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LOGIC AND QUANTIFIERS

Art-1. Introduction

Logic is the science of reasoning. It plays a vital role in any study involving reasoning. In fact it is a process by which we arrive at a conclusion from known statements with the use of laws of logic. The axiomatic approach to logic was first propounded by George Boole an Englishman. That is why logic relevant to mathematics *i.e.*, mathematical logic is called Boolean logic.

Art-2. Definitions

Sentence. It is sensible combination of words.

Example. Number of positive integers in infinite.

Propositions. (P.T.U. B.C.A. I 2007)

A sentence or proposition is an expression that is either true or false, but not both. The sentence "$3 + 3 = 6$" is true, while the sentence "π is rational" is false.

The difference between an ordinary sentence and a logical statement is that whereas it is not possible to say about truth or otherwise of an ordinary sentence, it is an essential requirement for a logical statement. Now we consider some sentences and see whether they are logical statements, true or false.

(*i*) "$3 + 3 = 8$." This is a statement, but is a false statement. Its false value will denoted by the letter F or 0.

(*ii*) "Sun is a heavenly body." This is a statement and is a true statement. Its truth value is denoted by the letter T or 1.

(*iii*) "Why are you going to Bombay ?" This is not a statement as the sentence is not declarative.

(*iv*) "May God bless you with happiness !" This is not a statement because of the exclamation mark.

(*v*) "$(x-1)^2 = x^2 - 2x + 1$." This is a statement and its truth value is T or 1.

It should be noted that mathematical identity is always a statement.

(*vi*) Consider the sentence : $x + 5 = 10$. The truth of the sentence is open till we are told what x stands for. Such a sentence is called an open sentence. An open sentence is, thus, not a statement.

Art-3. Logical Connectives and Compound Statements

Any statement whose truth or otherwise does not explicitly depend on another statement is said to be simple. For instance, 8 is an even number.

The set of real numbers is infinite are simple statements.

A compound statement is a combination of two or more simple statements.

The pharases or words which connect two simple statements are called *sentential connectives, logical connectives or simply connectives*. Some of the connectives are "and", "or", "not", "if then", "if and only if".

When simple statements are combined to make compound statements, then simple statements are called **components**. Our problem is to determine the truth value of a compound statement from the truth values of their components.

Art-4. Truth Tables

It is a table giving the truth values of a compound statement. It has a number of columns (vertical lines), and rows (horizontal lines). The number of columns depends upon the number of simple statements and how involved are their relationships. The number of rows in a truth table depends only upon the number of simple statements. *In case of n statements there are* 2^n *rows*. The truth tables are very helpful in finding out the validity of a report.

Art-5. Logical Operations

Let p and q denote sentences.

CONJUNCTION

We say that the sentence $p \wedge q$ (p and q) is true if the two sentences p, q are both true and is false otherwise.

Example : The sentence "3 + 3 = 6" and "3 + 4 = 7" is true.

Example : The sentence "3 + 3 = 6 and π is rational" is false.

Observe that in giving the truth table of $p \wedge q$ we need to look at four possible cases. This follows from the fact that each of p and q can be true or false. If we use the letter T to denote "true" and the letter "F" to denote "false", then the above definition can be summarised in the following "truth table"

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \wedge q$

For example, consider the two statements :

He is practical. He is sensitive.

The conjuction is "He is practical and sensitive."

The conjuction will be true when both the statements will be true and false even if one of the components is false.

EXAMPLES

1. Let $p : 5 + 7 = 12$

q : 2 is a prime number.

$\therefore$ $p \wedge q : 5 + 7 = 12$ and 2 is prime number.

Now p is true and q is true

$\therefore$ $p \wedge q$ is true.

2. Let p : Every even number is divisible by 2

q : 12 is an odd number.

$\therefore$ $p \wedge q$: Every even number is divisible by 2 and 12 is an odd number.

Now p is true and q is false.

$\therefore$ $p \wedge q$ is false.

DISJUNCTION

We say that the sentence $p \vee q$ (p or q) is true if at least one of two sentences p, q is true, and is false otherwise.

Example : The sentences "3 + 3 = 8 or 2 + 3 = 6" is false.

Example : The sentence "3 + 3 = 6 or π is rational" is true.

The truth values of $p \vee q$ are given in the truth table shown in the following table

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p \vee q$

For example, consider the two statements:

p : There is something wrong with the teacher

q : There is something wrong with the student.

Then $p \vee q$: There is something wrong with the teacher or with the student.

This 'or' is inclusive or, that is, there may be something wrong either with the teacher or with the student or with both.

The disjunction will be false when both the components are false.

Consider another two statements :

p : I shall watch the game on television

q : I shall go to college

$p \vee q$: I shall watch the game on television or go to college.

This is exclusive 'or', both p and q cannot happen to gather.

Exclusive OR or X-OR has the symbol $\overline{\vee}$.

Rule : $p \overline{\vee} q$ is true when either p or q is true, but not both.

Truth Table for V

p		q		$p \vee q$
T		T		T
T		F		T
F		T		T
F		F		F

Truth Table for $\overline{\vee}$

p	q	$p \overline{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

EXAMPLES

1. Let $p : 5 < 12$

$q : 8 + 3 = 12$

$\therefore$ $p \vee q : 5 < 12$ or $8 + 3 = 12$

Here p is true and q is false.

$\therefore$ $p \vee q$ is true.

2. Let p : Every even integer is prime

$q : 5 < 3$

$\therefore$ $p \vee q$: Every even integer is prime or $5 < 3$.

Now p is false and q is also false.

$\therefore$ $p \vee q$ is false.

3. Let $p : 3 + 5 = 8$

$q : 1 + 6 = 9$

$\therefore$ $p \vee q = (3 + 5 = 8) \vee (1 + 6 = 9)$

Now p is true and q is false.

$\therefore$ $p \vee q$ is true.

NEGATION

We say that the sentence $\sim p$ (not p) is true if the sentence p is false and is false if the sentence p is true.

Example : The negation of the sentence "Taj is in Delhi" is the sentence "Taj is not in Delhi".

The truth value of $\sim p$ may be defined equivalently by the following table :

Truth Table

p	$\sim p$
T	F
F	T

"not p"

Thus the truth value of the negation of p is always the opposite of the truth value of p.

To every statement, there corresponds a statement which is its negation. Negation refers to contradiction and not to a contrary statement. We should be very careful while writing the negation of the given statement. The best way is to put in the word "not" at the proper place or to put the phrase. "It is not the case that" in the beginning.

For example, if p stands for "He is a good student." Negation of p, denoted by $\sim p$ or $\rceil p$ is either "He is not a good student" or "It is not the case that he is a good student." We cannot say that "He is a bad student" is the negation of p.

TAUTOLOGIES AND CONTRADICTIONS

A tautology is a sentence which is true on logical ground only.

Example : The sentence $p \vee \sim p$ is a tautology.

Truth Table

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$p \vee \sim p$

A ***contradiction*** is a sentence which is false on logical ground only.

Example : The sentence $p \wedge \sim p$ is a contradiction.

Truth Table

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$p \wedge \sim p$

CONDITIONAL STATEMENT

CONDITIONAL

We say that the sentence $p \rightarrow q$ (if p, then q) is true if the sentence p is false or if the sentence q is true or both and is false otherwise. In other words the sentence p is true and the sentence q is false. The statement p is called the hypothesis and the statement q is called the conclusion. The connective if .... then is denoted by the symbol $\rightarrow$.

The conditional $p \rightarrow q$ is frequently read "p **implies** q" or "p **only if** q". The truth values of $p \rightarrow q$ are defined by the following table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$

CONVERSE, INVERSE AND CONTRAPOSTIVE

If $p \rightarrow q$ is a conditional statement, then $q \rightarrow p$ is called its ***converse***, $\sim p \rightarrow \sim q$ is called its inverse and $\sim q \rightarrow \sim p$ is called its ***contrapositive***.

p	q	$p \rightarrow q$	Converse of $p \rightarrow q$	inverse of $p \rightarrow q$	Contrapositive of $p \rightarrow q$
			$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

$p \rightarrow q$ is logically equivalent to $\sim p \vee q$.

i.e.

$$p \rightarrow q \equiv \sim p \vee q$$

p	p	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$\sim p \vee q$

EXAMPLES

1. Let $p : 2 + 3 = 6$

   $q : 2 + 2 = 4$

   $p \rightarrow q$: If $2 + 3 = 6$ then $2 + 2 = 4$

   Now p is false and q is true

   $\therefore$ $p \rightarrow q$ is true.

2. Let p : a is a vowel

   $q : 1 + 1 = 3$

   $\therefore$ $p \rightarrow q$: If a is a vowel then $1 + 1 = 3$

   Now p is true and q is false

   $\therefore$ $p \rightarrow q$ is false.

3. Let p : a is a vowel

   $q : 1 + 1 = 2$

   $p \rightarrow q$: If a is a vowel then $1 + 1 = 2$

   Now p is true and q is true

   $\therefore$ $p \rightarrow q$ is true.

4. Let p : Some odd numbers are divisible by 2

q : All prime numbers are odd

$p \to q$: If some odd numbers are divisible by 2 then all prime numbers are odd

Now p is false and q is false

$\therefore$ $p \to q$ is true.

5. Let $p : 3 + 2 = 7$

$q : 4 + 4 = 8$

$\therefore$ $p \to q : (3 + 2) = 7 \to (4 + 4 = 8)$

Now p is false and q is true

$\therefore$ $p \to q$ is true

$\therefore$ $(3 + 2 = 7) \to (4 + 4 = 8)$ is true.

6. Let p : All men are mortal

$q : 2 + 2 = 5$

$\therefore$ $p \to q$: All men are mortal $\to (2 + 2 = 5)$

Now p is true and q is false

$\therefore$ $p \to q$ is false.

$\therefore$ all men are mortal $\to 2 + 2 = 5$ is false.

Art-6. Biconditional

We say that the sentence $p \leftrightarrow q$ (p if and only if q) is true if the two sentences p, q are both true or both false, and is false otherwise. The $p \leftrightarrow q$ can also be stated as p is a necessary and sufficient condition for q.

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$

Art-7. Precedence to Logical Operators

We formed formulas are fully paranthesised, so there is no ambiguity in their interpretition. Often, however, it is more convenient to omit some of the paranthesis for the sake of readability. *e.g.* we would prefer to write

P $\to$ Q $\wedge$ R rather than

(P $\to$ ((~ Q) ^ R)).

The syntax rules given below define what an expression means when some of the paranthesis are omitted. These conventions are analogous to those of elementary algebra, as well as most programming languages, where there is a precedure value that says

$a + b \times c$ means $a + (b \times c)$ rather than $(a + b) \times c$.

The syntax values for propositional logic are straightforward.

1. The most tightly binding operator is ~. *e.g.* $\sim P \wedge Q$ means $(\sim P) \wedge Q$. Furthermore, $\sim P$ means $\sim (\sim P)$.

2. The second highest precedence is than $\wedge$ operator. In expressions combining $\wedge$ and $\vee$, the $\wedge$ operations comes first.

   e.g. $P \vee Q \wedge R$ means $P \vee (Q \wedge R)$.

   If there are several $\wedge$ operations in a sequence, they are performed left to right.

   e.g. $P \wedge Q \wedge R \wedge S$ means $(((P \wedge Q) \wedge R) \wedge S)$

3. The $\vee$ operator has the next level of precedence and it associates to the left.

4. The $\rightarrow$ operator has the next lower level of precedence.

   e.g. $P \wedge Q \rightarrow P \vee Q$ means $(P \wedge Q) \rightarrow (P \vee Q)$.

   The $\rightarrow$ operator associates to the right :

   thus $P \rightarrow Q \rightarrow R \rightarrow S$ means $(P \rightarrow (Q \rightarrow (R \rightarrow S)))$.

5. The $\leftrightarrow$ operator has the lowest level of precedence, and it associates to the right.

EXAMPLES

1. Let $p : 12 + 5 = 17$

$q : 5 + 2 = 7$

$\therefore$ $p \leftrightarrow q : 12 + 5 = 17$ iff $5 + 2 = 7$

Now p is true and q is true

$\therefore$ $p \leftrightarrow q$ is true.

2. Let $p : 5 = 4$

$q : 6 = 5$

$\therefore$ $p \leftrightarrow q : 5 = 4$ iff $6 = 5$

Now p is false and q is false

$\therefore$ $p \leftrightarrow q$ is true.

3. Let p : Only one even integer is prime

q : All odd integers are divisible by 5

$\therefore$ $p \leftrightarrow q$: Only one even integer is prime iff all odd integers are divisible by 5.

Now p is true and q is false.

$\therefore$ $p \leftrightarrow q$ is false.

Art-8. Prove that :

(i) $p \to q = (\sim p) \vee q$

(ii) $\sim (p \to q) = p \wedge \sim q$

Proof.

(i) Truth Table

p	q	$\sim p$	$p \to q$	$(\sim p) \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

A comparison of the last two columns in the truth table shows that they are identical.

$\therefore \quad p \to q = (\sim p) \vee q$

(ii) Truth Table

p	q	$\sim q$	$p \to q$	$\sim (p \to q)$	$p \wedge \sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

A comparison of the last two columns in the truth table shows that they are identical.

$\therefore \quad \sim (p \to q) = p \wedge \sim q.$

LOGICAL EQUIVALENCE

We say that the two propositions $p(p, q, ...)$ and $Q(p, q, ...)$ are logically equivalent, or simply equivalent or equal if they have identical truth tables, denoted by

$$P(p, q, ...) \equiv Q(p, q, ...).$$

Example : The sentences $p \to q$ and $\sim q \to \sim p$ are logically equivalent

p	p	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \to q$

p	q	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

$\sim q \to \sim p$

$p \to q \equiv \sim q \to \sim p.$

Art-9. Laws of the Algebra of Propositions

Here 0 stands for contradiction, 1 for tautology.

Commutative Laws

$p \vee q \Leftrightarrow q \vee p$ $\quad\quad$ $p \wedge q \Leftrightarrow q \wedge p$

Associative Laws

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $\quad\quad$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Distributive Laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $\quad\quad$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Identity Laws

$p \vee 0 \Leftrightarrow p$ $\quad\quad$ $p \wedge 1 \Leftrightarrow p$.

Negation Laws

$p \wedge \sim p \Leftrightarrow 0$ $\quad\quad$ $p \vee \sim p \Leftrightarrow 1$

Idempotent Laws

$p \vee p \Leftrightarrow p$ $\quad\quad$ $p \wedge p \Leftrightarrow p$

Null Laws

$p \wedge 0 \Leftrightarrow 0$ $\quad\quad$ $p \vee 1 \Leftrightarrow 1$

Absorbtion Laws

$p \wedge (p \vee q) \Leftrightarrow p$ $\quad\quad$ $p \vee (p \wedge q) \Leftrightarrow p$

DeMorgan's Laws

$\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$ $\quad\quad$ $\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$

Involution Laws

$\sim(\sim p) \Leftrightarrow p$

Art-10. Prove that

(i) $\sim(\sim p) = p$

(ii) $\sim(p \wedge q) = \sim p \vee \sim q$
(iii) $\sim(p \vee q) = \sim p \wedge \sim q$ ...[De-Morgan's Laws]

Proof :

(i) Truth Table

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

A comparison of the first and third column shows that they are identical.

$\therefore \quad \sim(\sim p) = p$.

(ii) Truth Table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

A comparison of the last two columns in the truth table shows that they are identical.

$\therefore \quad \sim(p \wedge q) = \sim p \vee \sim q$

(iii) Truth Table

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

A comparison of the last two columns in the truth table shows that they are identical.

$\therefore \quad \sim(p \vee q) = \sim p \wedge \sim q$

Art-11. Prove that $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ ...[Associative Law]

Proof :

Truth Table

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	F	T	F	F	F	F
F	T	T	F	T	F	F
F	F	T	F	F	F	F
T	T	F	T	F	F	F
T	F	F	F	F	F	F
F	T	F	F	F	F	F
F	F	F	F	F	F	F

A comparison of the last two columns show that they are identical.

Hence $(p \wedge q) \wedge r = p \wedge (q \wedge r)$.

Art-12. Prove that $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$

Proof :

Truth Table

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$	$p \leftrightarrow (q \leftrightarrow r)$
T	T	T	T	T	T	T
T	F	T	F	F	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T
T	T	F	T	F	F	F
T	F	F	F	T	T	T
F	T	F	F	F	T	T
F	F	F	T	T	F	F

The comparison of the last two columns shows that they are identical.

$\therefore \quad (p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r).$

Art-13. Duality

Let S be any given statement. Then the statement formed by interchanging

$\wedge$ with $\vee$

$\vee$ with $\wedge$

0 with 1

1 with 0

is called dual of S and is denoted by S*.

For example :

Dual of $p \vee 0 = p$ is $p \wedge 1 = p$

Dual of $p \wedge (p \vee q) = p$ is $p \vee (\bar{p} \wedge q) = p$.

EQUIVALENCE

Let S be a set of propositions and p, q be propositions generated by S. p and q are equivalent if $p \leftrightarrow q$ is a tautology. The equivalence of p and q is denoted by $p \Leftrightarrow q$.

IMPLICATION

Let S be a set of propositions and p, q be propositions generated by S. p implies q if $p \rightarrow q$ is a tautology. $p \Rightarrow q$ is written to indicate the implication.

Art-14. Converse, Inverse and Contrapositive (Pbi. U. M.Sc.-I.T. 2009)

Let $p \rightarrow q$ is a direct statement then

(*i*) $q \rightarrow p$ is called its converse.

(*ii*) $\sim p \rightarrow \sim q$ is called its inverse.

(*ii*) $\sim q \rightarrow \sim p$ is called its contrapositive.

ILLUSTRATIVE EXAMPLES

Example 1. For each of the following sentences, state whether it is a statement a indicate its truth value, if it is a statement.

(*i*) $3 \times 5 + 4 = 19$.

(*ii*) Moon is not a heavenly body.

(*iii*) Today is Sunday.

(*iv*) Why are you smoking ?

(*v*) Do you like reading ?

(*vi*) Logic is a very interesting subject.

(*vii*) There are only 100 positive integers.

(*viii*) The sum of three angles of triangle is two right angles.

(*ix*) May God bless you with success !

Sol. (*i*) This is a true statement.

(*ii*) This is a false statement.

(*iii*) This is a statement which is true on Sunday but false on other days.

(*iv*) This is not a statement as the sentence is declarative.

(*v*) This is not a statement.

(*vi*) This is not a statement as it is an open sentence.

(*vii*) This is a false statement as the number of positive integers is infinite.

(*viii*) This is a true statement.

(*ix*) This is not a statement.

Example 2. You are given the following statements :

$p : 5 \times 7 = 35$.

q : Moon is a heavenly body.

r : Jammu is the most populated city in India.

s : Food is necessary for life.

State the truth values of the followings :

(*i*) $p \wedge q, p \wedge r, p \wedge s, q \wedge r, q \wedge s, r \wedge s$.

(*ii*) $p \vee q, p \vee r, p \vee s, q \vee r, q \vee s, r \vee s$.

Sol. Here p, q, s are true statements and r is false.

$\therefore$ (*i*) $p \wedge q, p \wedge s, q \wedge s$ are true and others are false.

(*ii*) All statements are true as there is only one false statement.

Example 3. Let p be the statement "the south-west monsoon is very good this year" and q be the statement "the rivers are rising". Give the verbal translation for (*i*) $p \vee \sim q$ (*ii*) $\sim (\sim p \vee \sim q)$.

Sol. (*i*) The south-west monsoon is very good this year or the rivers are not rising.

(*ii*) It is not true that the south -west monsoon is not very good or the rivers are not rising.

or $\sim(\sim p \vee \sim q) = \sim(\sim p) \wedge \sim(\sim q) = p \wedge q$

$\therefore$ the above verbal translation can also be written as : The south-west monsoon is very good this year and rivers are rising.

Example 4. Write the following statements in symbolic form and give their negations :

(*i*) If you work hard, you will get the first division.

(*ii*) If it rains, he will not go to Kathua.

(*iii*) If Mahatma Gandhi was a saint then Sardar Patel was as iron man.

Sol. (*i*) Let the symbols for the statements be :

p : you work hard

q : first division

$\therefore$ the statements is $p \rightarrow q$

Its negation is $\sim(p \rightarrow q) = \sim(\sim p \vee q)$

$$= \sim(\sim p) \wedge \sim q$$

$$= p \wedge \sim q.$$

In words : Even if you work hard, you will not get first division.

(*ii*) Let p : It rains.

q : He will go to Kathua.

$\therefore$ symbolic expression is $p \rightarrow \sim q$

Its negation is $\sim(p \rightarrow \sim q) = \sim(p \rightarrow \sim q)$

$$= \sim(\sim p \vee \sim q)$$

$$= \sim(\sim p) \wedge \sim(\sim q)$$

$$= p \wedge q.$$

In words : Even if it rains , he will go to Kathua.

(*iii*) Let the symbols for the statements be :

p : Mahatma Gandhi was a saint

q : Sardar Patel was an iron man

$\therefore$ the statement is $p \rightarrow q$

Its negation is $\sim(p \rightarrow q) = \sim(\sim p \vee q)$

$$= \sim(\sim p) \wedge (\sim q) = p \wedge \sim q$$

In words : Even if Mahatma Gandhi was a saint man, Sardar Patel was not an iron man.

Example 5. Write the following statements in symbolic form :

"You can not ride the roller coaster if you are under 4 feet tall unless you are old than 16."

Sol p = "you can ride the roller coaster."

q = "you are under 4 feet tall."

r = "you are older than 16."

$(q \wedge \neg r) \to \neg p$

Example 6. Write the following statements in symbolic form

"Every student in this class has studied calculus."

Sol. $S(x)$: x is in this class, $C(x)$: x studied calculus

$\forall x(S(x) \to C(x))$

$\forall x\ (S(x) \wedge C(x))$ means "Every student is in this class and has studie calculus."

Example 7. Write the following statements in symbolic form

"Some student in this class has studied programming."

Sol. $S(x)$: x is in this class, $P(x)$: x studied programming.

$\exists\ x\ (S(x) \wedge P(x))$

Example 8. (*i*) Find the truth values of $\sim(\sim p \vee q)$ if p is true and q is false.

(*ii*) If p is true and q is false, find the truth values of $\sim(p \wedge \sim q)$.

Sol. (*i*) $\sim(\sim p \vee q) = \sim(\sim p) \wedge \sim q$

$= p \wedge \sim q$

Now p is true and q is false

$\therefore$ p is true and $\sim q$ is true

$\therefore$ $p \wedge \sim q$ is true.

$\therefore$ truth values of $\sim(\sim p \vee q)$ is T.

(*ii*) $\sim(p \wedge \sim q) = \sim p \vee \sim(\sim q) = \sim p \vee q$

Now p is true and q is false

$\therefore$ $\sim p$ is false and q is false.

$\therefore$ $\sim p \vee q$ is false.

Example 9. Determine which of the following statements are true or false :

(*i*) $[(6 < 8) \wedge (8 < 6)] \leftrightarrow 6 = 8$

(*ii*) $[(R \subseteq Q) \to (Q \subseteq R)] \to Q = R$

(*iii*) $[(\sqrt{2}$ is rational $) \vee (2$ is irrational$)] \to (1 = 0)$

Sol. (*i*) Let the symbols for the statements be

$p : 6 < 8$

$q : 8 < 6$

$r : 6 = 8$

$\therefore$ given statement is $(p \wedge q) \leftrightarrow r$

Now p is true and q is false

$\therefore$ $p \wedge q$ is false

Also r is false.

Now $p \wedge q$ is false and r is false.

$\therefore$ $(p \wedge q) \leftrightarrow r$ is true.

$\therefore$ $[(6 < 8) \wedge (8 < 6)] \leftrightarrow 6 = 8$ is true.

(*ii*) Let the symbols for the statements be

$p : \mathbf{R} \subseteq \mathbf{Q}$

$q : \mathbf{Q} \subseteq \mathbf{R}$

$r : \mathbf{Q} = \mathbf{R}$

$\therefore$ given statement is $(p \to q) \to r$

Now p is false, q is true and r is false.

$\therefore$ $p \to r$ is true.

$\therefore$ $(p \to q) \to r$ is false

$\therefore$ $[(\mathrm{R} \subseteq \mathrm{Q}) \to (\mathrm{Q} \subseteq \mathrm{R})] \to \mathrm{Q} = \mathrm{R}$ is false.

(*iii*) Let the symbols for the statements be :

$p : \sqrt{2}$ is rational

$q : 2$ is irrational

$r : 1 = 0$

$\therefore$ given statement is $(p \vee q) \to r$

Now p is false, q is false and r is false.

$\therefore$ $p \vee q$ is false

$\therefore$ $(p \vee q) \to r$ is true.

$\therefore$ $[(\sqrt{2}$ is rational$) \vee (2$ is irrational$)] \to (1 = 0)$ is true.

Example 10. Write down the truth table of the following statement

$[p \to (q \vee r)] \wedge [p \leftrightarrow \sim r]$

Sol.

Truth Table

p	q	r	$q \vee r$	$\sim r$	$p \rightarrow (q \vee r)$	$p \Leftrightarrow \sim r$	$[p \rightarrow (q \vee r)] \wedge [p \Leftrightarrow \sim r]$
T	T	T	T	F	T	F	F
T	F	T	T	F	T	F	F
F	T	T	T	F	T	T	T
F	F	T	T	F	T	T	T
T	T	F	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	F	T	T	T	F	F
F	F	F	F	T	T	F	F

Example 11. Prove that

$$p \Leftrightarrow q = (p \wedge q) \vee (\sim p \wedge \sim q).$$

Sol.

Truth Table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \wedge \sim q$	$p \Leftrightarrow q$	$(p \wedge q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

A comparison of the last two columns in the truth table shows that they are identical.

$$\therefore \quad p \Leftrightarrow q = (p \wedge q) \vee (\sim p \wedge \sim q).$$

Example 12. Prove that $(p \rightarrow r) \vee (q \rightarrow s) = (p \wedge q) \rightarrow (r \vee s)$.

Sol.

Truth Table

p	q	r	s	$p \rightarrow r$	$q \rightarrow s$	$p \wedge q$	$r \vee s$	$(p \rightarrow r) \vee (q \rightarrow s)$	$(p \wedge r) \rightarrow (r \vee s)$
T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T	T
F	T	T	T	T	T	F	T	T	T
F	F	T	T	T	T	F	T	T	T
T	T	F	T	F	T	T	T	T	T
T	F	F	T	F	T	F	T	T	T
F	T	F	T	T	T	F	T	T	T
F	F	F	T	T	T	F	T	T	T
T	T	T	F	T	F	T	T	T	T

T	F	T	F	T	T	F	T	T	T
F	T	T	F	T	F	F	T	T	T
F	F	T	F	T	T	F	T	T	T
T	T	F	F	F	F	T	F	F	F
T	F	F	F	F	T	F	F	T	T
F	T	F	F	T	F	F	F	T	T
F	F	F	F	T	T	F	F	T	T

A comparison of the last two columns in the truth table shows that they are identical.

$\therefore \quad (p \to r) \vee (q \to s) = (p \wedge q) \to (r \vee s)$

Example 13. Prove that

$$p \to (q \wedge r) = (p \to q) \wedge (p \to r).$$

Sol.

Truth Table

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T	T	T
T	F	T	F	F	F	T	F
F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	T
T	T	F	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	F	F	T	T	T	T
F	F	F	F	T	T	T	T

A comparison of the fifth and eighth columns shows that they are identical.

$\therefore \quad p \to (q \wedge r) = (p \to q) \wedge (p \to r).$

Example 14. Prove by means of a truth table that

$$p \leftrightarrow q = (p \to q) \wedge (q \to p).$$

Sol.

Truth Table

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

The like truth values of the last two columns prove the validity of the statement.

Example 15. Prove that $(p \wedge q) \rightarrow (p \wedge q)$ is a tautology but $(p \vee q) \rightarrow (p \wedge q)$ is not.

Sol.

Truth Table

p	q	$p \vee q$	$p \wedge q$	$(p \wedge q) \rightarrow (p \vee q)$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T	T
T	F	T	F	T	F
F	T	T	F	T	F
F	F	F	F	T	T

From the above table, it is clear that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology whil $(p \vee q) \rightarrow (p \wedge q)$ is not.

Example 16. Prove that $\{[(p \rightarrow q) \vee p] \wedge q\} \rightarrow q$ is a tautology.

Sol.

Truth Table

p	q	$p \rightarrow q$	$(p \rightarrow q) \vee p$	$[(p \rightarrow q) \vee p] \wedge q$	$\{[(p \rightarrow q) \vee p] \wedge q\} \rightarrow q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	T	F	T

From the above table, it is clear that

$$\{[(p \rightarrow q) \vee p] \wedge q\} \rightarrow q \text{ is a tautology.}$$

Example 17. Prove that if $p \rightarrow q$ and $q \rightarrow r$ then $p \rightarrow r$.

Sol. Here we are given that $p \rightarrow q$, $q \rightarrow r$ and we have to prove that $p \rightarrow r$. The resu will be established if we show that

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r) \text{ is a tautology.}$$

Truth Table

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$[(p \rightarrow q) \wedge q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	T	T

$\therefore$ if $p \to q$ and $q \to r$ then $p \to r$

$[\because [(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology$]$

:xample 18. Write down

(*i*) Contrapositive of $p \to \sim q$

(*ii*) Contrapositive of converse of $p \to \sim q$

(*iii*) Inverse of converse of $p \to q$.

ol. (*i*) Contrapositive of $p \to \sim q$ is

$$\sim(\sim q) \to \sim p$$

$$= q \to \sim p$$

(*ii*) Converse of $p \to \sim q$ is $\sim q \to p$

$\therefore$ contrapositive of $\sim q \to p$ is

$$\sim p \to \sim(\sim q)$$

$$= \sim p \to q$$

(*iii*) Converse of $p \to q$ is $q \to p$

Inverse of $q \to p$ is $\sim q \to \sim p$

Example 19. Construct truth table of $\sim(p \wedge q) \to \sim p \vee \sim q$.

Is it Contradiction or Tautology.

(P.T.U. B.C.A.-I 2007)

Sol.

Truth Table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$\sim(p \wedge q) \to \sim p \vee \sim q$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Given statement is a Tautology as result is always true.

Example 20. Prove that :

$p \to (\sim q \vee r) \equiv (p \wedge q) \to r$. (P.T.U. B.C.A.-I 2007)

Sol.

Truth Table

p	q	r	$\sim q$	$\sim q \vee r$	$p \to (\sim q \vee r)$ (I)	$p \wedge q$	$(p \wedge q) \to r$ (II)
T	T	T	F	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	T	F	T
F	T	F	F	F	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

From I and II

$$p \to (\sim q \vee r) \equiv (p \wedge q) \to r.$$

Example 21. State the Converse and Contrapositive of the implication "If it snows tonight, then I will stay at home". (Pbi.U., B.C.A. 2009)

Sol. Let p: It snows tonight

q: I will stay at home.

Given statement is $p \to q$

Converse of statement is $q \to p$

i.e. "If I stay at home then it Snows tonight".

Contrapositive of statement is $\sim q \to \sim p$

i.e. "If I do not stay at home then It will not Snow tonight".

Example 22. State Converse and Contrapositive of the implication "if today is Thursday, then I have a rest today".

(Pbi. U. B.C.A.-II Sept. 2009)

Sol. Let p : Today is Thursday q : I have a rest today

Given statement is $p \to q$

Converse is $q \to p$

i.e. If I have a rest today, then Today is Thursday.

Contrapositive is $\sim q \to \sim p$

i.e. If I do not have a rest today, then today is not Thursday.

Example 23. Give the converse and contrapositive of the implication "If today is Monday then tomorrow is Tuesday". (B.C.A. Pbi. U 2008)

Sol. Let p : Today is Monday

q = Tomorrow is Tuesday

Given statement is $p \to q$

Converse of statement is $q \to p$

i.e. "If tomorrow is Tuesday then today is Monday.

Contrapositive of statement is $\sim q \to \sim p$

i.e. "If tomorrow is not Tuesday then today is not Monday".

Example 24. Construct the Truth table for the proportion

$(p \to q) \to (q \to p)$ (Pbi. U. B.C.A.-II April 2010)

Sol.

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Example 25. If $p \to q$ is false, can you find the truth value of $(\sim (p \wedge q)) \to q$ without using truth table ? Explain your answer.

Sol. If $p \to q$ is false, it means p is true and q is false.

Also $p \wedge q$ will be false and $\sim (p \wedge q)$ will be true.

So $(\sim (p \wedge q)) \to q$ with $\sim (p \wedge q)$ as true and q as false will be false.

Hence truth value of $(\sim (p \wedge q)) \to q$ is false.

EXERCISE 2 (*a*)

1. Which of the following is a statement (or proposition) ? Justify your answer :
   (*i*) Listen to me, Krishna !
   (*ii*) 17 is a prime number.
   (*iii*) $x^2 + 5x + 6 = 0$.
   (*iv*) 6 has three prime factors.
   (*v*) Two non-empty sets have always a non-empty intersection.
   (*vi*) The real number x is less than 1.
   (*vii*) Two individuals are always related.
2. State the truth values of the following :
   (*i*) There are only finite number of rational numbers.
   (*ii*) $\sqrt{2}$ is a rational number.
   (*iii*) There is only one triangle apart from the triangles (congruent to it) with prescribed lengths for sides a, b, c with $a < b + c$.
   (*iv*) The quadratic equation $ax^2 + bx + c = 0, a \neq 0$, has always two real roots.

(v) A triangle one of whose vertices lies on a circle and whose side opposite to this vertex is a diameter of the circle is a right angled triangle.

(vi) There is always a real root for any quadratic equation.

(vii) The number of ways of selecting 2 persons in two chairs out of n persons is ${}^nP_2$.

(viii) $(\vec{a}+\vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b}$.

3. (i) Give the truth table for the statement $\sim p \vee q$

(ii) Write down the truth table for $\sim p \wedge \sim q$

(iii) Write down the truth table for the statement
$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$

(iv) Give the truth table for the statement
$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$.

(v) Write down the truth table for the statement
$(p \wedge q) \rightarrow \sim p$

(vi) Give the truth table for the statement
$(p \wedge q) \rightarrow (p \vee q)$.

(vii) Give the truth table for $l \leftrightarrow m$ where
$l = (p \rightarrow q) \wedge (q \rightarrow p)$ and $m = p \leftrightarrow q$.

4. (i) If p : lines l and m are perpendicular to each other
q : A is a point on m,

write down in symbols the statement r = A is a point on the line m which is perpendicular to l.

What is the negation of this statement ?

(ii) If p : I study
q : I fail,

What is the symbolism for the statement
r : I study or I fail.

What is the negation of this statement ?

(iii) 'Ram is smart and healthy'
'Ram is neither smart nor healthy'

Are these statements negations of each other ?

(iv) Are the following statements negation of each other ?
'x is not a rational number'
'x is not an irrational number.'

(v) Write down the statement. 'Two congruent triangles are precisely those which have corresponding sides equal' as an equivalence and write its negation also.

(vi) Write down the negation of the statement : 'All the sides of an equiangular triangle are of the same length'.

5. (i) If p stands for the statement, 'I do not like chocolates' and q for the statement. 'I like ice-cream', then what does $\sim p \wedge q$ stand for ?

(ii) If s stands for the statement, 'I will not go to school' and t for the statement, " I will watch a movie', then what does $\sim s \vee t$ stand for ?

(iii) If p stands for the statement, 'I like tennis and q stands for the statement, 'I like football', then what does $\sim p \wedge \sim q$ stand for ?

6. (a) $l : (p \to r) \vee (q \to r)$ and $m : (p \vee q) \to r$,

show that $l \neq m$.

(b) If p, q are two statements, show that

$$\sim(\sim p \wedge \sim q) = p \vee q$$

and $\sim(\sim p \vee \sim q) = p \wedge q$

(c) If p and q are two statements, show that

$$\sim(p \leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$$

7. Prove that

(i) $p \wedge q = q \wedge p$ (ii) $p \vee q = q \vee p$

(iii) $p \vee (q \vee r) = (p \vee q) \vee r$ (iv) $\sim(p \vee \sim q) = \sim p \wedge q$

(v) $\sim(p \wedge \sim q) = \sim p \vee q$

8. Write down the truth table for

(i) $(p \wedge q) \to p$ (ii) $p \to (p \to q)$

(iii) $(p \wedge q) \to (p \vee q)$ (iv) $p \wedge (q \to p)$

(v) $\sim(p \wedge q) \vee \sim(q \leftrightarrow p)$ (vi) $(p \to q) \vee \sim(p \leftrightarrow q)$

(vii) $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$ (viii) $p \wedge \sim q$

(ix) $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$ (Pbi.U., B.C.A., 2009)

(x) $(p \vee q) \to p$

9. Prove the following distributive laws :

(i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(iii) $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$

(iv) $(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r)$

10. (i) Show that $(\sim p \wedge \sim q) \to (p \to q)$ is a tautology

(ii) Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

(iii) Show that $(p \wedge q) \to (\sim p \vee q)$ is a tautology.

(iv) $(p \wedge q) \to (p \leftrightarrow q)$ is a tautology.

(v) $(p \wedge q) \to p$ is a tautology.

(vi) $(p \wedge q) \wedge (\sim p \vee \sim q)$ is a contradiction.

(vii) Show that $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$ is a tautology.

(viii) Show that $p \to (p \vee q)$ is a tautology.

(ix) Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

11. (i) Write down the truth table for $l \wedge m$ where $l = \sim q \to \sim r$, $m = \sim r \to \sim q$.

(ii) Write down the truth table for $l \leftrightarrow m$ where $l = \sim (p \vee q)$, $m = \sim p \wedge \sim q$.

12. Write down the truth table for the following statement patterns :

(i) $(p \to q) \leftrightarrow (\sim q \to \sim p)$ (ii) $(p \to q) \to [(q \to r) \to (p \to q)]$

13. Find the converse, inverse and contrapositive of the following statement :

"If $4x - 2 = 10$, then $x = 3$".

ANSWERS

1. (i) Not a statement (ii) Statement (iii) Not a statement
(iv) Statement (v) Statement (vi) Not a Statement (vii) Statement.

2. (i) F (ii) F (iii) T (iv) F (v) T (vi) F (vii) T (viii) T

3. (i)

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(ii)

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(iii)

p	q	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	T	T

(iv)

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(v)

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \to \sim p$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

(vi)

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(vii)

p	q	$p \to q$	$q \to p$	l	m	$l \leftrightarrow m$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

4. (i) r is $p \wedge q$; 'l is not perpendicular to m or A is not a point on m'

(ii) r is $p \vee q$; 'I do not study and I do not fail'.

(iii) No (iv) Yes

(v) $p \leftrightarrow q$;

'Two triangles are not congruent and have corresponding sides equal or two triangles are congruent and have a pair of corresponding sides equal.'

(vi) 'No all the sides of an equiangular triangle are of the same length'.

5. (*i*) I like chocolates and ice-cream.

(*ii*) Either I will go to school or I will watch a movie.

(*iii*) I like neither tennis nor football.

8. (*i*)

Truth Tables

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

(*ii*)

Truth Table

p	q	$p \rightarrow q$	$p \rightarrow (p \rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(*iii*)

Truth Tables

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(*iv*)

Truth Table

p	q	$q \rightarrow p$	$p \wedge (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	F

(v) **Truth Table**

p	q	$p \wedge q$	$q \leftrightarrow p$	$\sim (p \wedge q)$	$\sim (q \leftrightarrow p)$	$\sim (p \wedge q) \vee \sim (q \leftrightarrow p)$
T	T	T	T	F	F	F
T	F	F	F	T	T	T
F	T	F	F	T	T	T
F	F	F	T	T	F	T

(vi) **Truth Table**

p	q	$p \leftrightarrow q$	$\sim (p \leftrightarrow q)$	$p \rightarrow q$	$(p \rightarrow q) \vee \sim (p \leftrightarrow q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	T	T
F	F	T	F	T	T

(vii) **Truth Table**

p	q	$\sim q$	$p \rightarrow q$	$\sim (p \rightarrow q)$	$p \wedge \sim q$	$\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

(viii) **Truth Table**

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(ix) **Truth Table**

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$	$p \vee (q \wedge r) \leftrightarrow (p \vee r) \vee (p \vee r)$
T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T

F	F	T	F	F	T	F	F	T
T	T	F	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	F	F	T	F	F	F	T
F	F	F	F	F	F	F	F	T

11. (*i*)

q	r	l	m	$l \wedge m$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

(*ii*)

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$l \Leftrightarrow m$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

12. (*i*)

p	q	$p \to q$	$\sim q \to \sim p$	$(p \to q) \Leftrightarrow (\sim q \to \sim p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

(*ii*)

p	q	r	$p \to q$	$q \to r$	$(q \to r) \to (p \to q)$	$(p \to q) \to [(q \to r) \to (p \to q)]$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

13. (*i*) Converse : If $x = 3$, then $4x - 2 = 10$

(*ii*) Inverse : If $4x - 2 \neq 10$, then $x \neq 3$

(*iii*) Contrapositive : If $x \neq 3$, then $4x - 2 \neq 10$

Art-15. Arguments

Def. of argument : An argument is a statement which asserts that given set of propositions $p_1, p_2, p_3, \ldots, p_n$ taken together gives another proposition P.

These are expressed as $p_1, p_2, p_3, \ldots\ldots, p_n$ /– P. The sign "/-" is spoken at turnstile. The propositions $p_1, p_2, p_3, \ldots\ldots\ldots, p_n$ are called "premises" or "assumptions" and P is called the "conclusion".

Valid argument : An argument $p_1, p_2, p_3, \ldots\ldots p_n$ /–P is true if P is true whenever all the premises $p_1, p_2, p_3, \ldots\ldots, p_n$ are true, otherwise the argument is false. A true argument is called valid argument, and a false argument is called a fallacy.

Note. If is important to realise that the truth or the conclusion is irrelevant as far as the validity of argument is concerned. A true conclusion is neither necessary nor sufficient for the validity of argument.

The validity can also be judged by the relationship $p_1 \wedge p_2 \wedge p_3 \ldots\ldots \wedge p_n \rightarrow P$ provided it is a tautology.

ILLUSTRATIVE EXAMPLES

Example 1. Test the validity of : If he works hard then he will be successful. If he is successful then he will be happy. Therefore, hard work leads to happiness.

Sol. Let the s, mbols for the statements be :

p : he works hard.

q : he is successful

r : he is happy.

The argument is :

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow p \rightarrow r$$

Truth Table

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

From the table, it is clear that $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is tautology.

$\therefore$ given argument is valid.

Note. Another method for testing the validity of argument.

In the last three examples, there were only two or there statements and consequently 4 to 8 rows in the truth table. But if there are four or more statements then the truth table will have 16 or more rows and the chance of the making a mistake will be more. To over come this difficulty *i.e.*, to reduce the size of the table we have another method, infact above method stated in another way, which follows as :

"Assume that the conclusion is false. Now if $p_1 \wedge p_2 \wedge \ldots\ldots \wedge p_n$ is a fallacy, then the argument is valid, otherwise the argument is invalid.

Example 2. Test the validity of :

"If my brother stands first in the class, I will give him a watch. Either he stood first or I was out of station. I did not give my brother a watch this time. Therefore I was out of station."

Sol. Let the symbols for the statements be :

p : my brother stands first in the class.

q : I give him a watch.

r : I was out of station.

The argument is $p \to q, p \vee r, \sim q \mid\!- r$.

Assume that r is false.

Now there will be only four rows as there are only two variables p, q

Truth Table

p	q	r	$p \to q$	$p \vee q$	$\sim q$	$(p \to q) \wedge (p \vee r) \wedge (\sim q)$
T	T	F	T	T	F	F
T	F	F	F	T	T	F
F	T	F	T	T	F	F
F	F	F	T	F	T	F

Since $(p \to q) \wedge (p \vee r) \wedge (\sim q)$ is a fallacy.

$\therefore$ the argument is valid.

Example 3. Prove the validity of following arguments :

If man is a bachelor, he is unhappy.

If a man is unhappy, he dies young.

Therefore, bachelors die young. (P.T.U. B.C.A.-I 2005)

Sol. Let p ; man is a bachelor

q : man is unhappy

r : man dies young.

The given statement is $[(p \to q) \wedge (q \to r)] \to (p \to r)$

Truth Table

p	q	r	$p \to q$	$q \to r$	I: $(p \to q) \land (q \to r)$	II: $p \to r$	$\text{I} \to \text{II}$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since given statement is a tautology, so argument is valid.

Example 4. Check the validity of argument :

If I work, I cannot study. Either I work or pass mathematics.

I passed mathematics. Therefore, I study.

(P.T.U. B.C.A.-I 2004)

Sol. Let p : I work

q : I study

r : I pass mathematics

The given statement is

$$[(p \to \sim q) \land (p \lor r) \land (r)] \to q$$

Truth Table

p	q	r	$\sim q$	$p \to \sim q$	$p \lor r$	I: $(p \to \sim q) \land (p \lor r) \land (r)$	$\text{I} \to q$
T	T	T	F	F	T	F	T
T	T	F	F	F	T	F	T
T	F	T	T	T	T	T	F
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	T	F
F	F	F	T	T	F	F	T

The given statement is not a tautology.

So argument is not valid.

EXERCISE 2 (*b*)

1. Test the validity of :

   If it rains then crop will be good.

   It did not rain, therefore the crop will not be good.

2. Test the validity of :

   Unless we control population, all advances resulting from planning will be nullified. But this must not be allowed to happen. Therefore we must somehow control population.

3. Are the following arguments valid ? If valid, construct a formal proof ; if not valid, explain why.

   (*a*) If wages increase, then there will be inflation. The cost of living will not increase if there is no inflation. Wages will increase. Therefore, the cost of living will increase.

   (*b*) If the races are fixed or the casinos are crooked, then the tourist trade will decline. If the tourist trade decreases, then the police will be happy. The police force is never happy. Therefore, the races are not fixed.

ANSWERS

1. Not valid   2. Valid

3. (*a*) Not valid   (*b*) Valid

Art-16. Mathematical System

A mathematical system consists of

1. A set or universe, U.

2. **Definitions :** sentences that explain the meaning of concepts that relate to the universe. Any term used in describing the universe itself is said to be undefined. All definitions are given in terms of these undefined concepts of objects.

3. **Axions :** assertions about the properties of the universe and rules for creating and justifying more assertions. These rules always include the system of logic that we have developed to this point.

4. **Theorems :** the additional assertions mentioned above.

Example 1. In Euclidean geometry the universe consists of points and lines (two undefined terms). Among the definitions is a definition of parallel lines and among the axioms is the axiom that two distinct parallel lines never meet.

Example 2. In Propositional calculus, the universe consists of propositions. The axioms are the truth tables for the logical operators and the key definitions are those of implication and equivalence.

Theorem : A true proposition derived from axioms of mathematical system is called a theorem.

All theorems can be expressed in terms of a finite number of propositions p_1, p_2, ..., p_n , called the **premises** and the proposition C, called the **conclusion**. These theorems take the form $p_1 \wedge p_2 \wedge p_3 \wedge ...... \wedge p_n \Rightarrow C$.

Proof : A proof of a theorem is a finite sequence of logically valid steps that demonstrate that the premises of a theorem imply the conclusion.

There are two important types of proofs namely direct and indirect.

Direct Proof : It is a proof in which the truth of the premises of a theorem are shown to directly imply the truth of the theorem's conclusion.

Rules for Direct Proof

1. It must terminate in a finite number of steps.
2. Each step must be either a premise or a proposition that is implied from previous steps using any valid equivalence or implication.
3. The last step must be the conclusion of the theorem.

Indirect Poof

Negate the conclusion of the theorem and add this negation to the premises. If this set of propositions implies a contradiction, then the proof is complete.

Rules for Indirect Proof

1. The first step is the negated conclusions.
2. The last step must be a contradiction

ILLUSTRATIVE EXAMPLES

Example 1. Prove that the following are equivalences :

(*i*) $p \vee q \Leftrightarrow q \vee p$

(*ii*) $p \rightarrow q \Leftrightarrow \sim q \rightarrow p$

(*iii*) $(p \wedge q) \vee (\sim p \wedge q) \Leftrightarrow q$

Sol. (*i*) **Truth Table**

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \rightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Since $(p \vee q) \rightarrow (q \vee p)$ is a tautology

$\therefore$ $p \vee q$ and $q \vee p$ are equivalent

$\therefore$ $(p \vee q) \Leftrightarrow (q \vee p)$.

(*ii*) **Truth Table**

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since $(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$ is a tautology

$\therefore$ $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$ are equivalent

$\therefore$ $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$

(*iii*) **Truth Table**

p	q	$\sim p$	$p \wedge q$	$\sim p \wedge q$	$(q \wedge q) \vee (\sim p \wedge q)$	$(p \wedge q) \vee (\sim p \wedge q) \rightarrow q$
T	T	F	T	F	T	T
T	F	F	F	F	F	T
F	T	T	F	T	T	T
F	F	T	F	F	F	T

Since $(p \wedge q) \wedge (\sim p \wedge q) \rightarrow q$ is a tantology

$\therefore$ $(p \wedge q) \vee (\sim p \wedge q)$ and q are equivalent

$\therefore$ $(p \wedge q) \vee (\sim p \wedge q) \Leftrightarrow q.$

Example 2. Give direct proof of

$\sim p \vee q, s \vee p, \sim q \Rightarrow s$

Sol.

Step	**Proposition**	**Justification**
(1)	$\sim p \vee q$	Premise
(2)	$\sim q$	Premise
(3)	$\sim p$	Disjunctive simplification (1), (2).
(4)	$s \vee p$	Premise
(5)	s	Disjunctive simplification (3), (4). #

Note. Conditional Conclusion

The conclusion of a theorem is often a conditional proposition. The condition of the conclusion can be included as a premise in the proof of the theorem. Then we are to prove the consequence of the conclusion.

Example 3. Give indirect proof of

$$p \rightarrow r, q \rightarrow s, p \vee q \Rightarrow s \vee r$$

Sol.

Step	Proposition	Justification
(1)	$\sim (s \vee r)$	Negated conclusion
(2)	$\sim s \wedge \sim r$	De Morgan's Law, (1)
(3)	$\sim s$	Conjunctive simplification, (2)
(4)	$q \to s$	Premise
(5)	$\sim q$	Contrapositive (3), (4)
(6)	$\sim r$	Conjunctive simplification, (2)
(7)	$p \to r$	Premise
(8)	$\sim p$	Contrapositive, (6), (7)
(9)	$(\sim p) \wedge (\sim q)$	Conductive, (5), (8)
(10)	$\sim (p \vee q)$	De Morgan's Law, (9)
(11)	$P \vee q$	Premise
(12)	0	(10), (11) #

EXERCISE 2 (*c*)

1. Show that
   (*i*) $p \wedge q$ logically implies $p \leftrightarrow q$
   (*ii*) $p \leftrightarrow \sim q$ does not logically implies $p \to q$
2. Give direct proof of the theorem $p \to r, q \to s, p \vee q \Rightarrow s \vee r$.
3. Give direct proof of $p \to (q \to s), \sim r \vee p, q \Rightarrow r \to s$.
4. Give indirect proof of $a \to b, \sim (b \vee c) \Rightarrow \sim a$
5. Give direct and indirect proof of $p \to q, q \to r, \sim (p \wedge r), p \vee r \Rightarrow r$
6. Give direct and indirect proof of

$$(p \to q) \wedge (r \to s), (q \to t) \wedge (s \to u), \sim (t \wedge u), p \to r \Rightarrow \sim p$$

Art-17. Proposition over a Universe

Let U be a non-empty set. A proposition over U is a sentence that contains a variable that can take on any value in U and which has a definite truth value as a result of any such substitution.

Examples : Consider

(*i*) $7x^2 - 6x = 0$

$\Rightarrow x(7x - 6) = 0$

$\Rightarrow x = 0, \frac{7}{6}$

If we take **Q** as universe, then truth set (*i.e.*, solution set) of $7x^2 - 6x = 0$ is $\left\{0, \frac{7}{6}\right\}$.

If we take **Z** as universe, then truth set of $7x^2 - 6x = 0$ is $\{0\}$.

If we take **N** as universe, then truth set of $7x^2 - 6x = 0$ is ϕ.

(*ii*) $z^2 = 5$

If we take **Q** as universe, then truth set of $z^2 = 5$ is ϕ.

$\therefore$ $z^2 = 5$ is a contradiction over the rationals.

(*iii*) $(x+3)(x-3) = x^2 - 9$

If we take **Q** as universe, then truth set is **Q** as $(x+3)(x-3) = x^2 - 9$ is true for all rational numbers

$\therefore$ $(x+3)(x-3) = x^2 - 9$ is a tautology over the rationals.

Truth Set : If $p(n)$ is a proposition over U, then the truth set of $p(n)$ is

$$T_{p(n)} = \{a \in U / p(a) \text{ is true}\}$$

Example : Consider the set $\{1, 2, 3, 4\}$

Its power set is $\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$

Let proposition be $\{1, 2\} \cap A = \phi$

$\therefore$ truth set of proposition taken over the power se of $\{1, 2, 3, 4\}$ is

$$\{\phi, \{3\}, \{4\}, \{3, 4\}\}.$$

Tautology and contradiction : A proposition over U is a tautology if its truth set is U. It is a contradiction if its truth set is empty.

Equivalence : Two propositions are equivalent if $p \Leftrightarrow q$ is a tautology. In other words p and q are equivalent if $T_p = T_q$.

Example : $x + 7 = 12$ and $x = 5$ are equivalent propositions over the integers.

Implication : If p and q are propositions over U, then p implies q if $p \rightarrow q$ is a tautology. In other words $p \Rightarrow q$ when $T_p \subseteq T_q$.

Example : Over the natural numbers,

$$n \leq 3 \Rightarrow n \leq 8 \text{ as } \{0, 1, 2, 3\} \subseteq \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$$

Truth Set of compound Propositions

The truth sets of compound propositions can be expressed in terms of the truth sets of simple propositions. The following list gives the connection between compound and simple truth sets :

1. $T_{p \wedge q} = T_p \cap T_q$
2. $T_{p \vee q} = T_p \cup T_q$
3. $T_{\sim p} = T_p^{\,c}$
4. $T_{p \leftrightarrow q} = (T_p \cap T_q) \cup (T_p^c \cap T_q^c)$
5. $T_{p \rightarrow q} = T_p^c \cup T_q$.

ILLUSTRATIVE EXAMPLES

Example 1. If $U = P\{1, 2, 3, 4\}$, what are the truth sets of the following propositions ?

(*i*) $A \cap \{2, 4\} = \phi$

(*ii*) $3 \in A$ and $1 \notin A$

(*iii*) $A \cup \{1\} = A$

(*v*) A is a proper subset of $\{2, 3, 4\}$

(*vi*) $\# A^c = \# A$

Sol. (*i*) Truth set is

$\{\phi, \{1\}, \{3\}, \{1, 3\}\}$

(*ii*) Truth set is

$\{\{3), \{3, 2\}, \{3, 4\}, \{2, 3, 4\}\}$

(*iii*) Truth set is

$\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$

(*iv*) Truth set is

$\{\{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$

(*v*) Truth set is

$\{A \subseteq U, \# A = 2\}$

Example 2. Given the propositions over the natural numbers

$p : n \leq 4$

$q : 2n > 17$

and $r : n$ is a divisor of 18

What are the truth sets of

(*a*) q

(*b*) $p \wedge q$

(*c*) r

(*d*) $q \to r$

Sol. We have

$T_p = \{1, 2, 3\}, T_q = (9, 10, 11, 12, \ldots\ldots\}, T_r = \{1, 2, 3, 6, 9, 18\}$

(*a*) $T_q = \{9, 10, 11, 12, \ldots\ldots\}$

(*b*) $T_{p \wedge q} = T_p \cap T_q = \{1, 2, 3\} \cap \{9, 10, 11, 12, \ldots\} = \phi$

(*c*) $T_r = \{1, 2, 3, 6, 9, 18\}$

(*d*) $T_{q \to r} = T_q{}^c \cup T_r$

$= \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{1, 2, 3, 6, 9, 18\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 18\}$

Example 3. Let the universe be **Z**, the set of integers. Which of the following propositions are equivalent over **Z** ?

(*a*) $0 < n^2 \leq 4$.

(*b*) $0 < n^3 \leq 8$.

(*c*) $0 < n \leq 2$.

Sol. (*a*) Truth set is $\{-2, -1, 1, 2\}$

(*b*) Truth set is $\{1, 2\}$

(*c*) Truth set is $\{1, 2\}$

We know that two propositions are equivalent if their truth sets are equal

$\therefore$ (*b*) and (*c*) are equivalent.

EXERCISE 2 (*d*)

1. Over the universe of positive integers

$p(n):\ 4n^2 - 3n = 0$

$q(n):\ n$ is a perfect square and $n < 90$

$r(n):\ n$ is a divisor of 36

What are the truth sets of these propositions ?

2. Over the universe of positive integers :

$p(n) : n$ is prime and $n < 32$

$q(n) : n$ is a power of 3

$r(n) : n$ is a divisor of 27

(*a*) What are the truth sets of these propositions ?

(*b*) Which of the three propositions implies one of the others ?

3. If $U = \{0, 1, 2\}$, how many propositions over U could you list without listing two that are equivalent ?

4. (*a*) Determine the truth sets of the following propositions over the positive integers :

$p(n) : n$ is a perfect square and $n < 100$

$q(n) : n = \#\, P(A)$ for some set A.

(*b*) Determine $T_p \wedge q$ for p and q above.

ANSWERS

1. $\{1, 4, 9, 16, 25, 26, 49, 64, 81\}$, $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

2. (a) $T_p = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$
$T_q = \{1, 3, 6, 9, 12, 15, 18, 21, \ldots\ldots\}$
$T_r = \{1, 3, 9, 27\}$ (b) r implies q 3. 256

4. $T_p = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$
$T_q = \{1, 2, 4, 8, 16, 32, \ldots\ldots\}, \{1, 4, 16, 64\}$

Art-18. Quantifiers

If $p(n)$ is a propositions over U with $T_{p(n)} \neq \phi$, then we say "There exists an n in U such that $p(n)$ is true." We abbreviate this sentence as $(\exists n)_{U}, (p(n))$. $\exists$ is known as **existential quantifier.**

It is clear that if $p(n)$ is a propositions over a universe U, its truth set $T_{p(n)}$ is a subset of U.

Examples

(1) $(\exists k)_z, (5k = 100)$ means that there is an integer k such that 100 is a multiple of 5. This is true.

(2) $(\exists x)_Q (x^2 - 3 = 0)$ means that there is a rational number x such that $x^2 = 3$. This is false as the solution set of the equation $x^2 - 3 = 0$ over **Q** is empty. We write it as $((\nexists x)_Q, (x^2 - 3 = 0)$

If $p(n)$ is a propositions over U, with $T_{p(n)} = U$. Them we say "for all n in U, $p(n)$ is true." We abbreviate this as $(\forall n)_U, (p(n))$. $\forall$ is known as **universal quantifier.**

$\exists x : P(x)$ means, "There exists an x such that $P(x)$ holds."

$\forall x : P(x)$ means, "For all x, it is the case that $P(x)$ holds."

So for example, if x denotes a real number, then

$\exists x : x^2 = 9$ is true, since 3 is an x for which $x^2 = 9$.

On the other hand, $\forall x : x^2 = 9$ is clearly false; not all numbers, when squared, are equal to 3.

$\forall x : x^2 + 1 > 0$ is true, but $\forall x : x^2 > 2$ is false, since for example $x = 1$ does not satisfy the predicate. On the other hand, $\exists x : x^2 > 2$ is true, since x 2 is an example that satisfies it.

Negation of Quantified Proposition

When we negate a quantified proposition, then the universal and existential quantifiers become complement of one another. In simple words negation of an existentially quantified proposition is a universally quantified proposition and negation of a universally quantified proposition is an existentially quantified proposition. In symbols,

$$\sim (\forall n)_U (p(n)) \Leftrightarrow (\exists n)_U (\sim p(n))$$

and $$\sim (\exists n)_U (p(n)) \Leftrightarrow (\forall n)_U (\sim p(n))$$

Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other.

e.g. "For every real number, there is a real number larger than it."

This can be written as $\forall x \exists y : y > x$.

Example : $\forall x \exists y (x + y = 0)$

Example : $\forall x$ Q(x), where Q(x) is $\exists y$ P(x, y), where P(x, y) : $x + y = 0$.

Example : $\forall x \forall y (x + y = y + x)$ Commutative law for addition

Example : $\forall x \exists y (x + y = 0)$ Additive inverse property

Example : $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$ Associative law for addition

Example : $\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (x y < 0))$

where the domain of both variables is real number.

"For every real number x and every real number y, if x is positive and y is negative, then $x y$ is negative."

"The product of a negative real number and a positive real number is always negative."

Example : "The sum of two positive integers is always positive".

"For every two integers, if they are both positive then their sum is positive."

"For all positive integers x and y, $x + y$ is positive."

$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$, where domain is all integers.

$\forall x \forall y (x + y > 0)$, where domain is all positive integers.

Order of Quantifiers

The order of the quantifiers is important, unless all quantifiers are universal or all quantifiers are existential.

Example : $\forall x \forall y$ P(x, y) $\equiv \forall y \forall x$ P(x, y)

Example : $\exists x \exists y$ P(x, y) $\equiv \exists y \exists x$ P(x, y)

Example : $\forall x \exists y$ P(x, y) is NOT equivalent to $\exists y \forall x$ P(x, y)

Example : P(x, y) : $x + y = 0$, where the domain of both x and y consists of all real numbers. Find the truth values of $\forall x \exists y$ P(x, y) and $\exists y \forall x$ P(x, y).

$\exists y \forall x$ P(x, y) denotes

"There is a real number y such that for every real number x, $x + y = 0$"

This statement is False.

$\forall x \exists y$ P(x, y) denotes

"For every real number x, there is a real number y such that $x + y = 0$."

This statement is true.

ILLUSTRATIVE EXAMPLES

Example 1. Translate into your own words and indicate whether it is true or false that

$(\exists\, u)_Z\, (4\,u^2 - 9 = 0)$

Sol. Consider $4\,u^2 - 9 = 0$

$\therefore \quad 4\,u^2 = 9 \Rightarrow u^2 = \frac{9}{4} \Rightarrow u = \pm\frac{3}{2}$, which are not integers.

$\therefore$ the equation $4\,u^2 - 9 = 0$ has a solution in integers is false.

Example 2. Use quantifier to say that $\sqrt{3}$ is not a rational number.

Sol. $\sim (\exists\, x)_Q\, (x^2 = 3)$.

Example 3. Over the universe of Books, define the propositions B (x) : x has a blue cover, M (x) : x is a mathematics book, U(x) : x is published in the United States and R(x, y) : The bibliography of x includes y.

Translate into words.

(*a*) $(\exists\, x)\ (\mathrm{M}\,(x) \wedge \sim \mathrm{B}\,(x))$

(*b*) $(\forall\, x)\ (\mathrm{M}\,(x) \wedge \mathrm{U}\,(x) \rightarrow \mathrm{B}\,(x))$

(*c*) $(\exists\, x)\ (\sim \mathrm{B}\,(x))$

(*d*) $(\exists\, y)\ ((\forall\, x)\ (\mathrm{M}\,(x) \rightarrow \mathrm{R}\,(x, y))$

Express using quantifiers :

(*e*) Every book with a blue cover is a mathematics book.

(*f*) There are mathematics books that are published outside the United States

(*g*) Not all books have bibliographies.

Sol. We have

B (x) : x has a blue cover

M (x) : x is a mathematics book

U (x) : x is published in the United States

R (x, y) : The bibliography of x includes y.

(*a*) There exists a mathematics book which has not a blue cover.

(*b*) Every mathematics book that is published in the United States has a blue cover.

(*c*) There exists a book whose cover is not blue.

(*d*) There exist a book that appears in the bibliograph of every mathematics book.

(*e*) $(\forall\, x)\ (\mathrm{B}\ (x) \rightarrow \mathrm{M}\,(x))$ (*f*) $(\exists\, x)\ (\mathrm{M}\,(x) \wedge \sim \mathrm{U}\,(x))$

(*g*) $(\exists\, x)\ (\forall\, y)\ (\sim \mathrm{R}\,(x, y))$

Example 4. Let M (x) be "x is a mammal." Let A (x) be "x is an animal" and let W (x) be "x is warm blooded."

(*a*) Translate into formula : Every mammal is warm blooded.

(*b*) Translate into English $(\exists x)(A(x) \wedge (\sim M(x))$.

Sol. We have

M (x) : x is a mammal

A (x) : x is an animal

W (x) : x is warm blooded

(*a*) $(\forall x)(M(x) \wedge W(x))$

(*b*) There is an animal which is not mammal.

Example 5. Let P(x) be the statement "x can speak Russian" and let Q(x) be the statement "x knows the computer language C++". Express each of these sentences in terms of P(x), Q(x), quantifiers and logical connectives. The universe of discourse for quantifier consists of all students at your school.

(*i*) There is a student at your school who can speak Russian and who knows C++.

(*ii*) There is a student at your school who can speak Russian but who does not know C++.

(*iii*) Every student at your school either can speak Russian or knows C++.

(*iv*) No student at your school can speak Russian or knows C++.

(Pbi. U. M C.A 2008)

Sol. It is given that

P(x) : x can speak Russian

Q(x) : x knows the computer language C++.

(*i*) We can express "There is a student at your school who can speak Russian and who knows C++" as

$$\exists x(P(x) \wedge Q(x))$$

(*ii*) We can express "There is a student at your school who can speak Russian but who does not know C++."

$$\exists x(P(x) \wedge \neg Q(x))$$

(*iii*) We can express "Every student at your school either can speak Russian or knows C++"

$$\forall x(P(x) \vee Q(x))$$

(*iv*) We can express "No student at your school can speak Russian or knows C++."

$$\forall(x) \neg(P(x) \vee Q(x))$$

EXERCISE 2 (*e*)

1. Translate in your own words and indicate whether it is true or false that :

   $(\exists\, x)_Q\, (3\, x^2 - 12 = 0)$

2. Use quantifier to say that $\sqrt{5}$ is not a rational number.

3. Use universal quantifiers to state that the sum of two rational numbers is rational.

4. Use universal quantifiers to state that the sum of any two real numbers is real.

5. Over the universe of real numbers, use quantifiers to say that the equation $a + 2\,x = b$, has a solution for all values of a and b.

6. Let C (x) be "x is cold blooded." Let F (x) be "x is a fish and let S (x) be "x lives in the sea."

   (*a*) Translate into a formula : Every fish is cold blooded.

   (*b*) Translate into English :

   $(\exists\, x)\, (S\,(x) \wedge \sim F\,(x))$ and $(\forall\, x)\, (F\,(x) \rightarrow S\,(x))$.

ANSWERS

1. False 2. $\sim (\exists\, x)_Q\, (x^2 = 5)$

3. $(\forall\, a)_Q\, (\forall\, b)_Q$ ($a + b$ is a rational number)

4. $(\forall\, a)_R\, (\forall\, b)_R$ ($a + b$ is a real number)

5. $(\forall\, a)_R\, (\forall\, b)_R\, (\exists\, x)_R\, (a + 2\,x = b)$

6. (*a*) $(\forall\, x)\, (F\,(x) \wedge C\,(x))$

   (*b*) There exist animals that live in the sea that are not fish.

   Every fish lives in the sea.