4

INTERPOLATION

4.1 INTRODUCTION

Interpolation is the art of reading between the lines in a table and may be regarded as a special case of the general process of curve fitting. More precisely, we can define interpolation as follows :

Suppose that we are given the following tabulated values of the function $f(x)$ corresponding to a discrete set of values of x :

x :	x_0	x_1	x_2	....	x_n
$f(x)$:	y_0	y_1	y_2	....	y_n

Table 4.1

Then the process of finding the value of $f(x)$ corresponding to any untabulated value of x between x_0 and x_n is called *Interpolation.*

The process of finding the value of $f(x)$ for some value of x outside the given range $[x_0, x_n]$ is called *extrapolation.* Extrapolation assumes that the behaviour of $f(x)$ outside the given range is identical to the behaviour of $f(x)$ inside the given range and this may not always be valid.

Interpolation is not as important in Numerical Analysis as it was, now computers and calculators with built in functions are available and function values may often be obtained readily by an algorithm. However,

(*i*) interpolation is still important for functions that are available only in tabular form (*e.g.* the results of an experiment which are usually in tabular form).

(*ii*) interpolation provides the theoretical foundation for derivation of differentiation and integration formulae and for solution of differential equations.

4.2 BASIC PRINCIPLE IN INTERPOLATION

Suppose that the explicit form of the function is not given and function f is given in tabular form (as in Table 4.1).

The basic principle in interpolation is that we find a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points and using this function $\phi(x)$, we approximate the value of $f(x)$ corresponding to any untabulated value of x. This function $\phi(x)$ is called *interpolating function* or *smoothing function.*

Further, if the interpolating function $\phi(x)$ is a polynomial then interpolation is termed as *polynomial interpolation*. Similarly if $\phi(x)$ is a finite trigonometric series then we have *trigonometric interpolation*. In this chapter we shall concern only with polynomial interpolation.

4.3 FUNDAMENTAL ASSUMPTIONS IN INTERPOLATION

The following are the underlying assumptions for the validity of any method of interpolation

(*i*) There are no sudden changes in the values of the observations for a particular data under consideration.

e.g. if the given data refers to abnormal periods like the periods of famines, wars or epidemics etc. there may be sudden changes in the observations in the data. In such cases, interpolation can not be applied.

(*ii*) The rise or fall in the data is uniform. This means that the changes in the values of the observations should be in a uniform pattern.

(*iii*) The function behaves sufficiently smoothing between the tabular points for it to be approximated by a polynomial of fairly low degree.

4.4 FINITE DIFFERENCES

Suppose that the function f is tabulated at $(n+1)$ equidistant points $x_0, x_1, x_2, ..., x_n$ and the corresponding values of the function f are $y_0, y_1, y_2, ..., y_n$ respectively.

Then the *finite differences* are either the differences between the successive value of y or the differences between the successive values of the past differences.

Finite differences are of following three types :

1. Forward Differences : If the differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ are denoted by $\Delta y_0, \Delta y_1, ..., \Delta y_{n-1}$ respectively where Δ is forward difference operator then these differences are called *first forward differences.*

i.e. $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, ..., \Delta y_{n-1} = y_n - y_{n-1}$.

In general, $\Delta y_r = y_{r+1} - y_r$.

The differences of first order differences are called *second order differences* and are denoted $\Delta^2 y_0, \Delta^2 y_1, ..., \Delta^2 y_{n-2}$

i.e. $$\Delta^2 y_0 = \Delta y_1 - \Delta y_0,\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1, ..., \Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}.$$

In general, $$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r.$$

Similarly, we can define third order differences, fourth order differences etc.

Note (*i*). In general, the jth order forward difference is given by

$$\Delta^j y_r = \Delta^{j-1} y_{r+1} - \Delta^{j-1} y_r.$$

(*ii*) Any higher order difference can be expressed in terms of values of y.

e.g. $$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0,$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 .$$

Clearly the coefficients occurring on R.H.S. are binomial coefficients. So, in general,

$$\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3\, y_{n-3} + ... + (-1)^n\, {}^nC_n\, y_0 .$$

2. Backward Differences : If the differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ are denoted by $\nabla y_1, \nabla y_2, ...\ \nabla y_n$ respectively where ∇ is backward difference operator, then these differences are called *first backward differences.*

i.e. $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, ..., \nabla y_n = y_n - y_{n-1}$.

In general $\qquad \nabla y_r = y_r - y_{r-1}$

The differences of first order backward differences are called *second order backward differences* and are denoted by $\nabla^2 y_2, \nabla^2 y_3, ..., \nabla^2 y_n$.

i.e. $\qquad \nabla^2 y_2 = \nabla y_2 - \nabla y_1,\ \nabla^2 y_3 = \nabla y_3 - \nabla y_2, ..., \nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$

In general $\qquad \nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$.

Similarly, we can define third order backward differences, fourth order backward differences etc.

Note : (*i*) In general, the *j*th order backward difference is given by

$$\nabla^j y_r = \nabla^{j-1} y_r - \nabla^{j-1} y_{r-1} .$$

(*ii*) Any higher order backward difference can be expressed in terms of values of *y*.

e.g.
$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 ,$$
$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0 \text{ etc.}$$

Clearly the coefficients occurring on R.H.S. are binomial coefficients. So, in general

$$\nabla^n y_n = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - {}^nC_3\, y_{n-3} + ... + (-1)^n\, {}^nC_n\, y_0 .$$

3. Central Differences : If the differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ are denoted by $\delta y_{1/2}, \delta y_{3/2}, ..., \delta y_{(2n-1)/2}$ respectively where δ is central difference operator then these differences are called *first central differences.*

i.e. $\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, ..., \delta y_{(2n-1)/2} = y_n - y_{n-1}$. $\qquad \delta$

In general, $\qquad \delta y_r = y_{r+1/2} - y_{r-1/2}$

The differences of first central differences are called *second central differences* and are denoted by $\delta^2 y_1, \delta^2 y_2, ..., \delta^2 y_{n-1}$.

i.e. $\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2},\ \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}, ..., \delta^2 y_{n-1} = \delta y_{(2n-1)/2} - \delta y_{(2n-3)/2}$.

In general, $\delta^2 y_r = \delta y_{r+1/2} - \delta y_{r-1/2}$.

Similarly, we can define third order central difference, fourth order central difference etc.

In this chapter, we shall deal with forward differences and backward differences only.

<u>Note</u> Besides the above discussed operators, there is another useful operator namely shift operator. T shift operator is denoted by E and is defined as

$$E^n y_r = y_{r+n} \text{ where } n \text{ is any integer.}$$

For example, $E^2 y_0 = y_2,\ E^1 y_2 = y_3,\ E^{-2} y_3 = y_1,\ E^{-1} y_2 = y_1$ etc.

Also, $\Delta y_r = y_{r+1} - y_r = E\, y_r - y_r = (E-1)\, y_r$

i.e. $\Delta y_r = (E-1)\, y_r$ *i.e.* $\Delta \equiv E - 1$ or $E \equiv 1 + \Delta$

And, $\nabla y_r = y_r - y_{r-1} = y_r - E^{-1} y_r = (1 - E^{-1})\, y_r$

i.e. $\nabla y_r = (1 - E^{-1})\, y_r$ *i.e.* $\nabla \equiv 1 - E^{-1}$ or $E^{-1} \equiv 1 - \nabla$

4.4.1 DIFFERENCE DISPLAY

The differences of the function values and the differences of past differences can be listed in a ta known as *difference table*. In the difference table the value of x is called the argument and the value corresponding to that argument is called the entry.

The difference tables corresponding to each type of finite difference are given below :

(*i*) Forward Difference Table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Table 4.2

(ii) Backward Difference Table :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Table 4.3

(iii) Central Difference Table :

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Table 4.4

Notes (i) In forward difference table, the subscript remains constant along each forward diagonal of the table. In backward difference table, the subscript remains constant along each backward diagonal.

In central difference table, the subscript remains constant along horizontal lines of the table.

(ii) It is clear from the three tables that the same numbers occur in the same positions whether we use forward, backward or central differences.

i.e. $$\Delta y_0 = \nabla y_1 = \delta y_{1/2}, \Delta^3 y_1 = \nabla^3 y_4 = \delta^3 y_{5/2} \text{ etc.}$$

4.5 DIFFERENCES OF A POLYNOMIAL

Consider an nth degree polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0$$

tabulated for equidistant points at tabular interval h.

Now
$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
&= a_n[(x+h)^n - x^n] + a_{n-1}[(x+h)^{n-1} - x^{n-1}] + \ldots + a_1[(x+h) - h] \\
&= a_n nhx^{n-1} + \text{polynomial of degree } (n-2) \\
\Delta^2 f(x) &= \Delta f(x+h) - \Delta f(x) \\
&= a_n nh[(x+h)^{n-1} - x^{n-1}] + \ldots \\
&= a_n n(n-1)h^2 x^{n-2} + \text{polynomial of degree } n-3. \\
&\vdots \qquad\qquad \vdots \\
\Delta^n f(x) &= a_n n!\, h^n = \text{constant} \\
\Delta^{n+1} f(x) &= 0.
\end{aligned}$$

This shows that the first difference of a polynomial of nth degree is a polynomial of degree $(n-1)$ with leading coefficient $a_n nh$. Similarly, the second differences is a polynomial of degree $(n-2)$ with leading coefficient $a_n n(n-1)h^2$. Continuing in this way, the nth difference is $a_n n!h^n$ i.e. a constant.

So $(n+1)$th and higher differences are all zero.

To sum up, we can say that if $f(x)$ is a polynomial of degree n then the nth order differences are constant and $(n+1)$th and higher order differences are all zero.

Conversely, if the nth differences of a tabulated function are constant and $(n+1)$th and higher order differences are all zero, then the tabulated function represents a polynomial of degree n. This enables us to approximate a function by a polynomial of nth degree if its nth order differences become nearly constant.

4.6 DETECTION OF ERRORS BY USE OF DIFFERENCE TABLES

To understand how the difference tables can be used to check the error in any entry, suppose that there is an error of $+\varepsilon$ in the entry y_5. As higher differences are formed, the error spreads out in the difference table and is considerably magnified.

Let us look at the difference table (Table 4.5)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0 + \varepsilon$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1 + \varepsilon$	
		Δy_3		$\Delta^3 y_2 + \varepsilon$		$\Delta^5 y_1 - 5\varepsilon$
x_4	y_4		$\Delta^2 y_3 + \varepsilon$		$\Delta^4 y_2 - 4\varepsilon$	
		$\Delta y_4 + \varepsilon$		$\Delta^3 y_3 - 3\varepsilon$		$\Delta^5 y_2 + 10\varepsilon$
x_5	$y_5 + \varepsilon$		$\Delta^2 y_4 - 2\varepsilon$		$\Delta^4 y_3 + 6\varepsilon$	
		$\Delta y_5 - \varepsilon$		$\Delta^3 y_4 + 3\varepsilon$		$\Delta^5 y_3 - 10\varepsilon$
x_6	y_6		$\Delta^2 y_5 + \varepsilon$		$\Delta^4 y_4 - 4\varepsilon$	
		Δy_6		$\Delta^3 y_5 - \varepsilon$		$\Delta^5 y_4 + 5\varepsilon$
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5 + \varepsilon$	
		Δy_7		$\Delta^3 y_6$		$\Delta^5 y_5 - \varepsilon$
x_8	y_8		$\Delta^2 y_7$		$\Delta^4 y_6$	
		Δy_8		$\Delta^3 y_7$		
x_9	y_9		$\Delta^2 y_8$			
		Δy_9				
x_{10}	y_{10}					

Table 4.5

From table 4.5, following observations are made :

(*i*) As the order of differences increases, the effect of error increases.

(*ii*) The coefficients of ε in any column are binomial coefficients with alternating sign.

So, error in fifth differences are $^5C_0\,\varepsilon, -\,^5C_1\,\varepsilon, \,^5C_2\,\varepsilon, -\,^5C_3\,\varepsilon, \,^5C_4\,\varepsilon, -\,^5C_5\,\varepsilon$

i.e. $\varepsilon, -5\varepsilon, 10\varepsilon, -10\varepsilon, 5\varepsilon, -\varepsilon$

(*iii*) The algebraic sum of errors in any column is zero.

(*iv*) The maximum error occurs opposite of the entry containing the error.

The above facts can be used to detect errors by using difference tables.

CHECKPOINTS

1. What is interpolation? Explain with the help of suitable examples.
   (P.U. B.C.A. Sept. 2001; G.N.D.U. B.C.A. April 2007)
2. Explain the term extrapolation.
   (G.N.D.U. B.Sc. C.Sc. April 2003, P.U. B.C.A. April 2008)
3. What are the underlying assumptions for the validity of various methods used for interpolation?
   (G.N.D.U. B.Sc. I.T. 2007)
4. Define shift operator.
5. What are various types of differences? Give some advantages of difference table. Also discuss the error build up in difference table.
   (P.U. B.C.A. Sept. 2001, G.N.D.U. B.C.A. April 2006, Sept. 2006)
6. Explain difference tables in interpolation with examples. (P.U. B.C.A. April 2007)

ILLUSTRATIVE EXAMPLES

Example 1. Write forward difference table for the following data :

x :	10	20	30	40
y :	1.1	2.0	4.4	7.9

Sol. The forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	1.1			
		0.9		
20	2.0		1.5	
		2.4		−0.4
30	4.4		1.1	
		3.5		
40	7.9			

Example 2. Construct the backward difference table for the data :

x :	1	2	3	4
$f(x)$:	63	52	43	38

Sol. The backward difference table is :

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
1	63			
		-11		
2	52		2	
		-9		2
3	43		4	
		-5		
4	38			

Example 3. Form the table of backward difference of the function $f(x) = x^3 - 3x^2 - 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$.

Also evaluate $\nabla^4 f(4), \nabla^3 f(5)$ and $\nabla f(0)$.

Sol. The backward difference table is :

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-1	-6				
		-1			
0	-7		-6		
		-7		6	
1	-14		0		0
		-7		6	
2	-21		6		0
		-1		6	
3	-22		12		0
		$+11$		6	
4	-11		18		
		29			
5	18				

As we know that the subscript remains constant along each backward diagonal.

$\therefore \quad \nabla^4 f(4) = 0, \ \nabla^3 f(5) = 6, \ \nabla f(0) = -1.$

Example 4. Prove that $\Delta^3 y_2 = \nabla^3 y_5$. (P.U. B.C.A. April 200[2])

Sol. Consider
$$\Delta^3 y_2 = (E-1)^3 y_2$$
$$= (E^3 - 1 - 3E^2 + 3E)\, y_2$$
$$= E^3 y_2 - y_2 - 3E^2 y_2 + 3E\, y_2$$
$$= y_5 - y_2 - 3y_4 + 3y_3$$

So
$$\Delta^3 y_2 = y_5 - 3y_4 + 3y_3 - y_2 \quad \text{...(}$$

Also
$$\nabla^3 y_5 = (1-E^{-1})^3 y_5$$
$$= (1 - E^{-3} - 3E^{-1} + 3E^{-2})\, y_5$$
$$= y_5 - E^{-3} y_5 - 3E^{-1} y_5 + 3E^{-2} y_5$$
$$= y_5 - y_2 - 3y_4 + 3y_3$$

So
$$\nabla^3 y_5 = y_5 - 3y_4 + 3y_3 - y_2 \quad \text{...(i}$$

From (*i*) and (*ii*), we have, $\Delta^3 y_2 = \nabla^3 y_5$

Example 5. Prove that (*i*) $\Delta\nabla = \Delta - \nabla$ (*ii*) $\nabla = \Delta E^{-1}$

(P.U. B.C.A. April 200

Sol. (*i*) Consider
$$(\Delta\nabla)\, y_r = \Delta(\nabla y_r) = \Delta(y_r - y_{r-1})$$
$$= \Delta y_r - \Delta y_{r-1} = \Delta y_r - (y_r - y_{r-1})$$
$$= \Delta y_r - \nabla y_r = (\Delta - \nabla)\, y_r$$

i.e.
$$(\Delta\nabla)\, y_r = (\Delta - \nabla)\, y_r$$

So
$$\Delta\nabla = \Delta - \nabla$$

(*ii*) Consider
$$(\Delta E^{-1}) y_r = \Delta(E^{-1} y_r) = \Delta y_{r-1}$$
$$= y_r - y_{r-1} = \nabla y_r$$

i.e.
$$(\Delta E^{-1})\, y_r = \nabla y_r$$

So
$$\Delta E^{-1} = \nabla \quad \text{or} \quad \nabla = \Delta E^{-1}$$

Example 6. Find the degree of interpolating polynomial which might be suitable for the following tabu[lated] values.

x:	1	2	3	4	5
y:	4	13	34	73	136

Sol. The forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
		21		6	
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

From the difference table, we see that the 3rd order difference are constant and 4th order differences are zero. So degree of interpolating polynomial which is suitable for given data is 3.

Example 7. Find the missing term in the following table :

x :	0	1	2	3	4
$f(x)$:	1	3	9	–	81

Sol. Let the missing term in the data be y_3.

The forward difference table is :

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		y_3-19	
2	9		y_3-15		$124-4y_3$
		y_3-9		$105-3y_3$	
3	y_3		$90-2y_3$		
		$81-y_3$			
4	81				

Since only four entries are given in the data so y can be expressed as a polynomial of degree 3 in x. Thus fourth and higher order differences would be zero.

$$\therefore \quad \Delta^4 y_0 = 0 \quad \text{or} \quad 124-4y_3 = 0 \quad \text{or} \quad y_3 = 31$$

Example 8. The values of a polynomial of degree 5 are tabulated below. If $f(3)$ is known to be in error find the correct value.

x :	0	1	2	3	4	5	6
$f(x)$:	1	2	33	254	1025	3126	7777

Sol. The difference table is :

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
		31		160		
2	33		190		200	
		221		360		220
3	254		550		420	
		771		780		20
4	1025		1330		440	
		2101		1220		
5	3126		2550			
		4651				
6	7777					

Given $f(3)$ is known to be in error so let $y_3 + \varepsilon = 254$ where y_3 is true value of $f(3)$.

Also, $f(x)$ is a polynomial of degree 5 so $\Delta^5 y$ must be constant.

The sum of 5th difference is 240 so each entry under $\Delta^5 y$ must be $\frac{240}{2} = 120$.

So both the entries under $\Delta^5 y$ are in error.

Now, errors in $\Delta^5 y$ are $\varepsilon, -5\varepsilon, 10\varepsilon, -10\varepsilon, 5\varepsilon, -\varepsilon$ and the maximum error occur in the differences in the same horizontal line drawn in the table.

So, the differences under $\Delta^5 y$ can be written as

$$120 + 10\,(10),\ 120 - 10\,(10)$$

i.e. $\quad 120 + 10\,\varepsilon,\ 120 - 10\,\varepsilon$, where $\varepsilon = 10$.

$\therefore \quad y_3 + \varepsilon = 254$ gives $y_3 = 254 - 10 = 244$.

So the correct value of $f(3) = 244$.

Example 9. Find and correct by means of differences the error in the following data :

x :	3.60	3.61	3.62	3.63	3.64	3.65	3.66	3.67	3.68
y :	0.112046	0.120204	0.128350	0.136462	0.144600	0.152702	0.160788	0.168857	0.176908

(G.N.D.U. B.C.A. April 2005, April 2009, B.Sc. I.T. April 2007)

Sol. The difference table is :

x	y	$\Delta y \times 10^6$	$\Delta^2 y \times 10^6$	$\Delta^3 y \times 10^6$	$\Delta^4 y \times 10^6$
3.60	0.112046				
		8158			
3.61	0.120204		– 12		
		8146		– 22	
3.62	0.128350		– 34		82
		8112		60	
3.63	0.136462		26		– 122
		8138		– 62	
3.64	0.144600		– 36		82
		8102		20	
3.65	0.152702		– 16		– 21
		8086		– 1	
3.66	0.160788		– 17		0
		8069		– 1	
3.67	0.168857		– 18		
		8051			
3.68	0.176908				

From the difference table, we see that the second differences are quite irregular opposite to $x = 3.63$ and third differences are still more irregular. The irregularity begins in each column on the horizontal line corresponding to $x = 3.63$. Let $y_3 + \varepsilon = 0.136462$ where y_3 is the true value of 4th entry corresponding to $x = 3.63$. Further the algebraic sum of the third differences is small as compare to other differences. So the third differences are accumulated errors.

Now as the know that errors in $\Delta^3 y$ are $\varepsilon, -3\varepsilon, 3\varepsilon, -\varepsilon$ and the maximum error occur in the difference in same horizontal line drawn in the table.

On comparing these errors with $\Delta^3 y$ in the table, we get $-3\varepsilon = 60 \times 10^{-6}$

$\therefore \quad \varepsilon = -20 \times 10^{-6} = -0.000020.$

Hence $y_3 = 0.136462 - \varepsilon$

$= 0.136462 + 0.000020$

$= 0.136482$ which is the true value at 4th entry.

EXERCISE 4.1

1. Construct a forward difference table for the following data :

x :	0	10	20	30
y :	0	0.174	0.347	0.518

2. Construct backward difference table for the following data :

x :	0	1	2	3
y :	1	2	1	10

3. Construct the table of difference for the following data :

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(1)$.

4. Given the set of values

x :	10	15	20	25	30	35
y :	19.97	21.51	22.47	25.32	24.65	25.89

Form the difference table and write down the values of $\Delta^2 y_{10}$, Δy_{20}, $\Delta^3 y_{15}$ and $\Delta^5 y_{10}$.

5. Construct backward difference table for the following data :

x :	0	1	2	3
$f(x)$:	− 3	6	8	12

Evaluate $\nabla^3 f(3)$ and $\nabla^2 f(2)$.

6. If $u_0 = 3, u_1 = 12, u_2 = 81, u_3 = 2000, u_4 = 10$. Calculate $\Delta^4 u_0$.

7. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.

8. Given the table of values as

x :	2.0	2.1	2.2	2.3	2.4	2.5
$y(x)$:	13.000	14.261	15.648	17.167	18.824	20.65

Find the degree of the interpolating polynomial which might be suitable for above table of value

9. Find the missing values in the following data :

x :	0	5	10	15	20	25
y :	6	10	–	17	–	31

(P.U. B.C.A. April 20

10. Form the difference table of $f(x) = x^3 - 3x^2 + 5x + 7$ for the values of 0, 2, 4, 6, 8 and ext the table for calculations of $f(10)$.

(P.U. B.C.A. Sept. 20

11. One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error.

x :	0	1	2	3	4	5	6	7
y :	25	21	18	18	27	45	76	123

12. Find and correct by means of differences the error in the following table :

20736, 28561, 38416, 50625, 65540, 83521, 104976, 130321, 160000.

(G.N.D.U. B.Sc. I.T. April 2008)

13. Locate and correct the error in the following table of values.

x :	1	2	3	4	5	6	7	8
y :	3010	3424	3802	4105	4472	4771	5051	5315

ANSWERS

3. $\Delta^3 f(1) = 0.4$ 4. $\Delta^2 y_{10} = -0.58$, $\Delta y_{20} = 2.85$, $\Delta^3 y_{15} = -5.41$, $\Delta^5 y_{10} = 18.72$

5. $\nabla^3 f(3) = 9$, $\nabla^2 f(2) = -7$ 6. -7549 8. 5

9. 13.25, 22.5 10. $f(10) = 757$ 11. incorrect value = 18, corrected value = 19

12. incorrect value = 65540, corrected value = 65536

13. incorrect value = 4105, corrected value = 4151

4.7 INTERPOLATION WITH EQUAL INTERVALS

Newton derived general forward and backward difference interpolation formulae, corresponding to approximation by a polynomial of degree n for the tables of constant interval h. In this section, we shall discuss these formulae.

4.7.1 NEWTON'S FORWARD DIFFERENCE INTERPOLATION FORMULA

Procedure

Suppose that the function f is tabulated at $(n + 1)$ equidistant points $x_0, x_1, x_2, \ldots, x_n$ with spacing h and the corresponding values of the function f are $y_0, y_1, y_2, \ldots, y_n$ respectively.

In order to derive Newton's forward difference formula, we approximate the given function by a polynomial $\phi_n(x)$ of degree n such that $f(x)$ and $\phi_n(x)$ agree at the tabulated points.

So
$$f(x) \approx \phi_n(x) \qquad \ldots(4.1)$$

and
$$\phi_n(x_i) = y_i \text{ for } i = 0, 1, 2, 3, \ldots, n \qquad \ldots(4.2)$$

The conditions given by (5.2) are called *interpolatory conditions.*

Since $\phi_n(x)$ is a polynomial of degree n so $\phi_n(x)$ can be written as

$$\phi_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ \ldots\ldots + a_n(x - x_0)(x - x_1)(x - x_2) \ldots\ldots (x - x_{n-1}) \qquad \ldots(4.3)$$

Imposing conditions (4.2) on (4.3), we have

$$\phi_n(x_0) = y_0 \quad \Rightarrow \quad a_0 = y_0$$

$$\phi_n(x_1) = y_1 \quad \Rightarrow \quad a_0 + a_1(x_1 - x_0) = y_1$$

$$\phi_n(x_2) = y_2 \quad \Rightarrow \quad a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = y_2$$

and so on.

Solving above equations for $a_0, a_1, a_2, ..., a_n$, we have

$$a_0 = y_0$$

$$a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} \Rightarrow a_1 = \frac{y_1 - y_0}{h}$$

$$a_2 = \frac{y_2 - a_1(x_2 - x_0) - a_0}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - \frac{y_1 - y_0}{h}(2h) - y_0}{(2h)(h)} = \frac{y_2 - 2y_1 + 2y_0 - y_0}{2h^2}$$

$$\Rightarrow \quad a_2 = \frac{y_2 - 2y_1 + y_0}{2!h^2} \text{ and so on.}$$

Now, using forward differences, we have

$$a_0 = y_0, \quad a_1 = \frac{\Delta y_0}{h}, \quad a_2 = \frac{\Delta^2 y_0}{2!h^2} \text{ and so on.}$$

Continuing in this way, we have $a_n = \dfrac{\Delta^n y_0}{n!h^n}$.

Substituting the values of $a_0, a_1, a_2, ..., a_n$ in (4.3) we have

$$\phi_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2)$$

$$+ .... + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)(x - x_2).....(x - x_{n-1}) \quad ...(4.4)$$

Now, setting $x = x_0 + ph$, we have

$$x - x_0 = ph$$

$$x - x_1 = x - (x_0 + h) = x - x_0 - h = ph - h = (p - 1)h$$

$$x - x_2 = x - (x_0 + 2h) = x - x_0 - 2h = ph - 2h = (p-2)h$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$x - x_{n-1} = (p - \overline{n-1})h$$

Using these relations, equation (4.4) can be written as

$$\phi_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-\overline{n-1})}{n!}\Delta^n y_0 \quad \dots(4.5)$$

Using (4.1), equation (4.5) can also be written as

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-\overline{n-1})}{n!}\Delta^n y_0 \quad \dots(4.6)$$

Formula (4.5) (or (4.6)) is called *Newton's forward difference formula.*

This formula is useful for interpolation in the beginning of a set of tabulated values.

Note The formula (4.5) (or (4.6)) is derived on the assumption that $x_0 < x < x_1$.

4.7.2 NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

Procedure

Suppose that the function f is tabulated at $(n + 1)$ equidistant points $x_0, x_1, x_2, \dots, x_n$ with spacing h and the corresponding values of the function f are $y_0, y_1, y_2, \dots, y_n$ respectively.

In order to derive Newton's Backward difference formula, we approximate the given function by a polynomial $\phi_n(x)$ of degree n such that $f(x)$ and $\phi_n(x)$ agree at the tabulated points.

So $$f(x) \approx \phi_n(x) \quad \dots(4.7)$$

and $$\phi_n(x_i) = y_i \text{ for } i = 0, 1, 2, 3, \dots, n \quad \dots(4.8)$$

The conditions given by (4.8) are called *Interpolatory conditions*. Since $\phi_n(x)$ is a polynomial of degree n so $\phi_n(x)$ can be written as

$$\phi_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \dots + a_n(x-x_n)(x-x_{n-1})(x-x_{n-2})\dots\dots(x-x_1) \quad \dots(4.9)$$

Imposing conditions (4.8) on (4.9), we have

$$\phi_n(x_n) = y_n \Rightarrow a_0 = y_n$$

$$\phi_n(x_{n-1}) = y_{n-1} \Rightarrow a_0 + a_1(x_{n-1} - x_n) = y_{n-1}$$

$$\phi_n(x_{n-2}) = y_{n-2} \Rightarrow a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) = y_{n-2}$$

and so on.

Solving above equations for $a_0, a_1, a_2, \dots, a_n$, we have

$$a_0 = y_n$$

$$a_1 = \frac{y_{n-1} - a_0}{x_{n-1} - x_n} = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} \Rightarrow a_1 = \frac{y_n - y_{n-1}}{h}$$

$$a_2 = \frac{y_{n-2} - a_1(x_{n-2} - x_n) - a_0}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})} = \frac{y_{n-2} - \frac{y_n - y_{n-1}}{h}(-2h) - y_n}{(-2h)(-h)}$$

$$= \frac{y_{n-2} - 2y_{n-1} + y_n}{2h^2}$$

$$\Rightarrow \quad a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2!h^2} \text{ and so on.}$$

Now, using backward differences, we have

$$a_0 = y_n$$

$$a_1 = \frac{\nabla y_n}{h},$$

$$a_2 = \frac{\nabla^2 y_n}{2!h^2} \text{ and so on.}$$

Continuing in this way, we have $a_n = \dfrac{\nabla^n y_n}{n!h^n}$

Substituting the values of $a_0, a_1, a_2, ..., a_n$ in (4.9), we have

$$\phi_n(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \frac{\nabla^3 y_n}{3!h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2})$$

$$+ .... + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})(x - x_{n-2}) .... (x - x_1) \quad ...(4.1$$

Now setting $x = x_n + ph$, we have

$$x - x_n = ph,$$

$$x - x_{n-1} = x - (x_n - h) = x - x_n + h = ph + h = (p+1)h,$$

$$x - x_{n-2} = x - (x_n - 2h) = x - x_n + 2h = ph + 2h = (p+2)h$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x - x_1 = (p + \overline{n-1})h$$

Using these relations, equation (4.10) can be written as

$$\phi_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ ..... + \frac{p(p+1)(p+2)...(p+\overline{n-1})}{n!}\nabla^n y_n \quad ...(4.1$$

Using equation (4.7), equation (4.11) can be written as

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \dots + \frac{p(p+1)(p+2)\dots(p+\overline{n-1})}{n!}\nabla^n y_n \quad \dots(4.12)$$

Formula given by equation (4.11) (or (4.12)) is called *Newton's Backward difference formula.* This formula is useful for interpolation in the end of a set of tabulated values.

Note Formula (4.11) (or (4.12)) is derived on the assumption that $x_{n-1} < x < x_n$.

CHECKPOINTS

1. State and prove Newton forward difference interpolation formula. Also write its algorithm. (G.N.D.U. B.C.A. Sept. 2008)
2. When is the Newton forward difference formula convenient to use ?
3. When is the Newton backward difference formula convenient to use ?

ILLUSTRATIVE EXAMPLES

Example 1. Given a function in the form of a table as :

x :	2.0	2.1	2.2	2.3
$y(x)$:	11.000	12.201	13.648	15.167

Estimate the order of polynomial which might be suitable for interpolating function. Using that polynomial interpolate the value of $y(x)$ at $x = 2.05$. (G.N.D.U. B.Sc. C.Sc. April 2005)

Sol. The forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2.0	11.000			
		1.201		
2.1	12.201		0.246	
		1.447		-0.174
2.2	13.648		0.072	
		1.519		
2.3	15.167			

Since the highest order involving in difference table is 3 so the lowest possible degree of the polynomial is 3.

To find the polynomial take $x_0 = 2.0$ so that $y_0 = 11.0, \Delta y_0 = 1.201, \Delta^2 y_0 = 0.246, \Delta^3 y_0 = -0.174$

Also $h = 0.1$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{x-2}{0.1} = 10x - 20.$$

By Newton's forward difference formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0.$$

$$\therefore \quad y(x) = 11 + (10x-20)(1.201) + \frac{(10x-20)(10x-21)}{2} \times 0.246 + \frac{(10x-20)(10x-21)(10x-22)}{6} \times -0.174$$

$$= 11 + 12.01x - 24.02 + 0.123(100x^2 - 410x + 420) - 0.029(1000x^3 - 6300x^2 + 13220x - 9240)$$

$= -29x^3 + 195x^2 - 421.8x + 306.6$, which is the required cubic polynomial.

Also $\quad y(2.05) = -29(2.05)^3 + 195(2.05)^2 - 421.8(2.05) + 306.6$

$= -249.8386 + 819.4875 - 864.69 + 306.6$

$= 11.559$ (rounded off to three decimal places).

Example 2. Find the value of $f(1.6)$, using the following table :

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.50

Sol. The forward difference table is :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.60	
		0.54		
2.2	6.50			

Since $x = 1.6$ lies between 1.4 and 1.8 so take $x_0 = 1.4$ so that $y_0 = 4.82, \Delta y_0 = 1.14, \Delta^2 y_0 = -0.60$

Also $h = 0.4$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{1.6 - 1.4}{0.4} = 0.5$$

By Newton's forward difference formula,

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$\therefore \quad f(1.6) = 4.82 + 0.5 \times 1.14 + \frac{(0.5)(0.5-1)}{2}(-0.6)$$

$$= 4.82 + 0.57 + 0.075$$

$$= 5.465.$$

Example 3. Estimate from the following table the number of students who obtained marks between 40 and 45

Marks :	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of Students :	31	42	51	35	31

Sol. First we prepare cumulative frequency table as follows :

Marks above 30 but less than	No. of Students
x	y
40	31
50	73
60	124
70	159
80	190

The forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		– 25	
60	124		– 16		37
		35		12	
70	159		– 4		
		31			
80	190				

Here $x_0 = 40$ so that $y_0 = 31$, $\Delta y_0 = 42$, $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = -25$, $\Delta^4 y_0 = 37$

Also $h = 10$ and $x = 45$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

By Newton's forward interpolation formula,

$$y(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\therefore \quad y(45) = 31 + 0.5 \times 42 + \frac{(0.5)(0.5-1)}{2} \times 9 + \frac{(0.5)(0.5-1)(0.5-2)}{6} \times -25 + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$= 47.8672 \approx 48$ as number of students can not be in fractions.

$\therefore$ Number of students getting marks between 40 and 45 $= 48 - 31 = 17$.

Example 4. Given that $\sum_{11}^{20} f(x) = 44060$, $\sum_{14}^{20} f(x) = 38220$, $\sum_{17}^{20} f(x) = 27178$ and $f(20) = 8450$. Find the value of $f(11)$.

Sol. Let
$$u(x) = \sum_{x}^{20} f(x) \text{ so that}$$

$$u(11) = \sum_{11}^{20} f(x) = 44060$$

$$u(14) = \sum_{14}^{20} f(x) = 38220$$

$$u(17) = \sum_{17}^{20} f(x) = 27178$$

$$u(20) = \sum_{20}^{20} f(x) = f(20) = 8450$$

The forward difference table is :

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
11	44060			
		-5840		
14	38220		-5202	
		-11042		-2484
17	27178		-7686	
		-18728		
20	8450			

Now $$f(11) = \sum_{11}^{20} f(x) - \sum_{12}^{20} f(x) = u(11) - u(12)$$

To find $u(12)$, take $x_0 = 11$ so that $u_0 = 44060, \Delta u_0 = -5840, \Delta^2 u_0 = -5202, \Delta^3 u_0 = -2484.$

Also $h = 3$ and $x = 12$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{12-11}{3} = \frac{1}{3}$$

By Newton's forward interpolation formula ,

$$u(x) = u_0 + p\,\Delta u_0 + \frac{p(p-1)}{2!}\Delta^2 u_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 u_0$$

$$\therefore \quad u(12) = 44060 + \frac{1}{3}(-5840) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2}(-5202) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{6} \times (-2484)$$

$$= 44060 - 1946.67 + 578 - 153.33$$

$$= 42538$$

$$\therefore \quad f(11) = u(11) - u(12)$$

$$= 44060 - 42538 = 1522.$$

Example 5. Using Newton's backward difference formula, find a polynomial of degree 4 in x which satisfies the following data :

x :	1	2	3	4	5
y :	1	-1	1	-1	1

Sol. The backward difference table is :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

Take $x_n = 5$ so that $y_n = 1,\ \nabla y_n = 2,\ \nabla^2 y_n = 4,\ \nabla^3 y_n = 8,\ \nabla^4 y_n = 16$

Also $h = 1$

$$\therefore \quad p = \frac{x - x_n}{h} = x - 5.$$

By using Newton's backward difference formula,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 1 + (x-5)2 + \frac{(x-5)(x-4)}{2} \times 4 + \frac{(x-5)(x-4)(x-3)}{6} \times 8 + \frac{(x-5)(x-4)(x-3)(x-2)}{24} \times 1$$

$$= 1 + 2x - 10 + 2(x^2 - 9x + 20) + \frac{4}{3}(x^3 - 12x^2 + 47x - 60)$$

$$+ \frac{2}{3}(x^4 - 14x^3 + 71x^2 - 154x + 1$$

$$= \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56\,x + 31$$

which is the required polynomial of degree 4 in x.

Example 6. In the following table, the values of y are consecutive terms of a series of which 12.5 is the term. Find the first and tenth terms.

x :	3	4	5	6	7	8	9
y :	2.7	6.4	12.5	21.6	34.3	51.2	72.9

backw forwa

$p = \frac{x - x_n}{h}$ $p = \frac{x - x_0}{h}$

Sol. The difference table is :

x	y	1st Order	2nd Order	3rd Order	4th Order
		DIFFERENCES			
3	2.7				
		3.7			
4	6.4		2.4		
		6.1		0.6	
5	12.5		3.0		0
		9.1		0.6	
6	21.6		3.6		0
		12.7		0.6	
7	34.3		4.2		0
		16.9		0.6	
8	51.2		4.8		
		21.7			
9	72.9				

To find first term, we have to find value of y corresponding to $x = 1$. Since $x = 1$ is in the beginning of the table, so we shall use Newton's forward difference formula.

Take $x_0 = 3,\ y_0 = 2.7, \Delta y_0 = 3.7,\ \Delta^2 y_0 = 2.4,\ \Delta^3 y_0 = 0.6.$

Now $h = 1$ and $x = 1$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{1-3}{1} = -2.$$

So by Newton's forward difference formula,

$$y(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$\therefore \quad y(1) = 2.7 + (-2)(3.7) + \frac{(-2)(-2-1)}{2!}(2.4) + \frac{(-2)(-2-1)(-2-2)}{3!}(0.6)$$

$$= 2.7 - 7.4 + 7.2 - 2.4$$

$$= 0.1.$$

To find tenth term, we have to find the value of y corresponding to $x = 10$.

Since $x = 10$ is in the end of the table, so we shall use Newton's backward difference formula.

Take $x_n = 9,\ y_n = 72.9, \nabla y_n = 21.7,\ \nabla^2 y_n = 4.8,\ \nabla^3 y_n = 0.6.$

Now $h = 1$ and $x = 10$

$$\therefore \quad p = \frac{x - x_n}{h} = \frac{10-9}{1} = 1$$

By Newton's backward difference formula,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$\therefore \quad y(10) = 72.9 + (1)(21.7) + \frac{(1)(1+1)}{2!}(4.8) + \frac{(1)(1+1)(1+2)}{3!}(0.6)$$

$$= 72.9 + 21.7 + 4.8 + 0.6 = 100.$$

So first term = 0.1 and tenth term = 100.

EXERCISE 4.2

1. Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$. Find $\sin 52°$.

2. Given the table of values as :

x :	2.0	2.25	2.50	2.75	3.0
$y(x)$:	9.00	10.06	11.25	12.56	14.00

Find $y(2.35)$ using Newton's forward difference interpolation formula.

(G.N.D.U. B.Sc. I.T. April 200

3. From the following data estimate the number of students getting marks between 20 and 25

Marks below :	10	20	30	40	50
No. of Students :	20	45	115	210	325

(G.N.D.U. B.C.A. April 200

4. Find the number of men getting wages between Rs.10 and Rs.15 from the following data :

Wages (in Rs.) :	0 – 10	10 – 20	20 – 30	30 – 40
Frequency :	9	30	35	42

5. Given $\sum_{1}^{10} f(x) = 500426, \sum_{4}^{10} f(x) = 329240, \sum_{7}^{10} f(x) = 175212$ and $f(10) = 40365$ find $f(1)$.

6. Find $y(2.5)$ if the function $f(x)$ is given as

x :	0	1	2	3
$f(x)$:	0	2	8	27

(G.N.D.U. B.C.A. April 20

7. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

8. Find the cubic polynomial which takes the following values

$y(0) = 1$, $y(1) = 2$, $y(2) = 1$ and $y(3) = 10$.

Hence or otherwise obtain $y(4)$. (G.N.D.U. B.Sc. C.Sc. April 2003)

9. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978.

Year :	1941	1951	1961	1971	1981	1991
Population : (in thousands)	12	15	20	27	39	52

10. From the following table, find $f(0.7)$ approximately.

x :	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	2.68	3.04	3.38	3.68	3.96	4.21

11. From the following table, estimate the number of students who obtained marks between 75 and 80

Marks :	35 – 45	45 – 55	55 – 65	65 – 75	75 – 85
No. of Students :	18	40	64	50	28

(P.U. B.C.A. Sept. 2003)

12. The following table gives the values of y, which are consecutive terms of a series of which 36.2 is the 7th term. Find the first and tenth term of the series.

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

ANSWERS

1. 0.7880 2. 10.522 3. 32 4. 15
5. 58843.56 6. 15.3125 7. 8666 sq. units 8. $2x^3 - 7x^2 + 6x + 1$, 41
9. 2506 10. 4.32 11. 16
12. First term = 3.1, tenth term = 100

4.8 INTERPOLATION WITH UNEQUALLY SPACED POINTS

In section 4.6, we have discussed Newton's forward and backward difference interpolation formulae. But these formulae possess the disadvantage of requiring the values of arguments to be equally spaced. In this section, we shall discuss two such interpolation formulae which does not required function values at equal intervals of the arguments. These are :

(*i*) Lagrange interpolation formula.

(*ii*) Newton's divided difference interpolation formula.

4.8.1 LAGRANGE INTERPOLATION FORMULA

Procedure

Suppose that the function f is tabulated at $(n + 1)$ points $x_0, x_1, x_2, \ldots, x_n$ (not necessarily equidistant) and the corresponding values of the function f are $y_0, y_1, y_2, \ldots, y_n$.

Suppose that the function f is approximated by a polynomial

$$\phi_n(x) = a_0(x-x_1)(x-x_2)....(x-x_n) + a_1(x-x_0)(x-x_2).....(x-x_n) + a_2(x-x_0)(x-x_1)(x-x_3)$$
$$... (x-x_n) + ...+ a_n(x-x_0)(x-x_1)(x-x_2)\,.......\,(x-x_{n-1}) \quad ...(4.13)$$

of degree atmost n, such that $\phi_n(x)$ and $f(x)$ agree at the tabulated points. *i.e.* $f(x) \approx \phi_n(x)$ ...(4.14)

and $\quad \phi_n(x_j) = y_j$ for $j = 0, 1, 2, 3, ..., n$ ...(4.15)

The conditions given by (4.15) are called *interpolatory conditions.*

Now, imposing conditions (4.15) on equation (4.13), we have

$$\phi_n(x_0) = y_0 \Rightarrow a_0(x_0-x_1)(x_0-x_2)......(x_0-x_n) = y_0$$

$$\Rightarrow a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}$$

$$\phi_n(x_1) = y_1 \Rightarrow a_1(x_1-x_0)(x_1-x_2)....(x_1-x_n) = y_1$$

$$\Rightarrow a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)......(x_1-x_n)}$$

and so on.

Continuing in this way, we have

$$a_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)(x_n-x_2).....(x_n-x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2, ..., a_n$ in (4.13), we have

$$\phi_n(x) = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}\,y_1$$
$$+ \,.........\, + \frac{(x-x_0)(x-x_1)(x-x_2).......(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})}\,y_n \quad ...(4.16)$$

Using the product and summation notation, equation (4.15) can be written as

$$\phi_n(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \quad ...(4.17)$$

Now, using (4.14), we can write (4.17) as

$$f(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \quad ...(4.18)$$

The polynomial given by (4.16) (or (4.17)) or (4.18)) is called *Lagrangian polynomial* or *Lagrange interpolation formula.*

Inverse Interpolation

Rather than the value of a function $f(x)$ for a certain x, one might seek the value of x corresponding to a given value of $f(x)$; this is called *inverse interpolation.*

In Lagrange interpolation formula (4.17), interchanging x and y, we have

$$\phi_n(y) = \sum_{i=0}^{n} x_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(y - y_j)}{(y_i - y_j)} \qquad ...(4.19)$$

This polynomial given by (4.19) is called Lagrangian inverse polynomial and is used to find the value of x corresponding to any value of y.

Advantage of Lagrange Interpolation formula

(*i*) This formula is used to find the value of the function even when the arguments are not equally spaced.

(*ii*) This formula is used to find value of independent variable x corresponding to a given value of function (see equation 4.19).

Limitations of Lagrange Interpolation

(*i*) In the case of Newton interpolation formulae, the degree of the required approximating polynomial may be determined merely by computing terms until they no longer appear significant. But in the Lagrange interpolation, the degree of polynomial is chosen at the outset. So it is difficult to find the degree of approximating polynomial which is suitable for given set of tabulated points.

(*ii*) A change of degree in Lagrangian polynomial involves a completely new computation of all the terms.

(*iii*) For a polynomial of high degree, the formula involves a large number of multiplications which make the process quite slow.

Because of these limitations, Lagrange interpolation is rarely used in practice.

CHECKPOINTS

1. What is inverse interpolation ? (G.N.D.U. B.C.A. April 2007)
2. When is the Lagrange interpolation formula used in practical computation ?
3. What distinguishes the Lagrange formula from many other interpolation formulae ?
4. Why should the Lagrange formula be used in practice only with caution ?

ILLUSTRATIVE EXAMPLES

Example 1. From the following table, Interpolate the value of $y(x)$ using Lagrangian polynomial at (*i*) 2.8 (*ii*) 3.1

x :	2.0	3.0	4.0
$y(x)$:	6.6	9.2	8.6

(P.U. B.C.A. Sept. 2001; G.N.D.U. B.C.A. April 2007, B.Sc. C.Sc. Sept. 2007, B.Sc. I.T. April 2009)

$(-1)(-2)$
2

Sol. Here $x_0 = 2.0, \quad x_1 = 3.0, \quad x_2 = 4.0$

and $y_0 = 6.6, \quad y_1 = 9.2, \quad y_2 = 8.6.$

We know that Lagrangian polynomial is given by

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$x^2-4x-3x+12$
$x^2-7x+12$

$$= \frac{(x-3)(x-4)}{(2-3)(2-4)} \times 6.6 + \frac{(x-2)(x-4)}{(3-2)(3-4)} \times 9.2 + \frac{(x-2)(x-3)}{(4-2)(4-3)} \times 8.6$$

$$= 3.3\,(x^2 - 7x + 12) - 9.2(x^2 - 6x + 8) + 4.3(x^2 - 5x + 6)$$

$$= -1.6x^2 + x(-23.1 + 55.2 - 21.5) + (39.6 - 73.6 + 25.8)$$

$$= -1.6x^2 + 10.6x - 8.2$$

(*i*) $y(2.8) = -1.6\,(2.8)^2 + 10.6\,(2.8) - 8.2$

$= -12.544 + 29.68 - 8.2 = 8.936$

(*ii*) $y(3.1) = -1.6\,(3.1)^2 + 10.6\,(3.1) - 8.2$

$= -15.376 + 32.86 - 8.2 = 9.284.$

Example 2. Use Lagrange formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792

Sol. Here $x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 6$ and $y_0 = 648,\ y_1 = 704,\ y_2 = 729,\ y_3 = 792$

By Lagrange formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y$$

$$\therefore \quad f(x) = \frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)} \times 648 + \frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)} \times 704 + \frac{(x-0)(x-2)(x-6)}{(3-0)((3-2)(3-6)} \times 7$$

$$+ \frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)} \times 7$$

$$= -18\,(x^3 - 11x^2 + 36x - 36) + 88\,(x^3 - 9x^2 + 18x) + (-81)\,(x^3 - 8x^2 + 12x)$$

$$+ 11\,(x^3 - 5x^2 + 6$$

$= -x^2 + 30x + 648$, which is the required form of $f(x)$ as a polynomial in x.

Example 3. Certain corresponding values of x and $\log_{10}x$ are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find $\log_{10} 301$.

Sol. Let $y = f(x)$ where $f(x) = \log_{10} x$ Also $x_0 = 300$, $x_1 = 304$, $x_2 = 305$, $x_3 = 307$ and $y_0 = 2.4771$, $y_1 = 2.4829$, $y_2 = 2.4843$, $y_3 = 2.4871$.

By Lagrange formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\therefore \quad f(301) = \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)} \times 2.4771 + \frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)} \times 2.4829$$

$$+ \frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)} \times 2.4843$$

$$+ \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)} \times 2.4871$$

$$= \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)} \times 2.4771 + \frac{(1)(-4)(-6)}{(4)(-1)(-3)} \times 2.4829 + \frac{(1)(-3)(-6)}{(5)(1)(-2)} \times 2.4843$$

$$+ \frac{(1)(-3)(-4)}{(7)(3)(2)} \times 2.4871$$

$= 2.4786$ (rounded off to four decimal places)

Hence $\quad \log_{10} 301 = 2.4786$.

Example 4. Given the table of values :

x :	0.10	0.15	0.20	0.25
y :	0.200	0.300	0.500	0.600

Find x (0.150). (G.N.D.U. B.Sc. C.Sc. April 2004)

Sol. This is the problem of inverse interpolation.

Here Lagrangian inverse polynomial is given by

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$$\therefore \quad x(0.150) = \frac{(0.150)-0.300)(0.150-0.500)(0.150-0.600)}{(0.200-0.300)(0.200-0.500)(0.200-0.600)} \times 0.10$$

$$+\frac{(0.150-0.200)(0.150-0.500)(0.150-0.600)}{(0.300-0.200)(0.300-0.500)(0.300-0.600)} \times 0.15$$

$$+\frac{(0.150-0.200)(0.150-0.300)(0.150-0.600)}{(0.500-0.200)(0.500-0.300)(0.500-0.600)} \times 0.20$$

$$+\frac{(0.150-0.200)(0.150-0.300)(0.150-0.500)}{(0.600-0.200)(0.600-0.300)(0.600-0.500)} \times 0.25$$

$$= \frac{(-0.150)(-0.350)(-0.450)}{(-0.100)(-0.300)(-0.400)} \times 0.10 + \frac{(-0.050)(-0.350)(-0.450)}{(0.100)(-0.200)(-0.300)} \times 0.15$$

$$+\frac{(-0.050)(-0.150)(-0.450)}{(0.300)(0.200)(-0.100)} \times 0.20 + \frac{(-0.50)(-0.150)(-0.350)}{(0.400)(0.300)(0.100)} \times 0.25$$

$$= 0.196875 - 0.196875 + 0.1125 - 0.0546875$$

$= 0.058$ (rounded off to 3 decimal places).

Example 5. Using Lagrange interpolation formula, express the function $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ as sum of partial fractions.

Sol. Given function is $\dfrac{x^2+x-3}{x^3-2x^2-x+2} = \dfrac{x^2+x-3}{(x-1)(x-2)(x+1)}$.

We consider the numerator $f(x) = x^2 + x - 3$

Tabulating the values of $f(x)$ for $x = -1, 1, 2$, we get the following table :

x :	-1	1	2
$f(x)$:	-3	-1	3

Using Lagrange formula, we get

$$f(x) = \frac{(x-1)(x-2)}{(-1-1)(-1-2)} \times -3 + \frac{(x+1)(x-2)}{(1+1)(1-2)} \times -1 + \frac{(x+1)(x-1)}{(2+1)(2-1)} \times 3$$

$$\therefore \quad x^2 + x - 3 = -\frac{1}{2}(x-1)(x-2) + \frac{1}{2}(x+1)(x-2) + (x+1)(x-1)$$

Dividing both sides by $(x-1)(x-2)(x+1)$, we get

$$\frac{x^2+x-3}{x^3-2x^2-x+2} = -\frac{1}{2(x+1)} + \frac{1}{2(x-1)} + \frac{1}{x-2}$$

EXERCISE 4.3

1. Given $f(1) = 1, f(2) = 5, f(4) = 59$, obtain $f(3)$.

(G.N.D.U. B.Sc. I.T. April 2005)

2. Given the values of x and $f(x)$ as follows :

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using Lagrange formula.

3. Using Lagrange interpolation formula, find the value of y corresponding to $x = 10$ from the following table :

x :	5	6	9	11
y :	12	13	14	16

4. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Use Lagrange formula to find the value of $\log_{10} 656$.

5. If $y(1) = -3, y(3) = 9, y(4) = 30, y(6) = 132$, find the Lagrangian polynomial that takes the same values as y at the given points.

6. The following table gives the viscosity of oil as a function of temperature. Use Lagrange formula to find viscosity of oil at a temperature of 140°.

Temp. in °C :	110	130	160	190
Viscosity in poise :	10.8	8.1	5.5	4.8

7. Fit a polynomial of the second degree to the data points given in the following table :

x :	0	1.0	2.0
y :	1.0	6.0	17.0

(G.N.D.U. B.Sc. C.Sc. April 2002)

8. Find Lagrange interpolating polynomial when x_k and y_k are as given below :

x_k :	0	1	2	5
y_k :	2	3	12	147

(G.N.D.U. B.Sc. C.Sc. April 2006)

9. Given the table of values as

x :	20	25	30	35
$y(x)$:	0.342	0.423	0.500	0.650

Find $x(0.390)$.

10. Using Lagrange formula, express the function $\dfrac{x^2-10x+13}{(x-1)(x-2)(x-3)}$ as sums of partial fractions.

ANSWERS

1. 24.33
2. 810
3. 14.67
4. 2.8168
5. x^3-3x^2+5x-6
6. 7.03 poise
7. $3x^2+2x+1$
8. x^3+x^2-x+2
9. 22.841
10. $\dfrac{2}{x-1}+\dfrac{3}{x-2}-\dfrac{4}{x-3}$

4.8.2 NEWTON'S DIVIDED DIFFERENCE FORMULA

Since Lagrange interpolation formula is of little use in practice, so it is much more efficient to use Newton's divided difference formula to interpolate a tabulated function especially if the arguments are not equally spaced. Moreover, the use of divided difference is relatively safe since the degree of interpolating polynomial can be decided. First of all, we shall study divided differences.

Divided Differences

Suppose that the function f is tabulated at $(n+1)$ points $x_0, x_1, x_2, \ldots, x_n$ (not necessarily equidistant) and the corresponding values of function f are $y_0, y_1, y_2, \ldots, y_n$. Then $\dfrac{y_1-y_0}{x_1-x_0}, \dfrac{y_2-y_1}{x_2-x_1}, \dfrac{y_3-y_2}{x_3-x_2}, \ldots, \dfrac{y_n-y_{n-1}}{x_n-x_{n-1}}$ are called *first order divided differences* and are denoted by $\Delta_d y_0, \Delta_d y_1, \Delta_d y_2, \ldots, \Delta_d y_{n-1}$.

i.e. $\Delta_d y_0 = \dfrac{y_1-y_0}{x_1-x_0}, \Delta_d y_1 = \dfrac{y_2-y_1}{x_2-x_1}, \ldots, \Delta_d y_{n-1} = \dfrac{y_n-y_{n-1}}{x_n-x_{n-1}}$, where Δ_d is called the *divided difference operator.*

In general, $\Delta_d y_r = \dfrac{y_{r+1}-y_r}{x_{r+1}-x_r}$

The differences $\dfrac{\Delta_d y_1-\Delta_d y_0}{x_2-x_0}, \dfrac{\Delta_d y_2-\Delta_d y_1}{x_3-x_1}, \ldots, \dfrac{\Delta_d y_{n-1}-\Delta_d y_{n-2}}{x_n-x_{n-2}}$ are called *second order divided differences* and are denoted by $\Delta_d^2 y_0, \Delta_d^2 y_1, \ldots, \Delta_d^2 y_{n-2}$.

i.e.

$$\Delta_d^2 y_0 = \frac{\Delta_d y_1-\Delta_d y_0}{x_2-x_0}, \Delta_d^2 y_1 = \frac{\Delta_d y_2-\Delta_d y_1}{x_3-x_1}, \ldots, \Delta_d^2 y_{n-2}$$

$$= \frac{\Delta_d y_{n-1}-\Delta_d y_{n-2}}{x_n-x_{n-2}}.$$

In general $\Delta_d^2 y_r = \dfrac{\Delta_d y_{r+1} - \Delta_d y_r}{x_{r+2} - x_r}$

Similarly, higher order divided differences can be defined. In general, the jth order divided difference is given by

$$\Delta_d^j y_r = \frac{\Delta_d^{j-1} y_{r+1} - \Delta_d^{j-1} y_r}{x_{r+j} - x_r}.$$

Divided Difference Table

The table listing all the divided differences is given below :

x	y	$\Delta_d y$	$\Delta_d^2 y$	$\Delta_d^3 y$	$\Delta_d^4 y$
x_0	y_0				
		$\Delta_d y_0$			
x_1	y_1		$\Delta_d^2 y_0$		
		$\Delta_d y_1$		$\Delta_d^3 y_0$	
x_2	y_2		$\Delta_d^2 y_1$		$\Delta_d^4 y_0$
		$\Delta_d y_2$		$\Delta_d^3 y_1$	
x_3	y_3		$\Delta_d^2 y_2$		
		$\Delta_d y_3$			
x_4	y_4				

Table 4.6

Here each entry in the difference table is given by the difference between diagonally adjacent entries to its left divided by the difference between the values of x corresponding to the values of y intercepted by the diagonals passing through the calculated entry.

e.g. in Table 4.6, $\Delta_d^3 y_1$ is given by

$$\Delta_d^3 y_1 = \frac{\Delta_d^2 y_2 - \Delta_d^2 y_1}{x_4 - x_1}.$$

Another Notation for Divided Differences

We can also denote the first order divided differences

$\Delta_d y_0, \Delta_d y_1, \ldots, \Delta_d y_{n-1}$ by $f(x_0, x_1), f(x_1, x_2), \ldots, f(x_{n-1}, x_n)$

or $[x_0,x_1],[x_1,x_2],....,[x_{n-1},x_n]$

i.e. $\Delta_d y_0 = f(x_0,x_1) = [x_0,x_1]$, $\Delta_d y_1 = f(x_1,x_2) = [x_1,x_2]$ and so on.

Similarly second order divided differences can also be denoted as

$$\Delta_d^2 y_0 = f(x_0,x_1,x_2) = [x_0,x_1,x_2],\ \Delta_d^2 y_1 = f(x_1,x_2,x_3) = [x_1,x_2,x_3] \text{ and so on.}$$

The nth divided difference can also be denoted by

$$\Delta_d^n y_0 = f(x_0,x_1,x_2,....,x_n) = [x_0,x_1,x_2,....,x_n]$$

Relation between Divided Differences and forward differences

Let the function f be tabulated at $(n+1)$ equally spaced values $x_0,x_1,x_2,...,x_n$ and corresponding values of f be $y_0,y_1,y_2,....,y_n$.

Since the arguments are equally spaced

so $$x_1 - x_0 = x_2 - x_1 = ... = x_n - x_{n-1} = h \text{ (say)}$$

$$\therefore \quad \Delta_d y_0 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$\Delta_d^2 y_0 = \frac{\Delta_d y_1 - \Delta_d y_0}{x_2 - x_0} = \frac{1}{2h}\left[\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right] = \frac{1}{2h^2}\Delta^2 y_0$$

$$\therefore \quad \Delta_d^2 y_0 = \frac{1}{h^2\, 2!}\Delta^2 y_0.$$

In general, $$\Delta_d^n y_0 = \frac{1}{h^n \cdot n!}\Delta^n y_0. \quad ...(4.20)$$

Remark : We know that if the tabulated function is a polynomial of nth degree then $\Delta^n y_0$ would be a constant.

So, by using relation (4.20), the nth order divided difference would also be a constant.

Procedure for Newton's divided difference formula

Let the function f be tabulated at $(n+1)$ points $x_0,x_1,x_2,....,x_n$ (not necessarily equidistant) and the corresponding values of f be $y_0,y_1,y_2,....,y_n$.

Let $f(x)$ be approximated by a polynomial

$$\phi_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + ... + a_n(x-x_0)(x-x_1)....(x-x_{n-1}) \quad ...(4.21)$$

of degree n such that $f(x)$ and $\phi_n(x)$ agree at the tabulated points *i.e.* $f(x) \approx \phi_n(x)$ ...(4.22)

and $$\phi_n(x_i) = y_i \quad \text{for } i = 0, 1, 2, ..., n \quad ...(4.23)$$

Now, imposing conditions (4.23) on equation (4.21), we have

$$\phi_n(x_0) = y_0 \Rightarrow a_0 = y_0$$

$$\phi_n(x_1) = y_1 \Rightarrow a_0 + a_1(x_1 - x_0) = y_1$$

$$\phi_n(x_2) = y_2 \Rightarrow a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = y_2 \text{ and so on.}$$

Solving above equations for $a_0, a_1, a_2, \ldots\ldots, a_n$, we have

$$a_0 = y_0,$$

$$a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0},$$

$$a_2 = \frac{y_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - y_0 - \dfrac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{(y_2 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2 x_1 - y_2 x_0 - y_0 x_1 + x_0 y_0 - y_1 x_2 + y_1 x_0 + y_0 x_2 - x_0 y_0}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{(y_2 x_1 - y_2 x_0 - x_1 y_1 + y_1 x_0) - (y_1 x_2 - x_2 y_0 - x_1 y_1 + x_1 y_0)}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{(y_2(x_1 - x_0) - y_1(x_1 - x_0)) - (x_2(y_1 - y_0) - x_1(y_1 - y_0))}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{(y_2 - y_1)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_1)}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right] = \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

and so on.

Using divided differences, we have,

$$a_0 = y_0,$$

$$a_1 = \Delta_d y_0,$$

$$a_2 = \frac{\Delta_d y_1 - \Delta_d y_0}{x_2 - x_0} = \Delta_d^2 y_0 \text{ and so on.}$$

Continuing in this way, we have

$$a_n = \Delta_d^n y_0$$

Substituting the values of $a_0, a_1, a_2, \ldots, a_n$ in (4.21), we have

$$\phi_n(x) = y_0 + (x-x_0)\Delta_d y_0 + (x-x_0)(x-x_1)\Delta_d^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta_d^3 y_0 + \ldots + (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})\Delta_d^n y_0 \quad \ldots(4.24)$$

Using relation (4.22), we can also write (4.24) as

$$f(x) = y_0 + (x-x_0)\Delta_d y_0 + (x-x_0)(x-x_1)\Delta_d^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta_d^3 y_0 + \ldots + (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})\Delta_d^n y_0 \quad \ldots(4.25)$$

Formula given by (4.24) (or (4.25)) is called Newton's divided difference formula or Newton's general interpolation formula.

This formula is useful especially when the function is tabulated at unequal intervals.

Note This formula is derived on the assumption that $x_0 < x < x_1$.

4.9 CHOICE OF AN INTERPOLATION METHOD

In the preceding sections, we have derived some interpolation formulae which are of great importance. To get accurate results in interpolation, one should make proper choice of the formula to be used. Following are some points which would help us to make proper choice of the formula.

1. If the function is tabulated at arguments with equal interval then we can use Newton's forward difference formula or Newton's backward difference formula. Further
   (*a*) if interpolation is desired near the beginning of the table then we can use Newton's forward difference formula.
   (*b*) if interpolation is desired near the end of the table, then we can use Newton's backward difference formula.
   (*c*) if interpolation is desired near the middle of the table then central difference formulae like Gauss formula, Stirling's formula or Bessel's formula can be used. (Central difference formulae are out of scope of this book)
2. If the function is tabulated at arguments with unequal intervals then we can use either Lagrange interpolation formula or Newton's divided difference formula.

Further, as discussed earlier, Newton's divided difference formula is always preferred over Lagrange interpolation formula.

4.10 TRUNCATION ERROR IN INTERPOLATION

In the process of interpolation, we fit a polynomial of a finite degree on a set of tabulated values which introduces error in interpolation. This error is called the truncation error.

Suppose that the function f is tabulated at $(n+1)$ points $x_0, x_1, x_2, ..., x_n$ and corresponding values of the function f are $y_0, y_1, y_2, ..., y_n$.

If we approximate the function f by a polynomial $\phi_n(x)$ of degree n then truncation error is given by

$$\text{Truncation Error} = f(x) - \phi_n(x) = \frac{(x-x_0)(x-x_1)...(x-x_n)}{(n+1)!} f^{(n+1)}(\xi),\ x_0 < \xi < x_n$$

where $f^{(n+1)}(\xi)$ is the $(n+1)$th derivative of the function f at ξ.

(*a*) For Newton's forward difference formula,

$$\text{Truncation error} = \frac{p(p-1)(p-2).....(p-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi) \text{ where } p = \frac{x-x_0}{h}$$

(*b*) For Newton's backward difference formula,

$$\text{Truncation error} = \frac{p(p+1)(p+2).....(p+n)}{(n+1)\,!} h^{n+1} f^{(n+1)}(\xi) \text{ where } p = \frac{x-x_n}{h}$$

(*c*) For Lagrange interpolation formula,

$$\text{Truncation error} = \frac{(x-x_0)(x-x_1)...(x-x_n)}{(n+1)!} f^{(n+1)}(\xi)$$

(*d*) For Newton's divided difference formula,

$$\text{Truncation error} = \frac{(x-x_0)(x-x_1).....(x-x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

CHECKPOINTS

1. Define the term Divided Difference. (G.N.D.U. B.C.A. April 2007)
2. What major advantage has Newton's divided difference interpolation formula over Lagrange formula ?
3. Discuss how the choice of an interpolation formula should be made ?
4. Discuss truncation error involved in interpolation.

(G.N.D.U. B.C.A. April 2007)

ILLUSTRATIVE EXAMPLES

Example 1. Prepare the divided difference table for the following data :

x :	1	3	4	6	10
$f(x)$:	0	18	58	190	920

Sol. The divided difference table is :

x	$f(x)$	$\Delta_d f(x)$	$\Delta_d^2 f(x)$	$\Delta_d^3 f(x)$	$\Delta_d^4 f(x)$
1	0				
		$\frac{18-0}{3-1}=9$			
3	18		$\frac{40-9}{4-1}=10.333$		
		$\frac{58-18}{4-3}=40$		$\frac{8.667-10.333}{6-1}=-0.333$	
4	58		$\frac{66-40}{6-3}=8.667$		$\frac{1.536+0.333}{10-1}=0.208$
		$\frac{190-58}{6-4}=66$		$\frac{19.417-8.667}{10-3}=1.536$	
6	190		$\frac{182.5-66}{10-4}=19.417$		
		$\frac{920-190}{10-6}=182.5$			
10	920				

Example 2. Given the values of x and $f(x)$ as follows :

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Find $f(9)$ using Newton's divided difference formula.

Sol. The divided difference table is :

x	$f(x)$	$\Delta_d f(x)$	$\Delta_d^2 f(x)$	$\Delta_d^3 f(x)$	$\Delta_d^4 f(x)$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

By Newton's divided difference formula,

$$f(x) = f_0 + (x-x_0)\Delta_d f_0 + (x-x_0)(x-x_1)\Delta_d^2 f_0 + (x-x_0)(x-x_1)(x-x_2)\Delta_d^3 f_0$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta_d^4 f$$

$$\therefore \quad f(9) = 150 + (9-5)\,121 + (9-5)\,(9-7)\,24 + (9-5)\,(9-7)\,(9-11)\,1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 3. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ from the following data :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(G.N.D.U. B.Sc. C.Sc. April 2002, B.C.A. April 2004; P.U. B.C.A. Sept. 2005)

Sol. The divided difference table is :

x	$y=f(x)$	$\Delta_d y$	$\Delta_d^2 y$	$\Delta_d^3 y$	$\Delta_d^4 y$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

By Newton's divided difference formula,

$$f(x) = y_0 + (x-x_0)\Delta_d y_0 + (x-x_0)(x-x_1)\Delta_d^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta_d^3 y_0$$

$$= 48 + (x-4)\,52 + (x-4)(x-5)\,15 + (x-4)(x-5)(x-7)\,1$$

$$= 48 + 52x - 208 + 15(x^2 - 9x + 20) + (x^3 - 16x^2 + 83x - 140)$$

$$= x^3 - x^2$$

So $\quad f(x) = x^3 - x^2$

Hence $\quad f(8) = 8^3 - 8^2 = 448$

and $\quad f(15) = 15^3 - 15^2 = 3150$

EXERCISE 4.4

1. Prepare a divided difference table for the following data :

x :	1	2	4	7	12
$f(x)$:	22	30	82	106	216

2. Prepare a divided difference table for the following data :

x :	1	3	6	10	11
$f(x)$:	3	31	223	1011	1343

3. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

4. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$ $f(9) = 13104$, find $f(x)$ as a polynomial in x.

5. Determine $f(x)$ as a polynomial in x for the following data :

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

6. Use Newton's divided difference formula to find $f(x)$ from the following data :

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

7. Given the table of values as

x :	2.5	3.0	4.5	4.75	6.0
$f(x)$:	8.85	11.45	20.66	22.85	38.60

Find $f(3.5)$.

8. The following table is given :

x :	0	1	2	5
F (x) :	2	3	12	147

What is the form of F(x) ? (G.N.D.U. B.Sc. I.T. April 2007)

ANSWERS

3. 1
4. $x^5 - 9x^4 + 18x^3 - x^2 + 9x - 18$
5. $3x^4 - 5x^3 + 6x^2 - 14x + 5$
6. $x^3 - 9x^2 + 21x + 1$
7. 14.04
8. $x^3 + x^2 - x + 2$

MISCELLANEOUS EXERCISE

1. Prove that $e^x = \left(\frac{\Delta^2}{E}\right) e^x \frac{E e^x}{\Delta^2 e^x}$, the interval of differencing being unity.

(P.U. B.C.A. Sept. 2004)

2. From the following table, find y when $x = 1.85$ and 2.4 by Newton's interpolation formulae.

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

3. From the following data, estimate the values of $f(22)$ and $f(42)$

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

4. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$. Find u_2 and u_4.

5. Given $y_0 = -12$, $y_1 = 0$, $y_3 = 6$ and $y_4 = 12$. Find y_2.

6. If $y_1 = 4$, $y_3 = 12$, $y_4 = 19$ and $y_x = 7$, find x.

7. Using Lagrange formula, express the function $\frac{x^2 + 6x - 1}{(x^2 - 1)(x-4)(x-6)}$ as a sum of partial fractions.

8. Given the table of values

x :	150	152	154	156
$\sqrt{x}$:	12.24745	12.32883	12.40967	12.49000

Compute $\sqrt{155}$ using Lagrange interpolation formula. Can some other interpolation formula be used more profitably ? Is so, name the formula and the advantages you expect.

(P.U. B.C.A. April 2005; G.N.D.U. B.Sc. C.Sc. April 2007)

9. Find the form of the function F(x) and hence find F(5) by Lagrange interpolation formula. Given

x :	1	3	4	6	10
F (x) :	0	18	48	180	900

(G.N.D.U. B.Sc. I.T. April 2008)

10. Determine by Lagrange formula, the percentage number of criminals under age 35 years :

Age :	Under 25 years	Under 30 years	Under 40 years	Under 50 years
% number of Criminals :	52	67.3	84.1	94.4

(G.N.D.U. B.C.A. April 2009)

11. The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable ?

12. Find the equation of the cubic curve which passes through the points (4, −43), (7, 83), (9, 327) and (12, 1053).

13. Deduce Newton's forward difference interpolation formula as a special case of Newton's divided difference formula.

14. Find the polynomial of lowest possible degree which assumes the values 3, 12, 15, − 21 when x has the values 3, 2, 1 and − 1 respectively.

(G.N.D.U. B.Sc. I.T. April 2008, B.C.A. April 2009)

ANSWERS

2. $y(1.85) = 6.360,\ y(2.4) = 11.056$

3. $f(22) = 352.2, f(42) = 218.66$

4. $u_2 = 42,\ u_4 = 49$

5. 4

6. 1.86

7. $\frac{3}{35(x+1)} + \frac{1}{5(x-1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}$

8. 12.44990

9. $x^3 - x^2$, 100

10. 77.405

11. 147

12. $x^3 - 4x^2 - 7x - 15$

14. $x^3 - 9x^2 + 17x + 6$