$$\begin{aligned}
\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^2 \mathrm{d}x &= \int_0^\infty \left(\int_0^1 e^{-xy}\,\mathrm{d}y\right)^2 \mathrm{d}x \\
&= \int_0^\infty \left(\int_0^1 e^{-xy}\,\mathrm{d}y\right)\left(\int_0^1 e^{-xz}\,\mathrm{d}z\right)\mathrm{d}x \\
&= \int_0^\infty \int_0^1 \int_0^1 e^{-x(y+z)}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \int_0^1 \int_0^\infty e^{-x(y+z)}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^1 \ln(y+1) - \ln(y)\,\mathrm{d}y \\
&= \left((y+1)\ln(y+1) - y - y\ln(y) + y\right)\Big|_0^1 \\
&= \left((y+1)\ln(y+1) - y\ln(y)\right)\Big|_0^1 \\
&= 2\ln(2) + \lim_{y\to 0} y\ln(y) \\
&= 2\ln(2)
\end{aligned}$$