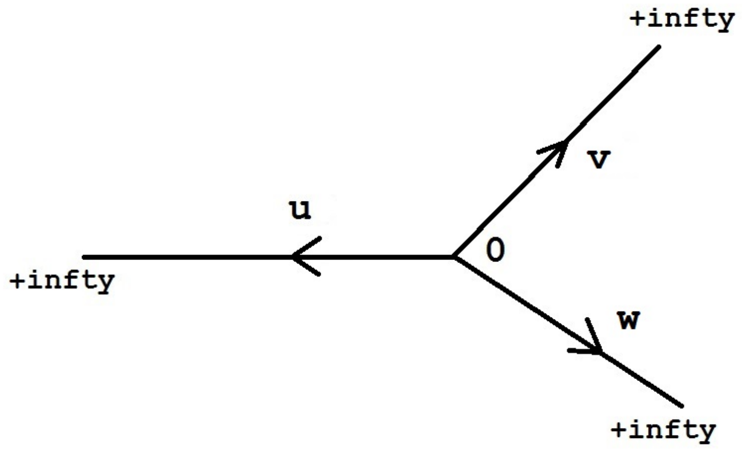
TERM PROJECT by Adilbek Kairzhan

1 Statement of the problem

The problem intended to be solved is Nonlinear Schrodinger Equation on the following star graph, say Γ graph, of Y-junction form:



The NLS equation on the graph Γ is

$$i\frac{\partial U}{\partial t}(x,t) + \frac{\partial^2 U}{\partial x^2}(x,t) + |U(x,t)|^2 U(x,t) = 0,$$

where U is a solution vector defined on $\mathbb{R}^+ \times \mathbb{R}^+$

$$U(x,t) = \begin{bmatrix} u(x,t) \\ v(x,t) \\ w(x,t) \end{bmatrix}$$

where each of the solutions u, v and w is complex-valued and its x variable is defined on the corresponding half-line as shown on the picture above.

All the operators in the NLS equation are applied to U vector componentwise.

The initial and boundary conditions are:

$$u(0,t) = v(0,t) = w(0,t),$$
$$\frac{\partial u}{\partial x}(0,t) + \frac{\partial v}{\partial x}(0,t) + \frac{\partial w}{\partial x}(0,t) = 0,$$
$$U(x,0) = F(x),$$

for some given vector $F(x)$.

Also, we assume $U(x,t) \in H^1(\Gamma)$. Therefore, for our future numerical approach we might say that

$$u(x,t) = v(x,t) = w(x,t) = 0$$

for all $x \geq M$, where M is some positive defined number. This assumption creates a boundary for the Γ graph, so one might assume that the space domain of the vector $U(x,t)$ is the bounded interval $[0, M]$.

2 Numerical approach

The space domain of our problem is bounded, and we will use Collocation approach based on Chebyshev approximation with Gauss-Lobatto-Chebyshev points.

The Chebyshev approximation formula used during the lectures is defined for the interval $[-1, 1]$, so we use a linear transformation

$$L : [-1, 1] \to [0, M],$$

where $L(x) = \frac{M}{2}x + \frac{M}{2} = y \in [0, M]$, to show a possible isomorphism between the initial and required intervals. Therefore, the initial NLS equation

$$i\frac{\partial U}{\partial t}(y,t) + \frac{\partial^2 U}{\partial y^2}(y,t) + |U(y,t)|^2 U(y,t) = 0,$$

where $y \in (0, M)$, is equivalent to

$$i\frac{\partial Z}{\partial t}(x,t) + \frac{4}{M^2}\frac{\partial^2 Z}{\partial x^2}(x,t) + |Z(x,t)|^2 Z(x,t) = 0$$

with $x \in (-1, 1)$. Here, $U(y,t) = U(\frac{M}{2}x + \frac{M}{2}, t) = Z(x,t)$, and the coefficient $\frac{4}{M^2}$ in the latter NLS equation occurs as a result of the Chain Rule. For simplicity, we say that

$$Z(x,t) = \begin{bmatrix} u(x,t) \\ v(x,t) \\ w(x,t) \end{bmatrix}.$$

Remark 1. *The solution functions u, v and w used in Z vector are different from those used in U vector. The decision to use the same notations is made in order to simplify the look of the future steps and calculations.*

Our problem now is to solve the NLS equation for $Z(x,t), x \in [-1, 1]$ with the following initial and boundary conditions:

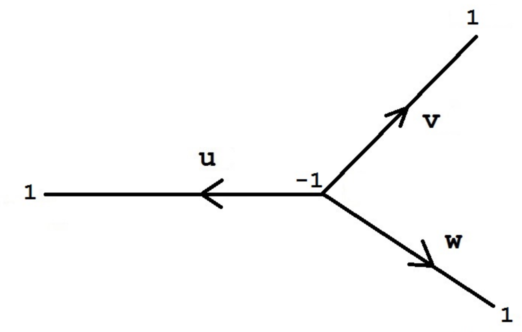
$$u(-1,t) = v(-1,t) = w(-1,t),$$

$$u(1,t) = v(1,t) = w(1,t) = 0,$$

$$\frac{\partial u}{\partial x}(-1,t) + \frac{\partial v}{\partial x}(-1,t) + \frac{\partial w}{\partial x}(-1,t) = 0,$$

$$U(x,0) = F(x),$$

for some given vector $F(x)$.



Consider an approximation of $u(x,t)$ (similarly for $v(x,t)$ and $w(x,t)$) in terms of a truncated Chebyshev series

$$u_N(x,t) = \sum_{k=0}^{N} \hat{u}_k(t) T_k(x),$$

where $T_k(x) = \cos(k \arccos(x))$. The residual $R_N(x,t)$ for $u(x,t)$ in our problem is

$$R_N = i\partial_t u_N + \frac{4}{M^2}\partial_{xx} u_N + |u_N|^2 u_N.$$

Let u_N^n be the approximation of u_N at time $t_n = n\Delta t$ for $n = 0, 1, ...$ Then, the time-discretization of the residual R_N is

$$R_N^n = i\frac{u_N^{n+1} - u_N^n}{\Delta t} + \frac{4}{M^2}\partial_{xx} u_N^{n+1} + |u_N^n|^2 u_N^n.$$

Based on the lecture slides, explicit treatment of the nonlinear term avoids costly iterations, and implicit treatment of the linear term allows to get better stability restrictions on the time step Δt.

The Gauss-Lobatto-Chebyshev collocation points on $[-1, 1]$ interval are $x_j = \cos(\frac{j\pi}{N})$ for $j = 0, ..., N$. Cancelling the residual at these collocation points we have

$$i\frac{u_N^{n+1}(x_j) - u_N^n(x_j)}{\Delta t} + \frac{4}{M^2}\partial_{xx} u_N^{n+1}(x_j) + |u_N^n(x_j)|^2 u_N^n(x_j) = 0.$$

To represent the nodal values of $\partial_{xx} u_N^{n+1}$ in terms of nodal values of u_N^{n+1} we use the Differentiation Matrix $D^{(2)} = \{d_{ml}\}_{1 \le m,l \le N+1}$, which was covered in the lectures. MatLab generates this matrix as $D^{(2)} = DD$, where the matrix D is the first differentiation matrix constructed by the function *cheb(N)* shown in the class.

Let $u_j^n = u_N^n(x_j)$. Then

$$\partial_{xx} u_j^{n+1} = \sum_{l=1}^{N+1} d_{jl} u_{l-1}^{n+1}.$$

Let $b_j^n = |u_j^n|^2 u_j^n$ be the nonlinear term. Then, after all these steps, the above equation of cancellation of the residual is

$$i\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{4}{M^2}\sum_{l=1}^{N+1} d_{jl} u_{l-1}^{n+1} + b_j^n = 0.$$

This is equivalent to

$$u_j^{n+1} + \frac{4\Delta t}{M^2 i}\sum_{l=1}^{N+1} d_{jl}u_{l-1}^{n+1} = u_j^n - \frac{\Delta t}{i}b_j^n.$$

Let $U^n = [u_0^n, u_1^n, ..., u_N^n]^T$ and $B_u^n = [b_0^n, b_1^n, ..., b_N^n]^T$, then we get an algebraic system

$$\Big(\mathbb{I} - i\frac{4\Delta t}{M^2}D^{(2)}\Big)U^{n+1} = U^n + i\Delta t B_u^n, \text{ or}$$

$$\Big(\frac{M^2 i}{\Delta t}\mathbb{I} + 4D^{(2)}\Big)U^{n+1} = \frac{M^2 i}{\Delta t}U^n - M^2 B_u^n.$$

To treat an aliasing error in the nonlinear term b_j^n, we should extend the spectrum as we did in the class for the quadratic nonlinearity. However, we have a cubic nonlinear term, so we should find first the value of the required extension $N' = N'(N)$.

Lemma 1. *Double extension of the spectrum to $N' = 2N+1$ helps to avoid the aliasing errors in the nonlinear term $b_j^n = |u_j^n|^2 u_j^n$.*

Proof.

$$b_j^n = |u_j^n|^2 u_j^n = u_j^n \bar{u}_j^n u_j^n = \sum_{k=0}^{N} \hat{b}_k^n \cos(\frac{k\pi j}{N}).$$

Since $u_j^n = \sum_{k=0}^N \hat{u}_k^n T_k(x_j) = \sum_{k=0}^N \hat{u}_k^n \cos(\frac{k\pi j}{N})$, we have

$$\hat{b}_k^n = \frac{2}{\bar{c}_k N}\sum_{j=0}^{N}\frac{1}{\bar{c}_j} b_j^n \cos(\frac{k\pi j}{N}) =$$

$$= \frac{2}{\bar{c}_k N}\sum_{j=0}^{N}\frac{1}{\bar{c}_j}\Big(\sum_{p=0}^{N}\hat{u}_p^n\cos(\frac{p\pi j}{N})\sum_{q=0}^{N}\bar{\hat{u}}_q^n\cos(\frac{q\pi j}{N})\sum_{r=0}^{N}\hat{u}_r^n\cos(\frac{r\pi j}{N})\Big)\cos(\frac{k\pi j}{N}) =$$

$$= \frac{2}{\bar{c}_k N}\sum_{j=0}^{N}\frac{1}{\bar{c}_j}\sum_{p,q,r=0}^{N}\hat{u}_p^n\bar{\hat{u}}_q^n\hat{u}_r^n\cos(\frac{p\pi j}{N})\cos(\frac{q\pi j}{N})\cos(\frac{r\pi j}{N})\cos(\frac{k\pi j}{N})$$

$$= \frac{2}{\bar{c}_k N}\sum_{p,q,r=0}^{N}\hat{u}_p^n\bar{\hat{u}}_q^n\hat{u}_r^n\sum_{j=0}^{N}\frac{1}{\bar{c}_j}\cos(\frac{p\pi j}{N})\cos(\frac{q\pi j}{N})\cos(\frac{r\pi j}{N})\cos(\frac{k\pi j}{N}).$$

Doing routine calculations, one can prove that

$$\sum_{j=0}^{N}\frac{1}{\bar{c}_j}\cos(\frac{p\pi j}{N})\cos(\frac{q\pi j}{N})\cos(\frac{r\pi j}{N})\cos(\frac{k\pi j}{N}) = 0$$

if $p \pm q \pm r \pm k \neq 2Nm$ for any choice of signs, and some integer m.

Therefore,

$$\hat{b}_k^n = \frac{1}{4\bar{c}_k}\sum_{\substack{p,q,r=0\\ p\pm q\pm r=\pm k}}^{N}\hat{u}_p^n\bar{\hat{u}}_q^n\hat{u}_r^n + \frac{1}{4\bar{c}_k}\sum_{\substack{p,q,r=0\\ p\pm q\pm r=\pm k+2Nm, m\neq 0}}^{N}\hat{u}_p^n\bar{\hat{u}}_q^n\hat{u}_r^n.$$

Notice that the second term represents aliasing errors in our case.

Similarly as in the lecture slides, we extend our spectrum to some N', and so

$$\hat{b}_k^n = \frac{1}{4\bar{c}_k} \sum_{\substack{p,q,r=0 \\ p \pm q \pm r = \pm k}}^{N} \hat{u}_p^n \bar{\hat{u}}_q^n \hat{u}_r^n + \frac{1}{4\bar{c}_k} \sum_{\substack{p,q,r=0 \\ p \pm q \pm r = \pm k + 2N'm, m \neq 0}}^{N} \hat{u}_p^n \bar{\hat{u}}_q^n \hat{u}_r^n.$$

Therefore, since $\hat{u}_p^n = 0$ for all $p > N$ we need (here, $k, q, r \leq N$)

$$\min(p) = \min(2N' \pm k \pm q \pm k) = 2N' - N - N - N > N,$$

so $N' > 2N$. $\square$

Notice that all the results above are also true for solution functions $v(x,t)$ and $w(x,t)$. Therefore, letting $V^n = [v_0^n, v_1^n, ..., v_N^n]^T$, $W^n = [w_0^n, w_1^n, ..., w_N^n]^T$, and B_v^n and B_w^n to be corresponding vector of nonlinear terms, we get the folowing algebraic systems

$$\mathbb{A}U^{n+1} = \frac{M^2 i}{\Delta t} U^n - M^2 B_u^n.$$

$$\mathbb{A}V^{n+1} = \frac{M^2 i}{\Delta t} V^n - M^2 B_v^n.$$

$$\mathbb{A}W^{n+1} = \frac{M^2 i}{\Delta t} W^n - M^2 B_w^n,$$

where $\mathbb{A} = \frac{M^2 i}{\Delta t}\mathbb{I} + 4D^{(2)}$.

Here, the nodal value u_0^n, v_0^n, w_0^n correspond to the values at the boundary point 1 on appropriate edges, and the values u_N^n, v_N^n, w_N^n correspond to the values at point -1, where all three edges meet. Hence, the boundary conditions are:

$$u_0^n = v_0^n = w_0^n = 0,$$

$$u_N^n = v_N^n = w_N^n,$$

$$\partial_x u_N^n + \partial_x v_N^n + \partial_x w_N^n = 0.$$

Now, we need to implement our boundary conditions into the algebraic system. First of all, since

$$u_0^n = v_0^n = w_0^n = 0,$$

we can ignore these elements in the algebraic system (typical for Chebyshev method), and so we remove the first row in each of the algebraic systems above, and to have appropriate matrix dimension, we also remove the first column of $\mathbb{A}$ matrix. Therefore, $U^n = [u_1^n, u_2^n, ..., u_N^n]^T$ for each $n = 0, 1, ...$ and new matrix $\mathbb{A}'$ is obtained from the previous matrix $\mathbb{A}$ by removing first row and first column. Similarly, for V^n and W^n cases.

Now, gathering all the algebraic systems in one, we create a new vector

$$Y^n = [(U^n)^T, (V^n)^T, (W^n)^T]^T$$

and the new algebraic system is

$$\mathbb{C}Y^{n+1} = \frac{M^2 i}{\Delta t} Y^n - M^2 B^n,$$

where $\mathbb{C} = diag(\mathbb{A}', \mathbb{A}', \mathbb{A}')$ is a $3N \times 3N$ matrix and $B^n = [(B_u^n)^T, (B_v^n)^T, (B_w^n)^T]^T$.

To implement the conditions

$$u_N^{n+1} = v_N^{n+1} = w_N^{n+1},$$
$$\partial_x u_N^{n+1} + \partial_x v_N^{n+1} + \partial_x w_N^{n+1} = 0$$

to the algebraic system, we notice that u_N^n does not obey the NLS equation ($x_N = 1$) and so we should replace the entries of the N-th row of the algebraic system with the new entries, which will give us the equation

$$u_N^{n+1} - v_N^{n+1} = 0.$$

So the entry $\{\mathbb{C}\}_{N,N} = 1$ and $\{\mathbb{C}\}_{N,2N} = -1$, all the other entries of N-th row are zeros. The corresponding N-th entry of the RHS vector in the algebraic system should be 0.

Similarly, we take $\{\mathbb{C}\}_{2N,2N} = 1$ and $\{\mathbb{C}\}_{2N,3N} = -1$, all other entries of $2N$-th row are zeros, to get

$$v_N^{n+1} - w_N^{n+1} = 0.$$

The corresponding $2N$-th entry of the RHS vector in the algebraic system should be 0.

To get the last condition

$$\partial_x u_N^{n+1} + \partial_x v_N^{n+1} + \partial_x w_N^{n+1} = 0$$

we first notice $\partial_x u_N^{n+1}$ might be obtained in terms of the nodal values $u_1^{n+1}, u_2^{n+1}, ..., u_N^{n+1}$ multiplying them by the last row of the Differentiation matrix $D = cheb(N)$ (similarly for v and w). Therefore, we replace the last row of $\mathbb{C}$ in the algebraic system with the three copies of last row of D. The corresponding $3N$-th entry of the RHS vector in the algebraic system should be 0.

The construction of the matrix, and the obtained discretized equation for moving to the new time level is given in MatLab code inside of *Solve˙NLS* function. Time stepping is performed using the first order explicit/implicit Euler scheme.

3 Computational results

To determine the initial condition of our problem, we use one of the most known solutions of the NLS equation on the star graph Γ called standing waves which is a stationary solution. The stationary solution formula on the interval $[-1, 1]$ is

$$u(x,t) = v(x,t) = w(x,t) = \sqrt{2} sech(\frac{M}{2}x + \frac{M}{2})e^{it}.$$

Therefore, the initial condition is

$$u(x,0) = v(x,0) = w(x,0) = \sqrt{2} sech(\frac{M}{2}x + \frac{M}{2}).$$

Notice that the exact solution is stationary, so we have

$$|u(x,t)| = |u(x,0)|.$$

We can plot the graph of the absolute values of the approximate solutions, and we can analyse the convergence of the approximate solution by plotting L_∞ norm of the relative errors with respect to number of collocation points.

So, $U^n = [u_1^n, ..., u_N^n]^T$ is the vector of nodal values of the approximation of u at time $t_n = n\Delta t$. Let

$$|U^n| = [|u_1^n|, ..., |u_N^n|]^T.$$

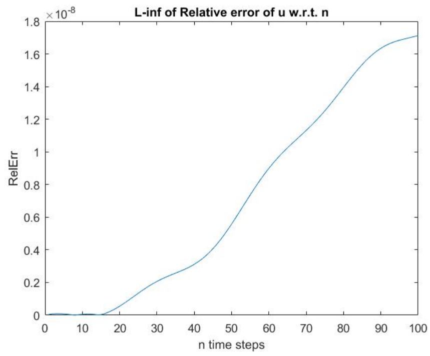
Then, since $|u(x,t)| = |u(x,0)|$ the vector of the relative errors of u at nodes at the time t_n is

$$RelErr_n(u) = [\frac{|u_1^n| - |u_1^0|}{|u_1^0|}, ..., \frac{|u_N^n| - |u_1^0|}{|u_1^0|}]^T.$$

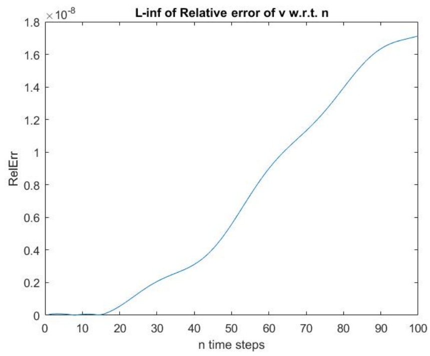
The L_∞ norm of the relative errors of u at time t_n is

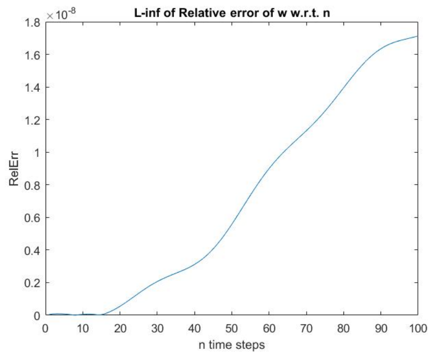
$$\|RelErr_n(u)\|_\infty.$$

The graph of $\|RelErr_n(u)\|_\infty$ as a function of n for $N = 129$ Gauss-Lobatto-Chebyshev collocation points is given by *Figure 4* in the MatLab code, see below:

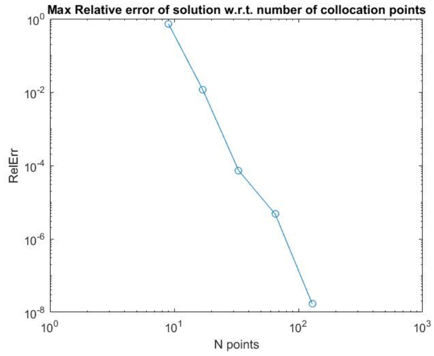


Similar graphs representing the relative errors for the the solution functions v and w are given below (*Figures 5 and 6* in MatLab code):





As we can see the error of the solutions on the edges is very small. To see the type of convergence of the full approximate solution, we plot LogLog graph of L_∞ norm of the relative errors on the all edges at all time steps with respect to number of collocation points, see *Figure 7* in the MatLab code:



It can be seen from the graph that relative error is decaying very fast, and this actually show the spectral convergence of the method.

Remark 2. *The solution graphs of u, v and w with respect to spatial variable x and time t are given in the Figures 1,2 and 3 in the MatLab code. It can be seen that the approximate solution is stationary, since the absolute values of solutions does not change in time.*

4 Conclusion

We used the Chebyshev Collocation method with Gauss-Lobatto˙Chebyshev collocation points to approximate the Nonlinear Schrodinger equation on the Y-junction graph as shown in the Introduction. The second section of this report presented the transformation of the initial continuous problem to the discretized algebraic system. The aliasing error of the nonlinear cubic term was treated by the method, similar to one from the class lectures, by extending the spectrum from N to $N' = 2N$. The computational results tested on the stationary solution of the problem has shown that the method gives a spectral convergence, and, in particular, the relative error for $N = 129$ collocation points is around 10^{-8}, which is very small.