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RECURSION AND RECURRENCE RELATIONS

Art-1. Introduction

Anyone interested in computer science must be able to think recursively. The ability to understand definitions, concepts, algorithms, etc. that are presented recursively and the ability to put thoughts into a recursive framework is essential in computer science.

Definition : *A technique of defining a function, a set or an algorithm in terms of itself is a recursion.*

(P.T.U. B.C.A. I 2004)

Art-2. The Many Faces of Recursion

Recursive definition of factorial : (P.T.U. B.C.A. I 2005)

$$\underline{|n} = n\,(n-1)\,(n-2)\,\ldots\ldots\,3.\,2.\,1 \text{ for } n \geq 1$$

The recursive definition of $\underline{|n}$ is

$$\underline{|0} = 1 \text{ and } \underline{|n} = n\,\underline{|n-1}$$

Here nth term is expressed as function of previous term. $\underline{|0} = 1$ is called basis.

(*ii*) We know that

$$C\,(n, r) = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$$

The recursive definition of binomial coefficient C (n, k) where $n \geq 0$, $k \geq 0$ and $n \geq k$ is given by

$$C\,(n, n) = 1,$$

$$C\,(n, 0) = 1,$$

and $C\,(n, k) = C\,(n-1, k) + C\,(n-1, k-1)$ if $n > k > 0$

Here nth term is expressed as function of previous terms. C $(n, n) = 1$, C $(n, 0) = 1$ are basis.

Polynomials and Their Evaluation

Def. of Polynomial Expression (Non-Recursive). A polynomial of the form

$$a_n x^n + a_{n-1} + x^{n-1} + \ldots + a_1 x + a_0 \text{ where}$$

(*i*) $a_n \neq 0$

(*ii*) $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants is called a polynomial of degree n.

For example, $f(n) = 4n^3 + 2n^2 - 8n + 9$ is a polynomial of degree 3. It is known as a cubic polynomial.

Def. of Polynomial Expression (Recursive). Let S be a set of coefficients.

(*i*) A zeroth degree polynomial expression is an element of S.

(*ii*) For $n \geq 1$, an nth degree polynomial expression is an expression of the form $p(x)\,x + a$ where $p(x)$ is an $(n-1)$th degree polynomial expression and $a \in S$.

Now we verify that $f(n)$ is a third degree polynomial expression.

$$\begin{aligned} f(n) &= 4n^3 + 2n^2 - 8n + 9 \\ &= (4n^2 + 2n - 8)\,n + 9 \\ &= ((4n + 2)\,n - 8)\,n + 9 \\ &= ((((4)\,n) + 2)\,n - 8)\,n + 9 \end{aligned}$$

Here 4 is a zeroth degree polynomial. So, $(4)\,n + 2$ is a first degree polynomial. Therefore $((4)\,n + 2)\,n + 8$ is a second degree polynomial

$\therefore$ $f(n)$ is a third degree polynomial. The final expression for $f(n)$ is called telescoping form. $f(n)$ in original form needed five multiplication and three additions/subtractions to find $f(7)$. But telescoping form needs only three multiplication and three addition/subtractions. This is called Horner's method for evaluating a polynomial expression.

Art-3. Recursion

An example is presented recursively if every object is described in one of two forms. One form is by a simple definition, which is generally called the basis for the recursion. The second form is by a recursive description in which objects are described in terms of themselves, with the following qualification. For proper use of recursion, the objects should be expressed in terms of simpler objects, where simpler means closer to the basis of the recursion. The basis must be reached after a finite number of applications of the recursion.

For example, the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,....... is the sequence F defined by

$F_0 = 1$, $F_1 = 1$ and $F_k = F_{k-2} + F_{k-1}$ for $k \geq 2$. Here the basis is the specification of the first two number F_0 and F_1.

Let us try to find F_3.

$$\therefore \quad F_3 = F_1 + F_2$$
$$= F_1 + (F_0 + F_1)$$
$$= 1 + (1 + 1) = 1 + 2 = 3$$

ITERATION

In this method, we start with basis terms and work forward.

Now $F_2 = F_0 + F_1 = 1 + 1 = 2$

$$F_3 = F_1 + F_2 = 1 + 2 = 3$$
$$F_4 = F_2 + F_3 = 2 + 3 = 5$$
$$F_5 = F_3 + F_4 = 3 + 5 = 8$$

$\therefore$ fifth term F_5 of Fiboniacci sequence is determined by iterative method. This is called iterative computation of the Fibonacci sequence. F_5 can also be determined by using the recursive definition for F. But that solution will be lengthy.

Art-4. Induction and Recursion

We have already studied the following Axioms which are as under :

Axiom 1. $1 \in N$ *i.e.,* 1 is a natural number

Axiom 2. For each $n \in N$, there exists a unique natural number n^*, called the successor of n, and is given by $n^* = n + 1$.

Axiom 3. 1 is not the successor of any natural number.

Axiom 4. If $m, n \in N$ and $m^* = n^*$, then $m = n$

Axiom 5. If $A \subset N$ such that

(*i*) $1 \in A$ (*ii*) and $n \in A \Rightarrow n^* \in A$ then $A = N$.

This axiom 5 is also called the **Principle of Mathematical Induction.**

This definition of the positive integers is a recursive definition. Here the number 1 is the basis element and recursion is that if n is a positive integer, then its successor is also positive integer. Here n is the simple object and the recursion is of a forward type.

We give below an example in which induction proof by recursion is given.

Example : Let B be a sequence of numbers given by

$B_0 = 100$ and $B_k = (1{\cdot}08)\, B_{k-1}$ for $k \geq 1$. Then

$B_k = 100\ (1{\cdot}08)^k,\ k \geq 0.$

Sol. Here $B_0 = 100, B_k = (1{\cdot}08)^k, k \geq 0$

If $k = 0$, then $B_0 = 100\,(1{\cdot}08)^0 = 100 \times 1 = 100$

$\therefore$ result is true for $k = 0$

Assume that for some $k > 0$, $B_k = 100\,(1{\cdot}08)^k$

$$\begin{aligned} B_{k+1} &= (1{\cdot}08)\, B_k \\ &= (1{\cdot}08)\,[100\,(1{\cdot}08)^k] \\ &= 100\,(1{\cdot}08)^{k+1} \end{aligned}$$

$\therefore$ result is true for $k + 1$

$\therefore$ result is true by mathematical induction.

Note. The result proved above for B is known as a closed from expression. It contains no recursion or summation signs.

ILLUSTRATIVE EXAMPLES

Example 1. Let $p(x) = x^5 + 3x^4 - 15x^2 + x - 10$. Write $p(x)$ in telescoping form.

Sol. We have

$$\begin{aligned} p(x) &= x^5 + 3x^4 - 15x^2 + x - 10 \\ &= (x^4 + 3x^3 - 15x + 1)\,x - 10 \\ &= ((x^3 + 3x^2 - 15)\,x + 1)\,x - 10 \\ &= (((x^2 + 3x)\,x - 15)\,x + 1)\,x - 10 \\ &= (((x\,(x + 3))\,x - 15)\,x + 1)\,x - 10 \\ &= (((x\,((x) + 3)))\,x - 15)\,x + 1)\,x - 10 \end{aligned}$$

which is required telescoping form.

Example 2. Apply recursion formula as well as method of iteration to find B(3) of the sequence

$$B(0) = 2 \text{ and } B(k) = B(k-1) + 3 \text{ for } k \geq 1.$$

Sol. By recursion formula

$$\begin{aligned} B(3) &= B(2) + 3 && [\because B(k) = B(k-1) + 3] \\ &= (B(1) + 3) + 3 \\ &= ((B(0) + 3) + 3) + 3 \\ &= ((2 + 3) + 3) + 3 && [\because B(0) = 2] \\ &= 11 \end{aligned}$$

By Iteration Method

$B(1) = B(0) + 3 = 2 + 3 = 5$

$B(2) = B(1) + 3 = 5 + 3 = 8$

$B(3) = B(2) + 3 = 8 + 3 = 11$

Example 3. Determine C (5, 2) by the recursive definition of binomial coefficient.

Sol. Recursive definition of binomial coefficient is

$C(n, k) = C(n-1, k) + C(n-1, k-1)$ ...(1)

where $n > k > 0$ and $C(n, n) = 1, C(n, 0) = 1$

Put $n = 5, k = 2$ in (1)

$\therefore$ $C(5, 2) = C(4, 2) + C(4, 1)$ ...(2)

Put $n = 4, k = 2$ in (1)

$\therefore$ $C(4, 2) = C(3, 2) + C(3, 1)$ ...(3)

Put $n = 3, k = 2$ in (1)

$\therefore$ $C(3, 2) = C(2, 2) + C(2, 1)$ ...(4)

Put $n = 2, k = 1$ in (1)

$\therefore$ $C(2, 1) = C(1, 1) + C(1, 0)$

$= 1 + 1 = 2$

$\therefore$ from (4), $C(3, 2) = 1 + 2 = 3$ ...(5)

Put $n = 3, k = 1$ in (1)

$C(3, 1) = C(2, 1) + C(2, 0)$

$= 2 + 1 = 3$ ...(6)

$\therefore$ from (3), (5), (6),

$C(4, 2) = 3 + 3 = 6$ ...(7)

Put $n = 4, k = 1$ in (1)

$\therefore$ $C(4, 1) = C(3, 1) + C(3, 0)$

$= 3 + 1 = 4$ ...(8)

From (2), (7), (8), we get,

$C(5, 2) = 6 + 4 = 10$

Example 4. Let a and b denote the positive integers. Suppose a function Q is defined as

$$Q(a, b) = \begin{cases} 0 & \text{if } a \leq b \\ Q(a-b, b)+1 & \text{if } b \leq a \end{cases}$$

Find Q(2, 3) and Q(14, 3)

Sol. We have

$$Q(a, b) = \begin{cases} 0 & \text{if } a \leq b \\ Q(a-b, b)+1 & \text{if } b \leq a \end{cases}$$

$\therefore \quad Q(2, 3) = 0$ ...(1)

[$\because$ Q $(a, b) = 0$ if $\leq b$]

Now put $a = 14, b = 3$ in (1)

$\therefore \quad Q(14, 3) = Q(14 - 3, 3) + 1$

$\therefore \quad Q(14, 3) = Q(11, 3) + 1$ ...(2)

Put $a = 11, b = 3$ in (1)

$\therefore \quad Q(11, 3) = Q(11 - 3, 3) + 1$

$\therefore \quad Q(11, 3) = Q(8, 3) + 1$ ...(3)

Put $a = 8, b = 3$ in (1)

$\therefore \quad Q(8, 3) = Q(8 - 3, 3) + 1$

$\therefore \quad Q(8, 3) = Q(5, 3) + 1$ ...(4)

Put $a = 5, b = 3$ in (1)

$\therefore \quad Q(5, 3) = Q(5 - 3, 3) + 1$

$\therefore \quad Q(5, 3) = Q(2, 3) + 1$

$= 0 + 1$

$\therefore \quad Q(5, 3) = 1$

$\therefore$ from (4), $Q(8, 3) = 1 + 1 = 2$

$\therefore$ from (3), $Q(11, 3) = 2 + 1 = 3$

$\therefore$ from (2), $Q(14, 3) = 3 + 1 = 4$

$\therefore$ we have

$Q(2, 3) = 0$ and $Q(14, 3) = 4$

EXERCISE 9 (*a*)

1. Let $p(x) = 5x^4 + 12x^3 - 6x^2 + x + 6$. Write $p(x)$ in telescoping form.
2. Determine C(5, 3) by the recursive definition of binomial coefficient.
3. Define the sequence L by $L_0 = 5$ and for $k \geq 1$, $L_k = 2L_{k-1} - 7$. Determine L_4. Prove by induction that $L_k = 7 - 2^{k+1}$.
4. Write short note on Recursion. (P.T.U. B.C.A. I 2004)

ANSWERS

1. $(((5(x) + 12)x - 6)x + 1)x + 6$ 2. 10

rt-5. Introduction to Recurrence Relation

Harish has four children Rajesh, Suresh, Deepak and Vikram. Rajesh is 6 years lder than Suresh, Suresh is 8 years older than Deepak and Deepak is 5 years older than karam. Vikram is 4 years old. Therefore Rajesh, Suresh, Deepak and Vikram are of 23 ars, 17 years, 9 years and 4 years. The ages of four children are known only when age one child is given.

From above, we make following observations :

1. Using our prior knowledge is a concise way to give information.
2. Some efforts are to be made to make use of the knowledge we already have.
3. We can refer to some prior knowledge in successive steps.

Such a chain of references can only be terminated when we reach a point when we ow explicitly what to do without referring to other prior knowledge.

't-6. Sequence

Students are already familiar with sequences. For their revision, we again define quence.

Let **N** denote the set of natural numbers 1, 2, 3,... and **Z** the set of integers...., – 3, – 1, 0, 1, 2, 3, ... A mapping $S : \mathbf{N} \to \mathbf{Z}$ is called a sequence of integers. The image any natural number k is generally written as $S(k)$ or S_k and is known as the kth term of k is called the index or argument. The first, second, third,......, nth,......terms of a quence are also denoted by $a_1, a_2, a_3, \ldots, a_n, \ldots$ Numeric function, discrete function . are also used for sequence.

When sequences of numeric functions can be expressed in a compact form, it is lled **closed form expression.** Sequence 0, 0, 2, 6, 12, 20,..., has closed from expression $(k) = k^2 - k$; $k \geq 0$.

A closed from of expression of $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots n^2$ is $\frac{n(n+1)(2n+1)}{6}$.

Consider the numeric function

$3^0, 3^1, 3^2, \ldots\ldots, 3^r, \ldots\ldots$

This function can be specified by a general expression for a_r, namely,

$a_r = 3^r, r \geq 0$

There is another way to represent this numeric function. This representation is

$a_r = 3\, a_{r-1}, a_0 = 1$

Again Fibonacci sequence of numbers *i.e.*, 1, 1, 2, 3, 5, 8, 13, 21, 34,...

can be described by the relations

$a_r = a_{r-1} + a_{r-2}\ ,\ \ a_0 = 1,\ a_1 = 1.$

Now we are in a position to define recurrence relation.

RECURRENCE RELATION

For a numeric function $(a_0, a_1, a_2, \ldots\ldots, a_r, \ldots)$, an equation relating a_r, for any r, to one or more of the a_i's, $i < r$, is called a recurrence relation. A recurrence relation is also called a **difference equation.**

From the definition of recurrence relation, it is clear that we can carry out a step-by-step computation to determine a_r from $a_{r-1}, a_{r-2}, \ldots\ldots$, to determine a_{r+1} from a_r, $a_{r-1}, \ldots$, and so on, provided that the value of the function at one or more points is given so that the computation can be initiated. These given values of the functions are called **boundary conditions.** Thus a numeric function can be described by a recurrence relation together with an appropriate set of boundary conditions. The numeric function is also known as the solution of the recurrence relation.

Order of the Recurrence Relation

The order of a recurrence relation is the difference between the largest and the smallest subscript appearing in the relation.

e.g. (*i*) $a_r - 4\, a_{r-1} + 4a_{r-2} = 0$ is a recurrence relation of order 2.

(*ii*) $a_{n+3} - a_{n+2} + a_{n+1} - a_n = 0$ is a recurrence relation of order 3.

Degree of the Recurrence Relation

The degree of a recurrence relation is the highest power of a_n occurring in that relation.

e.g. $a_r - 4\, a_{r-1} + 4a_{r-2} = 0$ is a recurrence relation of degree one.

Art-7. Linear Recurrence Relation with Constant Coefficients

A recurrence relation of the form

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \ldots + c_k a_{r-k} = f(r) \quad \ldots(1)$$

where c_i's are constants, is called a linear recurrence relation with constant coefficients. The recurrence relation (1) is known as kth-order recurrence relation provided that both c_0 and c_k are non-zero. In simple words, order of a recurrence relation is difference between the highest and lowest subscript.

For example.

(*i*) $2 a_r + 5 a_{r-1} = 3^r$

is a first order linear recurrence relation with constant coefficients.

(*ii*) $a_r + 8 a_{r-2} = r^2 + 4$

is a second order linear recurrence relation with constant coefficients.

(*iii*) $D_k + 3D(k-3) = 5k^2$ is a third order liner recurrence relation with constant coefficients.

Art-8. Determination of Recurrence Relation from Solutions

Consider the closed from expression

$$A(k) = 7 \cdot 5^k, \ k \geq 0$$

If $\quad k \geq 1, A(k) = 7 \cdot 5^k = 7 \cdot (5 \cdot 5^{k-1}) = 5(7 \cdot 5^{k-1}) = 5A(k-1)$

$\therefore \quad A(k) - 5A(k-1) = 0$ and $A(0) = 7$

defines linear recurrence relation.

$\therefore$ from a closed form expression (*i.e.*, solution), we have determined linear recurrence relation.

Students can easily prove the following results :

Closed form expression	Recurrence relation
$A(k) = 9 \cdot 2^k, \ k \geq 0$	$A(k) - 2A(k-1) = 0$
$A(k) = 5k + 7$	$A(k) - (k-1) = 5$

Art-9. Homogeneous Recurrence Relation

A recurrence relation is called a linear non-homogeneous relation if it is of the form

$$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = f(k)$$

where $f(k)$ is a function of k or a constant and $C_1, C_2, \ldots\ldots, C_n$ are constants. The order of recurrence relation is r.

If $f(k) = 0$, then recurrence relation is called **homogeneous linear relation.**

Art-10. Characteristic Equation

From the discussion in a previous article, it is clear that any closed form expression for a sequence that contains exponential expression 2^k, 3^k, 5^k, etc. and polynomial expression will be solution of finite order linear relation. Also a finite order linear relation defines a closed form expression.

The recurrence relation $A(k) - 2A(k-1) = 0$ *i.e.*, $A(k) = 2A(k-1)$ is a first order homogeneous relation and is obtained from an expression that contains powers of 2.

In general, the solution of $A(k) = a\,A(k-1)$ would contain a^k. Actually the solution is $A(k) = A(0)\,a^k$, where A(0) is the value of initial conditions.

Consider the function

$$S(k) - 5\,S(k-1) + 6\,S(k-2) = 0,\ S(0) = 3,\ S(1) = 5. \quad \ldots(1)$$

Its solutions will be of the form $b\,a^k$ where a, b are non-zero constants to be determined.

Now $S(k) = b\,a^k$, $S(k-1) = b\,a^{k-1}$, $S(k-2) = b\,a^{k-2}$

$\therefore$ from (1), we get,

$$b\,a^k - 5\,b\,a^{k-1} + 6\,b\,a^{k-2} = 0$$

Dividing both sides by $b\,a^{k-2} \neq 0$, we get, $a^2 - 5a + 6 = 0$ ...(2)

This equation is **characteristic equation** of the recurrence relation.

From (2), $(a-2)(a-3) = 0$

$\therefore$ $a = 2, 3$

2, 3 are known as **characteristic roots.**

$\therefore$ relation (1) is true for

$$S(k) = b_1\,2^k + b_2\,3^k$$

where b_1, b_2 are real numbers. ...(3)

(3) gives us the **general solution** of the recurrence relation (1).

If initial conditions are not given, then this is required solution.

Here initial condition $S(0) = 3$, $S(1) = 5$ are given.

Now $S(0) = 3 \Rightarrow b_1 + b_2 = 3$ ...(4)

$S(1) = 5 \Rightarrow 2\,b_1 + 3\,b_2 = 5$ ...(5)

Multiplying (4) by 2, we get

$$2\,b_1 + 2\,b_2 = 6 \quad \ldots(6)$$

Subtracting (6) from (5), we get

$$b_1 = -1$$

$\therefore$ from (4), $-1 + b_2 = 3$ or $b_2 = 4$

$\therefore$ solution is $S(k) = 4 \cdot 2^k - 1 \cdot 3^k$

Now we are in a position to define characteristic equation.

Def. If the nth order linear homogeneous relation is

$$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = 0$$

then the characteristic equation is

$$a^n + C_1 a^{n-1} + \ldots + C_{n-1} a + C_n = 0$$

Roots of characteristic equations are called characteristic roots. These roots may be real or imaginary.

Art-11. Method for finding Solutions of Recurrence Relations

Let the recurrence relation be

$$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = 0 \quad \ldots(1)$$

Step I. Write down the characteristic equation

$$a^n + C_1 a^{n-1} + C_2 a^{n-2} + \ldots + C_{n-1} a + C_n = 0 \quad \ldots(2)$$

Step II. Solve (2) and let its roots be $a_1, a_2, \ldots, a_n$.

Step III. Case I. If roots are different, then general solution is

$$S(k) = b_1 a_1^k + b_2 a_2^k + \ldots + b_n a_n^k$$

Case II. If two real roots a_1, a_2 are such that $a_1 = a_2$, then solution is

$$S(k) = (b_1 + b_2 k) a_1^k + b_3 a_3^k + \ldots + b_n a_n^k \text{ and so on.}$$

Art-12. Solution of Non-homogeneous Finite Order Linear Recurrence Relation

Let non-homogeneous recurrence relation of order k be

$$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = f(k) \quad \ldots(1)$$

where $C_1, C_2, \ldots, C_n$ are constants and $f(k)$ is a function of k.

The solution of (1) consists of two parts

(*i*) homogeneous solution

(*ii*) particular solution.

Let $S^{(h)}(k)$ be a homogeneous solution of

$$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = 0$$

and $S^{(p)}(k)$ be the particular solution containing no arbitrary constants of the relations $S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + S(k-n) = f(k)$.

Then the general solution is

$$S(k) = S^{(h)} + S^{(p)}(k)$$

To Find $S^p(k)$

Case I When $f(k)$ is a constant d

Let a particular solution be given by $S(k) = d$, then form (1),

$$d + C_1 d + C_2 d + \ldots + C_n d = f(k)$$

or $(1 + C_1 + C_2 + \ldots + C_n)\, d = f(k)$

$$\therefore \quad d = \frac{f(k)}{1 + C_1 + C_2 + \ldots + C_n}$$

If $1 + C_1 + C_2 + \ldots + C_n = 0$, then it is a case of failure and in this case particular solution is $S(k) = k\,d$

If this also fails, then particular solution is taken as

$S(k) = k^2 d$ and so on.

Case II. When $f(k)$ is a linear function *i.e.*, $f(k) = p_0 + p_1 k$

Let $S(k) = d_0 + d_1 k$ be a particular solution.

$\therefore$ from (1), we get,

$$(d_0 + d_1 k) + C_1 [d_0 + d_1 (k-1)] + C_2 [d_0 + d_1 (k-2)] + \ldots + C_n [d_0 + d_1 (k-n)] = f(k)$$

But $f(k) = p_0 + p_1 k$

$$\therefore \quad d_0 [1 + C_1 + C_2 + \ldots + C_n] - C_1 d_1 - 2 C_2 d_1 \ldots - n C_n d_1$$

$$+ d_2 [d_1 + C_1 d_1 + \ldots + C_n d_1] = p_0 + p_1 k$$

Equating coefficients of

Constant) $d_0 [1 + C_1 + C_2 + \ldots\ldots + C_n] - d_1 [C_1 + 2C_2 + \ldots + n C_n] = p_0$

k) $d_1 [1 + C_1 + C_2 + \ldots\ldots + C_n] = p_1$

$\therefore$ particular solution is known as d_0 and d_1 are known.

Case III. When $f(k)$ is an mth degree polynomial

Let $f(k) = p_0 + p_1 k + p_2 k^2 + \ldots\ldots + p_m k^m$

$\therefore$ particular solutions is

$$S(k) = d_0 + d_1 k + d_2 k^2 + \ldots\ldots + d_m k^m$$

It should be noted that when the particular solution contains any term similar to that of homogeneous solution, we multiply the particular solution by k.

Case IV. When $f(k)$ is an exponential function $f(k) = p\, a^k$

$\therefore$ particular solution is given by

$$S(k) = d\, a^k$$

Putting $S(k) = d\, a^k$ in (1), value of d can be determined.

If the homogeneous solution contains a term containing a^k, then we have

$$S(k) = d\, k\, a^k$$

It the homogeneous solution contains a term containing $k\, a^k$, then we have

$S(k) = d\, k^2\, a^k$ an so on.

Note. In stead of recurrence relation

$$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = 0$$

we can have recurrence relation as

$$s_n + C_1 s_{n-1} + C_2 s_{n-2} + \ldots + C_n s_{n-r} = 0$$

ILLUSTRATIVE EXAMPLES

Example 1. Solve the following recurrence equation using technique for linear recurrence relations with constant coefficients.

$$a_n = 6\, a_{n-1} - 8\, a_{n-2} \text{ and } a_0 = 4,\ a_1 = 10.$$ (P.T.U. B.C.A.I 2004)

Sol.

$$a_n - 6\, a_{n-1} + 8\, a_{n-2} = 0$$

Put $a_n = b^n$

$$b^n - 6\, b^{n-1} + 8\, b^{n-2} = 0$$

$$b^{n-2}(b^2 - 6\, b + 8) = 0$$

$$b^2 - 6\, b + 8 = 0$$

$$(b-2)(b-4) = 0$$

$$b = 2, \; b = 4$$

$$a_n = C_1 . 2^n + C_2 . 4^n \quad ...(1)$$

Put $n = 0$ in (1)

$$a_0 = C_1 . 2^0 + C_2 . 4^0$$

$$4 = C_1 + C_2 \quad ...(2)$$

Put $n = 1$ in (1)

$$a_1 = C_1 . 2^1 + C_2 . 4^1$$

$$10 = 2\,C_1 + 4\,C_2$$

$$5 = C_1 + 2\,C_2 \quad ...(3)$$

Subtract (2) from (3)

$$1 = C_2$$

Put in (2)

$$4 = C_1 + 1$$

$$C_1 = 3$$

Put in (1)

$$a_n = 3 . 2^n + 1 . 4^n$$

Example 2. Solve :

$$S(k) - 10\,S(k-1) + 9\,S(k-2) = 0 \quad S(0) = 3, \; S(1) = 11.$$

(P.T.U. B.C.A.-I 2006)

Sol. $S(k) - 10\,S(k-1) + 9\,S(k-2) = 0$

Put $S(k) = a^k$

$$a^k - 10\,a^{k-1} + 9\,a^{k-2} = 0$$

$$a^{k-2}(a^2 - 10\,a + 9) = 0$$

$$\Rightarrow \quad a^2 - 10\,a + 9 = 0$$

$$\Rightarrow \quad (a-1)(a-9) = 0$$

$$a = 1, \; a = 9$$

$$\therefore \quad S(k) = C_1 . 1^k + C_2 . 9^k$$

$$S(k) = C_1 + C_2 . 9^k \qquad \text{...(1)}$$

Put $k = 0$ in (1)

$$S(0) = C_1 + C_2 . 9^0$$

$$3 = C_1 + C_2 \qquad \text{...(2)}$$

Put $k = 1$ in (1)

$$S(1) = C_1 + C_2 . 9^1$$

$$11 = C_1 + 9 C_2 \qquad \text{...(3)}$$

Subtract (2) from (3)

$$8 = 8 C_2$$

$$C_2 = 1$$

Put in (2)

$$3 = C_1 + 1$$

$$C_1 = 2$$

Put in (1)

$$S(k) = 2 + 9^k$$

Example 3. Solve

$$s_n - 4 s_{n-1} - 11 s_{n-2} + 30 s_{n-3} = 0,$$

when $s_0 = 0, s_1 = -35, s_2 = -85$

Sol. The given recurrence relation is

$$s_n - 4 s_{n-1} - 11 s_{n-2} + 30 s_{n-3} = 0$$

Its characteristic equations is

$$a^n - 4 a^{n-1} - 11 a^{n-2} + 30 a^{n-3} = 0$$

or $\quad a^{n-3}(a^3 - 4a^2 - 11a + 30) = 0$

or $\quad a^3 - 4a^2 - 11a + 30 = 0 \qquad \text{...(1)}$

Put $a = 2$

$\therefore \quad 8 - 16 - 22 + 30 = 0$

$\therefore \quad 0 = 0$

$\therefore \quad a = 2$ is a root of (1)

2	1	– 4	– 11	30
		2	– 4	– 30
	1	– 2	– 15	0

Remaining roots of (1) are given by

$$a^2 - 2a - 15 = 0$$

$\therefore \quad (a - 5)(a + 3) = 0$

$\therefore \quad a = 5, -3$

$\therefore$ roots of (1) are 2, 5, – 3

$\therefore$ general solution is

$$s_n = C_1 2^n + C_2 5^n + C_3 (-3)^n \quad \text{...(2)}$$

Put $n = 0$ in (2)

$$\therefore \quad s_0 = C_1 . 2^0 + C_2 . 5^0 + C_3 (-3)^0$$

Put $n = 0$ in (2)

$$\therefore \quad s_0 = C_1 . 2^0 + C_2 . 5^0 + C_3 (-3)^0$$

$$\therefore \quad C_1 + C_2 + C_3 = 0 \quad \text{...(3)}$$

$[\because s_0 = 0]$

Put $n = 1$ in (2)

$$\therefore \quad s_1 = C_1 . 2^1 + C_2 . 5^1 + C_3 (-3)^1$$

$$\therefore \quad 2C_1 + 5C_2 - 3C_3 = -35 \quad \text{...(4)}$$

Put $n = 2$ in (2)

$$\therefore \quad s_2 = C_1 . 2^2 + C_2 . 5^2 + C_3 (-3)^2$$

$$\therefore \quad 4C_1 + 25C_2 + 9C_3 = -85 \quad \text{...(5)}$$

Multiplying (3) by 3, we get

$$3C_1 + 3C_2 + 3C_3 = 0 \quad \text{...(6)}$$

Adding (4) and (6), we get

$$5C_1 + 8C_2 = -35 \qquad \text{...(7)}$$

Multiplying (4) by 3, we get

$$6C_1 + 15C_2 - 9C_3 = -105 \qquad \text{...(8)}$$

Adding (5) and (8), we get

$$10C_1 + 40C_2 = -190$$

$$\therefore \quad C_1 + 4C_2 = -19 \qquad \text{...(9)}$$

Multiplying (9) by 5, we get

$$5C_1 + 20C_2 = -95 \qquad \text{...(10)}$$

Subtracting (11) from (7), we get

$$-12C_2 = 60 \Rightarrow C_2 = -5$$

$\therefore$ from (9), $C_1 - 20 = -19 \Rightarrow C_1 = 1$

$\therefore$ from (1), $1 - 5 + C_3 = 0 \Rightarrow C_3 = 4$

Putting values of C_1, C_2, C_3 in (2), we get

$$s_n = 2^n - 5 \cdot 5^n + 4 . (-3)^n$$

which is required solution.

ample 4. Solve the recurrence relation :

$$\sqrt{a_n} = \sqrt{a_{n-1}} + \sqrt{a_{n-2}} \qquad \text{(P.T.U. B.C.A.I 2005)}$$

l. Let $\sqrt{a_n} = b_n$

$\Rightarrow \quad a_n = b_n^2$

Put in $\sqrt{a_n} = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$

$$b_n = b_{n-1} + b_{n-2}$$

$$\Rightarrow \quad b_n - b_{n-1} + b_{n-2} = 0$$

Let $b_n = m^n$

$$m^n - m^{n-1} - m^{n-2} = 0$$

$$m^{n-2}(m^2 - m - 1) = 0$$

$$\Rightarrow \quad m^2 - m - 1 = 0$$

$$\Rightarrow \quad m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad m = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$\therefore \quad b_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solution of given relation is

$$a_n = b_n^2 = \left[C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n\right]^2$$

Example 5. Solve : $s_n - 2s_{n-1} + s_{n-2} = 1, s_0 = 2, s_1 = 5{\cdot}5$

Sol. The given non-homogeneous recurrence relation is

$$s_n - 2s_{n-1} + s_{n-2} = 1$$

It solution will consist of two parts *i.e.,* homogeneous solution and particular solution.

Homogeneous solution

The associated homogeneous relation is

$$s_n - 2s_{n-1} + s_{n-2} = 0$$

Its characteristic equation is

$$a^n - 2a^{n-1} + a^{n-2} = 0$$

or $\quad a^{n-2}(a^2 - 2a + 1) = 0$

or $\quad a^2 - 2a + 1 = 0$

$\therefore \quad (a-1)^2 = 0 \Rightarrow a = 1, 1$

$\therefore \quad$ homogeneous relation is

$$s_n^{(h)} = (C_1 + C_2 n).1^n$$

Particular solution

$$f(n) = 1$$

$\therefore$ particular solution is

$$s_n = p$$

Putting $s_n = p$ in given equation, we get

$$p - 2p + p = 1 \text{ or } 0 = 1, \text{ which is impossible}$$

We try $s_n = np$

Putting $s_n = np$ in given equation, we get,

$$np - 2(n-1)p + (n-2)p = 1$$

$$\therefore \quad np - 2np + 2p + np - 2p = 1$$

$$\therefore \quad 0 = 1, \text{ which is impossible.}$$

We try $s_n = n^2 p$

Putting $s_n = n^2 p$ in given equation, we get

$$n^2 p - 2(n-1)^2 p + (n-2)^2 p = 1$$

$$\therefore \quad n^2 p - 2n^2 p + 4np - 2p + np^2 - 4np + 4p = 1$$

$$\therefore \quad 2p = 1 \Rightarrow p = \frac{1}{2}$$

$\therefore$ particular solution is $s_n^{(p)} = \frac{1}{2} n^2$

$\therefore$ general solution of given equation is

$$s_n = s_n^{(h)} + s_n^{(p)}$$

or $\quad s_n = C_1 + C_2 n + \frac{1}{2} n^2$

Now $s_0 = 2 \quad \Rightarrow C_1 = 2$

Again $s_1 = \frac{11}{2} \quad \Rightarrow C_1 + C_2 + \frac{1}{2} = \frac{11}{2}$

$$\therefore \quad 2 + C_2 + \frac{1}{2} = \frac{11}{2} \quad \Rightarrow C_2 = 3$$

$\therefore$ general solution is

$$s_n = 2 + 3n + \frac{1}{2} n^2$$

Example 6. Solve the recurrence relation $a_k = 8\, a_{k-1} + 10\, k - 1$ with initial condition $a_0 = 1$.

(P.T.U. B.C.A.I 2003)

Sol.

$$a_k - 8\, a_{k-1} = 10\, k - 1 \qquad \ldots(1)$$

Homogeneous equation is

$$a_k - 8\, a_{k-1} = 0$$

Let $a_k = b^k$

$$b^k - 8\, b^{k-1} = 0$$

$$b^{k-1}(b - 8) = 0$$

$$\Rightarrow \quad b - 8 = 0$$

$$\Rightarrow \quad b = 8$$

$$a^h(k) = C_1 . 8^k$$

Now we find particular solution.

Since R.H.S. of (1) is linear.

So let $\quad a^p(k) = C + d\, k$

Put in (1)

$$C + d\, k - 8\, (C + d\, (k - 1)) = 10\, k - 1$$

$$C + d\, k - 8\, C - 8\, d\, k + 8\, d = 10\, k - 1$$

$$-7\, d\, k - 7\, C + 8\, d = 10\, k - 1$$

Comparing coefficients of k

$$-7\, d = 10$$

$$d = -\frac{10}{7}$$

Comparing constants

$$-7C + 8\, d = -1$$

$$-7C - \frac{80}{7} = -1$$

$$-7C = -1 + \frac{80}{7}$$

$$-7C = \frac{73}{7}$$

$$C = -\frac{73}{49}$$

$$a^p(k) = -\frac{73}{49} - \frac{10k}{7}$$

$$a(k) = a^h(k) + a^p(k)$$

$$= C_1 . 8^k - \frac{73}{49} - \frac{10}{7}k \quad \text{...(2)}$$

Put $k = 0$

$$a(0) = C_1 . 8^0 - \frac{73}{49} - 0$$

$$1 = C_1 - \frac{73}{49}$$

$$C_1 = 1 + \frac{73}{49} = \frac{122}{49}$$

Put in (2)

$$a(k) = \frac{122}{49} . 8^k - \frac{73}{49} - \frac{10}{7}k.$$

Example 7. Solve : $s_n - 4s_{n-1} + 4s_{n-2} = 3n + 2^n$, with $s_0 = s_1 = 1$.

Sol. The given non-homogeneous recurrence relation is

$$s_n - 4s_{n-1} + 4s_{n-2} = 3n + 2^n$$

Its solution will consist of two parts *i.e.*, homogeneous solution and particular solution.

Homogeneous Solution

The associated homogeneous relation is

$$s_n - 4s_{n-1} + 4s_{n-2} = 0$$

Its characteristic equation is

$$a^n - 4a^{n-1} + 4a^{n-2} = 0$$

or $\quad a^{n-2}(a^2-4a+4)=0$

or $\quad a^2-4a+4=0$

$\therefore \quad (a-2)^2=0$

$\therefore \quad a=2, 2$

$\therefore$ homogeneous solution is

$$s_n=(C_1+nC_2)2^n$$

Particular Solution

Here $f(n)=3n+2^n$

Since base 2 in 2^n is a characteristic root repeated twice

$\therefore$ particular solution is

$$s_n=cn+d+qn^2 2^n$$

$$s_{n-1}=c(n-1)+d+q(n-1)^2 2^{n-1}=cn+d-c+\frac{q}{2}(n^2-2n+1)2^n$$

$$s_{n-2}=c(n-2)+d+q(n-2)^2 2^{n-2}=cn+d-2c+\frac{q}{4}(n^2-4n+4)2^n$$

Putting value of s_n, s_{n-1}, s_{n-2} in given equation, we get

$$cn+d+qn^2 2^n-4cn-4d+4c-2q(n^2-2n+1)2^n$$
$$+4cn+4d-8c+q(n^2-4n+4)2^n=3n+2^n$$

$$\therefore \quad cn+d-4c+2^n(qn^2-2qn^2+4qn-2q+qn^2-4qn+4q)=3n+2^n$$

$$\therefore \quad cn+d-4c+2q.2^n=3n+2^n$$

Equating coefficients of like terms, we get,

$$c=3, d-4c=0, 2q=1$$

$$\therefore \quad c=3, d-12=0, q=\frac{1}{2}$$

$$\therefore \quad c=3, d=12, q=\frac{1}{2}$$

$\therefore$ particular solution is

$$s_n^{(p)}=3n+12+\frac{1}{2}n^2.2^n=3n+12+n^2 2^{n-1}$$

$\therefore$ general solution is

$$s(n) = s^h(n) + s^p(n)$$

$$s(n) = (C_1 + C_2 n)\, 2^n + 3n + 12 + n^2 2^{n-1} \quad \text{...(1)}$$

Now $s_0 = 1,\ s_1 = 1$

Put $n = 0$ and $n = 1$ in (1)

$$s(0) = (C_1 + 0)\, 2^0 + 0 + 12 + 0$$

$$1 = C_1 + 12$$

$$C_1 = -11$$

$$s(1) = (C_1 + C_2)\ 2 + 3 + 12 + 1$$

$$1 = (-11 + C_2)\, 2 + 16 = -22 + 2C_2 + 16$$

$$7 = 2C_2$$

$$C_2 = \frac{7}{2}$$

Put in (1), $$s(n) = \left(-11 + \frac{7}{2} n\right) 2^n + 3n + 12 + n^2 2^{n-1}$$

Example 8. Solve : $s_n - 4s_{n-1} + 3s_{n-2} = n^2$.

Sol. The given non-homogeneous recurrence relation is

$$s_n - 4s_{n-1} + 3s_{n-2} = n^2$$

Its solution will consist of two parts *i.e.,* homogeneous solution and particular solution.

Homogeneous Solution

The associated homogeneous relation is

$$s_n - 4s_{n-1} + 3s_{n-2} = 0$$

Its characteristic equation is

$$a^n - 4a^{n-1} + 3a^{n-2} = 0$$

Its characteristic equation is

$$a^n - 4a^{n-1} + 3a^{n-2} = 0$$

or $$a^{n-2}(a^2 - 4a + 3) = 0$$

or $$a^2 - 4a + 3 = 0$$

$\therefore$ $$(a-1)(a-3) = 0$$

$\therefore$ $$a = 1, 3$$

$\therefore$ homogeneous relation is

$$s_n^{(h)} = C_1 . 1^n + C_2 . 3^n$$

Particular Solution

Here $f(n) = n^2$

Let particular solution be

$$s_n = cn^2 + dn + f$$

This has a constant f. The homogeneous solution also has a constant C_1. Thus we multiply s_n by n and take this as s_n.

$\therefore$ $$s_n = cn^3 + dn^2 + fn$$

$$s_{n-1} = c(n-1)^3 + d(n-1)^2 + f(n-1)$$

$$s_{n-2} = c(n-2)^3 + d(n-2)^2 + f(n-2)$$

Putting s_n, s_{n-1}, s_{n-2} in given equation, we get,

$$cn^3 + dn^2 + fn - 4[c(n-1)^3 + d(n-1)^2 + f(n-1)] + 3[c(n-2)^3 + d(n-2)^2 + f(n-2)] = n^2$$

or $$cn^3 + dn^2 + fn - 4[c(n^3 - 3n^2 + 3n - 1) + d(n^2 - 2n + 1) + f(n-1)] + 3[c(n^3 - 6n^2 + 12n - 8) + d(n^2 - 4n + 4) + f(n-2)] = n^2$$

or $$cn^3 + dn^2 + fn - 4cn^3 + 12cn^2 - 12cn + 4c - 4dn^2 + 8dn - 4d - 4fn + 4f + 3cn^3 - 18cn^2 + 36cn - 24c + 3dn^2 - 12dn + 12d + 3fn - 6f = n^2$$

or $$-6cn^2 + 24cn - 4dn - 20c + 8d - 2f = n^2$$

or $$-6cn^2 + (24c - 4d)n + (-20c + 8d - 2f) = n^2$$

Equating coefficients,

n^2) $\quad -6c = 1 \quad \Rightarrow \quad c = -\frac{1}{6}$

n) $\quad 24c - 4d = 0 \quad \Rightarrow \quad -4 - 4d = 0 \quad \Rightarrow \quad d = -1$

Constant term) $\quad -20c + 8d - 2f = 0$

$\therefore \quad \frac{10}{3} - 8 - 2f = 0 \quad \Rightarrow \quad 2f = -\frac{14}{3} \quad \Rightarrow \quad f = -\frac{7}{3}$

$\therefore$ particular solution is

$$s_n^{(p)} = n\left(-\frac{1}{6}n^2 - n - \frac{7}{3}\right) = -\frac{1}{6}(n^3 + 6n^2 + 14n)$$

$\therefore$ general solution is

$$s_n = C_1 + C_2 . 3^n - \frac{1}{6}(n^3 + 6n^2 + 14n)$$

Example 9. Find particular solution of

$$a_r - 5a_{r-1} + 6a_{r-2} = 3r^2. \qquad \text{(P.T.U. B.C.A.I 2007)}$$

Sol. Let $a^p(r) = a + br + cr^2$ $\qquad$ [$\because$ R.H.S. is quadratic]

$$a_r = a + br + cr^2$$

$$a_{r-1} = a + b(r-1) + c(r-1)^2$$

$$a_{r-2} = a + b(r-2) + c(r-2)^2$$

Put in $a_r - 5a_{r-1} + 6a_{r-2} = 3r^2$

$$a + br + cr^2 - 5[a + b(r-1) + c(r-1)^2] + 6[a + b(r-2) + c(r-2)^2] = 3r^2$$

$$a + br + cr^2 - 5a - 5br + 5b - 5cr^2 - 5c + 10cr + 6a + 6br - 12b$$
$$+ 6cr^2 + 24c - 24cr = 3r^2$$

$$2cr^2 + 2br - 14cr + 2a - 7b + 19c = 3r^2$$

Comparing coefficients

$2c = 3 \qquad 2b - 14c = 0 \qquad 2a - 7b + 19c = 0$

$c = \frac{3}{2}$ Put $c = \frac{3}{2}$ Put $c = \frac{3}{2}$, $b = \frac{21}{2}$

$$2b - 14 \times \frac{3}{2} = 0 \qquad 2a - 7 \times \frac{21}{2} + 19 \times \frac{3}{2} = 0$$

$$2b = 21 \qquad 2a - \frac{147}{2} + \frac{57}{2} = 0$$

$$b = \frac{21}{2} \qquad 2a = \frac{90}{2}$$

$$2a = 45$$

$$a = \frac{45}{2}$$

$$a^P(r) = \frac{45}{2} + \frac{21}{2}r + \frac{3}{2}r^2$$

Example 10. If solution of recurrence relation $a\,s_n + b\,s_{n-1} + c\,s_{n-2} = 6$ is $3^n + 4^n + 2$, find a, b, c.

Sol. Given non-homogeneous recurrence relation is

$$a\,s_n + b\,s_{n-1} + c\,s_{n-2} = 6 \qquad ...(1)$$

Given solution is

$$s_n = 3^n + 4^n + 2$$

$\therefore$ $$s_{n-1} = 3^{n-1} + 4^{n-1} + 2 = \frac{3^n}{3} + \frac{4^n}{4} + 2$$

and $$s_{n-2} = 3^{n-2} + 4^{n-2} + 2 = \frac{3^n}{9} + \frac{4^n}{16} + 2$$

Putting values of s_n, s_{n-1}, s_{n-2} in (1) we get,

$$a(3^n + 4^n + 2) + b\left(\frac{3^n}{3} + \frac{4^n}{4} + 2\right) + c\left(\frac{3^n}{9} + \frac{4^n}{16} + 2\right) = 6$$

Equating coefficients, we get,

$$a + \frac{b}{3} + \frac{c}{9} = 0 \qquad ...(2)$$

$$a + \frac{b}{4} + \frac{c}{16} = 0 \qquad ...(3)$$

$$2a+2b+2c=6$$

or $$a+b+c=3 \quad ...(4)$$

Subtracting (3) from (2), we get,

$$b\left(\frac{1}{3}-\frac{1}{4}\right)+c\left(\frac{1}{9}-\frac{1}{16}\right)=0$$

or $$\frac{b}{12}+\frac{7c}{144}=0 \Rightarrow b=-\frac{7}{12}c \quad ...(5)$$

Putting this value of b in (4), we get,

$$a-\frac{7c}{12}+c=3, \text{ or } a=3-\frac{5c}{12}$$

$$\therefore \quad a=\frac{36-5c}{12} \quad ...(6)$$

$$\frac{36-5c}{12}-\frac{7c}{48}+\frac{c}{16}=0$$

$\therefore$ $144-20c-7c+3c=0 \Rightarrow 24c=144 \Rightarrow c=6$

$\therefore$ from (6),

$$a=\frac{36-30}{12}=\frac{6}{12}=\frac{1}{2}$$

From (5),

$$b=-\frac{7}{12}\times 6=-\frac{7}{2}$$

$\therefore$ we have, $a=\frac{1}{2}, b=-\frac{7}{2}, c=6$

EXERCISE 9 (*b*)

1. Find general solution of recurrence relation

$$s_n-3s_{n-1}+2s_{n-2}=0$$

2. Solve

(*i*) $s_n-6s_{n-1}+9s_{n-2}=0$ where $s_0=1,\ s_1=9$.

(*ii*) $s_n-8s_{n-1}+12s_{n-2}=0$ where $s_0=54,\ s_1=308$

(iii) $s_n - 8\, s_{n-1} + 16\, s_{n-2} = 0,\ s_2 = 16, s_3 = 80$

(iv) $s_n - 7\ s_{n-2} + 6\, s_{n-3} = 0,\ s_0 = 8, s_1 = 6,\ s_2 = 22$

(v) $s_n - 9\, s_{n-1} + 18\, s_{n-2} = 0, s_0 = 1, s_1 = 4$

(vi) $s_n - 20\, s_{n-1} + 100\, s_{n-2} = 0,\ s_0 = 2,\ s_1 = 30.$

3. Solve $S(k) - 7\,S(k-1) + 10\,S(k-2) = 0, S(0) = 4, S(1) = 17.$

4. Solve $s_n + 5 s_{n-1} = 9, s_0 = 6.$ (P.T.U. B.C.A.I 2007)

5. Solve

(i) $s_n - 2\, s_{n-1} + 4 = 0$ where $s_0 = 5$

(ii) $s_n - 5\, s_{n-1} + 6\, s_{n-2} = 2,\ s_0 = 1, s_1 = -1$

(iii) $s_n - s_{n-1} - 6\, s_{n-2} = -30,\ s_0 = 20,\ s_1 = -5$

(iv) $s_n - 6\, s_{n-1} + 8\, s_{n-2} = 2$

(v) $s_n - 2\, s_{n-1} + s_{n-2} = 12,\ s_0 = 7,\ s_1 = 11$

6. Solve $s_n - 4 s_{n-1} + 3 s_{n-2} = 5^n.$

7. Solve $s_n - 4 s_{n-1} + 3 s_{n-2} = 3^n.$

8. Solve $s_n - 4 s_{n-1} + 4 s_{n-2} = (n+1) 2^n.$ (Pbi. U. B.C.A. 2011)

9. Solve

(i) $s_n - 6\, s_{n-1} + 9\, s_{n-2} = 3^n$

(ii) $s_n - 5\, s_{n-1} + 6\, s_{n-2} = 4^n$

(iii) $s_n - 5\, s_{n-1} + 6\, s_{n-2} = 5^n$

(iv) $s_n - 3\, s_{n-1} - 4\, s_{n-2} = 4^n$

10. Solve

(i) $s_n - 7\, s_{n-1} + 10\, s_{n-2} = 6 + 8\, n$ with $s_0 = 1, s_1 = 2$

(ii) $s_n - 6\, s_{n-1} + 8\, s_{n-2} = n. 4^n,\ s_0 = 8, s_1 = 22$

11. Find a particular solution of $S(n) - 4\,S(n-1) + 4\,S(n-2) = 2^n.$

12. Find a particular solution of

(i) $s_n + 5\, s_{n-1} + 6\, s_{n-2} = 3\, n^2 - 2\, n + 1$

(ii) $a_n + 5\, a_{n-1} + 6\, a_{n-2} = 42{\cdot}4^n$

(iii) $a_n + a_{n-1} = 3\, n \cdot 2^n$

13. Solve : $S(k) - 4S(k-1) + 4S(k-2) = 3\,k + 2^k$ where $S(0) = 1, S(1) = 1.$

(Pbi. U. B.C.A. Sept. 2009)

ANSWERS

1. $s_n = C_1 + C_2\, 2^n$

2. (i) $s_n = (1 + 2n) \,.\, 3^n$ (ii) $s_n = 4.2^n + 50.6^n$

(iii) $s_n = (2+n).4^{n-1}$ (iv) $s_n = 5 + 2^{n+1} + (-3)^n$

(v) $s_n = 2.3^{n-1}(1 + 2^{n-1})$ (vi) $s_n = (2+n).10^n$

3. $S(k) = 2^k + 3 \cdot 5^k$

4. $s_n = \frac{9}{2}(-5)^n + \frac{3}{2}$

5. (i) $s_n = 2^n + 4$ (ii) $s_n = 2.2^n - 2.3^n + 1$

(iii) $s_n = 4.3^n + 11(-2)^n + 5$ (iv) $s_n = C_1.2^n + C_2.4^n + \frac{2}{3}$

(v) $s_n = 7 - 2n + 6n^2$

6. $s_n = C_1 + C_2.3^n + \frac{1}{8}5^{n+2}$

7. $s_n = C_1 + C_2.3^n + \frac{n}{2}3^{n+1}$

8. $s_n = (C_1 + C_2 n)2^n + \left(\frac{1}{6}n + 1\right)n^2\, 2^n$

9. (i) $s_n = C_1.3^n + C_2 n3^n + \frac{1}{2}n^2\, 3^n$ (ii) $s_n = C_1 \,.\, 2^n + C_2 \,.\, 3^n + 8.4^n$

(iii) $s_n = C_1.2^n + C_2.3^n + \frac{1}{6}5^{n+2}$ (iv) $s_n = C_1.4^n + C_2(-1)^n + \frac{4}{5}n4^n$

10. (i) $s_n = -9 \cdot 2^n + 2 \cdot 5^n + 2n + 8$ (ii) $s_n = 5 \cdot 2^n + 3 \cdot 4^n + (n^2 - n)\, 4^n$

11. $S^{(p)}(n) = n^2\, 2^{n-1}$

12. (i) $s_n^{(p)} = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$ (ii) $a_n^{(p)} = 16.4^n$

(iii) $a_n^{(p)} = 3\left(\frac{2}{3}n + \frac{2}{9}\right)2^n$

13. $S(k) = \left(-11 + \frac{7}{2}k\right)2^k + 3k + 12 + k^2\, 2^{k-1}$

Art-13. Generating Function

(P.T.U. B.C.A.-I, 2004, 2006)

Let S be a sequence with terms S_0, S_1, S_2...... . Generating function G(S, z) of sequence S is infinite series

$$G(S, z) = \sum_{n=0}^{\infty} S_n z^n = S_0 + S_1 z + S_2 z^2 + S_3 z^3 + \ldots\ldots\infty$$

Example : Let sequence S be $1^2, 2^2, 3^2, \ldots\ldots$

$\therefore$ generating function G (S, z) is

$$G(S, z) = 1^2 . z^0 + 2^2 . z^1 + 3^2 . z^2 + \ldots\ldots\infty$$

$$= \sum_{n=0}^{\infty} (n+1)^2 z^n$$

Art-14. Generating Functions of Some Standard Sequences

I. $S_n = a, n \geq 0$

$$\therefore \quad G(S, z) = \sum_{n=0}^{\infty} S_n z^n$$

$$= \sum_{n=0}^{\infty} a z^n = a \sum_{n-0}^{\infty} z^n$$

$$= a(1 + z + z^2 + \ldots\ldots\infty)$$

$$= a\left(\frac{1}{1-z}\right) \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$$

$$\therefore \quad G(S, z) = \frac{a}{1-z}$$

II. $S_n = b^n, n \geq 0$

$$\therefore \quad G(S, z) = \sum_{n=0}^{\infty} S_n z^n = \sum_{n=0}^{\infty} b^n z^n$$

$$= \sum_{n=0}^{\infty} (bz)^n$$

$$= 1 + bz + (bz)^2 + (bz)^3 + \ldots\ldots\infty$$

$$\therefore \quad G(S, z) = \frac{1}{1-bz}$$

III. $S_n = c\,b^n, n \geq 0$

$$\therefore \quad G(S, z) = \sum_{n=0}^{\infty} S_n z^n$$

$$= \sum_{n=0}^{\infty} c b^n z^n = c \sum_{n=0}^{\infty} (bz)^n$$

$$= c\,[1 + bz + (bz)^2 + (bz)^3 + \ldots \infty]$$

$$= c\left(\frac{1}{1-bz}\right)$$

$$\therefore \quad G(S, z) = \frac{c}{1-bz}$$

IV. $S_n = n$

$$\therefore \quad G(S, z) = \sum_{n=0}^{\infty} S_n z^n = \sum_{n=0}^{\infty} n z^n$$

$$= 0 + z + 2z^2 + 3z^3 + \ldots \infty$$

$$= z\,(1 + 2z + 3z^2 + \ldots \infty)$$

$$= z\,(1-z)^{-2}$$

$$\therefore \quad G(S, z) = \frac{z}{(1-z)^2}$$

Art-15. Prove that generating function of sum of two sequences is equal to sum of their generating functions

Or

If $s_n = a_n + b_n$, then $G(s, z) = G(a, z) + G(b, z)$

Proof. We have

$$s_n = a_n + b_n$$

$$\therefore \quad G(s, z) = \sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n$$

$$\therefore \quad G(s, z) = G(a, z) + G(b, z)$$

Art-16. Method to find Generating Function and Sequence from its Recurrence Relation

Proof. Let recurrence relation be

$$s_n + C_1 s_{n-1} + C_2 s_{n-2} + \ldots + C_r s_{n-r} = 0 \text{ for } n \geq r.$$

Step I. Multiply both sides by z^n and sum up terms from $n = r$ to ∞

$$\therefore \quad \sum_{n=r}^{\infty} s_n z^n + C_1 \sum_{n=0}^{\infty} s_{n-1} z^n + C_2 \sum_{n-r}^{\infty} s_{n-2} z^n + \ldots + C_r \sum_{n=r}^{\infty} s_{n-r} z^n = 0$$

Step II. If $G(s, z) = \sum_{n=0}^{\infty} s_n z^n$ be generating function then write each term in terms of $G(s, z)$.

Step III. Solve the equation for $G(s, z)$.

Then using standard generating functions, sequence s_n can be found.

Art- 17. Some important Operations on Sequence

I. If s, t are two sequences of natural number n, then

(i) $(s + t)(n) = s(n) + t(n)$

(ii) $c\, s(n) = (c\, s)(n)$, where c is constant.

(iii) $(s\, t)(n) = s(n)\, t(n)$

II. Convolution Operation

Convolution for two sequences s and t is defined as

$$(s * t)(n) = \sum_{r=0}^{\infty} s(r)\, t(n-r)$$

III. Pop Operation $s \uparrow$ (read as s pop)

$$(s \uparrow)(n) = s(n+1)$$

IV. Push Operation $s \downarrow$ (read as s push)

$$(s \downarrow)(n) = \begin{cases} s(n-1) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

Def. If s be a sequences of numbers, we define

$$s \uparrow n = (s \uparrow (n-1)] \uparrow, \text{ if } n > 1 \text{ and } s \uparrow 1 = s \uparrow$$

and $\quad s \downarrow n = [s \downarrow (n-1)] \downarrow$ and $s \downarrow 1 = s \downarrow$

te 1. $(s \uparrow 3)(n) = ((s \uparrow 2) \uparrow)(n)$ $[\because s \uparrow n = [s \uparrow (n-)] \uparrow$

$= (((s \uparrow 1) \uparrow) \uparrow (n)$ $[s \uparrow 2 = (s \uparrow 1) \uparrow]$

$= ((((s\uparrow) \uparrow) \uparrow)(n)$

$= ((s \uparrow) \uparrow)(n+1)$

$= (s \uparrow)(n+2)$ $[\because (s \uparrow)(n) = s(n+1)]$

$= s(n+3)$

$\therefore \quad (s \uparrow 3)(n) = s(n+3)$

In general $(s \uparrow n)(m) = s(n+m)$

and $(s \downarrow n)\, m = \begin{cases} 0, \text{ if } m < n \\ s(m-n), \text{ if } m \geq n \end{cases}$

te 2. **Some Important Results**

1. $G(s+t, z) = G(s, z) + G(t, z)$
2. $G(c\, s, z) = G(s, z)$ where c is constant
3. $G(s * t)(z) = G(s, z)\, G(t, z)$
4. $G(s \uparrow, z) = \dfrac{G(s, z) - s(0)}{z}$
5. $G(s \downarrow, z) = z\, G(s, z)$
6. $G(s \uparrow n, z) = \dfrac{G(s, z) - \sum_{r=0}^{n-1} s(r) z^r}{z^n}$
7. $G(s \downarrow n, z) = z^n\, G(s, z)$.

ILLUSTRATIVE EXAMPLES

ample 1. Write generating function of the sequence $s_n = 3 \cdot 4^n + 2(-1)^n + 7$.

l. We have

$$s_n = 3 \cdot 4^n + 2(-1)^n + 7 \cdot 1^n$$

$$\therefore \quad G(s, z) = 3\left(\frac{1}{1-4z}\right) + 2\left(\frac{1}{1-(-1)z}\right) + 7 \cdot \frac{1}{1-z}$$

$$= \frac{3}{1-4z} + \frac{2}{1+z} + \frac{7}{1-z}$$

Example 2. Find sequence whose generating function is $\dfrac{6-29z}{1-11z+30z^2}$.

Sol.
$$G(s, z) = \frac{6-29z}{1-11z+30z^2} = \frac{6-29z}{(1-6z)(1-5z)}$$

$$= \frac{6-\frac{29}{6}}{(1-6z)\left(1-\frac{5}{6}\right)} + \frac{6-\frac{29}{5}}{\left(1-\frac{6}{5}\right)(1-5z)}$$

$$\therefore \quad G(s, z) = \frac{7}{1-6z} - \frac{1}{1-5z}$$

$$\therefore \quad s_n = 7 \cdot 6^n - 1 \cdot 5^n$$

Example 3. Which sequence has the generating function $\dfrac{1}{1-z-z^2}$?

Sol. Here $G(s, z) = \dfrac{1}{1-z-z^2}$

Consider the equation

$$1 - z - z^2 = 0$$

$$\therefore \quad z^2 + z - 1 = 0$$

$$\therefore \quad z = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore \quad z = \frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}+1}{2}$$

Let
$$\alpha = \frac{\sqrt{5}-1}{2}, \beta = -\frac{\sqrt{5}+1}{2}$$

$$\therefore \quad z^2 + z - 1 = (z-\alpha)(z-\beta)$$

$$\Rightarrow \quad 1 - z - z^2 = -(z-\alpha)(z-\beta)$$

$$\therefore \quad \frac{1}{1-z-z^2} = \frac{-1}{(z-\alpha)(z-\beta)}$$

$$= \frac{-1}{(z-\alpha)(\alpha-\beta)} + \frac{-1}{(\beta-\alpha)(z-\beta)}$$

$$= -\frac{1}{(\alpha-\beta)(z-\alpha)} + \frac{1}{(\alpha-\beta)(z-\beta)}$$

$$= \frac{1}{\alpha-\beta}\left[\frac{1}{z-\beta}-\frac{1}{z-\alpha}\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha-z}-\frac{1}{\beta-z}\right]$$

$$\therefore \quad G(s,z) = \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha\left(1-\frac{z}{\alpha}\right)}-\frac{1}{\beta\left(1-\frac{z}{\beta}\right)}\right]$$

$$\therefore \quad s_n = \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)^n-\frac{1}{\beta}\left(\frac{1}{\beta}\right)^n\right]$$

$$= \frac{1}{\alpha-\beta}\left[\left(\frac{1}{\alpha}\right)^{n+1}-\left(\frac{1}{\beta}\right)^{n+1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{2}{\sqrt{5}-1}\right)^{n+1}-\left(\frac{-2}{\sqrt{5}+1}\right)^{n+1}\right]$$

$$\left[\because \quad \alpha-\beta = \frac{\sqrt{5}-1}{2}+\frac{\sqrt{5}+1}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

$$\left[\begin{array}{l}\because \quad \frac{2}{\sqrt{5}-1} = \frac{2}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2} \\ \text{and} \quad \frac{2}{\sqrt{5}+1} = \frac{2}{\sqrt{5}+1}\times\frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{2(\sqrt{5}-1)}{50-1} = \frac{\sqrt{5}-1}{2}\end{array}\right]$$

Example 4. Find generating function and sequence of recurrence relation

$$a_n + 2\,a_{n-1} = 0 \text{ with } a_0 = 5$$

Sol. The recurrence relation is $a_n + 2\,a_{n-1} = 0$

Multiplying both sides by z^n and summing up from $n = 1$ to ∞, we get,

$$\sum_{n=1}^{\infty} a_n z^n + 2\sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

or $$\sum_{n=0}^{\infty} a_n z^n - a_0 + 2\left(\sum_{n=1}^{\infty} a_{n-1} z^{n-1} . z\right) = 0$$

or $$\sum_{n=0}^{\infty} a_n z^n - a_0 + 2z \sum_{n=1}^{\infty} a_{n-1} z^{n-1} = 0$$

or $$G(s, z) - a_0 + 2z\, G(s, z) = 0 \qquad \left[\because\ G(s, z) = \sum_{n=0}^{\infty} a_n z^n\right]$$

$$\therefore\quad (1 + 2z)\, G(s, z) = 5 \qquad [\because\ a_0 = 5]$$

$$\therefore\quad G(s, z) = \frac{5}{1+2z} = \frac{5}{1-(-2)z}$$

$$\therefore\quad s_n = 5(-2)^n$$

Example 5. For the recurrence relation

$$s_n - 6s_{n-1} + 8s_{n-2} = 0 \text{ for } n \geq 2 \text{ and } s_0 = 10, s_1 = 25$$

(*i*) Find generating function

(*ii*) Find the sequence which satisfies it.

(Pbi. U. B.C.A. 2012)

Sol. The given recurrence relation is

$$s_n - 6s_{n-1} + 8s_{n-2} = 0$$

Multiplying both sides by z^n and summing up from $n = 2$ to ∞, we get,

$$\sum_{n=2}^{\infty} s_n z^n - 6\sum_{n=2}^{\infty} s_{n-1} z^n + 8\sum_{n=2}^{\infty} s_{n-2} z^n = 0$$

$$\therefore\quad \sum_{n=0}^{\infty} s_n z^n - s_0 - s_1 z - 6z\sum_{n=2}^{\infty} s_{n-1} z^{n-1} + 8z^2 \sum_{n=2}^{\infty} s_{n-2} z^n$$

$$\therefore\quad G(s, z) - 10 - 25z - 6z\left(\sum_{n=1}^{\infty} s_{n-1} z^{n-1} - s_0\right) + 8z^2 G(s, z) = 0$$

$$\left[\because\ G(s, z) = \sum_{n=0}^{\infty} s_n z^n\right]$$

$\therefore \quad G(s, z) - 10 - 25z - 6z\,[G(s, z) - 10] + 8z^2 G(s, z) = 0$

$\therefore \quad G(s, z) - 10 - 25z - 6z\,G(s, z) + 60z + 8z^2 G(s, z) = 0$

$\therefore \quad (1 - 6z + 8z^2)\,G(s, z) = 10 - 35z$

$$\therefore \quad G(s, z) = \frac{10 - 35z}{1 - 6z + 8z^2}$$

$$= \frac{10 - 35z}{(1-2z)(1-4z)} = \frac{10 - \frac{35}{2}}{(1-2z)\left(1 - \frac{4}{2}\right)} + \frac{10 - \frac{35}{4}}{\left(1 - \frac{2}{4}\right)(1-4z)}$$

$$= \frac{-15}{(1-2z)(-2)} + \frac{5}{(2)(1-4z)}$$

$$= \frac{15}{2(1-2z)} + \frac{5}{2(1-4z)}$$

$$\therefore \quad G(s, z) = \frac{5}{2}\left[\frac{3}{1-2z} + \frac{1}{1-4z}\right]$$

$$\therefore \quad s_n = \frac{5}{2}\,[3 \cdot 2^n + 1 \cdot 4^n]$$

Example 6. Solve

$$S(n) + 3\,S(n-1) - 4S(n-2) = 0, \quad n \geq 2 \text{ with } S(0) = 3,\ S(1) = -2$$

Sol. The given recurrence relation is

$$S(n) + 3S(n-1) - 4S(n-2) = 0$$

Multiplying both sides by z^n and summing up from $n = 2$ to ∞, we get,

$$\sum_{n=2}^{\infty} S(n)\,z^n + 3\sum_{n=2}^{\infty} S(n-1)\,z^n - 4\sum_{n=2}^{\infty} S(n-2)\,z^n = 0$$

$$\therefore \quad \sum_{n=0}^{\infty} S(n) z^n - S(0) - S(1) z + 3z \sum_{n=2}^{\infty} S(n-1) z^{n-1} - 4z^2 \sum_{n=2}^{\infty} S(n-2) z^{n-2} = 0$$

$\therefore \quad G(S, z) - 3 + 2z + 3z\,[G(S, z) - S(0)] - 4z^2 G(S, z) = 0$

$\therefore \quad G(S, z) - 3 + 2z + 3z\,[G(S, z) - 3] - 4z^2 G(S, z) = 0$

$\therefore \quad G(S, z) - 3 + 2z + 3z\,G(S, z) - 9z - 4z^2 G(S, z) = 0$

$\therefore \quad (1+3z-4z^2)\,G(S,z) = 3+7z$

$$\therefore \quad G(S,z) = \frac{3+7z}{1+3z-4z^2} = \frac{3+7z}{(1-z)(1+4z)}$$

$$= \frac{3+7}{(1-z)(1+4)} + \frac{3-\frac{7}{4}}{\left(1+\frac{1}{4}\right)(1+4z)}$$

$$= \frac{2}{1-z} + \frac{5}{1-(-4)z}$$

$\therefore \quad S(n) = 2.1^n + 5(-4)^n$

Example 7. If $s(n) = n^2+1$ and $t(n) = n+4$, find $(s * t)(n)$ and $(s \uparrow 2)(n)$, $(t \downarrow 4)(n)$.

Sol. We have

$$s(n) = n^2+1,\ t(n) = n+4$$

$$\therefore \quad (s*t)(n) = \sum_{r=0}^{n} s(r)\,t(n-2)$$

$$= \sum_{r=0}^{n} (r^2+1)(n-r+4) = \sum_{r=0}^{n} [(n+4)r^2 - r^3 - r + n + 4]$$

$$= (n+4)\sum_{r=0}^{n} r^2 - \sum_{r=0}^{n} r^3 - \sum_{r=0}^{n} r + (n+4)\sum_{r=0}^{n} 1$$

$$= (n+4)\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2} - (n+4)\,n$$

$$= \frac{1}{6}n(n+1)(n+4)(2n+1) - \frac{1}{4}n^2(n+1) - \frac{1}{2}n(n+1) - n(n+4)$$

$(s \uparrow 2)(n) = s(n+2) = (n+2)^2 + 1 = n^2 + 4n + 5$

$(t \downarrow 4)(n) = t(n-4)$ for $n \geq 4$

$= n - 4 + 4 = n$

Example 8. If $a(n) = n$, $b(n) = \frac{n}{2}$, $c(n) = 2^n$, find $G(a \uparrow 2, z)$, $G(b * b, z)$, $G(2c, z)$.

Sol. We have

$a(n) = n, \quad \therefore \quad (a \uparrow 2)(n) = n + 2$

$= s_1 + s_2$ where $s_1 = n$, $s_2 = 2$

$$\therefore \quad G(a \uparrow 2, z) = G(s_1, z) + G(s_2, z)$$

$$= \frac{z}{(1-z)^2} + \frac{2}{1-z}$$

$$b(n) = \frac{n}{2} \quad \Rightarrow \quad G(b, z) = \frac{1}{2} \frac{z}{(1-z)^2}$$

$$G(b * b, z) = G(b, z)\, G(b, z)$$

$$= \left[\frac{1}{2} \cdot \frac{z}{(1-z)^2}\right]\left[\frac{1}{2} \cdot \frac{z}{(1-z)^2}\right] = \frac{1}{4} \frac{z^2}{(1-z)^4}$$

$$c(n) = 2^n \quad \Rightarrow \quad G(c, z) = \frac{1}{1-2z}$$

$$\therefore \quad G(2c, z) = 2\, G(c, z) = \frac{2}{1-2z}$$

Example 9. By finding generating function of sequence $S(n)$, find solution of recurrence relation

$$S(n+2) - 7S(n+1) + 12\, S(n) = 0 \text{ for } n \geq 0$$

Given $\quad S(0) = 2, S(1) = 5$

Sol. The given recurrence relation is

$$S(n+2) - 7S(n+1) + 12\, S(n) = 0$$

$$\therefore \quad S(n) - 7S(n-1) + 12\, S(n-2) = 0$$

Multiplying both sides by z^n and summing up from $n = 2$ to ∞, we get,

$$\sum_{n=2}^{\infty} S(n) z^n - 7 \sum_{n=2}^{\infty} S(n-1) z^n + 12 \sum_{n=2}^{\infty} S(n-2) z^n = 0$$

$$\therefore \quad G(S, z) - S(0) - S(1) z - 7z \sum_{n=2}^{\infty} S(n-1) z^{n-1} + 12 \sum_{n=2}^{\infty} S(n-2) z^{n-2} = 0$$

$$\therefore \quad G(S, z) - 2 - 5z - 7z\, [G(S, z) - S(0)] + 12\, z^2\, G(S, z) = 0$$

$$\therefore \quad G(S, z) - 2 - 5z - 7z\, [G(S, z) - 2] + 12\, z^2\, G(S, z) = 0$$

$$\therefore \quad G(S, z) - 2 - 5z - 7z\, G(S, z) + 14z + 12\, z^2\, G(S, z) = 0$$

$$\therefore \quad (1 - 7z + 12\, z^2)\, G(S, z) = 2 - 9z$$

$$\therefore \quad G(S, z) = \frac{2-9z}{1-7z+12z^2} = \frac{2-9z}{(1-3z)(1-4z)}$$

$$= \frac{2-3}{(1-3z)\left(1-\frac{4}{3}\right)} + \frac{2-\frac{9}{4}}{\left(1-\frac{3}{4}\right)(1-4z)} = \frac{3}{1-3z} - \frac{1}{1-4z}$$

$$\therefore \quad S = (n) = 3.3^n - 4^n$$

$$= 3^{n+1} - 4^n$$

Example 10. If $S(n) - 6S(n-1) + 5S(n-2) = 0$ with

$S(0) = S(1) = 2$, find generating function

(*i*) Using definition of generating function

(*ii*) Using operation on sequences and their generating functions.

(*iii*) Write solution of recurrence relation

Sol. (*i*) The given recurrence relation is

$$S(n) - 6S(n-1) + 5S(n-2) = 0$$

Multiplying both sides by z^n and summing up from $n = 2$ to ∞, we get,

$$\sum_{n=2}^{\infty} S(n) z^n - 6 \sum_{n=2}^{\infty} S(n-1) z^n + 5 \sum_{n=2}^{\infty} S(n-2) z^n = 0$$

$$\therefore \quad \sum_{n=0}^{\infty} S(n) z^n - S(0) - S(1) z - 6z \sum_{n=2}^{\infty} S(n-1) z^{n-1} + 5z^2 \sum_{n=2}^{\infty} S(n-2) z^n = 0$$

$$\therefore \quad G(S, z) - 2 - 2z - 6z\left[G(S, z) - S(0)\right] + 5z^2 G(S, z) = 0$$

$$\therefore \quad G(S, z) - 2 - 2z - 6z\,[G(S, z) - 2] + 5z^2 G(S, z) = 0$$

$$\therefore \quad G(S, z) - 2 - 2z - 6z\, G(S, z) + 12z + 5z^2 G(S, z) = 0$$

$$\therefore \quad (1 - 6z + 5z^2)\, G(S, z) = 2 - 10z$$

$$\therefore \quad G(S, z) = \frac{2-10z}{1-6z+5z^2} = \frac{2(1-5z)}{(1-5z)(1-z)}$$

$$\therefore \quad G(S, z) = \frac{2}{1-z}$$

(*ii*) The given recurrence relation is

$$S(n) - 6S(n-1) + 5S(n-2) = 0$$

or $\quad S(n+2) - 6S(n+1) + 5S(n) = 0$

or $(S \uparrow 2)(n) - 6(S \uparrow 1)(n) + 5S(n) = 0$

$\Rightarrow$ $G(S \uparrow 2)(z) - 6G(S \uparrow, z) + 5G(S, z) = 0$

$$\Rightarrow \frac{G(S,z) - \sum_{r=0}^{1} S(r) z^r}{z^2} - 6\frac{G(S,z) - S(0)}{z} + 5G(S, z) = 0$$

$$\Rightarrow \frac{G(S,z) - [S(0) + S(1)z]}{z^2} - 6\frac{G(S,z) - S(0)}{z} + 5G(S, z) = 0$$

$$\Rightarrow \frac{G(S,z) - [2 + 2z]}{z^2} - 6\frac{G(S,z) - 2}{z} + 5G(S, z) = 0$$

$$\Rightarrow \frac{G(S,z) - 2 - 2z}{z^2} - 6\frac{G(S,z) - 2}{z} + 5G(S, z) = 0$$

$\Rightarrow$ $G(S, z) - 2 - 2z - 6z[G(S, z) - 2] + 5z^2G(S, z) = 0$

$\Rightarrow$ $G(S, z) - 2 - 2z - 6z\,G(S, z) + 12z + 5z^2 G(S, zi0 = 0$

$\Rightarrow$ $(1 - 6z + 5z^2)\,G(S, z) = 2 - 10z$

$$\therefore \quad G(S, z) = \frac{2 - 10z}{1 - 6z + 5z^2} = \frac{2(1-5z)}{(1-5z)(1-z)}$$

$$\therefore \quad G(S, z) = \frac{2}{1-z}$$

(*iii*) $S(n) = 2.1^n$

or $S(n) = 2$.

which is solution of given relation.

Example 11. Find the generating function for the following :

(Pbi. U. M.C.A. 2007)

(*a*) 0, 1, – 2, 4, – 8, ...

(*b*) 1, 1, 1, 1, 1, 1

(*c*) 2, 2, 2, 2, 2

(*d*) 2, – 2, 2, – 2, 2, – 2 ....

Sol. (*a*)
$$A(z) = 0 + z - 2z^2 + 4z^3 - 8z^4 + \ldots\ldots\ldots$$
$$= z(1 - 2z + 4z^2 - 8z^3 + \ldots\ldots)$$
$$= \frac{z}{1+2z}, |2z| < 1.$$

(b) $$A(z) = 1 + z + z^2 + z^3 + z^4 + z^5$$
$$= \frac{z^6 - 1}{z - 1}$$

(c) $$A(z) = 2 + 2z + 2z^2 + 2z^3 + 2z^4$$
$$= 2(1 + z + z^2 + z^3 + z^4)$$
$$= 2\left(\frac{z^5 - 1}{z - 1}\right)$$

(d) 2, – 2, 2, – 2, 2, – 2, ...

$$A(z) = 2 - 2z + 2z^2 - 2z^3 + 2z^4 - 2z^5 + \ldots\ldots$$
$$= 2(1 - z + z^2 - 2^3 + 2^4 - z^5 + \ldots\ldots)$$
$$= 2\left(\frac{1}{1+z}\right)$$

EXERCISE 9 (c)

1. Write generating function of sequence

(i) $2^{n+1} + 5^n$ (ii) $5 + (-1)^n$

(iii) $2^n [3 + 2(-1)^n]$ (iv) $\frac{1}{2}(-1)^n + \left(\frac{1}{3}\right)^{n-1} + 2$

2. Write sequence whose generating function is

(i) $\frac{6}{1+2z} + \frac{2}{1-3z}$ (ii) $\frac{1}{z^2-4}$

(iii) $\frac{3-5z}{1-2z-3z^2}$ (iv) $\frac{5+2z}{1-4z^2}$

3. Find generating function and sequence of recurrence relation

$a_n + 2a_{n-1} = 0$ with $a_0 = 3$

4. (a) Find sequence having $\frac{1}{1-5z+6z^2}$ as generating function.

(b) Find sequence whose generating function is $\frac{2}{1+z} + \frac{z}{(1-z)^2}$

5. For the recurrence relation $s_n + 3\,s_{n-1} - 4\,s_{n-2} = 0$; $n \geq 2$ and $s_0 = 3$, $s_1 = -2$.

   (i) Find generating function

   (ii) Find the sequence which satisfies it.

6. If $S(n) - 6S(n-1) + 5S(n-2) = 0$, $S(0) = 1$, $S(1) = 2$, find generating function $G(S, z)$.

7. Find the generating function of

$$S(n+2) = S(n+1) + S(n) \text{ where } S(0) = S(1) = 1 \text{ for } n \geq 0.$$

8. Solve $S(n) - 2\,S(n-1) - 3S(n-2) = 0$, $n \geq 2$ with $S(0) = 3$, $S(1) = 1$.

9. For the following sequences $s(n) = 4$, $t(n) = 3^n$, $a(n) = 3 \cdot 2^n$

   verify that

   (i) $G(s * a, z) = G(s, z)\,G(a, z)$

   (ii) $G(t\downarrow, z) = z\,G(t, z)$

10. If $S_n = 2^n$, $T_n = 3^n$. Then find convolution $S * T$ and verify that

$$G(S * T, z) = G(S, z).\,G(T, z)$$

11. Define the Febonacci sequence and find its generating function.

12. Solve recurrence relation

$$S(n) - 4S(n-2) = 0 \text{ for } n \geq 2 \text{ with } S(0) = 10 \text{ and } S(1) = 1$$

   using generating function.

13. Find generating function of Fibonacci sequence from its recurrence relation $S(n+2) = S(n+1) + S(n)$ for $n \geq 0$ using operations on sequences. Given $S(0) = S(1) = 1$.

14. If $S(n) + 3\,S(n-1) - 4S(n-2) = 0$ for $n \geq 2$ where $S(0) = 3$ and $S(1) = -2$, find solution using operations on sequences and their generating function.

ANSWERS

1. (i) $\dfrac{2}{1-2z} + \dfrac{1}{1-5z}$ (ii) $\dfrac{5}{1-z} + \dfrac{1}{1+z}$

   (iii) $\dfrac{3}{1-2z} + \dfrac{2}{1+2z}$ (iv) $\dfrac{1}{2(1+z)} + \dfrac{9}{3-z} + \dfrac{2}{1-z}$

2. (*i*) $s_n = 6(-2)^n + 2.3^n$ (*ii*) $s_n = -\frac{1}{8}\left[\frac{1}{2^n} + \frac{1}{(-2)^n}\right]$

(*iii*) $s_n = 2(-1)^n + 3^n$ (*iv*) $s_n = 3.2^n + 2.(-2)^n$

3. $G(s, z) = 3.\frac{1}{1-(-2)z}$, $s_n = 3(-2)^n$

4. (*a*) $s_n = -2^{n+1} + 3^{n+1}$ (*ii*) $s_n = 2(-1)^n + n$.

5. $G(s, z) = \frac{2}{1-z} + \frac{5}{1+4z}$, $s_n = 2.1^n + 5(-4)^n$

6. $G(S, z) = \frac{1-4z}{1-6z+5z^2}$ **7.** $G(S, z) = \frac{1}{1-z-z^2}$

8. $S(n) = 2(-1)^n + 3^n$ **11.** $G(F, z) = \frac{1}{1-z-z^2}$

12. $S(n) = \frac{1}{4}[21.2^n + 19(-2)^n]$ **13.** $G(S, z) = \frac{1}{1-z-z^2}$

14. $S(n) = 2 + (-4)^n$