CSE 431/531: Algorithm Analysis and Design — Fall 2021

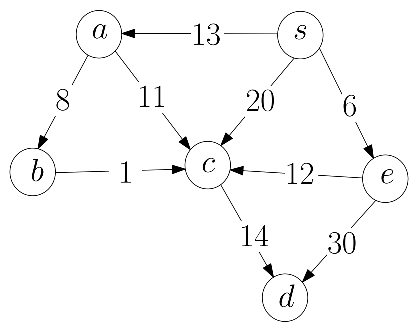
Homework 5

Instructor: Shi Li — **Deadline: 11/7/2021**

Your Name: ______________________ Your Student ID: ________________

Problems	1	2	3	Total
Max. Score	24	24	32	80
Your Score				

Problem 1 Consider the following directed graph G with non-negative edge weights. Use Dijkstra's algorithm to compute the shortest paths from s to all other vertices in G.



You just need to fill the following table.

iteration i	vertex added to S in iteration i	a		b		c		d		e	
		d	π	d	π	d	π	d	π	d	π
1	s	13	s	∞	$\perp$	20	s	∞	$\perp$	6	s
2	e	13	s	∞	$\perp$	18	e	36	e	.	.
3	a	.	.	21	a	18	e	36	e	.	.
4	c	.	.	21	a	.	.	32	c	.	.
5	b	.	.	.	.	.	.	32	c	.	.
6	d	.	.	.	.	.	.	.	.	.	.

Table 1: Dijkstra's algorithm for Shortest Paths

Recall that in the algorithm, we maintain a set S of vertices. For a vertex $v \in S$, $d[v]$ is the length of the shortest path from s to v, and $\pi[v]$ is the parent of v in the shortest path tree. For a vertex $v \notin S$, we have $d(v) = \min_{u \in S:(u,v)\in E} (d(u) + w(u,v))$, and $\pi(v)$ is the vertex $u \in S$ such that $d(v) = d(u) + w(u,v)$. When $d(v) = \infty$, we set $\pi(v) =$ "$\perp$".

You can also fill in the actual d and π values (instead of using ".") in the table for vertices not in S.

Problem 2 Consider the minimum spanning tree problem over the graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}_{\geq 0}$. Assume all the weights are distinct. Let T be the minimum spanning tree of $G = (V, E)$ w.r.t the weight function w. State if each of the following statements is correct or not. If your answer is no for some statement, you need to give a counter example.

(2a) Let C be a cycle in G and e^* be the lightest edge on C. Then e^* is in T.

No. Consider a graph with 4 vertices a, b, c, d, and edge weights $w(a, b) = 1, w(a, c) = 2, w(a, d) = 3, w(b, c) = 11, w(b, d) = 12, w(c, d) = 13$. The minimum spanning tree contains edges $(a, b), (a, c)$ and (a, d). (b, c) is the lightest edge on the cycle (b, c, d), but it is not in the MST.

(2b) Let C be a cycle in G and e^* be the heaviest edge on C. Then e^* is not in T.

Yes.

(2c) Let $U \subsetneq V, U \neq \emptyset$ be a proper subset of V and e^* be the lightest edge in E between U and $V \setminus U$. Then e^* is in T.

Yes.

(2d) Let $U \subsetneq V, U \neq \emptyset$ be a proper subset of V and e^* be the heaviest edge in E between U and $V \setminus U$. Then e^* is not in T.

No. The simplest example is a graph with just two vertices a and b and a single edge (a, b) of weight 1. Then (a, b) is the heaviest edge between $\{a\}$ and $\{b\}$, but it is in the MST.

If your answer is "no" for a question, you need to give a counter example. (If your answer is yes, then saying "yes" is sufficient.)

Problem 3 Suppose *there is* a negative cycle over a directed graph $G = (V, E)$ with edge weights $w : \mathbb{E} \to \mathbb{R}$. Show that for any array $d : V \to \mathbb{R}$ over vertices, there exists some edge $(u, v) \in E$ such that $d(u) + w(u, v) < d(v)$.

(So if we do not put an upper bound on the number of iterations the Bellman-Ford algorithm runs, it will run forever when there is a negative cycle.)

Assume towards contradiction that there is some $d : V \to \mathbb{R}$ such that for every $(u, v) \in E$ we have $d(u) + w(u, v) \geq d(v)$. Let $C = (v_1, v_2, \cdots, v_t)$, $t \geq 2$, be the negative cycle in G. So $w(v_1, v_2) + w(v_2, v_3) + \cdots + w(v_{t-1}, v_t) + w(v_t, v_1) < 0$.

By the assumption, we have

$$\begin{aligned}
d(v_1) + w(v_1, v_2) &\geq d(v_2) \\
d(v_2) + w(v_2, v_3) &\geq d(v_3) \\
&\cdots\cdots \\
d(v_{t-1}) + w(v_{t-1}, v_t) &\geq d(v_t) \\
d(v_t) + w(v_t, v_1) &\geq d(v_1)
\end{aligned}$$

Adding all the inequalities, we have

$$\begin{aligned}
&\big(d(v_1) + d(v_2) + \cdots + d(v_t)\big) + w(v_1, v_2) + w(v_2, v_3) + \cdots + w(v_{t-1}, v_t) + w(v_t, v_1) \\
&\geq d(v_1) + d(v_2) + \cdots + d(v_t),
\end{aligned}$$

which is equivalent to $w(v_1, v_2) + w(v_2, v_3) + \cdots + w(v_{t-1}, v_t) + w(v_t, v_1) \geq 0$. This contradicts that C is a negative cycle. Therefore the assumption does not hold.