Mamdani Fuzzy Systems

If-Then Rules and Rule Base

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- A fuzzy system with n inputs and m outputs (i.e., a multi-input multi-output, or MIMO fuzzy system) is equivalent to m fuzzy systems, each with n inputs and one output (i.e., multi-input single-output, or MISO fuzzy systems). Therefore, we will study only MISO fuzzy systems.
- Much of human decision making occurs in the form of "if-then" rules.

  R_i If $\tilde{x}$ is $\tilde{P}$, then $\tilde{y}$ is $\tilde{Q}$

  - $\tilde{x}$ is an input linguistic variable, $\tilde{P}$ is a linguistic value of $\tilde{x}$ and is a fuzzy set on $\tilde{x}$ universe, $\tilde{y}$ is a output linguistic variable, and $\tilde{Q}$ is a linguistic value of $\tilde{y}$ and is a fuzzy set on $\tilde{y}$ universe. A tilde over a symbol indicates a linguistic variable or value.
  - The first part of the statement, "$\tilde{x}$ is $\tilde{P}$" is called the *premise* of the rule, and the second part of the statement, "$\tilde{y}$ is $\tilde{Q}$" is called the *consequent* of the rule.
  - An example of stopping a car is,
    "If SPEED is FAST then BRAKE PRESSURE is HEAVY."
- In general, we use a number of rules to accomplish a task. A simple example for stopping a car might be

  R_1 If SPEED is SLOW, then BRAKE PRESSURE is LIGHT.
  R_2 If SPEED is MEDIUM, then BRAKE PRESSURE is MEDIUM.
  R_3 If SPEED is FAST, then BRAKE PRESSURE is HEAVY.

  - The collection of rules as a whole is called a *Rule base* .
- There may be more than one part to the premise, that is,

  R_j If $\tilde{x}_1$ is $\tilde{P}_1^k$ and $\tilde{x}_2$ is $\tilde{P}_2^l$ and ... and $\tilde{x}_n$ is $\tilde{P}_n^m$, then $\tilde{y}$ is $\tilde{Q}^j$

  - Another rule about stopping a car might be,
    "If SPEED is FAST and GRADE is DOWNHILL then BRAKE PRESSURE is VERY HEAVY."
- In general, we dispense with the tildes over linguistic variables and values. Thus, R_j is simply written

  R_j If x_1 is P_1^k and x_2 is P_2^l and ... and x_n is P_n^m, then y is Q^j

- The fuzzy system has the following structure (Fig. 3.1):

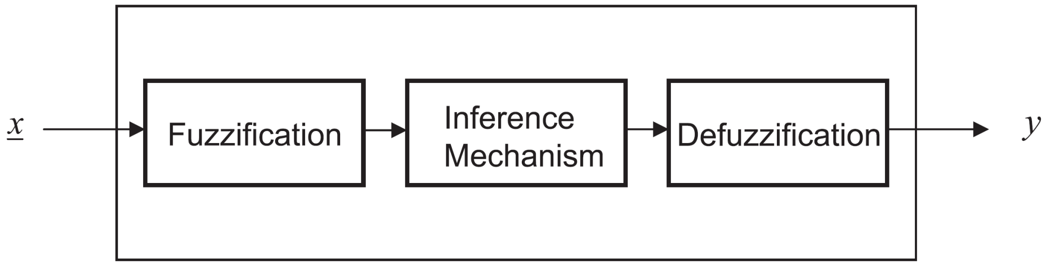

Figure 3.1. Structure of fuzzy systems.

The inputs $\underline{x}$ and output y are *crisp* (i.e., they are real numbers, not fuzzy sets).

— The fuzzification block converts the crisp inputs into fuzzy sets.

— The function of the inference stage is twofold: (1) to determine the degree of firing of each rule in the rule base, and (2) to create an implied fuzzy set for each rule corresponding to the rule's degree of firing.

— The defuzzification block combines these fuzzy recommendations to give a crisp output y.

Fuzzification

- The function of the fuzzification stage is to convert the measured quantities from the process (voltages, velocities, temperatures, etc.) into fuzzy sets to be used by the inference stage.

Inference

- The first function of the inference stage is to determine the degree of firing of each rule in the rule base.
- Consider the rule R_i, which has a single input x:

  R_i If $\tilde{x}$ is $\tilde{P}$, then $\tilde{y}$ is $\tilde{Q}$

  Let the fuzzy set P be characterized by the membership function $\mu^P(x)$, and the fuzzy set Q be characterized by the membership function $\mu^Q(y)$.
  - For a particular crisp input $x \in X$, we say rule R_i is *fired*, or is *on* (i.e., it is taken as true) to the extent $\mu^P(x)$, which is a real number in the interval [0, 1].
  - P is called the *premise fuzzy set for Rule i*.
  - $\mu^P(x)$ (or abbreviated to $\mu_i(x)$) is called the *premise membership function for Rule i*.
- Consider the rule R_j, which has n inputs $\underline{x} = (x_1, x_2, \ldots x_n)$:

  R_j If $\tilde{x}_1$ is $\tilde{P}_1^k$ and $\tilde{x}_2$ is $\tilde{P}_2^l$ and ... and $\tilde{x}_n$ is $\tilde{P}_n^m$, then $\tilde{y}$ is $\tilde{Q}^j$

  Let the fuzzy set $P_1^k \times P_2^l \times \cdots \times P_n^m$ be characterized by the membership function $\mu^{P_1^k \times P_2^l \times \cdots \times P_n^m}(\underline{x})$, and the fuzzy set Q^j be characterized by the membership function $\mu^{Q^j}(y)$.
  - For a particular crisp input $\underline{x} = (x_1, x_2, \ldots x_n) \in X_1 \times X_2 \times \ldots \times X_n$, we say rule R_j is *fired* to the extent $\mu^{P_1^k \times P_2^l \times \cdots \times P_n^m}(\underline{x}) = \mu_1^k(x_1) * \mu_2^l(x_2) * \cdots \mu_n^m(x_n)$, which is a real number in the interval [0, 1]. (see Cartesian product)
  - $P_1^k \times P_2^l \times \cdots \times P_n^m$ is called the *premise fuzzy set for Rule j*.
  - $\mu^{P_1^k \times P_2^l \times \cdots \times P_n^m}(\underline{x})$ (or abbreviated to $\mu_j(\underline{x})$) is called the *premise membership function for Rule j*.
- The second function of the inference stage is to determine the degree to which each rule's recommendation is to be weighted in arriving at the final decision and to determine an implied fuzzy set corresponding to each rule.

- From the discussion above, we know rule R_j is fired to the degree $\mu_j(\underline{x})$. Therefore we attenuate the recommendation of rule R_j, which is fuzzy set Q_j characterized by $\mu^{Q^j}(y)$, by $\mu_j(\underline{x})$.
  - This produces an *implied fuzzy set* $\hat{Q}^j$ defined on Y, characterized by the membership function $\mu^{\hat{Q}^j}(y) = \mu_j(\underline{x}) * \mu^{Q^j}(y)$.
  - If there are R rules, each rule has its own premise membership function $\mu_j(\underline{x})$, j=1, 2, ..., R, so that produce R membership functions $\mu^{\hat{Q}^j}(y)$.

Defuzzification

- The function of the defuzzification stage is to convert the collection of recommendations of all rules into a crisp output. There are several ways to do this, the two most common are center of gravity (COG) defuzzification and center average (CA) defuzzification.
- Center of Gravity (COG) Defuzzification:

  Define the point q_i is the *center of area* of $\mu^{Q^j}(y)$ in the universe Y. Then, the crisp output of the fuzzy system is calculated as

$$y^{crisp} = \frac{\sum_{i=1}^{R} q_i \int \mu^{\hat{Q}^i}}{\sum_{i=1}^{R} \int \mu^{\hat{Q}^i}}$$

  where $\int \mu^{\hat{Q}^i}$ is the area under $\mu^{\hat{Q}^i}$.

- Center Average (CA) Defuzzification:

  Define the point q_i is the *center of area* of $\mu^{Q^j}(y)$ in the universe Y. Then, the crisp output of the fuzzy system is calculated as

$$y^{crisp} = \frac{\sum_{i=1}^{R} q_i \mu_i(\underline{x})}{\sum_{i=1}^{R} \mu_i(\underline{x})}$$

  where $\mu_i(\underline{x})$ is the *premise membership function.*

Example: Fuzzy System for Wind Chill

- Consider a fuzzy system to determine what the temperature feels like to a person under certain weather conditions. This is known as “wind chill.”
  - Wind chill is determined by the actual temperature and the wind speed.
- Suppose there are four fuzzy sets for TEMPERATURE: COLD, COOL, WARM, and HOT (characterized in Fig. 3.2), and three fuzzy sets for WIND SPEED: LOW, GENTLE, and HIGH (characterized in Fig. 3.3).

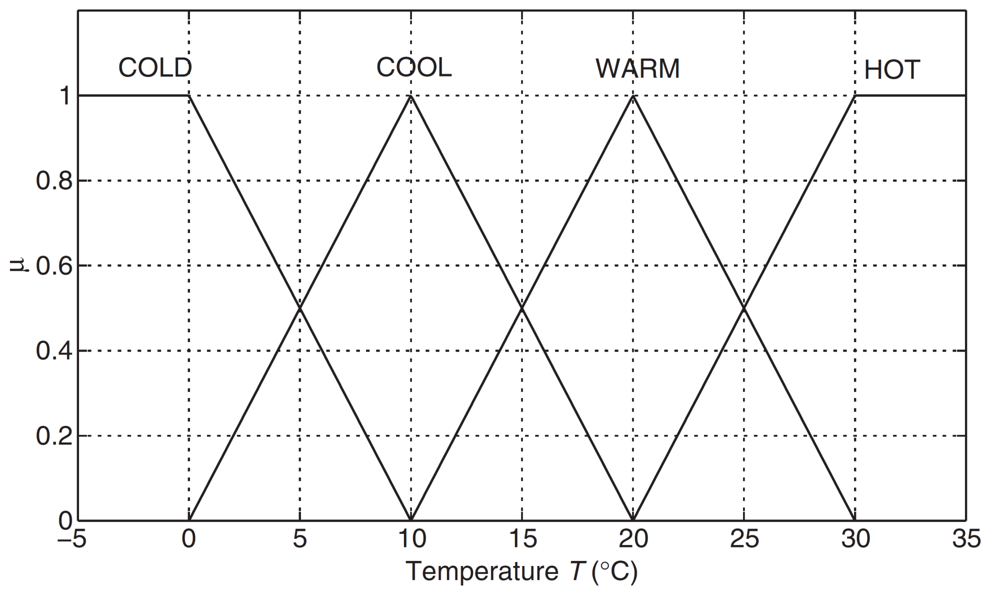

Figure 3.2. Fuzzy sets on the TEMPERATURE universe.

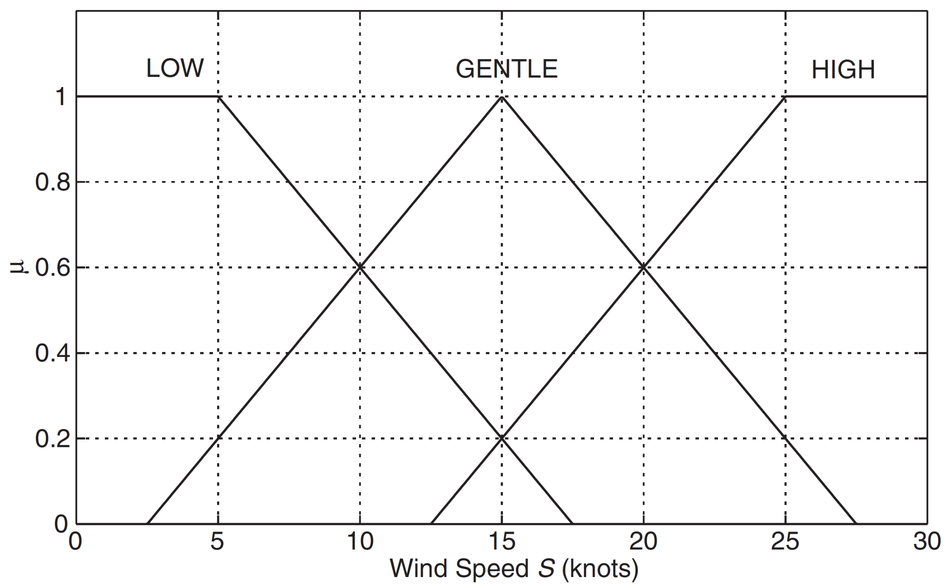

Figure 3.3. Fuzzy sets on the WIND SPEED universe.

- Suppose there are also five fuzzy sets for WIND CHILL: SEVERE, BAD, BEARABLE, MILD, and UNNOTICEABLE, characterized on the output universe of discourse as in Figure 3.4 .

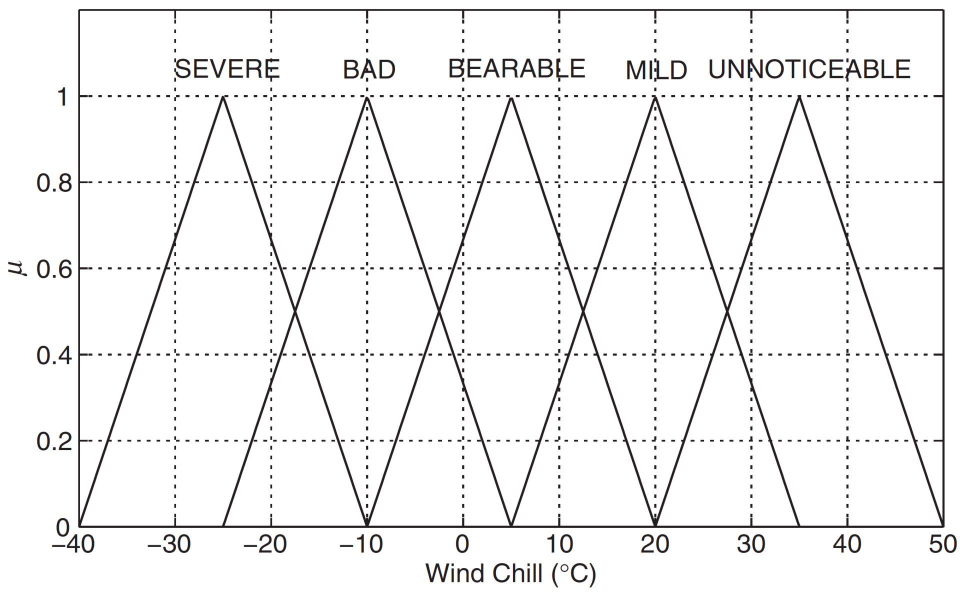

Figure 3.4. Fuzzy sets on the WIND CHILL universe.

— The end memberships on the output universe (i.e., SEVERE and UNNOTICEABLE) do not saturate at 1 as the end memberships on the input universes do.
  - ✧ The defuzzification methods require the calculation of the areas of the implied fuzzy sets $\int \mu^{\hat{Q}^i}$ (for COG defuzzification) and the centers of area q_i of the output fuzzy sets (for both COG and CA). Both of these calculations are ill-defined if the output fuzzy sets saturate at 1.
  - ✧ We must allow for the inputs to be any values, but the outputs are prescribed by the fuzzy system, hence their range is restricted.

— The lowest wind chill is $-25\,°\mathrm{C}$ because this is the center of area of the SEVERE output fuzzy set; the highest wind chill is 35°C because this is the center of area of the UNNOTICEABLE output fuzzy set.

- For conciseness, the rule base can be tabulated as in Table 3.1:

TABLE 3.1 Tabulation of Rule Base

Wind Chill		Wind Speed		
		LOW	GENTLE	HIGH
Temperature	COLD	BEARABLE	BAD	SEVERE
	COOL	MILD	BEARABLE	BAD
	WARM	UNNOTICEABLE	MILD	BEARABLE
	HOT	UNNOTICEABLE	UNNOTICEABLE	MILD

It is equivalent to the following rule base:

1. If TEMPERATURE is COLD and WIND SPEED is LOW, then WIND CHILL is BEARABLE.
2. If TEMPERATURE is COLD and WIND SPEED is GENTLE, then WIND CHILL is BAD.
3. If TEMPERATURE is COLD and WIND SPEED is HIGH, then WIND CHILL is SEVERE.
4. If TEMPERATURE is COOL and WIND SPEED is LOW, then WIND CHILL is MILD.
5. If TEMPERATURE is COOL and WIND SPEED is GENTLE, then WIND CHILL is BEARABLE.
6. If TEMPERATURE is COOL and WIND SPEED is HIGH, then WIND CHILL is BAD.
7. If TEMPERATURE is WARM and WIND SPEED is LOW, then WIND CHILL is UNNOTICEABLE.
8. If TEMPERATURE is WARM and WIND SPEED is GENTLE, then WIND CHILL is MILD.
9. If TEMPERATURE is WARM and WIND SPEED is HIGH, then WIND CHILL is BEARABLE.
10. If TEMPERATURE is HOT and WIND SPEED is LOW, then WIND CHILL is UNNOTICEABLE.
11. If TEMPERATURE is HOT and WIND SPEED is GENTLE, then WIND CHILL is UNNOTICEABLE.
12. If TEMPERATURE is HOT and WIND SPEED is HIGH, then WIND CHILL is MILD.

— Now the fuzzy system is completely specified (i.e., all premise and consequent fuzzy sets are specified, together with the rule base). This kind of rule base is said to be *complete*.

- **Wind Chill Calculation, *Minimum* T-norm**

— ***Inference***:

- Let us calculate the wind chill corresponding to a temperature of 7°C and a wind speed of 22 knots:

  Referring to Figure 3.2 , we have

  $\mu^{COLD}(7) = 0.3$

  $\mu^{COOL}(7) = 0.7$

  $\mu^{WARM}(7) = 0$

  $\mu^{HOT}(7) = 0$

  Referring to Figure 3.3 , we have

  $\mu^{LOW}(22) = 0$

  $\mu^{GENTLE}(22) = 0.44$

  $\mu^{HIGH}(22) = 0.76$

- Using *minimum* T-norm, we obtain the following degrees of firing for the 12 rules in the rule base:

$$R_1 : \mu_1(7,22) = \mu^{COLD \cap LOW} = \min(0.3, 0) = 0$$

$$R_2 : \mu_2(7,22) = \mu^{COLD \cap GENTLE} = \min(0.3, 0.44) = 0.3$$

$$R_3 : \mu_3(7,22) = \mu^{COLD \cap HIGH} = \min(0.3, 0.76) = 0.3$$

$$R_4 : \mu_4(7,22) = \mu^{COOL \cap LOW} = \min(0.7, 0) = 0$$

$$R_5 : \mu_5(7,22) = \mu^{COOL \cap GENTLE} = \min(0.7, 0.44) = 0.44$$

$$R_6 : \mu_6(7,22) = \mu^{COOL \cap HIGH} = \min(0.7, 0.76) = 0.7$$

$$R_7 : \mu_7(7,22) = \mu^{WARM \cap LOW} = \min(0, 0) = 0$$

$$R_8 : \mu_8(7,22) = \mu^{WARM \cap GENTLE} = \min(0, 0.44) = 0$$

$$R_9 : \mu_9(7,22) = \mu^{WARM \cap HIGH} = \min(0, 0.76) = 0$$

$$R_{10} : \mu_{10}(7,22) = \mu^{HOT \cap LOW} = \min(0, 0) = 0$$

$$R_{11} : \mu_{11}(7,22) = \mu^{HOT \cap GENTLE} = \min(0, 0.44) = 0$$

$$R_{12} : \mu_{12}(7,22) = \mu^{HOT \cap HIGH} = \min(0, 0.76) = 0$$

- Since only four rules (Rules 2, 3, 5, and 6) are fired for this input, we will create four nonzero implied fuzzy sets. Using *minimum* T-norm, the membership function characterizing Rules 3, 5, and 6's implied fuzzy sets are

$$\mu_2^{implied}(y) = \min_y \left\{ \mu_2(7,22), \mu^{BAD}(y) \right\}$$

$$\mu_3^{implied}(y) = \min_y \left\{ \mu_3(7,22), \mu^{SEVERE}(y) \right\}$$

$$\mu_5^{implied}(y) = \min_y \left\{ \mu_5(7,22), \mu^{BEARABLE}(y) \right\}$$

$$\mu_6^{implied}(y) = \min_y \left\{ \mu_6(7,22), \mu^{BAD}(y) \right\}$$

✧ The resulting implied fuzzy sets' membership functions are shown in Figure 3.5.

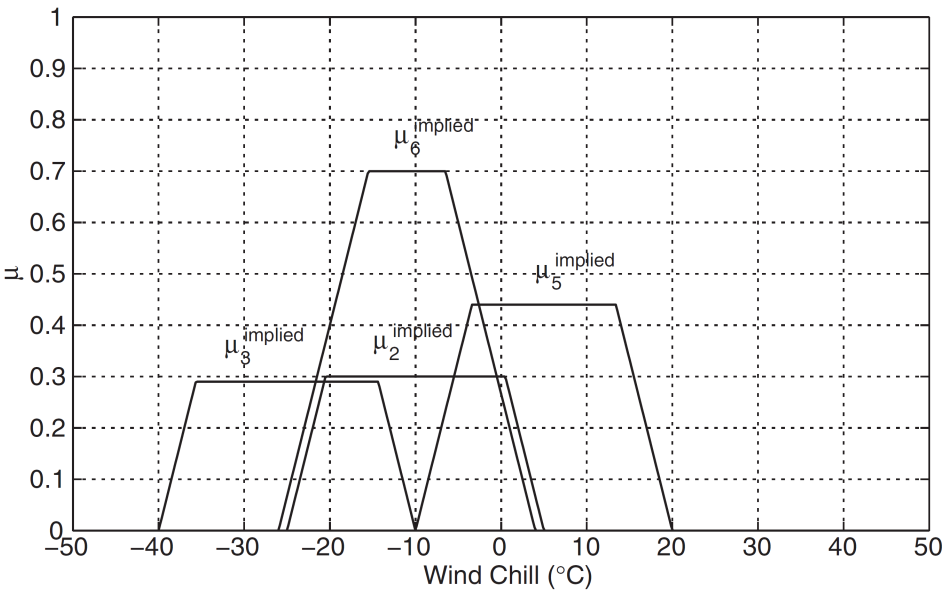

Figure 3.5. Implied fuzzy sets, *minimum* T-norm (slightly displaced to improve clarity).

— ***COG Defuzzification***:

✧ The centers of area of the memberships characterizing the output fuzzy sets and the areas of the four trapezoidal membership functions characterizing the implied fuzzy sets in Figure 3.5 are

$q_2 = -10$

$q_3 = -25$

$q_5 = 5$

$q_6 = -10$

$$\int \mu_2^{implied} = 7.65$$

$$\int \mu_3^{implied} = 7.65$$

$$\int \mu_5^{implied} = 10.296$$

$$\int \mu_6^{implied} = 13.65$$

✧ The crisp output of the fuzzy system corresponding to the crisp input $(T,S) = (7,22)$ is calculated using COG defuzzification to be

$$y^{crisp} = \frac{-10(7.65) - 25(7.65) + 5(10.296) - 10(13.65)}{7.65 + 7.65 + 10.296 + 13.65} = -8.9887°\text{C}$$

✧ The input-output characteristic is shown in Figure 3.6.

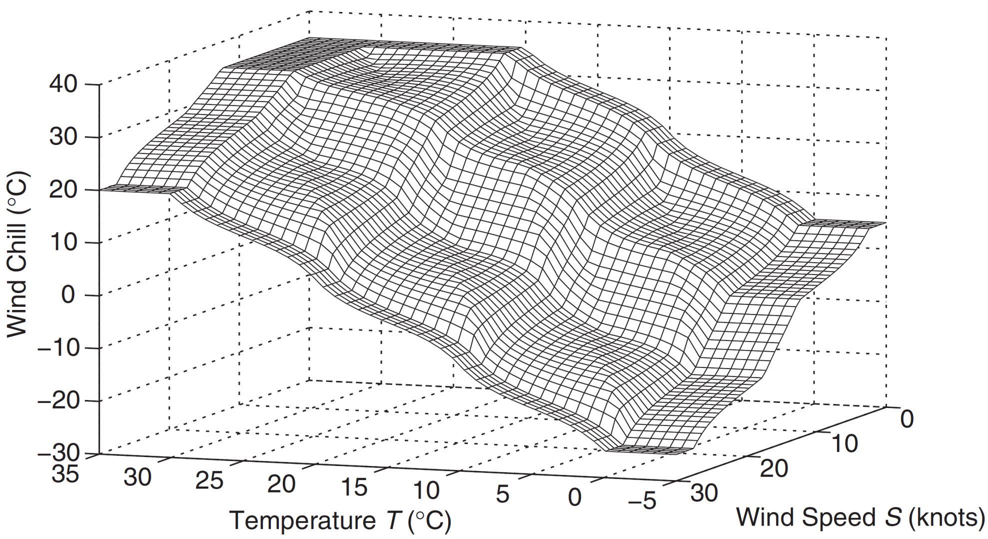

Figure 3.6. Input-output characteristic of *Wind Chill* fuzzy system, *min* T-norm, COG.

– ***CA Defuzzification***:

✧ The crisp output of the fuzzy system corresponding to the crisp input $(T,S) = (7,22)$, is calculated using CA defuzzification to be

$$y^{crisp} = \frac{-10(0.3) - 25(0.3) + 5(0.44) - 10(0.7)}{0.3 + 0.3 + 0.44 + 0.7} = -8.7931°\text{C}$$

✧ The input-output characteristic is shown in Figure 3.7. Its shape is slightly different from that of Figure 3.6 due to the use of CA defuzzification rather than COG.

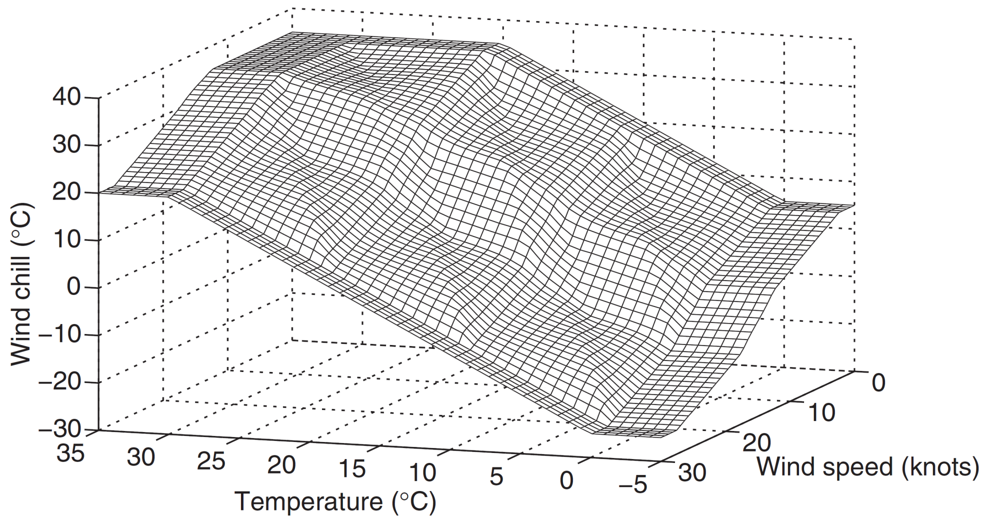

Figure 3.7. Input-output characteristic of *Wind Chill* fuzzy system, *min* T-norm, CA defuzzification.

- **Wind Chill Calculation, *Product* T-norm**
  - ***Inference***:
    - ✧ Let us calculate the crisp output corresponding to the same crisp input, but this time using *product* T-norm throughout. We obtain the following degrees of firing for the 12 rules in the rule base:

$$R_1 : \mu_1(7,22) = \mu^{COLD \cap LOW} = (0.3)(0) = 0$$

$$R_2 : \mu_2(7,22) = \mu^{COLD \cap GENTLE} = (0.3)(0.44) = 0.132$$

$$R_3 : \mu_3(7,22) = \mu^{COLD \cap HIGH} = (0.3)(0.76) = 0.228$$

$$R_4 : \mu_4(7,22) = \mu^{COOL \cap LOW} = (0.7)(0) = 0$$

$$R_5 : \mu_5(7,22) = \mu^{COOL \cap GENTLE} = (0.7)(0.44) = 0.308$$

$$R_6 : \mu_6(7,22) = \mu^{COOL \cap HIGH} = (0.7)(0.76) = 0.532$$

$$R_7 : \mu_7(7,22) = \mu^{WARM \cap LOW} = (0)(0) = 0$$

$$R_8 : \mu_8(7,22) = \mu^{WARM \cap GENTLE} = (0)(0.44) = 0$$

$$R_9 : \mu_9(7,22) = \mu^{WARM \cap HIGH} = (0)(0.76) = 0$$

$$R_{10} : \mu_{10}(7,22) = \mu^{HOT \cap LOW} = (0)(0) = 0$$

$$R_{11} : \mu_{11}(7,22) = \mu^{HOT \cap GENTLE} = (0)(0.44) = 0$$

$$R_{12} : \mu_{12}(7,22) = \mu^{HOT \cap HIGH} = (0)(0.76) = 0$$

Again, only Rules 2, 3, 5, and 6 are fired.

✧ Using *product* T-norm, the membership function characterizing Rules 2, 3, 5, and 6's implied fuzzy sets are

$$\mu_2^{implied}(y) = (\mu_2(7,22))(\mu^{BAD}(y))$$

$$\mu_3^{implied}(y) = (\mu_3(7,22))(\mu^{SEVERE}(y))$$

$$\mu_5^{implied}(y) = (\mu_5(7,22))(\mu^{BEARABLE}(y))$$

$$\mu_6^{implied}(y) = (\mu_6(7,22))(\mu^{BAD}(y))$$

✧ The resulting implied fuzzy sets' membership functions are shown in Figure 3.8.

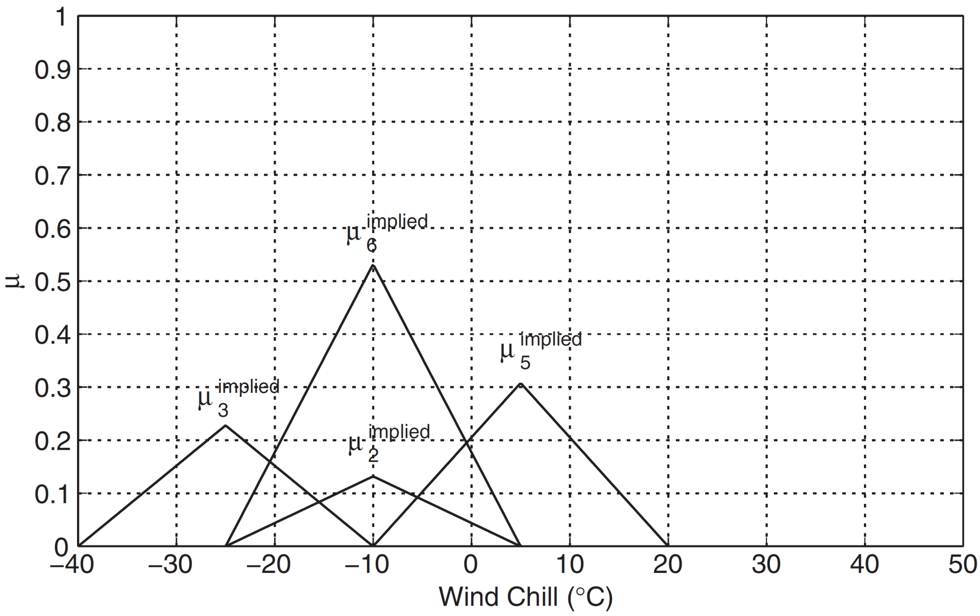

Figure 3.8. Implied fuzzy sets, *product* T - norm.

— ***COG Defuzzification***:

✧ The areas of the four triangular membership functions characterizing the implied fuzzy sets in Figure 3.8 are

$$\int \mu_2^{implied} = 1.98$$

$$\int \mu_3^{implied} = 3.42$$

$$\int \mu_5^{implied} = 4.62$$

$$\int \mu_6^{implied} = 7.98$$

✧ The crisp output of the fuzzy system corresponding to the crisp input $(T,S)=(7,22)$ is calculated using COG defuzzification to be

$$y^{crisp} = \frac{-10(1.98)-25(3.42)+5(4.62)-10(7.98)}{1.98+3.42+4.62+7.98} = -9.0°\text{C}$$

✧ The input-output characteristic is shown in Figure 3.9.

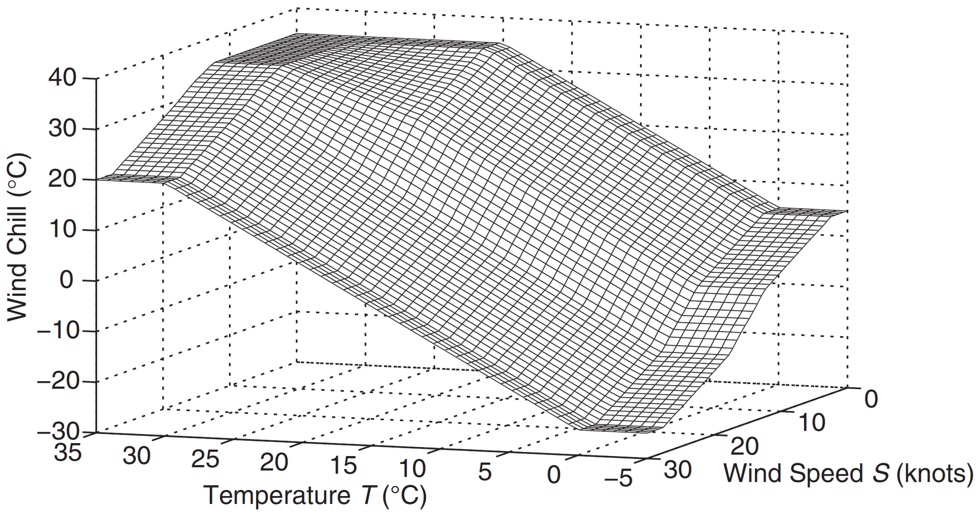

Figure 3.9. Input-output characteristic of *Wind Chill* fuzzy system, *product* T-norm, COG defuzzification.

- ***CA Defuzzification***:
  - ✧ If CA defuzzification is used instead of COG, the crisp output of the fuzzy system corresponding to the crisp input $(T,S)=(7,22)$ is calculated to be

$$y^{crisp} = \frac{-10(0.132)-25(0.228)+5(0.308)-10(0.532)}{0.132+0.228+0.308+0.532} = -9.0°\text{C}$$

  - ✧ The input-output characteristic is identical to that of Figure 3.9.

- Singleton Output Fuzzy Sets:
  - The *singleton* output fuzzy sets are characterized by membership functions that are zero except for one point in the output universe, where they are 1.
  - For example, the output fuzzy sets SEVERE, BAD, BEARABLE, MILD, and UNNOTICEABLE could be characterized by the membership functions

shown in Figure 3.10.

— Defuzzification must be CA since there is no area to compute.

— Any result for triangular output memberships, *product* T-norm, and CA defuzzification would also be obtained if the output fuzzy sets were singletons.

  ✧ In general, if *product* T-norm is used and the output memberships are symmetrical and normal, the output fuzzy sets may be replaced by singletons.

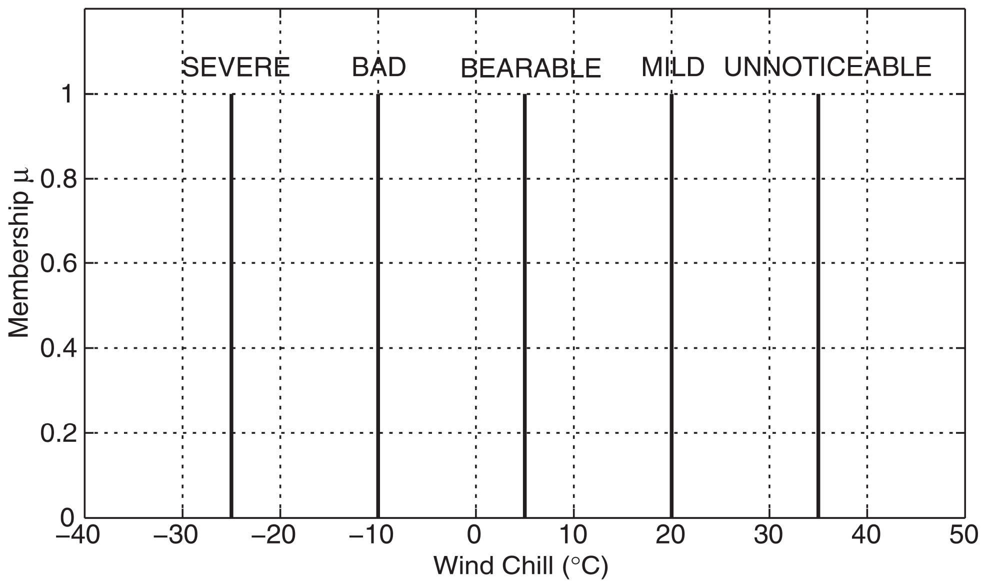

Figure 3.10. Membership functions characterizing singleton output fuzzy sets.