Week 10: Heteroskedasticity

Marcelo Coca Perraillon

University of Colorado
Anschutz Medical Campus

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Outline

- The problem of (conditional) unequal variance: heteroskedasticity
- Correcting and testing for heteroskedasticity
- The sandwich estimator
- Examples

Big picture

- Heteroskedasticity is so **common that we should just assume it exists**
- We can perform some tests to detected it
- The solutions depend on the **source** of heteroskedasticity
- The problem is not about the bias or consistency of the OLS estimates; the issue is that **SEs are not correct** in the presence of heteroskedasticity
- We will follow Chapter 8 of Wooldridge

Homoskedasticity

- In the linear model $y_i = \beta_0 + \beta_1 x1_i + \cdots + xp_p + \epsilon_i$ we assumed that $\epsilon_i \sim N(0, \sigma^2)$
- That is, the error terms have all the same variance conditional on all explanatory variables: $var(\epsilon_i | x1, ..., xp) = \sigma^2$
- If this is not the case, then we need to add an index to the variance to denote that some observations have a different error variance that depends on values of x
- To simplify, we will focus on the simple linear model (only one covariate). In the presence of heteroskedasticity: $var(\epsilon_i | x_i) = \sigma_i^2$

Homoskedasticity

- In the SLR model, we can write the variance of $\hat{\beta}_1$ as
  $var(\hat{\beta}_1) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sigma_i^2}{\sum_{i=1}^{n}(x_i - \bar{x})^4}$
- If we have homoskedasticity the formula reduces to the one we saw in Chapter 2 (2.22):
  $var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$
- But in the presence of heteroskedasticity we can't no longer simplify that formula
- White (1980) introduced a rather simple solution to calculate the variance in the presence of unknown heteroskedasticity: **estimate** σ_i^2:
  $var(\hat{\beta}_1) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 \hat{\epsilon_i}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^4}$

Huber-White robust standard errors

- In the previous equation, $\hat{\epsilon_i}^2$ is the **estimated residual** of the regression
- So the estimation proceeds in two steps: **1)** Estimate the regression and **2)** Obtain the residuals to calculate the robust variance
- In matrix notation, the variance-covariance matrix is $var(\hat{\beta}) = \sigma^2(X'X)^{-1}$
- The **Huber-White robust variance-covariance** matrix is
- $var(\hat{\beta}_{rob}) = (X'X)^{-1}X'\hat{\Sigma}(X'X)^{-1}$
- The way the formula looks is the reason why Huber-White robust standard errors are (affectionately?) referred to as the **sandwhich estimator**

Example

- Using the mksp1 dataset we saw that it's likely there is a hetoskedasticity problem

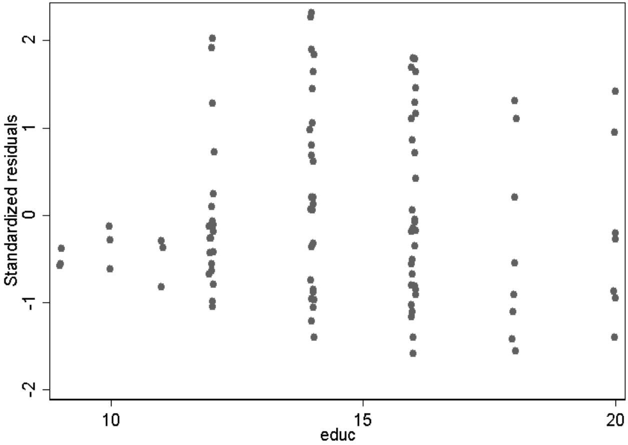
```
* Load data
webuse mksp1

* Regress educ on income
reg income educ
    Source |       SS           df       MS      Number of obs   =       100
-------------+----------------------------------   F(1, 98)        =     10.34
       Model |  2.7896e+09         1  2.7896e+09   Prob > F        =    0.0018
    Residual |  2.6433e+10        98   269719984   R-squared       =    0.0955
-------------+----------------------------------   Adj R-squared   =    0.0862
       Total |  2.9222e+10        99   295173333   Root MSE        =     16423

------------------------------------------------------------------------------
      income |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   2001.493   622.3571     3.22   0.002     766.4461    3236.541
       _cons |   14098.23   9221.392     1.53   0.130    -4201.327    32397.78
------------------------------------------------------------------------------
predict incres, rstandard

scatter incres educ, xline(0) jitter(1)
```

Example



- Some evidence of unequal variances conditional on education

Huber-White robust SEs in Stata

- The option vce(robust) or simply robust uses the sandwich estimator

```
reg income educ, vce(robust)

Linear regression                               Number of obs     =        100
                                                F(1, 98)          =      13.84
                                                Prob > F          =     0.0003
                                                R-squared         =     0.0955
                                                Root MSE          =      16423

------------------------------------------------------------------------------
             |               Robust
      income |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   2001.493   538.0771     3.72   0.000     933.6971     3069.29
       _cons |   14098.23   7680.933     1.84   0.069    -1144.337    29340.79
------------------------------------------------------------------------------
```

- Conclusions won't change but notice that CIs are narrower. SEs went down

Huber-White robust SEs in Stata

- Compare models; some tests will of course change now that we have different SEs

```
qui reg income educ
est sto m1
test educ= 900
 ( 1)  educ = 900
       F(  1,    98) =    3.13
            Prob > F =    0.0799
qui reg income educ, robust
est sto m2
test educ= 900
 ( 1)  educ = 900
       F(  1,    98) =    4.19
            Prob > F =    0.0433
est table m1 m2, se stats(N F)
----------------------------------------
    Variable |     m1           m2
-------------+--------------------------
        educ |  2001.4935    2001.4935
             |  622.35711    538.07705
       _cons |  14098.225    14098.225
             |   9221.392    7680.9332
-------------+--------------------------
           N |        100          100
           F |  10.342584    13.836282
----------------------------------------
```

- Note that Stata calculates a different F statistics

The good and the bad of the sandwich

- Good: We do **not need to know the source** of unequal variance
- Great: The sandwhich estimator is **asymptotically unbiased**. This has an important but subtle implication
- If we most often than not suspect some form of heteroskedasticity and the sandwich estimator is asymptotically valid *even in the presence of equal variance*, **why not just always use the robust SEs?**
- In fact, many applied researchers add the option robust to every single model for "insurance"
- The only drawback is that if the assumptions of the linear model are valid, in **smaller samples the robust SEs may not be unbiased**

Testing for heteroskedasticity

- If small samples and unequal variance in doubt, useful to have a test for heteroskedasticity rather than just assume it
- The null hypothesis is $H_0 : var(\epsilon|x_1, x_2, ..., x_p) = \sigma^2$ (that is, homoskedasticity)
- As usual with hypothesis testing, we will look at the data to provide evidence that the variance is not equal conditional on $x_1, x_2, ..., x_p$
- Recall the basic formula of the variance: $var(X) = E[(X - \bar{X})^2] = E[X^2] - (E[X])^2$
- Since $E[\epsilon] = 0$ we can rewrite the null as: $H_0 : E(\epsilon^2|x_1, x_2, ..., x_p) = E[\epsilon^2] = \sigma^2$
- If you see the problem this way, it looks a lot easier. We need to figure out if the $E[\epsilon^2]$ is **related** to one or more of the explanatory variables. If not, we can't reject the null

Testing for heteroskedasticity

- By related, it could be in any functional form, but start with a linear relationship
- $\epsilon^2 = \gamma_0 + \gamma_1 x_1 + \cdots + \gamma_p x_p + u$
- If we reject $H_0 : \gamma_0 = \gamma_1 = ... = \gamma_p = 0$ then there is **evidence of unequal variance**
- Of course, we do not observe ϵ^2 so we need to work with $\hat{\epsilon}^2$
- The test is an F-test of the **overall significance of the model**
- As you probably suspect, Stata has a command for that

Testing for heteroskedasticity, example

- Let's go back to the income, education, and age dataset and estimate the model

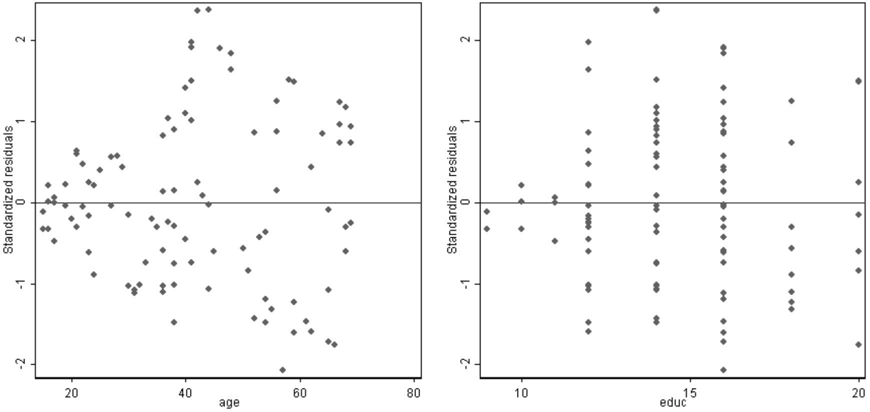
  $income = \beta_0 + \beta_1 educ + \beta_2 age + \epsilon$

```
* Get residuals
qui reg income age edu
predict incress, rstandard

* Combine the plots
scatter incress age, yline(0) legend(off) saving(r1.gph, replace)
scatter incress educ, yline(0) legend(off) saving(r2.gph, replace)

* Export plot
graph combine r1.gph r2.gph, row(1) ysize(10) xsize(20)
graph export rall.png, replace
```

Testing for heteroskedasticity, example



- Clearly, we suspect unequal variance conditional on age and education

Testing for heteroskedasticity, example

- We use the post-estimation command hettest and confirm that we do reject the null:

```
 reg income age edu

      Source |       SS           df       MS      Number of obs   =       100
-------------+----------------------------------   F(2, 97)        =     14.71
       Model |  6.8005e+09         2  3.4002e+09   Prob > F        =    0.0000
    Residual |  2.2422e+10        97   231151328   R-squared       =    0.2327
-------------+----------------------------------   Adj R-squared   =    0.2169
       Total |  2.9222e+10        99   295173333   Root MSE        =     15204

------------------------------------------------------------------------------
      income |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   440.2441   105.6871     4.17   0.000     230.4845    650.0037
        educ |   706.8841   654.6241     1.08   0.283    -592.3636    2006.132
       _cons |   14800.35   8538.327     1.73   0.086     -2145.86    31746.57
------------------------------------------------------------------------------

estat hettest, rhs
Breusch-Pagan / Cook-Weisberg test for heteroskedasticity
         Ho: Constant variance
         Variables: age educ

         chi2(2)      =     9.86
         Prob > chi2  =   0.0072
```

By hand

- Not exactly the same as the Breusch-Pagan but close (p-value of F test: 0.0012)

```
qui reg income age edu

* Get square of residuals
predict r1, res
gen r12 = r1^2

*  Regress
reg r12 age edu

      Source |       SS           df       MS      Number of obs   =       100
-------------+----------------------------------   F(2, 97)        =      7.21
       Model |  9.9160e+17         2  4.9580e+17   Prob > F        =    0.0012
    Residual |  6.6689e+18        97  6.8752e+16   R-squared       =    0.1294
-------------+----------------------------------   Adj R-squared   =    0.1115
       Total |  7.6605e+18        99  7.7379e+16   Root MSE        =    2.6e+08

------------------------------------------------------------------------------
         r12 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |    6024159    1822704     3.31   0.001      2406596     9641722
        educ |     872384   1.13e+07     0.08   0.939    -2.15e+07    2.33e+07
       _cons |  -3.72e+07   1.47e+08    -0.25   0.801    -3.29e+08    2.55e+08
------------------------------------------------------------------------------
```

- We do gain some intuition: as suspected, the problem is age and not so much education

Using Breusch-Pagan

- We can also test for age or education separately

```
qui reg income age edu

estat hettest age

Breusch-Pagan / Cook-Weisberg test for heteroskedasticity
         Ho: Constant variance
         Variables: age

         chi2(1)      =     9.86
         Prob > chi2  =   0.0017

estat hettest edu

Breusch-Pagan / Cook-Weisberg test for heteroskedasticity
         Ho: Constant variance
         Variables: educ

         chi2(1)      =     2.39
         Prob > chi2  =   0.1219
```

- Age is the source of heteroskedasticity

Testing for heteroskedasticity, example

- Correcting does change SEs but not by a lot

```
* Regular
qui reg income age edu
est sto reg

* Robust
qui reg income age edu, robust
est sto rob

* Compare
est table reg rob, se p stats(N F)

----------------------------------------
    Variable |    reg          rob
-------------+--------------------------
         age |  440.24407    440.24407
             |  105.68708    94.815869
             |     0.0001       0.0000
        educ |  706.88408    706.88408
             |  654.62413    612.81005
             |     0.2829       0.2515
       _cons |  14800.355    14800.355
             |  8538.3265    7245.2375
             |     0.0862       0.0438
-------------+--------------------------
           N |        100          100
           F |   14.71002    21.294124
----------------------------------------
                          legend: b/se/p
```

Back to transformations

- Remember that taking the log(y) tends to help with OLS assumptions? Could it fix the heteroskedastic problem? Yep, mostly

```
reg lincome age edu
...
------------------------------------------------------------------------------
     lincome |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0093932   .0024094     3.90   0.000     .0046113    .0141752
        educ |   .0217054   .0149237     1.45   0.149     -.007914    .0513248
       _cons |   9.895059   .1946512    50.83   0.000      9.50873    10.28139
------------------------------------------------------------------------------
estat hettest, rhs

Breusch-Pagan / Cook-Weisberg test for heteroskedasticity
         Ho: Constant variance
         Variables: age educ
         chi2(2)      =     5.00
         Prob > chi2  =   0.0821

estat hettest age
         chi2(1)      =     4.98
         Prob > chi2  =   0.0256

estat hettest educ
         chi2(1)      =     0.87
         Prob > chi2  =   0.3500
```

Back to transformations

- Since taking the log has helped with heteroskedasticity, the original and the robust model should be similar

```
* Log income, no robust
qui reg lincome age edu
est sto lm1

* Log income, robust
qui reg lincome age edu, robust
est sto lm1rob

* Compare
est table lm1 lm1rob, se p stats(N F)
----------------------------------------
    Variable |    lm1          lm1rob
-------------+--------------------------
         age |  .00939325    .00939325
             |  .00240939    .00215669
             |     0.0002       0.0000
        educ |  .02170542    .02170542
             |  .01492369    .01349306
             |     0.1491       0.1109
       _cons |  9.8950586    9.8950586
             |   .1946512    .16247044
             |     0.0000       0.0000
-------------+--------------------------
           N |        100          100
           F |  14.651729    21.599741
----------------------------------------
                          legend: b/se/p
```

Alternative: White test

- An alternative test that is popular is the **White test**
- It does use more degrees of freedom. The logic is similar to the other test
- White showed that the errors are homokedastic if ϵ^2 is uncorrelated with all the covariates, their squares, and cross products
- With three covariates, the White test will use 9 predictors rather than 3
- Easy to implement in Stata (of course)

White

- White test in Stata

```
qui reg income age edu

 estat imtest, white

White's test for Ho: homoskedasticity
         against Ha: unrestricted heteroskedasticity

         chi2(5)      =     23.77
         Prob > chi2  =    0.0002

Cameron & Trivedi's decomposition of IM-test

---------------------------------------------------
              Source |       chi2     df      p
---------------------+-----------------------------
  Heteroskedasticity |      23.77      5    0.0002
            Skewness |       3.77      2    0.1518
            Kurtosis |       2.29      1    0.1302
---------------------+-----------------------------
               Total |      29.83      8    0.0002
---------------------------------------------------
```

- Same conclusion, we reject the null

Big picture

- With large samples, robust SEs buy you insurance but with smaller samples it would be a good idea to test for heteroskedasticity
- Of course, with small samples, the power of the heteroskedasticity test is itself compromised
- No hard rules. Researchers follow different customs; some always add the robust option (I don't)
- **Careful with likelihood ratio tests in the presence of heteroskedasticity**
- Stick to robust F tests to compare nested model (use the test command in Stata)

Summary

- Robust SEs are asymptotically valid even if no heteroskedasticity
- Always suspect unequal variance; very common
- Taking the log transformation may help
- Next class, dealing with unequal variance when we know the source: weighted models
- Weighted models for dealing with heteroskedasticity is sort of old fashioned. I do want to cover weighted models because they are used a lot in survey data analysis and lately in propensity scores