The Art of Computer Simulation

Akira Kageyama
Graduate School of System Informatics, Kobe University

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C3-101 9:40 - 10:35

"The Art of Doing Science and Engineering" by Richard W. Hamming

Hamming code,

Hamming window,

Hamming distance,

etc.

"The Art of Doing Science and Engineering" by Richard W. Hamming

- Contents

  ...

  Chap 18 Simulation I

  Chap 19 Simulation II

  Chap 20 Simulation III

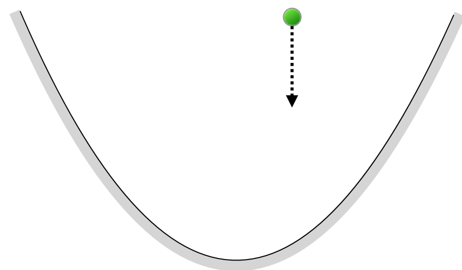
  ...

"A simulation is the answer to the question, "What if...?"

R. Hamming

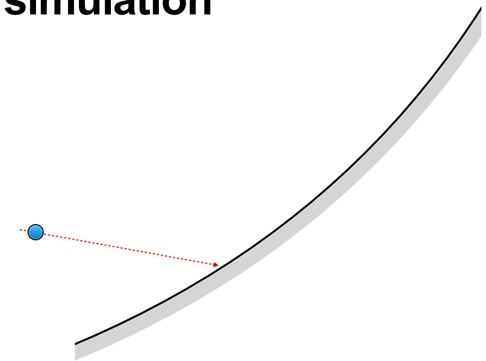
Let's make a simple simulation

- What if we drop an elastic ball onto a parabola-curved floor?



Let's make a simple simulation

- Collision detection is a key.
  - Video games



Let's make a simple simulation

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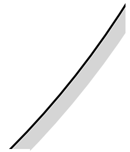
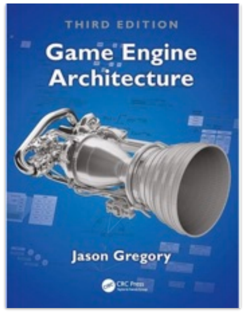
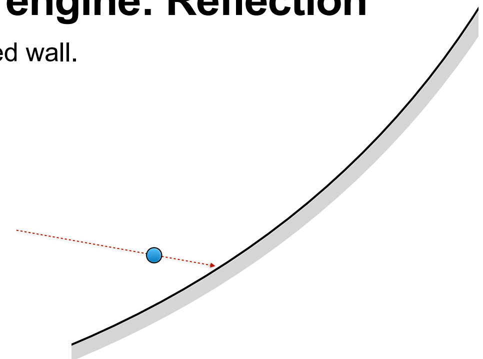




An element of game engine: Reflection

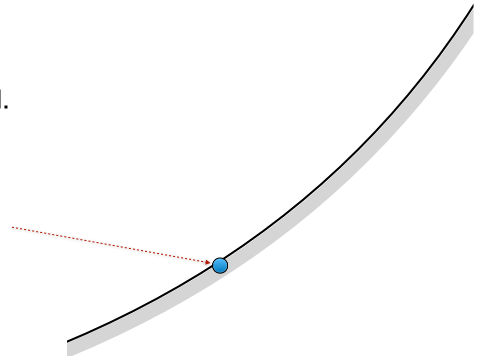
- A particle located outside a curved wall.



Reflection

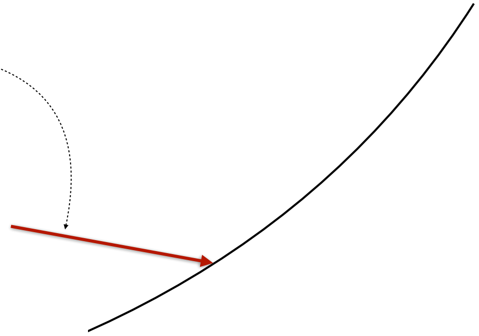
- It impinges on the wall.
  - It (slightly) invades into the wall.

    → Collision



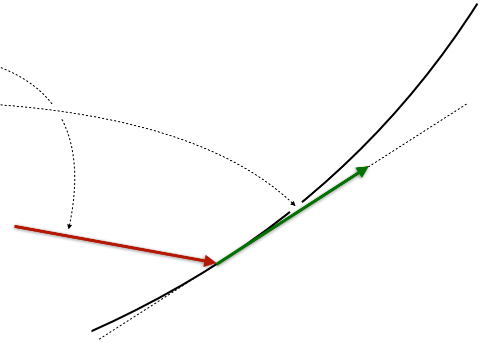
Reflection

- Incoming velocity vector



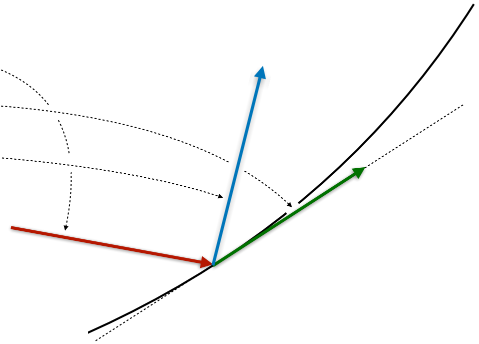
Reflection

- Incoming velocity vector
- Tangential vector



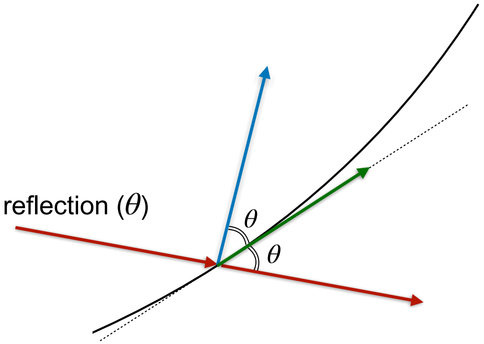
Reflection

- Incoming velocity vector
- Tangential vector
- Reflected velocity vector



Reflection

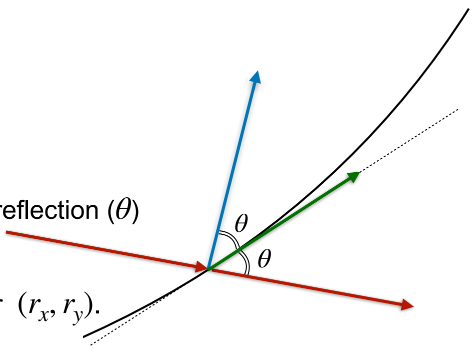
- Incoming velocity vector
- Tangential vector
- Reflected velocity vector
- Angle of incidence (θ) = angle of reflection (θ)



Reflection

- Incoming velocity vector
- Tangential vector
- Reflected velocity vector
- Angle of incidence (θ) = angle of reflection (θ)
- Reflected (blue) vector (b_x, b_y) is 2θ rotation of incident (red) vector (r_x, r_y).

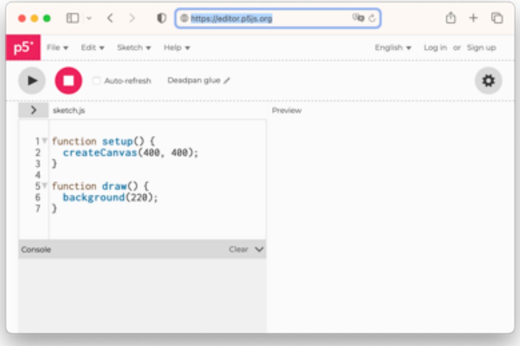
$$\begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} r_x \\ r_y \end{pmatrix}$$



Let's try with P5.js on Web Editor

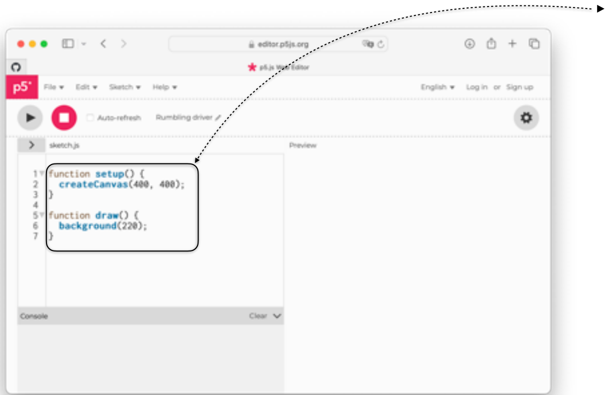
`https://editor.p5js.org/`

N A



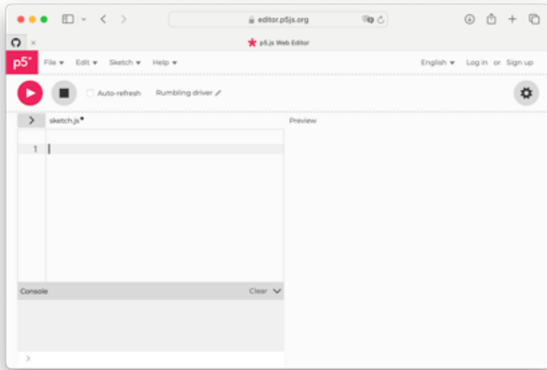
Q k _ p r   N f m l c

Let's try with P5.js Web Editor



- Clear the default code and leave this area blank by select all & delete.

Let's try with P5.js Web Editor



- Preparation is done.

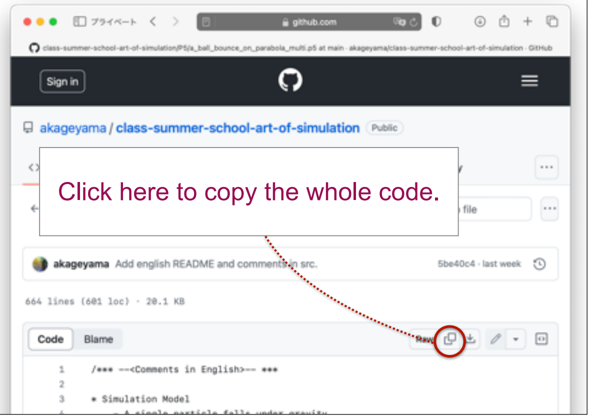
Get a sample code

https://tinyurl.com/kobe-sc01

17

Get a sample code

https://tinyurl.com/kobe-sc01



Get a sample code

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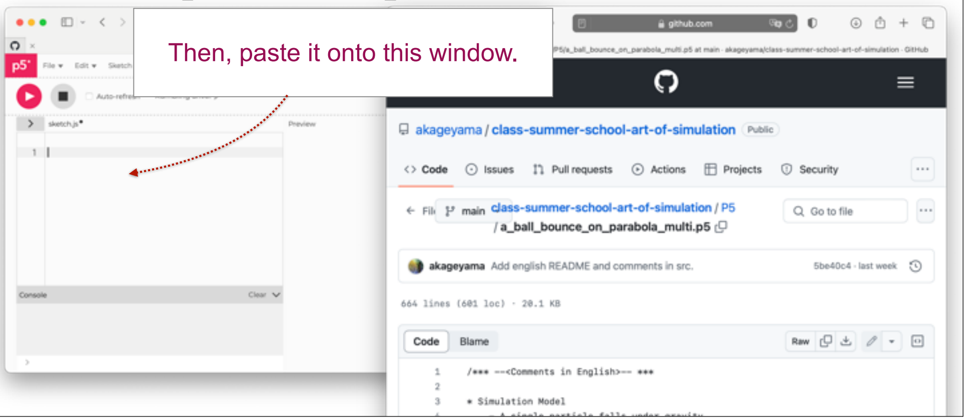
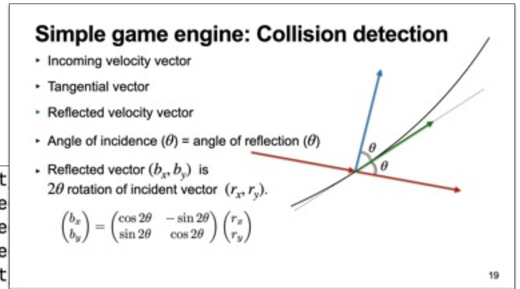


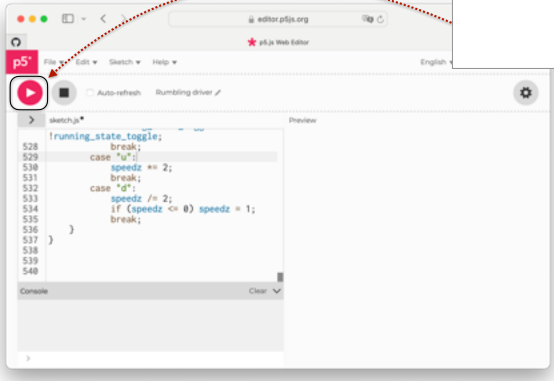
Reflection part




```
let dot_product
    vecIx_norme
    vecIy_norme
let angle_betwe
    dot_product
);
let angle_for_reflection =
    2 * angle_between_vecI_normed_and_vecT_normed;
let cos_angle = cos(angle_for_reflection);
let sin_angle = sin(angle_for_reflection);
let vecRx = cos_angle * vecIx - sin_angle * vecIy;
let vecRy = sin_angle * vecIx + cos_angle * vecIy;
ball.vx = vecRx;
ball.vy = vecRy;
```

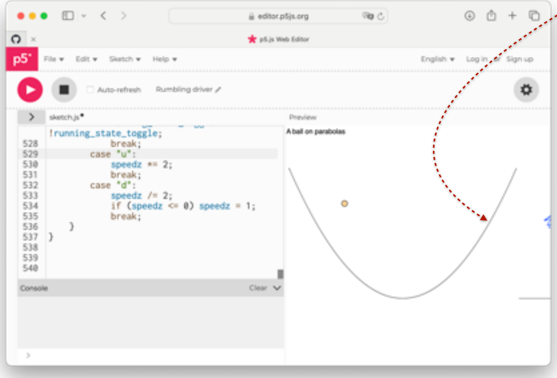
20

Sample code



Click here to start.

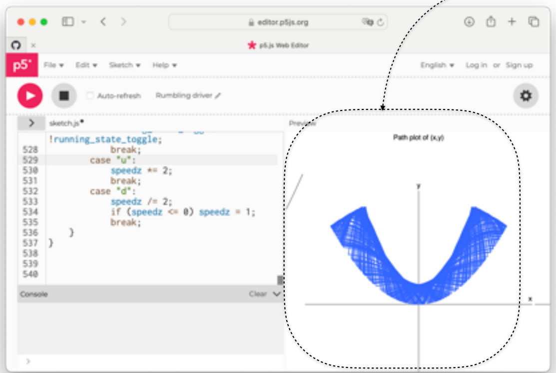
Sample code



$$y = ax^2$$

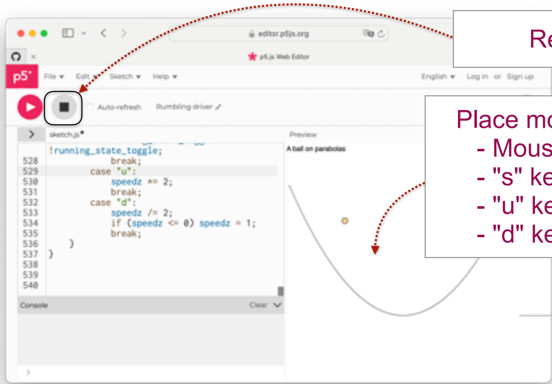
```
let VERTICAL_MARGIN = 75;
let HORIZONTAL_MARGIN = 5;
let PARABOLA_FACTOR_A = 0.5;
let timez = 0.0;
let step = 0;
let dt = 0.001;
let running_state_toggle = true;
let x_coord_min = -2.0;
let x_coord_max = 2.0;
let y_coord_min = -2.0;
let y_coord_max = 2.0;
```


Let's try with P5.js Web Editor https://editor.p5js.org/



- This Preview Window is wide.
- You can scroll right.
- This part shows the particle trajectory.

Usage



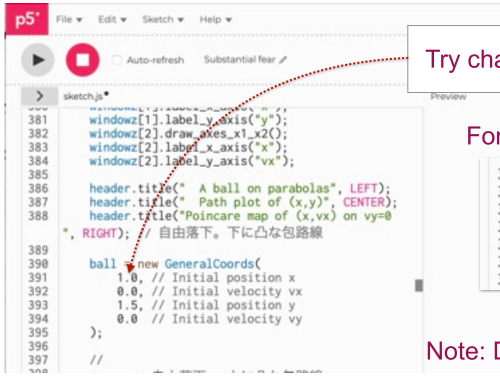
Reset.

Place mouse cursor in this window and ...
- Mouse click → Pause / Restart
- "s" key → Pause / Restart
- "u" key → Speed up
- "d" key → Speed down

Code explanation

Comments

Numerical experiments



Try changing the initial conditions.

For example,

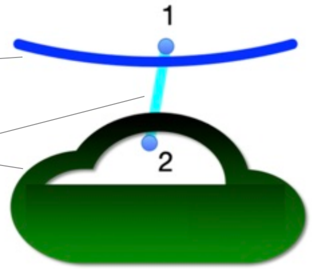
```
389
390     ball = new GeneralCoords(
391         0.0, // Initial position x
392         2.0, // Initial velocity vx
393         1.5, // Initial position y
394         0.0  // Initial velocity vy
395     );
```

Note: Do not place the ball blow the parabola.

Simulation of chaotic motion

Problem setting

- Bent metal rail
- A heavy object with a handle
- Connected by a rubber-rope
- Both ends (1 and 2) are pulleys

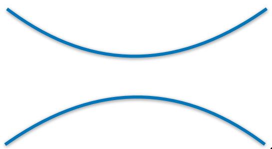


- The pulleys will show a kind of oscillating motion driven by the tension force.
- Problem: How do they move?

Simplification

> *"Everything should be made as simple as possible, but not simpler."*
>
> A. Einstein

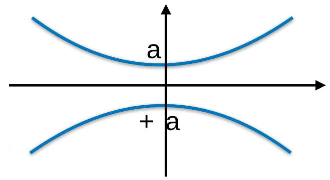
- No twisting motion (motion in a plane, i.e., 2-D simulation)
- Same pulleys.
- No friction.
- Rubber rope is a linear spring with no mass.
- The cable and handle are fixed.
- Symmetric parabolas.



Mathematical model

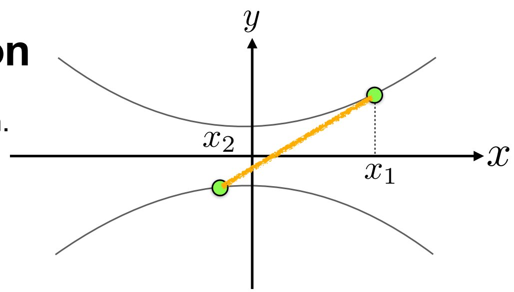
- m : Mass of Pulley 1 and pulley 2
- k : Spring constant of the rubber-rope. (Natural length is zero.)
- $2c$: Closest distance between the cable and handle.

$$y(x) = \begin{cases} y_1 = +x^2/c + c \\ y_2 = -x^2/c - c \end{cases}$$



Equations of motion

- See Appendix for the derivation.



$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = v_1,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = v_2,$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{-1}{4x_1^2 + c^2}\left\{4\,x_1\,v_1^2 + \frac{kc^2}{m}\left(x_1 - x_2 + 2\,x_1\frac{x_1^2 + x_2^2 + 2c^2}{c^2}\right)\right\},$$

$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{-1}{4x_2^2 + c^2}\left\{4\,x_2\,v_2^2 + \frac{kc^2}{m}\left(x_2 - x_1 + 2\,x_2\frac{x_1^2 + x_2^2 + 2c^2}{c^2}\right)\right\}.$$

Before you compute...

- Dimensional analysis
- Linear analysis
- Symmetry analysis
- Conservation quantities

Dimensional analysis

- This system contains three parameters, c, k, and m.
- k and m appear always in the form k/m.
- The time scale of the system is determined by $\omega = \sqrt{\dfrac{k}{m}}$.
  - No other quantity with the dimension of time (measured by second).
- If you can find an oscillation solution in this problem under a specific circumstance, its frequency would be $\mathcal{O}(\omega)$.

Variable normalization

- We re-scale positions by c.

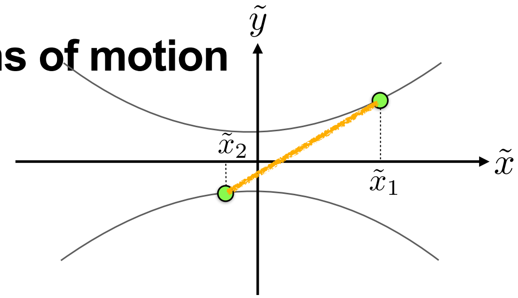
$$x_i = c\,\tilde{x}_i$$

$$v_i = c\,\tilde{v}_i$$

$\tilde{x}$: Non-dimensional position

$\tilde{v}_i$: Re-scaled velocity

Re-scaled equations of motion



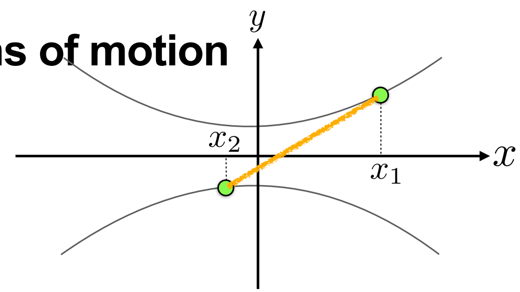
$$\frac{\mathrm{d}\tilde{x}_1}{\mathrm{d}t} = \tilde{v}_1,$$

$$\frac{\mathrm{d}\tilde{x}_2}{\mathrm{d}t} = \tilde{v}_2,$$

$$\frac{\mathrm{d}\tilde{v}_1}{\mathrm{d}t} = \frac{-1}{1+4\tilde{x}_1^2}\left\{4\,\tilde{x}_1\,\tilde{v}_1^2 + \omega^2\big(\tilde{x}_1 - \tilde{x}_2 + 2\,\tilde{x}_1(\tilde{x}_1^2+\tilde{x}_2^2+2)\big)\right\},$$

$$\frac{\mathrm{d}\tilde{v}_2}{\mathrm{d}t} = \frac{-1}{1+4\tilde{x}_2^2}\left\{4\,\tilde{x}_2\,\tilde{v}_2^2 + \omega^2\big(\tilde{x}_2 - \tilde{x}_1 + 2\,\tilde{x}_2(\tilde{x}_1^2+\tilde{x}_2^2+2)\big)\right\}.$$

Re-scaled equations of motion



$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = v_1,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = v_2,$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{-1}{1+4x_1^2}\left\{4\,x_1\,v_1^2 + \omega^2\big(x_1 - x_2 + 2\,x_1(x_1^2+x_2^2+2)\big)\right\},$$

$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{-1}{1+4x_2^2}\left\{4\,x_2\,v_2^2 + \omega^2\big(x_2 - x_1 + 2\,x_2(x_1^2+x_2^2+2)\big)\right\}.$$

Linear analysis

- If x_i is small, then we can ignore nonlinear terms of x_i.

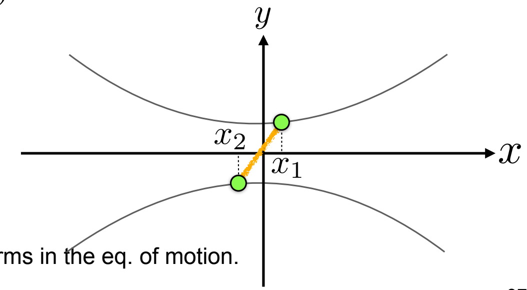
  Example:

  When $(x_1, x_2) = (0.1, -0.05)$

  $|x_1|^2 = 0.01$

  $|x_2|^2 = 0.0025$

  $|x_1 x_2| = 0.005$



- Linearization ' Leave only linear terms in the eq. of motion.

Linear analysis

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = v_1,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = v_2,$$
$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{-1}{1+\cancel{4x_1^2}}\left\{4\cancel{x_1 v_1^2} + \omega^2\left(x_1 - x_2 + 2\,x_1\left(\cancel{x_1^2 + x_2^2} + 2\right)\right)\right\},$$
$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{-1}{1+\cancel{4x_2^2}}\left\{4\cancel{x_2 v_2^2} + \omega^2\left(x_2 - x_1 + 2\,x_2\left(\cancel{x_1^2 + x_2^2} + 2\right)\right)\right\}.$$

- G e l m p c l m l j g l c _ p r c p k q ,

Linear analysis

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = v_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = v_2$$
$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = -\omega^2\left(5x_1 - x_2\right)$$
$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = -\omega^2\left(5x_2 - x_1\right)$$

Linear analysis

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = v_1$$
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$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = -\omega^2\left(5x_2 - x_1\right)$$

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} = -\omega^2\left(5x_1 - x_2\right)$$
$$\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} = -\omega^2\left(5x_2 - x_1\right)$$

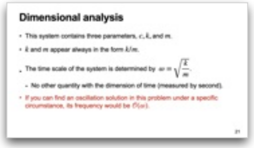
Linear analysis

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -\omega^2 (5x_1 - x_2)$$
$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -\omega^2 (5x_2 - x_1)$$

- Solutions

$$x_1(t) = c_1 \cos(2\omega t + c_2) + c_3 \cos(\sqrt{6}\,\omega t + c_4)$$
$$x_2(t) = c_1 \cos(2\omega t + c_2) - c_3 \cos(\sqrt{6}\,\omega t + c_4)$$

- Harmonic oscillations with frequencies 2ω and $\sqrt{6}\,\omega$


Dimensional analysis

- This system contains three parameters, c, k, and m.
- k and m appear always in the form k/m.
- The time scale of the system is determined by $\omega = \sqrt{\frac{k}{m}}$.
  - No other quantity with the dimension of time (measured by second).
- If you can find an oscillation solution in this problem under a specific circumstance, its frequency would be $\mathcal{O}(\omega)$.

21

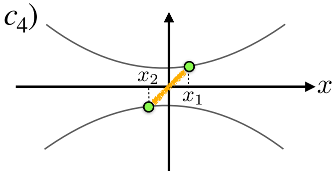
Linear solutions

$$x_1(t) = c_1 \cos(2\omega t + c_2) + c_3 \cos(\sqrt{6}\,\omega t + c_4)$$
$$x_2(t) = c_1 \cos(2\omega t + c_2) - c_3 \cos(\sqrt{6}\,\omega t + c_4)$$

- Case $c_1 = 0$

$$x_1(t) = -x_2(t) = c_3 \cos(\sqrt{6}\,\omega t + c_4)$$

Harmonic oscillation of $\sqrt{6}\,\omega$



Linear solutions

$$x_1(t) = c_1 \cos(2\omega t + c_2) + c_3 \cos(\sqrt{6}\,\omega t + c_4)$$
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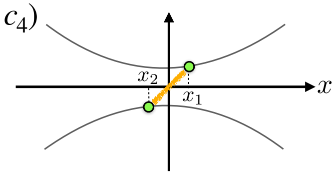
- Case $c_1 = 0$

$$x_1(t) = -x_2(t) = c_3 \cos(\sqrt{6}\,\omega t + c_4)$$

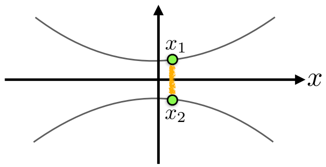
Harmonic oscillation of $\sqrt{6}\,\omega$



- Cse $c_3 = 0$

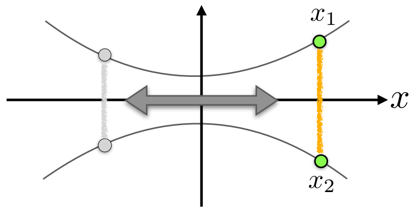
$$x_1(t) = x_2(t) = c_1 \cos(2\omega t + c_2)$$

Harmonic oscillation of 2ω



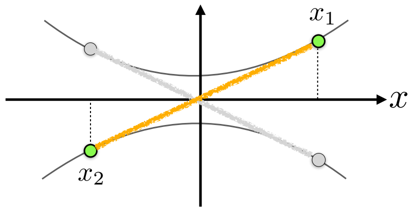
Nonlinear (large amplitude) symmetric solution 1

- If $(x_1, v_1) = (x_2, v_2)$ at $t = 0$



Nonlinear (large amplitude) symmetric solution 2

- if $(x_1, v_1) = -(x_2, v_2)$ at $t = 0$,



Conservation quantity

- Total energy $H = K + U$
- K : Kinetic energy of the two pulleys
- U : Potential energy of the rubber rope

$$H = \frac{m}{2}\{\dot{x}_1^2(1+4x_1^2) + \dot{x}_2(1+4x_2^2)\} + \frac{k}{2}\left\{(x_1 - x_2)^2 + (x_1^2 + x_2^2 + 2)^2\right\}$$

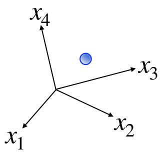
Simulation model

- State of the system
- Discretization
- Visualization

State of the system

- (x_1, x_2, v_1, v_2)
  - x_1, x_2 : Position of point mass 1 and 2
  - v_1, v_2 : Velocity of point mass 1 and 2

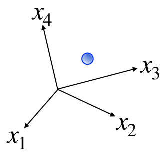
State of the system



- (x_1, x_2, v_1, v_2)
  - x_1, x_2 : Position of point mass 1 and 2
  - v_1, v_2 : Velocity of point mass 1 and 2
- Combine them → General coordinates
- 4-dimensional space ("phase space")
- "State" ⇔ A point in the 4-D space

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

State of the system



- (x_1, x_2, v_1, v_2)
  - x_1, x_2 : Position of point mass 1 and 2
  - v_1, v_2 : Velocity of point mass 1 and 2
- Combine them → General coordinates
- 4-dimensional space ("phase space")
- "State" ⇔ A point in the 4-D space
- "Time development of state"

  ⇔ Directed curve in the 4-D space

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

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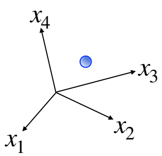

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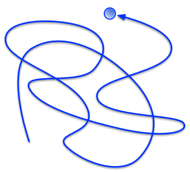


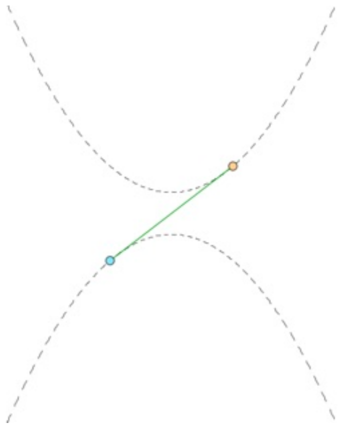
Visualization 1

- Parabola y_1 and parabola y_2

  Dashed curve
- Point mass 1 and 2 $(x, y) = (x_i, y_i)$

  Small circles
- Rubber rope

  Green line



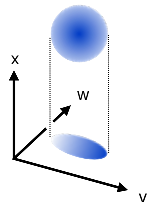
Visualization 2

- We want to ``see'' the time development of the system state,

  i.e., a curve in 4-D phase space.
- Trick: Project 4-D to 2-D (screen).

  "Seeing" is a projection (from 3-D to 2-D).
  - An example of projection:

$$\big(x_1(t), x_2(t), x_3(t), x_4(t)\big) \Rightarrow \big(x_1(t), x_2(t)\big)$$



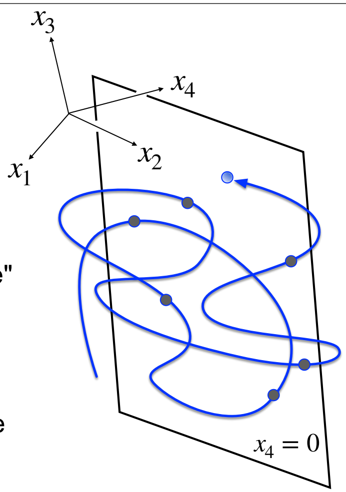
N p m h c a r g m l   d p m k

Visualization 3

- A cross section of a 4-D directed curve
  - For example, take $x_4 = 0$ plane.
- Focus on cross points when the curve crosses the plane ($x_4 = 0$), from "back side" ($x_4 < 0$) to "front side" ($x_4 > 0$).

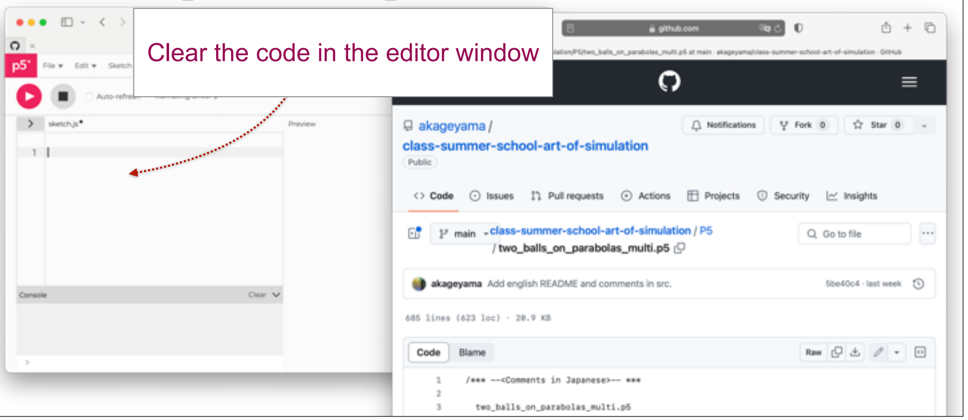
$$(x_1, x_2, x_3, x_4 = 0)$$

- Place a point on, say (x_1, x_3) , on x_1-x_3 plane
- This is called "Poincare map."



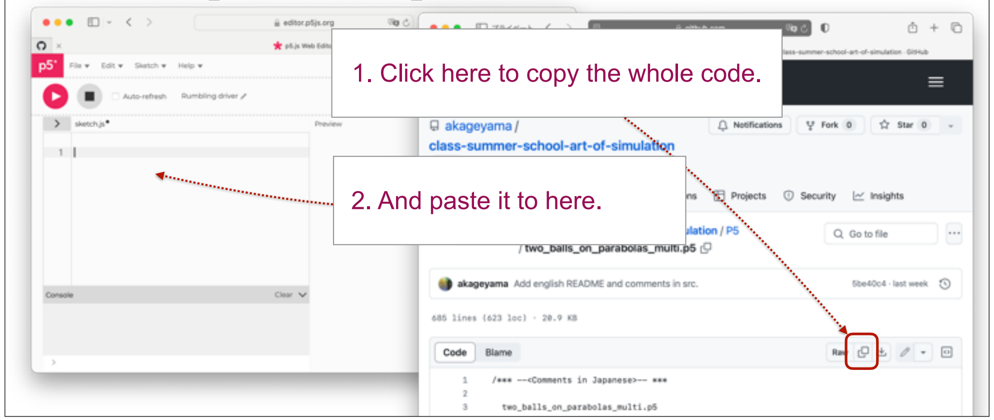
Sample code

https://tinyurl.com/kobe-sc01b



Sample code

https://tinyurl.com/kobe-sc01b



c o s _ r g m l] m d] k m r g m l

- J g l c 3 . / _ l b

```
let x1 = b.x1;
let v1 = b.v1;
let x2 = b.x2;
let v2 = b.v2;
let dx = x1 - x2;
let x1sq = x1 * x1;
let v1sq = v1 * v1;
let x2sq = x2 * x2;
let v2sq = v2 * v2;
let dy = x1sq + x2sq + 2;
let f1 = OMEGA_SQ * (dx + 2 * x1 * dy);
let f2 = OMEGA_SQ * (-dx + 2 * x2 * dy);
db.x1 = v1 * dt;
db.v1 = (-1.0 / (1 + 4 * x1sq)) * (4 * x1 * v1sq + f1) * dt;
db.x2 = v2 * dt;
db.v2 = (-1.0 / (1 + 4 * x2sq)) * (4 * x2 * v2sq + f2) * dt;
```

$$\frac{dx_1}{dt} = v_1, \quad dx_1 = v_1\,dt$$

$$\frac{dx_2}{dt} = v_2,$$

$$\frac{dv_1}{dt} = \frac{-1}{1+4x_1^2}\{4x_1v_1^2 + \omega^2(x_1 - x_2 + 2x_1(x_1^2+x_2^2+2))\},$$

$$\frac{dv_2}{dt} = \frac{-1}{1+4x_2^2}\{4x_2v_2^2 + \omega^2(x_2 - x_1 + 2x_2(x_1^2+x_2^2+2))\}.$$

Initial condition

```
header.title("  Path plot of (x1,x2)", CENTER);
header.title("Poincare map of (x1,v1) on v2=0  ", RIGHT); //// 線形（微小）単振動
//      balls = new GeneralCoords( x_coord_max*0.05,0,  // x1 & v1
//                                 x_coord_max*0.05,0); // x2 & v2
//// 線形（微小）振動
//    balls = new GeneralCoords( x_coord_max*0.1,0,  // x1 & v1
//                              -x_coord_max*0.05,0); // x2 & v2
//// 非線形単一周期運動
//    balls = new GeneralCoords( x_coord_max*0.4,0,  // x1 & v1
//                              -x_coord_max*0.4,0); // x2 & v2
//// 比較的単純な非線形運動
//    balls = new GeneralCoords( x_coord_max*0.4,0,  // x1 & v1
//                              -x_coord_max*0.2,0); // x
// 複雑な運動
balls = new GeneralCoords(
    x_coord_max * 0.4,
    0, // x1 & v1
    -x_coord_max * 0.1,
    0  // x2 & v2
);
```

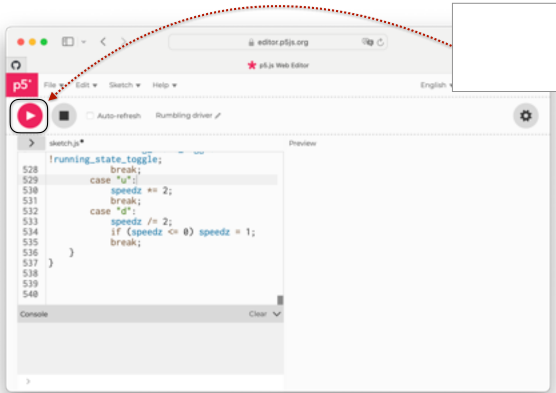
x_1 : x-coord of particle 1

v_1 : velocity of particle 1

x_2 : x-coord of particle 2

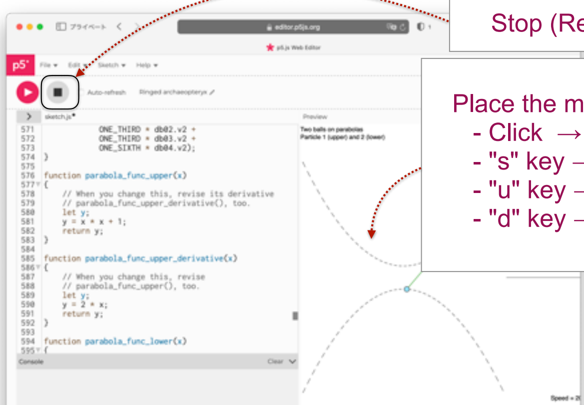
v_2 : velocity of particle 2

Execution of the simulation



Click this button.

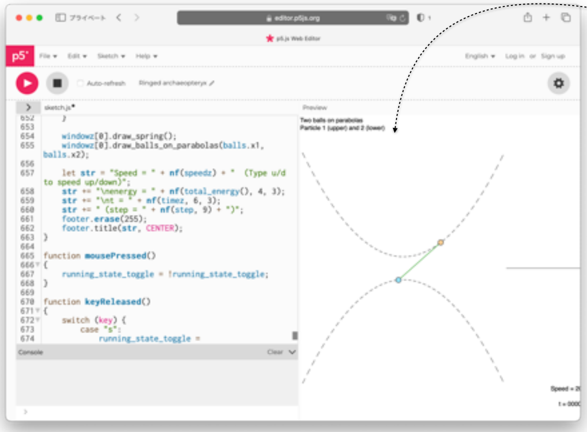
Usage (same as before)



Stop (Reset)

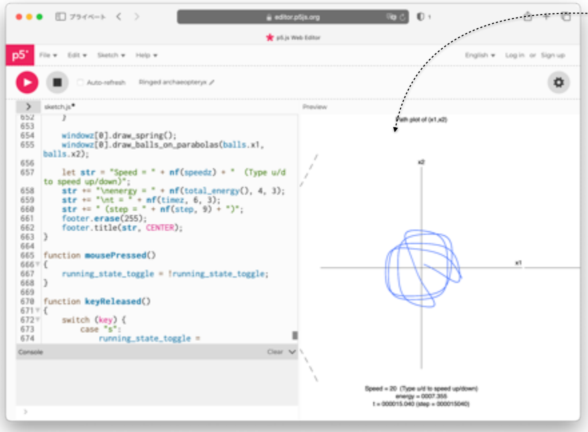
Place the mouse cursor.
- Click → Pause / Restart
- "s" key → Pause / Restart
- "u" key → Speed up
- "d" key → Speed down

Visualization 1



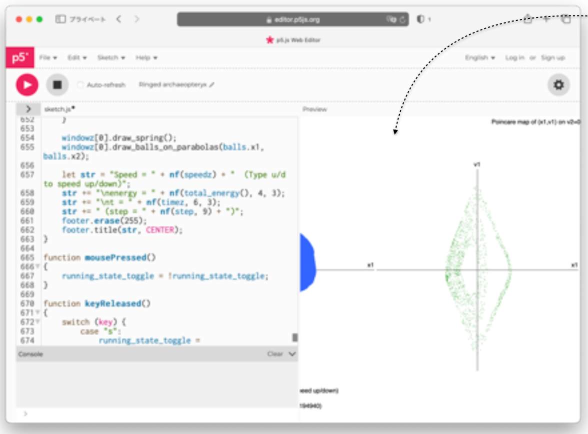
- This window is wide.
- You can scroll right.

Visualization 2

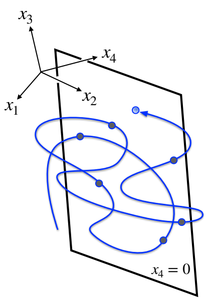


- Trajectory of x-coord of particle 1, x_1, and x-coord of particle 2, x_2.

Visualization 3



- Poincare map (x_1, v_1) when $v_2 = 0$.



Let's try with different initial condition

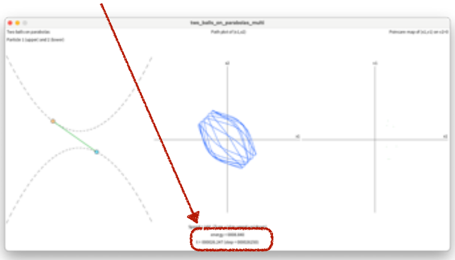
```
    header.title("  Path plot of (x1,x2)", CENTER);
    header.title("Poincare map of (x1,v1) on v2=0  ", RIGHT); //// 線形（微小）単振動
    //      balls = new GeneralCoo          ord_max*0.05,0,         & v1
    //                                      ord_max*0.05,0);        & v2
    //// 線形（微小）振動
    //    balls = new GeneralCoords( x_coord_max*0.1,0,   // x1 & v1
    //                              -x_coord_max*0.05,0); // x2 & v2
    // 非線形単一周期運動
        balls = new GeneralCoords( x_coord_max*0.4,0,  // x1 & v1
                                  -x_coord_max*0.4,0); // x2 & v2
    //// 比較的単純な非線形運動
    //    balls = new GeneralCoords( x_coord_max*0.4,0,   // x1 & v1
    //                              -x_coord_max*0.2,0); // x2 & v2
    // 複雑な運動
    //balls = new GeneralCoords(
    //     x_coord_max * 0.4,
    //    0, // x1 & v1
    //    -x_coord_max * 0.1,
    //    0  // x2 & v2
    //);
```

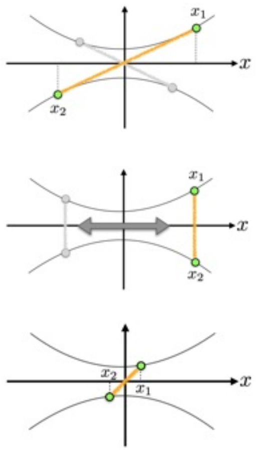
x_1 v_1 x_2 v_2

Comment out

You can try various oscillations

- Symmetric nonlinear oscillation
- Small amplitude oscillation of $\sqrt{6}\,\omega$
- You can check the conservation of total energy.





Summary

- Start with a model as simple as possible, but not simpler.
- Think before compute.
  - Linearized solutions, symmetry, conservation law, etc.
- Cultivate your intuition about the subject by quick experiments.
  - Then, move on to more complex models.
  - Visualization is a key.

> "*The purpose of computing is insight, not numbers.*"
>
> R. Hamming

Appendix: Derivation of the Equations of Motion

In the followings, $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes the dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{a}\cdot\boldsymbol{a}$ is written as $\boldsymbol{a}^2$.

Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ represent the velocities of pulleys 1 and 2, respectively. The total kinetic energy of the system is

$$K = \frac{m}{2}(\boldsymbol{v}_1^2 + \boldsymbol{v}_2^2), \tag{1}$$

Neglecting the gravitation, the potential energy U of the system comes solely from the tension of the rubber band connecting the two pulleys. Assuming the natural length is 0 and the elastic coefficient (spring constant) k, the potential energy is given by

$$U = \frac{k}{2}(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2. \tag{2}$$

Because we are assuming that the two curves near the pulleys of the cable and hanger are specified by one parameter c as

$$y(x) = \begin{cases} y_1 = +x^2/c + c & \text{(cable)}, \\ y_2 = -x^2/c - c & \text{(hanger)}, \end{cases} \tag{3}$$

Appendix: Derivation of the Equations of Motion

$$(\boldsymbol{x}_1-\boldsymbol{x}_2)^2=(x_1-x_2)^2+(y_1-y_2)^2=(x_1-x_2)^2+\left(\frac{x_1^2}{c}+\frac{x_2^2}{c}+2c\right)^2, \tag{4}$$

$$\boldsymbol{v}_i^2=\dot{\boldsymbol{x}}_i^2=\left\{1+\left(\frac{\mathrm{d}y_i}{\mathrm{d}x_i}\right)^2\right\}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2=\left(1+\frac{4x_i^2}{c^2}\right)\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2 \quad (i=1,2). \tag{5}$$

The Lagrangian $L(x_1,x_2,\dot{x}_1,\dot{x}_2)=K-U$ of this system is

$$L=\frac{m}{2}\left\{\dot{x}_1^2\left(1+\frac{4x_1^2}{c^2}\right)+\dot{x}_2\left(1+\frac{4x_2^2}{c^2}\right)\right\}-\frac{k}{2}\left\{(x_1-x_2)^2+\left(\frac{x_1^2}{c}+\frac{x_2^2}{c}+2c\right)^2\right\}. \tag{6}$$

The Lagrange's equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right)=\frac{\partial L}{\partial x_i} \quad (i=1,2), \tag{7}$$

is written as follows:

Appendix: Derivation of the Equations of Motion

$$m\left(1+\frac{4x_1^2}{c^2}\right)\frac{\mathrm{d}\dot{x}_1}{\mathrm{d}t}+8m\frac{x_1\dot{x}_1^2}{c^2}=4m\frac{x_1\dot{x}_1^2}{c^2}-k\left\{x_1-x_2+\frac{2x_1}{c}\left(\frac{x_1^2}{c}+\frac{x_2^2}{c}+2c\right)\right\}, \tag{8}$$

$$m\left(1+\frac{4x_2^2}{c^2}\right)\frac{\mathrm{d}\dot{x}_2}{\mathrm{d}t}+8m\frac{x_2\dot{x}_2^2}{c^2}=4m\frac{x_2\dot{x}_2^2}{c^2}-k\left\{x_2-x_1+\frac{2x_2}{c}\left(\frac{x_1^2}{c}+\frac{x_2^2}{c}+2c\right)\right\}. \tag{9}$$

They are coupled second-order differential equations of x_1 and x_2. We split them into coupled first-order differential equations for the variables x_1, v_1, x_2, v_2 as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=v_1, \tag{10}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t}=v_2, \tag{11}$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}t}=\frac{-1}{4x_1^2+c^2}\left\{4\,x_1\,v_1^2+\frac{kc^2}{m}\left(x_1-x_2+2\,x_1\frac{x_1^2+x_2^2+2c^2}{c^2}\right)\right\}, \tag{12}$$

$$\frac{\mathrm{d}v_2}{\mathrm{d}t}=\frac{-1}{4x_2^2+c^2}\left\{4\,x_2\,v_2^2+\frac{kc^2}{m}\left(x_2-x_1+2\,x_2\frac{x_1^2+x_2^2+2c^2}{c^2}\right)\right\}. \tag{13}$$