2

SOLUTION OF TRANSCENDENTAL EQUATIONS

2.1 INTRODUCTION

One of the most commonly occurring problems of applied mathematics and engineering is to find one or more roots of an equation of the form

$$f(x) = 0 \qquad \ldots(2.1)$$

The equation (2.1) can be any of the following two types :

(a) Algebraic Equation : If the equation (2.1) is of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n = 0 \qquad \ldots(2.2)$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants, $a_n \neq 0$ and n is a positive integer then it is called an *algebraic equation* with degree n.

Further if $n = 1$ then it is called a *linear algebraic equation* and if $n > 1$ then it is called a *non-linear algebraic equation.*

(b) Transcendental Equation : If the equation (2.1) involves transcendental functions such as e^x, $\log x$, $\sin x$ etc. then it is called a *transcendental equation.*

e.g. $\qquad x + \cos x = 0,\ x e^x - 1 = 0$ etc.

Now returning back to the main subject of the chapter, if equation (2.1) is a quadratic, cubic or a biquadratic equation then algebraic methods are available to find the roots of the equation.

e.g. the algebraic formula for finding the roots of a quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Similar algebraic formulae are available for finding the roots of a general cubic equation or a general biquadratic equation. Such algebraic methods for finding the roots of an equation are also termed as *direct methods*. The direct methods give the exact value of the roots in a fixed number of steps. Also the direct methods give all the roots at the same time.

It is important to mention here that the algebraic formula for finding the roots of a general cubic equation is somewhat more complicated and the formula for the roots of a general biquadratic equation usually takes several pages to describe! Besides this there is no such formula available for finding the roots of algebraic equations having degree higher than four and transcendental equations. Further when the numbers are substituted in the algebraic formula (if available), rounding errors arise which reduce the accuracy in the roots obtained.

So in practice, for finding the roots of an algebraic equation with degree higher than two or a transcendental equation, we have to use numerical methods. The numerical methods for finding the roots of a nonlinear equations are called *iterative methods* and these will remain the main subject of this chapter.

2.2 ITERATIVE METHODS–MEANING AND THEIR CHARACTERISTICS

An *iterative method* starts with an approximate solution and uses this approximate solution in a recurrence formula to provide another approximate solution ; by repeatedly applying the formula, a sequence of solutions is obtained which, under suitable conditions, converges to the exact solution. Iterative methods make us able to find a root to any specified degree of accuracy. Iterative methods are also called *trail and error methods*. Generally iterative methods give one root at a time. Iterative methods are relatively insensitive to the propagation of errors. Iterative methods are quite general in technique and have the advantage of simplicity of operations but these are disliked by human problem solver because these are very cumbersome and time consuming. However these methods have the ease of implementation on computers due to following reasons :

(*a*) In iterative methods, same sequence of steps is repeated time and again. Such a procedure can be concisely expressed as a computational algorithm.

(*b*) It is possible to develop the algorithms which can handle a class of similar problems.

e.g. a general algorithm can be developed to solve a class of algebraic equations with degree n for some fixed positive integer n.

(*c*) Rounding errors are negligible in iterative methods as compared to the direct methods.

2.3 PROCEDURE TO FIND INITIAL APPROXIMATION

Let us suppose that our problem is to find some or all the roots of the nonlinear equation $f(x) = 0$ Before we use a numerical method, we should have some idea about the number, nature and approximate location of the roots. This initial approximation to the root is generally obtained by any of the following two methods.

1. Analytic Method : An approach to find initial approximation to the root of equation $f(x) = 0$ is the construction of a table of values of the function f and then use *Intermediate Value Theorem* which states :

"If f is a continuous function on interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs then there exists atleast one real root of $f(x) = 0$ in the interval (a, b)."

After obtaining the interval containing the root, we may take any point in the interval as the initial approximation to the root. (refer tables 2.1 and 2.2)

2. Graphical Method : In this method, we first sketch $y = f(x)$. The point where this sketch intersects x-axis gives the root of equation $f(x) = 0$, so any point in the neighbourhood of this point may be taken as initial approximation to the root. (refer fig. 2.1 and fig. 2.2)

Further if the equation $f(x) = 0$ can be easily written in the form $g(x) = h(x)$, then the point of intersection of the sketch of $y = g(x)$ and $y = h(x)$ gives the root of the equation $f(x) = 0$, so any point in the neighbourhood of this point may be taken as initial approximation to the root. (refer fig. 2.3)

We now illustrate both of these methods by a few examples :

(*i*) Consider the equation $x^2 + 2x - 1 = 0$

Analytic Method : We construct a table of values of $f(x) = x^2 + 2x - 1$ corresponding to different values of x as follows :

x	-3	-2	-1	0	1	2	3
$f(x)$	2	-1	-2	-1	2	7	14

Table 2.1

From table 2.1, we see that $f(-3)$ and $f(-2)$ have opposite signs so a root of $f(x) = 0$ lies in the interval $(-3, -2)$.

Similarly the equation $f(x) = 0$ has a root in the interval $(0, 1)$. We may take any point in the interval containing the root as the initial approximation.

Graphical Method :

We sketch the curve $y = x^2 + 2x - 1$ (see fig 2.1)

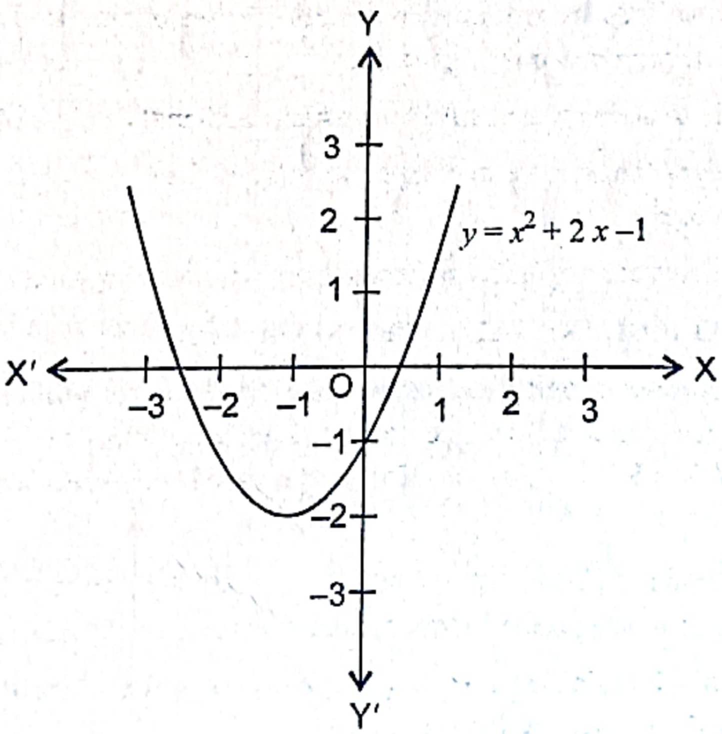


Fig. 2.1

We deduce from the sketch that the equation $x^2 + 2x - 1 = 0$ has a root near $x = -2.5$ and a root near $x = 0.5$. So $x = -2.5$ (or $x = 0.5$) can be taken as initial approximation to the root of $f(x) = 0$.

(*ii*) Consider the equation $\cos x - x\,e^x = 0$

Analytic Method : We construct a table of values of $f(x) = \cos x - x e^x$ corresponding to different values of x as follows :

x :	0	0.5	1	1.5
$f(x)$:	1	0.0532	– 2.1780	– 6.6518

Table 2.2

From table 2.2, we see that $f(0.5)$ and $f(1)$ have opposite signs so atleast one root of $f(x) = 0$ lies in the interval (0.5, 1). We may take any point in this interval as the initial approximation.

Graphical Method :

We sketch the curve $y = \cos x - x e^x$ (see fig. 2.2)

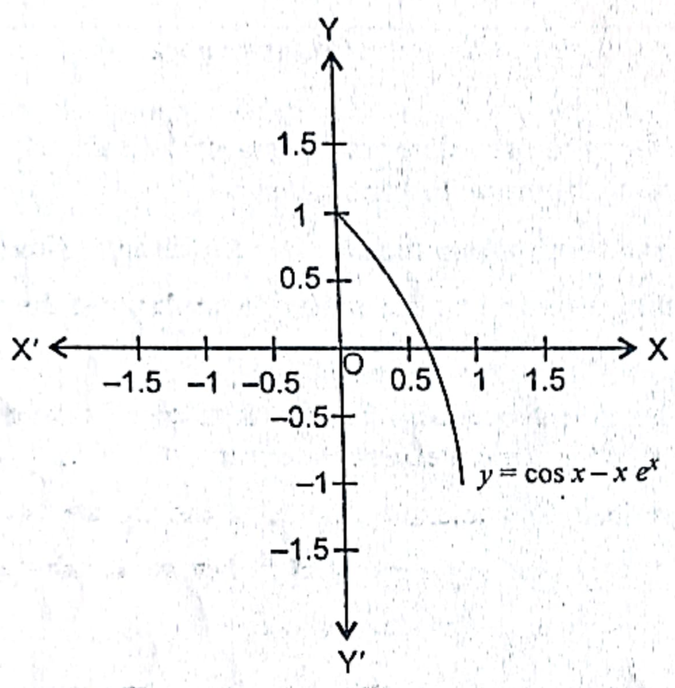


Fig. 2.2

We deduce from the sketch that the equation $\cos x - x e^x = 0$ has a root near $x = 0.5$.

So $x = 0.5$ can be taken as initial approximation to the root.

Alternatively, since the equation $\cos x - x e^x = 0$ is equivalent to $x = e^{-x} \cos x$ so we sketch $y = x$ and $y = e^{-x} \cos x$ on one set of axes and see where they intersect (See fig. 2.3).

We deduce from the sketch that when two curves intersect, the corresponding value of x is near 0.5. So $x = 0.5$ can be taken as initial approximation to the root of equation $f(x) = 0$.

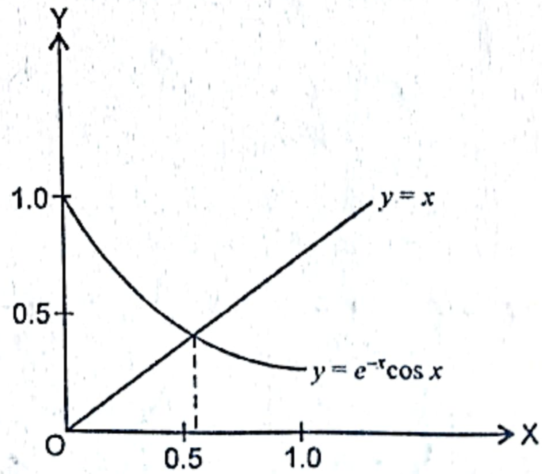


Fig. 2.3

2.4 TERMINATING CRITERIA FOR ITERATIVE METHODS

Let us suppose that our problem is to find some or all the roots of the nonlinear equation $f(x) = 0$. The general technique in almost all the iterative methods is that we begin with an initial approximation say x_0 to the exact value of the root say ξ and then find the better approximations or iterates $x_1, x_2, x_3, \ldots$ successively by repeating the same method. If each time after applying the same method, the successive approximations approach the root ξ more and more closely then we say that the sequence of approximations or iterates converges to the root ξ. In practice, except in some rare cases, the exact value ξ of the root is not known. Also it is not possible to find the root ξ which satisfies the given equation exactly.

So the basic question which is relevant to all iterative methods is that when to terminate or stop the iterative procedure ?

Following are some *stopping rules* or *terminating criteria* which provides us the answer of above said question :

(*a*) Terminate the iterative procedure when the absolute value of $f(x)$ at a particular approximation becomes less than or equal to the prescribed error tolerance.

i.e. if ε is the prescribed error tolerance and x_k is the kth approximation to a root of equation $f(x) = 0$ such that $|f(x_k)| \leq \varepsilon$ then we terminate the iterative procedure and the approximation x_k is taken as the approximate solution.

(*b*) Terminate the iterative procedure when the difference between two successive approximations becomes less than or equal to the prescribed error tolerance.

i.e. if ε is the prescribed error tolerance and x_{k-1} and x_k are two successive approximations to a root of equation $f(x) = 0$ such that $|x_k - x_{k-1}| \leq \varepsilon$ then we terminate the iterative procedure and the approximation x_k is taken as the approximate solution.

(*c*) Terminate the iterative procedure when the relative error using two successive approximations becomes less than or equal to the prescribed error tolerance.

i.e. if ε is the prescribed error tolerance and x_{k-1} and x_k are two successive approximations to a root of equation $f(x) = 0$ such that $\left|\dfrac{x_k - x_{k-1}}{x_k}\right| \leq \varepsilon$ then we terminate the iterative procedure and the approximation x_k is taken as the approximate solution.

It should be noted that each of above said criterion has their own advantages and disadvantages.

e.g. there may be some cases in which $|f(x_k)|$ is very small but x_k remains too far from the exact root (See fig. 2.4). Similarly there may be some cases in which $|x_k - x_{k-1}|$ is very small but x_k still remains too far from the exact root.

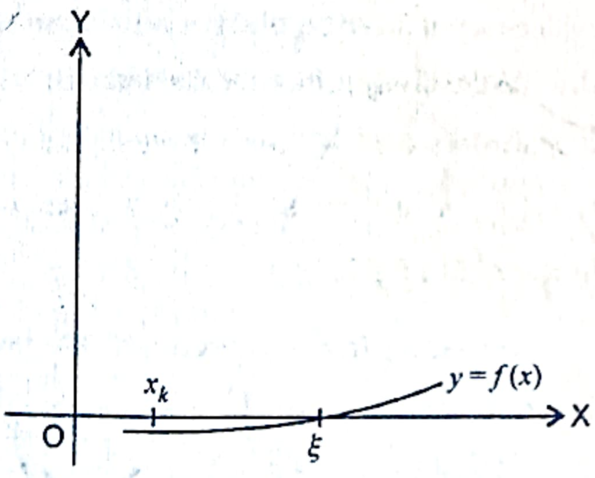


Fig. 2.4

So in practice it is better to look simultaneously at the size of $|f(x_k)|$ and at the difference between the successive approximations obtained from an iterative method. Alternatively, the third criteria can also be used to terminate the iterative procedure as it is known to be the most reliable criteria. Further, one should specify a maximum number of iterates to be computed to avoid an infinite loop if that is possible.

2.5. ORDER OF CONVERGENCE OF ITERATIVE METHODS

The speed or fastness of convergence of an iterative method is judged by its *order of convergence*. Higher the order of convergence means the error in the successive approximations obtained decreases more rapidly. The order of convergence of an iterative method is defined as follows :

An iterative method is said to be of *order p* or has the *order of convergence p* if p of the largest positive real number (≥ 1) such that

$$|e_{k+1}| \leq A\,|e_k|^p \quad (k \to \infty) \qquad ...(2.3)$$

for some constant $A \neq 0$, where $e_k = x_k - \xi$ and $e_{k+1} = x_{k+1} - \xi$ are the errors in kth and $(k+1)$th approximation respectively.

The constant A is called *asymptotic error constant.*

Notes **1.** It follows form the above definition that number of significant digits in each approximation increases p times than that in previous approximation.

2. It may be seen that any iterative process of order exceeding unity certainly will yield convergence to ξ if the iteration is initiated sufficiently near to ξ (see section 2.8). On the other hand, when $p = 1$, this statement generally cannot be made unless the asymptotic error constant A associated with ξ is smaller than 1 in absolute value (see sec. 2.6 and sec. 2.7). In other words, if $p = 1$ the sequence of approximations is said to converge *linearly* to ξ and in that case it is necessary that $|A| < 1$ and the constant A in that case is called the *rate of linear convergence* of approximations to ξ.

It should be noted that for linearly convergent methods, $|e_{k+1}| \leq A\,|e_k|$ so smaller the value of A means that error in the next approximation decreases more rapidly. So smaller the magnitude of A means higher the speed of convergence.

e.g. if one method converges linearly with a rate of 0.146 and the second method converges linearly with a rate of 0.5 then the later method has much poorer rate than the former.

2.6 BISECTION (OR BOLZANO) METHOD

Bisection method for finding the root of the equation $f(x) = 0$ is based on the repeated application of intermediate value theorem which states that if f is a continuous function on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs then there exists atleast one real root of $f(x) = 0$ in the interval (a, b).

Procedure

Without loss of generality, suppose that a continuous function f is negative at a and positive at b, so there exists atleast one real root between a and b. (We may also take f as positive at a and negative at b).

Let the approximate value of root be $x_1 = \dfrac{a+b}{2}$ *i.e.* the point of bisection of the interval (a, b). Now, if we evaluate $f(x_1)$, there are three possibilities :

(*i*) $f(x_1) = 0$, in which case x_1 is the root.

(ii) $f(x_1) < 0$, in which case the root lies in the interval (x_1, b)

(iii) $f(x_1) > 0$, in which case the root lies in the interval (a, x_1)

Presuming there is just one root, if case (i) occurs, the process is terminated. If either case (ii) or case (iii) occurs, the process of bisection of the interval containing the root can be repeated until the root is obtained to the desired accuracy. The bisection method is shown graphically in fig. 2.5 in which the successive points of bisection are denoted by x_1, x_2 and x_3.

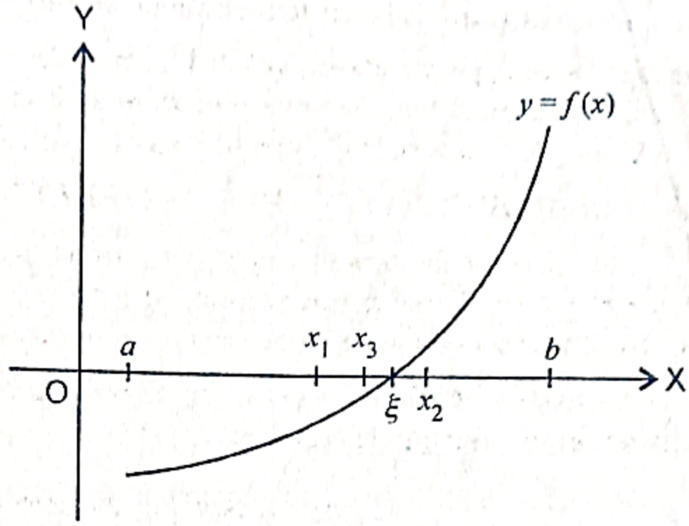


Fig. 2.5

<u>**Number of iterations required to achieve an accuracy ε**</u>

In bisection method, the length of interval at each step (or iteration) is reduced to half of its length in previous iteration. So at the end of nth step, the length of new interval will be $\frac{|b-a|}{2^n}$.

If it is required to achieve an accuracy ε *i.e.* the permissible error in the root is ε then the approximate number of iterations required may be determined from the relation

$$\frac{|b-a|}{2^n} \leq \varepsilon$$

or $$\frac{|b-a|}{\varepsilon} \leq 2^n$$

or $$\log_e\left(\frac{|b-a|}{\varepsilon}\right) \leq \log_e 2^n$$

or $$\log_e\left(\frac{|b-a|}{\varepsilon}\right) \leq n \log_e 2$$

or $$n \geq \frac{\log_e\left(\frac{|b-a|}{\varepsilon}\right)}{\log_e 2} \qquad ...(2.4)$$

From relation (2.4), we can have minimum number of iterations required to achieve an accuracy ε.

The following table (table 2.3) gives the minimum number of iterations required to achieve an accuracy ε with length of initial interval unity (using inequality (2.4)).

ε	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
n	7	10	14	17	20

Table 2.3

Advantages of Bisections Method

(*i*) The bisection method is simple to use.

(*ii*) Convergence is assured in the bisection method for any $f(x)$ which is continuous in the interval containing the root.

(*iii*) Computational effort is normally measured in terms of number of evaluations of $f(x)$ as this is the most time consuming part of the solution. In bisection method, each iteration involves only one function evaluation so computational effort is less as compared to other methods.

(*iv*) This method is suitable for implementation on a computer.

Drawbacks of Bisection Method

(*i*) The bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the root (see table 2.3). In other words, we can say that the bisection method is rather slow.

(*ii*) Another drawback of the bisection method is that it applies only for roots of f about which $f(x)$ changes sign. In particular, double roots (refer sec. 2.10) can be overlooked.

Order of Convergence

In bisection method, a new interval is obtained in each iteration whose length is equal to the half of the length of the interval obtained in previous iteration. Since we take the mid point of intervals as the successive approximations so the upper bound of the error in any approximation is the half of the error in previous approximation

i.e. $$|e_{k+1}| \leq \frac{1}{2}|e_k|, \quad k = 0, 1, 2, 3, \ldots$$

So the order of convergence of bisection method is one *i.e.* bisection method converges linearly with rate 1/2 (as A = 1/2).

In other words, we can say that the average rate of decrease in the error in successive approximations is 1/2 *i.e.* at each step, the error (on an average) decreases by a factor of 1/2.

CHECKPOINTS

1. How does a transcendental equation differ from an algebraic equation ?
2. Why are numerical methods used in solving nonlinear equations ?
3. What do you mean by iterative methods ?
4. What are the characteristics of iterative methods ? (G.N.D.U. B.C.A. April 2006)
5. What is the technique used to find the initial approximation for solution of polynomial equation ? Take an illustrative example to explain the concept. (G.N.D.U. B.C.A. Sept. 2008)
6. Discuss the criteria to terminate an iterative process. (G.N.D.U. B.Sc. C.Sc. April 2005)
7. When may the bi-section method be used to find a root of the equation $f(x) = 0$?
8. What are the three possible choices after a bisection value is calculated ?
9. What is the maximum error after an iterations of the bisection method ?
10. Write short note on bi-section method. (Pbi. U. B.C.A. Sept. 2006)

ILLUSTRATIVE EXAMPLES

Example 1. Find a real root of the equation $x^4 - x - 10 = 0$ using bisection method correct to three decimal places.

Sol. The given equation is $x^4 - x - 10 = 0$.

Let $\quad f(x) = x^4 - x - 10$

Now $\quad f(1.5) = (1.5)^4 - 1.5 - 10 = -6.4375 < 0$

and $\quad f(2) = 2^4 - 2 - 10 = 4 > 0$

So a real root of given equation lies in the interval (1.5, 2)

Iteration 1. Taking $a = 1.5$ and $b = 2$.

The first approximation to the root is given by $x_1 = \dfrac{a+b}{2} = \dfrac{1.5+2}{2} = 1.75$

Now $\quad f(1.75) = (1.75)^4 - 1.75 - 10 = -2.3711 < 0$

and $\quad f(2) = 4 > 0$

So a real root of given equation lies in the interval (1.75, 2)

Iteration 2. Taking $a = 1.75$ and $b = 2$.

The second approximation to the root is given by $x_2 = \dfrac{a+b}{2} = \dfrac{1.75+2}{2} = 1.875$

3.622

Now $\quad f(1.875) = (1.875)^4 - 1.875 - 10 = 0.4846 > 0$

and $\quad f(1.75) = -2.3711 < 0$

3.622

So a real root of given equation lies in the interval (1.75, 1.875)

Iteration 3. Taking $a = 1.75$ and $b = 1.875$

The third approximation to the root is given by $x_3 = \dfrac{a+b}{2} = \dfrac{1.75+1.875}{2} = 1.8125$

Now $\quad f(1.8125) = (1.8125)^4 - 1.8125 - 10 = -1.10202 < 0$

and $\quad f(1.875) = 0.4846 > 0$

So a real root of given equation lies in the interval (1.8125, 1.875)

Iteration 4. Taking $a = 1.8125$ and $b = 1.875$

The fourth approximation to the root is given by $x_4 = \dfrac{a+b}{2} = \dfrac{1.8125+1.875}{2} = 1.8438$

Now $\quad f(1.8438) = (1.8438)^4 - 1.8438 - 10 = -0.2865 < 0$

and $\quad f(1.875) = 0.4846 > 0$

So a real root of given equation lies in the interval (1.8438, 1.875)

Iteration 5. Taking $a = 1.8438$ and $b = 1.875$

The fifth approximation to the root is given by $x_5 = \dfrac{a+b}{2} = \dfrac{1.8438+1.875}{2} = 1.8594$

Now $\quad f(1.8594) = (1.8594)^4 - 1.8594 - 10 = 0.094 > 0$

and $\quad f(1.8438) = -0.2865 < 0$

So a real root of given equation lies in the interval (1.8438, 1.8594)

Iteration 6. Taking $a = 1.8438$ and $b = 1.8594$

The sixth approximation to the root is given by $x_6 = \dfrac{a+b}{2} = \dfrac{1.8438+1.8594}{2} = 1.8516$

Now $\quad f(1.8516) = (1.8516)^4 - 1.8516 - 10 = -0.0975 < 0$

and $\quad f(1.8594) = 0.094 > 0$

So a real root of given equation lies in the interval (1.8516, 1.8594)

Iteration 7. Taking $a = 1.8516$ and $b = 1.8594$

The seventh approximation to the root is given by $x_7 = \dfrac{a+b}{2} = \dfrac{1.8516+1.8594}{2} = 1.8555$

Now $\quad f(1.8555) = (1.8555)^4 - 1.8555 - 10 = -0.002 < 0$

and $\quad f(1.8594) = 0.094 > 0$

So a real root of given equation lies in the interval (1.8555, 1.8594)

Iteration 8. Taking $a = 1.8555$ and $b = 1.8594$.

The eight approximation to the root is given by $x_8 = \dfrac{a+b}{2} = \dfrac{1.8555+1.8594}{2} = 1.8575$

Now $\quad f(1.8575) = (1.8575)^4 - 1.8575 - 10 = 0.0471 > 0$

and $\quad f(1.8555) = -0.002 < 0$

So a real root of the given equation lies in the interval (1.8555, 1.8575).

Iteration 9. Taking $a = 1.8555$ and $b = 1.8575$.

The ninth approximation to the root is given by $x_9 = \dfrac{1.8555+1.8575}{2} = 1.8565$

Now $\quad f(1.8565) = (1.8565)^4 - 1.8565 - 10 = 0.0225 > 0$

and $\quad f(1.8555) = -0.002 < 0$

So a real root of the given equation lies in the interval (1.8555, 1.8565).

Iteration 10. Taking $a = 1.8555$ and $b = 1.8565$

The tenth approximation to the root is given by $x_{10} = \dfrac{a+b}{2} = \dfrac{1.8555+1.8565}{2} = 1.856$

From 9th and 10th iteration, we see that there is no change in the successive approximations to the root upto first three decimal places.

So a real root of the given equation is given by $x = 1.856$ (correct to 3 decimal places).

Example 2. Find a real root of the equation $x^3 - x - 1 = 0$ by using the bisection method correct to three decimal places.

(G.N.D.U. B.Sc. I.T. April 2007, B.C.A. April 2009; P.U. B.C.A. April 2008)

Sol. The given equation is $x^3 - x - 1 = 0$.

Let $f(x) = x^3 - x - 1$.

Now $f(1.25) = (1.25)^3 - 1.25 - 1 = -0.2969 < 0$

and $f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875 > 0$

So a real root of given equation lies in the interval (1.25, 1.5)

Iteration 1. Taking $a = 1.25$ and $b = 1.5$

The first approximation to the root is given by $x_1 = \frac{a+b}{2} = \frac{1.25+1.5}{2} = 1.375$

Now $f(1.375) = (1.375)^3 - 1.375 - 1 = 0.2246 > 0$

and $f(1.25) = -0.2969 < 0$

So a real root of the given equation lies in the interval (1.25, 1.375).

Iteration 2. Taking $a = 1.25$ and $b = 1.375$

The second approximation to the root is given by $x_2 = \frac{a+b}{2} = \frac{1.25+1.375}{2} = 1.3125$

Now $f(1.3125) = (1.3125)^3 - 1.3125 - 1 = -0.0515 < 0$

and $f(1.375) = 0.2246 > 0$

So a real root of the given equation lies in the interval (1.3125, 1.375)

Iteration 3. Taking $a = 1.3125$ and $b = 1.375$.

The third approximation to the root is given by $x_3 = \frac{a+b}{2} = \frac{1.3125+1.375}{2} = 1.3437$

Now $f(1.3437) = (1.3437)^3 - 1.3437 - 1 = 0.0824 > 0$

and $f(1.3125) = -0.0516 < 0$.

So a real root of the given equation lies in the interval (1.3125, 1.3437)

Similarly by performing subsequent iterations, the successive approximations to the root are given by

$x_4 = 1.3281,\ x_5 = 1.3203,\ x_6 = 1.3242,\ x_7 = 1.3262,\ x_8 = 1.3252,\ x_9 = 1.3247, x_{10} = 1.3249$

From 9th and 10th iteration, we see that there is no change in the successive approximations to the root upto first three decimal places.

So a real root of given equation is given by $x = 1.324$ (correct to first three decimal places).

Example 3. Find a real root of the equation $x^3 - x - 11 = 0$ between 2 and 3 correct to four decimal places using the bisection method. (G.N.D.U. B.C.A. April 2002)

Sol. The given equation is $x^3 - x - 11 = 0$

Let $f(x) = x^3 - x - 11$

Given a real root of the given equation lies in the interval (2, 3).

Also $f(2) = 2^3 - 2 - 11 = -5 < 0$

and $f(3) = 3^3 - 3 - 11 = 13 > 0$

8 − 2 − 11

Iteration 1. Taking $a = 2$ and $b = 3$

The first approximation to the root is given by $x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

Now $f(2.5) = (2.5)^3 - 2.5 - 11 = 2.125 > 0$

and $f(2) = -5 < 0$

So a real root of the given equation lies in the interval (2, 2.5).

Iteration 2. Taking $a = 2$ and $b = 2.5$

The second approximation to the root is given by $x_2 = \frac{a+b}{2} = \frac{2+2.5}{2} = 2.25$

Now $f(2.25) = (2.25)^3 - 2.25 - 11 = -1.8594 < 0$

and $f(2.5) = 2.125 > 0$

So a real root of given equation lies in the interval (2.25, 2.5)

Iteration 3. Taking $a = 2.25$ and $b = 2.5$

The third approximation to the root is given by $x_3 = \frac{a+b}{2} = \frac{2.25+2.5}{2} = 2.375$

Now $f(2.375) = (2.375)^3 - 2.375 - 11 = 0.0215 > 0$

and $f(2.25) = -1.8594 < 0$

So a real root of given equation lies in the interval (2.25, 2.375)

Similarly by performing subsequent iterations, the successive approximations are given by

$x_4 = 2.3125,\quad x_5 = 2.34375,\quad x_6 = 2.35938, x_7 = 2.36719,\quad x_8 = 2.37110,\quad x_9 = 2.37305,$
$x_{10} = 2.37402,\quad x_{11} = 2.37354,\quad x_{12} = 2.37378, x_{13} = 2.37366, x_{14} = 2.37360$

From 13th and 14th iteration, we see that there is no change in first four decimal places of the successive approximations to the root.

So a real root of the given equation is given by $x = 2.3736$ (correct to four decimal places).

Example 4. Find a root of the equation $x - \cos x = 0$ by using the bisection method correct to three decimal places.

Sol. The given equation is $x - \cos x = 0$.

Let $f(x) = x - \cos x$.

Now $f(0.5) = 0.5 - \cos 0.5 = -0.3776 < 0$

and $f(1) = 1 - \cos 1 = 0.4597 > 0$

So a real root of given equation lies in the interval (0.5, 1)

Iteration 1. Taking $a = 0.5$ and $b = 1$.

The first approximation to the root is given by $x_1 = \frac{a+b}{2} = \frac{0.5+1}{2} = 0.75$

Now $f(0.75) = 0.75 - \cos 0.75 = 0.01831 > 0$

and $f(0.5) = -0.3776 < 0$

So a real root of given equation lies in the interval (0.5, 0.75)

Iteration 2. Taking $a = 0.5$ and $b = 0.75$

The second approximation to the root is given by $x_2 = \frac{a+b}{2} = \frac{0.5+0.75}{2} = 0.625$

Now $f(0.625) = 0.625 - \cos 0.625 = -0.1859 < 0$

and $f(0.75) = 0.01831 > 0$

So a real root of given equation lies in the interval (0.625, 0.75)

Iteration 3. Taking $a = 0.625$ and $b = 0.75$

The third approximation to the root is given by $x_3 = \frac{a+b}{2} = \frac{0.625+0.75}{2} = 0.6875$

Now $f(0.6875) = 0.6875 - \cos 0.6875 = -0.08533 < 0$

and $f(0.75) = 0.01831 > 0$

So a real root of given equation lies in the interval (0.6875, 0.75)

Similarly by performing subsequent iterations, the successive approximations to the root are given by

$x_4 = 0.7187,\ x_5 = 0.7343,\ x_6 = 0.7421,\ x_7 = 0.7382,\ x_8 = 0.7402,\ x_9 = 0.7392,\ x_{10} = 0.7388,\ x_{11} = 0.7385$

From 10th and 11th iteration, it is clear that there is no change in the successive approximations to the root upto first three decimal places.

So a real root of given equation is given by $x = 0.738$ (correct to three decimal places).

EXERCISE 2.1

1. Find a root of the equation $x^3 - 2x - 5 = 0$ in the interval [2, 3] correct to three decimal places by using bisection method. (G.N.D.U. B.Sc. C.Sc. April 2002, B.Sc. I.T. April 2009)
2. Find a real root of the equation $2x^3 + x^2 - 20x + 12 = 0$ by using the bisection method correct to three decimal places.
3. Find a root of the equation $x^3 - 5x - 3 = 0$ using bisection method correct to first three decimal places. (G.N.D.U. B.Sc. C.Sc. April 2006)
4. Find a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places using the bisection method. (G.N.D.U. B.Sc. C.Sc. April 2008)
5. Find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places using the bisection method.
6. Find a real root of equation $x^4 + 2x^2 - 16x + 5 = 0$ correct to three decimal places using the bisection method. (G.N.D.U. B.Sc. C.Sc. Sept. 2006)

ANSWERS

1. 2.094	2. 2.522	3. 2.490
4. 1.834	5. 2.706	6. 0.326

2.7 THE METHOD OF INTERPOLATION (OR THE METHOD OF FALSE POSITION)

The method of false position is the oldest method and dates back to the ancient Egyptians. This method is also known as the *method of linear interpolation* or the *Regula-falsi method*. It remains an effective alternative to the bisection method for solving nonlinear equation of the form $f(x) = 0$ for a real root in the interval (a, b), given that f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs.

Procedure (G.N.D.U. B.Sc. C.Sc. April 2004)

Without loss of generality, suppose that a continuous function f is negative at a and positive at b. So there exists atleast one real root between a and b. (We may also take f as positive at a and negative at b).

Now we connect the two points $(a, f(a))$ and $(b, f(b))$ on the sketch of the curve $y = f(x)$ by a straight line segment (see fig. 2.6).

The equation of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

The point of intersection of this chord with the x-axis is given by putting $y = 0$ in the above equation.

i.e. $$0 - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or $$x - a = -f(a)\frac{(b - a)}{f(b) - f(a)}$$

or $$x = a - \frac{(b - a)}{f(b) - f(a)}f(a)$$

or $$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

This value of x is taken as the approximate value of the root.

∴ Hence the first approximate value of the root say x_1 is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \quad ...(2.5)$$

As in the bisection method, there are three possibilities :

(*i*) $f(x_1) = 0$, in which case x_1 is the root.

(*ii*) $f(x_1) < 0$, in which case the root lies in the interval (x_1, b)

(*iii*) $f(x_1) > 0$, in which case the root lies in the interval (a, x_1)

Now if case (*i*) occurs, the process terminates, if either case (*ii*) or case (*iii*) occurs, the same process is repeated until the root is obtained to the desired accuracy.

The method of false position is shown graphically in fig. 2.6 in which the successive points where the straight lines intersect the x-axis are denoted by x_1 and x_2.

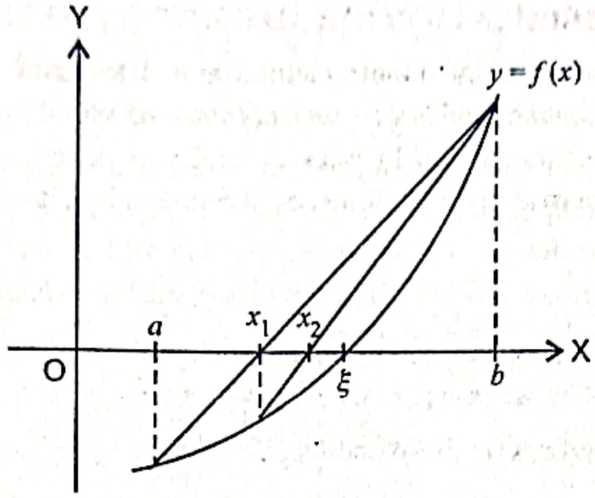


Fig. 2.6

General Formula for Regula-Falsi Method

Denoting the initial interval containing the root of the equation $f(x) = 0$ by (x_0, x_1) instead of (a, b), the next approximation to the root say x_2 is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general,
$$x_{k+1} = \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} \quad \text{...(2.6)}$$

or
$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k) \quad (k = 1, 2, 3, \ldots.) \quad \text{...(2.7)}$$

where x_{k-1} and x_k are two previous approximations to the root.

Geometrical Interpretation

Geometrically, the method of false position consists of replacing a part of curve $y = f(x)$ between the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ by means of a straight line joining these points and taking the point of intersection of this straight line with the x-axis as the next approximation x_{k+1} to the root of the equation $f(x) = 0$ assuming that root lies between x_{k-1} and x_k.

Advantages of Regula-falsi method

(*i*) This method is suitable for implementation on a computer.

(*ii*) Like bisection method, the method of false position has almost assured convergence and it may converge to the root faster than the bisection method.

(*iii*) Since $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ are known before starting the next iteration for x_{k+1}, the false position method requires one function evaluation per iteration *i.e.* $f(x_{k+1})$. So computational effort is less.

Limitations of Regula-falsi method

In Regula-falsi method, it may happen that most or all the computed approximations are on the same side of the root (refer fig. 2.6). Consequently the convergence may be slow.

Note Another consequence which results from having the calculated approximations on the same side of the root is that as iterations of this method are performed although the length of the interval containing the root becomes smaller yet it may not go to zero. Thus the length of the interval may be unsuitable for use as termination criteria. So for developing algorithm, instead one should check the value of $|f(x)|$ at successive approximations to terminate the iterative procedure.

Order of Convergence

Let us assume that ξ is a simple root of equation $f(x) = 0$.

The general iterative formula for Regula-falsi method is

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \cdot f(x_k) \qquad \text{...(2.8)}$$

If e_{k-1}, e_k and e_{k+1} are the errors in the approximations x_{k-1}, x_k and x_{k+1} respectively then

$$e_{k-1} = x_{k-1} - \xi,\ e_k = x_k - \xi \text{ and } e_{k+1} = x_{k+1} - \xi$$

i.e. $x_{k-1} = \xi + e_{k-1}, x_k = \xi + e_k$ and $x_{k+1} = \xi + e_{k+1}$

Substituting these values in formula (2.8), we have

$$\xi + e_{k+1} = \xi + e_k - \frac{(\xi + e_k) - (\xi + e_{k-1})}{f(\xi + e_k) - f(\xi + e_{k-1})} \cdot f(\xi + e_k)$$

$$\Rightarrow \quad e_{k+1} = e_k - \frac{(e_k - e_{k-1}) f(\xi + e_k)}{f(\xi + e_k) - f(\xi + e_{k-1})} \qquad \text{...(2.9)}$$

Expanding $f(\xi + e_k)$ and $f(\xi + e_{k-1})$ using Taylor's series about the point ξ and substituting $f(\xi) = 0$ in (2.9), we have

$$e_{k+1} = e_k - \frac{(e_k - e_{k-1})\left[e_k f'(\xi) + \frac{1}{2} e_k^2 f''(\xi) + \ldots\ldots\right]}{(e_k - e_{k-1}) f'(\xi) + \frac{1}{2}(e_k^2 - e_{k-1}^2) f''(\xi) + \ldots\ldots}$$

$$= e_k - \frac{e_k f'(\xi) + \frac{1}{2} e_k^2 f''(\xi) + \ldots\ldots}{f'(\xi) + \frac{1}{2}(e_k + e_{k-1}) f''(\xi) + \ldots\ldots}$$

$$= e_k - \frac{e_k + \frac{1}{2} e_k^2 \frac{f''(\xi)}{f'(\xi)} + \ldots\ldots}{1 + \frac{1}{2}(e_k + e_{k-1}) \frac{f''(\xi)}{f'(\xi)} + \ldots\ldots}$$

(by dividing numerator and denominator of second term in R.H.S. by $f'(\xi) \neq 0$

$$= e_k - \left[e_k + \frac{1}{2}e_k^2\frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]\left[1 + \frac{1}{2}(e_k + e_{k-1})\frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]^{-1}$$

$$= e_k - \left[e_k + \frac{1}{2}e_k^2\frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]\left[1 - \frac{1}{2}(e_k + e_{k-1})\frac{f''(\xi)}{f'(\xi)} - \ldots\ldots\right]$$

(using binomial theorem for any index)

$$= e_k - \left[e_k - \frac{1}{2}e_k(e_k + e_{k-1})\frac{f''(\xi)}{f'(\xi)} + \frac{1}{2}e_k^2\frac{f''(\xi)}{f'(\xi)} + \text{terms having higher orders of errors}\right]$$

$$= \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}e_k e_{k-1} + \text{terms having higher orders of errors.}$$

so $$e_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}e_k e_{k-1} \qquad \ldots(2.10)$$

(neglecting the terms having higher orders of errors)

The equation (2.10) in terms of errors in called *error equation.*

Assuming the straight line joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ lies above the curve $y = f(x)$ (see fig. 2.6), one of the points x_0 or x_1 is always fixed and other point varies with k. If the point x_0 is fixed then in each iteration, $f(x)$ is approximated by the straight line joining the points $(x_0, f(x_0))$ and $(x_k, f(x_k))$ $k = 1, 2, 3, \ldots$

So the error equation (2.10) becomes

$$e_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}e_0 e_k$$

or $$e_{k+1} = A\, e_k \qquad \ldots(2.11)$$

where $$A = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \cdot e_0 \text{ is asymptotic error constant.}$$

From equation (2.11) it is clear that the regular-falsi method is linear order convergent i.e. order of convergence of this method is 1.

CHECKPOINTS

1. Discuss Regula-Falsi method. (G.N.D.U. B.Sc. C.Sc. April 2004)
2. When may the method of false position be used to find a root of the equation $f(x) = 0$?
3. On what geometric construction is the method of false position based ?

ILLUSTRATIVE EXAMPLES

Example 1. Find a real root of the equation $x^3 - 4x - 9 = 0$ between 2.625 and 2.75 with error < 0.0001

(G.N.D.U. B.C.A. April 200

Sol. The given equation is $x^3 - 4x - 9 = 0$

Here, we have to find a root with error < 0.0001, so we stop the iterative procedure when the differe between the successive approximations becomes less than 0.0001

Let $\quad f(x) = x^3 - 4x - 9$

Now $\quad f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.41211$

and $\quad f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.79688$

Iteration 1. Taking $a = 2.625$ and $b = 2.75$ so that $f(a) = -1.41211$ and $f(b) = 0.79688$

The first approximation to the root is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.625(0.79688) - 2.75(-1.41211)}{0.79688 - (-1.41211)} = 2.70491$$

Now $f(2.70491) = (2.70491)^3 - 4(2.70491) - 9 = -0.02906 < 0$

and $\quad f(2.75) = 0.79688 > 0$

So a real root of the given equation lies in the interval (2.70491, 2.75)

Iteration 2. Taking $a = 2.70491$ and $b = 2.75$ so that $f(a) = -0.02906$ and $f(b) = 0.79688$

The second approximation to the root is given by

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.70491(0.79688) - 2.75(-0.02906)}{0.79688 - (-0.02906)} = 2.70650$$

Now $\quad f(2.70650) = (2.70650)^3 - 4(2.70650) - 9 = -0.0005 < 0$

and $\quad f(2.75) = 0.79688 > 0$

So a real root of the given equation lies in the interval (2.70650, 2.75)

Iteration 3. Taking $a = 2.70650$ and $b = 2.75$ so that $f(a) = -0.0005$ and $f(b) = 0.79688$

The third approximation to the root is given by

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.70650(0.79688) - 2.75(-0.0005)}{0.79688 - (-0.0005)} = 2.70654$$

Now $\quad |x_2 - x_3| = |2.70650 - 2.70654| = 0.00004 \ (< 0.0001)$

So a real root of the given equation with the permissible error < 0.0001 is given by $x = 2.70654$.

Example 2. Given one root of equation $x^6 - x^4 - x^3 - 1 = 0$ lies between 1.4 and 1.5. Find it correct four decimal places.

(G.N.D.U. B.Sc. I.T. April 2008, B.C.A. April 20

Sol. The given equation is $x^6 - x^4 - x^3 - 1 = 0$.

Let $\quad f(x) = x^6 - x^4 - x^3 - 1$

Now $\quad f(1.4) = (1.4)^6 - (1.4)^4 - (1.4)^3 - 1 = -0.05606$

and $\quad f(1.5) = (1.5)^6 - (1.5)^4 - (1.5)^3 - 1 = 1.95312$

Iteration 1. Taking $a = 1.4$ and $b = 1.5$ so that $f(a) = -0.05606$ and $f(b) = 1.95312$

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.4\,(1.95312) - 1.5\,(-0.05606)}{1.95312 - (-0.05606)} = 1.40279$$

Now $\quad f(1.40279) = (1.40279)^6 - (1.40279)^4 - (1.40279)^3 - 1 = -0.01274 < 0$.

and $\quad f(1.5) = 1.95312 > 0$

So a real root of given equation lies in the interval (1.40279, 1.5)

Iteration 2. Taking $a = 1.40279$ and $b = 1.5$ so that $f(a) = -0.01274$ and $f(b) = 1.95312$

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.40279\,(1.95312) - 1.5\,(-0.01274)}{1.95312 - (-0.01274)} = 1.40342$$

Now $\quad f(1.40342) = (1.40342)^6 - (1.40342)^4 - (1.40342)^3 - 1 = -0.00286 < 0$

and $\quad f(1.5) = 1.95312 > 0$

So a real root of given equation lies in the interval (1.40342, 1.5)

Iteration 3. Taking $a = 1.40342$ and $b = 1.5$ so that $f(a) = -0.00286$ and $f(b) = 1.95312$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.40342\,(1.95312) - 1.5\,(-0.00286)}{1.95312 - (-0.00286)} = 1.40356$$

Now $\quad f(1.40356) = (1.40356)^6 - (1.40356)^4 - (1.40356)^3 - 1 = -0.00066 < 0$

and $\quad f(1.5) = 1.95312 > 0$

So a real root of given equation lies in the interval (1.40356, 1.5)

Iteration 4. Taking $a = 1.40356$ and $b = 1.5$ so that $f(a) = -0.00066$ and $f(b) = 1.95312$

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.40356\,(1.95312) - 1.5\,(-0.00066)}{1.95312 - (-0.00066)} = 1.40359$$

From 3rd and 4th iteration, we see that there is no change in the successive approximations to the root upto first four decimal places.

So a real root of given equation is given by $x = 1.4035$ (correct to four decimal places)

Example 3. Find the fourth root of 32 correct to three decimal places using the method of false position.

Sol. Let $\quad x = (32)^{1/4}$

$$\Rightarrow \quad x^4 = 32 \quad \Rightarrow \quad x^4 - 32 = 0 \qquad \text{...(1)}$$

Let $\quad f(x) = x^4 - 32$

Now $f(2.25) = (2.25)^4 - 32 = -6.3 < 0$

and $f(2.5) = (2.5)^4 - 32 = 7.0625 > 0$

So a real root of equation (1) lies in the interval (2.25, 2.5)

Iteration 1. Taking $a = 2.25$ and $b = 2.5$ so that $f(a) = -6.3$ and $f(b) = 7.0625$

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.25\,(7.0625) - 2.5\,(-6.3)}{7.0625 - (-6.3)} = 2.3574$$

Now $f(2.3574) = (2.3574)^4 - 32 = -1.1160 < 0$

and $f(2.5) = 7.0625 > 0$

So a real root of the equation (1) lies in the interval (2.3574, 2.5)

Iteration 2. Taking $a = 2.3574$ and $b = 2.5$ so that $f(a) = -1.1160$ and $f(b) = 7.0625$

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.3574\,(7.0625) - 2.5\,(-1.1160)}{7.0625 - (-1.1160)} = 2.3768$$

Now $f(2.3768) = (2.3768)^4 - 32 = -0.0868 < 0$

and $f(2.5) = 7.0625 > 0$

So a real root of the equation (1) lies in the interval (2.3768, 2.5)

Iteration 3. Taking $a = 2.3768$ and $b = 2.5$ so that $f(a) = -0.0868$ and $f(b) = 7.0625$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.3768\,(7.0625) - 2.5\,(-0.0868)}{7.0625 - (-0.0868)} = 2.3783$$

Now $f(2.3783) = (2.3783)^4 - 32 = -0.0061 < 0$

and $f(2.5) = 7.0625 > 0$

So a real root of the equation (1) lies in the interval (2.3783, 2.5)

Iteration 4. Taking $a = 2.3783$ and $b = 2.5$ so that $f(a) = -0.0061$ and $f(b) = 7.0625$

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.3783\,(7.0625) - 2.5\,(-0.0061)}{7.0625 - (-0.0061)} = 2.3784$$

From 3rd and 4th iteration, we see that there is no change in the successive approximations to the root upto first three decimal places so the root of equation (1) is given by $x = 2.378$.

Hence the fourth root of $32 = 2.378$.

Example 4. Find a real root of the equation $e^x - x^3 = 0$ correct to four decimal places using the method of false position.

(G.N.D.U. B.Sc. I.T. April 2005)

Sol. The given equation is $e^x - x^3 = 0$.

Let $f(x) = e^x - x^3$

Now $f(1.75) = e^{1.75} - (1.75)^3 = 0.39523 > 0$

and $f(2) = e^2 - 2^3 = -0.61094 < 0$

So a real root of given equation lies in the interval (1.75, 2).

Iteration 1. Taking $a = 1.75$ and $b = 2$ so that $f(a) = 0.39523$ and $f(b) = -0.61094$

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.75\,(-0.61094) - 2\,(0.39523)}{-0.61094 - 0.39523} = 1.84820$$

Now $f(1.84820) = e^{1.84820} - (1.84820)^3 = 0.03522 > 0$

and $f(2) = -0.61094 < 0$

So a real root of the equation lies in the interval (1.84820, 2).

Iteration 2. Taking $a = 1.84820$ and $b = 2$ so that $f(a) = 0.03522$ and $f(b) = -0.61094$

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.84820\,(-0.61094) - 2\,(0.03522)}{-0.61094 - 0.03522} = 1.85647$$

Now $f(1.85647) = e^{1.85647} - (1.85647)^3 = 0.00281 > 0$

and $f(2) = -0.61094 < 0$

So a real root of the equation lies in the interval (1.85647, 2).

Iteration 3. Taking $a = 1.85647$ and $b = 2$ so that $f(a) = 0.00281$ and $f(b) = -0.61094$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.85647\,(-0.61094) - 2\,(0.00281)}{-0.61094 - 0.00281} = 1.85713$$

Now $f(1.85713) = e^{1.85713} - (1.85713)^3 = 0.00021 > 0$

and $f(2) = -0.61094 < 0$

So a real root of the equation lies in the interval (1.85713, 2).

Iteration 4. Taking $a = 1.85713$ and $b = 2$ so that $f(a) = 0.00021$ and $f(b) = -0.61094$

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.85713\,(-0.61094) - 2\,(0.00021)}{-0.61094 - 0.00021} = 1.85718$$

From 3rd and 4th iteration, we see that there is no change in the successive approximations to the root upto first four decimal places.

So a real root of the given equation is given by $x = 1.8571$ (correct to four decimal places)

Example 5. Find a real root of the equation $x \log_{10} x = 1.2$ by the method of false position correct to four decimal places.

Sol. Rewriting the given equation as $x \log_{10} x - 1.2 = 0$

Let $\quad f(x) = x \log_{10} x - 1.2$

Now $\quad f(2.7) = 2.7 \log_{10} 2.7 - 1.2 = -0.035318 < 0$

and $\quad f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 > 0$

So a real root of given equation lies in the interval (2.7, 3)

Iteration 1. Taking $a = 2.7$ and $b = 3$ so that $f(a) = -0.035318$ and $f(b) = 0.23136$

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.7\,(0.23136) - 3\,(-0.035318)}{0.23136 - (-0.035318)} = 2.73973$$

Now $\quad f(2.73973) = 2.73973 \log_{10} 2.73973 - 1.2 = -0.00080 < 0$

and $\quad f(3) = 0.23136 > 0$

So a real root of the equation lies in the interval (2.73973, 3)

Iteration 2. Taking $a = 2.73973$ and $b = 3$ so that $f(a) = -0.00080$ and $f(b) = 0.23136$

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.73973\,(0.23136) - 3\,(-0.00080)}{0.23136 - (-0.00080)} = 2.74063$$

Now $\quad f(2.74063) = 2.74063 \log_{10} 2.74063 - 1.2 = -0.00001 < 0$

and $\quad f(3) = 0.23136 > 0$

So a real root of the equation lies in the interval (2.74063, 3)

Iteration 3. Taking $a = 2.74063$ and $b = 3$ so that $f(a) = -0.00001$ and $f(b) = 0.23136$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.74063\,(0.23136) - 3\,(-0.00001)}{0.23136 - (-0.00001)} = 2.74064$$

From second and third iteration, it is clear that there is no change in the successive approximations to the root upto first four decimal places.

So a real root of the given equation is given by $x = 2.7406$ (correct to 4 decimal places)

Example 6. Use the method of false position to obtain a root of $x\,e^x = \cos x$, correct to three decimal places.

(Pbi. U. B.C.A. April 2005)

Sol. Let $\quad f(x) = \cos x - x\,e^x,$

$f(0) = 1$ and $f(1) = -2 \cdot 17798$

Since $\quad f(0)\,f(1) < 0$

$\therefore \quad f(x) = 0$ has atleast one real root between 0 and 1

1st iteration. Take $x_0 = 0$, $f(x_0) = 1$

$$x_1 = 1, \ f(x_1) = -2.17798$$

By Method of False position,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0-(1)(1)}{-2.17798-1} = 0.31467$$

$$f(0.31467) = 0.51987$$

Since $f(0.31467) f(1) < 0$

$\therefore$ the root of $f(x) = 0$ lies between 0.31467 and 1

2nd iteration. Take $x_0 = 0.31467$, $f(x_0) = 0.51987$

$$x_1 = 1, \ f(1) = -2.17798$$

$$x_2 = \frac{(0.31467)(-2.17798) - (1)(0.51987)}{-2.17798 - 0.51987} = 0.44673$$

$$f(0.44673) = 0.20356$$

Since $f(0.44673)\ f(1) < 0$

$\therefore$ the root lies between 0.44673 and 1

3rd iteration. Take $x_0 = 0.44673$, $f(x_0) = 0.20356$

$$x_1 = 1, \ f(x_1) = -2.17798$$

$$x_2 = \frac{(0.44673)(-2.17798) - (1)(0.20356)}{-2.17798 - 0.20356} = 0.49402$$

$$f(0.49402) = 0.07083$$

since $f(0.49402)\ f(1) < 0$

$\therefore$ the root lies between 0.49402 and 1

4th iteration. Take $x_0 = 0.49402$, $f(x_0) = 0.07083$

$$x_1 = 1, \quad f(x_1) = -2.17798$$

$$x_2 = \frac{(0.49402)(-2.17798) - (1)(0.07083)}{-2.17798 - 0.07083} = 0.50995$$

$$f(0.50995) = 0.02364$$

since $f(0.50995)\ f(1) < 0$

$\therefore$ root lies between 0.50995 and 1

5th iteration. Take $x_0 = 0.50995$, $f(x_0) = 0.02364$

$$x_1 = 1, \ f(x_1) = -2.17798$$

$$x_2 = \frac{(0.50995)(-2.17798) - (1)(0.02364)}{-2.17798 - 0.02364} = 0.5152$$

$$f(0.5152) = 0.007809$$

Since $f(0.5152)\, f(1) < 0$

$\therefore$ root lies between 0.5152 and 1

6th iteration. Take $x_0 = 0.5152$, $f(x_0) = 0.007809$

$x_1 = 1$, $f(x_1) = -2.17798$

$$x_2 = \frac{(0.5152)(-2.17798)-(1)(0.007809)}{-2.17798-0.007809} = 0.51692$$

$\therefore$ $f(0.51692) = 0.002592$

Since $f(0.51692) f(1) < 0$

$\therefore$ the root lies between 0.51692 and 1

7th iteration. Take $x_0 = 0.51692$, $f(x_0) = 0.002592$

$x_1 = 1, \quad f(x_1) = -2.17798$

$$x_2 = \frac{(0.51692)(-2.17798)-(1)(0.002592)}{-2.17798-0.002592} = 0.51749$$

$\therefore$ the required root up to three decimal places is 0.517.

Example 7. Use the method of False position to obtain a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places. (Pbi. U. B.C.A. Sept. 2006)

Sol. Let $f(x) = x^3 - 2x - 5 = 0$

$f(2) = 8 - 4 - 5 = -1$ and $f(3) = 27 - 6 - 5 = 16$

Since $f(2)\, f(3) < 0$,

$\therefore$ $f(x) = 0$ has atleast one real root between 2 and 3.

1st iteration. Take $x_0 = 2$, $f(x_0) = -1$

$x_1 = 3$, $f(x_1) = 16$

By Method of False Position, we have,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(2)(16)-(3)(-1)}{16-(-1)} = 2 + \frac{1}{17} = 2.0588$$

$f(2.0588) = -0.3908$

Since $f(2.0588) f(3) < 0$

$\therefore$ the root lies between 2.0588 and 3

2nd iteration. Take $x_0 = 2.0588$, $f(x_0) = -0.3908$

$x_1 = 3$, $f(x_1) = 16$

$$x_2 = \frac{(2.0588)(16)-(3)(-0.3908)}{16-(-0.3908)} = 2.0813$$

$f(x_2) = f(2.0813) = -0.1468$

Since $f(2.0813)\, f(3) < 0$

$\therefore$ the root lies between 2.0813 and 3

3rd iteration. Take $x_0 = 2.0813$, $f(x_0) = -0.1468$

$x_1 = 3$, $f(x_1) = 16$

By Method of False Position, we have,

$$x_2 = \frac{(2.0813)(16)-(3)(-0.1468)}{16-(-0.1468)} = 2.0903$$

$$f(x_2) = f(2.0903) = -0.04734$$

Since $f(2.0903) f(3) < 0$

$\therefore$ the root lies between 2.0903 and 3

4th iteration. Take $x_0 = 2.0903$, $f(x_0) = -0.04734$

$x_1 = 3$, $f(x_1) = 16$

$$x_2 = \frac{(2.0903)(16)-(3)(-0.04734)}{16-(-0.04734)} = 2.0928$$

Hence the root is correct to three decimal places is 2.093.

EXERCISE 2.2

1. Find a real root of the equation $x^3 + x - 1 = 0$ correct to three decimal places by using false position method.
2. Find a root of the equation $x^3 + x^2 - 3x - 3 = 0$ correct to three decimal places using false-position method. (G.N.D.U. B.Sc. C.Sc. April 2005)
3. Given that the equation $x^3 - 2x - 5 = 0$ has a root between 1.75 and 2.5. Find the root correct to four significant digits. (G.N.D.U. B.C.A. April 2007)

ANSWERS

1. 0.682 2. 1.732 3. 2.094

2.8 THE NEWTON-RAPHSON ITERATIVE METHOD

The Newton-Raphson method is the process for determination of a real root of an equation $f(x) = 0$, given just one point close to the desired root.

(G.N.D.U. B.Sc. C.Sc. Sept. 2007)

Procedure

Let x_0 denote the known approximate value of the root of the nonlinear equation $f(x) = 0$ and let h be the difference between the exact value ξ and the approximate value x_0 of the root ;

i.e. $h = \xi - x_0$ or $\xi = x_0 + h$.

Since ξ is the exact value of the root of the equation $f(x) = 0$

$$\therefore \quad f(\xi) = 0 \quad \text{or} \quad f(x_0 + h) = 0 \qquad \text{...(2.12)}$$

By using second degree terminated Taylor expansion of $f(x_0 + h)$ about x_0, we have

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0+\theta h), \; 0<\theta<1 \qquad \text{...(2.13)}$$

Using equation (2.12) and (2.13) and ignoring the remainder terms,

$$f(x_0)+h\,f'(x_0) \approx 0$$

or $$h \approx -\frac{f(x_0)}{f'(x_0)}$$

and consequently, $x_1 = x_0 + h$ *i.e.* $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ should be a better approximation to the root than x_0.

Similarly starting with the approximation x_1, a better approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

So by repetition of the above procedure, we can obtain a sequence of approximations $x_0, x_1, x_2, \ldots$ such that each approximation is a better estimate of the root ξ than the previous one.

In general, starting with the approximation x_k, the next approximation x_{k+1} is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\;;\; k = 0, 1, 2, 3, \ldots.. \qquad \text{...(2.14)}$$

The formula (2.14) is called *Newton-Raphson formula* or *Newton's iterative formula.*

The procedure developed above is called the *Newton-Raphson iterative method* or *Newton's method* or *Newton's method of tangents*.

Geometrical Interpretation

Let x_0 be the initial approximation to the root ξ of the equation $f(x) = 0$. Let it corresponds to the point P_0 on the curve $y = f(x)$ (see fig. 2.7). Suppose we draw a tangent from point P_0. This tangent will intersect the x-axis at some point. Let the value of x *i.e.* abscissa corresponding to this point be x_1 which corresponds to the point P_1 on the curve. Now we draw a tangent to the curve at point P_1 which will intersect the x-axis at some point. Let the value of x *i.e.* abscissa corresponding to this point be x_2.

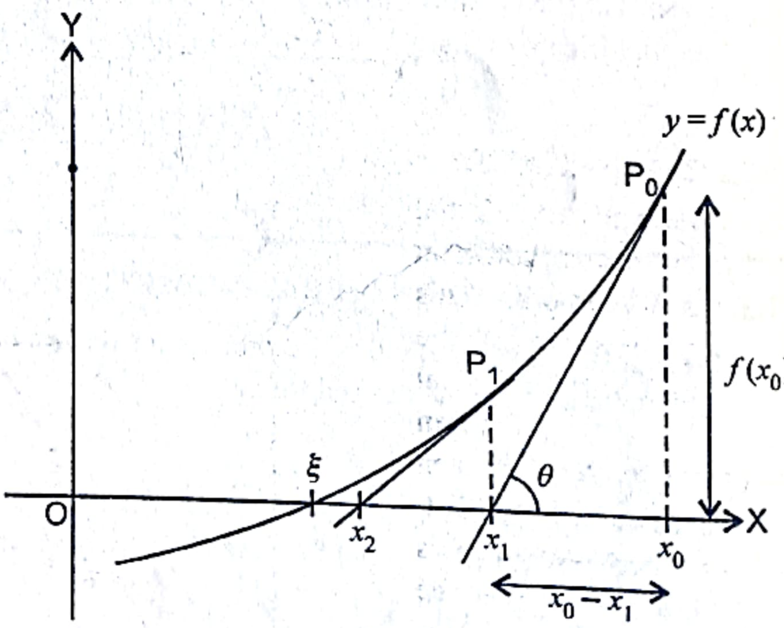


Fig. 2.7

Now we repeat the same procedure as above time and again.

It is evident that the points $x_0, x_1, x_2, ...$ will approach the point ξ as a limit. But ξ represents the exact real root of the equation $f(x) = 0$. Hence the quantities $x_0, x_1, x_2, ...$ are successive approximations to the desired root. This is geometrical significance of Newton Raphson method. In other words, geometrically, the Newton-Raphson method consists of replacing a part of curve $y = f(x)$ between the point $(x_k, f(x_k))$ and x-axis by means of a tangent line at the point $(x_k, f(x_k))$ and taking the point of intersection of this tangent line with the x-axis as the next approximation x_{k+1} to the root of the equation $f(x) = 0$.

In order to derive the fundamental formula from fig. 2.7, let the tangent at point P_0 makes an angle θ with positive direction of the x-axis.

$\therefore$ slope of tangent at P_0, $\tan\theta = f'(x_0)$ ...(2.15)

Also, from fig. 2.7, $\tan\theta = \dfrac{f(x_0)}{x_0 - x_1}$ ...(2.16)

From equations (2.15) and (2.16), we have

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1} \quad \text{or} \quad x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

or

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

which gives the next approximation to the root.

In general, starting with the approximation x_k, the next approximation x_{k+1} is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, 3, ...$$

Choosing initial approximation

Proper choice of initial approximation is very important in Newton-Raphson method. This method is very sensitive to the initial approximation. If the initial approximation is chosen sufficiently close to the root then this method converges very fast. But if the initial approximation is not sufficiently close then the procedure may fall in an endless loop. The importance of selecting the initial approximation is shown in figure 2.8.

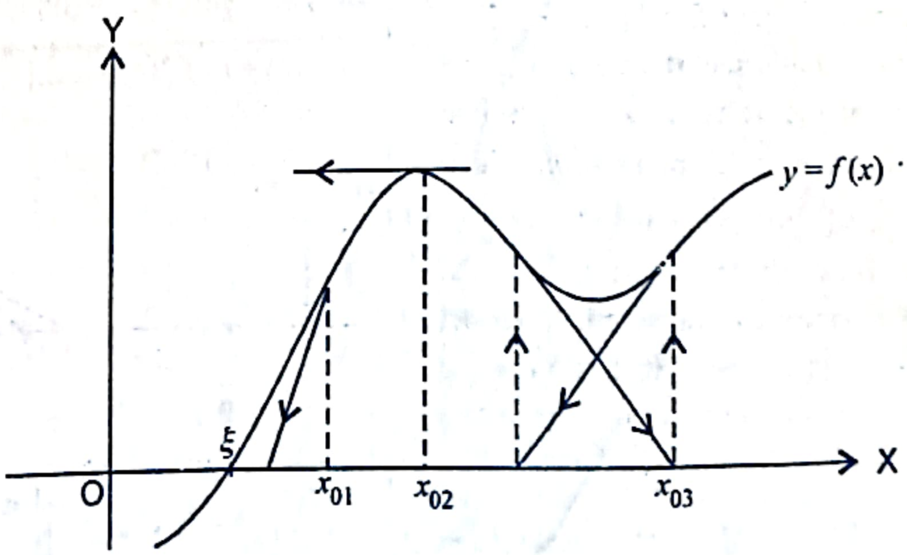


Fig. 2.8

In fig. 2.8, it is clear that for an initial approximation x_{01} close to root ξ, the method converges very fast. For the initial approximation x_{02}, slope of tangent at the corresponding point *i.e.* $f'(x_{02})$ becomes zero so the method diverge. Also for the initial approximation x_{03} which is too far from the root ξ, the method will fall into an endless loop. This illustrates the importance of selecting the initial approximation for Newton-Raphson Method.

Order of Convergence (G.N.D.U. B.Sc. C.Sc. April 200

Let us assume that ξ is a simple root of equation $f(x) = 0$.

The general iteration formula for Newton-Raphson method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad \text{...(2.1}$$

If e_k and e_{k+1} are the errors in the approximations x_k and x_{k+1} respectively then

$$e_k = x_k - \xi \quad \text{and} \quad e_{k+1} = x_{k+1} - \xi$$

or

$$x_k = \xi + e_k \quad \text{and} \quad x_{k+1} = \xi + e_{k+1}$$

Substituting these values in equation (2.17), we have

$$\xi + e_{k+1} = \xi + e_k - \frac{f(\xi + e_k)}{f'(\xi + e_k)}$$

or

$$e_{k+1} = e_k - \frac{f(\xi + e_k)}{f'(\xi + e_k)}$$

Expanding $f(\xi + e_k)$ and $f'(\xi + e_k)$ using Taylor's series about ξ and using $f(\xi) = 0$, we get

$$e_{k+1} = e_k - \frac{\left[e_k f'(\xi) + \frac{1}{2} e_k^2 f''(\xi) + \ldots\ldots\right]}{[f'(\xi) + e_k f''(\xi) + \ldots\ldots]}$$

or

$$e_{k+1} = e_k - \frac{\left[e_k + \frac{1}{2} e_k^2 \frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]}{\left[1 + e_k \frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]}$$

[by dividing numerator and denominator by $f'(\xi) \neq$

or

$$e_{k+1} = e_k - \left[e_k + \frac{1}{2} e_k^2 \frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]\left[1 + e_k \frac{f''(\xi)}{f'(\xi)} + \ldots\ldots\right]^{-1}$$

Using binomial theorem for any index, we have

$$e_{k+1} = e_k - \left[e_k + \frac{e_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + \dots\right]\left[1 - e_k\frac{f''(\xi)}{f'(\xi)} - \dots\right]$$

or $$e_{k+1} = e_k - \left[e_k - e_k^2\frac{f''(\xi)}{f'(\xi)} + \frac{e_k^2}{2}\frac{f''(\xi)}{f'(\xi)} + \text{terms containing higher powers of errors}\right]$$

or $$e_{k+1} = \frac{1}{2}e_k^2\frac{f''(\xi)}{f'(\xi)}$$ [by neglecting the terms containing higher powers of errors]

or $$e_{k+1} = C\,e_k^2 \qquad \dots(2.18)$$

where $C = \dfrac{1}{2}\dfrac{f''(\xi)}{f'(\xi)}$

From equation (2.18), it is clear that the Newton-Raphson method is *second order convergent*.

In other words, in each iteration, the number of significant digits in the approximation doubles than that in previous iteration.

Advantages of the Newton-Raphson Method

(*i*) The Newton-Raphson method is the best known procedure for finding the roots of an equation. It has been generalized in many ways for the solution of other more difficult nonlinear problems *e.g.* non-linear systems and non-linear differential and integral equations.

(*ii*) The order of convergence of this method is 2 which makes it fast as compare to other methods.

So this method is first preference in attempting to solve a nonlinear equation.

(*iii*) This method is easy for implementation on a computer.

(*iv*) It is clear from the formula (2.14) that the greater the numerical value of derivative $f'(x)$, the smaller is the correction that must be applied to get the correct value of the root. If the sketch of the curve $y = f(x)$ is nearly vertical when it crosses the x-axis, the correct value of the root can be found with great speed and a little labour.

Drawbacks of the Newton-Raphson Method

(*i*) This method is not applicable to solve the equations whose sketch is nearly horizontal when it crosses the x-axis as it would make $f'(x)$ zero. So in that case, the method fails.

(*ii*) This method is very sensitive to the choice of initial approximation. If the initial approximation is not properly selected then this method can fall in an endless loop and might even fail altogether in some cases.

(*iii*) This method requires two function evaluation per iteration *i.e.* we have to evaluate the value of $f(x)$ and $f'(x)$ at each step. So the computation per iteration is more as compared to other methods.

(*iv*) This method is difficult to apply when the derivative of $f(x)$ is not a simple expression.

Note For developing the algorithm of this procedure, one should have the check of the value of $f'(x)$. Th value of $f'(x)$ is checked not only at the initial approximation but also at each calculated approximatio This is because of the fact that if the value of $f'(x)$ becomes too small then this method may fail. So w should have a safety exit in the algorithm which would provide us to exit from the algorithm when th value of $f'(x)$ becomes less than or equal to a prescribed number.

Further we should put a limit on the maximum number of iterations permitted to achieve a specifi accuracy as we know that in some cases, this method may fall in an endless loop.

CHECKPOINT

1. Write a note on Newton-Raphson method for solving nonlinear equations. (G.N.D.U. B.Sc. C.Sc. April 200
2. What is geometrical interpretation of Newton-Raphson iterative procedure ?
3. What is the convergence criterion for the Newton-Raphson method ?
4. What major advantage has the Newton-Raphson method over some other methods ?
5. What are different parameters to be taken into account while selecting an iterative method ? (G.N.D.U. B.C.A. April 200
6. Compare Bisection method and Newton-Raphson method. (P.U. B.C.A. April 200
7. Discuss comparison of iterative methods *i.e.* Bisection method, false position method an Newton-Raphson method. (G.N.D.U. B.Sc. C.Sc. April 200

ILLUSTRATIVE EXAMPLES

Example 1. Given the equation $x^3-4x-9=0$ has a root near 2.625. Find it correct to four significa digits. (G.N.D.U. B.C.A. April 200

Sol. The given equation is $x^3-4x-9=0$

Let $\quad f(x) = x^3-4x-9$

So $\quad f'(x) = 3x^2-4$

Since it is required to find a root near 2.625 so take the initial approximation x_0 to the root as 2.625

Iteration 1. The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3-4x_0-9}{3x_0^2-4} = 2.625 - \frac{(2.625)^3-4(2.625)-9}{3(2.625)^2-4} = 2.709$$

Iteration 2. The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3-4x_1-9}{3x_1^2-4} = 2.7097 - \frac{(2.7097)^3-4(2.7097)-9}{3(2.7097)^2-4}$$

$$= 2.7065$$

Iteration 3. The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2^3 - 4x_2 - 9}{3x_2^2 - 4} = 2.7065 - \frac{(2.7065)^3 - 4(2.7065) - 9}{3(2.7065)^2 - 4}$$

$$= 2.7065$$

From 2nd and 3rd iteration, we see that there is no change in the successive approximations to the root upto first four significant digits.

So a real root of given equation near 2.625 is given by $x = 2.706$ (correct to four significant digits)

Example 2. Find a root of the equation $x^4 + x^3 - 7x^2 - x + 5 = 0$ correct to three decimal places which lies between 2 and 3, using Newton's method.

Sol. The given equation is $x^4 + x^3 - 7x^2 - x + 5 = 0$

Let $\quad f(x) = x^4 + x^3 - 7x^2 - x + 5$

So $\quad f'(x) = 4x^3 + 3x^2 - 14x - 1$

Now $\quad f(2) = (2)^4 + (2)^3 - 7(2)^2 - 2 + 5 = -1$

and $\quad f(3) = (3)^4 + (3)^3 - 7(3)^2 - 3 + 5 = 47$

Since $f(2)$ is nearer to zero than $f(3)$ so take the initial approximation x_0 to the root as 2.

Iteration 1. The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^4 + x_0^3 - 7x_0^2 - x_0 + 5}{4x_0^3 + 3x_0^2 - 14x_0 - 1}$$

$$= 2 - \frac{[(2)^4 + (2)^3 - 7(2)^2 - 2 + 5]}{4(2)^3 + 3(2)^2 - 14 \times 2 - 1} = 2.0667$$

Iteration 2. The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^4 + x_1^3 - 7x_1^2 - x_1 + 5}{4x_1^3 + 3x_1^2 - 14x_1 - 1}$$

$$= 2.0667 - \frac{[(2.0667)^4 + (2.0667)^3 - 7(2.0667)^2 - 2.0667 + 5]}{4(2.0667)^3 + 3(2.0667)^2 - 14(2.0667) - 1} = 2.0609$$

Iteration 3. The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2^4 + x_2^3 - 7x_2^2 - x_2 + 5}{4x_2^3 + 3x_2^2 - 14x_2 - 1}$$

$$= 2.0609 - \frac{[(2.0609)^4 + (2.0609)^3 - 7(2.0609)^2 - 2.0609 + 5]}{4(2.0609)^3 + 3(2.0609)^2 - 14(2.0609) - 1} = 2.0608$$

From second and third iteration, it is clear that there is no change in the successive approximations the root upto first three decimal places.

So a root of given equation is given by $x = 2.060$ (correct to three decimal places)

Example 3. Find a real root of the equation $3x - \cos x - 1 = 0$ correct to three decimal places usi Newton-Raphson method.

(Pbi. U. B.C.A. 2005, 20

Sol. The given equation as $3x - \cos x - 1 = 0$.

Let $f(x) = 3x - \cos x - 1$

So $f'(x) = 3 + \sin x$

Now $f(0.5) = 3(0.5) - \cos 0.5 - 1 = -0.3776 < 0$

and $f(1) = 3(1) - \cos 1 - 1 = 1.4597 > 0$

So a real root of given equation lies between 0.5 and 1.

Also $f(0.5)$ is nearer to zero than $f(1)$ so take the initial approximation x_0 to the root as 0.5

Iteration 1. The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{3x_0 - \cos x_0 - 1}{3 + \sin x_0} = 0.5 - \frac{3(0.5) - \cos 0.5 - 1}{3 + \sin 0.5} = 0.6085$$

Iteration 2. The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1} = 0.6085 - \frac{3(0.6085) - \cos 0.6085 - 1}{3 + \sin 0.6085}$$

$$= 0.6071$$

Iteration 3. The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{3x_2 - \cos x_2 - 1}{3 + \sin x_2} = 0.6071 - \frac{3(0.6071) - \cos 0.6071 - 1}{3 + \sin 0.6071}$$

$$= 0.6071$$

From 2nd and 3rd iteration, it is clear that there is no change in the successive approximations to root upto first three decimal places.

So a real root of given equation is given by $x = 0.607$ (correct to three decimal places)

Example 4. Find the iterative formula for finding $\sqrt{N}$ for some positive real number N using Newt Raphson method. Hence evaluate each of the following correct to four decimal places.

(*a*) $\sqrt{28}$ (*b*) $\sqrt{12}$ (Pbi. U. 2005, 2006; G.N.D.U. B.C.A. 20

(*c*) $\sqrt{32}$

Sol. Let $x = \sqrt{N}$ or $x^2 - N = 0$

Let $f(x) = x^2 - N$

So $f'(x) = 2x$

Using Newton's iterative formula, $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$, $k = 0, 1, 2, 3, \ldots\ldots$

we have,
$$x_{k+1} = x_k - \frac{x_k^2 - N}{2x_k}$$

or
$$x_{k+1} = \frac{1}{2}\left(2x_k - \frac{x_k^2}{x_k} + \frac{N}{x_k}\right)$$

or
$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{N}{x_k}\right), \quad k = 0, 1, 2, 3, \ldots \qquad \ldots(i)$$

which is required iterative formula.

(a) Taking $N = 28$ in (i), we have

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{28}{x_k}\right), \quad k = 0, 1, 2, 3, \ldots \qquad \ldots(ii)$$

Since $\sqrt{28} \approx \sqrt{25} = 5$

So take the initial approximation $x_0 = 5$.

The successive approximations using the iterative formula (ii) are calculated as follows :

Iteration 1. The first approximation is given by

$$x_1 = \frac{1}{2}\left(x_0 + \frac{28}{x_0}\right) = \frac{1}{2}\left(5 + \frac{28}{5}\right) = 5.3$$

Iteration 2. The second approximation is given by

$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1}\right) = \frac{1}{2}\left(5.3 + \frac{28}{5.3}\right) = 5.29151$$

Iteration 3. The third approximation is given by

$$x_3 = \frac{1}{2}\left(x_2 + \frac{28}{x_2}\right) = \frac{1}{2}\left(5.29151 + \frac{28}{5.29151}\right) = 5.29150$$

From 2nd and 3rd iteration, it is clear that there is no change in the successive approximations upto first four decimal places.

So $\sqrt{28} = 5.2915$ (correct to four decimal places).

(b) Taking $N = 12$ in (i), we have

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{12}{x_k}\right), \quad k = 0, 1, 2, 3, \ldots \qquad \ldots(iii)$$

Since $\sqrt{12} \approx \sqrt{9} = 3$

So take the initial approximation $x_0 = 3$.

Now using the iterative formula (*iii*), the successive approximations are calculated as follows :

Iteration 1. The first approximation is given by

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right) = \frac{1}{2}\left(3 + \frac{12}{3}\right) = 3.5$$

Iteration 2. The second approximation is given by

$$x_2 = \frac{1}{2}\left(x_1 + \frac{12}{x_1}\right) = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.46428$$

Iteration 3. The third approximation is given by

$$x_3 = \frac{1}{2}\left(x_2 + \frac{12}{x_2}\right) = \frac{1}{2}\left(3.46428 + \frac{12}{3.46428}\right) = 3.46410$$

Iteration 4. The fourth approximation is given by

$$x_4 = \frac{1}{2}\left(x_3 + \frac{12}{x_3}\right) = \frac{1}{2}\left(3.46410 + \frac{12}{3.46410}\right) = 3.46410.$$

From 3rd and 4th iteration, it is clear that there is no change in the successive approximations to t root upto first four decimal places.

So $\sqrt{12} = 3.4641$ (correct to four decimal places).

(*c*) Taking $N = 32$ in (*i*), we have

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{32}{x_k}\right), \quad k = 0, 1, 2, 3, ... \qquad ...($$

Since $\sqrt{32} \approx \sqrt{36} = 6$

So take the initial approximation $x_0 = 6$.

Now using the iterative formula (*iv*), the successive approximations are calculated as follows :

Iteration 1. The first approximation is given by

$$x_1 = \frac{1}{2}\left(x_0 + \frac{32}{x_0}\right) = \frac{1}{2}\left(6 + \frac{32}{6}\right) = 5.66667$$

Iteration 2. The second approximation is given by

$$x_2 = \frac{1}{2}\left(x_1 + \frac{32}{x_1}\right) = \frac{1}{2}\left(5.66667 + \frac{32}{5.66667}\right) = 5.65686$$

Iteration 3. The third approximation is given by

$$x_3 = \frac{1}{2}\left(x_2 + \frac{32}{x_2}\right) = \frac{1}{2}\left(5.65686 + \frac{32}{5.65686}\right) = 5.65685$$

From 2nd and 3rd iteration, it is clear that there is no change in the successive approximations upto first four decimal places.

So $\sqrt{32} = 5.6568$ (correct to four decimal places).

Example 5. Find the iterative formula for finding $\frac{1}{N}$ for some real number N using Newton Raphson method. Hence find the value of each of the following correct to four decimal places.

(*a*) $\frac{1}{31}$ (*b*) $\frac{1}{18}$

Sol. Let $x = \frac{1}{N}$ or $\frac{1}{x} - N = 0$

Let $f(x) = \frac{1}{x} - N$

So $f'(x) = -\frac{1}{x^2}$

Using Newton's iterative formula, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, $k = 0, 1, 2, 3, ...$

we have,

$$x_{k+1} = x_k - \frac{\frac{1}{x_k} - N}{-\frac{1}{x_k^2}}$$

or

$$x_{k+1} = x_k + x_k^2\left(\frac{1}{x_k} - N\right)$$

or

$$x_{k+1} = x_k + x_k - N x_k^2$$

or

$$x_{k+1} = 2x_k - N x_k^2$$

or

$$x_{k+1} = x_k(2 - N x_k), \quad k = 0, 1, 2, 3, ... \qquad ...(i)$$

which is required iterative formula.

(*a*) Taking $N = 31$ in (*i*), we have

$$x_{k+1} = x_k(2 - 31x_k), \quad k = 0, 1, 2, 3, ... \qquad ...(ii)$$

Since $\frac{1}{31} \approx \frac{1}{30} \approx 0.03$

So take $x_0 = 0.03$.

Now using the iterative formula (*ii*), the successive approximations can be calculated as follows :

Iteration 1. The first approximation is given by

$$x_1 = x_0\,(2-31\,x_0) = 0.03\,(2-31\times 0.03) = 0.0321$$

Iteration 2. The second approximation is given by

$$x_2 = x_1\,(2-31\,x_1) = 0.0321(2-31\times 0.0321) = 0.03226.$$

Iteration 3. The third approximation is given by

$$x_3 = x_2\,(2-31\,x_2) = 0.03226\,(2-31\times 0.03226) = 0.03226.$$

From 2nd and 3rd iteration, it is clear that there is no change in the successive approximations upto first four decimal places.

So $$\frac{1}{31} = 0.0322 \text{ (correct to four decimal places).}$$

(*b*) Taking $N = 18$ in the equation (*i*), we have

$$x_{k+1} = x_k\,(2-18\,x_k),\quad k = 0, 1, 2, 3, \ldots \qquad \ldots(iii)$$

Since $$\frac{1}{18} \approx \frac{1}{20} = 0.05$$

So take $x_0 = 0.05$

Now using the iterative formula (*iii*), the successive approximations can be calculated as follows :

Iteration 1. The first approximation is given by

$$x_1 = x_0\,(2-18\,x_0) = 0.05\,(2-18\times 0.05) = 0.055$$

Iteration 2. The second approximation is given by

$$x_2 = x_1\,(2-18\,x_1) = 0.055\,(2-18\times 0.055) = 0.05555$$

Iteration 3. The third approximation is given by

$$x_3 = x_2\,(2-18\,x_2) = 0.05555\,(2-18\times 0.05555) = 0.05556$$

From 2nd and 3rd iteration, it is clear that there is no change in the successive approximations upto first four decimal places.

So $$\frac{1}{18} = 0.0555 \text{ (correct to four decimal places).}$$

Example 6. Find a root of the equation $e^x - x^3 = 0$ correct to four significant digits using Newton-Raphson method.

(G.N.D.U. B.Sc. I.T. April 2005)

Sol. The given equation is $e^x - x^3 = 0$.

Let $f(x) = e^x - x^3$

So $f'(x) = e^x - 3x^2$

Now $f(1) = e^1 - 1^3 = 1.71828 > 0$

and $f(2) = e^2 - 2^3 = -0.61094 < 0$

So a root of given equation lies between 1 and 2.

Also $f(2)$ is nearer to zero than $f(1)$ so take the initial approximation x_0 to the root as 2.

Iteration 1. The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{e^{x_0} - x_0^3}{e^{x_0} - 3x_0^2} = 2 - \frac{e^2 - (2)^3}{e^2 - 3(2)^2} = 1.8675$$

Iteration 2. The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{e^{x_1} - x_1^3}{e^{x_1} - 3x_1^2}$$

$$= 1.8675 - \frac{e^{1.8675} - (1.8675)^3}{e^{1.8675} - 3(1.8675)^2} = 1.8572$$

Iteration 3. The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{e^{x_2} - x_2^3}{e^{x_2} - 3x_2^2}$$

$$= 1.8572 - \frac{e^{1.8572} - (1.8572)^3}{e^{1.8572} - 3(1.8572)^2} = 1.8572$$

From 2nd and 3rd iteration, we observe that there is no change in the successive approximations to the root upto first four significant digits.

So a root of the given equation is given by $x = 1.857$ (correct to four significant digits)

Example 7. Obtain the formula using Newton-Raphson technique to calculate $x^{1/3}$ for $x > 0$.

(P.U. B.C.A. April 2001)

Sol. Let $z = x^{1/3},\ x > 0$

$\therefore$ $z^3 = x$ or $z^3 - x = 0$

Let $f(z) = z^3 - x$

So $f'(z) = 3z^2$

Using Newton-Raphson method, $z_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}$, $k = 0, 1, 2, 3, \ldots\ldots$

or
$$z_{k+1} = z_k - \frac{z_k^3 - x}{3z_k^2} = \frac{1}{3}\left(3z_k - \frac{z_k^3 - x}{z_k^2}\right)$$

$$= \frac{1}{3}\left(3z_k - \frac{z_k^3}{z_k^2} + \frac{x}{z_k^2}\right) = \frac{1}{3}\left(3z_k - z_k + \frac{x}{z_k^2}\right)$$

$\therefore$
$$z_{k+1} = \frac{1}{3}\left(2z_k + \frac{x}{z_k^2}\right), \quad k = 0, 1, 2, 3, \ldots\ldots$$

which is the required iterative formula for calculating the value of $x^{1/3}$.

Example 8. Find the fifth root of 3 correct to five decimals using Newton's method.

(P.U. B.C.A. Sept. 2003)

Sol. Let $x = 3^{1/5}$

$\therefore \quad x^5 = 3$ or $x^5 - 3 = 0$

Let $f(x) = x^5 - 3$

So $f'(x) = 5x^4$

Using Newton's iterative formula, $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$, $k = 0, 1, 2, 3, \ldots\ldots$, we have

$$x_{k+1} = x_k - \frac{x_k^5 - 3}{5x_k^4} = \frac{1}{5}\left(5x_k - \frac{x_k^5}{x_k^4} + \frac{3}{x_k^4}\right)$$

or
$$x_{k+1} = \frac{1}{5}\left(5x_k - x_k + \frac{3}{x_k^4}\right)$$

or
$$x_{k+1} = \frac{1}{5}\left(4x_k + \frac{3}{x_k^4}\right), \quad k = 0, 1, 2, 3, \ldots\ldots$$

Let us take initial approximation $x_0 = 1$.

Iteration 1. The first approximation is given by

$$x_1 = \frac{1}{5}\left(4x_0 + \frac{3}{x_0^4}\right) = \frac{1}{5}\left(4(1) + \frac{3}{(1)^4}\right) = 1.4$$

$x_0 - \frac{f(x_0)}{f'(x_0)}$

Iteration 2. The second approximation is given by

$$x_2 = \frac{1}{5}\left(4x_1 + \frac{3}{x_1^{\,4}}\right) = \frac{1}{5}\left(4(1.4) + \frac{3}{(1.4)^4}\right) = 1.276185$$

Iteration 3. The third approximation is given by

$$x_3 = \frac{1}{5}\left(4x_2 + \frac{3}{x_2^{\,4}}\right) = \frac{1}{5}\left(4(1.276185) + \frac{3}{(1.276185)^4}\right) = 1.247150$$

Iteration 4. The fourth approximation is given by

$$x_4 = \frac{1}{5}\left(4x_3 + \frac{3}{x_3^{\,4}}\right) = \frac{1}{5}\left(4(1.247150) + \frac{3}{(1.247150)^4}\right) = 1.245734$$

Iteration 5. The fifth approximation is given by

$$x_5 = \frac{1}{5}\left(4x_4 + \frac{3}{x_4^{\,4}}\right) = \frac{1}{5}\left(4(1.245734) + \frac{3}{(1.245734)^4}\right) = 1.245731$$

From 4th and 5th iteration, we see that there is no change in the successive approximations upto first five decimal places.

Hence $\qquad 3^{1/5} = 1.24573$ (correct of five decimal places)

EXERCISE 2.3

1. Find a root of the equation $x^2 - 8 = 0$ correct to three decimal places using Newton-Raphson method. (G.N.D.U. B.Sc. C.Sc. Sept. 2007)
2. Find a root of the equation $x^3 - 3x + 1 = 0$ using Newton-Raphson method, correct to three decimal places.
3. Find a root of the equation $3x^3 - 9x^2 + 8 = 0$ by using Newton-Raphson method correct to three decimal places.
4. Find a root of the equation $x^4 - x - 10 = 0$ correct to four decimal places using Newton's method.
5. Use Newton-Raphson method to find a root of the equation $x \sin x + \cos x = 0$ near π correct to three decimal places.
6. Find a real root of the equation $\log x - x + 3 = 0$ correct to four decimal places using Newton's method.

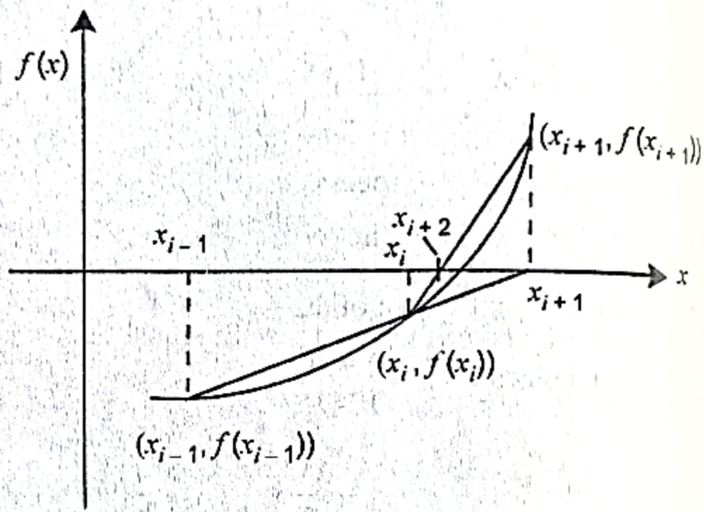
   (G.N.D.U. B.C.A. April 2002)
7. Find the iterative formula for finding $\frac{1}{\sqrt{N}}$ for some positive real number N. Hence evaluate each of the following correct to three decimal places.

   (a) $\frac{1}{\sqrt{14}}$ $\qquad$ (b) $\frac{1}{\sqrt{15}}$

ANSWERS

1. 2.828 2. 0.347 3. 1.226 4. 1.8555
5. 2.798 6. 0.0524
7. $x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{N x_k}\right)$, $k = 0, 1, 2, 3, \ldots\ldots$ (a) 0.267 (b) 0.258

2.9 SECANT METHOD [CHORD METHOD]



The secant method is similar to the Regula-falsi method except that two most recent approximations to the root are used to find the new approximation instead of using only those two approximations which bound the interval enclose the root. Also it is not necessary to determine initial approximations to enclose the root. Thus secant method is faster than the Regula-falsi method but the serious disadvantage with this method is that convergence is not always assured.

Let us suppose that x_{i-1} and x_i are two initial approximations to the root. A secant is drawn connecting $f(x_{i-1})$ and $f(x_i)$. The point where it cuts the x-axis gives the next approximation to the root. Let this intersection point be x_{i+1}. Now x_i and x_{i+1} are taken as starting approximations for the next iteration. Another secant is drawn through $f(x_i)$ and $f(x_{i+1})$ to get intersecting point with x-axis is x_{i+2} as shown in fig. This procedure is repeated till $f(x)$ is nearly equal to zero.

To calculate x_{i+1} the formula used is :

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1}) f(x_i)}{(f(x_i) - f(x_{i-1}))}$$

ILLUSTRATIVE EXAMPLES

Example 1. Find the root of the equation $x\,e^x = \cos x$, using the secant method to four decimal places.

Sol. Given equation $f(x) = \cos x - x\,e^x = 0$

Take initial approximations $x_0 = 0$ and $x_1 = 1$ such that $f(x_0) = 1$

$$f(x_1) = \cos 1 - e = -2.17798$$

Using the secant method we get

$$x_2 = x_1 - \frac{(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1-(1-0)(-2.17789)}{-2.17789-1} = 1 + \frac{(-2.17789)}{3.17789}$$

$$= 0.31467$$

Now $f(x_2) = f(0.31467) = 0.51987$. Therefore the root lies between 0.31467 and 1.

Take $x_0 = 0.31467$, $x_1 = 1$, $f(x_0) = 0.51987$, $f(x_1) = -2.17789$, we get next iteration.

$$x_3 = \frac{1-(1-0.31467)(-2.17789)}{-2.17789-0.51987} = 0.53171$$

Repeating this process, the successive approximations are as $x_5 = 0.51690$, $x_6 = 0.51775$, $x_7 = 0.51776$ etc. hence the root is 0.5178 correct to four decimal places.

Example 2. Find the root of the equation $x = 0.5 + \sin x$ using the secant method to four decimal places.

Sol. Given equation $f(x) = 0.5 + \sin x - x = 0$. Take initial approximations $x_0 = 1$ and $x_1 = 1.5$ such that $f(x_0) = 0.5 + \sin 1 - 1 = 0.3415, f(x_1) = 0.5 + \sin 15 - 1.5 = -0.025$. Using the secant method we get

$$x_2 = x_1 - \frac{(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0)}$$

$$x_2 = 1.5 - \frac{(1.5-1)\times(-0.0025)}{(-0.0025)-0.3415} = 1.4964$$

Now $f(x_2) = f(1.4964) = 0.000834$. Therefore the root lies between 1.4964 and 1.5.

Take $x_0 = 1.4964, x_1 = 1.5, f(x_0) = 0.000834, f(x_1) = -0.0025$, we get next iteration

$$x_3 = \frac{1.5-(1.5-1.4964)\times(-0.0025)}{(-0.0025)-0.000834} = 1.4973$$

The value of $f(x_3) = f(1.4973) = 0.00000036$

Therefore the root is 1.4973.

EXERCISE 2.4

1. Describe Secant Method to solve non-linear equation.
2. Compute real root of $f(x) = x^3 - 5x + 3 = 0$ in the interval (1, 2) by Secant Method. Perform four iterations.
3. Find $\sqrt{25}$ by Secant Method.

4. Find a root of the following equations using chord (Secant) Method.

(*i*) $x^3 + x^2 + x + 7 = 0$ (*ii*) $x - e^{-x} = 0$

ANSWERS

2. 1.831141 3. 5.00 4. (*i*) -2.0625 (*ii*) 0.567

2.10 COMPARISON OF ITERATIVE METHODS

The different parameters which are taken into account for comparison and hence selection of an iterative method are as follows :

(*a*) **Order of Convergence :** The order of convergence of the bisection method and the false position method is 1. In some cases asymptotic error constant A for false position method is less than 1/2. In such cases false position method has faster convergence than the bisection method otherwise the bisection method is faster that the false position method. The order of convergence of Newton's method is 2. So Newton-Raphson method is much faster than the other two methods. Thus when it is required to find a root rapidly then Newton-Raphson method is preferred.

(*b*) **Reliability of Convergence :** The bisection method and the false-position method have assured convergence. These two methods always converge to the root provided the initial interval enclose the root. On the other hand, Newton-Raphson method does not have guarantee of convergence. This method is very sensitive to the initial approximation. If the initial approximation is not chosen properly then this method may fall in an endless loop or may fail altogether. So if it is required to have assured convergence then one should select bisection or false position method.

(*c*) **The amount of computation required per iteration :** The bisection and the false position method require one function evaluation per iteration while the Newton-Raphson method requires two function evaluations per iteration *i.e.* we have to evaluate $f(x)$ and $f'(x)$ in each iteration.

So if less computation per iteration is required then one should use either the bisection or the false position method.

MISCELLANEOUS EXERCISE

1. Find a root of the equation $x \log_{10} x = 1.2$ by using the bisection method correct to four significant digits.

2. Find a real root of the equation $\sin x = \frac{1}{x}$ correct to four significant digits using the bisection method.

3. Determine two smallest positive roots of the equation $x \sin x + \cos x = 0$ correct to three significant digits using bisection method.

   (P.U. B.C.A. Sept. 2006, 2007)

4. Find a real root of the equation $x e^x = 2$ correct to 4 decimal places using Regula-Falsi method.

5. Find a root of the equation $x e^x = \cos x$ correct to four decimal places using the method of false position.

   (G.N.D.U. B.C.A. April 2003)

6. Find a real root of the equation $\cos x = 3x - 1$ correct to four decimal places using the false position method.

   (P.U. B.C.A. Sept. 2003, 2004)

7. Find a negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places using Newton-Raphson method.

8. Find three roots of the equation $x^3 - 4x + 1 = 0$ using Newton-Raphson method correct to 3 significant digits.

   (P.U. B.C.A. Sept. 2006, April 2008)

9. Using Newton's iterative method, find a real root of $x \log_{10} x = 1.2$ correct to six decimal places.

   (P.U. B.C.A. April 2003)

10. Find the smallest positive root of the equation $e^x \sin x = 1$ using Newton's method correct to three decimal places.

11. The bacteria concentration C in a reservoir varies with time t as $C = 4e^{-2t} + e^{-0.1t}$. Newton-Raphson method to calculate the time required for the bacteria concentration to be units.

12. Find the iteration formula for finding $\sqrt[k]{N}$ for some positive real number N. Hence evaluate of the following correct to four decimal places

(a) $\sqrt[3]{24}$ (b) $\sqrt[3]{41}$ (c) $\sqrt[4]{32}$ (d) $(28)^{-1/4}$ (e) $(30)^{-1/5}$

ANSWERS

1. 2.740
2. 1.114
3. 2.80, – 2.80
4. 0.8525
5. 0.5177
6. 0.6071
7. – 15.64
8. – 2.11, 0.254, 1.8
9. 2.740646
10. 0.588
11. 6.931
12. $x_{n+1} = \frac{1}{k}\left((k-1)\,x_n + \frac{N}{x_n^{\,k-1}}\right)$, $n = 0, 1, 2, 3, \ldots$; (a) 2.8845

(b) 3.4482 (c) 2.3784 (d) 0.4347 (e) 0.5065