1

SET THEORY

Art-1. Introduction

Set is a basic and unifying idea of mathematics. In fact all mathematical ideas can be expressed in terms of sets. In almost whole of the business mathematics, the set theory is applied in one form or the other.

Art-2. Def. of a Set

A set is a collection of well-defined and different objects.

By the words 'well defined' we mean that we are given a rule with the help of which we can say whether a particular object belongs to the set or not. The word 'different' implies that repetition of objects is not allowed.

The words 'family', 'class', 'collection' are also used as synonyms for the word set when the elements are themselves sets.

Element of a Set

Each object of the set is called an element of the set.

Examples of Sets

(*i*) The set of days of a week. (*ii*) The set of integers from 1 to 100000.

(*iii*) The set of even integers. (*iv*) The set of all states of India.

(*v*) The set of all solutions of equation $x^2 = 1$.

Set Notations

Sets are generally denoted by capital letters A, B, C,……

The elements of sets are denoted mostly by small letters *a, b, c*……

Some Standard Sets

N = Set of all natural numbers 1, 2, 3, 4,……

W = Set of all whole numbers 0, 1, 2, 3, 4,……

I or **Z** = Set of all integers

Q = Set of all rational numbers

R = Set of all real numbers

Methods of Designating a Set

A set can be specified in two ways :

(1) Tabular, Roster or Enumeration Method :

When we represent a set by listing all its elements within curly brackets { separated by commas, it is called the tabular, roaster or enumeration method.

(*i*) A set of vowels : $A = \{a, e, i, o, u\}$,

(*ii*) A set of positive even integers upto 10 : $B = \{2, 4, 6, 8, 10\}$

(*iii*) A set of odd natural numbers : $C = \{1, 3, 5, \ldots\ldots\}$.

(2) Selector, Set-builder or Rule Method :

In this method, we do not list all the elements but the set is represented by specifying the defining property.

For example,

$A = \{x : x$ is a vowel in English alphabets $\}$

$B = \{x : x$ is a positive even integer up to 10 $\}$

$C = \{x/x$ is an odd positive integer $\}$

Here (:) or (/) means such that.

Note 1. The order of elements in a set is immaterial.

Thus $\{2, 5, 9, 11\}, \{2, 9, 5, 11\}, \{11, 5, 2, 9\}$ represent the same set.

2. Repetition of elements is not allowed in a set.

Membership of a Set

If an object x is a member of the set A, we write $x \in A$, which can be read as 'x belongs to A' or 'A contains x'. Similarly we write $x \notin A$ to show that x is not a member of the set A.

Example. Let $A = \{1, 2, 5, 7, 9, 10\}$.

Here $5 \in A$, but $6 \notin A$.

TYPES OF SETS

1. FINITE SET

A set is said to be finite if it has finite number of distinct elements.

Examples :

$A = \{2, 4, 6, 8\}$

$B = \{x : x$ is a student of Modi College, Patiala$\}$

Set of months of the year

Set of even natural numbers less than 100.

2. INFINITE SET

A set is said to be infinite if it has an infinite number of elements.

Examples :

$A = \{1, 2, 3\ldots\ldots\}$

$B = \{x : x$ is an odd integer $\}$

$C = \{x : x$ is a multiple of 6 $\}$

3. EMPTY SET (P.T.U. B.C.A. I 2004)

A set which contains no element, is called a null set or void set or empty set. It is denoted by ϕ (read as phi) or { }.

Examples : $A = \left\{ x : x \text{ is a positive integer satisfying } x^2 = \frac{1}{4} \right\}$

$B = \{ x : x \text{ is a fraction satisfying } x^2 = 9 \}$

4. SINGLETON SET

A set containing only single element is called a singleton set or a unit set.

Example : $A = \{x : x \text{ is a perfect square and } 30 \le x \le 40 \} = \{6\}$

$B = \{x : x \text{ is a positive integer satisfying } x^2 = 4 \} = \{2\}$

$C = \{3\}$

5. SUB-SET, SUPER-SET (P.T.U. B.C.A. I 2004)

If every element of a set A is a element of a set B, then A is called sub-set of B and B is called super-set of A.

Or if $x \in A \Rightarrow x \in B$, then A is a sub set of B and B is a super set of A.

We write these as $A \subset B$ and $B \supset A$.

Thus $A \subset B$ means A is contained in B or B contains A.

Note 1. Since every element of A belongs A

$\therefore$ $A \subset A$ $\Rightarrow$ *every set is sub set of itself.*

2. The empty set ϕ is considered to be a Subset of every set

3. If set A has n elements then number of subsets of A is 2^n.

Example. Let $A = \{1, 2, 3, 4, 5, 6, 8, 10\}$

and $B = \{2, 4, 6, 10 \}$, $C = \{ 1, 2, 7, 8 \}$, $D = \{ 2, 7, 8, 1 \}$

Now every element of B is an element of A

$\therefore$ $B \subset A$

Again $7 \in C$, but $7 \notin A$

$\therefore$ $C \not\subset A$ *i.e.,* C is not a sub-set of A.

Now every member of D is a member of C and every member of C is a member of D.

$\therefore$ $C \subset D$ and $D \subset C$

In this case we can also write $C \subseteq D$ and $D \subseteq C$.

6. PROPER SUB-SET

A non-empty set A is said to be a proper subset of B, if $A \subset B$ and $A \neq B$.

Note 1. ϕ and A are called improper subsets of A.

7. EQUAL SETS

Two sets A and B are said to be equal if both have the same elements. In other words, two sets A and B are equal when every element of A is an element of B and every element of B is an element of A.

i.e., If $A \subset B$ and $B \subset A$, then $A = B$.

Example. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$B = \{x : x \text{ is a natural number and } 1 \leq x \leq 10\}$

Here $A = B$.

8. EQUIVALENT SETS

Two finite sets A and B are said to be equivalent sets if the total number of elements in A is equal to the total number of elements in B.

Example. Let $A = \{1, 2, 3, 4, 6\}$, $B = \{1, 2, 7, 9, 12\}$

$\therefore$ A and B are equivalent sets.

We write the above fact as $A \sim B$

Note : Equal sets are equivalent but equivalent sets are not always equal.

9. POWER SET

(P.T.U. B.C.A.-I 2004; B.Tech. Dec. 2003; Pbi. U. M.Sc. 2011)

The power set of a finite set is the set of all sub-sets of the given set. Power set of A is denoted by P(A).

Example. Take $A = \{1, 2, 3\}$

$$P(A) = \{\phi, \{1\}, (2\} \{3), \{1, 2), \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Note : If set A has n elements then P(A) has 2^n elements.

10. DISJOINT SETS

Two sets A and B are disjoint if they have not element in common.

Example. Let A = Set of odd numbers

B = Set of even numbers.

Then A and B are disjoint sets.

11. UNIVERSAL SET

(P.T.U. B.C.A. I 2004)

If all the sets under consideration are sub-sets of a fixed set U, then U is called a universal set.

Example. In Plane geometry, the universal set consists of points in a plane.

12. COMPARABLE AND NON-COMPARABLE SETS

Two sets are said to be comparable if one of the two sets is a sub-set of the other.

Example . Let $A = \{2, 3, 5\}$, $B = \{2, 3, 5, 6\}$, $C = \{1, 5\}$

Here $A \subset B$

$\therefore$ A and B are comparable sets. On the other hand $A \not\subset C$, $C \not\subset A$

$\therefore$ A, C are Non-comparable.

13. ORDER OF A FINITE SET

The number of different element of a finite set A is called the order of A and is denoted by O(A).

14. CARDINALITY

Number of different elements in a set is known as its cardinality.

Example If $A = \{2, 3, 6, 8\}$, then $O(A) = 4$

ILLUSTRATIVE EXAMPLES

Example 1. List the elements in each of the following sets using braces and ellipses where necessary

(a) $\{x : x$ is a natural number divisible by $5\}$

(b) $\{x : x$ is a negative odd integer$\}$

(c) $\{x : x$ is an even prime number$\}$

(d) $\{x : x-1$ is an integer divisible by $4\}$

(e) $\{x : x$ is an integer divisible by 2 and by $5\}$

Sol. (a) Let $A = \{x : x$ is a natural number divisible by $5\} = \{5, 10, 15, 20, \ldots\}$

(b) Let $A = \{x : x$ is a negative odd integer$\} = \{-1, -3, -5, -7, \ldots\}$

(c) Let $A = \{x : x$ is an even prime number$\} = \{2\}$.

(d) Let $A = \{x : x-1$ is an integer divisible by $4\}$
$= \{\ldots, -7, -3, 1, 5, 9, 13, \ldots\}$

(e) Let $A = \{x : x$ is an integer divisible by 2 and by $5\}$
$= \{\ldots -20, -10, 0, 10, 20, \ldots\}$

Example 2. Write the following sets in Roster form :

(i) $\{x : x$ is a vowel before g in the English alphabet$\}$

(ii) $\{x \in \mathbf{N} : x$ is a prime number between 6 and $30\}$

(iii) $\{x \in \mathbf{N} : 3x + 5 < 31\}$

(iv) $\{x : x^2 + 5x + 6 = 0\}$

Sol. (i) Letters before g in the English alphabet are a, b, c, d, e. Out of these a and e are vowels.

$\therefore$ required set is $\{a, e\}$

(ii) Prime numbers between 6 and 30 are 7, 11, 13, 17, 19, 23, 29

$\therefore$ required set is $\{7, 11, 13, 17, 19, 23, 29\}$

(iii) $3x + 5 < 31 \quad \Rightarrow 3x < 26 \quad \Rightarrow x < \dfrac{26}{3} \quad \Rightarrow x < 8\dfrac{2}{3}$

$\therefore$ $x = 1, 2, 3, 4, 5, 6, 7, 8$ as $x \in \mathbf{N}$

$\therefore$ required set is $\{1, 2, 3, 4, 5, 6, 7, 8\}$

(iv) $x^2 + 5x + 6 = 0 \quad \Rightarrow (x+2)(x+3) = 0 \quad \Rightarrow x = -2, -3$

$\therefore$ required set is $\{-2, -3\}$

Example 3. Write the following sets in Roster form :

(i) $A = \{x \in \mathbf{N} : x^2 = 25\}$

(ii) $B = \{x \in \mathbf{N} : |x| \le 4\}$

(iii) $C = \{x : x^2 - 3x + 2 = 0\}$

(iv) $D = \{x : x$ is a positive multiple of 3 and 7 but less than $28\}$

Sol. (*i*) $x^2 = 25 \Rightarrow x = 5, -5$

But $x \in N$, $\therefore x = 5$

$\therefore$ A = {5}

(*ii*) $|x| = x$ as $x \in N$

$\therefore$ $|x| \leq 4 \Rightarrow x \leq 4 \Rightarrow x = 1, 2, 3, 4$

$\therefore$ B = {1, 2, 3, 4}

(*iii*) $x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) \Rightarrow x = 1, 2$

$\therefore$ C = {1, 2}

(*iv*) Since x is a positive multiple of 3 and 7

$\therefore$ x is a multiple of 21

$\therefore$ D = {21}

Example 4. Redefine each of the following sets, using set builder notation :

(*a*) $\{-2, -4, -6, \ldots\ldots\}$ (*b*) $\{0, 3, -3, 6, -6, 9, \ldots\ldots\}$

(*c*) $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\ldots\}$

(*d*) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$ (*e*) $\{0, 1, 2, \ldots\ldots, 99, 100\}$

Sol. (*a*) Let $A = \{-2, -4, -6, \ldots\ldots\}$
$= \{-2x : x \in I\}$

(*b*) Let $A = \{0, 3, -3, 6, -6, 9 \ldots\ldots\}$
$= \{3x : x \in I\}$

(*c*) Let $A = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\ldots\}$
$= \{x : x = 2n \text{ or } x = 3n : n \in N\}$

(*d*) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

$= \left\{x : x = \frac{1}{n} : n \in \mathbf{N}\right\}$

(*e*) Let $A = \{0, 1, 2, \ldots\ldots, 99, 100\}$
$= \{x : x \in I \text{ s.t. } 0 \leq x \leq 100\}$

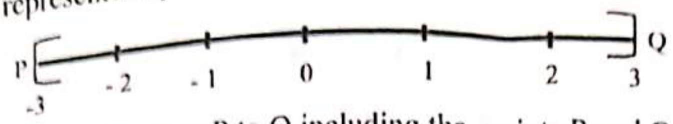
Example 5. Define geometrically, the following set :

(*a*) $\{x \in \mathbf{R} : |x| \leq 3\}$ (*b*) $\{x \in \mathbf{Z}, : |x| \leq 3\}$

(*c*) $\{x \in N, |x| \leq 3\}$ (*d*) $\{(x, y), x, y \in \mathbf{R}, x^2 + y^2 = 25\}$

Sol. (*a*) Let $A = \{x \in \mathbf{R} : |x| \leq 3\} = \{x \in \mathbf{R} : -3 \leq x \leq 3\}$

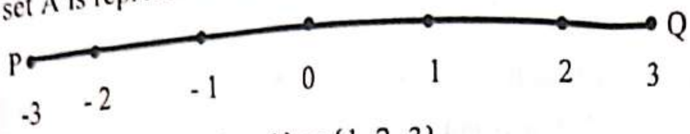
The set A is represented by



by the points on the line from P to Q including the points P and Q.

(*b*) Let $A = \{x \in \mathbf{Z}, : |x| \leq 3\} = \{-3, -2, -1, 0, 1, 2, 3\}$

The set A is represented by the points marked by dot on the line PQ.

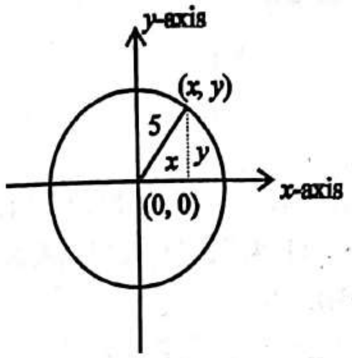


(*c*) Let $A = \{x \in \mathbf{N}, |x| \leq 3\} = \{1, 2, 3\}$

The set A is represented by the points marked by dot on the line PQ

(*d*) Let $A = \{(x, y) ; x, y \in \mathbf{R} ; x^2 + y^2 = 25 \}$

The set A is represented by all the points which lie on the circle whose centre is at the point (0, 0) and radius as shown below :



Example 6. Determine which of the following pairs of sets are equal :

(*a*) $S = \{x: x$ is an integer divisible by both 3 and 2$\}$ and $Q = \{6, 12, 18, 24, \ldots\ldots\}$

(*b*) $X = \{x: x$ is real and $x^2 < x\}$ and $T = \{x : x$ is real and $0 < x < 1\}$

Sol. (*a*) $S = \{x: x$ is an integer divisible by both 3 and 2$\}$

$= \{x : x = 3n$ or $x = 2n$ where $n \in I\}$

$= \{\ldots\ldots -9, -8, -6, -4, -3, -2, 0, 2, 3, 4, 6, 8, 9, \ldots\ldots\}$

$Q = \{6, 12, 18, 24, \ldots\ldots\}$

Since $-6 \in S$ and $-6 \notin Q$

$\therefore \quad S \neq Q$

(*b*) $X = \{x : x$ is real and $x^2 < x\}$

Since $x^2 < x \quad \Rightarrow x^2 - x < 0$

$\Rightarrow x(x-1) < 0$

$\Rightarrow 0 < x < 1$

$\therefore \quad X = \{x : x \text{ is real and } 0 < x < 1\}$

$= T$

$\therefore \quad X = T$

Example 7. Is the set $A = \{x : x^2 - 5x + 6 = 0 \text{ and } x^2 - 3x + 2 = 0\}$ empty ? Justify.

Sol. $A = \{x : x^2 - 5x + 6 = 0 \text{ and } x^2 - 3x + 2 = 0\}$

Now $x^2 - 5x + 6 = 0 \quad \Rightarrow (x-2)(x-3) = 0 \quad \Rightarrow x = 2, 3$

Also $x^2 - 3x + 2 = 0 \quad \Rightarrow (x-1)(x-2) = 0 \quad \Rightarrow x = 1, 2$

$\therefore$ there is a value 2 of x which satisfies $x^2 - 5x + 6 = 0$ and $x^2 - 3x + 2 = 0$

$\therefore \quad A = \{2\} \quad \Rightarrow$ A is not an empty set.

Example 8. Find the cardinal number of each set :

(*i*) $A = \{x : x^2 = 25, 3x = 6\}$

(*ii*) Power set P(B) of $B = \{1, 4, 5, 9\}$

(*iii*) $A = \{x : x \in N, x^2 = 5\}$

(*iv*) B {6, 7, 8, 9, ...} (Pbi.U., B.C.A. 2000)

Sol. (*i*) $A = \{x : x^2 = 25, 3x = 6\}$

Since $x^2 = 25 \Rightarrow x = \pm 5$ and $3x = 6 \Rightarrow x = 2$

$\therefore \quad A = \phi$

Card (A) = 0 *i.e.,* # (A) = 0

(*ii*) Here $B = \{1, 4, 5, 9\}$

$\therefore$ P (B) = $\{\phi, \{1\}, \{4\}, \{5\}, \{9\}, \{1, 4\}, \{1, 5\}, \{1, 9\}, \{4, 5\}, \{4, 9\}, \{5, 9\},$
$\{1, 4, 5\}, \{1, 4, 9\}, \{4, 5, 9\}, \{1, 5, 9\}, \{1, 4, 5, 9\}\}$

$\therefore$ Cardinal number of (P (B)) = 16

(*iii*) As $x^2 = 5 \quad \Rightarrow \quad x \pm\sqrt{5} \not\subset N$

$\therefore$ Cardinal number is ϕ

(*iv*) Cardinal number is ∞

Example 9. List all the members of the power set of each of the following sets :

(*a*) $A = \{a, b, 2, 3\}$, (*b*) $C = \{\{a\}, \{b\}\}$, (*c*) $D = \{\phi, \{\phi\}\}$

Sol. Here $A = \{a, b, 2, 3\}$

$\therefore$ P (A) = $\{\phi, \{a\}, \{b\}, \{2\}, \{3\}, \{a, b\}, \{a, 2\}, \{a, 3\}, \{b, 2\}, \{b, 3\}, \{2, 3\},$
$\{a, b, 2\}, \{a, b, 3\}, \{b, 2, 3\}, \{a, 2, 3\}, \{a, b, 2, 3\}\}$

(*b*) $C = \{\{a\}, \{b\}\}$

$\therefore$ P (C) = $\{\phi, \{\{a\}\}, \{\{b\}\}, \{\{a\}, \{b\}\}\}$

(*c*) $D = \{\phi, \{\phi\}\}$

$\therefore$ P (D) = $\{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Example 10. Give the power set of the set $A = \{1, 2, \phi\}$.

(Pbi. U. M.C.A. 2007)

Sol. $P(A) = \{\phi, \{1\}, \{2\}, \{\phi\}, \{1, 2\}, \{1, \phi\}, \{2, \phi\}, \{1, 2, \phi\}\}$.

Example 11. Find the power set of $A = \{1, 2, 3, 4, 5\}$.

(Pbi. U. B.C.A. 2010)

Sol. $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\},$
$\{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\},$
$\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\},$
$\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$.

Example 12. Let $A = \{r, s, t, u, v, w\}$, $B = \{u, v, w, x, y, z\}$, $C = \{s, u, y, z\}$, $D = \{u, v\}$, $E = \{s, u\}$ and $F = \{s\}$. Let X be an unknown set.

Determine which sets A, B, C, D, E or F con equal X if we are given

(*i*) $X \subset A$ and $X \subset B$

(*ii*) $X \not\subset B$ and $X \subset C$

(*iii*) $X \not\subset A$ and $X \not\subset C$

(*iv*) $X \subset B$ and $X \not\subset C$.

(Pbi.U. B.C.A. 2000)

Sol. (*i*) The only set which is a subset of both A and B is D. Notice that C, E, and F are not subsets of B since $s \in C, E, F$ and $s \notin B$.

(*ii*) Set X can equal C, E or F since they are subsets of C and these are not subsets of B.

(*iii*) Only B is not a subset of either A or C. D and A are subsets of A ; C, E, F are subsets of C. Thus $X = B$.

(*iv*) Both B and D are subsets of B and are not subsets of C. Hence $X = B$ or $X = D$.

Example 13. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of m and n.

Sol. Let A and B be two sets having m and n elements respectively. Then,

Number of subsets of $A = 2^m$,

Number of subsets of $B = 2^n$.

It is given that $2^m - 2^n = 56$

$\Rightarrow \quad 2^n (2^{m-n} - 1) = 2^3(2^3 - 1)$

$\Rightarrow \quad n = 3$ and $m - n = 3$

$\Rightarrow \quad n = 3$ and $m = 6$.

Example 14. Prove that a set containing n distinct elements has 2^n subsets.

(Pbi.U. B.C.A. 19...

OR

Find the number of subsets of a set A containing n elements.

(P.T.U. B.Tech. May 200...

OR

If $|A| = n$, then prove that $|P(A)| = 2^n$

(Pbi.U. B.C.A. April 2008; M.Sc. I.T. 200...

Sol. Let $A = \{a_1, a_2, a_3, ..., a_n\}$

where a_i's are distinct.

A selection of r objects from the elements of the set A can be made in ${}^nC_r$ wa... $0 \le r \le n$. Hence, there are ${}^nC_r$ subsets of a A which contains r elements.

$\therefore$ Number of subsets of A containing no elements is ${}^nC_0$.

Number of subsets of A containing 1 element is ${}^nC_1$.

Number of subsets of A containing 2 element is ${}^nC_2$.

....

....

....

Number of subsets of A containing n elements is ${}^nC_n$.

Hence, the total number of subsets of A

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n.$$

Hence, if A is a finite set of order n, then A has 2^n subsets.

EXERCISE 1 (*a*)

1. Enumerate the element, in the following sets :
   (*a*) $\{x \in \mathbf{R} / x^2 - 3x + 2 = 0\}$ (*b*) $\{x \in \mathbf{R} / x^2 + 1 = 0\}$
   (*c*) $\{x \in \mathbf{C} / x^2 + 1 = 0\}$
2. Describe the following sets in set-builder form :
   (*i*) $A = \{5, 6, 7, 8, 9, 10, 11\}$
   (*ii*) $B = \{2, 4, 6, 8, 10\}$
   (*iii*) $C = \{18, 27, 36, 45, 54, 63, 72, 81, 90\}$

3. Describe the following sets using set builder notation

(a) $\{3, 5, 7, 9, \ldots\ldots, 77, 79\}$

(b) The rational numbers that are strictly between -1 and 1

(c) The even integers

4. Which of the following sets are null sets ?

(i) $A = \{x : x \in \mathbf{N}, x < 1\}$ (ii) $B = \{x : x + 4 = 4\}$

(iii) $C = \{x : x < 1 \text{ and } x > 3\}$

5. What is the set $\{x : x \in \mathbf{R}, x^2 = 9, 2x = 4\}$?

6. What is the set $\{x : x \in \mathbf{R}, x^2 = 4, x^2 - 3x + 2 = 0\}$?

7. Is the set $A = \{x : x^3 = 8 \text{ and } 2x + 3 = 0\}$ empty ? Justify

8. Let $A = \{x : x \text{ is rational and } 0 \leq x \leq 2\}$ and let

$B = \{x : x \text{ is rational and } 1 \leq x \leq 3\}$.

Indicate whether each of the following is true or false.

(a) $0{\cdot}5 \in A$, (b) $2^{1/2} \in B$

(c) $2{\cdot}6 \in A$. or $2{\cdot}6 \in B$ (d) $\exists\ x/x \in A$ and $x \in B$

9. Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$ and $C = \{1, 5, 9\}$. Determine which of the following statements are true ? Give reason for your answer.

(a) $3 \in A$ (b) $\{3\} \in A$ (c) $\{3\} \subseteq A$ (d) $B \subseteq A$ (e) $A \subseteq B$

(f) $\phi \subseteq C$ (g) $\phi \in A$

10. Find the cardinal number of each set :

(i) $A = \{x : x \in \mathbf{N}, x^2 = 5\}$ (ii) $B = \{6, 7, 8, 9, \ldots\ldots\}$

11. List some of the elements of the power set of the integer. Use set builder notation for at least two of these elements.

ANSWERS

1. (a) $\{1, 2\}$ (b) ϕ (c) $\{i, -i\}$

2. (i) $\{x : x$ is positive integer from 5 to 11$\}$

(ii) $\{x : x = 2n,\ n = 1, 2, 3, 4, 5\}$

(iii) $\{x : x$ is a two digit number in which the sum of two digits is 9$\}$

3. (a) $\{2k + 1 : k \in \mathbf{I}$ such that $1 \leq k \leq 39\}$

(b) $\{x \in \mathbf{Q} : -1 < x < 1\}$ (c) $\{x : x = 2n : n \in \mathbf{I}\}$

4. A, C 5. ϕ 6. $\{2\}$ 7. Yes

8. (a) True (b) True (c) True (d) True

9. (a) True (b) False (c) True (d) True

(e) True (f) True (g) False

10. (i) 0 (ii) ∞

11. ϕ, $\{1, 2, 3, 4, 5\}$, E, N, W ; $E = \{x : x$ is a natural number$\}$,

$W = \{x : x$ is whole number$\}$

Operation on Sets

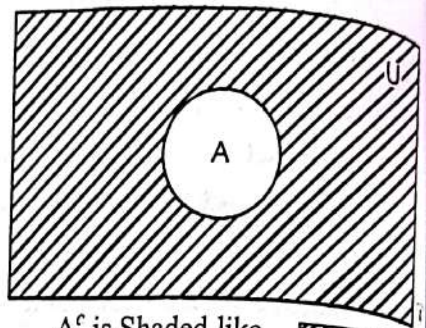
(P.T.U. B.C.A. I 2

Art-3. Venn Diagrams

The relations between sets can be illustrated by certain diagrams called V diagrams. In a Venn diagram, universal set U is represented by a rectangle and any s set of U is represented by a circle within a rectangle U.

Art-4. Complement of a Set

Let A be a subset of universal set U. Then the complement of A is the set of all those elements of U which do not belong to A and we denote complement of A by A^c or A'.



A^c is Shaded like

The complement of A is the shaded region

We can write

$$A^c = \{x : x \in U, x \notin A\}$$

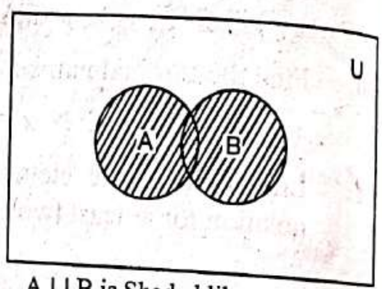
Example : If $U = \{2, 4, 6, 8, 10\}$ $A = \{4, 8\}$ then $A^c = \{2, 6, 10\}$

Note : $U^c = \phi$ and $\phi^c = U$, $(A^c)^c = A$

Art-5. Union of Two Sets

(P.T.U. B.C.A. I 200

If A and B be two given sets, then their union is the set consisting of all the elements of A together with all the elements in B. We should not repeat the elements. The union of two sets A and B is written as $A \cup B$.



$A \cup B$ is Shaded like

The union of two sets A and B

In symbols, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Example : Let $A = \{1, 2, 3, 5, 8\}$,

$B = \{2, 4, 6\}$

$\therefore$ $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

Note : If $A_1, A_2, ..., A_n$ is a family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ o $A_1 \cup A_2 \cup A_3 ..... \cup A_n$.

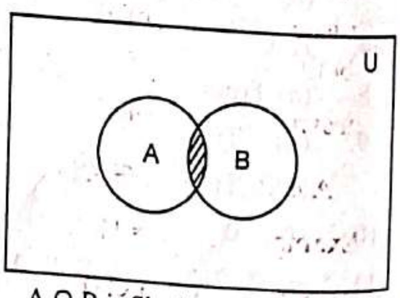
Art-6. Intersection of Two Sets

(P.T.U. B.C.A. I 2005)

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements, which are common to A and B.

In symbols,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



$A \cap B$ is Shaded like

The intersection of two sets A and B

Example. Let $A = \{2, 4, 6, 8, 10, 12\}$,

$B = \{2, 3, 5, 7, 11\}$

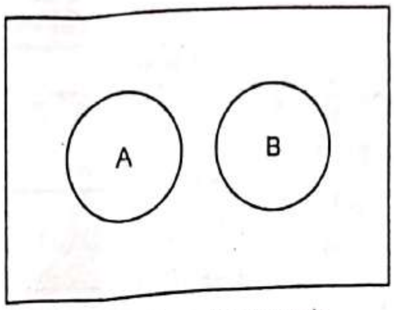
$A \cap B = \{2\}$

ote : If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by

$$\bigcap_{i=1}^{n} A_i \quad \text{or} \quad A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_i$$

isjoint Sets

If A and B two given sets such that $A \cap B = \phi$, hen the sets A and B are said to be **disjoint.**



A and B are disjoint sets

xample. Let $A = \{a, b, c, d\}$,

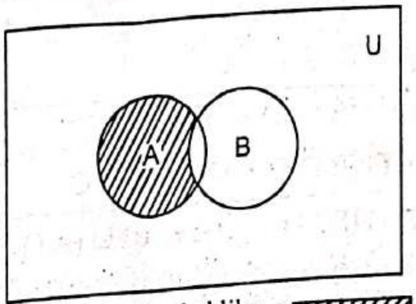
$B = \{l, m, n, p\}$,

$\therefore \; A \cap B = \phi$

Thus A and B are disjoint sets.

Art-7. Difference of Two Sets

The difference of two set A and B is the set of those elements of A which do not belong to B. We denote this by $A - B$.



A − B is Shaded like

In symbols, we write

$A - B = \{x : x \in A \text{ and } x \notin B\}$

$A - B$ is also sometimes written as $A \setminus B$.

Example . Let $A = \{a, b, c, d, e\}$, $B = \{c, d, e, f, g\}$

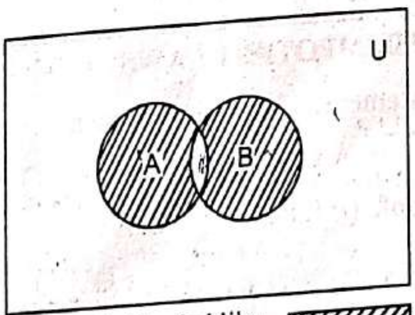
Then $A - B = \{a, b\}$

Note. $B - A \neq A - B$ (P.T.U. B.Tech. May 2007)

Symmetric Difference of Two Sets

If A and B are any two sets, then the set $(A - B) \cup (B - A)$ is called symmetric difference of A and B and is denoted by $A \Delta B$.

In other words, the symmetric difference of A and B consists of all the elements that belong to exactly one of the sets A and B and not to both.



A Δ B is Shaded like

In symbols, we write

$$A \Delta B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\} = (A \cup B) - (A \cap B)$$

Example . Let $A = \{1, 2, 4\}$, $B = \{1, 2, 3, 5, 6\}$

$\therefore A \Delta B = (A \setminus B) \cup (B \setminus A) = (A - B) \cup (B - A) = \{4\} \cup \{3, 5, 6\} = \{3, 4, 5, 6\}$

Art-8. Fundamental Laws of Algebra of Sets

There are a number of general laws about sets which follow the definitions of s theoretic operations, subsets etc. A useful selection of these is shown below. They a grouped under their traditional names. The equations given below hold for any sets A, B, (

Table : Basic Laws of Set Theory

Identity	Name
$A \cup \phi = A$ $A \cap U = A$	Identity Laws
$A \cup U = U$ $A \cap \phi = \phi$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Law
$\overline{(\overline{A})} = A$	Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	De Morgan's Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Complement Laws

I. IDEMPOTENT LAWS

Statement. If A is any set, then

(*i*) $A \cup A = A$ (*ii*) $A \cap A = A$

Proof. (*i*) L.H.S. $= A \cup A$

$= \{x : x \in A \cup A\} = \{x : x \in A \text{ or } x \in A\}$

$= \{x : x \in A\} = A$

$=$ R.H.S.

(*ii*) L.H.S. $= A \cap A$

$= \{x : x \in A \cap A\} = \{x : x \in A \text{ and } x \in A\}$.

$= \{x : x \in A\} = A =$ R.H.S.

II. IDENTITY LAWS

Statement. If A is any set, then

(*i*) $A \cup \phi = A$ (*ii*) $A \cap U = A$

Proof. (*i*) L.H.S. $= A \cup \phi = \{x : x \in A \cup \phi\}$

$= \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\}$

$= A$

$=$ R.H.S.

(*ii*) L.H.S. $= A \cap U = \{x : x \in A \cap U\}$

$= \{x \in A \text{ and } x \in U\} = \{x : x \in A\}$

$= A =$ R.H.S.

(P.T.U. B.C.A. I 2005)

III. COMMUTATIVE LAWS

Statement. If A and B are any two sets, then

(P.T.U. B.Tech. Dec. 2002)

(P.T.U. B.Tech. May 2006)

(*i*) $A \cup B = B \cup A$

(*ii*) $A \cap B = B \cap A$

Proof. (*i*) L.H.S. $= A \cup B$

$= \{x : x \in A \cup B\} = \{x : x \in A \text{ or } x \in B\}$

$= \{x : x \in B \text{ or } x \in A\} = \{x : x \in B \cup A\} = B \cup A$

$=$ R.H.S.

(*ii*) L.H.S. $= A \cap B$

$= \{x : x \in A \cap B\} = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\} = \{x : x \in B \cap A\} = B \cap A$

$=$ R.H.S.

IV. ASSOCIATIVE LAWS

Statement. If A, B and C are any three sets, than

(*i*) $A \cup (B \cup C) = (A \cup B) \cup C$ (P.T.U. B.Tech. Dec. 2007)

(*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$

(Pbi.U. B.C.A. April 2011; Pbi. U. M.Sc. 2011)

Proof. (*i*) L.H.S. $= A \cup (B \cup C)$

$= \{x : x \in A \cup (B \cup C)\} = \{x : x \in A \text{ or } x \in (B \cup C)\}$

$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$

$= \{x : x \in (A \cup B) \text{ or } x \in C\}$

$= \{x : x \in (A \cup B) \cup C\} = (A \cup B) \cup C$

$=$ R.H.S.

(ii) L.H.S. $= A \cap (B \cap C)$

$= \{x : x \in A \cap (B \cap C)\} = \{x : x \in A \text{ and } x \in (B \cap C)\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in (A \cap B) \text{ and } x \in C\}$

$= \{x : x \in (A \cap B) \cap C\} = (A \cap B) \cap C$

= R.H.S.

V. DISTRIBUTIVE LAWS (P.T.U. B.C.A. I 200

Statement. If A, B, C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. L.H.S. $= A \cup (B \cap C)$

$= \{x : x \in A \cup (B \cap C)\}$

$= \{x : x \in A \text{ or } x \in (B \cap C)\}$

$= \{x : x \in A \text{ or } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x : x \in (A \cup B) \text{ and } x \in (A \cup C)\}$

$= \{x : x \in (A \cup B) \cap (A \cup C)\}$

$= \{(A \cup B) \cap (A \cup C)\}$

= R.H.S.

$\therefore \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Note. We can also prove above result by showing that

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

(ii) L.H.S. $= A \cap (B \cup C)$

$= \{x : x \in A \cap (B \cup C)\}$

$= \{x : x \in A \text{ and } x \in (B \cup C)\}$

$= \{x : x \in A \text{ and } (x \in B \text{ or } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\}$

$= \{x : x \in (A \cap B) \text{ or } x \in (A \cap C)\}$

$= \{x : x \in (A \cap B) \cup (A \cap C)\}$

$= (A \cap B) \cup (A \cap C)$

= R.H.S.

$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

VI. DE MORGAN'S LAWS

(Pbi.U., M.Sc.-I.T. 2007, 2010; P.T.U. B.Tech. May 2004; Pbi. U. B.C.A. 2012)

Statement. If A and B are two sub-sets of U, then

(*i*) $(A \cup B)^c = A^c \cap B^c$

OR

Complement of union of two sets is equal to the intersection of complements of two sets.

(P.T.U. B.Tech. May 2006)

(*ii*) $(A \cap B)^c = A^c \cup B^c$

OR

Complement of intersection of two sets is equal to the union of complements of two sets.

Proof. (*i*) $\text{L.H.S.} = (A \cup B)^c$

$$= \{x : x \in (A \cup B)^c\}$$

$$= \{x : x \notin (A \cup B)\}$$

$$= \{x : x \notin A \text{ and } x \notin B\}$$

$$= \{x : x \in A^c \text{ and } x \in B^c\}$$

$$= \{x : x \in (A^c \cap B^c)\}$$

$$= A^c \cap B^c = \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $(A \cup B)^c = A^c \cap B^c$

(*ii*) $\text{L.H.S.} = (A \cap B)^c$

$$= \{x : x \in (A \cap B)^c\}$$

$$= \{x : x \notin (A \cap B)\}$$

$$= \{x : x \notin A \text{ or } x \notin B\}$$

$$= \{x : x \in A^c \text{ or } x \in B^c\}$$

$$= \{x : x \in (A^c \cup B^c)\}$$

$$= A^c \cup B^c$$

$$= \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $(A \cap B)^c = A^c \cup B^c$

VII. COMPLEMENTATION LAW

(P.T.U. B.Tech. Dec. 2007)

Statement : If A is any set then $\overline{(\overline{A})} = A$ or $(A^c)^c = A$.

Proof : Let $x \in (A^c)^c$

$\Rightarrow \quad x \notin A^c$

$\Rightarrow \quad x \in A$

$\therefore \quad (A^c)^c \subseteq A$ ...(1)

Again, let $y \in A$

$\Rightarrow \quad y \notin A^c$

$\Rightarrow \quad y \in (A^c)^c$

$\therefore \quad A \subseteq (A^c)^c$ ...(2)

From (1) and (2), we have, $(A^c)^c = A$

Art-9. If A, B are two sets, then prove that $B - A = B \cap A^c$

(P.T.U. B.Tech. May 2008)

Proof : L.H.S. $= B - A$

$= \{x : x \in B - A)\}$

$= \{x : x \in B \text{ and } x \notin A\}$

$= \{x : x \in B \text{ and } x \in A^c\}$

$= \{x : x \in (B \cap A^c)\}$

$= B \cap A^c$

$=$ R.H.S.

$\therefore \; B - A = B \cap A^c$

Art-10. If A, B, C are any sets, prove that

(*i*) $A - (B \cup C) = (A - B) \cap (A - C)$

(*ii*) $A - (B \cap C) = (A - B) \cup (A - C)$

Proof. (*i*) L.H.S. $= A - (B \cup C)$

$= A \cap (B \cup C)^c$ $\quad [\because B - A = B \cap A^c]$

$= A \cap (B^c \cap C^c)$

$= (A \cap B^c) \cap (A \cap C^c)$

$= (A - B) \cap (A - C)$

$=$ R.H.S.

$\therefore$ L.H.S. $=$ R.H.S.

$\therefore \; A - (B \cup C) = (A - B) \cap (A - C)$

(ii) L.H.S. $= A-(B\cap C) = A\cap(B\cap C)^c$

$= A\cap(B^c\cup C^c)$

$= (A\cap B^c)\cup(A\cap C^c)$

$= (A-B)\cup(A-C)$

$=$ R.H.S.

$\therefore$ L.H.S. $=$ R.H.S.

$\therefore$ $A-(B\cap C) = (A-B)\cup(A-C)$

ILLUSTRATIVE EXAMPLES

Example 1. Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$ and $C = \{1, 5, 9\}$ and let the universal set $U = \{0, 1, 2, \ldots\ldots, 9\}$. Determine

(a) $A\cap B$ (b) $A\cup B$ (c) $B\cup A$ (d) $A\cup C$ (e) $A\cap C$

(f) $A-B$ (g) $B-A$ (h) A^c (i) C^c

Sol. (a) $A\cap B = \{x : x\in A \text{ and } x\in B\} = \{2, 3\}$

(b) $A\cup B = \{x: x\in A \text{ or } x\in B\} = \{0, 2, 3\}$

(c) $B\cup A = \{x: x\in B \text{ or } x\in A\} = \{0, 2, 3\}$

(d) $A\cup C = \{x : x\in A \text{ or } x\in C\} = \{0, 1, 2, 3, 5, 9\}$

(e) $A\cap C = \{x: x\in A \text{ and } x\in C\} = \{\ \}$ or ϕ

(f) $A-B = \{x: x\in A \text{ and } x\notin B\} = \{0\}$

(g) $B-A = \{x : x\in B \text{ and } x\notin A\} = \{\ \}$ or ϕ

(h) $A^c = \{x : x\in U \text{ and } x\notin A\} = \{1, 4, 5, 6, 7, 8, 9\}$

(i) $C^c = \{x: x\in U \text{ and } x\notin A\} = \{0, 2, 3, 4, 6, 7, 8\}$

Example 2. Let $A = \{x : x \text{ is an integer}\}$ and let $B = \{x : x \text{ is an integer divisible by } 6\}$. Let $C = \{x: x \text{ is an integer divisible by 2 or 3}\}$, and let $D = \{x: x \text{ is an integer divisible by 2 and 3}\}$. Determine which of the following relations hold. If containment holds determine whether it is proper

(a) $A\subset B$ (b) $B\subset C$ (c) $C\subset B$ (d) $D\subset B$ (e) $A\subset D$ (f) $D\subset C$ (g) $C\subset D$.

Sol. Here $A = \{x : x \text{ is an integer}\} = \{\ldots\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

$B = \{x: x \text{ is an integer divisible by } 6\} = \{\ldots\ldots -12, -6, 0, 6, 12, \ldots\}$

$C = \{x: x \text{ is an integer divisible by 2 or 3}\}$

$= \{\ldots\ldots -6, -4, -3, -2, 0, 2, 3, 4, 6, \ldots\ldots\}$

$D = \{x: x \text{ is an integer divisible by 2 and 3}\}$

$= \{\ldots\ldots -12, -6, 0, 6, 12, \ldots\ldots\}$

(a) $A \subset B$ does not hold as $-3 \in A$ but $-3 \notin B$

(b) $B \subset C$ holds as any integer which is divisible by 6 is also divisible by 2 or 3.

(c) $C \subset B$ does not hold as $2 \in C$ but $2 \notin B$.

(d) $D \subset B$ holds since $D = B$

(e) $A \subset D$ does not hold as $1 \in A$ but $1 \notin D$

(f) $D \subset C$ holds as any integer which is divisible by 2 and 3 is also divisible by 2 or 3.

(g) $C \subset D$ does not hold as $2 \in C$ but $2 \notin D$.

Example 3. Prove that $A \cup A^c = U$ and $A \cap A^c = \phi$

Sol. Let x be any element $A \cup A^c$

$\Rightarrow$ $x \in A$ or $x \in A^c$

$\Rightarrow$ $x \in A$ or $x \notin A$

$\Rightarrow$ $x \in U$

$\therefore$ $x \in A \cup A^c \Rightarrow x \in U$

$\Rightarrow$ $A \cup A^c \subseteq U$ ...(1)

Conversely, let x be any element of U

$\Rightarrow$ either $x \in A$ or $x \notin A$

$\Rightarrow$ either $x \in A$ or $x \in A^c$

$\Rightarrow$ $x \in A \cup A^c$

$\therefore$ $x \in U \Rightarrow x \in A \cup A^c$

$\therefore$ $U \subseteq A \cup A^c$ ...(2)

From (1) and (2) we get

$A \cup A^c = U$

Further let x be any element of $A \cap A^c$

$\Rightarrow$ $x \in A$ and $x \in A^c$

$\Rightarrow$ $x \in A$ and $x \notin A$.

$\Rightarrow$ $x \in \phi$

$\therefore$ $A \cap A^c \subset \phi$

But $\phi \subset A \cap A^c$ always

$\therefore$ $A \cap A^c = \phi$.

Example 4. For sets A and B, prove that $A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$.

(Pbi.U., B.C.A., Sept. 2006)

Sol. R.H.S. $= (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$

$= (A - B) \cup [(B - A) \cup (A \cap B)]$

$= (A \cap B^c) \cup [(B \cap A^c)] \cup (A \cap B)]$ $[A - B = A \cap B^c]$

$= (A \cap B^c) \cup [(B \cap A^c) \cup (B \cap A)]$ [Commutative Law]

$= (A \cap B^c) \cup [B \cap (A^c \cup A)]$ [Distributive Law]

$= (A \cap B^c) \cup (B \cap X)$ $[A \cup A^c = X]$

$= (A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B)$ [Distributive Law]

$= (A \cup B) \cap X$

$= A \cup B$

$=$ L.H.S.

Example 5. Let A and B be two sets. Prove that (Pbi.U., B.C.A., Sept. 2007)

$$A \Delta B = (A - B) \cup (B - A).$$

Sol. $A \Delta B$ is symmetric difference of sets A and B. It is defined as the set of elements that belong to set A or set B but not to both.

$$A \Delta B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$
$$= \{x : (x \in A - B) \text{ or } (x \in B - A)\}$$
$$= \{x : (x \in A - B) \cup (x \in B - A)\}$$
$$= (A - B) \cup (B - A)$$

Hence proved.

Example 6. For sets A, B and C using properties of sets, prove that

(*i*) $A - (B \cup C) = (A-B) \cap (A-C)$

(*ii*) $A - (B \cap C) = (A-B) \cup (A-C)$

(*iii*) $(A \cup B) - C = (A - C) \cup (B - C)$.

Sol. (*i*) $A - (B \cup C) = A \cap (B \cup C)'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cap C')$ $[\because (B \cup C)' = B' \cap C]$

$= (A \cap B') \cap (A \cap C')$

$= (A-B) \cap (A - C)$

(*ii*) $A - (B \cap C) = A \cap (B \cap C)'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cup C')$ $[\because (B \cap C)' = B' \cup C']$

$= (A \cap B') \cup (A \cap C')$ [$\because \cap$ is distribution over $\cup$]

$= (A-B) \cup (A-C)$

(*iii*) $(A \cup B)-C = (A \cup B) \cap C'$ $[\because X - Y = X \cap Y']$

$= (A \cap C') \cup (B \cap C')$

$= (A-C) \cup (B-C)$.

Example 7. Prove that

$$A \cup B = A \cap B \text{ iff } A = B$$

Sol. (*i*) Assume that $A \cup B = A \cap B$ ...(1)

Let x be any element of A

$\therefore \quad x \in A \quad \Rightarrow x \in A \cup B$

$\Rightarrow x \in A \cap B$ [$\because$ of (1

$\Rightarrow x \in B$

$\therefore \quad x \in A \quad \Rightarrow x \in B$

$\therefore \quad A \subset B$ ...(2

Similarly $B \subset A$ ...(3

From (2) and (3), $A = B$.

$\therefore \quad A \cup B = A \cap B \quad \Rightarrow A = B$

(*ii*) Assume that $A = B$

$\therefore \quad A \cup B = A \cup A = A$

$A \cap B = A \cap A = A$

$\therefore \quad A \cup B = A \cap B$

$\therefore \quad A = B \quad \Rightarrow \quad A \cup B = A \cap B$

Example 8. Let A and B be the following subsets of the real numbers :

$A = \{x : 0 < x < 5\}$ and $B = \{x : 2 < x < 8\}$. Express $A \cup B$ as the union of thre disjoint sets.

Sol. $A = \{x : 0 < x < 5\}$, $B = \{x : 2 < x < 8\}$

We know that

$A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$

Here $(A \backslash B) = \{x : 0 < x \leq 2\}$

$(B \backslash A) = \{x : 5 \leq x < 8\}$

and $A \cap B = \{x : 2 < x < 5\}$

$\therefore \quad A \cup B = \{x : 0 < x \leq 2\} \cup \{x : 5 \leq x < 8\} \cup \{x : 2 < x < 5\}$.

Example 9. Let $X = \{1, 2, 3, 4\}$

If $R = \{<x, y> / x \in X \wedge y \in X \wedge (x - y)$ is an integral non-zero multiple of 2$\}$

$S = \{<x, y> | x \in X \wedge y \in X \wedge (x - y)$ is an integral non-zero multiple of 3$\}$

Find $R \cup S$ and $R \cap S$.

Sol. $X = \{1, 2, 3, 4\}$

$R = \{<x, y> / x \in X \wedge y \in X \wedge (x - y)$ is an integral non-zero multiple of 2$\}$

$= \{<1, 3>, <2, 4>\}$

$S = \{<x, y> | x \in X \wedge y \in X \wedge (x - y)$ is an integral non-zero multiple of 3$\}$

$= \{<1, 4>\}$

$\therefore \quad R \cup S = \{<1, 3>, <2, 4>, <1, 4>\}$

and $R \cap S = \{ \}$ or ϕ.

Example 10. For $A = \{1, 2, \{1, 3\}, \phi\}$, determine the following sets :

(i) $A - \{1\}$ (ii) $A - \phi$

(iii) $A - \{\phi\}$ (iv) $A - \{1, 2\}$ (P.T.U. B.C.A. I 2006)

Sol. We know $A - B = \{x : x \in A \text{ and } x \notin B\}$

(i) $A - \{1\} = \{2, \{1, 3\}, \phi\}$

(ii) $A - \phi = A$

(iii) $A - \{\phi\} = \{1, 2, \{1, 3\}\}$

(iv) $A - \{1, 2\} = \{\{1, 3\}, \phi\}$

Example 11. $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Find $A \cup B$, $A \cap B$, $A - B$, $\overline{A}$. (P.T.U. B.C.A. I 2004)

Sol.

$$\begin{aligned} A \cup B &= \{1, 2, 3\} \cup \{3, 4, 5\} \\ &= \{1, 2, 3, 4, 5\} \\ A \cap B &= \{1, 2, 3\} \cap \{3, 4, 5\} \\ &= \{3\} \\ A - B &= \{1, 2, 3\} - \{3, 4, 5\} \\ &= \{1, 2\} \\ \overline{A} &= U - A \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{1, 2, 3\} \\ &= \{4, 5, 6, 7, 8, 9\} \end{aligned}$$

Example 12. Find $A \cup (B \backslash A) = A \cup B$. (P.T.U. B.C.A. I 2005)

Sol.

$$\begin{aligned} \text{L.H.S.} &= A \cup (B \backslash A) \\ &= A \cup (B - A) \\ &= A \cup (B \cap A^c) && [A - B = A \cap B^c] \\ &= (A \cup B) \cap (A \cup A^c) && [\text{Distributive Law}] \\ &= (A \cup B) \cap X && [A \cup A^c = X] \\ &= A \cup B && [A \cap X = A] \\ &= \text{R.H.S.} \end{aligned}$$

Example 13. Prove that $A \cup (B$ [illegible] $(A \cup C)$, where A, B and C are any three sets. (Pbi. U. B.C.A. April, 2011)

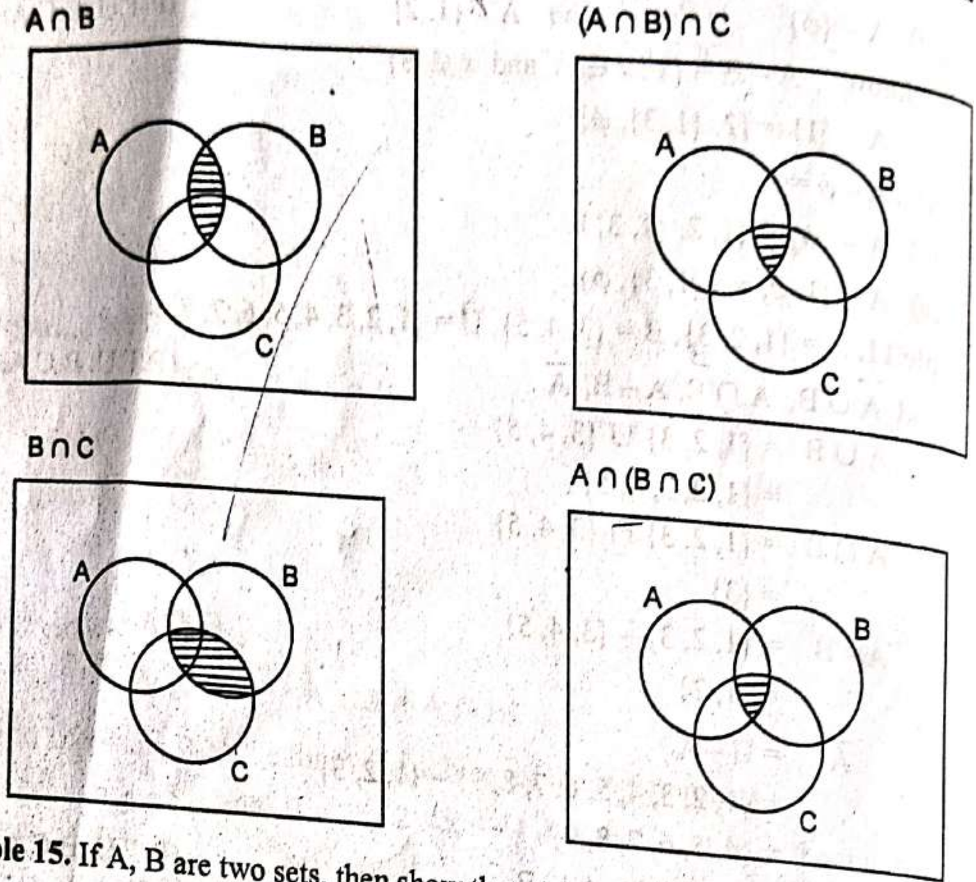
Sol.

$$\begin{aligned} \text{R.H.S.} &= (A \cup B) \cup (A \cup C) \\ &= (B \cup A) \cup (A \cup C) && [\text{Commutative Law}] \\ &= B \cup (A \cup (A \cup C)) && [\text{Associative Law}] \\ &= B \cup ((A \cup A) \cup C) && [\text{Associative Law}] \\ &= B \cup (A \cup C) && [\text{Idempotent Law}] \\ &= (B \cup A) \cup C && [\text{Associative Law}] \\ &= (A \cup B) \cup C && [\text{Commutative Law}] \\ &= A \cup (B \cup C) && [\text{Associative Law}] \end{aligned}$$

Example 14. Draw Venn diagram of $(A \cap B) \cap C$ and $A \cap (B \cap C)$.

(B.C.A. I 2

Sol.



Example 15. If A, B are two sets, then show that $A \cup B = \phi \Leftrightarrow A = \phi, B = \phi$.

Sol. Let $A \cup B = \phi$ (P.T.U. B.C.A. I 20

We show $A = \phi,\ B = \phi$

Let $x \in A$

$\Rightarrow \quad x \in A \cup B$ [$\because$ $A \subset A \cup$

$\Rightarrow \quad x \in \phi$ [$A \cup B =$

$\therefore \quad A \subseteq \phi$

Also $\quad \phi \subseteq A$

$\therefore \quad A = \phi$

Similarly, we can show $B = \phi$

$\therefore \quad A \cup B = \phi \Rightarrow A = \phi,\ B = \phi$ ...(

Again, let $A = \phi,\ B = \phi$

We show $\quad A \cup B = \phi$

Let $\quad x \in A \cup B$

$\Rightarrow \quad x \in A$ or $x \in B$

$\Rightarrow \quad x \in \phi$ or $x \in \phi$

$\Rightarrow \quad x \in \phi$

$\therefore \quad A \cup B \subseteq \phi$

Also $\quad \phi = A \cup B$

$\therefore \quad A \cup B = \phi$

$\therefore \quad A = \phi,\ B = \phi \Rightarrow A \cup B = \phi$ ...(2)

From (1) and (2), we have

$$A \cup B = \phi \Rightarrow A = \phi,\ B = \phi.$$

Example 16. Let $A = \{1, 2, 4\}$, $B = \{4, 5, 6\}$,

Find $A \cup B$, $A \cap B$ and $A - B$. (Pbi.U., B.C.A., 2006)

Sol. $A = \{1, 2, 4\}$ and $B = \{4, 5, 6\}$

(i) $A \cup B = \{1, 2, 4\} \cup \{4, 5, 6\} = \{1, 2, 4, 5, 6\}$

(ii) $A \cap B = \{1, 2, 4\} \cap \{4, 5, 6\} = \{4\}$

(iii) $A - B = \{1, 2, 4\} - (4, 5, 6\} = \{1, 2\}$.

Example 17. Is it true that power set of $A \cup B$ is equal to union of power sets of A and B ? Justify. (Pbi.U. BCA, 2000)

Sol. Let $A = \{a, b\}$, $B = \{c\}$

then $A \cup B = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$P(B) = \{\phi, \{C\}\}$

$P(A) \cup P(B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}\}$

where as

$P(A \cup B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

showing that $P(A \cup B) \neq P(A) \cup P(B)$

Example 18. Prove that $P(A \cap B) = P(A) \cap P(B)$.

Sol. Let $X \in P(A \cap B)$

then $X \subseteq (A \cap B)$

$\Rightarrow \quad X \subseteq A$ and also $X \subset B$ $(\because A \cap B \subseteq A$ and also $A \cap B \subseteq B)$

$\Rightarrow \quad X \in P(A)$ and also $X \in P(B)$

$\Rightarrow \quad X \in P(A) \cap P(B)$

Hence $P(A \cap B) \subseteq P(A) \cap P(B)$ ...(1)

Conversely, let $Y \in P(A) \cap P(B)$

$\Rightarrow \quad Y \in P(A)$ and $Y \in P(B)$

$\Rightarrow \quad Y \subseteq A$ and $Y \subseteq B$

$\Rightarrow \quad$ each element of Y is contained in both A and B

$\Rightarrow$ each element of Y is contained in $A \cap B$

$\Rightarrow$ $Y \subseteq A \cap B$

$\Rightarrow$ $Y \in P(A \cap B)$

Hence $P(A) \cap P(B) \subseteq P(A \cap B)$ ...(2)

From (1) and (2); we get

$$P(A \cap B) = P(A) \cap P(B)$$

Example 19. Find power set $P(A)$ of $A = \{1, 2, 3, 4\}$.

Sol. $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$

Example 20. Let A and B be two sets. Prove that

$$A - B = A \cap B^c$$ (Pbi. U. April, 2007)

Sol. Let $x \in A - B$

$\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \in B^c$

$\Rightarrow$ $x \in A \cap B^c$

$\therefore$ $A - B \subseteq A \cap B^c$ ...(i)

Conversely let $x \in A \cap B^c$, then

$\Rightarrow$ $x \in A$ and $x \in B^c$

$\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A - B$

$\therefore$ $A \cap B^c \subseteq A - B$ ...(ii)

$\therefore$ $A - B = A \cap B^c$.

Example 21. If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.

Sol. We have $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

Case I : When $n(A \cap B)$ is minimum, *i.e.* $n(A \cap B) = 0$. This is possible only when $A \cap B = \phi$. In this case, $n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$. So, maximum number of elements in $A \cup B$ is 9.

Case II : When $n(A \cap B)$ is maximum.

This is possible only when $A \subseteq B$. In this case, $n(A \cap B) = 3$.

$\therefore$ $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 3 = 6$

so, minimum number of elements in $A \cup B$ is 6.

EXERCISE 1 (*b*)

1. Let A = {0, 2, 3}, B = {2, 3} and C = {1, 5, 9}. Let D = {3, 2} and let E = {2, 3, 2}. Determine which of the following are true. Give reason for your decisions.

   (*a*) A = B (*b*) B = C (*c*) B = D (*d*) B = E (*e*) $A \cap B = B \cap A$

   (*f*) $A \cup B = B \cup A$ (*g*) A − B = B − A

2. Determine whether each of the following inclusions is proper :

   (*a*) $A \subset B$ where A = {x : x is an odd prime} and
   B = {x : x is an integer not divisible by 2}

   (*b*) $S \subset T$, where S = {x : x is a real number with *a* finite decimal expansion}, and T = {x : x is a rational number}

   (*c*) $X \subset Y$, where X = {x : x^2 is integer divisible by 9} and
   Y = {x : x is an integer divisible by 3}

3. Prove that each of the following relations holds

   (*a*) $A \subset B$ where A = {x : x is an integer multiple of 10} and
   B = {x : x is an integer multiple of 5}

   (*b*) A = B, where A = {x : x is even integer} and
   B = {x : x^2 is an even integer}

4. Let U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {$x \in U$: x multiple of 3},
   B = {$x \in U$: $x^2 - 5 \geq 0$}.

   Determine (*a*) $A \cup B$ (*b*) $A \cap B$ (*c*) B^c

5. Let A and B be subsets of natural numbers defined as follows :
   A = {x: if p is prime and if x is divisible by p, then x is divisible by p^2}
   and B = {x : there is an integer y such that $x = y^2$}.
   Prove that $B \subset A$. Show that the containment is proper.

6. Let U be the set of letters of the alphabet. Let A = {$a, b, c, \ldots\ldots, l$}, B = {$h, i, j \ldots\ldots, q$} and C = {$o, p, q, \ldots\ldots z$}. Find the elements in each of the following set :

   (*a*) $A \cap B$ (*b*) $A \cup C$ (*c*) $A \cap (B \cup C)$ (*d*) $(A \cap B) \cup C$ (*e*) $A^c \cap B^c$

   (*f*) $(A \cap B)^c$ (*g*) A\B (*h*) B\A (*i*) A\(B\C) (*j*) A\(C\B)

7. Let U be the set of integers and let A = {x : x is divisible by 3}, let B = {x : x is divisible by 2}. Let C = {x : x is divisible by 5} Find the elements in each of the following set :

   (*a*) $A \cap B$ (*b*) $A \cup C$ (*c*) $A \cap (B \cup C)$ (*d*) $(A \cap B) \cup C$ (*e*) $A^c \cap B^c$

   (*f*) $(A \cap B)^c$ (*g*) A\B (*h*) B\A (*i*) A\(B\C) (*j*) A\(C\B)

8. Answer true or false

(*a*) $A^c \cup B^c = (A \cup B)^c$ (*b*) $A^c = U \backslash A$ (*c*) $A \cup (B \cup C) = (A \cup B) \cup C$

(*d*) $A \cup (B \cap C) = (A \cup B) \cap C$ (*e*) $A \backslash (B \backslash C) = (A \backslash B) \backslash C$

9. Let $A = \{a, b, c, d, e\}$, $B = \{a, b\}$, $C = \{B, \phi\}$, $D = \{a, b, \{a, b\}\}$. Find $A \cap B$, C ... $A \cap D$, $C \cap P(A)$ and $D \cap P(A)$. Indicate whether, each of the following is ... or false :

(*a*) $A \in P(A)$ (*b*) $C \subset P(A)$ (*c*) $D \subset P(A)$ (*d*) $B \subset D$

(*e*) $B \in D$ (*f*) $\{a, b\} \in C$

10. Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.
11. Prove that $A \backslash B$ and $B \backslash A$ are disjoints.
12. Prove that if $A \subset B$, then $P(A) \subset P(B)$.
13. Let A and B sets, then $(A \cap B) \cup (A \cap B^c) = A$.
14. Let A, B, C be sets. If $A \subseteq B$ and $B \cap C = \phi$, then $A \cap C = \phi$.
15. Prove that $A' - B' = B - A$.

ANSWERS

1. (*a*) False (*b*) False (*c*) True (*d*) True
   (*e*) True (*f*) True (*g*) False
2. (*a*) Proper (*b*) Not proper (*c*) Not proper
4. (*a*) $\{0, 3, 4, 5, 6, 7, 8, 9\}$ (*b*) $\{3, 6, 9\}$ (*c*) $\{0, 1, 2\}$
6. (*a*) $\{h, i, j, k, l\}$ (*b*) $\{a, b, c, \ldots\ldots, j, k\, l, o, p, q \ldots\ldots X\}$
   (*c*) $\{n, i, j, k, l\}$ (*d*) $\{h, i, j, k, l, o, p, q, \ldots\ldots, z\}$
   (*e*) $\{r, s, t, \ldots\ldots, z\}$ (*f*) $\{a, b, c, d, e, f, g, m, n, o, p, \ldots, z\}$
   (*g*) $\{a, b, c, d, e, f, g\}$ (*h*) $\{m, n, o, p, q\}$
   (*i*) $\{a, b, c, d, e, f, g\}$ (*j*) $\{a, b, c, \ldots\ldots\ldots, l\}$
7. (*a*) $\{\ldots\ldots -12, -6, 0, 6, 12, \ldots\ldots\}$
   (*b*) $\{\ldots\ldots -9, -6, -5, -3, 0, 3, 5, 6, \ldots\ldots\}$
   (*c*) $\{\ldots\ldots -15, -12, -6, 0, 6, 12, 15, \ldots\ldots\}$
   (*d*) $\{\ldots\ldots -12, -10, 6, -5, 0, 5, 6, 10, 12, \ldots\ldots\}$
   (*e*) $\{\ldots\ldots -11, -7, -5, -1, 1, 5, 7, 11, \ldots\ldots\}$
   (*f*) $\{-7, -5, -3, -2, -1, 1, 2, 3, 4, 5, \ldots\ldots\}$
   (*g*) $\{\ldots\ldots\ldots -15, -9, -3, 3, 9, 15, \ldots\ldots\}$
   (*h*) $\{\ldots\ldots -10, -8, -4, -2, 2, 4, 8, 10 \ldots\ldots\}$
   (*i*) $\{\ldots\ldots -15, -9, -3, 3, 9, 15, \ldots\ldots\}$
   (*j*) $\{\ldots\ldots -18, -12, -9, -6, -3, 0, 3, 9, 12, 18 \ldots\ldots\}$
8. (*a*) False (*b*) True (*c*) True (*d*) False (*e*) False
9. (*a*) True (*b*) True (*c*) False (*d*) False (*e*) True (*f*) True

Art-11. Some Important Problems

We give below some other important problems on union and intersection.

ILLUSTRATIVE EXAMPLES

Example 1. Give examples of three sets A, B, C for which

$$A - (B - C) = (A - B) - C$$

Sol. Take $A = \{1, 2, 3\}, B = \{3, 4, 5\}, C = \{6, 7\}$

$\therefore \quad B - C = \{3, 4, 5\} - \{6, 7\} = \{3, 4, 5\}$

$A - (B - C) = \{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$ ...(1)

Also $A - B = \{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$

$(A - B) - C = \{1, 2\} - \{6, 7\} = \{1, 2\}$ ...(2)

From (1) and (2), we get,

$$A - (B - C) = (A - B) - C$$

Example 2. Give an example of three sets A, B and C such that

$A \cap B \neq \phi, B \cap C \neq \phi, A \cap C \neq \phi$ but $A \cap B \cap C = \phi$

Sol. Let $A = \{1, 2, 3\}, B = \{3, 4, 5\}, C = \{4, 5, 2\}$

$\therefore \quad A \cap B = \{3\} \neq \phi, B \cap C = \{4, 5) \neq \phi, C \cap A = \{2\} \neq \phi$

But $A \cap B \cap C = \phi$

Example 3. Prove that $A \subset B \Leftrightarrow B^c \subset A^c$ for all sets A, B.

Sol. (*i*) Assume that $A \subset B$ ...(1)

We are to prove that $B^c \subset A^c$

Let x be an element of B^c

$\therefore \quad x \in B^c$

$\therefore \; x \notin B$

$\Rightarrow \quad x \notin A$ $[\because$ of (1)]

$\Rightarrow \quad x \in A^c$

$\therefore \quad B^c \subset A^c$

$\therefore \quad A \subset B \quad \Rightarrow \quad B^c \subset A^c$

(*ii*) Assume that $B^c \subset A^c$

We are to prove that $A \subset B$.

Let y by any element of A. ...(2)

$\therefore \quad y \in A \Rightarrow y \notin A^c$

$\Rightarrow \quad y \notin B^c$ [$\because$ of

$\Rightarrow \quad y \in B$

$\therefore \quad A \subset B$

$\therefore \quad B^c \subset A^c \Rightarrow A \subset B$

Combining the results proved in (*i*) and (*ii*), we get,

$$A \subset B \Leftrightarrow B^c \subset A^c$$

Example 4. If A, B and C are any sets, prove that

$$A \cup B = A \cup C \text{ and } A \cap B = A \cap C \Rightarrow B = C$$

Sol. Let x be any element of B

$\therefore \quad x \in A$ or $x \notin A$.

Case I. $x \in A$

$\therefore \quad x \in A \cap B$ [$\because$ $x \in$

$\Rightarrow \quad x \in A \cap C$ [$\because$ $A \cap B = A \cap$

$\Rightarrow \quad x \in C$

But x is any element of B

$\therefore \quad B \subset C$

Case II. $x \notin A$

$\therefore \quad x \in A \cup B$

$\Rightarrow \quad x \in A \cup C$ [$\because$ $A \cup B = A \cup$

$\Rightarrow \quad x \in C$ [$\because$ $x \notin$

But x is any element of B

$\therefore \quad B \subset C$

$\therefore$ from both the cases, it is clear that

$$B \subset C$$

Similarly $\quad C \subset B$

From (1) and (2), $B = C$

$\therefore \quad A \cup B = A \cup C$ and $A \cap B = A \cap C \Rightarrow B = C$.

Example 5. For any sets A and B, prove that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Sol. R.H.S. $= (A \cup B) - (A \cap B)$

$= (A \cup B) \cap (A \cap B)'$ [$\because$ $A - B = A \cap B'$

$= (A \cup B) \cap (A' \cup B') = [(A \cup B) \cap A'] \cup [(A \cup B) \cap B']$

$= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')]$

$$= [\phi \cup (B \cap A')] \cup [(A \cap B') \cup \phi]$$
$$= (B \cap A') \cup (A \cap B') = (A \cap B') \cup (B \cap A') = (A - B) \cup (B - A)$$
$$= \text{L.H.S.}$$

Example 6. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

Sol. R.H.S. $= (A \cap B) - (A \cap C)$

$$= (A \cap B) \cap (A \cap C)^c = (A \cap B) \cap (A^c \cup C^c)$$
$$= [(A \cap B) \cap A^c] \cup [(A \cap B) \cap C^c]$$
$$= [(A \cap A^c) \cap B] \cup [A \cap (B \cap C^c)] = (\phi \cap B) \cup [A \cap (B - C)]$$
$$= \phi \cup [A \cap (B - C)] = A \cap (B - C)$$
$$= \text{L.H.S.}$$

EXERCISE 1 (c)

1. Verify the following identities :
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   where, A, B, C are three sets defined by
   $$A = \{1, 2, 4, 5\}, B = \{2, 3, 5, 6\}, C = \{4, 5, 6, 7\}$$
2. If X and Y are two sets, then find $X \cap (X \cup Y)^c$.
3. Show that (i) $A \subset A \cup B$ (ii) $A \cap B \subset A$.
4. Prove the following :
   (i) $B \subset A \cup B$ (ii) $A \cap B \subset B$
   (iii) $B \subset A \Leftrightarrow A \cap B = B$
   (iv) $A \subset C$ and $B \subset D \Rightarrow A \cup B \subset C \cup D$
   (v) $B \subset C \Rightarrow A \cap B \subset A \cap C$
   (vi) $A = B \Leftrightarrow A \subset B$ and $B \subset A$.
5. For any two sets A and B, prove that $A \cap B = \phi \Rightarrow A \subset B^c$.
6. Prove that
   (i) $A \cap (A' \cup B) = A \cap B$ (ii) $A - (A - B) = A \cap B$
7. Prove that $A^c \setminus B^c = B \setminus A$.
8. Prove the following :
   (i) $(A - B) \cap B = \phi$ (ii) $A \cap (B - A) = \phi$
   (iii) $A \cup (B - A) = A \cup B$ (iv) $A - B = A - (A \cap B)$
   (v) $(A - C) \cup (B - C) = (A \cup B) - C$
   (vi) $(A - B) - C = A - (B \cup C) = (A - B) \cap (A - C)$
   (vii) If $A \subset B$, then $B - (B - A) = A$.
9. Prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

ANSWERS

2. ϕ

Art-12. Inclusion–Exclusion Principle

We know that number of elements of a finite set A is denoted by n (A). Followi results of number of elements should be kept in mind for doing problems :

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint sets.
3. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
4. $n(A) = n(A - B) + n(A \cap B)$
5. $n(B) = n(B - A) + n(A \cap B)$
6. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
   $- n(C \cap A) + n(A \cap B \cap ($
7. $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$
8. $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$
9. $n(A \cap B' \cap C') = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

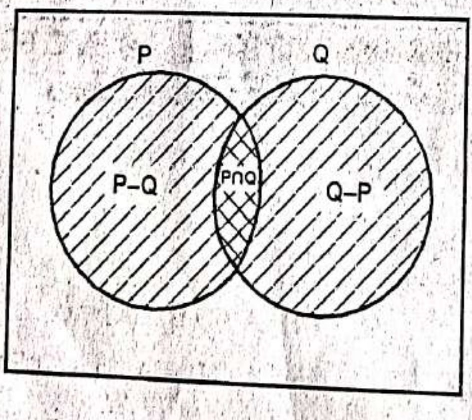
Art-13. State and prove inclusion-exclusion principle for two sets. Also extended fo three sets. (Pbi. U. 2012

Result I. If P and Q be any two non-disjoint sets, then

$$n(P \cup Q) = n(P) + n(Q) - n(P \cap Q).$$

Proof : From Venn Diagram, we see that

P ∪ Q is the union of three disjoint sets P–Q, Q–P and P ∩ Q.



$\therefore \quad n(P \cup Q) = n(P-Q) + n(Q-P) + n(P \cap Q)$ ...(1)

Also, $\quad n(P) = n(P-Q) + n(P \cap Q)$ ...(2)

and $\quad n(Q) = n(Q-P) + n(P \cap Q)$ ...(3)

Adding (2) and (3), we get

$$n(P) + n(Q) = n(P-Q) + n(Q-P) + 2\,n(P \cap Q)$$

$\therefore \quad n(P) + n(Q) = n(P \cup Q) + n(P \cap Q)$

$$[\because \; n(P \cup Q) = n(P-Q) + n(Q-P) + n(P \cap Q)]$$

$\therefore \quad n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

Hence the result.

Result II. If P, Q and R are three finite sets, then

$$n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(P \cap R) - n(Q \cap R) + n(P \cap Q \cap R)$$

Proof : We have

$$n((P \cup (Q \cup R)) = n(P) + n(Q \cup R) - n((P \cap (Q \cup R))$$
$$= n(P) + n(Q) + n(R) - n(Q \cap R) - n((P \cap (Q \cup R))$$

Now $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

$\therefore \quad n((P \cap (Q \cup R)) = n(P \cap Q) + n(P \cap R) - n((P \cap Q) \cap (P \cap R))$

$$= n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)$$

$\therefore \quad n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(P \cap R)$

$$- n(Q \cap R) + n(P \cap Q \cap R)$$

Hence the result.

Note : Inclusion-Exclusion Principle in General

Let $P_1, P_2, \ldots, P_n$ are finite sets. Then $n(P_1 \cup P_2 \cup \ldots P_n)$

$$= \sum_{1 \le i \le n} n(P_i) - \sum_{1 \le i < j \le n} n(P_i \cap P_j)$$
$$+ \sum_{1 \le i < j < k \le n} n(P_i \cap P_j \cap P_k) \ldots + (-1)^{n-1}\, n(P_1 \cup P_2 \cup \ldots P_n)$$

ILLUSTRATIVE EXAMPLES

Example 1. Let A and B be two finite disjoint sets such that $n(A \cup B) = 500$ and $n(A) = 425$. Find $n(B)$.

Sol. Here $n(A \cup B) = 500, \; n(A) = 425$

Since A and B are disjoint sets

$\therefore \quad n(A \cap B) = 0$

Now $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad 500 = 425 + n(B) - 0$

$\Rightarrow \quad n(B) = 500 - 425 = 75$

Example 2. Let A and B be two finite sets such that $n(A-B)=15$, $n(A\cup B)=9$
$n(A\cap B)=30$. Find $n(B)$.

Sol. Here $n(A-B)=15,\quad n(A\cup B)=90,\quad n(A\cap B)=30$

Now, $n(A\cup B)=n(A-B)+n(A\cap B)+n(B-A)$

$\Rightarrow\quad 90=15+30+n(B-A)$

$\Rightarrow\quad n(B-A)=90-15-30=45$

Again $n(B)=n(A\cap B)+n(B-A)=30+45=75$

Example 3. A and B are two sets such that $n(A-B)=14+x$, $n(B-A)=3x$ and $n(A\cap B)=x$.

Illustration the information by a Venn's diagram.

Calculate, given that $n(A)=n(B)$

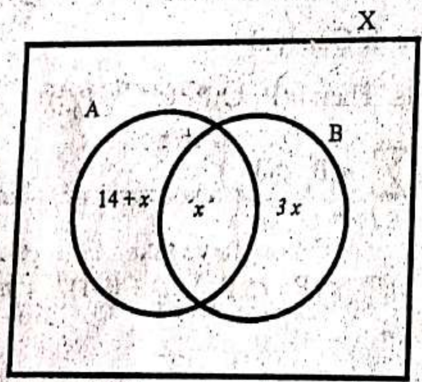
(*i*) The numerical value of x, (*ii*) $n(A\cup B)$ (Pbi. U. B.C.A. 201

Sol. Since $n(A\cup B)=n(A-B)+n(B-A)+n(A\cap B)$

$=(14+x)+(3x)+x=5x+14$

But $n(A)=n(A-B)+n(A\cap B)=14+x+x=14+2x.$

Also $n(B)=14+2x$ (as $n(A)=n(B)$ given)



Also $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$\therefore\quad 5x+14=(14+2x)+(14+2x)-x$

$5x+14=28+3x$

$2x=14$

$x=7$

(*ii*) $n(A\cup B)=5x+14=5(7)+14=35+14=49$

Example 4. In a class of 60 boys, there are 45 boys who play cards and 30 boys play carrom. Find

(*i*) how many boys play both games ?

(*ii*) how many boys play cards only ?

(*iii*) how many boys play carrom only ?

Sol. Let A denote the set of boys playing cards and B denote the set of boys playing carrom.

Here $n(A \cup B) = 60,\ n(A) = 45,\ n(B) = 30$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow\ 60 = 45 + 30 - n(A \cap B)$

$\Rightarrow\ n(A \cap B) = 45 + 30 - 60 = 15$

$n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow\ 45 = n(A - B) + 15$

$\Rightarrow\ n(A - B) = 45 - 15 = 30$

$n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow\ 30 = n(B - A) + 15$

$\Rightarrow\ n(B - A) = 30 - 15 = 15$

(*i*) Number of boys who play both games $= n(A \cap B) = 15$

(*ii*) Number of boys who play cards only $= n(A - B) = 30$

(*iii*) Number of boys who play carrom only $= n(B - A) = 15$

Example 5. In a class of 25 students 12 have taken economics, 8 have taken economics but not politics. Find the number of students who have taken economics and politics and those who have taken politics but not economics.

Sol. Let $n(A)$ = number of students who have taken economics.

$n(B)$ = number of students who have taken politics.

$\therefore$ $n(A \cap B^c)$ = number of student who have taken economics but not politics

$n(A \cap B)$ = number of students who have taken both economics and politics

$n(B \cap A^c)$ = number of students who have taken politics but not economics

Here $n(A) = 12,\ n(A \cap B^c) = 8, n(A \cup B) = 25$

Now $n(A) = n(A \cap B^c) + n(A \cap B)$

$\therefore$ $12 = 8 + n(A \cap B)$

$\Rightarrow\ n(A \cap B) = 4$

Also $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore$ $25 = 12 + n(B) - 4$

$\Rightarrow\ n(B) = 17$

Also $n(B) = n(B \cap A^c) + n(A \cap B)$

$17 = n(B \cap A^c) + 4$

$n(B \cap A^c) = 13$

Example 6. In a survey of 60 people, it was found that 25 read Newsweek magazine, 26 read Times, 26 read Fortune, 9 read both Newsweek and Fortune, 11 read both Times and Newsweek, 8 read both Times and Fortune, 3 read all three magazines.

Find (*i*) the number of people who read at least one of the three magazines ?

(*ii*) The number of people who read exactly one magazine ?

Sol. n (X) = Number of person to be surveyed

n (A) = Number of person who read News week magazine

n (B) = Number of person who read Times

n (C) = Number of person who read Fortune

n (A ∩ C) = Number of person who read News week & Fortune

n (A ∩ B) = Number of person who read News week & Time

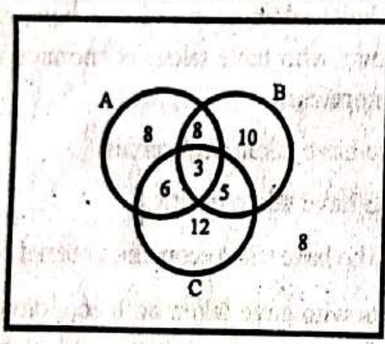
n (B ∩ C) = Number of person who read Time & Fortune

n (A ∩ B ∩ C) = Number of person who read all the three magazine.

Here $n(X) = 60$, $n(A) = 25$, $n(B) = 26$, $n(C) = 26$, $n(A \cap C) = 9$

$n(A \cap B) = 11$, $n(B \cap C) = 8$, $n(A \cap B \cap C) = 3$.

The number of element shown is the Venn's diagram



(*i*) Number of person who read atleast one of the three magazine

$= 8 + 8 + 10 + 6 + 3 + 5 + 12 = 52$

(*ii*) Number of people who read exactly one magazine

$= 8 + 10 + 12 = 30$

Example 7. Each student in a class of 40; studies atleast one of the subjects English, Mathematics and Economics. 16 study English, 22 Economics and 26 Mathematics, 5 study English and Economics, 14 Mathematics and Economics and 2 English, Economics and Mathematics. Find the number of students who study

(*i*) English and Mathematics

(*ii*) English, Mathematics but not Economics

Sol. Let A, B, C denote the set of students who study English, Economics and Mathematics respectively.

$\therefore$ $n(A \cup B \cup C) = 40$, $n(A) = 16$, $n(B) = 22$, $n(C) = 26$

$n(A \cap B) = 5$, $n(B \cap C) = 14$, $n(A \cap B \cap C) = 2$

Now $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A)$
$+ n(A \cap B \cap C)$

$\therefore \quad 40 = 16 + 22 + 26 - 5 - 14 - n(C \cap A) + 2$

$\Rightarrow \quad n(C \cap A) = 16 + 22 + 26 - 5 + 2 - 40 = 7$

$\therefore$ number of students who study English and Mathematics $= n(A \cap C) = 7$

Number of students who study English, Mathematics but not Economics

$$= n(A \cap C) - n(A \cap B \cap C) = 7 - 2 = 5$$

Example 8. In a town of 10,000 families, it was found that 40% families buy newspaper A, 20% buy newspaper B and 10% buy newspaper C. 5 % families buy A and B, 3% buy B and C, and 4 % buy A and C. If 2% families buy all the newspapers, find the number of families which buy (*i*) A only (*ii*) B only (*iii*) none of A, B, and C.

Sol. Total number of families = 10,000.

Let E, F, G denote the sets of families who buy newspapers A, B, C respectively.

$$\therefore \quad n(E) = \frac{40}{100} \times 10000 = 4000$$

$$n(F) = \frac{20}{100} \times 10000 = 2000$$

$$n(G) = \frac{10}{100} \times 10000 = 1000$$

$$n(E \cap F) = \frac{5}{100} \times 10000 = 500$$

$$n(F \cap G) = \frac{3}{100} \times 10000 = 300$$

$$n(E \cap G) = \frac{4}{100} \times 10000 = 400$$

$$n(E \cap F \cap G) = \frac{2}{100} \times 10000 = 200$$

Number of families buying newspaper A alone $= n(E \cap F^c \cap G^c)$

$= n(E) - n(E \cap F) - n(E \cap G) + n(E \cap F \cap G)$

$= 4000 - 500 - 400 + 200 = 3300$

Number of families buying newspaper B alone $= n(F \cap E^c \cap G^c)$

$= n(F) - n(F \cap E) - n(F \cap G) + n(E \cap F \cap G)$

$= 2000 - 500 - 300 + 200 = 1400$

Number of families who buy at least one of the newspapers

$= n(E \cup F \cup G)$

$= n(E) + n(F) + n(G) - n(E \cap F) - n(F \cap G) - n(G \cap E) + n(E \cap F \cap G)$

$= 4000 + 2000 + 1000 - 500 - 300 - 400 + 200 = 6000$

$\therefore$ number of families who do not buy newspapers A, B, C $= 10000 - 6000 = 4000$.

Example 9. Suppose that 100 of the 120 students at a college take at least one of the languages Hindi, English and Mathematics. Also suppose 65 study Hindi ; 45 study English ; 42 study Mathematics, 20 study Hindi and English, 25 study Hindi and Mathematics, 15 study English and Mathematics. Find number of students studying all subjects. Also find number of students studying exactly one subject.

(P.T.U. B.C.A. I 2004)

Sol. Let Mathematics : A

Hindi : B

English : C

$n(A) = 42, \quad n(B) = 65, \quad n(C) = 45$

$n(A \cap B) = 25 \quad n(B \cap C) = 20 \quad n(A \cap C) = 15$

$n(A \cup B \cup C) = 100 \quad n(U) = 120$

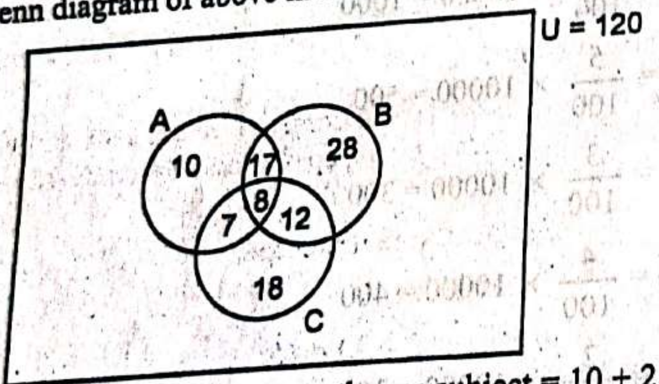
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$100 = 42 + 65 + 45 - 25 - 20 - 15 + n(A \cap B \cap C)$$

$$n(A \cap B \cap C) = 100 - 92 = 8$$

8 students study all subjects.

The Venn diagram of above information is



Number of students studying exactly one subject = 10 + 28 + 18 = 56

Example 10. A survey of 500 television watchers produced the following information :

285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basket-ball, 50 do not watch any of the three games.

How many watch all the three games ? How many watch exactly one of the three games ?

Sol. Let F, H, B denote the sets of viewers who watch football, hockey, basketball respectively.

$\therefore \quad n(F) = 285, n(H) = 195, n(B) = 115, n(F \cap B) = 45,$

$n(F \cap H) = 70, \ n(H \cap B) = 50, \ n(F \cup H \cup B)^c = 50$

Also total number of viewers = 500

Now $n(F \cup H \cup B)^c = 50$

$\Rightarrow \quad 500 - n(F \cup H \cup B) = 50$

$\Rightarrow \quad n(F \cup H \cup B) = 450$

$\Rightarrow \quad n(F) + n(H) + n(B) - n(F \cap H) - n(H \cap B) - n(B \cap F) + n(F \cap H \cap B) = 450$

$\Rightarrow \quad 285 + 195 + 115 - 70 - 50 - 45 + n(F \cap H \cap B) = 450$

$\Rightarrow \quad n(F \cap H \cap B) = 20$

$\therefore$ number of viewers watching all the three games = 20.

Number of viewers watching foot- ball alone $= n(F \cap H^c \cap B^c)$

$= n(F) - n(F \cap H) - n(F \cap B) + n(F \cap H \cap B)$

$= 285 - 70 - 45 + 20 = 190$

Number of viewers watching hockey alone $= n(H \cap F^c \cap B^c)$

$= n(H) - n(H \cap F) - n(H \cap B) + n(F \cap H \cap B)$

$= 195 - 70 - 50 + 20 = 95$

Number of viewers watching basket- ball alone $= n(B \cap H^c \cap F^c)$

$= n(B) - n(B \cap H) - n(B \cap F) + n(F \cap H \cap B)$

$= 115 - 50 - 45 + 20 = 40$

$\therefore$ number of viewers watching exactly one of the three games

$= 190 + 95 + 40 = 325$

EXERCISE 1 (*d*)

1. In a college, department A has 7 teachers and department B has 5 teachers. How many teachers department A or B have ?

2. In a particular college, 300 students are randomly selected. 180 students read the newspaper A, 147 students read the newspaper B and 84 students read both A and B.

   Find how many students read atleast one newspaper.

3. Let A and B be two finite sets such that

   $n(A \cap B) = 12, \; n(A - B) = 24, \; n(B - A) = 45.$

   Find $n(A)$ and $n(B)$.

4. Let A and B be two finite sets such that

   $n(A) = 115, n(B) = 326$ and $n(A - B) = 47$. Find $n(A \cup B)$ and $n(A \cap B)$.

5. Let A and B be two sets such that $n(A) = 20, \; n(A \cup B) = 42, \; n(A \cap B) = 4$

   Find (*i*) $n(B)$ (*ii*) $n(A - B)$ (*iii*) $n(B - A)$.

6. A survey shows that 76% of the students like oranges, whereas 62% like bananas. What percentage of the students like both oranges and bananas ?

7. A town has total population 25,000 out of which 13,000 read "The Tribune" and 10,500 read "The Indian Express" and 2,500 read both papers. Find the percentage of population who read neither of these newspapers.

8. A town have a total population of 60000. Out of it 32000 read 'The Hindustan Times' paper and 35000 read 'Time of India' paper, while 7500 both the newspapers. Indicate how many read neither the Hindustan Times nor Times of India ?

9. In a joint family of 12 persons, 7 take tea, 6 take milk and 2 take neither. H many members take both tea and milk ?

10. In a group of 950 persons, 750 can speak Hindi and 460 can speak English. Fi

    (*i*) how many can speak both Hindi and English ?

    (*ii*) how many can speak Hindi only ?

    (*iii*) how many can speak English only ?

11. In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

    (*i*) how many drink tea and coffee both ?

    (*ii*) how many drink coffee but not tea ?

12. Of the members of three athletic teams in a certain college, 21 are in basketball team, 26 in hockey team and 29 in football team. 14 play hockey a basketball, 15 play hockey and football, 12 play football and basketball and play all the three games. How many members are there in all ?

13. The report of one survey of 100 students stated that the various languages wer

Sanskrit, Hindi and Tamil	5
Hindi and Sanskrit	10
Tamil and Sanskrit	8
Hindi and Tamil	20
Sanskrit	30
Hindi	23
Tamil	50

The surveyor who prepared this report was fired. Why ?

14. A class has 175 students. The following table shows the number of stude studying one or more of the following subjects in this class ;

Subjects	Number of Students
Mathematics	100
Physics	70
Chemistry	46
Mathematics and Physics	30
Mathematics and Chemistry	28
Physics and Chemistry	23
Mathematics, Physics and Chemistry	18

How many students are enrolled in Mathematics alone, Physics alone an Chemistry alone ? Are there students who have not offered any of these thr subjects ?

ANSWERS

1. 12
2. 243
3. 36 ; 57
4. 373 ; 68
5. (i) 26 (ii) 16 (iii) 22
6. 38%
7. 16%
8. 500
9. 3
10. (i) 260 (ii) 490 (iii) 200
11. (i) 16 (ii) 20
12. 43
13. Surveyor prepared the report on the base of 70 students and not on all the 100 students
14. 60 ; 35 ; 13 ; 22

Art-14. Cartesian Product of Sets

Ordered Pair. By an ordered pair of elements, we mean a pair (a, b), $a \in A, b \in B$ in that order. The ordered pairs $(a, b), (b, a)$ are different unless $a = b$. Also $(a, b) = (c, d)$ iff $a = c, b = d$.

Cartesian Product of Two Sets. The *Cartesian product* of the sets A and B, denoted by $A \times B$, is the set of all possible ordered pairs whose first component is a member of A and whose second component is a member of B.

In symbols, $A \times B = \{(a, b) : a \in A, b \in B\}$

Note 1. $A \times B$ and $B \times A$ are different sets if $A \neq B$.

2. $A \times B = \phi$ when one or both of A, B are empty.

Cartesian Product of n Sets. The set of all ordered n-tuple $(a_1, a_2, \ldots\ldots, a_n)$ of elements $a_1 \in A_1, \ldots, a_n \in A_n$ is called the Cartesian product of the n sets $A_1, A_2, \ldots\ldots, A_n$ and is denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$ or briefly by $\prod_{i=1}^{n} A_i$

Note : $\prod$ is the symbol used for expressing a product just as Σ is the symbol used to express a sum.

Art-15. Prove that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(Pbi.U., M.Sc. I.T., 2006; Pbi. B.C.A. 2012)

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (P.T.U. B.Tech. May 2005, 2007)

Proof : (i) Let (a, b) be an arbitrary element of $A \times (B \cup C)$. Then,

$(a, b) \in A \times (B \cup C)$

$\Rightarrow$ $a \in A$ and $b \in (B \cup C)$

$\Rightarrow$ $a \in A$ and $(b \in B$ or $b \in C)$

$\Rightarrow$ $(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in (A \times B)$ or $(a, b) \in (A \times C)$

$\Rightarrow \quad (a, b) \in (A \times B) \cup (A \times C)$

$\therefore \quad A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Again let (x, y) be an arbitrary elements of $(A \times B) \cup (A \times C)$. Then

$(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow \quad (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow \quad x \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad (x, y) \in A \times (B \cup C)$

$\therefore \quad (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ ...

Hence from (*i*) and (*ii*)

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(*ii*) Let (a, b) be an arbitrary element of $A \times (B \cap C)$.

Then, $(a, b) \in A \times (B \cap C)$

$\Rightarrow \quad a \in A$ and $b \in (B \cap C)$

$\Rightarrow \quad a \in A$ and $(b \in B$ and $b \in C)$

$\Rightarrow \quad (a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$

$\Rightarrow \quad (a, b) \in (A \times B)$ and $(a, b) \in (A \times C)$

$\Rightarrow \quad (a, b) \in (A \times B) \cap (A \times C)$

$\therefore \quad A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ...

Again, let (x, y) be an arbitrary element of $(A \times B) \cap (A \times C)$. Then,

$(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow \quad (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $y \in (B \cap C)$

$\Rightarrow \quad (x, y) \in A \times (B \cap C)$

$\therefore \quad (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ...(

Hence from (*i*) and (*ii*),

$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Theorem : Let A be a non-empty set such that

$A \times B = A \times C$. Show that $B = C$.

Proof : Let b be an arbitrary element of B. Then,

$(a, b) \in A \times B \ \forall \ a \in A$

$\Rightarrow \quad (a, b) \in A \times C \ \forall a \in A \qquad [\because A \times B = A \times C]$

$\Rightarrow \quad b \in C$

Thus, $b \in B \Rightarrow b \in C$

$\therefore \qquad B \subset C \qquad ...(i)$

Now, let c be an arbitrary element of C. Then,

$(a, c) \in A \times C \ \forall \ a \in A$

$\Rightarrow \quad (a, c) \in A \times B \ \forall \ a \in A \qquad [\because A \times B = A \times C]$

$\Rightarrow \quad c \in B.$

Thus, $c \in C \Rightarrow c \in B$

$\therefore \qquad C \subset B \qquad ...(ii)$

From (*i*) and (*ii*), we have

$B = C.$

Theorem : For any three sets A, B, C, prove that $A \times (B-C) = (A \times B) - (A \times C)$

(Pbi. U M.Sc.I.T. April 2010)

Proof : Let (a, b) be an arbitrary elements of $A \times (B-C)$.

Then, $(a, b) \in A \times (B-C)$

$\Rightarrow \quad a \in A$ and $b \in (B-C)$

$\Rightarrow \quad a \in A$ and ($b \in B$ and $b \notin c$)

$\Rightarrow \quad$ ($a \in A$ and $b \in B$) and ($a \in A$ and $b \notin c$)

$\Rightarrow \quad (a, b) \in (A \times B)$ and $(a, b) \notin (A \times C)$

$\Rightarrow \quad (a, b) \in (A \times B) - (A \times C)$

$\therefore \quad A \times (B-C) \subseteq (A \times B) - (A \times C) \qquad ...(i)$

Again, let (x, y) be an arbitrary elements of $(A \times B) - (A \times C)$. Then,

$(x, y) \in (A \times B) - (A \times C)$

$\Rightarrow \quad (x, y) \in A \times B$ and $(x, y) \notin A \times C$

$\Rightarrow \quad$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \notin C$)

$\Rightarrow \quad x \in A$ and ($y \in B$ and $y \notin C$)

$\Rightarrow \quad x \in A$ and $y \in (B-C)$

$\Rightarrow \quad (x, y) \in A \times (B-C)$

$\therefore \quad (A \times B) - (A \times C) \subseteq A \times (B-C)$

Hence, from (*i*) and (*ii*), $\qquad ...(ii)$

$A \times (B-C) = (A \times B) - (A \times C).$

Art-16. Graphical Representation of A × B

Draw two perpendicular lines $x'Ox$ and $y'Oy$ intersecting at O, where $x'Ox$ is horizontal and $y'Oy$ is vertical. Now on horizontal line $x'Ox$ represent the elements of A and on vertical line $y'Oy$, represent the elements of B.

Now if $a \in A$, $b \in B$, draw a vertical line through a and a horizontal line through b. The point where they meet represents the ordered pair (a, b), The set of all such points obtained graphically represents A × B.

Note 1. If A is the set of all numbers, then A consists of all points in a line. A × A will consist of all points in the plane.

Note 2. The ordered pair (a, b) represents a point whose co-ordinates are (a, b).

Note 3. Let n (A) denote the number of elements of A.

Then $n(A \times B) = n(A) \times n(B)$.

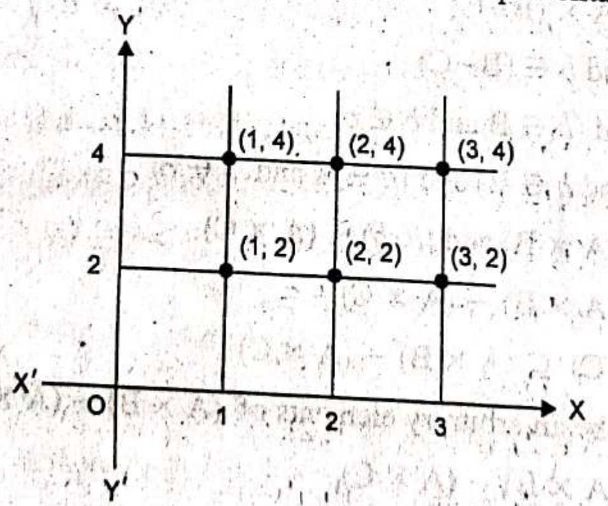
ILLUSTRATIVE EXAMPLES

Example 1. Let A = {1, 2, 3}, B = {2, 4}. Find A × B and show it graphically.

Sol. Here A = {1, 2, 3}, B = {2, 4}

$$A \times B = \{1, 2, 3\} \times \{2, 4\} = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}.$$

Now to represent (1, 2), we draw a vertical line through 1 and a horizontal line through 2. These two lines meet in the point which represents (1, 2). Similarly we can represent the other points in A × B and get the graphical representation of A × B.



Example 2. How many different elements does A × B have if A has m elements and B has n elements ?

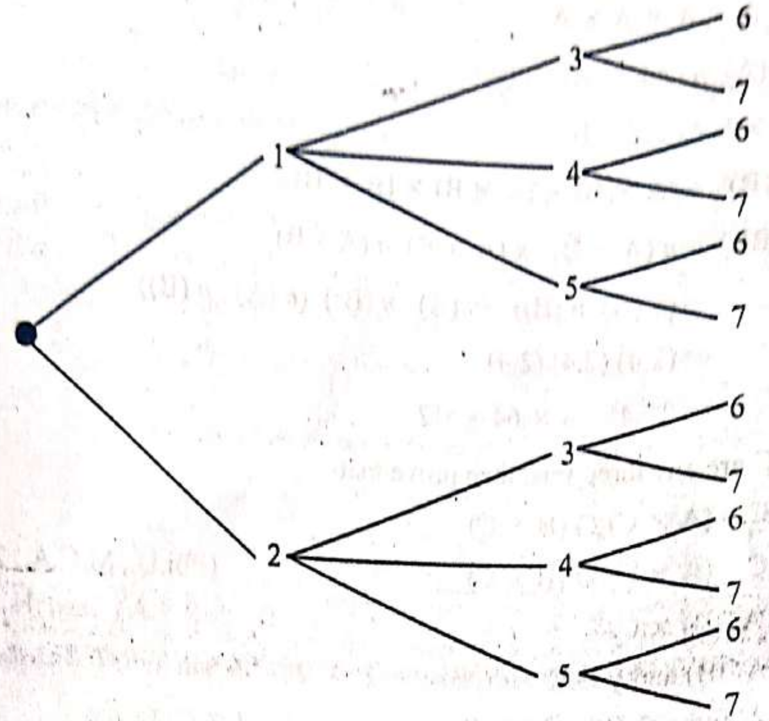
Sol. $m\,n$.

Example 3. Let A {1, 2}, B = {3, 4, 5} and C = {6, 7}. Then,

A × B × C consists of all ordered triple (a, b, c), where $a \in A$, $b \in B$ and $c \in C$.

Hence

A × B × C = {(1, 3, 6), (1, 3, 7), (1, 4, 6), (1, 4, 7), (1, 5, 6), (1, 5, 7), (2, 3, 6), (2, 3, 7), (2, 4, 6), (2, 4, 7), (2, 5, 6), (2, 5, 7)}



Example 4. Let A = {1, 2}, B = {a, b, c] and C = {x, y}, List all the elements in each of the following sets :

$A \times C, B \times B, A \times B \times C, (A \times A) \times (C \times C)$

Sol. $A \times C = \{(a, c) : a \in A, C \in C\} = \{(1, x), (1, y), (2, x), (2, y)\}$

$B \times B = \{(x, y) : x, y \in B\}$

$= \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$

$A \times B \times C = \{(x, y, z) : x \in A, y \in B, z \in C\}$

$= \{(1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y),$
$(2, a, x), (2, a, y), (2, b, x), (2, b, y), (2, c, x), (2, c, y)\}$

$(A \times A) \times (C \times C) = \{(1, 1), (1, 2), (2, 1), (2, 2)\} \times \{(x, x), (x, y), (y, x), (y, y)\}$

$= \{(a, b) : a \in A \times A, b \in C \times C\}$

$= \{[((1, 1), (x, x))], [((1, 1), (x, y))], [((1, 1), (y, x))],$
$[((1, 1), (y, y))], [((1, 2), (x, x))], [((1, 2), (x, y))], \{((1, 2), (y, x))]$
$[((1, 2), (y, y))], \{((2, 1), (x, x))], [((2, 1), (x, y)], [((2, 1), (y, x))],$
$[((2, 1), (y, y))], [((2, 2), (x, x))], [((2, 2), (x, y))], [((2, 2), (y, x))], [(2, 2), (y, y)]\}$

Example 5. Let A = [+, –], and B = [00, 01, 10, 11]

(*a*) List the elements of $A \times B$

(*b*) How many elements do A^4 and $(A \times B)^3$ have ? (P.T.U. B.Tech. May 2005)

Sol. A = {+, – }, B {00, 01, 10, 11}

(*a*) $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

$= \{(+, 00), (+, 01), (+, 10), (+, 11), (-, 00), (-, 01), (-, 10), (-, 11)\}$

(b) Since $A^4 = A \times A \times A \times A$

$\therefore \quad n(A^4) = n(A) . n(A) . n(A) . n(A)$

$= 2 \times 2 \times 2 \times 2 = 16$

(c) Since $(A \times B)^3 = (A \times B) \times (A \times B) \times (A \times B)$

$\therefore \quad n((A \times B)^3) = n(A \times B) . n(A \times B) . n(A \times B)$

$= (n(A) . n(B)) . (n(A) . n(B)) . (n(A) . n(B))$

$= (2.4)(2.4)(2.4)$

$= 2^3 . 4^3 = 8 \times 64 = 512$

Example 6. If A, B, C are any three sets, then prove that

(i) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(ii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ (Pbi.U., M.C.A., 2

Sol. (i) $\forall (x, y) \in (A \cap B) \times C$

$\Leftrightarrow \quad x \in (A \cap B)$ and $y \in C$

$\Leftrightarrow \quad (x \in A$ and $x \in B)$ and $y \in C$

$\Leftrightarrow \quad (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in C)$

$\Leftrightarrow \quad (x, y) \in (A \times C)$ and $(x, y) \in (B \times C)$

$\Leftrightarrow \quad (x, y) \in (A \times C) \cap (B \times C)$

Hence $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(ii) $\forall (x, y) \in (A \cup B) \times C$

$\Leftrightarrow \quad x \in (A \cup B)$ and $y \in C$

$\Leftrightarrow \quad (x \in A$ or $x \in B)$ and $y \in C$

$\Leftrightarrow \quad (x \in A$ and $y \in C)$ or $(x \in B$ and $y \in C)$

$\Leftrightarrow \quad (x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$

$\Leftrightarrow \quad (x, y) \in (A \times C) \cup (B \times C)$

Hence $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Example 7. If A and B are any two non-empty sets, then prove that

$A \times B = B \times A \Leftrightarrow A = B$ (Pbi.U. B.C.A., 2003, 2(

Sol. First, let us assume that $A = B$.

Then we have to prove that $A \times B = B \times A$.

Now $A = B$

$\Rightarrow \quad A \times B = A \times A$ and $B \times A = A \times A$ $[\because B =$

$\Rightarrow \quad A \times B = B \times A.$

Conversely, let $A \times B = B \times A$.

Then we have to prove that $A = B$.

Let x be an arbitrary element of A. Then

$$x \in A \Rightarrow (x, b) \in A \times B \ \forall b \in B.$$
$$\Rightarrow (x, b) \in B \times A \qquad [\because A \times B = B \times A]$$
$$\Rightarrow x \in B$$
$$\therefore \quad A \subseteq B$$

Again, let y be an arbitrary element of B. Then

$$y \in B \Rightarrow (a, y) \in A \times B \ \forall \ a \in A$$
$$\Rightarrow (a, y) \in B \times A \qquad [\because A \times B = B \times A]$$
$$\Rightarrow y \in A$$
$$\therefore \quad B \subseteq A$$

Hence, $A = B$.

Example 8. Prove that if A, B, C, D are arbitrary sets, then

$$(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D).$$

(P.T.U. B.Tech. Dec. 2003)

Sol. Let (x, y) be any element of $(A \cap C) \times (B \cap D)$.

Then $(x, y) \in (A \cap C) \times (B \cap D)$

$\Leftrightarrow$ $x \in (A \cap C)$ and $y \in (B \cap D)$

$\Leftrightarrow$ $(x \in A$ and $x \in C)$ and $(y \in B$ and $y \in D)$

$\Leftrightarrow$ $(x \in A$ and $y \in B)$ and $(x \in C$ and $y \in D)$

$\Leftrightarrow$ $(x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$

$\Leftrightarrow$ $(x, y) \in (A \times B) \cap (C \times D)$.

$\therefore$ $(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$.

Example 9. For any three sets A, B, C, prove that

$$(A - B) \times C = (A \times C) - (B \times C).$$

Sol. Let (a, b) be an arbitrary element of $(A - B) \times C$.

Then $(a, b) \in (A - B) \times C$

$\Rightarrow$ $a \in (A - B)$ and $b \in C$

$\Rightarrow$ $(a \in A$ and $a \notin B)$ and $b \in C$

$\Rightarrow$ $(a \in A$ and $b \in C)$ and $(a \notin B$ and $b \in C)$

$\Rightarrow$ $(a, b) \in (A \times C)$ and $(a, b) \notin (B \times C)$

$\Rightarrow$ $(a, b) \in (A \times C) - (B \times C)$

$\therefore$ $(A - B) \times C \subseteq (A \times C) - (B \times C)$ ...(*i*)

Again, let (x, y) be an arbitrary element of $(A \times C) - (B \times C)$. Then,

$$(x, y) \in (A \times C) - (B \times C)$$

$\Rightarrow \quad (x, y) \in (A \times C)$ and $(x, y) \notin (B \times C)$

$\Rightarrow \quad (x \in A$ and $y \in C)$ and $(x \notin B$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $x \notin B$ and $y \in C$

$\Rightarrow \quad x \in (A - B)$ and $y \in C$

$\Rightarrow \quad (x, y) \in (A - B) \times C$

$\therefore \quad (A \times C) - (B \times C) \subseteq (A - B) \times C$ ...(i

Hence, from (*i*) and (*ii*)

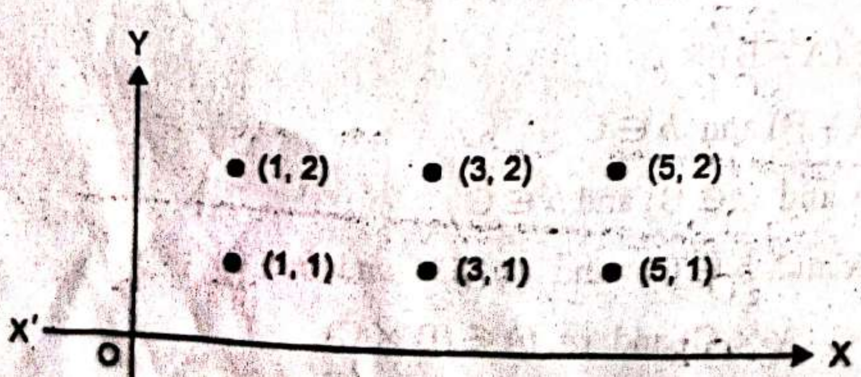
$$(A - B) \times C) = (A \times C) - (B \times C)$$

EXERCISE 1 (*e*)

1. Let $A = \{1, 3, 5\}$, $B = \{1, 2\}$. Find $A \times B$ and show it graphically.
2. Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$ and $C = \{1, 5, 9\}$ and let the universal set $U = \{0, 1, 2, \ldots\ldots, 9\}$. Determine
   (*i*) $A \times B$ (*ii*) B^2 (*iii*) B^3
3. Show that $n(A \times B) = n(A) \cdot n(B)$, where A, B are finite sets and n (A denotes numbet of elements of A. State the general rule.
4. Prove that $A \subset B$ and $C \subset D \Rightarrow (A \times C) \subset (B \times D)$.
5. A, B, C are any three sets, then prove that
   $$(A \cap B) \times C = (A \times C) \cap (B \times C)$$
6. If A and B be non-empty subsets, then show that $A \times B = B \times A$ iff $A = B$.

ANSWERS

1. $A \times B = \{1, 3, 5\} \times \{1, 2\} = \{(1, 1), (1, 2), (3, 1), (3, 2), (5, 1), (5, 2)\}$



2. (i) $\{(0, 2), (0, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$

(ii) $\{(2, 2), (2, 3), (3, 2), (3, 3)\}$

(iii) $\{(2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3), (3, 2, 2), (3, 2, 3), (3, 3, 2), (3, 3, 3)\}$

3. $n(A \times B \times C \times .....) = n(A) \cdot n(B) \cdot n(C) .....$

Art-16. Partition of Sets

A partition of a non-empty set A is a collection $P = \{A_1, A_2, A_3, ...\}$ of subsets of A if and only if

(i) $A = A_1 \cup A_2 \cup A_3 \cup ....$

and (ii) $A_i \cap A_j = \phi$ for $i \neq j$

$A_1, A_2, A_3, ...$ are called cells or blocks of the partition P.

Example (i) Let $A = \{a, b, c\}$ be any set. Then

$P_1 = \{\{a\}, \{b\}, \{c\}\}, P_2 = \{\{a\}, \{b, c\}\}$; $P_3 = \{\{b\}, \{a, c\}\}$;

$P_4 = \{\{c\}, \{a, b\}\}$; $P_5 = \{\{a, b, c\}\}$

are partitions of the set A.

Example (ii) Let $\mathbf{Z}$ = set of integers. Then the collection

$P = \{\{n\} : n \in \mathbf{Z}\}$ is a partition of $\mathbf{Z}$.

Art-17. Minimum Set or Minset or Minterm

Let A be any non-empty set and $B_1, B_2, ..., B_n$ be any subsets of A. Then the minimum set generated by the collection $\{B_1, B_2, ..., B_n\}$ is a set of the type $D_1 \cap D_2 \cap ...... \cap D_n$, where each $D_1, D_2, ..., D_n$ is B_i

or B_i' for $i = 1, 2, 3, ..., n$.

For example (i) The minsets generated by two sets B_1 and B_2 are

$A_1 = B_1 \cap B_2'$

$A_2 = B_1 \cap B_2$

$A_3 = B_1' \cap B_2$

$A_4 = B_1' \cap B_2'$

(ii) The minsets generated by three sets B_1, B_2 and B_3 are

$A_1 = B_1 \cap B_2 \cap B'_3$

$A_2 = B_1 \cap B'_2 \cap B_3$

$A_3 = B_1 \cap B'_2 \cap B'_3$

$A_4 = B'_1 \cap B_2 \cap B_3$

$A_5 = B'_1 \cap B'_2 \cap B'_3$

$A_6 = B'_1 \cap B'_2 \cap B_3$

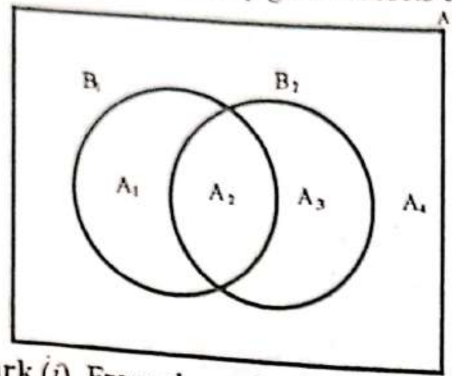
$A_7 = B'_1 \cap B_2 \cap B'_3$

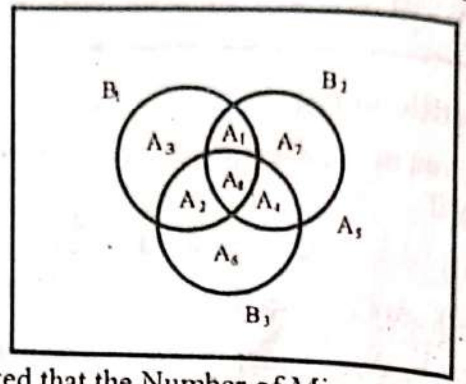
$A_8 = B_1 \cap B_2 \cap B_3$

Remark : From above, it is easily checked that the number of minsets generated sets is 2^n.

Art-18. Venn diagram of Minset

Minsets generated by given subsets can be represented through Venn diagram





Remark (*i*) From above, it is easily checked that the Number of Minsets generated sets is 2^n.

(*ii*) The collection of all the non-empty minsets generated by given subsets rise the partition of the set.

Art-19. Normal form (or Cannonical Form)

A set F is said to be in minset normal (or cannonical) form when it is expresse the union of distinct non-empty minsets or it is ϕ

i.e., either $F = \phi$ or $F = \underset{\lambda \in \Delta}{\cup} A_i$, where A_λ's are non-empty minsets.

Art-20. Principle of Duality for Sets

(P.T.U. B.Tech. Dec. 20

Let S be any identity in set theory involving the operation union ($\cup$), intersec ($\cap$). Then the statement S* obtained from S by changing union to intersect intersection to union and empty set ϕ to universal set U is also an identity called the d of the statement S.

example (*i*) Dual of $A \cup A' = X$ is $A \cap A' = \phi$

(*ii*) Dual of $(A \cup B)' = A' \cap B'$ is $(A \cap B)' = A' \cup B'$

(*iii*) Dual of $A \cup (B \cap A) = A$ is $A \cap (B \cup A) = A$

Art-21. Max Set or Maximum Set or Maxterm

The dual of the Minset is the Maxset *i.e.*, A maxset is the maximum or the larg set that is obtained by the union of any two or more subsets of a partition set.

ILLUSTRATIVE EXAMPLES

Example 1. State the dual of

(*a*) $A \cup (B \cap A) = A$

(*b*) $A \cup [(B^c \cup A) \cap B]^c = U$

(*c*) $(A \cup B^c)^c \cap B = A^c \cap B$

(*d*) $(A \cap U) \cap (\phi \cup A^c) = \phi$

(Pbi.U., B.C.A. 200

1. (a) The dual of $A \cup (B \cap A) = A$ is $A \cap (B \cup A) = A$.
(b) The dual of $A \cup [(B^c \cup A) \cap B]^c = \cup$ is $A \cap [(B^c \cap A) \cup B]^c = \phi$.
(c) The dual of $(A \cup B^c)^c \cap B = A^c \cap B$ is $(A \cap B^c)^c \cup B) = A^c \cup B$.
(d) The dual of $(A \cap U) \cap (\phi \cup A^c) = \phi$ is $(A \cup \phi) \cup (U \cap A^c) = U$.

xample 2. Write the dual of each set of equations :
(a) $(U \cap A) \cup (B \cap C) = A$
(b) $(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$

ol. (a) The dual of $(U \cap A) \cup (B \cap C) = A$ is $(\phi \cup A) \cap (B \cup C) = A$.
(b) The dual of $(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$ is
$(A \cap B \cap C)^c = (A \cap C)^c \cup (A \cap B)^c$.

xample 3. Let $X = \{1, 2, 3, \ldots\ldots, 8, 9\}$. Determine whether or not each of the following s a partition of X.
(a) $\{\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}\}$
(b) $\{\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}\}$
(c) $\{\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}\}$
(d) $\{\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}\}$

Sol. (a) $\{\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}\}$ is not a partition of X as
$\{1, 3, 6\} \cup \{2, 8\} \cup \{5, 7, 9\} = \{1, 2, 3, 5, 6, 7, 8, 9\} \neq X$.
(b) $\{\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}\}$ is a partition of the X as
$\{2, 4, 5, 8\} \cup \{1, 9\} \cup \{3, 6, 7\} = X$ and
$\{2,4,5,8\} \cap \{1,9\} = \phi$, $\{1,9\} \cap \{3,6,7\} = \phi$, $\{2,4,5,8\} \cap \{3,6,7\} = \phi$
(c) $\{\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}\}$ is not a partition of the set X as
$\{1, 5, 7\} \cap \{3, 5, 6\} = \{5\} \neq \phi$
(d) $\{\{1, 2, 7\}, \{3, 7\}, \{4, 6, 8, 9\}, \{3, 5\}\}$ is not a partition of the set X as
$\{3, 7\} \cap \{3, 5\} = \{3\} \neq \phi$

Example 4. Find all the partitions of the set $A = \{a, b, c, d\}$. (Pbi.U., B.C.A., 2000)

Sol. Let P (A) denote power set of A. Then

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$$

All the possible partitions of the set A are
$P_1 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$
$P_2 = \{\{a\{, \{b, c, d\}\}$
$P_3 = \{\{b\}, \{c, d, a\}\}$
$P_4 = \{\{c\}, \{a, b, d\}\}$

$P_5 = \{\{d\}, \{a, b, c\}\}$

$P_6 = \{\{a, b\}, \{c, d\}\}$

$P_7 = \{\{a, c\}, \{b, d\}\}$

$P_8 = \{\{a, d\}, \{b, c\}\}$

$P_9 = \{\{a\}, \{b\}, \{c, d\}\}$

$P_{10} = \{\{a\}, \{c\}, \{b, d\}\}$

$P_{11} = \{\{a\}, \{d\}, \{b, c\}\}$

$P_{12} = \{\{b\}, \{c\}, \{a, d\}\}$

$P_{13} = \{\{b\}, \{d\}, \{a, c\}\}$

$P_{14} = \{\{c\}, \{d\}, \{a, b\}\}$

$P_{15} = \{\{a, b, c, d\}\}$.

Remark From the above examples, we observe that the number of partitions of set is equal to the possible number of its disjunctions.

Example 5. Let $X = \{1, 2, 3, \ldots\ldots, 8, 9\}$. Find the cross partition P of the following partition of X :

$$P_1 = \{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}\} \text{ and } P_2 = \{\{1, 2, 3, 4,\}, \{5, 7\}, \{6, 8, 9\}\}$$

Sol. Here $X = \{1, 2, 3, \ldots\ldots, 8, 9\}$ and

$$P_1 = \{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}\} \text{ and } P_2 = \{\{1, 2, 3, 4,\}, \{5, 7\}, \{6, 8, 9\}\}$$

be the partition of X. Then

$\{1, 3, 5, 7, 9\} \cap \{1, 2, 3, 4\} = \{1, 3\}$

$\{1, 3, 5, 7, 9\} \cap \{5, 7\} = \{5, 7\}$

$\{1, 3, 5, 7, 9\} \cap \{6, 8, 9\} = \{9\}$

$\{2, 4, 6, 8\} \cap \{1, 2, 3, 4\} = \{2, 4\}$

$\{2, 4, 6, 8\} \cap \{5, 7\} = \phi$

$\{2, 4, 6, 8\} \cap \{6, 8, 9\} = \{6, 8\}$

$\therefore$ the cross partition P of X is

$$P = \{\{1, 3\}, \{5, 7\}, \{9\}, \{2, 4\}, \{6, 8\}\}.$$

Example 6. Partition the set $A = \{1, 2, 3, 4, 5, 6\}$ using the minsets generated by $B_1 = \{1, 3, 5\}$ $B_2 = \{1, 2, 3\}$. Also, represent the minsets thus generated through a Venn diagram.

Sol. The Number of minsets generated by B_1, B_2 is $2^2 = 4$. And these are

$A_1 = B_1 \cap B'_2 = \{1, 3, 5\} \cap \{4, 5, 6\} = \{5\}$

$A_2 = B_1 \cap B_2 = \{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$

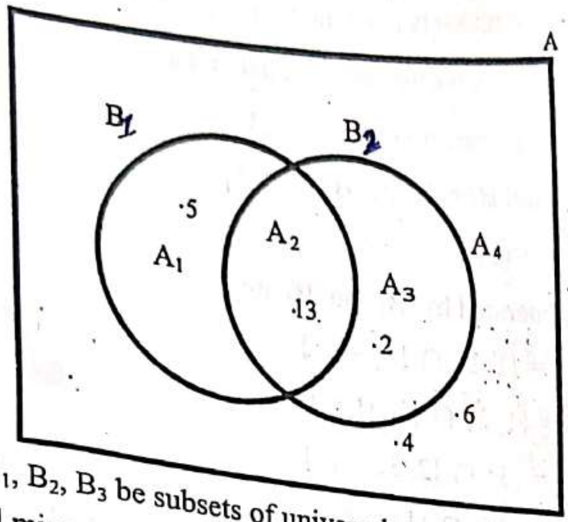
$A_3 = B'_1 \cap B_2 = \{2, 4, 6\} \cap \{1, 2, 3\} = \{2\}$

$A_4 = B'_1 \cap B'_2 = \{2, 4, 6\} \cap \{4, 5, 6\} = \{4, 6\}$

Since $A = A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_i \cap A_j = \phi$ for $i \neq j$

$\therefore$ $\{A_1, A_2, A_3, A_4\}$ is a partition of the set A.

Minsets generated by the sets B_1, B_2 can be represented by a Venn diagram as



Example 7. Let B_1, B_2, B_3 be subsets of universal set U

(*a*) Find all minsets generated by B_1, B_2, B_3.

(*b*) Illustrate via. *a* Venn diagram all minsets obtained in part (*a*)

(*c*) Express the following sets in minset normal form B'_1, $B_1 \cap B_2$, $B'_1 \cap B'_2$.

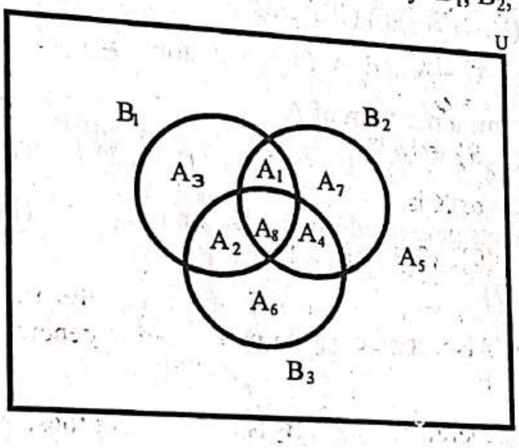
Sol. (*a*) Minsets generated by B_1, B_2, B_3 are

$A_1 = B_1 \cap B_2 \cap B'_3$, $A_2 = B_1 \cap B'_2 \cap B_3$, $A_3 = B_1 \cap B'_2 \cap B'_3$

$A_4 = B_1' \cap B_2 \cap B_3$, $A_5 = B'_1 \cap B'_2 \cap B'_3$, $A_6 = B'_1 \cap B'_2 \cap B_3$

$A_7 = B'_1 \cap B_2 \cap B'_3$, $A_8 = B_1 \cap B_2 \cap B_3$

(*b*) Venn's diagram of the minsets generated by B_1, B_2, B_3 is



(*c*) The minset normal form of B'_1 is

$$(B'_1 \cap B_2 \cap B_3) \cup (B'_1 \cap B_2 \cap B'_3) \cup (B'_1 \cap B'_2 \cap B_3) \cup (B'_1 \cap B'_2 \cap B'_3)$$

The minset normal form of $B_1 \cap B_2$ is

$$(B_1 \cap B_2 \cap B_3) \cup (B_1 \cap B_2 \cap B'_3)$$

The minset normal form of $B'_1 \cap B'_2$ is

$$(B'_1 \cap B'_2 \cap B'_3) \cup (B'_1 \cap B'_2 \cap B_3)$$

Example 8. Let $A = \{1, 2, 3\}$, $B_1 = \{1, 2\}$, $B_2 = \{2, 3\}$

(a) Find minsets and maxsets generated by B_1 and B_2.

(b) Write B'_2 as minsets normal form. What is maxset normal form of B'_2?

(c) Do minsets form a partition of A. Check.

Sol. Here $A = \{1, 2, 3\}$ and $B_1 = \{1, 2\}, B_2 = \{2, 3\}$.

$\therefore \quad B'_1 = \{3\}, B'_2 = \{1\}$

(a) The minsets generated by B_1 and B_2 are

$A_1 = B_1 \cap B'_2 = \{1, 2\} \cap \{1\} = \{1\}$

$A_2 = B_1 \cap B_2 = \{1, 2\} \cap \{2, 3\} = \{2\}$

$A_3 = B'_1 \cap B_2 = \{3\} \cap \{2, 3\} = \{3\}$

$A_4 = B'_1 \cap B'_2 = \{3\} \cap \{1\} = \phi$

The maxsets generated by B_1 and B_2 are

$M_1 = B_1 \cup B'_2 = \{1, 2\} \cup \{1\} = \{1, 2\}$

$M_2 = B_1 \cup B_2 = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$

$M_3 = B'_1 \cup B_2 = \{3\} \cup \{2, 3\} = \{2, 3\}$

$M_4 = B'_1 \cup B'_2 = \{3\} \cup \{1\} = \{1, 3\}$

(b) Minset normal form of B'_2 is

$B_1 \cap B'_2$

and Maxset normal form of B'_2 is

$M_1 \cap M_4$ *i.e.,* $(B_1 \cup B'_2) \cap (B'_1 \cup B'_2)$

(c) Since $A_1 \cup A_2 \cup A_3 = A$ and $A_i \cap A_j = \phi$ for $i \neq j$

$\therefore$ $\{A_1, A_2, A_3\}$ forms a partition of A.

Example 9. Let $A = \{2, 3, 4, 5, 6, 7\}$; $B_1 = \{5, 6, 7\}$; $B_3 = \{3, 4\}$.

Find minsets and maxsets generated by B_1, B_2 and B_3. (Pbi. U. B.C.A., 201

Sol. $A = \{2, 3, 4, 5, 6, 7\}$

$B_1 = \{5, 6, 7\}$ $\qquad B_1^c = \{2, 3, 4\}$

$B_2 = \{2, 4, 5\}$ $\qquad B_2^c = \{3, 6, 7\}$

$B_3 = \{3, 4\}$ $\qquad B_3^c = \{2, 5, 6, 7\}$

Minsets generated by B_1, B_2, B_3 are given by :

$B_1 \cap B_2 \cap B_3 = \phi$

$B_1 \cap B_2 \cap B_3^c = \{5\}$

$B_1 \cap B_2^c \cap B_3 = \phi$

$$B_1^c \cap B_2 \cap B_3 = \{4\}$$

$$B_1 \cap B_2^c \cap B_3^c = \{6, 7\}$$

$$B_1^c \cap B_2 \cap B_3^c = \{2\}$$

$$B_1^c \cap B_2^c \cap B_3 = \{3\}$$

$$B_1^c \cap B_2^c \cap B_3^c = \phi$$

Maxsets generated by B_1, B_2, B_3 are given by

$$B_1 \cup B_2 \cup B_3 = \{2, 3, 4, 5, 6, 7\}$$

$$B_1 \cup B_2 \cup B_3^c = \{2, 4, 5, 6, 7\}$$

$$B_1 \cup B_2^c \cup B_3 = \{3, 4, 5, 6, 7\}$$

$$B_1^c \cup B_2 \cup B_3 = \{2, 3, 4, 5\}$$

$$B_1 \cup B_2^c \cup B_3^c = \{2, 3, 5, 6, 7\}$$

$$B_1^c \cup B_2 \cup B_3^c = \{2, 3, 4, 5, 6, 7\}$$

$$B_1^c \cup B_2^c \cup B_3 = \{2, 3, 4, 6, 7\}$$

$$B_1^c \cup B_2^c \cup B_3^c = \{2, 3, 4, 5, 6, 7\}$$

Example 10. Let S be a set of words or string of length ≤ 2

i.e., $S = \{0, 1, 00, 01, 10, 11\}$ and $A = \{0, 00, 01\}$, $B = \{00, 01, 10, 11\}$

Find a partition of S using minsets generated by A and B.

Sol. Here $S = \{0, 1, 00, 01, 10, 11\}$ and $A = \{0, 00, 01\}$, $B = \{00, 01, 10, 11\}$

$\therefore$ $A' = \{1, 10, 11\}$ and $B' = \{0, 1\}$

$\therefore$ minsets generated by A and B are

$A_1 = A \cap B' = \{0, 00, 01\} \cap \{0, 1\} = \{0\}$

$A_2 = A \cap B = \{0, 00, 01\} \cap \{00, 01, 10, 11\} = \{00, 01\}$

$A_3 = A' \cap B = \{1, 10, 11\} \cap \{00, 01, 10, 11\} = \{10, 11\}$

$A_4 = A' \cap B' = \{1, 10, 11\} \cap \{0, 1\} = \{1\}$

Clearly $A_1 \cup A_2 \cup A_3 \cup A_4 = S$

and $A_i \cap A_j = \phi$ for $i \neq j$

$\therefore$ $\{A_1, A_2, A_3, A_4\}$ forms a partition of S.

EXERCISE 1 (*f*)

1. State duality principle for sets. Write the dual of the following :

   (*a*) $A \cup B = (B' \cap A)'$

   (*b*) $A = (B' \cap A) \cup (A \cap B)$

   (*c*) $A \cup (A \cap B) = A$

   (*d*) $(A \cap B) \cup (A' \cap B) \cup (A \cap B') \cup (A' \cap B') = X$

2. Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following partition of S :

   (*a*) $P_1 = \{\{1, 2, 3\}, \{1, 4, 5, 6\}\}$ (*b*) $P_2 = \{\{1, 2\}, \{3, 5, 6\}\}$

   (*c*) $P_3 = \{\{1, 3, 5\}, \{2, 4\}, \{6\}\}$ (*d*) $P_4 = \{\{1, 3, 5\}, \{2, 4, 6, 7\}\}$

3. Determine whether or not each of the following is a partition of the set positive integers :

   (*a*) $\{\{n : n > 5\}, \{n : n < 5\}\}$

   (*b*) $\{\{n : n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}\}$

   (*c*) $\{\{n : n^2 > 11\}, \{n : n^2 < 11\}\}$

4. Let $\{A_1, A_2, \ldots\ldots, A_m\}$ and $\{B_1, B_2, \ldots\ldots B_n\}$ be partition of *a* set X. Show the collection of sets

   $$P = \{A_i \cap B_j : i = 1, 2, \ldots\ldots, m; j = 1, 2, \ldots\ldots, n\}/\phi$$

   is also a partition (called the cross partition) of X.

5. Let $\{A_1, A_2, \ldots\ldots, A_n\}$ be a partition of set A, let B be any non-empty subse A. Prove that $\{A_i \cap B : A_i \cap B \neq \phi\}$ is a partition of $A \cap B$.

6. Partition the set $A = \{1, 2, 3, \ldots\ldots, 10\}$ using the minsets generated by $B_1 = \{1, 7, 8\}$, $B_2 = \{1, 6, 9, 10\}$, $B_3 = \{1, 9, 10\}$. Also represent the min thus generated through a Venn diagram.

7. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B_1 = \{1, 3, 5\}$, $B_2 = \{1, 2, 3\}$.

   (*a*) Find minsets and mexset generated by B_1 and B_2

   (*b*) Do minsets form a partition of A ?

   (*c*) Do maxset form a partition of A ?

8. Let $A = \{1, 2, 3, \ldots\ldots, 9\}$ and $B_1 = \{5, 6, 7\}$, $B_2 = \{2, 4, 5, 9\}$, and $B_3 = \{3, 4, 5, 6, 8, 9\}$. Find

   (*i*) A partition of A using minsets

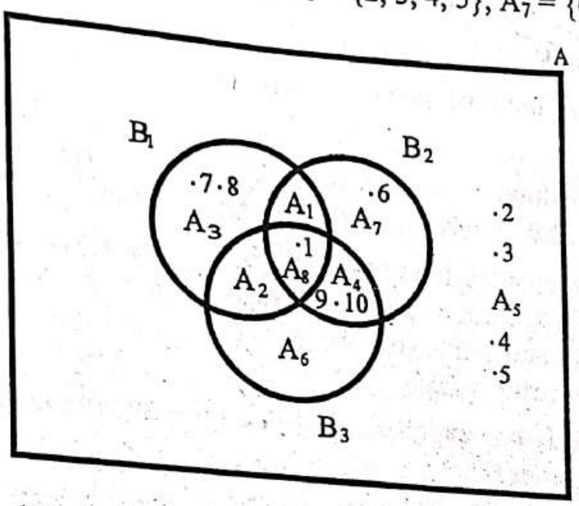
   (*ii*) The number of distinct subsets of A using B_1, B_2 and B_3

   (*iii*) Normal form of $B'_2 \cap B'_3$, B'_1, $B_1 \cap B_2$, $B'_1 \cap B'_2$

9. Let $A = \{0, 1, 2, 3, 4, 5\}$, $B_1 = \{0, 2, 4\}$ and $B_2 = \{1, 5\}$. Find the maxsets generated by B_1 and B_2.

ANSWERS

1. (a) $(A \cap B) = (B' \cup A)'$ (b) $A = (B' \cup A) \cap (A \cup B)$
   (c) $A \cap (A \cup B) = A$
   (d) $(A \cup B) \cap (A' \cup B) \cap (A \cup B') \cap (A' \cup B') = \phi$
2. (a) Not a partition (b) Not a partition
   (c) Partition (d) Not a partition
3. (a) Not a partition (b) Not a partition (c) Partition
6. $\{A_3, A_4, A_5, A_7, A_8\}$ is a partition of the set A where
   $A_3 = \{7, 8\}$, $A_4 = \{9, 10\}$, $A_5 = \{2, 3, 4, 5\}$, $A_7 = \{6\}$, $A_8 = \{1\}$



7. (a) {5}, {3}, {2}, {4, 6} ; {1, 3, 4, 5, 6}, {1, 2, 3, 5}, {1, 2, 3, 4, 6}, {2, 4, 5, 6}
   (b) Yes (c) No
8. (i) $P = \{A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ is a partition where
   $A_2 = \{6\}$, $A_3 = \{7\}$, $A_4 = \{4, 9\}$, $A_5 = \{1\}$, $A_6 = \{3, 8\}$, $A_7 = \{2\}$, $A_8 = \{5\}$
   (ii) 128
   (iii) (a) $(B_1 \cap B'_2 \cap B'_3) \cup (B'_1 \cap B'_2 \cap B'_3)$
   (b) $(B'_1 \cap B_2 \cap B_3) \cup (B'_1 \cap B'_2 \cap B'_3)$
   $\cup (B'_1 \cap B'_2 \cap B_3) \cup (B'_1 \cap B_2 \cap B'_3)$
   (c) $(B_1 \cap B_2 \cap B_3)$
   (d) $(B'_1 \cap B'_2 \cap B'_3) \cup (B'_1 \cap B'_2 \cap B_3)$
9. {0, 1, 2, 4, 5}, {0, 2, 3, 4}, {1, 3, 5}, {0, 1, 2, 3, 4, 5}