Probabilities in Blackjack

Akaash Parthasarathy

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1 Introduction and Rules

Blackjack is a card game that is symmetric for the player and the dealer and one of the few games where the player can obtain an edge over the house with optimal play. In the casino version of blackjack, the house acts as the dealer. The objective of the game is to beat the dealer by obtaining a total card value (count) as close to 21 as possible but not exceeding 21. While a standard 52-card deck is used, it is often the case that several decks are shuffled together. As far as the card values for each card go, aces are worth 1 or 11 based on the individual player's choice, face cards are worth 10, and the remaining cards are worth their numbered value.

Once players have placed their bets, the dealer deals one card face-up to each player including themselves in clockwise rotation. The dealer then deals a second round of cards with each of the players receiving a second card face up. In the second round, the dealer's card is face down. If the dealer's face up card has value 10 or is an ace, they check their face down card to see if they have a natural or a "blackjack" (the first two cards they received have a total value of 21). If the dealer has a natural, they immediately collect the bets of all players who do not have naturals. Excluding naturals, play proceeds in clockwise rotation with each player deciding either to "stand" (not ask for another card) or "hit" (ask for another card) in each round. The player may thus decide to stand on the two cards initially dealt to them or hit, once each round, until they eventually stand on a total card value less than or equal to 21 or "bust" by obtaining a total card value over 21.

After every player has either decided to stand or hit, the dealer turns their face-down card up. If the dealer's total is 17 or higher, they must stand. On the other hand, if the dealer's total is 16 or lower, they must take another card and continue to do so until their total card value is at least 17. If the dealer has an ace and counting the ace as 11 causes their total to be at least 17 but not exceed 21, they must stand. Notice that the dealer does not have any choice with respect to standing or hitting and must follow pre-defined rules whereas the player has a choice in each round.

Suppose that the player places a bet B at the beginning of a hand. The player makes a profit of $\frac{3}{2}B$ if they are dealt a natural (except when the dealer is also dealt a natural). If neither the player nor the dealer has a natural, the player makes a profit of B when they have a final total higher than the dealer's total without having busted. The player loses their bet if they bust or have a lower total than the dealer.

1.1 Doubling Down

There is an additional decision that a player can make when dealt the initial hand. A player can choose to "double down" by doubling their initial bet. In such a case, the dealer deals the player exactly one more card, but they are not allowed to subsequently hit. We will not be dealing with doubling down in depth.

1.2 Soft Hands and Splits

A soft hand is a hand that contains an ace (a hard hand is one that does not). For instance, a soft 17 refers to the case when a player's initial hand consists of an ace and a 6 whereas a hard 17 refers to the case when a player's initial hand has a total card value of 17 without an ace. Since an ace is treated as either a 1 or an 11 depending on the player's card total, this case must be treated separately.

A player's initial hand might also have two cards of the same value (for instance, a 9 and a 9). In such a case, a player may choose to split the hand by placing an additional bet equal to their initial bet and play two different hands. We will not be dealing with splits in depth.

2 Probabilities in Blackjack

For simplicity, assume you are the only player apart from the dealer. Assume that cards are drawn from a single standard 52-card deck (we will later generalize our discussion to n decks). Suppose also that the cards are numbered from 1 to 52 in the following manner. The hearts cards $A, 2, \ldots, K$ are labeled from 1 to 13 in order, the diamonds cards $A, 2, \ldots, K$ are labeled from 14 to 26 in order, the spades cards $A, 2, \ldots, K$ are labeled from 27 to 39 in order, and the clubs cards $A, 2, \ldots, K$ are labeled from 40 to 52. Note that the actual suits of the cards do not matter.

Let $C = \{1, 2, \ldots, 52\}$ represent the above set of cards. Then, the set of possible outcomes for your hand after the first two cards are dealt is some subset of $C' = \{(\omega_1, \omega_2) \mid \omega_1, \omega_2 \in S, \omega_1 \neq \omega_2\}$. Notice that, because of the mechanics of blackjack, the sum of the values of the initial 2 cards can never exceed 21. The values of each card can be modeled by the function $v : C \to \mathbb{N}$ by

$$v(c) = \begin{cases} 11 \text{ if } c = 1 \\ c \text{ if } 1 < c < 10 \\ 10 \text{ if } c \geq 10 \end{cases}$$

As the game progresses, your hand changes. Supposing the number of cards in your hand is k. Then, the set of possible outcomes of your hand becomes a subset of $C'' = \{(\omega_1, \omega_2, \ldots, \omega_k) \mid \omega_1, \omega_2, \ldots, \omega_k \in C$ and $\omega_i \neq \omega_j, i \neq j$ and $\omega_1 + \omega_2 + \cdots + \omega_k \leq 21\}$. Of course, given that you are playing against the dealer and the dealer's hand consists of cards from the same deck, your actual hand also necessarily depends on the cards in the dealer's hand.

We then define probability measures on the measure spaces C' and C''. Let $\pi_1 : \mathcal{P}(C') \to [0, 1]$ by $\pi_1(\{(\omega_1, \omega_2)\}) = \frac{1}{52} \cdot \frac{1}{51}$ and $\pi_2 : \mathcal{P}(C'') \to [0, 1]$ by $\pi_2(\{(\omega_1, \omega_2)\}) = \frac{1}{52} \cdot \frac{1}{51} \cdot \cdots \cdot \frac{1}{52-k+1}$. Note that, given a single 52-card deck, the maximum value of k (the maximum number of cards that a player could be dealt without busting) is 11 (4 aces, 4 2s, and 3 3s). This number increases when multiple decks are used. However, casinos typically implement a rule called "Charlie Rule" that causes a player to automatically win once they obtain some fixed k number of cards in their hand without busting.

2.1 Favorable Initial Hands

An initial hand where the two card values sum up to a total of at least 18 is generally considered favorable.

We first determine the probability of obtaining a natural. The only way to obtain a score of 21 is if one of the cards is an ace and the other is a face card or a 10. Since there are 4 aces and 16 face cards or 10s and the order in which these cards are dealt does not matter, there are $2! * 4 * 16 = 128$ ways to obtain a natural. In particular, the set modeling the outcomes for obtaining a natural is given by

$$S_1 = \{(\omega_1, \omega_2) \in C' \mid v(\omega_1) + v(\omega_2) = 21\}$$

with $|S_1| = 128$. Notice that since $S_1 \subset C'$, we have $\pi_1(S_1) = |S_1| \cdot \pi_1(\omega) = \frac{128}{52*51} = \frac{32}{663} \approx 0.048265 \approx 4.83\%$. You would thus expect to obtain a natural about 4.83% of the time or about once in every 21 hands.

Similarly, a total of 20 can be obtained by either two cards of value 10 or a 9 and an ace. There are 4 aces, 16 face cards or 10s, and 4 9s and the order in which the cards are dealt does not matter. Thus, there are $2! * (4 * 4 + \binom{16}{2}) = 272$ ways to obtain a total of 20. In this situation, the set modeling the outcomes for obtaining a card total of 20, S_2, must satisfy $|S_2| = 272$. Again, since $S_2 \subset C'$, we have $\pi_1(S_2) = \frac{|S_2|}{52*51} = \frac{68}{663} \approx 0.10256 \approx 10.3\%$.

We can similarly determine the probabilities of obtaining card totals of 19, 18, and 17. The probability of obtaining a card total of 19 is $\frac{40}{663}$ or about 6.03%, the probability of obtaining a card total of 18 is $\frac{43}{663}$ or about 6.49%, and the probability of obtaining a card total of 17 is $\frac{16}{221}$ or about 7.24%.

2.2 The Plot Thickens

Probabilities in blackjack are necessary conditional. As the game progresses, the probabilities of each outcome must be calculated based on the cards that have been played so far. Again, assuming you are the only player, this will require counting cards that show for the dealer and keeping track of the cards that you have been previously dealt. As such, the probability of the player winning a hand changes with each card dealt.

Suppose that n cards have been seen (we do not consider the dealer's face-down card for the calculation since we do not know its value) so far and suppose that these cards are given by $S_1 = \{c_1, c_2, \ldots, c_n\}$, where $1 \le c_i \le 52$ for each c_i. Then, the remaining cards are given by the set $S_2 = S \setminus S_1$. We define a probability measure on measure space S_2 by $\pi : \mathbb{P}(S_2) \to [0,1]$ so that $\pi(\{\omega\}) = \frac{1}{|S_2|} = \frac{1}{52-n}$. Suppose that $n(x)$ is the number of x-valued cards already dealt. Then, there is a simple formula for determining the probability that the next card dealt has value x. Let $p(x)$ represent this probability. When x is 10, we have

$$p(x) = (16 - n(x)) \cdot \pi(\{x\}) = \frac{16 - n(x)}{52 - n}$$

and when x is not 10, we have

$$p(x) = (4 - n(x)) \cdot \pi(\{x\}) = \frac{4 - n(x)}{52 - n}$$

Note that if m is the number of complete decks being used, we can modify the measure space to reflect all $52m - n$ cards. In such a case, these formulae thus change to

$$p(x) = \frac{16m - n(x)}{52m - n}$$

when x is 10 and

$$p(x) = \frac{4m - n(x)}{52m - n}$$

when x is not 10.

Let us consider an example of the application of the above formulae. Suppose, for simplicity, that the game is being played with one deck and you are the only player other than the dealer. Suppose you hold the cards A, 3, 10, and 4 (totaling 18), and the dealer's face-up card is a 3. We may now calculate the probability $p(3)$ of obtaining a total of 21 (getting a 3) using $n = 5$ and $n(3) = 2$ as

$$p(3) = \frac{4 - n(3)}{52 - n} = \frac{2}{47} \approx 0.043$$

Similarly, since $n(2) = 0$, the probability $p(2)$ of obtaining a total of 20 (getting a 2) is

$$p(2) = \frac{4 - n(2)}{52 - n} = \frac{4}{47} \approx 0.085$$

Thus, the probability of obtaining a total of either 20 or 21 points is the sum of these probabilities $p(2)+p(3) = \frac{6}{47} \approx 0.128$. In other words, there is a 12.8% chance of obtaining a total of 20 or 21 points when the next card is dealt.

Although these formulae provide a heuristic to determine whether or not you should hit or stand, real games are often played using a random cut from several decks of cards. As such, you will likely be unaware of the actual number of x-valued cards in the stack of cards. This ties in nicely with the idea of approximation-based approaches to playing blackjack, which we discuss subsequently.

3 Basic Strategy

The intuition behind the basic strategy or the optimal fixed strategy (and, in fact, any strategy) is to always make plays that result in positive expected value (a net profit). When initially dealt their cards, the player must make a number of decisions regarding their next play. In general, the player's play depends on the cards they hold, the dealer's face-up card and all the other cards that have been dealt (seen) so far. The basic strategy focuses specifically on making decisions based on the player's hand and the dealer's up card. Note that calculating the probabilities and expectations necessary to explain the basic strategy by hand is quite difficult. The probabilities and expectations in this section have largely been calculated via simulations using Python (see Appendix for more details).

Table 1: Simulated probabilities of results for the dealer

Result	17	18	19	20	21	natural	bust
Probability	0.145412	0.137212	0.134406	0.176096	0.07427	0.047934	0.28467

Table 1 lists the probabilities of all possible results for the dealer over 500000 simulations. Notably, the dealer busts about 28.4% of the time (slightly more than once every four hands). If a player mimics the prescribed rules for the dealer, hitting until they reach a total of 17 or greater, the probability distribution for the final value of their hand is the same as that in 1. However, since the rules for the dealer and the player are not the same, the expectation for the player following this strategy is not 0. Specifically, the player loses if they bust regardless of the outcome of the dealer. Based on simulations, the proportion of player wins (excluding naturals) is 0.366494, naturals is 0.045674, losses is 0.492204, and ties is 0.095628. The expectation for the player using this strategy can thus be approximated by $\sum_i x_i P(x_i) = 1 \cdot 0.366494 + 1.5 \cdot 0.095628 + (-1) \cdot 0.492204 + 0 \cdot 0.103204 \approx -0.0572$.

In making decisions, the player should thus use the information given by their own hand and the dealer's face-up card and compare the expected profit/loss for each possible choice (hitting or standing, doubling down or not, splitting or not). The highest expectation calculated in this manner must give the best choice for the player. Note that in our analysis, we make two primary assumptions. First, the player uses a fixed strategy that is not modified based on the distribution of cards at the beginning of each hand. That is, the player always follows the same strategy regardless of the cards seen so far. Second, the effect of previously dealt cards on the deck is negligible. This is commonly known as the "infinite-deck" approximation, which assumes the following:

1. Each card with value 10 has probability 4/13 of showing
2. Each card with value not 10 has probability 1/13 of showing
3. Cards already dealt (not in the player's or dealer's hand) are ignored

Of course, such an approximation is only adequate when the deck has an infinite number of cards for otherwise the probabilities of cards in the deck showing would change as the game progressed. Given a sufficient number of decks of cards (as is the case in several casinos), however, the strategy devised using this assumption approximates the optimal fixed strategy.

Notice that the dealer's strategy is pre-determined and can be approximated using a Markov process when we ignore the effect of the previously dealt cards. This is because the dealer's subsequent state then becomes dependent only on the current state and is independent of how the current state was achieved. We can model the dealer's strategy as follows. Suppose that $p_{i,j}$ represents the one-step transition probability from state i to state j and $\pi_{i,j}$ represents the (absorption) probability of the dealer eventually reaching state j starting from state i. Let D be the dealer's state space (the set of all possible total values of the dealer's hand). Then, π satisfies the following equations

$$\pi_{i,j} = \sum_k p_{i,k} \pi_{k,j} \ \forall i, j \in D$$

$$\pi_{i,i} = 1 \ \forall i \in \{17, 18, 19, 20, 21, \text{natural}, \text{bust}\}$$

$$\sum_j \pi_{i,j} = 1 \ \forall i \in D$$

The state space D consists of the following elements:

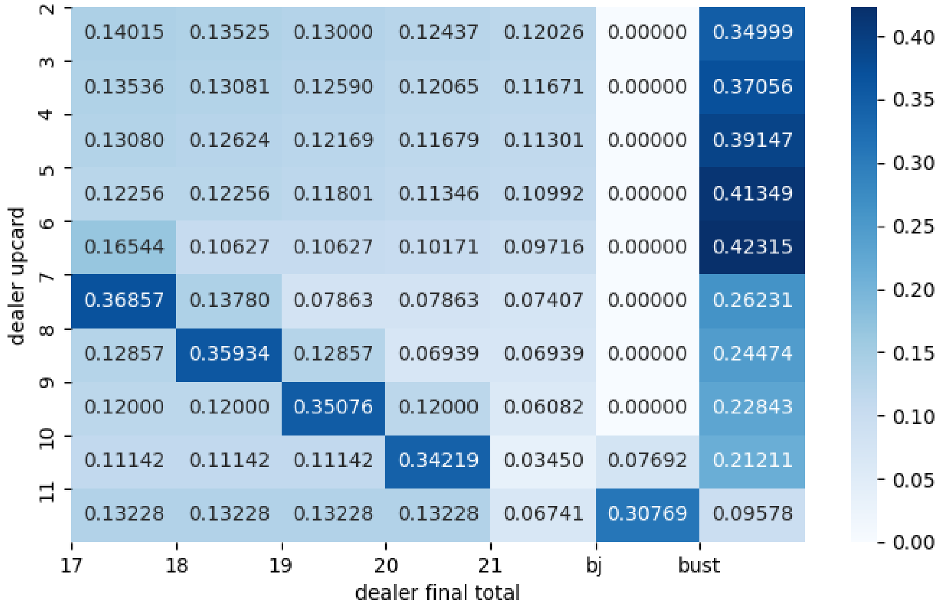

Figure 1: Probabilities of dealer's results conditioned on the dealer's face-up card

$\{f_i$ for $i \in \{2, \ldots, 11\}\}$: the dealer holds only one card of value i (this is the dealer's face-up card).
$\{h_i$ for $i \in \{4, \ldots, 16\}\}$: the dealer holds a hard total of i.
$\{s_i$ for $i \in \{12, \ldots, 16\}\}$: the dealer holds a soft total of i.
$\{d_i$ for $i \in \{17, \ldots, 21\}\}$: the dealer stands with a total of i.
bj: the dealer holds a natural.
bust: the dealer's total is greater than 21 (the dealer has busted).

The dealer's strategy then corresponds to a random walk on D. The initial state for this random walk is the dealer's up-card and each card drawn from the deck corresponds to a transition in the state space.

The heatmap in 1 provides the probabilities of the dealer's final total based on the dealer's face-up card. More details about how these probabilities were calculated are provided in the Appendix. Notice that a clear pattern emerges. When the dealer's face-up card is low (between 2 and 6), the most likely outcome for the dealer is to bust. In such situations, the player must play as though the dealer will bust. On the other hand, for face-up cards with values between 7 and 11, the most likely dealer total is the sum of the value of their face-up card and 10 (that is, the dealer will likely obtain a high total without busting). In such a scenario, it is in the player's interest to assume that the dealer's face-down card is a 10 and take more risks, attempting to achieve a higher total.

Now, the player's strategy may be modeled as a Markov decision problem. Suppose that $p_{st}(a)$ is the transition probability from state to s to t given that the player takes action $a \in A_s$, where A_s represents all possible actions that the player may take in state s. Suppose also that S is the state space for the player. Let $V(s)$ be the player's profit when they play optimally starting from state s. Then, V must satisfy

$$V(s) = \max_{a \in A_s} \sum_t p_{st}(a) V(t) \ \forall s \in S$$

We can formulate this as a linear program as follows:

$$min \sum_s V(s) \text{ subject to}$$

$$V(s) \geq \sum_t p_{st}(a) V(t) \ \forall s \in S, a \in A_s$$

	2	3	4	5	6	7	8	9	10	ace
4	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
5	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
6	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
7	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
8	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
9	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
10	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
11	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
12	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
13	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
14	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
15	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
16	-0.30004	-0.25887	-0.21706	-0.17302	-0.1537	-0.47539	-0.51052	-0.54315	-0.57576	-0.80844
17	-0.15989	-0.12351	-0.08626	-0.05046	0.01174	-0.10682	-0.38195	-0.42315	-0.46434	-0.67616
18	0.11551	0.14266	0.17078	0.19466	0.28345	0.39955	0.10596	-0.18315	-0.2415	-0.4116
19	0.38076	0.39937	0.41871	0.43523	0.49599	0.61598	0.59387	0.28761	-0.01866	-0.14704
20	0.63513	0.64592	0.65719	0.6667	0.70397	0.77324	0.79183	0.75837	0.43495	0.11752
21	0.87976	0.88328	0.88699	0.89008	0.90284	0.92594	0.93061	0.93919	0.88856	0.6249

Figure 2: Expectations of player's profit when player stays conditioned on the dealer's face-up card

Hard	2	3	4	5	6	7	8	9	10	ace
4	-0.121261	-0.088389	-0.054892	-0.017745	0.0111316	-0.088278	-0.159329	-0.240659	-0.335099	-0.465418
5	-0.134629	-0.101148	-0.06704	-0.029379	-0.001185	-0.119446	-0.188088	-0.266608	-0.357743	-0.483642
6	-0.147233	-0.113185	-0.07851	-0.040348	-0.013005	-0.151931	-0.217237	-0.292634	-0.380507	-0.50194
7	-0.115637	-0.082453	-0.048559	-0.012646	0.0291867	-0.06881	-0.2106	-0.285359	-0.365072	-0.520453
8	-0.028056	0.0023227	0.03344	0.0656248	0.1149633	0.0822068	-0.059892	-0.210177	-0.301774	-0.44696
9	0.0684088	0.0957912	0.1238482	0.1530693	0.1960241	0.1718696	0.0983852	-0.052169	-0.21343	-0.358417
10	0.1768379	0.20096	0.2256733	0.2516315	0.2878	0.2569133	0.1979622	0.1165391	-0.044992	-0.239852
11	0.233291	0.2557354	0.2787041	0.3031808	0.3336931	0.2921518	0.2299884	0.1582668	0.0596875	-0.122765
12	-0.257607	-0.23752	-0.216987	-0.196605	-0.170524	-0.212846	-0.27157	-0.340007	-0.420696	-0.533617
13	-0.31145	-0.29453	-0.277213	-0.260219	-0.235624	-0.269071	-0.323601	-0.38715	-0.462075	-0.56693
14	-0.365293	-0.35154	-0.337439	-0.323833	-0.300724	-0.321281	-0.371915	-0.430925	-0.500498	-0.597864
15	-0.419136	-0.40855	-0.397665	-0.387447	-0.365824	-0.369761	-0.416778	-0.471573	-0.536177	-0.626588
16	-0.472979	-0.46556	-0.457892	-0.451061	-0.430924	-0.414778	-0.458437	-0.509318	-0.569307	-0.65326
17	-0.537603	-0.532982	-0.528179	-0.524102	-0.50875	-0.483484	-0.505979	-0.553691	-0.610512	-0.678171
18	-0.623412	-0.620879	-0.618239	-0.615999	-0.607477	-0.591142	-0.591053	-0.616525	-0.668858	-0.723432
19	-0.729624	-0.728523	-0.727371	-0.726402	-0.722553	-0.715448	-0.713658	-0.715572	-0.744345	-0.789045
20	-0.855403	-0.855132	-0.854847	-0.854609	-0.853628	-0.851851	-0.851492	-0.850832	-0.854726	-0.875008
21	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Figure 3: Expectations of player's profit when player draws on a hard hand conditioned on the dealer's face-up card

Now, the expectations for the player's profit when the player stays (based on the dealer's face-up card) are provided in the heatmap in 2. Further, the expectations for the player's profit when the player draws on a hard hand and on a soft hand (based on the dealer's face-up card) are provided in the heatmaps in 3 and 4 respectively. Assuming that the player may only hit or stand, the optimal player decision at each hand total is determined by the maximum of the corresponding stand and hit expectations for that hand total. This determines the optimal fixed strategy provided in 5. Solving the aforementioned linear program simply provides the expected player profits based on the player's and dealer's starting hand assuming optimal play.

Since the idea behind the basic strategy is to make decisions that result in positive expectation, it is sufficient for a player to know the optimal plays in every scenario (the optimal action to take in each state $s \in S$). Notice that the strategy provided aligns with our intuition of playing safely for low dealer face-up cards and playing more aggressively for high dealer face-up cards. In particular, for hard hands, the player should not draw cards after 11 when the dealer's face-up card is 5 or 6, and after 12 when the dealer's face-up card is 2, 3, or 4. In addition, the player should only stop drawing after a total of 17 or higher when the dealer's face-up card is between a 7 and an ace. For soft hands, the strategy is different. To maximize their profit, the player should always draw on soft hands 17 or below and on soft 18s when the dealer's face-up card is between a 9 and an ace.

Note that, typically, the player cannot obtain an advantage over the house using just basic strategy.

Soft	2	3	4	5	6	7	8	9	10	ace
12	0.0767613	0.0989082	0.1216149	0.1516446	0.1859569	0.1654762	0.0951224	7.442E-05	-0.128084	-0.287588
13	0.0407541	0.0687776	0.0974082	0.1284542	0.1616957	0.1223888	0.0540641	-0.037686	-0.160808	-0.314679
14	0.0164117	0.0453742	0.0749462	0.1069204	0.1391674	0.0795105	0.013284	-0.075155	-0.193305	-0.341535
15	-0.006192	0.0236425	0.0540886	0.0869246	0.1182483	0.0370311	-0.027048	-0.112181	-0.225441	-0.368051
16	-0.027181	0.003463	0.0347208	0.0683572	0.0988234	-0.004887	-0.066789	-0.148636	-0.257102	-0.394139
17	-0.006628	0.0233992	0.0541079	0.0861331	0.1280546	0.0538226	-0.07291	-0.14978	-0.249412	-0.41919
18	0.0569256	0.0848231	0.1134249	0.1426992	0.1907569	0.1706772	0.0396845	-0.100735	-0.201096	-0.361957
19	0.1181372	0.1440638	0.1706395	0.1982154	0.2398046	0.2206226	0.1522792	0.0079021	-0.149671	-0.300905
20	0.1768379	0.20096	0.2256733	0.2516315	0.2878	0.2569133	0.1979622	0.1165391	-0.044992	-0.239852
21	0.233291	0.2557354	0.2787041	0.3031808	0.3336931	0.2921518	0.2299884	0.1582668	0.0596875	-0.122765

Figure 4: Expectations of player's profit when player draws on a soft hand conditioned on the dealer's face-up card

Hard	2	3	4	5	6	7	8	9	10	ace
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	H	H	H	H	H	H	H	H	H
10	H	H	H	H	H	H	H	H	H	H
11	H	H	H	H	H	H	H	H	H	H
12	H	H	H	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
16	S	S	S	S	S	H	H	H	H	H
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S
Soft										
12	H	H	H	H	H	H	H	H	H	H
13	H	H	H	H	H	H	H	H	H	H
14	H	H	H	H	H	H	H	H	H	H
15	H	H	H	H	H	H	H	H	H	H
16	H	H	H	H	H	H	H	H	H	H
17	H	H	H	H	H	H	H	H	H	H
18	S	S	S	S	S	S	S	H	H	H
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

Figure 5: Optimal Fixed Strategy

However, using this strategy brings the average loss of the player to approximately 0.5% of the initial bet. In addition to following the basic strategy, player should employ a combination of card counting, deviations, and advanced betting strategy. We briefly explore a form of card-counting called High-Low card counting.

4 High-Low Card Counting

Professional players make use of a strategy called card counting to increase their odds of winning. At a high level, card counting involves keeping track of the cards that have been dealt as play proceeds. As discussed, if the dealer's face-up card is 2 through 6, the dealer is likely to bust and the player is likely to win. On the other hand, if the dealer's face-up card is a 10 or an ace, the player is less likely to win. Thus, if more lower-valued cards have already been dealt,

This means that if the low valued cards are gone, the player has an advantage. If the high values cards are gone, the dealer has an advantage. This is the principle for the hi-low counting system.

In particular, Edward Thorp was one of the first to investigate whether using the information about the cards that have already been dealt could increase the player's expectation significantly and formulated a strategy called the high-low system based on his results. High-low card counting involves assigning values to each card value played. The particular values that are assigned to each card are +1 for 2 through 6 (low cards); 0 for 7 through 9 (neutral cards); and -1 for 10, face cards, and aces. The total of these values for all cards played so far gives the count at any given moment. Given the current count C, the true count is calculated as $\frac{C}{n}$, where n is the number of remaining decks rounded to the nearest 0.5. The player chooses whether to raise their bet, hit, stand, double, or split depending on the value of the true count.

Thorp proved that, although the high-low counting system is based on an approximation of the actual cards dealt, it exactly reflected changes in winning expectation. A more detailed explanation is provided in Thorp's book "Beat the Dealer: A Winning Strategy for the Game of Twenty-One."

5 Conclusion

I have provided a brief explanation of the mechanics of and some of the probabilities involved in Blackjack. I explained the basic strategy for Blackjack (up to hitting and standing) and briefly discussed a popular form of card counting aimed at increasing the player's expectation against the house. Notice that the basic strategy depends on the specific rules of the game of Blackjack being played. For instance, typical games allow the player to double down, split, and even make side bets such as insurance. To account for these possible player actions, one must calculate the expectations of using these strategies when possible and find the best course for each hand as required. Other versions of the game require the dealer to hit on a soft 17, requiring a modification of the entire analysis. The analysis provided here however should provide a foundation for understanding analyses of different variations of Blackjack.

6 Exercises

1. (Naturals) Assuming a one-deck game, what is the probability that neither you nor the dealer is dealt a natural? How many games do you have to play so that the probability of getting at least one blackjack is greater than 0.5?

2. (Charlie Rule) What is the probability of getting a 5-card Charlie? What about a 6-card Charlie? A 7-card Charlie?

3. (Card Totals) Suppose you are playing a game of blackjack with 8 decks. Suppose that, so far, three 2s, five 10s, and two As have been dealt. Given that you hold the cards A, A, 6, and 2 and that the dealer's up-card is an A, what is the probability that you obtain a total greater than or equal to 18 with the next card dealt?

4. (Strategy) Suppose you hold a soft 18 and the dealer's face-up card is an 8. Assuming cards are drawn from an infinite deck, should you hit or stand? Why or why not?

7 References

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8 Appendix

Simulations were used for my initial explorations of strategies and the mechanics of blackjack. I utilized simulated draws from decks of cards to find approximate probabilities for the dealer's final results (regardless of the dealer's face-up card). After modeling the dealer's pre-determined strategy as a Markov process, I used Python to fill the corresponding transition matrix and calculate the transition probabilities between each state and the absorption probabilities for the absorbing states accordingly. Finally, I used Excel to calculate the expectations for the player's profit depending on the player's choice at each possible player state and to generate the optimal fixed strategy chart. All the relevant material is provided in the following GitHub repository: https://github.com/akaashrp/blackjack-simulations.