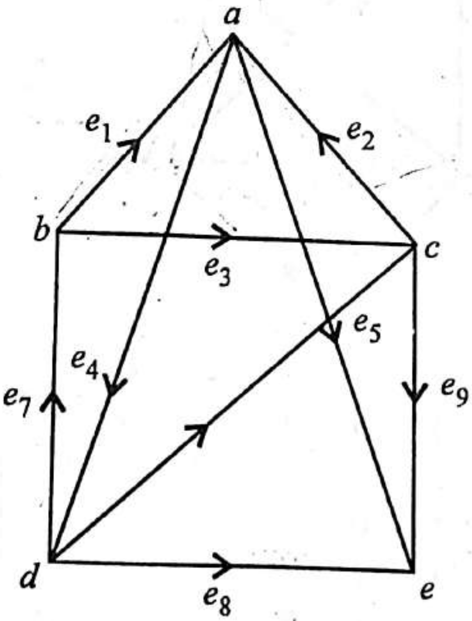


Example 4. Draw the Directed graph G whose incidence matrix M_1 is

(I)

$$M_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{array}$$

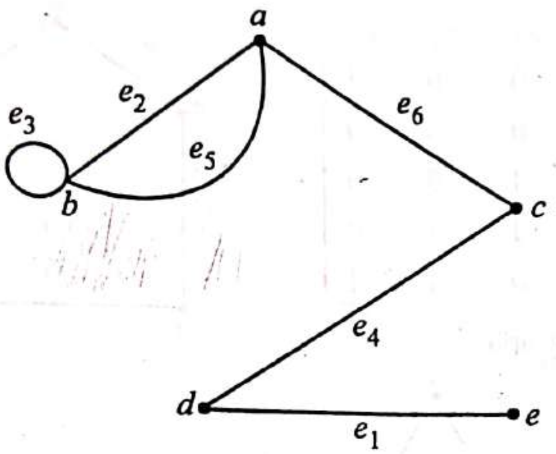
Directed graph is



(II)

$$M_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

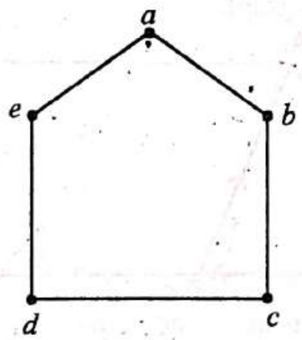
Undirected graph is



Adjacency list : In adjacency list of a graph we list each vertex followed by the vertices adjacent to it.

First write vertices of graph in a vertical column, then after each vertex write the vertices adjacent to it.

Example 5. For a graph



I. Write the adjacency list,

II. Find the adjacency matrix

III. Find the incidence matrix

IV. Draw complement graph.

Sol. I. adjacency list is

a ; e, b

b ; a, c

c ; b, d

d ; c, e

e ; a, d

II $V = \{a, b, c, d, e\}, |V| = 5$

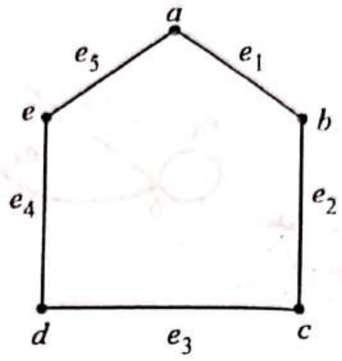
adjacency matrix will be a square matrix ;

$$M = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

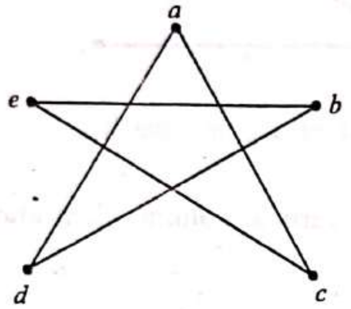
III. $|V| = 5, \ |E| = 5$

$\therefore$ Incidence matrix is

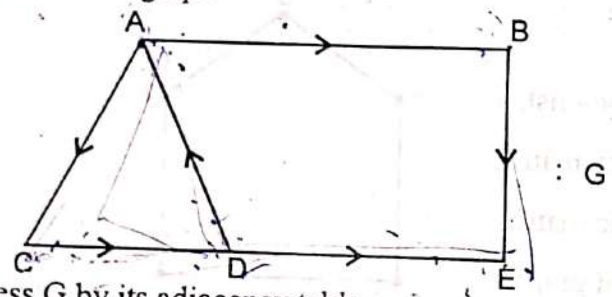
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$



IV. Complement of graph



Example 6. Consider the graph



(*i*) Express G by its adjacency table.

(*ii*) Find all the simple paths from A to E.

(*iii*) Find all cycles in G.

(*iv*) Show that G is unilaterally connected by exhibiting a spanning path of G.

(*v*) Is G strongly connected ?

I. $$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

II. A – B – E, A – C – D – E

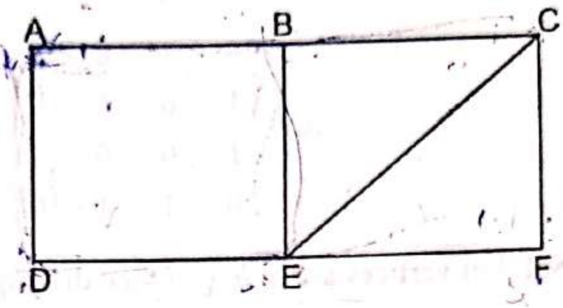
III. A – C – D – A

IV. G is unilaterally connected.

V. G is not strongly connected.

[$\because$ adjacency matrix of strongly connected digraphs has all entries = 1]

xample 7. Consider the graph G. Find



I. All simple paths from A to F

II. d(A, F)

III. diam(G)

IV. All cycles which include vertex A,

V. All cycles in G.

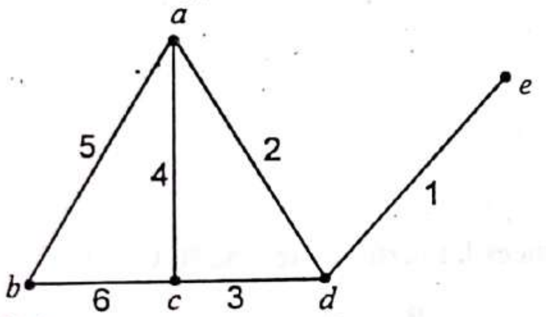
ol. (1) A – B – C – F, A – B – E – F, ADEF, A – B – EF etc.

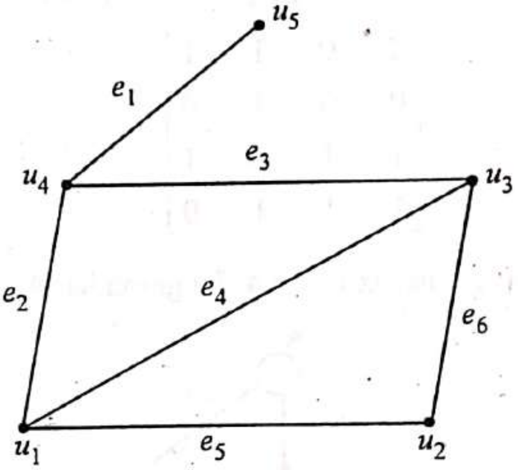
(2) d(A, F) = 3 (no. of edges)

(3) diam(G) = size of G = no. of edges = 8

(4) A – B – E – D – A , A – B – C – E – D – A etc.

xample 8. Show that the graphs G, G′ are isomorphic.





Let $f : G \to G'$ s.t

$$f(a) = u_1,\ f(b) = u_2,\ f(c) = u_3,\ f(d) = u_4,\ f(e) = u_5$$

The adjacency matrix for G for a, b, c, d, e

and adjacency matrix for G′ for the ordering

$a \to u_1,\ b \to u_2,\ c \to u_3,\ d \to u_4,\ e \to u_5$ is

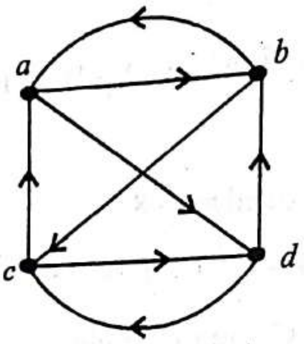
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

∴ G and G′ are isomorphic.

Example 9. Draw the Diagraph with given matrix as adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

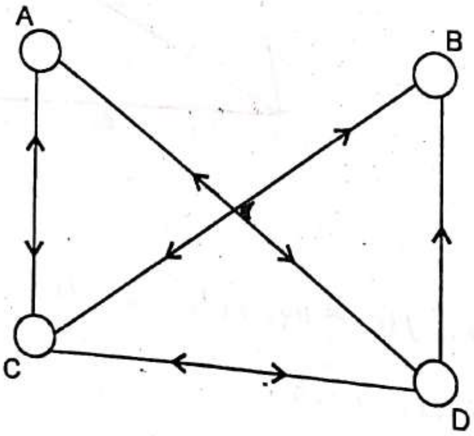
Sol. Let vertices are a, b, c, d then diagraph is shown below



Example 10. Draw a graph with adjacency matrix :

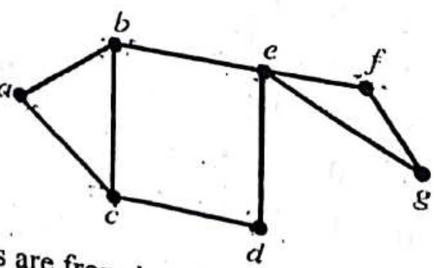
$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Sol. Given matrix is 4×4. So graph has 4 vertices let vertices are {A, B, C, D}



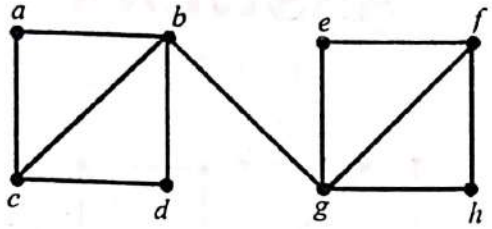
EXERCISE 7 (*b*)

1. For the graph



find how many paths are from b to f.

2. Let G = (V, E) be undirected graph.



Find how many paths are there in G from a to h ? How many of these have length 5 ?

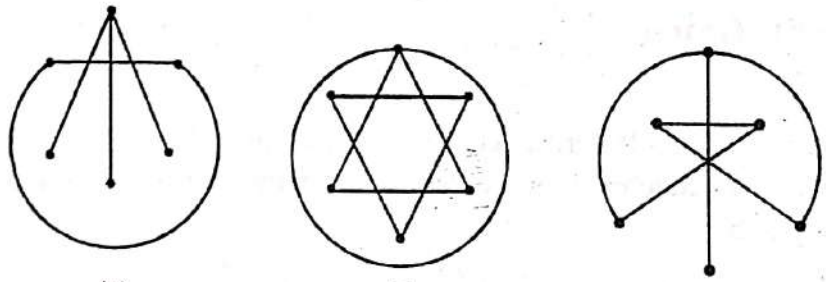
3. Let G = (V, E) be undirected graph. Define a relation R on V by a R b If there is a path in G from a to b. Prove that R is an equivalence relation.

4. Find the number of connected graphs with four vertices and draw them.

5. (*i*) Which of the following multigraphs are connected ?

   (*ii*) Which are cycle free ?

   (*iii*) Which are simple graphs ?



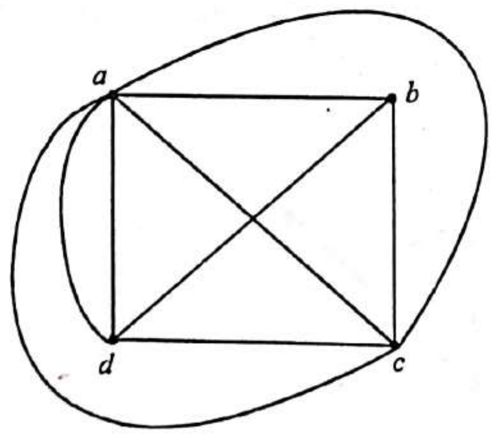
(a) (b) (c)

6. A graph has adjacency matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Is the graph connected ?

7. Write adjacency matrix of multigraph

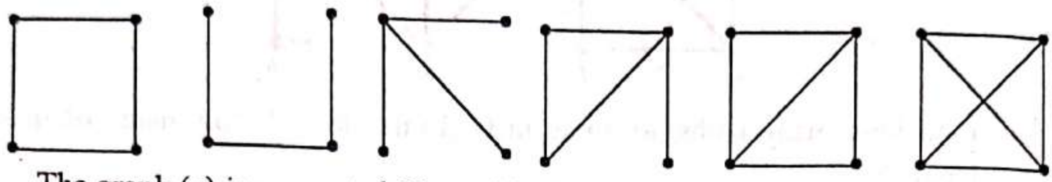


ANSWERS

1. 6

2. 9 ; 3

4. 6



5. The graph (c) is connected. N; (c) is simple graphs. No Graph is cycle free.

6. not strongly but unilaterlly

7. $\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$

Art-8. Bipartite Graph

(P.T.U. B.C.A.-I 2007)

Let G be any graph. If vertex set V can be partitioned into two disjoint subsets A and B such that every edge in G joins a vertex in A with a vertex in B, then graph is said to be Bipartite graph.

Example :

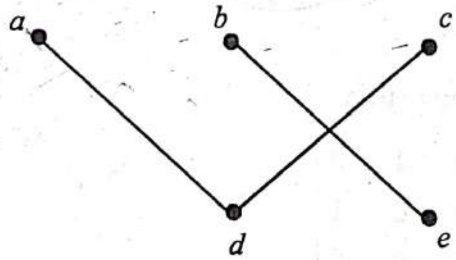
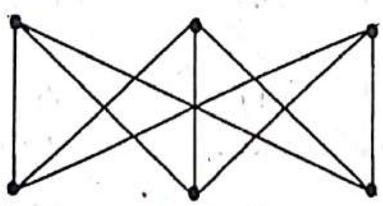


Fig. Show bipartite graph of 5 vertices.

Complete Bipartite Graph

Bipartite graph is said to be complete if every vertex in A is joined to every vertex in B. It is denoted by $k_{m,n}$. Where m, n are number of vertices in sets A and B respectively.

Example : Draw $k_{3,3}$



$k_{3,3}$

rt-9. Planar Graphs

A Planar graph is a graph drawn in the plane in such a way that no two edges tersect (cross) each other.

lanar graph : A Planar graph is a graph which is isomorphic to a plane graph *i.e.*, it can ; redrawn as a plane graph.

A graph which is not a planar graph is called **non-planar graph.**

or example. (*i*) The complete graph with four vertices K_4 is usually drawn with crossing lges see fig (*a*). But it can also be drawn with non-crossing edges see fig (*b*)

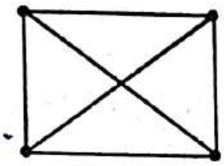


Fig. (a)

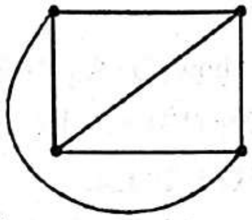


Fig. (b)

Hence K_4, is a planar graph.

(*ii*) A complete graph of five vertices is non-planar *i.e.*, K_5 is non-planar.

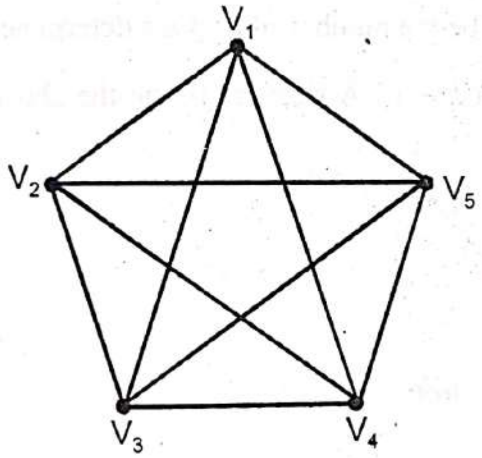


Fig (*a*)

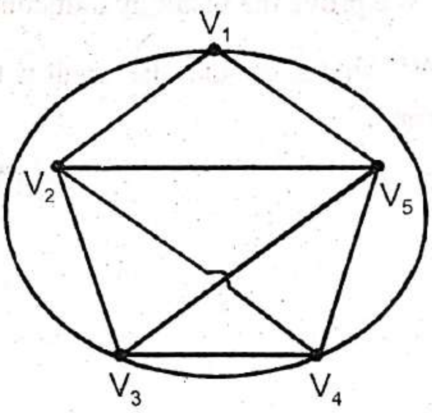


Fig (*b*)

Since the graph shown in fig. (*a*) cannot be drawn in plane without crossing edges see fig. (*b*). Hence K_5 is non-planar graph.

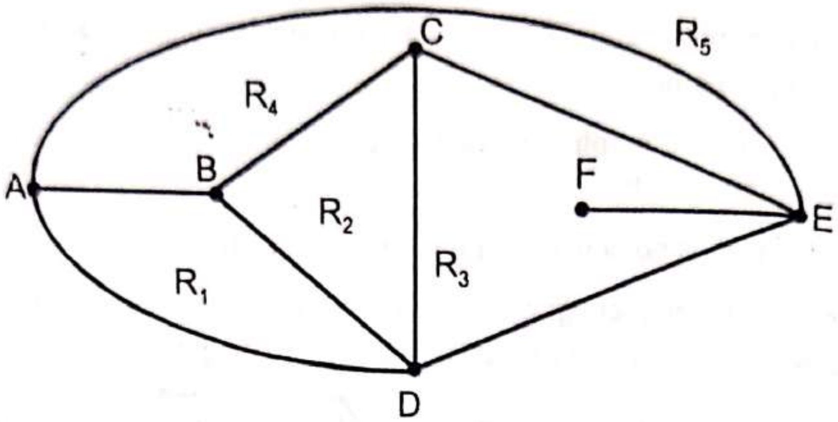
Region : A plane graph partitions the plane into several regions. These regions are called faces. Each region is depicted by the set of edges.

Cycle : The boundary of the region R of graph G is cycle if the boundary of R contains no cut edges of G. *i.e.*, contain no edge such that on removing any edge in R it will not be a closed circuit.

Degree of face : If G be graph and g be its face, then the number of edges in the boundary of g with cut edges counting twice is defined as the degree of face g.

Cut Edge : Cut edge in a graph is an edge whose removal results n a disconnected graph.

For example. Consider the following plane graph



Various regions are shown by R_1, R_2, R_3, R_4, R_5

Here deg $(R_1) = 3$, deg $(R_2) = 3$, deg $(R_3) = 5$, deg $(R_4) = 4$, deg $(R_5) = 3$

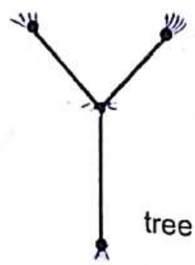
Theorem 1. EULER'S FORMULA

Let G = (V, E) be a connected planar graph and let R be the number of regions defined by any planar depiction of G. Then

$$R = |E| - |V| + 2$$

Proof. We prove the result by induction let k be the number of regions determined by G.

We first show that the result is true for $k = 1$. A tree determine the above region, for example



No. of vertices $= 4$, No. of edges $= 3$. Also from the formula, we have

$$1 = |E| - |V| + 2 \Rightarrow |E| = |V| - 1$$

i.e., No. of edges = No. of vertices -1, which is always true for a tree.

$\therefore$ The result is true for $k = 1$.

Let us assume that the result is true for all $k \geq 1$. Let G be a connected plane graph determining $(k+1)$ regions. Remove an edge which is common to the boundary of two regions. We obtain a graph G′ having k–regions.

Let $|V'|$, $|E'|$, R′ denote respectively the number of vertices number of edges and regions of G′, then

$$R' = |E'| - |V'| + 2 \quad \ldots (1)$$

Also, we have

$$|V'| = |V|, |E'| = |E| - 1, R' = R-1$$

$\therefore$ $|E| - |V| + 2 = |E'| + 1 - |V'| + 2 = (|E'| - |V'| + 2) + 1$

$= R' + 1$ [$\because$ of (1)]

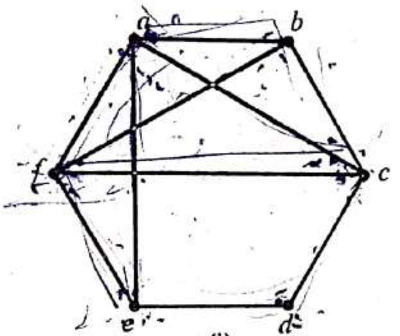
$= R$

$\therefore$ result is true for $k+1$

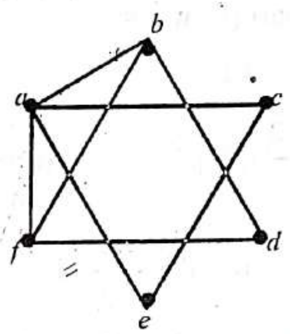
$\therefore$ result follows by induction for all connected graphs.

ILLUSTRATIVE EXAMPLES

xample 1. Verify Euler's formula for the following graphs.



(i)



(ii)

ol. (*i*) In fig. (*i*) $|V| = 6$, $|E| = 10$, $R = 6$

$\therefore$ By Euler's formula, $R = |E| - |V| + 2$

i.e., $6 = 10 - 6 + 2$, which is true

(*ii*) In fig. (*ii*) $|V| = 6$, $|E| = 8$, $R = 4$

$\therefore$ By Euler's formula, $R = |E| - |V| + 2$

i.e., $5 = 9 - 6 + 2$, which is true

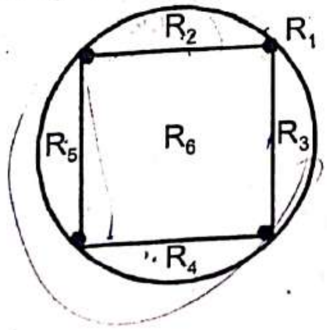
xample 2. Determine the number of regions defined by a connected planar graph with 4 odes and 8 edges. Draw such a graph. (Pbi. U. B.C.A.-II 2012)

ol. Here $|V| = 4$, $|E| = 8$

$\therefore$ By Euler's formula.

$$R = |E| - |V| + 2$$
$$= 8 - 4 + 2 = 6$$

$\therefore$ The given connected graph has 6 regions. The required graph is



Theorem 2. Let G = (V, E) be a simple, connected Planar graph with more than one edge, then the following inequalities holds.

(*ii*) $2|E| \geq 3R$ (*ii*) $|E| \leq 3|V| - 6$

(*iii*) There is a vertex v of G such that $\deg(v) \leq 5$.

Proof. (*i*) Since G has more then one edge $\therefore |E| > 1$

If G defines only one region, then R = 1

$\therefore \quad |E| > 1 \Rightarrow 2|E| > 2 > 3 \Rightarrow 2|E| > 3R$ holds.

So, let R > 1, then each region is bounded by atleast 3-edges. But each edge in a planar graph touches almost 3 region. Thus we have $2|E| \geq 3R$

(*ii*) By part (*i*), we have

$$2|E| \geq 3R$$

or $R \leq \frac{2}{3}|E|$, Adding | V | both sides

$$|V| + R \leq |V| + \frac{2}{3}|E| \quad \text{...(1)}$$

By Euler's formula $R = |E| - |V| + 2$

$$\therefore \quad |V| + R = |E| + 2 \quad \text{...(2)}$$

$\therefore$ From (1) and (2) we have

$$|E| + 2 \leq |V| + \frac{2}{3}|E|$$

$$\Rightarrow \quad 3|E| + 6 \leq 3|V| + 2|E|$$

$$\Rightarrow \quad |E| \leq 3|V| - 6$$

(*iii*) Let each vertex of G of degree ≥ 6.
Also by first theorem on graph theory

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$\Rightarrow \quad 6|V| \leq 2|E| \quad [\because \text{L.H.S} \geq 6|V| \text{ and R.H.S} = 2|E|]$$

$$\Rightarrow \quad |V| \leq \frac{1}{3}|E| \quad \text{... (3)}$$

Also, By part (*i*) $R \leq \frac{2}{3}|E|$ ...(4)

Adding (3) and (4) went

$$|V| + R \leq \frac{1}{3}|E| + \frac{2}{3}|E| = |E|$$

$$\Rightarrow \quad |V| + R \leq |E| \quad \text{...(5)}$$

But by Euler's formula, we have

$$|V| + R = |E| + 2 \qquad \ldots(6)$$

$\therefore$ from (5) and (6) we have

$$|E| + 2 \leq |E|$$

$\Rightarrow$ $2 \leq 0$, not possible

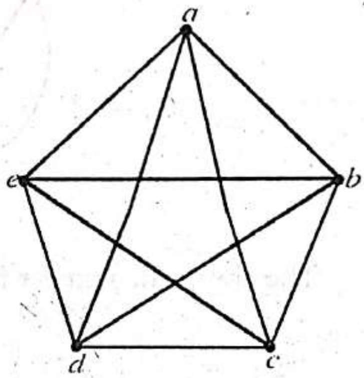
$\therefore$ Our supposition is wrong

$\therefore$ Each vertex of G cannot have a degree ≥ 6

Hence, there exist a vertex of G with degree ≤ 5.

ample 3. Prove that the graph K_5 is not planar.

l. Number of vertices in $K_5 = 5$



Number of edges in $K_5 = |E| = 10$

for planar graph

$$|E| \leq 3|V| - 6$$

$\Rightarrow$ $10 \leq 3 \times 5 - 6$

$\Rightarrow$ $10 \leq 9$,

which is contradiction.

$\therefore$ K_5 is not planar graph.

t-10. Coloring

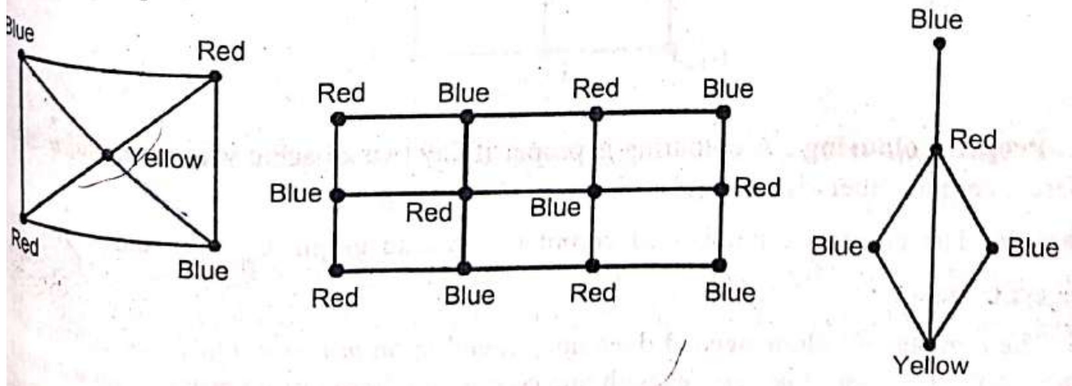
(P.T.U. B.C.A.-I 2007)

Suppose G be a simple graph with n vertices, we are to paint all its vertices such that two adjacent vertices have the same colour.

romatic Number :

The **minimum number** of colours needed to paint all the vertices of the graph such t no two adjacent vertices have the same colour is called **chromatic number** of G and noted by C(G).

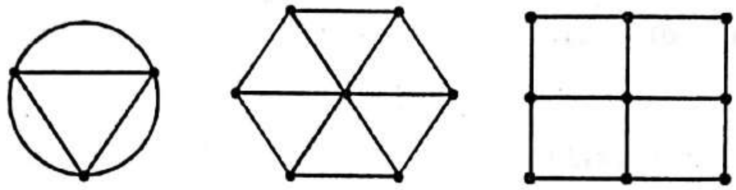
For example.



The above graphs are 3-chromatic, 2-chromic and 3-chromatic receptively.

Remark. A complete graph of n vertices is n - chromatic, as all its vertices are adjacent.

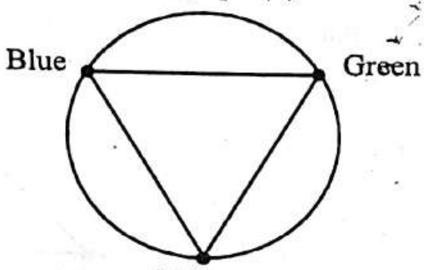
Example. Find the chromatic number for the following graphs.



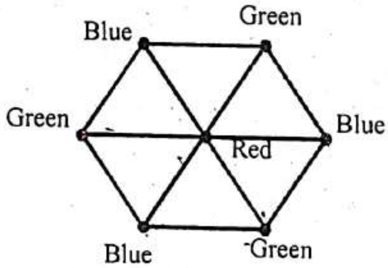
(a) (b) (c)

Sol. Chromatic number of graph G is the minimum number of colour required to paint all the vertices of the graph so that no two adjacent vertices have the same colour.

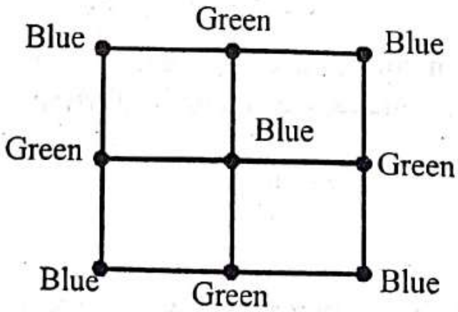
The chromatic colour for the graph (a) is 3 as shown below



The chromatic number for the graph (b) is 3 as shown below.



The chromatic number for the graph (c) is 2 as shown below



Proper Colouring : A colouring is proper if any two adjacent vertices u and v have different colours otherwise it is called improper colouring.

Example. The chromatic number of complete bipartite graph $k_{m, n}$ m and n are +ve integers is two.

Sol. The number of colour needed does not depend upon m and n. Only two colours are needed colour the set of m vertices with one colour and the set of n vertices with a second colour. Edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same colour.

Theorem : Prove that following statements are equivalent for a graph G.

(*a*) G is 2-colorable.

(*b*) G is bipartite

(*c*) G is contains no odd cycle.

Proof : $a \Rightarrow b$.

If G be 2-colorable then graph G has two sets of vertices V_1 and V_2 with different colours say red and blue.

As no vertices of V_1 and V_2 are adjacent

$\therefore$ $\{V_1, V_2\}$ is partition of G.

$\therefore$ G is bipartite.

$b \Rightarrow c$

Let G be bipartite and $\{V_1, V_2\}$ be partition of vertices of G.

Let a vertex $x \in V_1$ and cycle begins at x.

Let it joined to vertex $y \in V_2$ and then to a vertex in V_1 and so on.

When cycle gets completed *i.e.* It returns to x in V_1 then it will be of even length

($\because$ G is bipartite)

$\therefore$ G has no odd cycle.

$c \Rightarrow a$

Let each cycle in G be even let some vertex be coloured while then its adjacent vertex will have different colour black and its adjacent vertex will have colour white because every cycle has even length.

$\therefore$ sequence of vertices of even cycle is WBW ; WBWBW so on.

Only two colours are used to colour the graph.

$\therefore$ G is 2 colourable.

Four Colour Theorem : If G is any planar graph then $C(G) \leq 4$.

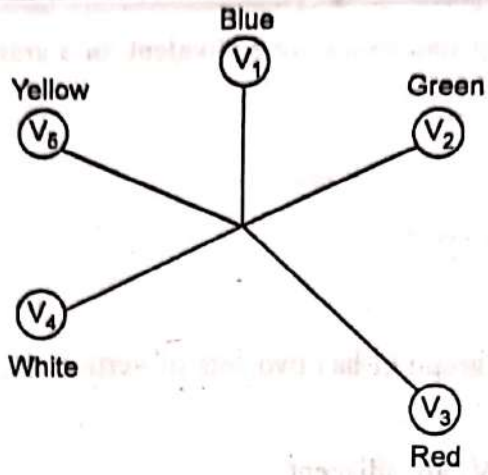
Theorem : Five Colour Theorem : If G is planar graph then

$$C(G) \leq 5.$$

Proof : Basis : A graph with one vertex has chromatic number of one.

(Clearly)

Induction : Let us assume that all planar graphs with $n - 1$ vertices have a chromatic number of 5 or less. Let G be a planar graph with n vertices.



$\therefore$ $\exists$ a vertex V with degree (V) ≤ 5.

Let G-V be the planar obtained by deleting V and all edges that connect V to other vertices in G.

Now by the Induction Hypothesis G-V has a 5-colouring. Let us assume that we use the colours red, white, blue, green and yellow.

(*i*) If deg(V) < 5 then we can produce a 5 colouring of G by selecting a colour *i.e.* not used in colouring the vertices that are connected to V with an edge in G.

(*ii*) If deg(V) = 5, then we apply same technique if the five vertices that are adjacent to V are not coloured differently.

Now we have possible condition is that V_1, V_2, V_3, V_4, V_5 are all connected to V be an edge and they are all coloured differently. Let us assume that they are red coloured, white, blue, yellow, and green.

If V_1 and V_3 are not connected to one another using only blue and red vertices in G-V. If we take all paths that begin at V_1 and go through only blue and red vertices. Then we can not reach V_3. When we exchange the colours of the vertices in these paths, including V_1, we still have a 5 colouring of G-V. As V_1 is now red, we can colour V-blue.

Now we assume that V_1 is connected to V_3 employing only blue and red vertices.

Then a path from V_1 to V_3 by employing Blue and red vertices followed by the edges (V_3, V) and (V,V_1) complete a circuit that either encloses V_2 or encloses V_4 and V_5.

$\therefore$ No path from V_2 to V_4 exist employing only green and white vertices. We can then repeat the same process as in the previous paragraph with V_2 and V_4, which will allow us to colour V-green. Hence G is 5 colourable.

Example 1. Determine the chromatic number of the complete graphs k_6, k_{10} and in general k_n.

Sol. It would take six colours to colour a k_6 graph since every vertex is adjacent to every other vertex, we need different colour for every one. Similarly it takes ten colour to colour the graph k_{10} and n-colour to colour the graph k_n.

$$\therefore \quad c(k_6) = 6$$

$$c(k_{10}) = 10$$

$$c(k_n) = n$$

Example 2. A tree with two or more vertices is 2-chromatic. (P.T.U. B.C.A.-I 2007)

Sol. Let T be any tree. Suppose three arbitrary vertices V_1, V_2, V_3 of tree. If V_1 is connected to V_2 and V_3 then V_2 , V_3 are not conntected. (Otherwise cycle will be formed). If V_1 is cloured Red V_2 is coloured Blue then V_3 can be coloured Blue. This is true for all vertices. So maximum colours needed are 2.

Therefore chromatic number of graph is 2.

Again, if T has only two vertices then result is true.

Example 3. What will be chromatic number of complete graph with n-vertices ? Explain.

(P.T.U. B.C.A.-I 2007)

Sol. Let G be a graph containing n vertices.

Then a vertex V is connected to exactly $n - 1$ vertices.

So all these vertices must have different colours.

$\therefore$ number of colours required = n

i.e. graph is n-chromatic.

SHORTEST PATH PROBLEM

Let G be a connected graph whose edges are assigned unique weights (taken as distances). We want to determine shortest possible path between a pair of vertices. Method for this was developed by Dijkstra and is know as Dijkstra's algorithm.

(P.T.U. B.C.A.-I 2004)

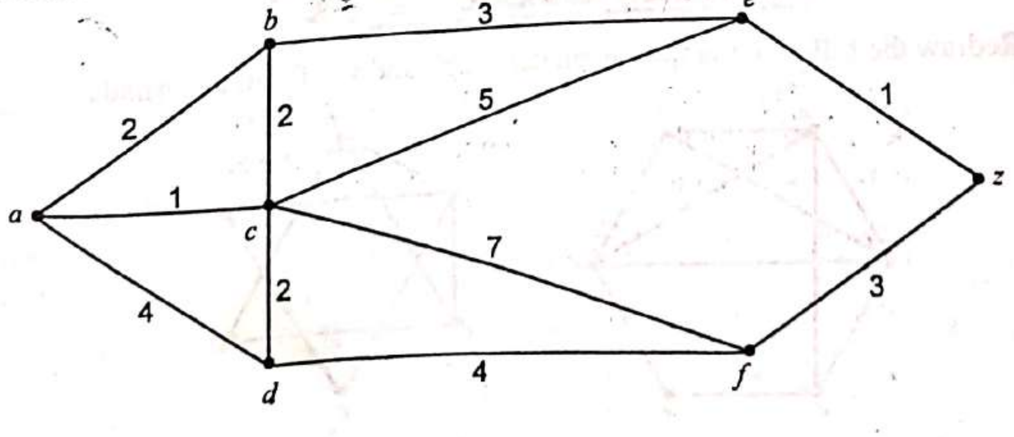
DIJKSTRA'S ALGORITHM

This algorithm maintains a sets of vertices whose shortest path from source is already known. If there is no path from source vertex do any other vertex then it is represented by $+\infty$. All weights must be positive.

Following points are considered.

1. Initially there is no vertex in sets.
2. Include source vertex V_S in S. Determine all the paths from V_S to all other vertices without going through any other vertex.
3. Include that vertex in S which is nearest to V_S find shortest paths to all the vertices through this vertex, give the values.
4. Repeat the process until $(n - 1)$ vertices are not included in S.

Example. Find the shortest path between a and z.



Step I : Include the vertex a in S and determine all the direct paths from a to all other vertices without going through any other vertices.

Distance to all other vertices

	a	b	c	d	e	f	z
a	0	2 (a)	1 (a)	4 (a)	∞	∞	∞

Step II : Include vertex in S, nearer to a and determine shortest path to all the vertices through this vertex. The nearest vertex is c.

Distance to all other vertices

	a	b	c	d	e	f	z
$a, c,$	0	2 (a)	1 (a)	3 (a, c)	6 (a, c)	8(a, c)	∞

Step III : Second nearest vertex is b

Distance to all other vertices

	a	b	c	d	e	f	z
a, c, b	0	2 (a)	1 (a)	3 (a, c)	5 (a, b)	8(a, c)	∞

Step IV : Next vertex is d.

	a	b	c	d	e	f	z
a, c, b, d	0	2 (a)	1 (a)	3 (a, c)	5 (a, b)	7(a, c)	∞

Step V : Next vertex is e

	a	b	c	d	e	f	z
a, c, b, d, e	0	2 (a)	1 (a)	3 (a, c)	5 (a, b)	7(a, c)	6(a, b, e)

Step VI : Next vertex is z .

	a	b	c	d	e	f	z
a, c, b, d, e, z	0	2 (a)	1 (a)	3 (a, c)	5 (a, b)	7(a, c)	6(a, b, e)

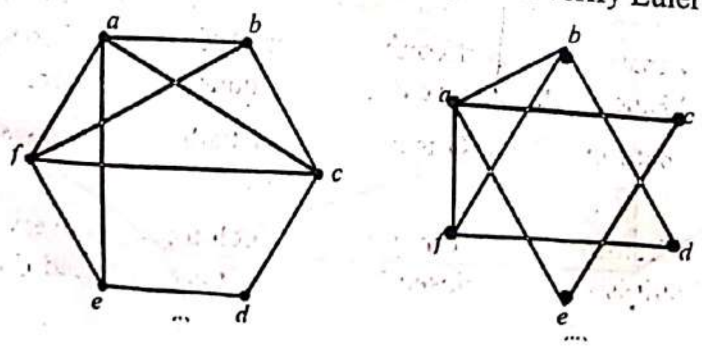
$n - 1$ vertices are included in S.

So minimum path between a and z is 6.

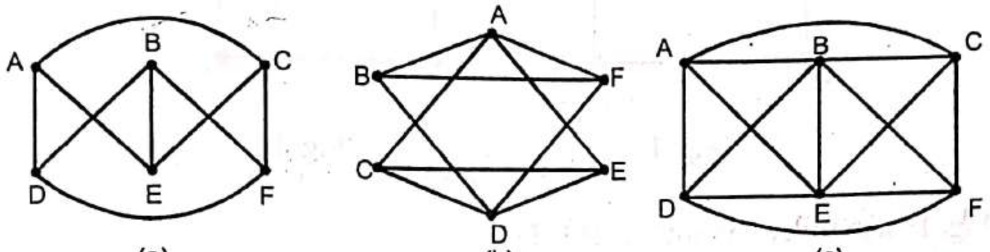
Path is $a \to b \to e \to z$.

EXERCISE 7(c)

1. Redraw the following graphs as planar graph and verify Euler formula :

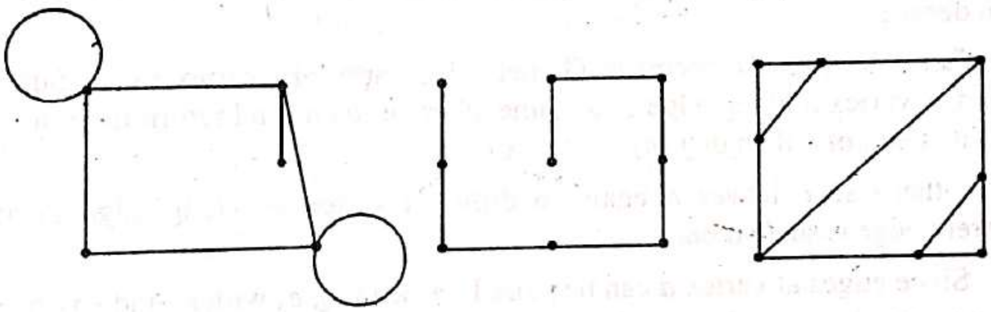


2. Show that $K_{3,3}$ satisfies the inequality $|E| \leq 3|V| - 6$ but is non-planar.
3. Find number of regions defined by connected planar graph with 6 nodes and 10 edges. Draw a simple and non-simple example.
4. How many edges must be drawn in order to obtain a planar graph with 5 nodes that define 7 regions. Draw such a graph.
5. Draw a simple planar graph with 6 nodes and 11 edges.
6. Draw a planar representation of the following graphs :



(a) (b) (c)

7. How many regions must a planar graph define if it has 11 edges and 7 nodes ?
8. How many edges must a planar graph have if it define 5 regions and has 6 nodes?
9. Draw $K_{2,2}$, $K_{1,4}$, $K_{2,3}$ and K_4.
10. Verify Euler's formula



ANSWERS

3. 6 4. 10 7. 6 8. 9

Art-11. Euler Paths and Circuits (Pbi.U. 2009)

In this section, we discuss an important application of graph theory. Suppose we are given a geometrical figure. We want to traverse all the edges of graph by traversing each edge exactly once. This may or may not be possible. The solution of such type of problems was given by mathematician Leonhard Euler (1707-1783). First we discuss the terms Euler Circuit and Euler Path, then we discuss methods to find them.

Euler Path : A simple path in a graph G is called Euler Path if it traverses every edge of graph exactly once.

Euler Circuit : Euler Circuit is a circuit in graph G which traverses every edge of graph exactly once. Euler Circuit is simply a closed Euler path. It is also called Euler line.

Eulerian Graph (P.T.U. B.C.A.-I 2005)

A graph which contain either Euler Path or Euler Circuit is called Eulerian Graph.

Example :

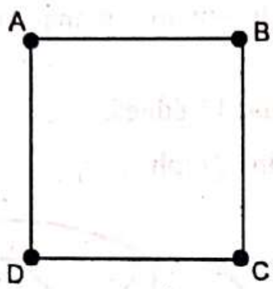


Fig. I

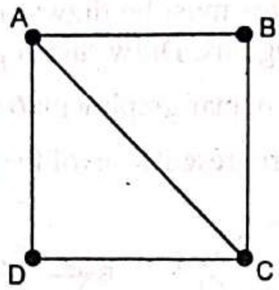


Fig. II

Fig. I has Euler Circuit A B C D A

Fig. II has Euler Path A B C D A C

Remark : Circuit starts and ends at same vertex whereas path starts and ends at different vertices. (P.T.U. B.C.A.-I 2007)

Theorem 1. A connected graph G is a Euler Graph if all the vertices of G are of even degree.

Proof : G is a connected graph having Euler Circuit. We have to show every vertex is of even degree.

Let c be Euler's circuit in G. Let 'a' be arbitrary vertex of G. Suppose circuit c begins at vertex a along edge e_1 to some other vertex a_1 and return to vertex a along edge e_2. If it is end of c then deg $(a) = 2$ (even).

Otherwise c leaves a again to different vertex a_2 along edge e_3 and return on different edge e_4 and so on.

Since edges at vertex a can be paired : e_1 with e_2, e_3 with e_4 and so on. So there must be even number of edges incident on each vertex. $\therefore$ deg(a) must be even. As a is arbitrary vertex in G.

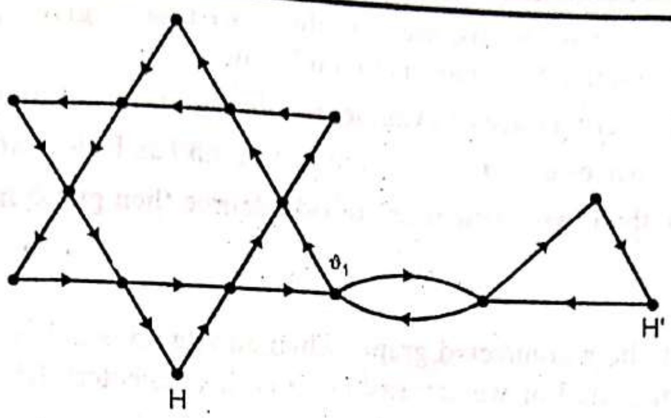
$\therefore$ every vertex in G is of even degree.

Theorem 2. (Converse of theorem 1)

If G is a connected graph and every vertex of G has even degree, then prove that G has a Euler Circuit. (Pbi.U. B.C.A.II Apirl 2008)

Proof : Given every vertex of the graph G is of even degree. To show that G is a Euler graph we show G has Euler circuit.

We construct a walk starting with any arbitrary vertex v and going through all the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since each vertex is of even degree, therefore we can exit from every vertex we entered. Now the tracing cannot stop at any vertex but v. Since v is also of even degree we shall reach at v again. When the tracing comes to an end if this closed walk W that we have just traced, contains all the edges of G, then it is a Euler circuit, otherwise remove from G these edges and obtain a subgraph H of G.



Let H′ be the subgraph of G formed by the remaining edges. Since the degree of each vertices of H or H′ is even and the graph G is given to be connected. Therefore H must touch H′ at least at one vertex (say) at v_1. Now starting from v_1 we can again construct a new walk in H′. Since all the vertices of H′ are also of even degree this walk in H′ must terminate at the vertex v_1. Now if we have traversed all edges then Euler circuit has been obtained and if not we can repeat the above process again. Continue in this way combined the subgraph H, H′,....., formed, we get a closed walk that transverse all the edges of the graph G and thus get a Euler circuit in G.

Hence G is an Euler graph.

Theorem 3. If a graph has Euler Path then it has either no vertex of odd degree or two vertices of odd degree.

Proof : Let graph G has Euler Path starting at vertex a and ending at vertex b. If a and b are same then path is a Euler circuit, so by theorem 1 all vertices are of even degree.

Suppose $a \neq b$. Draw a new edge e_1 joining a and b. New graph is $G \cup \{e_1\}$. This new graph has Euler circuit obtained by Euler path plus new edge e_1. So all the vertices are of even degree.

Now remove the edge e_1. Then graph $G \cup \{e_1\} - \{e_1\} = G$ has only two vertices a and b of odd degree.

Hence the proof.

Theorem 4. (Converse of Theorem 3) : If G is a connected graph having either zero or two vertices of odd degree, then G has a Euler path.

Proof : G be a connected graph. If G has zero vertices of odd degree *i.e.*, all the vertices of G are of even degree then by Theorem 2, G posseses an Euler circuit.

Now, suppose G has two vertices of odd degree say v_1 and v_2. We construct an Euler Path by starting at one of the two vertices v_1 or v_2 and going through the edges in such a way that no edge will be traced more than once. For a vertex of even degree, whenever we enter the vertex through an edge, we can leave the vertex through another edge that has not been traced before. Therefore, when the construction eventually comes to an end, we must have reached the other vertex of odd degree. If all the edges in the graph were traced, we would get an Euler Path other wise remove those edges that has been traced out and obtain a subgraph formed by the remaining edges. The degree of vertices of this subgraph are all even. Again by theorem 2, this subgraph has an Euler Circuit. Since the original graph is connected. So there must a path between the two subgraph and thus we obtain a path that contain all the edges of G exactly once. Hence, we get an Euler Path.

Remarks : From above theorems, we can check weather a given graph contains Euler Circuit, Euler Path or none. Summary is given below :

(*a*) If all the vertices are of even degree then graph has Euler Circuit.

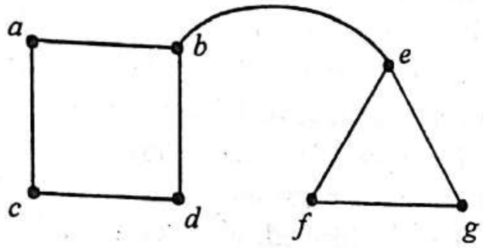
(*b*) If two vertices are of odd degree, the graph has Euler path but no circuit.

(*c*) If more than two vertices are of odd degree then graph has neither Euler Path nor Euler Circuit.

Definition

Bridge : Let G be a connected graph. Then an edge e is called bridge if by deleting e, G becomes disconnected or we can say G$-e$ is disconnected. Bridge is also called cut edge.

Example :



In figure edge '*be*' is a bridge.

Art-12. Fleury's Algorithm for Euler Circuit

Let G be a connected graph with each vertex of even degree. Choose any vertex, say v_1 of G to start the circuit.

Step 1 : Select an edge e_1 from v_1 such that e_1 must not be a Bridge. Let this edge be (V_1, V_2). Then Path is specified by P : e_1. Remove e_1 from graph.

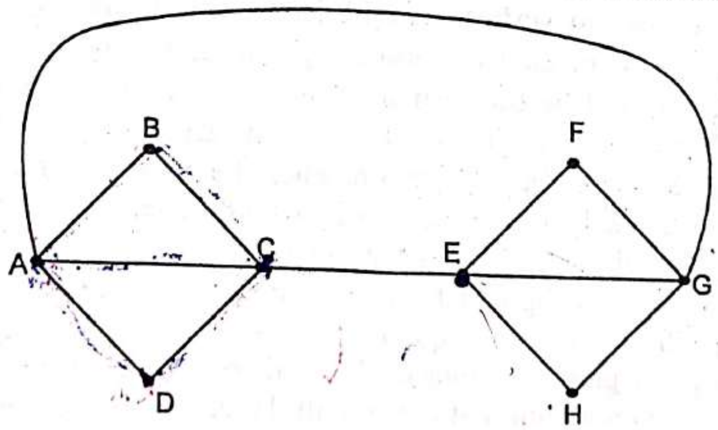
Step 2 : Now check all the edges from V_2. Again choose an edge from v_2 such that this edge must not be a bridge. Also remove this edge from G.

Step 3 : Repeat step 2 until no edge remains in Graph. P will give us required Euler circuit.

Art-13. Fleury's Algorithm for Euler Path

Let G be a connected graph with exactly two vertices of odd degree. For Euler Path, we have to start from a vertex of odd degree and path will end at other vertex of odd degree. The rest of steps are same as that of Euler Circuit.

Example 1. Use Fleury's algorithm to construct Euler Path or Circuit in following graph.



Sol. First we check degree of each vertex

deg (A) = 4 deg (B) = 2 deg (C) = 4 deg (D) = 2

deg (E) = 4 deg (F) = 2 deg (G) = 4 deg (H) = 2

Since every vertex has even degree, So Euler circuit exist. We can begin from anywhere. Let us start from vertex A.

Detailed steps are shown below :

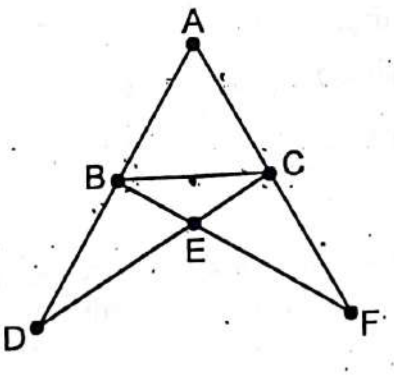
Sr.No.	Current Path (P)	Next Edge	Description
1.	A	{A, B}	No edge is a cut edge choose any.
2.	A,B	{B, C}	Only one edge from B remains.
3.	A,B,C	{C, D}	No edge from C is a cut edge. So choose any.
4.	A,B,C,D	{D, A}	Only one edge from D remains.
5.	A,B,C,D,A	{A, C}	Neither AG or AC is cut edge. Chose any.
6.	A,B,C,D,A,C	{C, E}	Only one edge from C remains.
7.	A,B,C,D,A,C,E	{E, F}	No edge from E is cut edge So choose any.
8.	A,B,C,D,A,C,E,F	{F, G}	Only one edge from F remains.
9.	A,B,C,D,A,C,E,F,G	{G, H}	GA is cut edge so choose GH or GE.
10.	A,B,C,D,A,C,E,F,G,H	{H, E}	Only one edge from H remains
11.	A,B,C,D,A,C,E,F,G,H,E	{E, G}	Only one edge from E remains
12.	A,B,C,D,A,C,E,F,G,H,E,G	{G, A}	Only one edge from G remains
13.	A,B,C,D,A,C,E,F,G,H,E,G,A	No edge	

The required Euler circuit is

A, B, C, D, A, C, E, F, G, H, E, G, A.

Example 2. Apply Fleury's algorithm to construct an Euler circuit for following graph.

(B.C.A. April 2006)



Sol. deg (A) = 2 deg (B) = 4 deg (C) = 4

deg (D) = 2 deg (E) = 4 deg (F) = 2

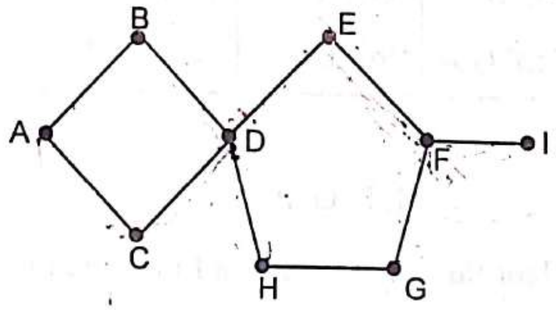
Since every vertex of Graph has even degree and graph is connected so Euler circuit exists. We can begin from anywhere. Let us begin from A. Detailed steps are :

Sr. No.	Current Path (P)	Next Edge	Description
1.	A	{A, B}	Neither AB nor AC is cut edge choose any.
2.	A,B	{B, D}	No edge from D is cut edge choose any.
3.	A,B,D	{D, E}	Only one edge from D remains.
4.	A,B,D,E	{E, B}	No edge from E is cut edge choose any.
5.	A,B,D,E,B	{B, C}	Only one edge from B remains.
6.	A,B,D,E,B,C	{C, E}	CA is cut edge so choose CE or CF.
7.	A,B,D,E,B,C,E	{E, F}	Only one edge from E remain.
8.	A,B,D,E,B,C,E,F	{F, C}	Only one edge from F remains.
9.	A,B,D,E,B,C,E,F,C	{C, A}	Only one edge from C remains.
10.	A,B,D,E,B,C,E,F,C,A	No edge	

so required Euler circuit is

A, B, D, E, B, C, E, F, C, A.

Example 3. Use Fleury's algorithm to construct Euler Path or Circuit in following graph.



Sol. First we check degree of all vertices

deg (A) = 2 deg (B) = 2 deg (C) = 2 deg (D) = 4

deg (E) = 2 deg (F) = 3 deg (G) = 2 deg (H) = 2

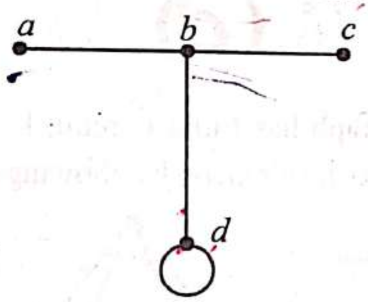
deg (I) = 1

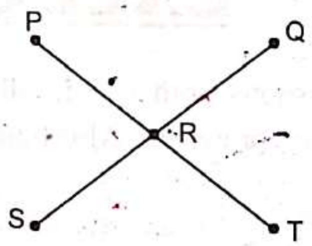
Two vertices F and I have odd degree and graph is connected. So we can find Euler Path (no circuit). Path must begin form F(or I) and ends at I (or F). Steps shown below :

Sr. No.	Current Path (P)	Next Edge	Description
1.	F	{F, E}	FI is cut edge choose FE or FG.
2.	F, E	{E, D}	Only one edge from E remains.
3.	F,E,D	{D, B}	No edge from D is a cut edge choose any.
4.	F,E,D,B	{B, A}	Only one edge from B remains.
5.	F,E,D,B,A	{A, C}	Only one edge from A remains.
6.	F,E,D,B,A,C	{C, D}	Only one edge from C remains.
7.	F,E,D,B,A,C,D	{D, H}	Only one edge from D remains.
8.	F,E,D,B,A,C,D,H	{H, G}	Only one edge from H remains.
9.	F,E,D,B,A,C,D,H,G	{G, F}	Only one edge from G remains.
10.	F,E,D,B,A,C,D,H,G,F	{F, I}	Only one edge from F remains
11.	F,E,D,B,A,C,D,H,G,F,I	No edge	

So required Euler Path is F, E, D, B, A, C, D, H, G, F, I.

Example 4. Find Euler Path or circuit (if exists)

(*i*) 

(*ii*) 

Sol. (*i*) deg(a) = 1

deg(b) = 3

deg(c) = 1

deg(d) = 3

In this graph four vertices are of odd degree. So Neither Euler Path, nor Euler Circuit exists.

(*ii*) deg(P) = 1

deg(Q) = 1

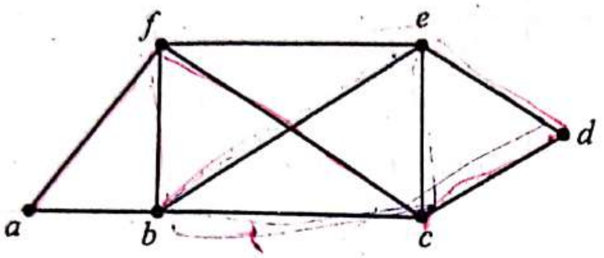
deg(S) = 1

deg(T) = 1

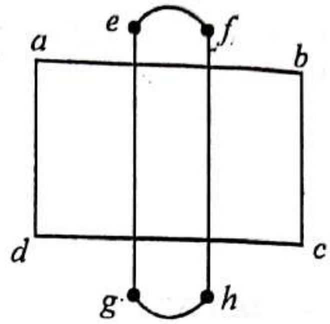
deg(R) = 4

Again four vertices are of odd degree, so neither Euler Path, nor Euler Circuit exists.

Example 5. Which of the following graphs are transversable :



I



II

(P.T.U. B.C.A.-I 2004)

Sol. Graph I is traversable as it contain Euler circuit. Euler circuit is shown below :

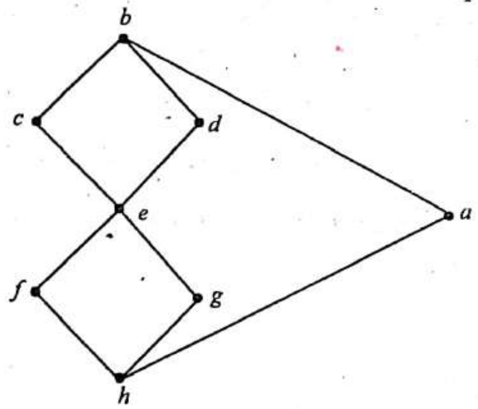
$$a \to f \to c \to d \to e \to f \to b \to e \to c \to b \to a$$

Graph II is not traversable because it is not connected. It contain two components ($a\ b\ c\ d$) and ($e\ f\ g\ h$).

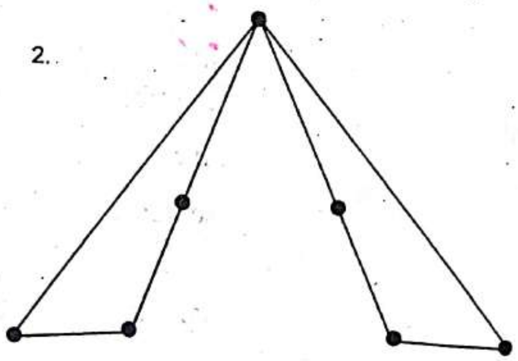
EXERCISE 7 (*d*)

In Questions from 1 to 5, tell whether graph has Euler Circuit, Euler Path but no circuit or neither. Also find path or circuit (if exists) by showing all steps.

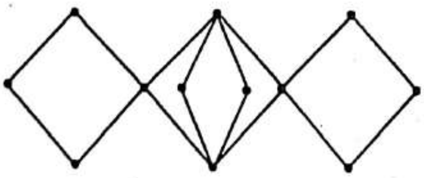
1.



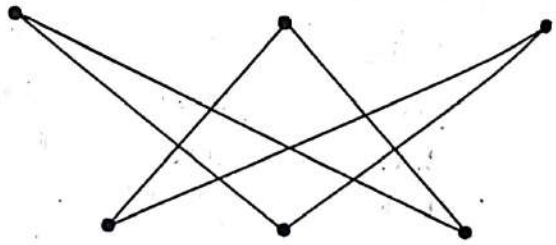
2.



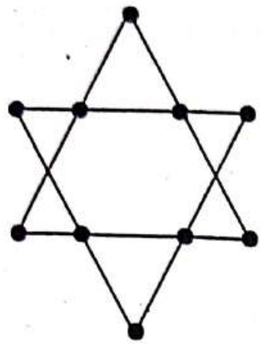
3.



4.



5.



6. Does Complete graph K_n have Euler Circuit ? Euler Path ? Justify your answer.
7. Draw a graph that has exactly one Euler Circuit ? Characterise all such graphs ?
8. Show that an undirected graph possesses an Eulerian circuit if and only if it is connected and its vertices are all of even degree.

(Pbi.U. M.Sc.-I.T. 2010, 2011)

rt-14. Hamiltonian Paths and Circuits (Pbi.U. B.C.A. 2009)

In previous section, we have traversed all the edges of graph exactly once (vertices ere repeated in some cases). Now we discuss another application of graph theory in hich we have to traverse all the vertices of graph exactly once. This theory was roposed by Irish mathematician **William Hamilton** (1805-1865).

Hamiltonian Path : (P.T.U. B.C.A.-I 2007)

A Hamiltonian Path in a connected graph is a path which contains each vertex of raph exactly once.

Hamiltonian Circuit : A Hamiltonian circuit is a circuit that contains each vertex f graph exactly once except for the first vertex, which is also the last.

Hamiltonain Graph : (P.T.U. B.C.A.-I 2005)

A graph which possesses either Hamiltonian circuit or Hamiltonian path is called a Hamiltonian graph.

Remarks : I In Hamiltonian circuit or path we have to visit all the vertices. There may be some unvisited edges.

II. If G has n vertices, then Hamiltonian circuit will contain n edges where as Hamiltonian Path will contain $n-1$ edges.

III. There may be more than one Hamiltonian path and circuit in a graph.

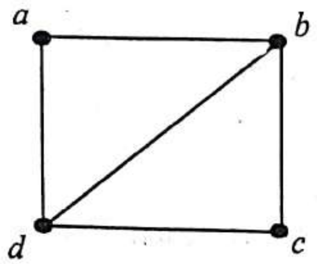
Theorems on Hamiltonian Graphs (Without proof)

Theorem I. Let G be a connected simple graph with n vertices, $n > 2$. Let U and V are any two non-adjacent vertices in G and deg (U) + deg(V) $\geq n$, then G is Hamiltonian.

Theorem II. Let G be a connected simple graph with n vertices, $n > 2$. If deg(V) $\geq \frac{n}{2}$ for every V $\in$ G then G is Hamiltonian.

Theorem III. Let m be the number of edges in Graph G. If $m \geq \frac{1}{2}(n^2 - 3n + 2)$ where n is number of vertices of G then G is Hamiltonian.

Consider the graph shown below



Here $n = 4$

Graph satisfy all above theorems.

Here non-adjacent vertices are a and c

$$\deg(a) = 2 \quad \deg(c) = 2$$

$$\deg(a) + \deg(c) = 4(n)$$

so theorem I is true.

also $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 2$, $\deg(d) = 3$

so all vertices have degrees ≥ 2 which is $\frac{n}{2}$

$\therefore$ theorem II also true.

Similarly for theorem III we have $m = 5$ and $n = 4$

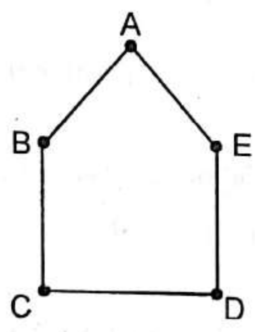
so $\frac{1}{2}(n^2 - 3n + 2) = \frac{1}{2}(16 - 12 + 2) = 3$

$\therefore \quad m \geq \frac{1}{2}(n^2 - 3n + 2)$

$\therefore$ graph is Hamiltonian and Hamiltonian circuit is a, b, c, d, a.

Note : The converse of these theorems is not true. We have some graphs that do not satisfy these theorems but still they are Hamiltonian.

For example in graph



$$\deg(A) + \deg(C) = 2 + 2 = 4$$

which is not greater than or equal to $n(5)$. Still graph is Hamiltonian.

and circuit is (A, B, C, D, E, A).

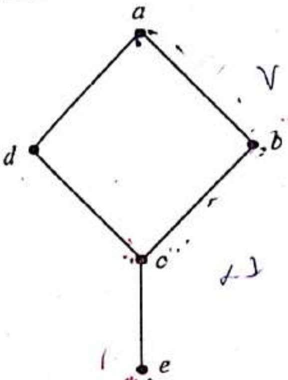
There are no hard and fast rules for constructing Hamiltonian paths and circuits. However, to find Hamiltonian paths and circuits following points should be kept in mind :

1. If Graph G with n vertices has less than n edges then no Hamiltonian circuit is possible.

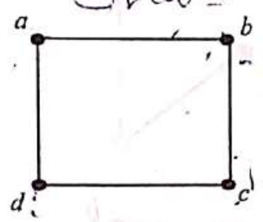
2. In Hamiltonian circuit, any vertex can't have degree more than 2.

3. Whenever we enter and leaves a vertex we delete all other edges incident on that vertex.

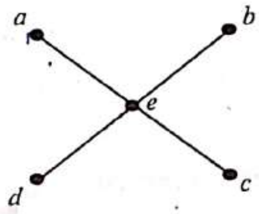
Example 1. Consider the following graphs



I



II



III

Graph in Fig. I has Hamiltonian path e, c, d, a, b but no Hamiltonian circuit.

Graph in Fig. II has both Hamiltonian path and circuit path is a, b, c, d and circuit is a, b, c, d, a.

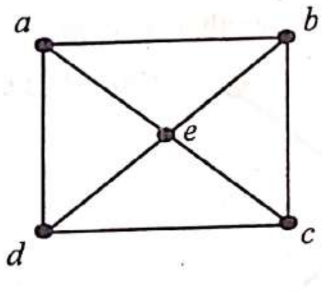
Graph in Fig. III has neither Hamiltonian path nor Hamiltonian circuit.

Remarks : I. A graph can have both Hamiltonian path as well as Hamiltonian circuit.

II. A graph can't have both Euler Path and Euler Circuits.

III. If a graph has Hamiltonian circuit then it also has Hamiltonian path converse is not true.

Example 2. Does the graph G given below have Hamiltonian circuit ? Justify your answer. (Pbi.U. B.C.A. April 2006, Sept. 2006)



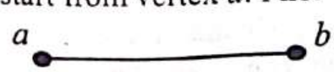
Sol. Number of vertices $(n) = 5$

Number of edges = 8

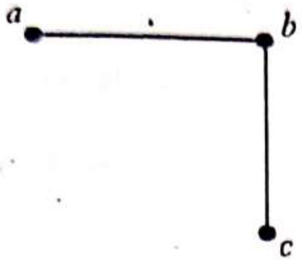
To construct Hamiltonian circuit we have to include 5 edges.

Let us start from vertex a. First we insert edge ab.

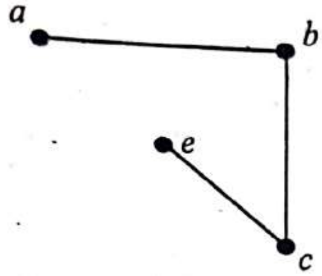
(*i*)



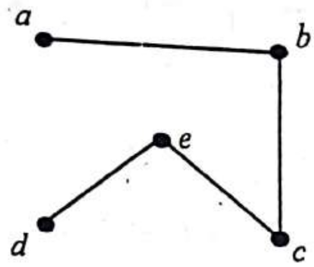
(*ii*) Insert *bc*



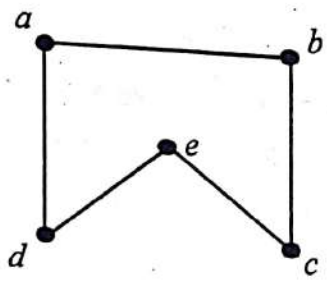
(*iii*) Insert *ce*



(*iv*) Insert *ed*



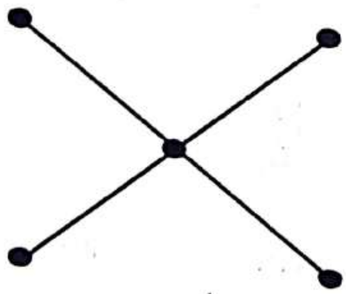
(*v*) Insert *da*



Required Hamiltonian circuit is *a, b, c, e, d, a*.

Example 3. Does the graph shown below has Hamiltonian circuit ? Justify your answer.

(Pbi. U. B.C.A. Sept. 2007, 2008)

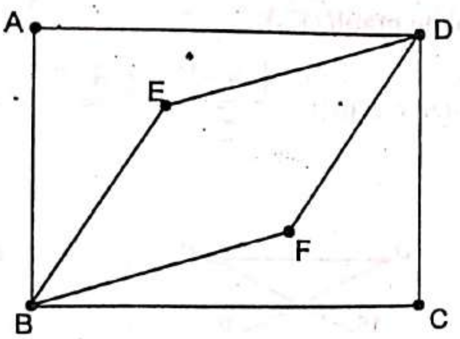


Sol. Number of vertices $(n) = 5$

Number of edges = 4

We know, Hamiltonian circuit of n vertices must contain n edges. But in given graph we have only $n-1$ edges. So Hamiltonian circuit is not possible.

Example 4. Is the graph given below a Hamiltonian ? Justify your answer. (Pbi. U. B.C.A. April 2007)



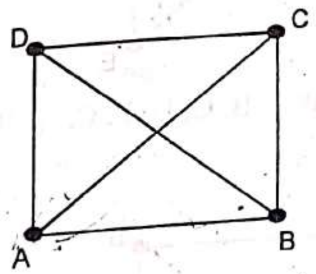
Sol. Number of vertices (n) = 6 number of edges = 8. To construct Hamiltonian circuit we have to take 6 edges from 8 edges with condition that every vertex will have degree 2. We can start from anywhere. Suppose we start from A. From A we can go to either D or B. Let us go to D. Then from D if we go to C then degree of D becomes 2. So we have to delete DE and DF. But by deleting these two edges, degrees of E and F become 1. So it will not be possible to include E and F in circuit. (In circuit a vertex must have degree 2). So Hamiltonian circuit will not exist.

Similarly we can show Hamiltonian path will also not exist. As we move from A to D, D to C, C to B. After B if we go to E or F then there is no way to go ahead. Other side if we go to A then E and F becomes isolate. So Hamiltonian path will also not exist.

Hamiltonian Circuit in Complete Graph :

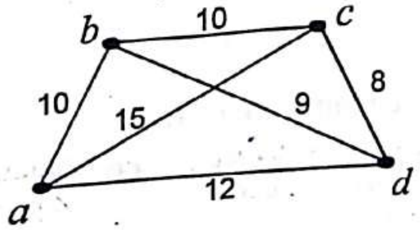
Let K_n be complete graph of n vertices, $n \geq 3$. Then K_n will definitely contain a Hamiltonian circuit. Infact K_n will contain $\frac{|\underline{n-1}}{2}$ Hamiltonian circuits.

For example : Consider the graph K_4.



Then by above result, K_4 contains $\frac{|\underline{4-1}}{2} = 3$. Hamiltonian circuits which are ABCDA, ABDCA and ADBCA.

Example 5. Find Hamiltonian circuit of minimal weight for the graph show below.

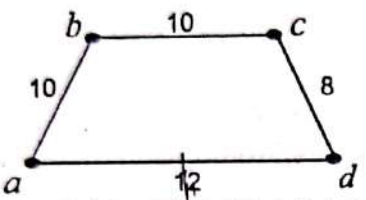


Sol. Number of vertices (n) = 4

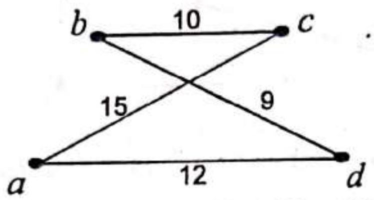
Graph shown above is complete graph (K_4)

So total number of Hamiltonian circuits $= \frac{|\underline{n-1}}{2} = \frac{|\underline{4-1}}{2} = 3$

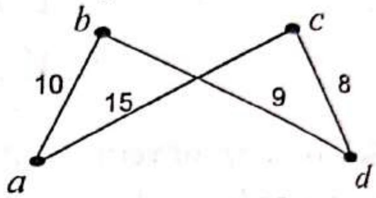
These 3 circuits are



weight = 10 + 10 + 8 + 12
= 40



weight = 10 + 9 + 15 + 12
= 46



weight = 10 + 8 + 15 + 9
= 42

So circuit of minimum weight is a, b, c, d, a.

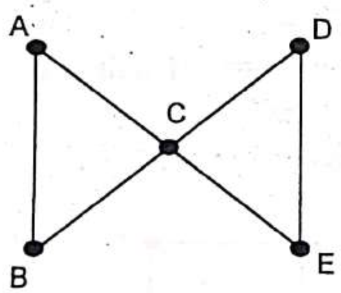
Example 6. Give an example of graph that has

(i) Euler circuit but not Hamiltonian circuit. (Pbi. U. B.C.A. Sept. 2007)

(ii) Hamiltonian Circuit but not Euler circuit.

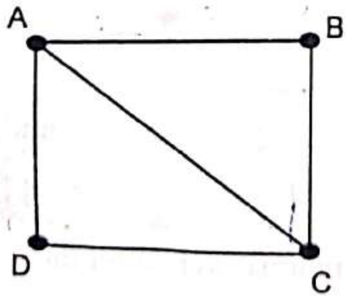
Sol. Euler circuit contains all the edges of graph exactly once where as Hamiltonian circuit is a circuit which contains all the vertices of graph exactly once. (except for first vertex which is also last)

(i) Consider the graph



This graph has a Euler circuit A, B, C, D, E, C, A. But no Hamiltonian circuit.

(ii) Consider the graph



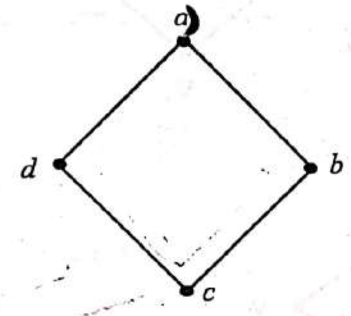
This graph has Hamiltonian circuit A, B, C, D, A

But no Euler circuit (as deg (A) = 3, which is odd)

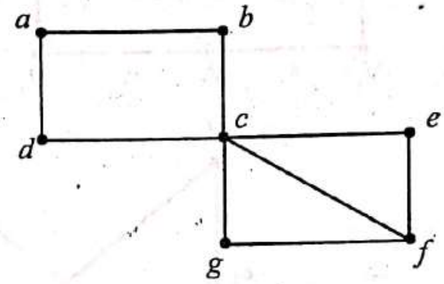
EXERCISE 7 (*e*)

In following graphs (1−4) determine whether the graph shown has a Hamiltonian circuit, Hamiltonian Path or neither. Justify you answer. Also give Hamiltonian path or circuit or both (if exists)

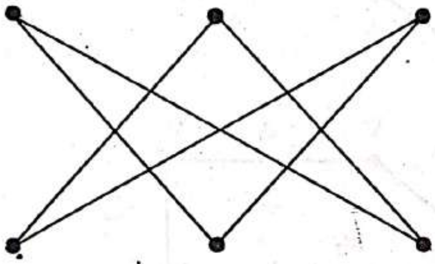
1.



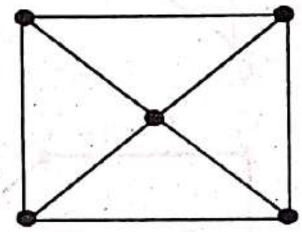
2.



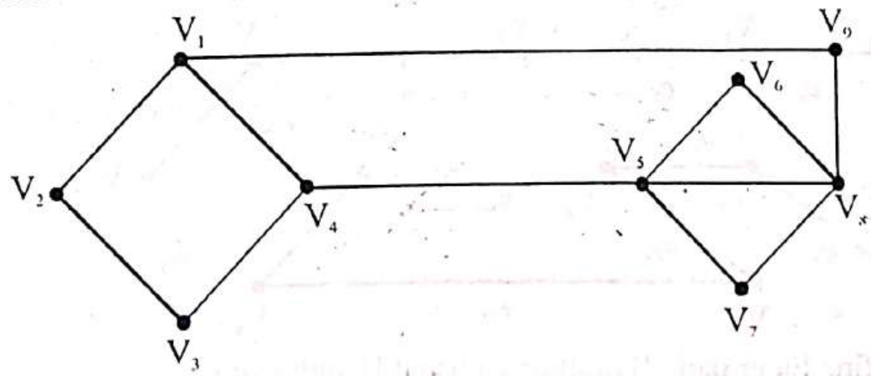
3.



4.

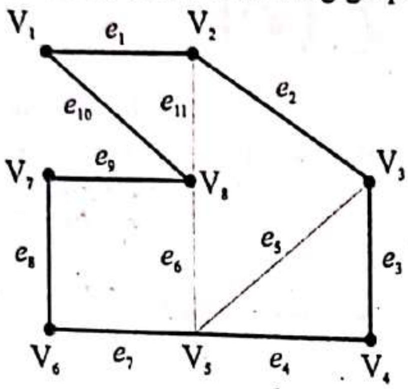


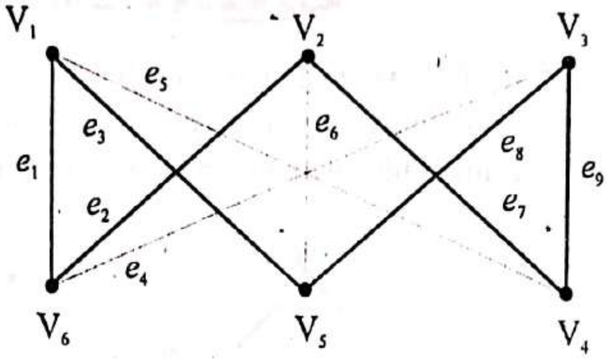
5. How many edges must a Hamiltonian cycle in K_n contains ?
6. How many Hamiltonian cycles does K_n have ?
7. Give an example of connected graph that has
   (*a*) Neither Euler circuit nor Hamiltonian circuit.
   (*b*) An Euler circuit but no Hamiltonian cycle.
   (*c*) A Hamiltonian cycle but no Euler circuit.
   (*d*) Both Hamiltonian cycle and Euler circuit. (Pbi.U. M.Sc.I.T. 2010)
8. Is it possible to trace the figure given below without lifting the pencil ? Give reasons.

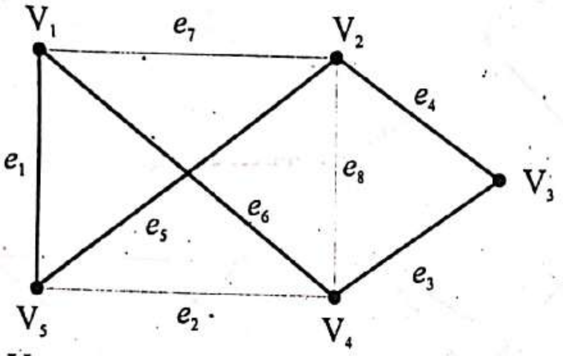


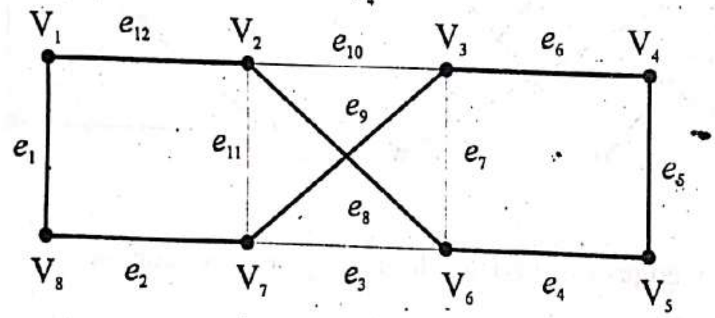
(Pbi. U. B.C.A. 2008)

9. Which of the following graph have a Hamiltonian circuit ?

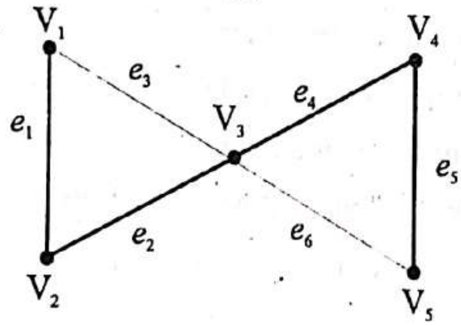


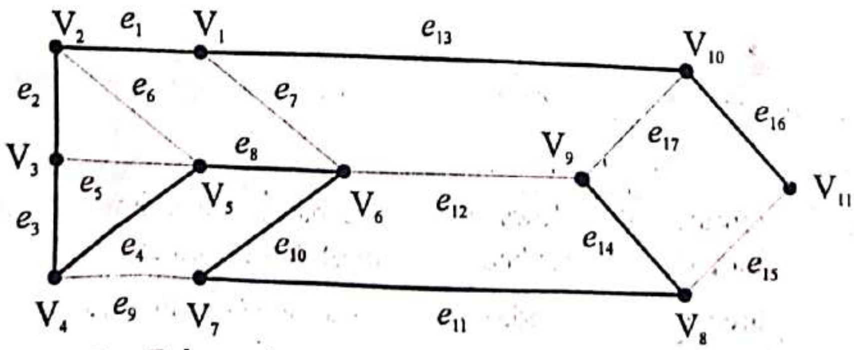






10. Find Hamiltonian paths for each of the following graphs and show that no Hamiltonian circuit exits.





11. Define Euler path, Hamiltion path and Hamilton circuit.

(Pbi. U. B.C.A. 2011)